Transfer Paradox and Bargaining Solutions

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We investigate whether bargaining solutions are immune to the transfer paradox for n-person bargaining problems. We show that two families of bargaining solutions, one generalizing the Nash solution and another generalizing the Kalai-Smorodinsky solution, are immune to the strong transfer paradox requiring the donor gains and the recipient loses. Also, we present examples demonstrating that the Nash and the Kalai-Smorodinsky solutions are subject to the weak transfer paradox so that the donor and the recipient could gain together or lose together. (*JEL* Classification: C78)

I. Introduction

The transfer problem, which concerns the welfare consequences of a transfer payment from one agent to another, has been studied extensively in the international trade literature.[1] It is well-known that the Walrasian mechanism is subject to two forms of transfer paradoxes in more than two-agent or two-country setting: (a) the *strong transfer paradox*,[2] i.e., the donor gains and the recipient loses, and (b) the *weak transfer paradox*, i.e., the donor and the recipient gain together or lose together.

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[1]See, for example, Aumann and Peleg (1974), Bhagwati, Brecher and Hatta (1983), Chichilnisky (1980), Gale (1974), Guesnerie and Laffont (1978), Johnson (1960), Jones (1970,1984), Samuelson (1952), and Yano (1983).

[2]These terminology is due to Yano (1983).

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Here we study the transfer problem in the context of bargaining theory. A bargaining problem, as formulated by Nash (1950), consists of a feasible set and a disagreement point; the feasible set represents the set of alternatives in the utility space available to agents through cooperation and the disagreement point gives the maximal utility level attainable to each agent on his/her own. When bargaining theory is applied to exchange economies, it is natural to choose the feasible set as the set of utility vectors attainable by all possible redistribution of the total endowment and the disagreement point as the utility level of each agent from his/her own endowment. A transfer of part of the endowment from one agent to another would result in a certain change in the disagreement point without affecting the feasible set; the utility level of the donor at the disagreement point (weakly) decreases,[3] that of recipient increases, whereas that of each other agent is unchanged.

Thomson (1987), as a by-product of his investigation of monotonicity properties of bargaining solutions with respect to changes in the disagreement point, has shown that the Nash (1950) and the Kalai-Smorodinsky (1975) bargaining solutions are immune to the transfer paradox for 2-person problems.[4] Although he does not address this question for the case of more than 2 people, a careful examination of his counter-examples[5] reveals that they can be used to demonstrate that the bargaining solutions are subject to the weak transfer paradox, so that both the donor and the recipient may gain together or lose together (for a detail,see our Theorems 1 and 2 below). However, the question remained whether these bargaining solutions are subject to the strong transfer paradox for more than 2-person problems. In this paper, we will show that two families of bargaining solutions, one generalizing the Nash solution and another generalizing the Kalai-Smorodinsky solution, are immune to the strong transfer paradox for n-person problems. This is in contrast with what we know of the Walrasian mechanism.[6]

The paper is organized as follows. Section II contains some preliminaries, defines the families of bargaining solutions that we will consid-

[3]We assume that the preferences of agents are weakly monotonic.

[4]See his remark in p.55.

[5]These are used in the proofs of Theorems 4 and 5 in Thomson (1987, 55-6).

[6]However, a related statement could be made for the Walrasian mechanism. Under the Jones (1970) presumption, the strong paradox is expected to occur very rarely. See Yano (1983, 286) for details. However, for the bargaining solutions studied here, the strong paradox never occurs.

er, and states properties of bargaining solutions. Section III formally introduces our main conditions requiring that the transfer paradox should not occur, and investigates whether the bargaining solutions satisfy the conditions.

II. Preliminaries

An n-person bargaining problem, or simply a *problem*, is a pair (S, d), where S is a subset of $\mathbb{R}^n$ and d is a point in S, such that

(1) S is convex and closed,
(2) $a_i(S, d) \equiv \max\{x_i \mid x \geqq d, x \in S\}$ exists for all i,[7]
(3) S is *comprehensive*, i.e., for all $x \in S$ and for all $y \in \mathbb{R}^n$, if $y \leq x$, then $y \in S$,
(4) there exists $x \in S$ with $x > d$.

The *feasible set* is denoted by S. Each point x of S is a *feasible alternative*. The coordinates of x are the von Neumann-Morgenstern utility levels attained by the agents through the choice of some joint action. The point d is the *disagreement point*. The intended interpretation of (S, d) is as follows: the agents can achieve any point of S if they unanimously agree on it. If they do not agree on any point, they end up at d. Let Σ^n be the class of all problems.

A bargaining solution, or a *solution*, is a function $F : \Sigma^n \to \mathbb{R}^n$ such that for all $(S, d) \in \Sigma^n$, $F(S, d) \in S$. The value taken by the solution F when applied to the problem (S, d), $F(S, d)$, is called the *solution outcome of* (S,d).

The following notation and terminology will be used frequently. A *positive, independent person-by-person, and linear transformation of order n*, or *positive linear transformation*, is a function $\lambda : \mathbb{R}^n \to \mathbb{R}^n$, given by $a \in \mathbb{R}^n_{++}$ and $b \in \mathbb{R}^n$, such that for all $x \in \mathbb{R}^n$, $\lambda(x) \equiv (a_1x_1 + b_1, \cdots, a_nx_n + b_n)$. Let Λ^n be the class of all such transformations. Given $x^1, \cdots, x^k \in \mathbb{R}^n$, *concomp* $\{x^1, \cdots, x^k\}$ is the *convex and comprehensive hull* of these points (the smallest convex and comprehensive set containing them). Finally, we denote by e the vector whose coordinates are all equal to 1.

Now we define two families of solutions: the first family, introduced by Lensberg (1987), generalizes the Nash (1950) solution, and the second family, introduced by Peters and Tijs (1985), generalizes the Kalai-

[7]Vector inequalities: given $x, y \in \mathbb{R}^n$, $x \geqq y$, $x \geq y$, $x > y$.

Smorodinsky (1975) solution.[8]

Definition 1: A *strictly concave and separably additive solution relative to f*, N^f : let $f \equiv \{f_i\}_{i \in N}$ be functions such that for each i, $f_i : \mathbb{R}_+^{\{i\}} \rightarrow \mathbb{R} \cup \{-\infty\}$ is strictly increasing and strictly concave; for each $(S, d) \in \Sigma^n$, $N^f(S, d) = \text{argmax} \{\Sigma f_i (x_i - d_i) \mid x \in S, x \geqq d\}$.

If $f_i(x_i - d_i) = a_i \log (x_i - d_i)$ for some $\alpha_i > 0$ and for all i, then it is a *weighted Nash solution*.

If $f_i(x_i - d_i) = \log (x_i - d_i)$ for all i, then it is the *Nash solution*.

Definition 2: A *monotone path* is the image G of a continuous and strictly increasing function $\psi : [0, 1] \rightarrow [0, 1]^n$, which satisfies $\psi(0) = 0$ and $\psi_i([0, 1]) = [0, 1]$ for at least one i.

The *normalized monotone path solution relative to* G, K^G : for each $(S, d) \in \Sigma^n$, let $\lambda \in \Lambda^n$ be a positive linear transformation such that $\lambda(d) = 0$ and $\lambda(a(S, d)) = e$; $K^G(S, d)$ is the inverse image under the transformation of the maximal point $\overline{x}$ of $\lambda(S)$ along G, i.e, $K^G(S, d) = \lambda^{-1}(\overline{x})$.[9]

Given $y \in \mathbb{R}^n_{++}$ such that $y \leqq e$ and that $y_i = 1$ for at least one i, if $G = [0, y]$, then it is a *weighted Kalai-Smorodinsky solution* (Thomson 1984).

If $G = [0, e]$, then it is the *Kalai-Smorodinsky solution.*

We will use the following properties of solutions.

Individual Rationality (*I. R*). For all $(S, d) \in \Sigma^n$, $F(S,d) \geqq d$.

Scale Invariance (*S.INV*). For all (S, d) Σ^n and for all positive linear transformations $\lambda \in \Lambda^n$, $F(\lambda(S), \lambda(d)) = \lambda(F(S, d))$.

Monotonicity with Respect to the Disagreement Point (*d-MON*). For all (S, d), $(S', d') \in \Sigma^n$, and for all i, if $S' = S$, $d'_i > d_i$, and $d'_j = d_j$ for all $j \neq i$, then $F_i(S', d') \geqq F_i(S, d)$.

Each N^f satisfies *I.R*; each K^G satisfies *I.R* and *S.INV*. The fact that both N^f and K^G satisfy *d-MON* will be shown in this paper (see Remark

[8]Our families are slightly different from those introduced by Lensberg and by Peters and Tijs. Lensberg required each f_i to be strictly increasing and continuous and Σf_i to be strictly quasi-concave. Peters and Tijs required G to be weakly increasing with the end-point e.

[9]Note that $0 < \overline{x} \leqq e$

2 below).

III. Results

When bargaining problems are applied to economies, it is natural to choose the disagreement point as the utility level of each agent from his/her initial endowment. If a transfer occurs, then the utility level of the donor at the disagreement point decreases, that of the recipient increases, and that of each other agent is unchanged. In addition, the transfer does not affect the feasible set.

As discussed in the introduction, two forms of transfer paradoxes can occur if there are more than two agents: (i) for the *strong transfer paradox*, the donor gains and the recipient loses, and (ii) for the *weak transfer paradox*, the donor and the recipient gain together or lose together. Consequently, we introduce the following conditions to study the transfer problem in the context of bargaining theory. The first condition, introduced by Thomson (1987) in that context,[10] prevents both paradoxes, while the second prevents the strong transfer paradox.

No Transfer Paradox (*N.T.P*). For all (S, d), $(S', d') \in \Sigma^n$ and for all i, j, if $S' = S$, $d_i' > d_i$, $d_j' < d_j$, and $d_k' = d_k$ for all $k \notin \{i, j\}$, then $F_i(S', d') \geqq F_i(S, d)$ and $F_j(S', d') \leqq F_j(S, d)$.

No Strong Transfer Paradox (*N.S.T.P*). For all (S, d), $(S', d') \in \Sigma^n$ and for all i, j, if $S' = S$, $d_i' > d_i$, $d_j' < d_j$, and $d_k' = d_k$ for all $k \notin \{i, j\}$, then one of the following three holds: (a) $F_i(S', d') > F_i(S, d)$, (b) $F_j(S', d') < F_j(S, d)$, or (c) $F_i(S', d') = F_i(S, d)$ and $F_j(S', d') = F_j(S, d)$.

Now we investigate whether the Nash (1950) and the Kalai-Smorodinsky (1975) solutions satisfy the conditions. As shown by Thomson (1987), these solutions are immune to the two transfer paradoxes for 2-person problems. Consequently, our attention is given to the more than 2-person case. We begin with negative results.

***Theorem* 1**
The Nash solution is subject to the weak transfer paradox on Σ^n for $n \geqq 3$.

***Proof*:** See the example in the Remark 1 of Chun and Thomson (1990).

Q.E.D.

[10]Also, Moulin (1985) investigated the implication of the condition in a different context.

Theorem 2

The Kalai-Smorodinsky solution is subject to the weak transfer paradox on Σ^n for $n \geqq 3$.

Proof: We adopt the example given in the proof of Theorem 5 in Thomson (1987). Let $n \equiv 3$. Also, let $d \equiv (0, 0, .25)$, $d' \equiv (1, 0, 0)$, and $S \equiv$ *concomp* {(0, 2, 2), (2, 1, 2)}.

We have $a(S, d) = (2, 2, 2)$ and $K(S, d) = (1.333, 1.333, 1.416)$. On the other hand, $a(S, d') = (2, 1.5, 2)$ and $K(S, d') = (1.75, 1.125, 1.5)$. Note that agents 1 and 3 gain together at the expense of agent 2. Its extension to $n > 3$ is obvious.

Q.E.D.

Remark 1

The negative results in Theorems 1 and 2 carry over even if we restrict our domain to the economic problem of fair division.[11] (See Cho and Chun (1991) for details.)

Next, we will present our first positive result showing that the family of strictly concave and separably additive solutions, which contains the Nash solution, is immune to the strong transfer paradox. Although our results generalize those of Thomson (1987), the techniques used in the proofs of Theorems 3 and 4 are similar to his.

Theorem 3

Strictly concave and separably additive solutions are immune to the strong transfer paradox.

Proof: Let F be a strictly concave and separably additive solution relative to $f \equiv \{f_i\}_{i \in N}$ and (S, d), $(S, d') \in \Sigma^n$ be two problems satisfying the hypotheses of *N.S.T.P.* Without loss of generality, we can assume that $d' = (d_1 + \alpha, d_2 - \beta, d_3, \cdots, d_n)$ for $\alpha > 0$ and $\beta > 0$. Let $F(S, d) \equiv x$ and $F(S, d') \equiv y$.

We claim that, for each i, $f_i(x_i - d_i)$, $f_i(x_i - d_i')$, $f_i(y - d_i)$ and $f_i(y_i - d_i')$, can be assumed to be finite. Since f_i 's are strictly increasing, $z_i > 0$ implies that $f_i(z_i) > -\infty$ for all i. From the assumption on the problem, there exist z, z' in S such that $z > d$ and $z' > d'$. Moreover, $x \geqq d$ and $y \geqq d'$. Therefore, $\Sigma f_i(x_i - d_i) \geqq \Sigma f_i(z_i - d_i) > -\infty$ and $\Sigma f_i(y_i - d_i') \geqq \Sigma f_i(z_i' - d_i') > -\infty$ which imply that $f_i(x_i - d_i) > -\infty$ and $f_i(y_i - d_i') > -\infty$ for all

[11]The application of bargaining solutions to the economic problem of fair division was first studied by Chun and Thomson (1988).

i. Remaining terms are: $f_1(y_1 - d_1)$, $f_2(y_2 - d_2)$, $f_1(x_1 - d_1 - \alpha)$ and $f_2(x_2 - d_2 + \beta)$.

(a) $f_1(y_1 - d_1)$ and $f_2(x_2 - d_2 + \beta)$: Since F satisfies *I.R*, $y_1 \geqq d_1 + \alpha$, which implies that $y_1 > d_1$. Consequently, we have $f_1(y_1 - d_1) > -\infty$. Similarly, we can show that $f_2(x_2 - d_2 + \beta) > -\infty$.

(b) $f_2(y_2 - d_2)$ and $f_1(x_1 - d_1 - \alpha)$: If $y_2 \leqq d_2$, together with $x_2 \geqq d_2$, we have $y_2 \leqq x_2$. If $y_2 < x_2$, then *N.S.T.P.* holds so that we are done. If $y_2 = x_2$, then $f_2(d_2 - d_2) = f_2(x_2 - d_2) > -\infty$. On the other hand, if $y_2 > d_2$, it is clear that $f_2(y_2 - d_2) > -\infty$. Therefore, we may assume that $f_2(y_2 - d_2) > -\infty$. Similarly, we can also assume that $f_1(x_1 - d_1 - \alpha) > -\infty$.

$F(S, d) = x$ implies that (i) $\Sigma f_i(y_i - d_i) \leqq \Sigma f_i(x_i - d_i)$. Also, $F(S, d') = y$ implies that (ii) $f_1(x_1 - d_1 - \alpha) + f_2(x_2 - d_2 + \beta) + \Sigma_{i \neq 1,2} f_i(y_i - d_i) f_1(y_1 - d_1 - \alpha) + f_2(y_2 - d_2 + \beta) + \Sigma_{i \neq 1,2} f_i(y_i - d_i)$. By adding up (i) and (ii) side by side and deleting the repeated terms, we have (iii) $f_1(y_1 - d_1) + f_2(y_2 - d_2) + f_1(x_1 - d_1 - \alpha) + f_2(x_2 - d_2 + \beta) \leqq f_1(y_1 - d_1 - \alpha) + f_2(y_2 - d_2 + \beta) + f_1(x_1 - d_1) + f_2(x_2 - d_2)$.

It is enough to show that neither $[y_1 \leqq x_1$ and $y_2 > x_2]$ nor $[y_1 < x_1$ and $y_2 \geqq x_2]$ can happen. Suppose now, by way of contradiction, that the first case holds, i.e., $y_1 \leqq x_1$ and $y_2 > x_2$. Since both f_1 and f_2 are strictly increasing and strictly concave, they imply that

$$f_1(y_1 - d_1) - f_1(y_1 - d_1 - \alpha) \geqq f_1(x_1 - d_1) - f_1(x_1 - d_1 - \alpha) \text{ and}$$
$$f_2(x_2 - d_2 + \beta) - f_2(x_2 - d_2) > f_2(y_2 - d_2 + \beta) - f_2(y_2 - d_2),$$

or equivalently,

$$f_1(y_1 - d_1) + f_1(x_1 - d_1 - \alpha) \geqq f_1(y_1 - d_1 - \alpha) + f_1(x_1 - d_1) \text{ and}$$
$$f_2(y_2 - d_2) + f_2(x_2 - d_2 + \beta) > f_2(y_2 - d_2 + \beta) + f_2(x_2 - d_2).$$

By adding up these two inequalities, we obtain a contradiction to (iii).

Similarly, we can show that the second case, $y_1 < x_1$ and $y_2 \geqq x_2$, is impossible.

Q.E.D.

Now we show that the normalized monotone path solutions, which generalize the Kalai-Smorodinsky solution, are immune to the strong transfer paradox.

Theorem 4

Normalized monotone path solutions are immune to the strong transfer paradox.

***Proof*:** Let F be a normalized monotone path solution and (S, d), (S, d')

$\in \Sigma^n$ be two problems satisfying the hypotheses of *N.S.T.P.* Since F satisfies *S.INV*, we may assume that $d = 0$ and a $(S, d) = e$. Without loss of generality, we can also assume that $d' \equiv (\alpha, -\beta, 0, \cdots, 0)$ for $\alpha > 0$ and $\beta > 0$. Let $x \equiv F(S, d)$ and $x' \equiv F(S, d')$. Note that $a_1(S, d') \geqq a_1(S, d) = 1$ and $a_2(S, d') \leqq a_2(S, d) = 1$.

Let $\lambda \in \Lambda^n$ be a positive linear transformation such that $\lambda(d') = 0$ and $\lambda(a(S, d')) = e$, and $y \equiv \lambda(x')$. Since both x and y are points on the monotone path, either $y > x$, or $y = x$, or $y < x$. Moreover, it can be easily shown that

(i) $x_1' = (a_1(S, d') - \alpha)\, y_1 + \alpha$ and $x_2' = (a_2(S, d') + \beta)\, y_2 - \beta$.

It is enough to show that neither $[x_1' \leqq x_1$ and $x_1' > x_2]$ nor $[x_1' < x_1$ and $x_2' \geqq x_2]$ can happen. Suppose now, by way of contradiction, that the first case holds, i.e., $x_1' \leqq x_1$ and $x_2' > x_2$. Then, from (i), we have

(ii) $(a_1(S, d') - \alpha)\, y_1 + \alpha \leqq x_1$ and $(a_2(S, d') + \beta)\, y_2 - \beta > x_2$.

Since $y_1 > 0$, $y_2 > 0$, $a_1(S, d') > \alpha$, and $a_2(S, d') > -\beta$, together with $a_1(S, d')\, \gamma\, a_1(S, d) = 1$ and $a_2(S, d') \leqq a_2(S, d') = 1$, (ii) gives

$$(1 - \alpha)\, y_1 + \alpha \leqq x_1 \text{ and } (1 + \beta)\, y_2 - \beta > x_2,$$

or equivalently,

$$\alpha\,(1 - y_1) \leqq x_1 - y_1 \text{ and } \beta\,(y_2 - 1) > x_2 - y_2.$$

Since $0 < y \leqq e$, $\alpha\,(1 - y_1) \geqq 0$ and $\beta\,(y_2 - 1) \leqq 0$, which imply that (iii) $y_1 \leqq x_1$ and (iv) $y_2 > x_2$. As noted earlier, either $y > x$, or $y = x$, or $y < x$. Consequently, (iii) and (iv) cannot be satisfied at the same time, which is the desired contradiction.

Similarly, we can show that the second case, $x_1' < x_1$ and $x_2' \geqq x_2$, is impossible.

Q.E.D.

Remark 2

It can easily be checked that *Pareto optimality*,[12] *d-continuity*, and *N.S.T.P* together imply *d-MON*. Since N^f satisfies the three conditions, it

[12]*Pareto optimality* requires that the solution outcome be a Pareto optimal point of the feasible set. *d-continuity* requires that a small change in the disagreement point cause only a small change in the solution outcome. *Weak Pareto optimality* requires that the solution outcome be a weakly Pareto optimal point of the feasible set. Finally, *S-continuity* requires that a small change in the feasible set cause only a small change in the solution outcome.

satisfies *d-MON*. Also it can easily be checked that *weak Pareto optimality, d-continuity, S-continuity*, and *N.S.T.P* together imply *d-MON*. Since K^G satisfies the four conditions, it satisfies *d-MON*. However, it remains an open question whether *d-MON* together with other minor conditions implies *N.S.T.P.*

***Remark* 3**

Although *N.S.T.P* is a weak condition, it is not difficult to find examples of bargaining solutions that do not satisfy the condition. In fact, we can use examples in Thomson (1987, 56-7). Note that, these examples are similar to N^f, except that their f_i's are required only to be strictly quasi-concave, not strictly concave.

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