Examples on ‘Note on Differential Regulator Equation for Non-minimum Phase Linear Systems with Time-varying Exosystems’

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1 Introduction

In this article we present two examples for [1], which has introduced a solution to the output regulation problem for non-minimum phase linear time-varying systems with time-varying exosystems.

2 Examples

Two illustrative examples are given; the first one is the case that the analytic solution to the DRE is achievable, while the second deals with the time-varying exosystem which becomes time-invariant after a finite time \( T > t_0 \). In the latter case, it will be observed that the convergence is achieved far before the time \( T \).

Example 1. Consider the 2nd order plant and the exosystem given by

\[
\begin{align*}
\dot{x} &= \begin{bmatrix} 1 & \lambda_t \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\
\dot{w} &= \rho_t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} w, \\
e &= x_2 - w_1,
\end{align*}
\]

where \( \lambda_t = 1 + (2 + 0.1 \cos t)(2 + 0.05 \cos t - 0.05 \sin t) \) and \( \rho_t = 2 + 0.1 \cos t \). Note that the system is of non-minimum phase.

To solve the problem, we need to find the solution of the DRE in \{9\} and the gains \( K_t \) and \( J_t \) in \{11\}. Before solving the DRE, the system (1) is put into the form in \{12a’\} and \{12c’\}:

\[
\begin{bmatrix} \dot{z} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} 1 & \lambda_t \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z \\ \zeta \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} \lambda_t & 0 \\ 0 & -\rho_t \end{bmatrix} w,
\]

where \( z = x_1 \) and \( \zeta = x_2 - w_1 \). Then, the solution \{15\} is obtained as, by integrating by parts, \( \Pi_t = [-1 - 2 - 0.05 \cos t + 0.05 \sin t] \), and this results in

\[
\Pi_t = \begin{bmatrix} -1 & -2 - 0.05 \cos t + 0.05 \sin t \\ 1 & 0 \end{bmatrix},
\]

\[
R_t = \begin{bmatrix} 1 & 4 + 0.15 \cos t - 0.05 \sin t \end{bmatrix}.
\]
For the gain $K_t$, we used the backstepping design to obtain $K_t = [−3 − \lambda_t − 1 − \lambda_t]$, which guarantees the condition \{10a\}. On the other hand, to find the output injection gain $J_t$ in \{11\} is more involved. Since the system (1) is uniformly observable, one may obtain the output injection gain $J_t$ by employing the coordinate transformation that converts the system (1) into time-varying observer canonical form, in theory. However, this approach is time-consuming and accompanies tedious calculation due to the inherent time-varying nature of the system. Rather than doing that, the linear matrix inequality (LMI) based approach, proposed in [2], is used under slight modification. We consider the following augmented system that is obtained from (1) when $u \equiv 0$,

$$\dot{\eta} = \begin{bmatrix} 1 & \lambda_t & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_t \\ 0 & 0 & -\rho_t & 0 \end{bmatrix} - J_t[0 \ 1 \ -1 \ 0] \eta$$

$$=: (A_0 + \varepsilon_{1t}A_1 + \varepsilon_{2t}A_2 - J_tC_0)\eta,$$

where $\eta = \text{col}(x, w)$, $C_0 = [0 \ 1 \ -1 \ 0]$, $\varepsilon_{1t} = \lambda_t - 1$, $\varepsilon_{2t} = \rho_t - 2$, and other matrices are appropriately defined. Under these settings, the theorem in [2] can be modified as follows:

**Lemma 1.** If there exist a symmetric positive definite matrix $P$ and a vector $Y$ such that $U_i^TP + PU_i - C_0^TY^T - YC_0 < 0$ for $i = 1, \ldots, 4$, where $U_1 = A_0 + \varepsilon_{1m}A_1 + \varepsilon_{2m}A_2$, $U_2 = A_0 + \varepsilon_{1m}A_1 + \varepsilon_{2M}A_2$, $U_3 = A_0 + \varepsilon_{1M}A_1 + \varepsilon_{2m}A_2$, $U_4 = A_0 + \varepsilon_{1M}A_1 + \varepsilon_{2M}A_2$, $\varepsilon_{jm}$ and $\varepsilon_{jM}$ are lower and upper bound for $\varepsilon_{jt}$ respectively, then the system (3) is exponentially stable with the output injection gain $J_t = P^{-1}Y$.

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Figure 1: The time responses for $w_1$ and $x_2$. 

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1The braces are used to explicitly designate that the equation numbers in the braces are those in [1], which causes no confusion with the equation numbers in this article.
Figure 2: The time responses for $w_1$ and $x_3$

The proof of this lemma is straightforward and is omitted. Using Lemma 1 and the LMI toolbox [3], we finally obtain a (constant) output injection gain $J_t = [54.7 \ 16.7 \ 6.7 \ -5.9]^T$ which guarantees the condition (10b). The simulation result with the designed dynamic output feedback controller of the form (11) is given in Fig. 1.

Example 2. Suppose the plant and the exosystem are given by

$$\dot{x} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} w,$$

$$e = x_3 - w_1,$$

$$\dot{w} = \rho_t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} w,$$

where

$$\rho_t = \begin{cases} 2.5 + 0.75t, & 0 \leq t \leq 10, \\ 10, & t > 10. \end{cases}$$

Since the system (4) eventually becomes time-invariant after $t = 10$, the scheme at the end of the Section 3 in [1] can be used. In fact, the solution $\Pi_t^{\text{as}}$ to (13) is obtained by integrating backward for the time interval $[0, 10] \ a \ priori$, where the initial condition is set to the solution of the (static) Sylvester equation (19). On the other hand, $\Pi_t^{\text{as}}$ is obtained on-line by running (12) and (17).

Next, the gains $K_t$ and $J_t$ need to be found to solve the problem. The gain $K_t$ is obtained as, by using pole placement, $K_t = [12 \ -1 \ -5]$ since the plant is time-invariant. For the gain

\[ K_t = \begin{bmatrix} 12 & -1 & -5 \end{bmatrix}, \quad \text{and} \quad \text{the initial condition for the controller is set to zero.} \]
$J_t$, the LMI approach in Example 1 is again used and results in $J_t = [58.8 \ 7.4 \ 24.1 \ 3.0 \ -4.0]^T$. The simulation result is depicted in Fig. 2.\textsuperscript{3} Note that the state trajectory for $x_3$ reaches its steady-state about $t = 6$, while the system becomes time-invariant at $t = 10$.

References


\textsuperscript{3}The initial condition for $x$ is set to $x(0) = [1 \ -1 \ 1]^T$ and others are the same as in Example 1.