Examples on 'Note on Differential Regulator Equation for Non-minimum Phase Linear Systems with Time-varying Exosystems'

H. Shim, Jung-Su Kim, Hongkeun Kim, and J. Back

1 Introduction

In this article we present two examples for [1], which has introduced a solution to the output regulation problem for non-minimum phase linear time-varying systems with time-varying exosystems.

2 Examples

Two illustrative examples are given; the first one is the case that the analytic solution to the DRE is achievable, while the second deals with the time-varying exosystem which becomes time-invariant after a finite time $T > t_0$. In the latter case, it will be observed that the convergence is achieved far before the time T.

Example 1. Consider the 2nd order plant and the exosystem given by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & \lambda_t \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad \dot{w} = \rho_t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} w, \\ e &= x_2 - w_1, \end{aligned} \tag{1}$$

where $\lambda_t = 1 + (2 + 0.1\cos t)(2 + 0.05\cos t - 0.05\sin t)$ and $\rho_t = 2 + 0.1\cos t$. Note that the system is of non-minimum phase.

To solve the problem, we need to find the solution of the DRE in {9}[1] and the gains K_t and J_t in {11}. Before solving the DRE, the system (1) is put into the form in {12a'} and {12c'}:

$$\begin{bmatrix} \dot{z} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} 1 & \lambda_t \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z \\ \zeta \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} \lambda_t & 0 \\ 0 & -\rho_t \end{bmatrix} w, \tag{2}$$

where $z = x_1$ and $\zeta = x_2 - w_1$. Then, the solution {15} is obtained as, by integrating by parts, $\Pi_t^z = [-1 \quad -2 - 0.05\cos t + 0.05\sin t]$, and this results in

$$\begin{aligned} \Pi_t &= \begin{bmatrix} -1 & -2 - 0.05\cos t + 0.05\sin t \\ 1 & 0 \end{bmatrix}, \\ R_t &= \begin{bmatrix} 1 & 4 + 0.15\cos t - 0.05\sin t \end{bmatrix}. \end{aligned}$$

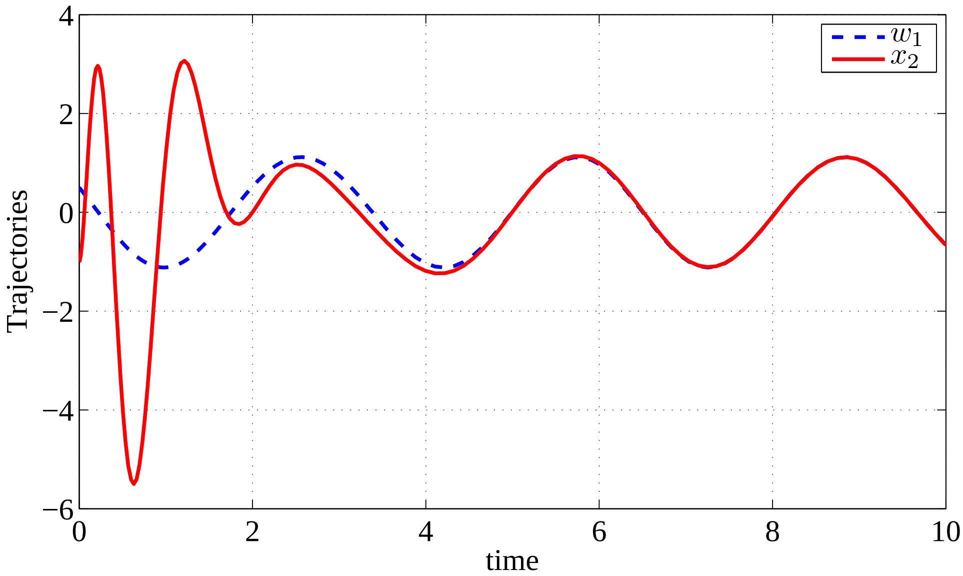

Figure 1: The time responses for w_1 and x_2.

For the gain K_t, we used the backstepping design to obtain $K_t = [-3-\lambda_t \quad -1-\lambda_t]$, which guarantees the condition {10a}. On the other hand, to find the output injection gain J_t in {11} is more involved. Since the system (1) is uniformly observable, one may obtain the output injection gain J_t by employing the coordinate transformation that converts the system (1) into time-varying observer canonical form, in theory. However, this approach is time-consuming and accompanies tedious calculation due to the inherent time-varying nature of the system. Rather than doing that, the linear matrix inequality (LMI) based approach, proposed in [2], is used under slight modification. We consider the following augmented system that is obtained from (1) when $u \equiv 0$,

$$\begin{aligned}\dot{\eta} &= \left(\begin{bmatrix} 1 & \lambda_t & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_t \\ 0 & 0 & -\rho_t & 0 \end{bmatrix} - J_t[0 \ 1 \ \ -1 \ 0] \right) \eta \\ &=: (A_0 + \varepsilon_{1t}A_1 + \varepsilon_{2t}A_2 - J_tC_0)\eta, \end{aligned} \tag{3}$$

where $\eta = \mathrm{col}(x, w)$, $C_0 = [0 \ 1 \ \ -1 \ 0]$, $\varepsilon_{1t} = \lambda_t - 1$, $\varepsilon_{2t} = \rho_t - 2$, and other matrices are appropriately defined. Under these settings, the theorem in [2] can be modified as follows:

Lemma 1. *If there exist a symmetric positive definite matrix P and a vector Y such that $U_i^TP + PU_i - C_0^TY^T - YC_0 < 0$ for $i = 1, \cdots, 4$, where $U_1 = A_0 + \varepsilon_{1m}A_1 + \varepsilon_{2m}A_2$, $U_2 = A_0 + \varepsilon_{1m}A_1 + \varepsilon_{2M}A_2$, $U_3 = A_0 + \varepsilon_{1M}A_1 + \varepsilon_{2m}A_2$, $U_4 = A_0 + \varepsilon_{1M}A_1 + \varepsilon_{2M}A_2$, ε_{jm} and ε_{jM} are lower and upper bound for ε_{jt} respectively, then the system (3) is exponentially stable with the output injection gain $J_t = P^{-1}Y$.*

[1]The braces are used to explicitly designate that the equation numbers in the braces are those in [1], which causes no confusion with the equation numbers in this article.

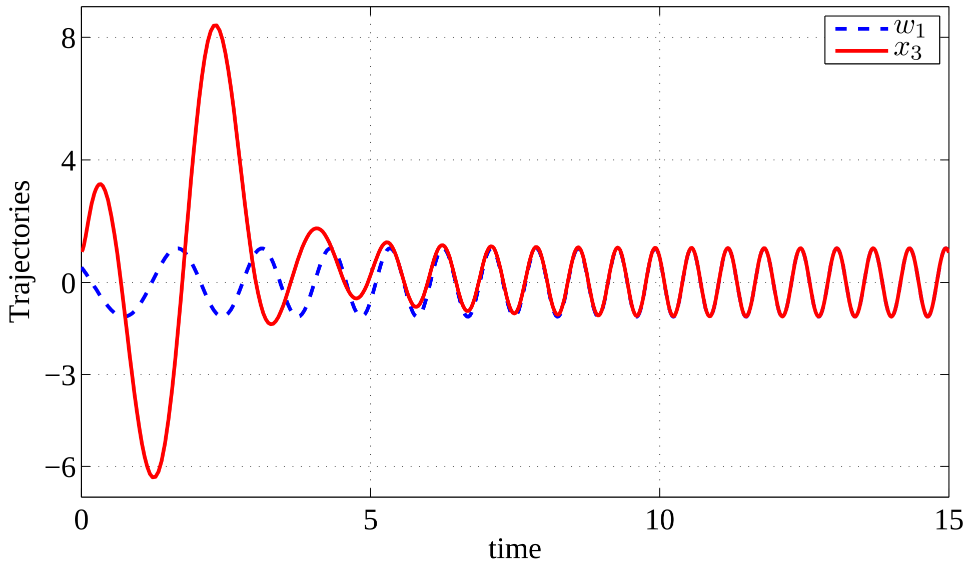

Figure 2: The time responses for w_1 and x_3

The proof of this lemma is straightforward and is omitted. Using Lemma 1 and the LMI toolbox [3], we finally obtain a (constant) output injection gain $J_t = [54.7 \;\; 16.7 \;\; 6.7 \;\; -5.9]^T$ which guarantees the condition {10b}. The simulation result with the designed dynamic output feedback controller of the form {11} is given in Fig. 1.[2]

Example 2. Suppose the plant and the exosystem are given by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} w, \\ e &= x_3 - w_1, \\ \dot{w} &= \rho_t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} w, \end{aligned} \tag{4}$$

where

$$\rho_t = \begin{cases} 2.5 + 0.75t, & 0 \le t \le 10, \\ 10, & t > 10. \end{cases}$$

Since the system (4) eventually becomes time-invariant after $t = 10$, the scheme at the end of the Section 3 in [1] can be used. In fact, the solution Π_t^{za} to {13} is obtained by *integrating backward* for the time interval $[0, 10]$ *a priori*, where the initial condition is set to the solution of the (static) Sylvester equation {19}. On the other hand, Π_t^{zs} is obtained *on-line* by running {12} and {17}.

Next, the gains K_t and J_t need to be found to solve the problem. The gain K_t is obtained as, by using pole placement, $K_t = [12 \;\; -1 \;\; -5]$ since the plant is time-invariant. For the gain

[2]Here, $x(0) = [1 \;\; -1]^T$, $w(0) = [0.5 - 1]^T$, and the initial condition for the controller is set to zero.

J_t, the LMI approach in Example 1 is again used and results in $J_t = [58.8\ \ 7.4\ \ 24.1\ \ 3.0\ \ -4.0]^T$. The simulation result is depicted in Fig. 2.[3] Note that the state trajectory for x_3 reaches its steady-state about $t = 6$, while the system becomes time-invariant at $t = 10$.

References

[1] H. Shim, J.-S. Kim, H. Kim, and J. Back, "Note on differential regulator equation for non-minimum phase linear systems with time-varying exosystems," to appear in *Automatica*, 2010.

[2] L.A. Zheng, S.H. Chen, and J.H. Chou, "LMI robust stability condition for linear systems with time-varying elemental uncertainties, norm-bounded uncertainties and delay perturbations," *JSME International Journal Series C*, vol. 47, pp. 275–279, 2004.

[3] P. Gahinet, A. Nemirovski, A.J. Laub, and M. Chilali, *LMI control toolbox*, The Math Works Inc., Massachusetts, 1995.

[3] The initial condition for x is set to $x(0) = [1\ \ -1\ 1]^T$ and others are the same as in Example 1.