Note:
Does Uncertain Future Hamper Cooperation?

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In this paper, a counter example is constructed to show that the equilibrium strategy of Fudenberg and Maskin (1986) can not be applied when players have incomplete information about the future stage game. When the future stage game is uncertain, players have ex-post incentive to deviate from the equilibrium strategy of Fudenberg and Maskin since the ex-ante future punishment is not sufficiently severe. (*JEL* classifications: C72, C73)

I. Motivation

It is well known that strategic interaction in a long-term relationship may allow cooperative outcomes that would not occur in a one-shot game. The essence of this phenomenon can be summarized in the "Folk Theorem" for repeated games, which asserts that any individually rational outcome can arise as a perfect equilibrium in infinitely repeated games with sufficiently little discounting. Repeated play allows players to respond to each other's actions, so that each player must consider the reactions of his opponents in making his decision. Thus the possibility of future retaliation allows cooperative outcome to occur once repetition is allowed.

Using Abreu (1988)'s simple strategy profiles, Fudenberg and Maskin (1986) prove the Folk Theorem by constructing an equilibrium strategy which can support, under *complete* information, any individually ratio-

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nal payoff vector of an one-shot game as a perfect equilibrium payoff of the infinitely repeated game when players are sufficiently patient. The idea behind their construction of equilibrium strategy is simple. If a player deviates from initial path, he is minmaxed by the other players long enough to wipe out any gain from his deviation. To induce the other players to go through with minmaxing him, they are ultimately given a "reward" in the future. If a further deviation occurs during the punishment phase, the punishment phase is begun again.

In this paper, I show that there are some situations in which the equilibrium strategy in Fudenberg and Maskin (1986) can not support some level of payoff as a subgame perfect equilibrium payoff. More specifically, I raise uncertainty of future stage game as a source which may hamper a cooperation in infinitely repeated games. Consider a class of infinitely repeated games where nature chooses the stage game at the beginning of each period according to a certain distribution. The game is repeated stochastically and the players have *incomplete* information about the stage game they will play until nature chooses one.

There are a few examples for this class of game. Rotemberg and Saloner (1986) analyze the behavior of oligopolies in the repeated game situation where there are observable shifts in industry demand at the beginning of each period. They argue that oligopolies find implicit collusion is more difficult in the period when demand is relatively high. When today's demand is relatively high and price is the strategic variable, a single firm can get large benefit from undercutting ex-post the price that maximizes joint profits. However, the punishment from deviating is not severe since oligopolies can carry out only ex-ante punishments. Thus, when demand is high, the benefit of deviating from the joint profit-maximizing output may exceed the punishment a deviating firm can expect.

When the stage game is chosen stochastically as in Rotemberg and Saloner (1986), players make a decision after nature has chosen a stage game that will be played in this period. Thus perfection should be checked for ex-post decision by the players. When the players consider a deviation from the equilibrium strategy, they need to compare the ex-post gain from deviation and the ex-ante punishment which is weighted by a certain probability distribution. Thus it is possible in some stage game that this ex-post gain may outweigh the ex-ante punishment and future punishment may be non-credible. This ex-post incentive consideration complicates the analysis of Folk theorem in Fudenberg and Maskin (1986).

II. A Counter Example

According to Fudenberg and Maskin (1986), when either there are only two players or a "full dimensionality" condition holds, any individually rational payoff vector of a one-shot game of complete information can arise in a perfect equilibrium of the infinitely-repeated game if players are sufficiently patient. Formally, the following theorem applies for two-player games.[1]

Theorem 1 of Fudenberg and Maskin (1986)

For any individually rational payoff $(v_1, v_2) \in V^*$, there exists $\underline{\delta} \in (0, 1)$ such that, for all $\delta \in (\underline{\delta}, 1)$, there exists a subgame perfect equilibrium of the infinitely repeated game in which player *i*'s average payoff is v_i when players have discount factor δ.

The equilibrium strategies constructed in the proof of this theorem has the following simple feature. After a deviation by either players, each player minmaxes the other for a certain number of periods, after which they return to the original path. If a further deviation occurs during the punishment phase, the phase is begun again. Notice that they used punishment strategies which is "simple" in the sense of Abreu (1988). Since there is no loss in restricting attention to simple punishments when players discount future, it was possible for them to construct the equilibrium strategies without relying on arbitrarily severe punishments as in Aumann and Shapley (1976) and Rubinstein (1979).

When the stage is repeated stochastically, the uncertainty in the future stage game may complicate the analysis of Folk theorem. This is true because the perfection in the stochastically repeated game should be satisfied for ex-post decision by the players. I want to show that it is possible in some stage game that the ex-post gain from deviating the equilibrium strategies is outweighing the expected ex-ante punishment since the future since the future punishment is weighted by probability distribution over states.

In order to explain the idea more clearly, consider the following two-person stochastic game;

[1]A general argument for the n player game can be similarly extended from two-player case.

$g(\phi_1)$		Player B	
		b_{11}	b_{12}
Player A	a_{11}	4,4	2+ε, 2
	a_{12}	2,2+ε	2, 2

$g(\phi_2)$		Player B	
		b_{21}	b_{22}
Player A	a_{21}	4, 4	3+ε, 2
	a_{22}	2, 3+ε	2, 2

where $0 < \varepsilon < 1$. At the beginning of each period, nature chooses g (ϕ_1) as stage game with probability 1/2 and g (ϕ_2) with probability 1/2. These two games are similar in their payoff structure. However, minmaxing the other player in state ϕ_2 is more costly than in state ϕ_1.

In this stochastically repeated situation, we can consider as if two players are playing the following ex-ante stage game G.

		Player B			
		(b_{11}, b_{21})	(b_{11}, b_{22})	(b_{21}, b_{21})	(b_{12}, b_{22})
Player A	(a_{11}, a_{21})	8/2, 8/2	(7+ε)2, 6/2	(6+ε)/2, 6/2	(5+2ε)2, 4/2
	(a_{11}, a_{22})	6/2, (7+ε)/2	6/2, 6/2	(4+ε)/2, (5+ε)/2	(4+ε)/2, 4/2
	(a_{12}, a_{21})	6/2, (6+ε)/2	(5+ε)/2, (4+ε)/2	6/2, 6/2	(5+ε)/2, 4/2
	(a_{12}, a_{22})	4/2, (5+2ε)/2	4/2, (4+ε)/2	4/2, (5+ε)/2	4/2, 4/2

Before nature chooses the state of this period, players specify contingent actions in each state. For example, (a_{11}, a_{21}) represents player A's contingent action plan where he will choose a_{11} if ϕ_1 is realized and a_{21} if ϕ_2. The payoff of this ex-ante stage game is one-period expected payoff which depends on the probability distribution of ϕ's. Notice that player A and B, respectively, have (a_{12}, a_{22}) and (b_{12}, b_{22}) as minmaxing strategy so that their minmax values are {(5 + 2ε)/2, (5 + 2ε)/2}.

According to the Theorem 1 of Fudenberg and Maskin, any payoff vector greater than minmax value can be supported as a subgame perfect equilibrium of infinitely repeated game. However, this theorem cannot be applied if we have stochastic stage game since each player considers incentive to deviate ex-post in each subgame.

Consider the ex-ante individually rational discounted expected average payoff of {(11 + 4ε)/4, (11 + 4ε)/4} which is larger than minmax value {(5 + 2ε)/2, (5 + 2ε)/2}. I want to check perfection for player B in the subgame of state ϕ_2 deviation phase. According to Theorem 1 of Fudenberg and Maskin, the equilibrium strategy profile would give player B the payoff of $2 + \delta(1 - \delta^{t-1})4/2(1 - \delta) + \delta^t\,(11 + 4\varepsilon)/4(1 - \delta)$ if he follows equilibrium. However, player B will get get $(3 + \varepsilon) + \delta\,(1 - \delta^t)4/2(1 - \delta) + \delta^{t+1}\,(11 + 4\varepsilon)/4(1 - \delta)$ by deviating from equilibrium.

Notice that the one-period ex-post gain from deviation, $1 + \varepsilon$, is large in state ϕ_2 but the ex-ante future punishment, $(3 + 4\varepsilon)\,\delta^t/4$, is weighted by probability distribution over ϕ's. We can easily show that there does not exist $\delta < 1$ and t such that deviation from the equilibrium is not profitable. Thus player B does not have an ex-post incentive to follow the equilibrium strategy. I summarize this result in the following proposition.

Proposition

For some infinitely repeated stochastic games, there are some level of individually rational ex-ante payoff vectors for which Folk Theorem of Fudenberg and Maskin (1986) can not apply.

This example shows that the simple punishment of "mutual minmaxing" in each state may not be credible in the infinitely repeated *stochastic* game. When the stage is determined by nature at the beginning of each period, players have to compare ex-post gain from deviation and ex-ante expected punishment. These extra incentive constraints complicate the analysis and may reduce the possible range of payoffs that can be sustained as subgame perfect equilibria.

III. Comments

Several sources for this problem are possible. First, the simple strategy profile is not enough to characterize all the subgame perfect equilibria in the stochastically repeated game, so that we need to consider more general history-dependent strategies such as those in Aumann-Shapley/Rubinstein (1976, 1979) which use successively longer punishments. Second, the "mutual minmaxing" strategy of Fudenberg and Maskin (1986) may not be used for stochastically repeated game and we have to find the worst perfect equilibrium for each player in infinitely repeated stochastic game.

In general, we can charaterize the range of all subgame perfect equilibria if we find optimal simple penal code in infintely repeated game. As shown in the previous example, Fudenberg and Maskin's strategy can not be applied for a certain level of payoff.[2] It is an interesting

[2]Instead of finding explicit equilibrium strategy, it is possible to characterize perfect equilibria by using the dynamic-programming approach (Fudenberg, Levine and Maskin 1990). See Fudenberg and Tirole (1991) for the related literature.

research topic to construct an equilibrium strategy and find the range of ex-ante payoffs that can be supported as subgame perfect equilibria in infinitely repeated stochastic game. I expect that players may have a tendency to give up "mutual minmaxing" and return, for example, to a Nash equilibrium of each stage game if the ex-post incentive to deviate is too high for players. Thus the set of payoffs for subgame perfect equilibria may be smaller than that of the ex-ante individually rational payoffs. Since players take an action after realization of stage game, perfection requires all the ex-post incentive constraints to be satisfied. These extra constraints may restrict the range of payoffs that can be supported as subgame perfect equilibria.

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