General Analysis of Horizontal Merger

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A new method of proof is presented for the existence of a unique Cournot equilibrium in oligopoly with a merged entity and independent firms. The method is then applied to derive price-enhancing effect of an increase in the number of merged firms. On the basis of the existence proof and stability condition, a sufficient condition is derived for an increase in the number of merged firms to be profitable. This condition is illustrated for a simple case of linear demand and identical quadratic cost functions. Finally, a numerical example is given. (*JEL* Classification: L13)

I. Introduction

Quite a few economists have analyzed the economic effects of horizontal merger in relationship to anti-trust policies. Salant, Switzer and Reynolds (1983) have shown that if firms have identical constant average costs and behave as Cournot oligopolists, and if, in addition, the demand function is linear, merger is profitable only in two firm industry in the sense that profits per firm after merger are larger than the sum of two firm's profits before merger. Merger in their sense is nothing but a decrease in the number of identical firms. Hence all firms remain identical after merger. Perry and Porter (1985) have considered two cases of horizontal merger. In one case an industry consists of identical oligopolists who behave as dominant firms, and of a competitive fringe. Oligopolists as well as a competitive fringe are assumed to have capital stock and merger occurs if a fraction of the competitive

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fringe is transformed into a large firm having identical fixed capital as the oligopolists. In the second case two groups of oligopolists exist and all large oligopolists in one group have twice as much capital stock as all small oligopolists in another group. Merger in this case is taken to be an integration of two small oligopolists into one large oligopolist. Perry and Porter have derived profitability conditions for merger assuming a linear demand function, and quadratic cost functions which depend on firm's output and capital stock. However, they have not analyzed the welfare effects of merger, welfare being defined as the sum of all firm's profits and the consumer surplus. Farrell and Shapiro (1990) have examined the price and welfare effects of horizontal merger in a model with general demand and cost functions. They have proven that if merger generates no "synergies", then the industry output decreases and the price rises. They have also shown a possibility of merger to lead to lower price. Besides they have derived a sufficient set of conditions for merger to be profitable as well as welfare-enhancing. Levin (1990) also has derived a sufficient condition for merger to increase price as well as welfare, assuming that all firms have constant average costs and that outsiders act as Cournot oligopolists after merger, regardless of the merged entity's behavior, which needs not be Cournotian. Earlier contributions by Szidarovszky and Yakowitz (1982) and Fluck, Okuguchi and Szidarovszky (1987) have made numerical computations regarding the effects of profitability of merger on the basis of simple demand and cost functions.

In this paper we will analyze output and profitability effects of horizontal merger in Cournot oligopoly without product differentiation assuming general demand and cost functions. Except Perry and Porter (1985), the authors mentioned above have not analyzed the effects of number of firms in a merger. In our paper we will analyze output and profitability effects in relationship to an increase in the number of merged firms. To do so we will present in section II a new way of determining the equilibrium industry output and the total output of insiders.[1] In section III we will be concerned with profitability effects of an increase in the number of merged firms. Our result will be diagrammatically illustrated for a simple case of linear demand and identical quadratic cost functions. Section IV concludes.

1For somewhat related approach in other context, see Okuguchi (1990, 1993)

II. The Equilibrium Industry and Merged Entity's Output

Let there be n firms in oligopoly product differentiation. Firm i's profit π_i is defined by

$$\pi_i \equiv x_i f(\sum_j x_j) - C_i(x_i), \quad i = 1,2,\ldots,n, \tag{1}$$

where if p is the price of a homogeneous good, $p = f(\Sigma x_j)$ with $f' < 0$ is the inverse demand function, and x_i and $C_i(x_i)$ are firm i's output and cost function, respectively. Let I be a subset of $N = \{1,2,\ldots,n\}$ and J be its complement. I or J may be an empty set. Suppose that firms in I (outsiders), independently maximize their profits and those in J (insiders) are merged into a single entity. The merged entity's profit π_J is defined as

$$\pi_J \equiv \sum_{j\in J}(x_j f(\sum_k x_k) - C_j(x_j)). \tag{2}$$

The first and second order conditions for individual profit maximization under the Cournot behavioristic assumption are:

$$\frac{\partial \pi_i}{\partial x_i} = f(Q) + x_i f'(Q) - C_i'(x_i) = 0, \quad i \in I, \tag{3}$$

$$\frac{\partial^2 \pi_i}{\partial x_i^2} = f'(Q) + x_i f''(Q) + f'(Q) - C_i''(x_i) < 0, \quad i \in I, \tag{4}$$

where $Q \equiv \sum_j x_j$ is the industry output.

We assume the following.

***Assumption* 1 (A1):** $f' + Qf'' < 0$.

***Assumption* 2 (A2):** $f' < C_i''$, $\quad i \in N$.

If A1 holds, the marginal revenues of any individual firm of the merged entity and of the industry as a whole all decrease if respective outputs(x_i, $Q_J \in \sum_{j\in J} x_j$, and Q) increase. A2 is satisfied, for example, if the demand function is linear and if, in addition, the marginal costs are either constant or increasing. The second order condition (4) holds under A1 and A2. Solving (3) with respect to x_i, we have

$$x_i = \varphi_i(Q), \quad i \in I \tag{5}$$

such that

$$\varphi_i' = -\frac{f' + x_i f''}{f' - C_i''} < 0, \quad i \in I. \tag{6}$$

Assuming that the merged entity behaves also as a Cournot oligopolist, the first order condition for profit maximization for the merged entity is given by

$$\frac{\partial \pi_J}{\partial x_j} = f(Q) + x_j f'(Q) - C_j'(x_j) + (Q_J - x_j) f'(Q) = 0, \quad j \in J. \tag{7}$$

Since the Hessian matrix of π_J must be negative definite for the second order condition to hold, the following inequality must hold.

$$\frac{\partial^2 \pi_J}{\partial x_j^2} = f'(Q) + Q_J f''(Q) + f'(Q) - C_j''(x_j) < 0, \quad j \in J. \tag{8}$$

The above inequality is satisfied if A1 and A2 hold. A1 does rule out neither $C_i'' = 0$ nor $C_i'' < 0$. However, we impose the following restriction on the cost functions.

***Assumption* 3 (A3):** $C_i'' > 0, \quad i \in N.$

Solving (7) with respect to x_j, we have

$$x_j = \psi_j(Q, Q_J), \quad j \in J, \tag{9}$$

where by A1 and A3 the partial derivatives have the following signs.

$$\frac{\partial \psi_J}{\partial Q} = \frac{f' + Q_J f''}{C_j''} < 0, \quad j \in J, \tag{10.1}$$

$$\frac{\partial \psi_J}{\partial Q_J} = \frac{f'}{C_j''} < 0, \quad j \in J. \tag{10.2}$$

Rewriting (7),

$$f(Q) + x_j f'(Q) - C_j'(x_j) = -(Q_J - x_j) f'(Q), \quad j \in J. \tag{7'}$$

Given Q, the LHS of (7′) is strictly decreasing in x_j and becomes zero at $x_j = \varphi_j(Q)$; given Q and Q_J, the RHS becomes a straight line with negative slope. Hence given Q and Q_J, the solution of (7′) is given by the intersection E of the two curves corresponding to the LHS and RHS of (7′) as shown in Figure 1. From the figure,

$$\psi_j(Q, Q_J) < \varphi_j(Q), \quad j \in J. \tag{11}$$

By definition,

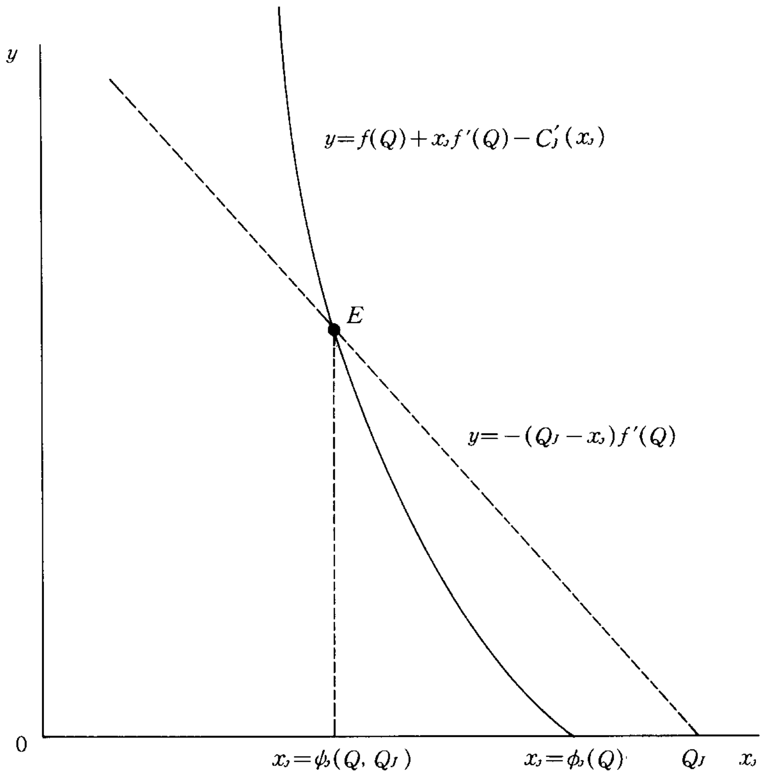

FIGURE 1
SOLUTION OF (7)

$$Q = \sum_{i \in I} \varphi_i(Q) + \sum_{j \in J} \psi_j(Q, Q_J) \equiv F_1(Q, Q_J), \tag{12}$$

$$Q_J = \sum_{j \in J} \psi_j(Q, Q_J) \equiv F_2(Q, Q_J), \tag{13}$$

where with the notation $\frac{\partial F_1}{\partial Q} \equiv F_{11}$, $\frac{\partial F_1}{\partial Q_J} \equiv F_{12}$, etc,

$$F_{11} < 0, \quad F_{12} < 0, \tag{12'}$$

$$F_{21} < 0, \quad F_{22} < 0, \tag{13'}$$

This fact is shown by

$$Q = G_1(Q_J), \tag{14}$$

where with the notation $\partial\psi_j / \partial Q \equiv \psi_{j1}$, $\partial\psi_j / \partial Q_J \equiv \psi_{j2}$,

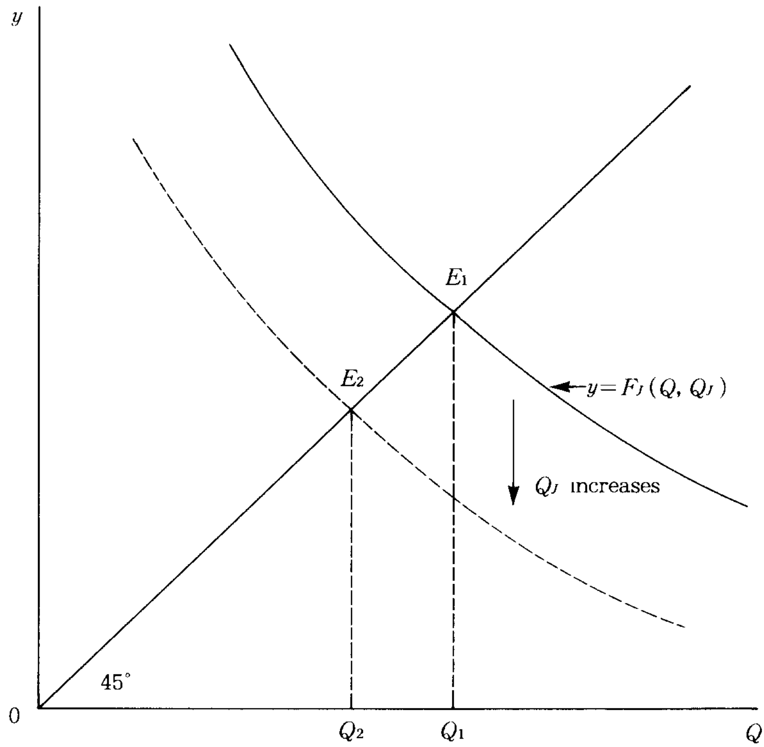

FIGURE 2

DETERMINATION OF Q AS A FUNCTION OF Q_J

$$G_1' = \frac{F_{12}}{1-F_{11}} = \frac{\sum_{j\in J} \psi_{j2}}{1-\sum_{i\in I} \varphi_i' - \sum_{j\in J} \psi_{j1}} < 0. \tag{15}$$

Likewise, solving (13) with respect to Q_J, we get

$$Q_J = G_2(Q), \tag{16}$$

where

$$G_2' = \frac{F_{21}}{1-F_{22}} = \frac{\sum_{j\in J} \psi_{j1}}{1-\sum_{j\in J} \psi_{j2}} < 0.$$

It is clear from (12′), (13′), $F_{12} = F_{22}$ and $F_{11} < F_{21}$ that $G_1' > 1/G_2'$. Hence (12) and (13), that is, (14) and (16) have a unique solution (Q^*, Q_J^*) as shown in Figure 3. This fact is stated as

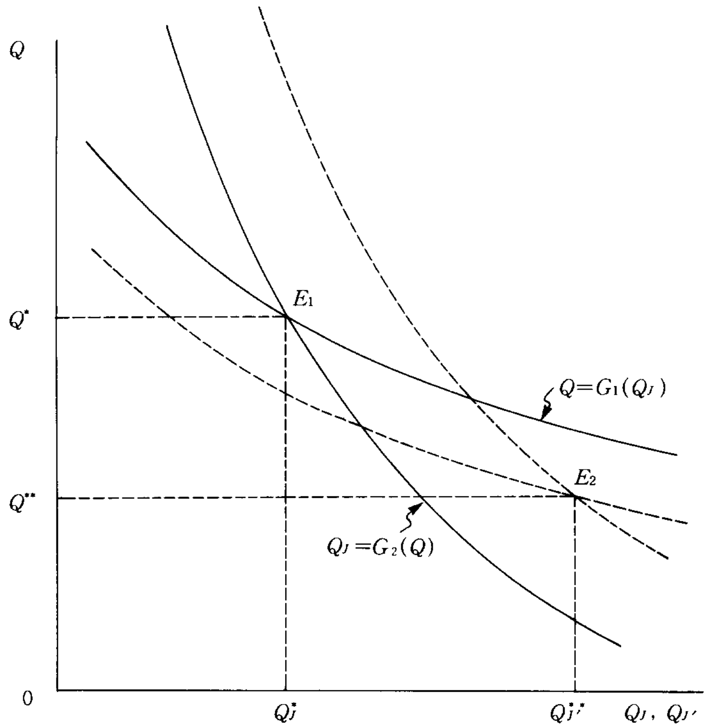

FIGURE 3

DETERMINATION OF THE EQUILIBRIUM INDUSTRY AND THE MERGED ENTITY'S OUTPUT

***Proposition* 1**

Given a partition of N into two disjoint subsets I (outsiders) and J (insiders), there exists a unique equilibrium if A1, A2 and A3 are satisfied.

We are now in a position to analyze the effect of an increase in the number of firms in the merged entity. So let one firm, say firm i_0, be merged. Then $N = I' \cup J'$, $I' = I \backslash i_0$, $J' \in J \cup \{i_0\}$.
Since

$$\varphi_{i_o}(Q) > \psi_{i_o}(Q, Q_{J'}) \tag{17}$$

and

$$Q = \sum_{i \in I'} \varphi_i(Q) + \sum_{j \in J'} \psi_j(Q, Q_{J'}), \tag{18}$$

the curve for $y = F_1(Q, Q_{J'})$ shifts downward. This implies that $Q \equiv H_1(Q_{J'})$ which satisfies (18) is smaller than that satisfying (12), namely

$$G_1(Z) > H_1(Z), \text{ for any } Z. \tag{19}$$

From

$$\begin{aligned} Q_{J'} &= \sum_{j \in J} \psi_j(Q, Q_{J'}) + \psi_{\iota_o}(Q, Q_{J'}) \\ &= F_2(Q, Q_{J'}) + \psi_{\iota_o}(Q, Q_{J'}). \end{aligned} \tag{20}$$

We derive $Q_{J'} \equiv H_2(Q)$. Given Q, the curve for $F_2(\cdot) + \psi_{\iota_o}(\cdot)$ lies above that for $F_2(\cdot)$. Hence

$$G_2(Q) < H_2(Q), \text{ for any } Q. \tag{21}$$

As $H_1' > 1/H_2'$, there exists a unique equilibrium (Q^{**}, Q_J^{**}) for the new merger. From (19) the curve for $H_1(Q_{J'})$ lies below that for $G_1(Q_J)$; from (21) the curve for $H_2(Q)$ lies above that for $G_2(Q)$. Hence

$$Q_{J'}^{**} > Q_J^{*}, \; Q^{**} < Q^{*}. \tag{22}$$

We record this results as

Proposition 2
Under our assumptions, if a number of firms in the merged entity increases, the equilibrium industry output decreases and the sum of the outsider's outputs increases.

Corollary
Under our assumptions, the equilibrium industry output is less for all firm's joint maximization than for non-cooperative maximization.

III. Effects of Number of Merged Firms

In this section we will derive a sufficient condition for an increase in the number of firms in the merged entity to be profitable. Out result is stated as

Proposition 3
If the stability condition[2] is satisfied and if, in addition,

[2]See Appendix.

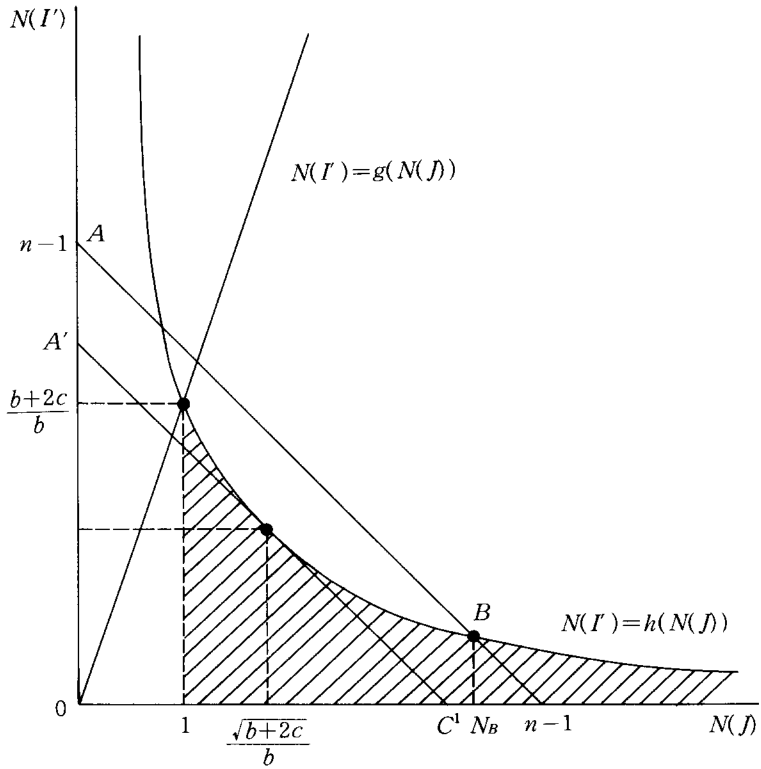

FIGURE 4
REGION FOR $d\pi_{J'} > 0$

$$-\sum_{i\in I'}\varphi_i' \sum_{j\in J}\frac{1}{C_j''} < \frac{1}{C_{i_0}''} < -\frac{1}{\sum_{i\in I'}\varphi_i'}\sum_{j\in J}\frac{1}{C_j''} \tag{23}$$

holds, the merged entity's profits increase if the number of merged firms increases.

Proof: Omitted (the proof is available upon request to the first author).
Q.E.D.

The implications of (23) will become clearer in the case of the following linear demand and identical quadratic cost functions:

$$\begin{aligned} p &= 1 - bQ, \quad b > 0, \\ C_i &= cx_i^2, \quad c > 0, \quad i \in I. \end{aligned} \tag{24}$$

In this case (23) is shown to be equivalent to the following set of two inequalities.

$$N(I') < \frac{b+2c}{b} N(J) \equiv g(N(J);b,c),$$
$$N(I') < \frac{b+2c}{bN(J)} \equiv h(N(J);b,c). \tag{25}$$

The shaded region in Figure 4 satisfies (25). If the line for

$$N(I') + N(J) = n - 1,$$

connects A and B as depicted, an increase in the number of firms in the merged entity raise profits, that is, $d\pi_{J'} = d\pi_{i_0} + d\pi_J > 0$ if $N(J) > N_B$. If the line coincides with the one connecting A′ and C′ or lies below it, we have $d\pi_{J'} > 0$ invariably. The same is true if $N(I') = 0$. Other things being equal, $d\pi_{J'} > 0$ is more likely to hold, the larger c/b or the smaller n. If

$$n \leq 2\sqrt{\frac{b+2c}{'b}} + 1, \tag{26}$$

we have $d\pi_{J'} > 0$ for any $N(J) \geq 1$.

Let us be more specific and let $b = 0.1$, $c = 1$. We then have from (26), $n \leq 1 + 2\sqrt{21} \doteqdot 10.165$. Hence, if the number of firms is equal to or less than ten, an increase in the number of a merged entity is always profitable.

IV. Conclusion

We have proven that a unique equilibrium exists for Cournot oligopoly with a merged entity and non-cooperative oligopolists if the assumptions A1, A2 and A3 hold. Our novel way of existence proof has enabled us to easily derive the industry output-reducing (price-enhancing) effect of an increase in the number of merged firms. We have derived also a condition which is sufficient for an increase in the number of firms in the merged entity to be profitable. In the case of linear demand and identical quadratic cost functions, the condition for the profitability is given by (25), which leads to $n \leq 10$ in the case of $b = 0.1$ and $c = 1$.

Appendix

In order to derive the stability condition, we assume that outputs are adjusted according to

$$\frac{dx_i}{dt} = \frac{k_i \partial \pi_i}{\partial x_i}, \quad i \in I, \tag{A1}$$

$$\frac{dx_J}{dt} = \frac{k_j \partial \pi_J}{\partial x_J}, \quad j \in J, \tag{A2}$$

where k_i's and k_j's are constant and positive, and t denotes time. Adjustment system (A1) is common in the literature on the stability of non-cooperative Cournot oligopoly equilibrium (see Okuguchi 1976; Okuguchi and Szidarovszky 1990). However, (A2) is rather ad hoc. The equilibrium is globally stable if the Jacobian matrix of the above system is negative dominant diagonal, that is,

$$f' + x_i f'' + C_i'' < 0, \quad i \in I, \tag{A3}$$

$$f' + Q_J f'' + f' - C_J'' < 0, \quad j \in J, \tag{A4}$$

$$f' + x_i f'' + f' - C_i'' < (n-1)(f' + x_i f''), \quad i \in I, \tag{A5}$$

$$f' + Q_J f'' + f' - C_J'' < N(I)(f' + Q_J f'') + (N(J) - 1)(f' + Q_J f'' + f'), \quad j \in J, \tag{A6}$$

where $N(I)$ and $N(J)$ are cardinal numbers of I and J, respectively. (A3) and (A4) hold under A1-3; (A4) implies that $\sum_{i \in I} \varphi_i' > -1$ if $N(I) \leq n-2$. Note that if $N(I) \leq n-2$ a merger occurs.

The adjustment system (A1-2) takes into account firm's behaviors over time. However, we can formulate an alternative dynamic system, which gives an iterative process for computing the equilibrium. Let therefore

$$Q(t+1) = F_1(Q(t), Q_J(t)), \tag{A7}$$

$$Q_J(t+1) = F_2(Q(t), Q_J(t)). \tag{A8}$$

Taking the maximum vector norm, we can prove that (A7) and (A8) are contractive and stable if

$$-F_{11} - F_{12} < 1,$$

that is,

$$-\sum_{i \in I} \frac{f' + x_i f''}{f' - C_i''} + \sum_{j \in J} \frac{f' + Q_J f''}{C_J''} + \sum_{j \in J} \frac{f'}{C_J''} + 1 > 0. \tag{A9}$$

Let f be linear. Then other things being equal, (A9) is more likely to be satisfied, the larger C_j'' for $j \in J$.

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