Economic Characterizations of the Second-Order Sufficient Conditions

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The second-order sufficient conditions for optimization in terms of Hessian matrix is equivalent to the law of diminishing marginal productivity for one factor case, and the second-order conditions in terms of the bordered Hessian matrix is equivalent to the law of diminishing marginal rate of substitution for the two-factor case. But in general, the whole implications of the second-order conditions cannot be replaced by these laws.

In this paper, we introduce the conditions of generalized diminishing marginal productivity and of generalized diminishing marginal rate of substitution, and show that these conditions are equivalent to the second order sufficient conditions. (*JEL* Classification: C62)

I. Introduction

Founders of economics invented the concepts of the law or conditions of diminishing marginal productivity and of diminishing marginal rate of substitution, and used them for the stability conditions of some optimization problems in the production theory. But, unfortunately, the conditions of diminishing marginal productivity fits for the case of one input only, and the conditions of diminishing marginal rate of substitution for the case of two inputs only. In other words, the laws as the stability conditions do not fit for the general cases.

In the course of theoretical developments, the conditions were replaced by the mathematical concepts of concavity and quasiconcavity of the production functions, which are generally expressed as the second-order sufficient conditions in terms of Hessian and bordered Hessian matrices.

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It is well-know that the second-order sufficient conditions for optimization in terms of Hessian matrix is equivalent to the law of diminishing marginal productivity for one factor case, and the second-order conditions in terms of the bordered Hessian matrix is equivalent to the law of diminishing marginal rate of substitution for the two-factor case. But, in general, the implications of the second-order conditions are richer than those of the laws. Therefore, the second-order conditions cannot be replaced by these laws. Further, no economic literature have revealed the full economic implications of the second-order sufficient conditions yet, it seems.

In this paper, we will introduce the conditions of generalized diminishing marginal productivity and of generalized diminishing marginal rate of substitution. Then, we will show that these conditions are equivalent to the second order sufficient conditions.

II. The Conditions of Generalized Diminishing Marginal Productivity

Consider the profit maximization problem with a production function,

$$y = f(x_1, x_2, \ldots, x_n) \equiv f(\boldsymbol{x}) \tag{1}$$

and the objective function,

$$\pi = \pi(\boldsymbol{x}) = f(\boldsymbol{x}) - \boldsymbol{w}' \boldsymbol{x} \tag{2}$$

where $\boldsymbol{x}$ is the n-vector of factor inputs, and $\boldsymbol{w}$ is the n-vector of factor prices, respectively. The first-order condition of this profit maximization problem is given by

$$f_{\boldsymbol{x}}(\boldsymbol{x}) - \boldsymbol{w} = 0, \tag{3}$$

and the second-order sufficient condition is given by

$$\frac{|\boldsymbol{H}_i|}{|\boldsymbol{H}_{i-1}|} < 0, \quad i = 1, 2, \ldots, n, \tag{4}$$

where $\boldsymbol{H}_i$ are the Hessian submatrices of the form,

$$\boldsymbol{H}_i = \begin{bmatrix} f_{11} & f_{12} & \cdot\cdot & f_{1i} \\ f_{21} & f_{22} & \cdots & f_{2i} \\ \cdots & \cdots & \cdots & \cdots \\ f_{i1} & f_{i2} & \cdot\cdot & f_{ii} \end{bmatrix}, \quad i = 1, 2, \ldots, n, \tag{5}$$

and $|\boldsymbol{H}_0| = 1$, by convention.

Conventionally, the marginal productivity function of input x_k is defined by

$$MP_k(\boldsymbol{x}) \equiv f_k(\boldsymbol{x}), \quad k = 1, 2, \ldots, n, \tag{6}$$

Here, we want to define the generalized marginal productivity function of input x_k as

$$GMP_k(\boldsymbol{x}_u) \equiv f_k(\boldsymbol{x}_c(\boldsymbol{x}_u); \boldsymbol{x}_u), \quad k = 1, 2, \ldots, n, \tag{7}$$

where $\boldsymbol{x}_c$ and $\boldsymbol{x}_u$ are subvectors of x such that,

$$\boldsymbol{x}_c = (x_1, x_2, \ldots, x_{k-1}) \text{ and } \boldsymbol{x}_u = (x_k, x_{k+1}, \ldots, x_n);$$

respectively, and the vector-valued function

$$\boldsymbol{x}_c = \boldsymbol{x}_c(\boldsymbol{x}_u), \tag{8}$$

is the solution of the suboptimal problem,

$$\underset{\boldsymbol{x}_u}{\text{Max}}\ \pi(\boldsymbol{x}_c, \boldsymbol{x}_u), \tag{9}$$

with $\boldsymbol{x}_u$ parametric at the optimal solution of the original problem (2).

A process of derivation given in Appendix A leads to the relation,

$$\frac{\partial GMP_k(\boldsymbol{x}_u)}{\partial x_k} = \frac{|\boldsymbol{H}_k|}{|\boldsymbol{H}_{k-1}|}, \quad k = 1, 2, \ldots, n, \tag{10}$$

The left-hand side of (10) measures the rate of change of the generalized marginal productivity function of input x_k. Therefore, the negativity of this side, or,

$$\frac{\partial GMP_k(\boldsymbol{x}_u)}{\partial x_k} < 0, \quad k = 1, 2, \ldots, n, \tag{11}$$

may be called the generalized diminishing marginal productivity. From (4), (10), and (11), we derive the conclusion that the second-order conditions (4) is equivalent to the conditions of the generalized diminishing marginal productivity.

III. Generalized Diminishing Marginal Rate of Substitution

Consider the output maximization problem with constrained total cost C.

$$\begin{aligned} &\text{Max } f(\boldsymbol{x}) \\ &\text{s.t. } C = \boldsymbol{w}'\boldsymbol{x}. \end{aligned} \tag{12}$$

Its Lagrangean function is,

$$L(\lambda, \boldsymbol{x}) = f(\boldsymbol{x}) + \lambda(C - \boldsymbol{w}'\boldsymbol{x}),$$

where λ is the Lagrange multiplier. The first order conditions of this optimization is given by

$$\begin{aligned} C - \boldsymbol{w}'\boldsymbol{x} = 0 \\ f_{\boldsymbol{x}}(\boldsymbol{x}) - \lambda \boldsymbol{w} = 0, \end{aligned}$$

and the second-order sufficient condition is

$$\frac{|\overline{\boldsymbol{H}}_i|}{|\overline{\boldsymbol{H}}_{i-1}|} < 0, \quad i = 2, 3, \ldots, n, \tag{13}$$

where the bordered Hessian submatrices are defined by

$$\overline{\boldsymbol{H}}_i = \begin{bmatrix} 0 & f_1 & f_2 & \cdots & f_i \\ f_1 & f_{11} & f_{12} & \cdots & f_{1i} \\ f_2 & f_{21} & f_{22} & \cdots & f_{2i} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_i & f_{i1} & f_{i2} & \cdots & f_{ii} \end{bmatrix}, \quad i = 1, 2, \ldots, n. \tag{14}$$

Conventionally, the marginal rate of substitution of input x_k with respect to input x_1 is defined by

$$MRS_{1k}(\boldsymbol{x}) \equiv \frac{f_k(\boldsymbol{x})}{f_1(\boldsymbol{x})}, \quad k = 1, 2, \ldots, n.$$

Here, we want to define the generalized marginal rate of substitution of input x_k with respect to input x_1 as

$$GMRS_{1k}(\boldsymbol{x}_u) \frac{f_k(\boldsymbol{x}_c(\boldsymbol{x}_u); \boldsymbol{x}_u)}{f_1(\boldsymbol{x}_c(\boldsymbol{x}_u); \boldsymbol{x}_u)}, \quad k = 2, 3, \ldots, n, \tag{15}$$

where $\boldsymbol{x}_c$ and $\boldsymbol{x}_u$ are subvectors of $\boldsymbol{x}$, as before, and the vector-valued function

$$\boldsymbol{x}_c = \boldsymbol{x}_c(\boldsymbol{x}_u), \tag{16}$$

is the solution of the suboptimal problem,

$$\underset{\boldsymbol{x}_c}{\text{Max}}\, f(\boldsymbol{x}_c, \boldsymbol{x}_u) \tag{17}$$

$$\text{s.t. } C - \boldsymbol{w}'_u \boldsymbol{x}_u = \boldsymbol{w}'_c \boldsymbol{x}_c,$$

with $\boldsymbol{x}_u$ parametric at the optimal solution of the original problem (12). Note that, in equation (15), x_1 is an element of the subvector $\boldsymbol{x}_c$, and x_k is an element of the subvector $\boldsymbol{x}_u$. Note, also, that the conventional marginal rate of substitution function is the same as (15) when $k = 2$.

A process of derivation given in Appendix B leads to

$$\frac{\partial GMRS_{1k}(\boldsymbol{x}_u)}{\partial x_k} = \frac{|\bar{\boldsymbol{H}}_k|}{f_1 \, |\bar{\boldsymbol{H}}_{k-1}|}, \quad k = 2, 3, \ldots, n. \tag{18}$$

The left-hand side of (18) measures the rate of change of the k-th order marginal rate of substitution between x_k and x_1, with respect to the change of x_k. Therefore, we may call its negativity, or,

$$\frac{\partial GMRS_{1k}(\boldsymbol{x}_u)}{\partial x_k} < 0, \quad k = 2, 3, \ldots, n, \tag{19}$$

the conditions of generalized diminishing marginal rate of substitution. From (13), (18), and (19), we derive the conclusion that the second-order sufficient condition (13) is equivalent to this set of conditions.

IV. Interpretations

We showed in Section II that the second-order sufficient condition (4) is equivalent to the conditions of generalized diminishing marginal productivity (11), and in Section III that the second-order sufficient condition (13) is equivalent to the conditions of generalized diminishing marginal rate of substitution, (19). If we use Samuelson's (1983), or Henderson and Quandt's (1980) terminology, the two sets of the generalized conditions are equivalent to regular strict concavity and regular strict quasiconcavity, respectively, of a production function.

The conditions of generalized diminishing marginal productivity (11) say that, for all $k = 1, 2, \ldots, n$, the marginal productivity of the k-th input is diminishing if the first $k - 1$ inputs change *optimally* and affect the marginal productivity indirectly according to the optimality rule. Since ordering of inputs is arbitrary, (11) may be interpreted as saying that the marginal productivity of each input is diminishing irrespective of the number of dependent inputs which affect this marginal productivity indirectly.

The conditions of generalized diminishing marginal rate of substitution (19) say that for all $k = 2, 3, \ldots, n$, the marginal rate of substitution between the first and the k-th inputs is diminishing if the k-th input changes independently and the first $k - 1$ inputs change *optimally* and affect the marginal rate of substitution indirectly according to the optimality rule. Since ordering of inputs is arbitrary, (19) may be interpreted as saying that the marginal rate of substitution of any pair of inputs is diminishing irrespective of the number of dependent inputs which affect this marginal rate of substitution indirectly.

According to the conventional definition of the terms, diminishing marginal productivity is neither necessary nor sufficient for diminishing marginal rate of substitution. But, fortunately, according to our definition of the terms, the conditions of generalized diminishing marginal productivity are sufficient for the conditions of generalized diminishing marginal rate of substitution.

Our derivation of the results are based on the context of production. But the main results do not depend on that specific context. Similar results can be derived from the utility maximization problem, cost minimization problem, etc., without any difficulty.

Appendix

A. Derivation of (10)

Equation (8) is a rearrangement of the first-order condition of the sub-optimal problem (9), or,

$$f_c(\boldsymbol{x}_c, \boldsymbol{x}_u) - w_c = 0, \tag{A1}$$

where f_c is $k - 1$ dimensional vector of partial derivatives of f with respect to vector x_c. If we differentiate (A1) with respect to x_k, the first element of $\boldsymbol{x}_u$, we get

$$\boldsymbol{H}_{k-1} \frac{\partial \boldsymbol{x}_c(\boldsymbol{x}_u)}{\partial \boldsymbol{x}_k} + \boldsymbol{f}_{ck} = \boldsymbol{0}, \tag{A2}$$

so that

$$\frac{\partial \boldsymbol{x}_c(\boldsymbol{x}_u)}{\partial \boldsymbol{x}_k} = -[\boldsymbol{H}_{k-1}]^{-1} f_{ck}. \tag{A3}$$

Using this result we can evaluate the generalized marginal productivity function of input x_k with respect to x_k as follows:

$$\frac{\partial GMRS_k(\boldsymbol{x}_u)}{\partial x_k} = -\boldsymbol{f}'_{kc}[\boldsymbol{H}_{k-1}]^{-1}\boldsymbol{f}_{ck} + \boldsymbol{f}_{kk} = \frac{|\boldsymbol{H}_k|}{|\boldsymbol{H}_{k-1}|}, \quad k = 2, 3, \ldots, n. \tag{A4}$$

B. Derivation of (18)

Equation (16) is a rearrangement of the first-order condition of the sub-optimal problem (17), or,

$$\begin{aligned} &\operatorname*{Max}_{x_u} f(\boldsymbol{x}_c, \boldsymbol{x}_u) \\ &\text{s.t.} \quad C - \boldsymbol{w}'_c\boldsymbol{x}_c - \boldsymbol{w}'_u\boldsymbol{x}_u = 0, \end{aligned} \tag{B1}$$

where x_u is parametric at the optimal solution of the original optimal problem (12). The first-order condition is

$$\begin{aligned} L_\lambda &= C - \boldsymbol{w}'_c\boldsymbol{x}_c - \boldsymbol{w}'_u\boldsymbol{x}_u = 0 \\ L_c &= \boldsymbol{f}_c(\boldsymbol{x}_c, \boldsymbol{x}_u) - \lambda\boldsymbol{w}_c = \mathbf{0}, \end{aligned} \tag{B2}$$

where L is the Lagrangean function with λ the Lagrange multiplier:

$$L(\lambda, \boldsymbol{x}_c) = f(\boldsymbol{x}_c, \boldsymbol{x}_u) - \lambda(C - \boldsymbol{w}'_c\boldsymbol{x}_c - \boldsymbol{w}'_u\boldsymbol{x}_u). \tag{B3}$$

To obtain the effect of independent change of x_k, the first element of $\boldsymbol{x}_u$, we differentiate the first-order condition (B2) totally to get

$$\begin{bmatrix} 0 & -\boldsymbol{w}'_c \\ -\boldsymbol{w}_c & \boldsymbol{H}_{k-1} \end{bmatrix}\begin{bmatrix} d\lambda \\ d\boldsymbol{x}_c \end{bmatrix} = \begin{bmatrix} w_k \\ -f_{ck} \end{bmatrix} dx_k. \tag{B4}$$

We also know from the first-order condition of the original optimization problem that $\boldsymbol{w}_c = \boldsymbol{f}_c/\lambda$ and $w_k = f_k/\lambda$, so that (B4) becomes

$$\begin{bmatrix} 0 & -\boldsymbol{f}'_c \\ \boldsymbol{f}_c & \boldsymbol{H}_{k-1} \end{bmatrix}\begin{bmatrix} -\dfrac{d\lambda}{\lambda} \\ d\boldsymbol{x}_c \end{bmatrix} = -\begin{bmatrix} f_k \\ \boldsymbol{f}_{ck} \end{bmatrix} dx_k. \tag{B5}$$

If we evaluate the partial derivatives at the original optimal point, then the $k \times k$ coefficient matrix of (B5) is nothing but $\overline{\boldsymbol{H}}_{k-1}$ and if we partition its inverse as

$$[\overline{\boldsymbol{H}}_{k-1}]^{-1} = \begin{bmatrix} 0 & -\boldsymbol{f}'_c \\ \boldsymbol{f}_c & \boldsymbol{f}_{cc} \end{bmatrix}^{-1} = \begin{bmatrix} g_{00} & \boldsymbol{g}'_{01} \\ \boldsymbol{g}_{10} & \boldsymbol{G}_{11} \end{bmatrix}, \tag{B6}$$

then we get the partial derivative of $\boldsymbol{x}_c$ with respect to x_k as follows:

$$\frac{\partial \boldsymbol{x}_c}{\partial x_k} = -[\boldsymbol{g}_{10} \quad \boldsymbol{G}_{11}]\begin{bmatrix} f_k \\ \boldsymbol{f}_{ck} \end{bmatrix}. \tag{B7}$$

According to (15), the generalized marginal rate of substitution of input x_k with respect to input x_1 is defined as

$$GMRS_{1k}(\boldsymbol{x}_u) \equiv \frac{f_k(\boldsymbol{x}_c(\boldsymbol{x}_u);\boldsymbol{x}_u)}{f_1(\boldsymbol{x}_c(\boldsymbol{x}_u);\boldsymbol{x}_u)}, \quad k = 2, 3, \ldots, n. \tag{B8}$$

Differentiation of (B8) with respect to x_k leads to

$$\frac{\partial GMRS_{1k}(\boldsymbol{x}_u)}{\partial x_k} = \frac{1}{f_1^2}\left\{f_1\left(\boldsymbol{f}'_{\boldsymbol{kc}}\frac{\partial \boldsymbol{x}_c}{\partial x_k} + f_{kk}\right) - f_k\left(\boldsymbol{f}_{\boldsymbol{1c}}\frac{\partial \boldsymbol{x}_c}{\partial x_k} + f_{1k}\right)\right\}. \tag{B9}$$

A process of modification using (B7) leads to

$$\begin{aligned}\frac{\partial GMRS_{1k}(\boldsymbol{x}_u)}{\partial x_k} &= \frac{1}{f_1}\left\{-\boldsymbol{f}'_{\boldsymbol{kc}}[\boldsymbol{g}_{10} \vdots \boldsymbol{G}_{11}]\begin{bmatrix} f_k \\ \boldsymbol{f}_{\boldsymbol{ck}} \end{bmatrix} + f_{kk}\right\} \\ &\quad + \frac{1}{f_1^2} f_k\left\{\boldsymbol{f}'_{\boldsymbol{1c}}[\boldsymbol{g}_{10} \vdots \boldsymbol{G}_{11}]\begin{bmatrix} f_k \\ \boldsymbol{f}_{\boldsymbol{ck}} \end{bmatrix} - f_{1k}\right\},\end{aligned} \tag{B10}$$

and the last expression of the right-hand side is

$$f'_{1c}[\boldsymbol{g}_{10} \vdots \boldsymbol{G}_{11}]\begin{bmatrix} f_k \\ f_{ck} \end{bmatrix} + f_{1k} = \boldsymbol{f}'_{\boldsymbol{1c}}\boldsymbol{g}_{\boldsymbol{10}} f_k + \boldsymbol{f}'_{\boldsymbol{1c}}\boldsymbol{G}_{11}\boldsymbol{f}_{\boldsymbol{ck}} - f_{1k}. \tag{B11}$$

But from the definition of (B6),

$$\begin{aligned}\boldsymbol{f}_{\boldsymbol{c}} g_{00} + \boldsymbol{f}_{\boldsymbol{cc}}\boldsymbol{g}_{10} &= \boldsymbol{0} \\ \boldsymbol{f}_{\boldsymbol{c}}\boldsymbol{g}'_{01} + \boldsymbol{f}_{\boldsymbol{cc}}\boldsymbol{G}_{11} &= \boldsymbol{I}_{k-1},\end{aligned}$$

or, remembering that the first element of the $(k-1) \times 1$ vector f_c is f_1, and the first row of the $(k-1) \times (k-1)$ matrix $\boldsymbol{f}_{\boldsymbol{cc}}$ is the $1 \times (k-1)$ vector $\boldsymbol{f}'_{\boldsymbol{1c}}$, we get

$$\begin{aligned}f_1 g_{00} + \boldsymbol{f}'_{\boldsymbol{1c}}\boldsymbol{g}_{10} &= 0 \\ f_c\boldsymbol{g}'_{01} + \boldsymbol{f}'_{\boldsymbol{1c}}\boldsymbol{G}_{11} &= [1, 0, 0, \ldots, 0],\end{aligned}$$

or,

$$\begin{aligned}\boldsymbol{f}'_{1c}\boldsymbol{g}_{10} &= -f_1 g_{00} \\ \boldsymbol{f}'_{1c}\boldsymbol{G}_{11} &= [1, 0, 0, \ldots, 0] - f_1\boldsymbol{g}'_{01}.\end{aligned} \tag{B12}$$

Then we can substitute (B12) into (B11), to get

$$\boldsymbol{f}'_{\boldsymbol{1c}}[\boldsymbol{g}_{10} \vdots \boldsymbol{G}_{11}]\begin{bmatrix} f_k \\ \boldsymbol{f}_{\boldsymbol{ck}} \end{bmatrix} + f_{1k} = \boldsymbol{f}'_{\boldsymbol{1c}}\boldsymbol{g}_{10} f_k + \boldsymbol{f}'_{\boldsymbol{1c}}\boldsymbol{G}_{11}\boldsymbol{f}_{\boldsymbol{ck}} - f_{1k}$$

$$= -f_1 g_{00} f_k + f_{1k} - f_1 \boldsymbol{g}_{01} \boldsymbol{f}_{ck} - f_{1k} \tag{B13}$$

$$= -f_1 \left\{ [g_{00} \vdots \boldsymbol{g}'_{01}] \begin{bmatrix} f_k \\ \boldsymbol{f}_{ck} \end{bmatrix} \right\}.$$

Inserting (B13) into (B10), we get

$$\frac{\partial GMRS_{1k}(\boldsymbol{x}_u)}{\partial x_k} = \frac{1}{f_1} \left\{ f_{kk} - f_k [g_{00} \vdots \boldsymbol{g}'_{01}] \begin{bmatrix} f_k \\ \boldsymbol{f}_{ck} \end{bmatrix} - \boldsymbol{f}'_{kc} [\boldsymbol{g}_{10} \vdots \boldsymbol{G}_{11}] \begin{bmatrix} f_k \\ \boldsymbol{f}_{ck} \end{bmatrix} \right\}$$

$$= \frac{1}{f_1} \left\{ f_{kk} - [f_k \quad \boldsymbol{f}'_{kc}] \begin{bmatrix} g_{00} & \boldsymbol{g}_{01} \\ \boldsymbol{g}_{10} & \boldsymbol{G}_{11} \end{bmatrix} \begin{bmatrix} f_k \\ \boldsymbol{f}_{ck} \end{bmatrix} \right\}$$

$$= \frac{|\overline{\boldsymbol{H}}_k|}{f_1 |\overline{\boldsymbol{H}}_{k-1}|},$$

or,

$$\frac{\partial GMRS_{1k}(\boldsymbol{x}_u)}{\partial x_k} = \frac{|\overline{\boldsymbol{H}}_k|}{f_1 |\overline{\boldsymbol{H}}_{k-1}|}. \tag{B14}$$

The last equality follows from the fact that

$$\overline{\boldsymbol{H}}_k = \begin{bmatrix} 0 & \boldsymbol{f}'_0 & f_k \\ \boldsymbol{f}_c & \boldsymbol{f}_{cc} & \boldsymbol{f}_{ck} \\ f_k & \boldsymbol{f}'_{kk} & f_{kk} \end{bmatrix} = \begin{bmatrix} [\overline{\boldsymbol{H}}_{k-1}] & \begin{bmatrix} f_k \\ \boldsymbol{f}_{ck} \end{bmatrix} \\ [f_k \quad \boldsymbol{f}'_{kc}] & f_{kk} \end{bmatrix}, \tag{B15}$$

and that, in general,

$$|\overline{\boldsymbol{H}}_k| = |\overline{\boldsymbol{H}}_{k-1}| \left\{ f_{kk} - [f_k \quad \boldsymbol{f}'_{kc}][\overline{\boldsymbol{H}}_{k-1}]^{-1} \begin{bmatrix} f_k \\ \boldsymbol{f}_{ck} \end{bmatrix} \right\}. \tag{B14}$$

C. A Note on the Literature

Debreu (1952) shows rigorously the sign definiteness of quadratic forms in terms of signs of principal minors of Hessian or bordered Hessian matrices. Diewert (1981) and Otani (1980) give mathematical characterizations of various quasiconcave functions.

As for a diminishing marginal productivity interpretation of the second-order condition, Silberberg (1983) says as follows:

"[For two-factor case,] diminishing marginal productivity in each factor does not, by itself, guarantee that a maximum profit position will be achieved. Condition $f_{11}f_{22} - f_{12}f_{21} > 0$ is also required. This relation, though less intuitive than diminishing marginal productivity, arises from the fact that changes in one factor affect the marginal products of the other factors as well as its own marginal product, and the overall effect on all marginal products must be akin to diminishing marginal productivity." (Silberberg 1993, p.109)

But his interpretation is graphical, and does not try any generalization to higher dimensions. He simply says, "... there are all the remaining principal minors to consider; these are not easily given intuitive explanations." (p.151) Smith says similar things as follows:

> "Unfortunately this sufficient condition for maximization ... does not translate into a simple condition analogous to the condition (of a diminishing marginal productivity) in the case of a function of more than one variable." (Smith 1982, p.29)

As for a diminishing marginal rate of substitution interpretation of the second-order conditions for n-dimensional case, Samuelson (1947 or 1983) and (Hicks 1946) say as follows:

> "The isoquants must also be convex to the origin in all directions in order that its contact with the isocost plane represent a true proper minimum." (Samuelson 1983, p. 61)

> "In order that equilibrium should be stable, ..., it is necessary that no possible substitution of equal market values should lead the consumer to a preferred position. This means not only that we must have a diminishing marginal rate of substitution between each pair of commodities, but also that more complicated substitutions (of some X for some Y and some Z) must be ruled out in the same way. We may express this by saying that the marginal rate of substitution must diminish for substitutions in every direction." (Hicks 1946, p. 25)

> "These conditions [of diminishing marginal rate of substitution] have got to hold not only for single substitutions ... but also for group substitutions ... The marginal rate of substitution ... between any pair of groups of factors must diminish." (Hicks 1946, p. 87)

But neither Samuelson nor Hicks shows any rigorous links which connect their "every direction" and "group substitutions" with the second-order conditions. Silberberg (1978) and Smith (1982) express impossibility of diminishing MRS interpretation of the second-order conditions in general case:

> "The notion that the indifference hypersurface is convex to the origin is a much stronger assumption, in an n-good world, than simply diminishing MRS between any pair of goods, other goods held constant. Only in the case of only two goods, wherein there are no other goods to be held constant, is quasi-concavity equivalent to diminishing MRS." (Silberberg 1993, p. 222)

> "When there are more than two goods, quasi-concavity is still a sufficient condition, but it cannot be checked in a simple way like [the property of diminishing marginal rate of substitution.]" (Smith 1982, p. 94)

Arrow and Enthoven (1961) establish an equivalence between quasi-concavity and a kind of diminishing marginal rate of substitution for general case. But their kind of diminishing marginal rate of substitution is defined along a ray which prohibits conventional interpretation.

As for the relation between diminishing marginal productivity and diminishing marginal rate of substitution, Archibald and Lipsey (1976) say:

> "Since convexity may be interpreted as diminishing marginal substitutability, what we have found is that diminishing marginal [productivity] is neither necessary nor sufficient for diminishing marginal substitutability." (Archibald and Lipsey 1976, p. 256)

Although this statement is correct, considering that concavity implies quasi-concavity in general, it disturbs our intuition.

The basic idea of this paper was originally developed by this author in Jeong (1984), and this is an extension of that idea.

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