Coase Theorem in Two-sided Matching Marriage Games

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This paper demonstrates the Coase Theorem via a two-sided matching framework. We show that in a residual claimer sharing rule economy, regardless of which party gets the right to claim the residual, the equilibrium marital status maximizes the total output and is identical to the optimal marital status in a central planner dictatorial economy. (*JEL* Classifications: J12, C78)

I. Introduction

Choosing a lifetime partner is an important decision. Before people get married, they have to enter the marriage market to signal their existences and reveal their information strategically to other parties. In principle, any occasion that involves more than one person is a potential marriage market. Players in the marriage market are both searching and being searched. If some basic conditions are satisfied, 'dating' starts.

Dating exists because marriage is not totally unbinding. The purposes of dating are for people to reveal their own information and search for partners'. Usually, the dating process takes the form of one partner at a time in order to avoid instability of partnership. The costs for dating are sizable, but they can be regarded as investments whose returns are a stream of temporary partial spousal services, and the avoiding of a lifetime mismatch.

After dating, people decide whether to break up or get married with their dating partners. The benefits of getting married are:

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Imposing costs on competitors; monopolization and monopsonization of spousal services; economies of scale in household production; producing loves and cares that can rarely be purchased in the outside market; giving one's offsprings a stable and safe environment of parental loves and cares, and so on. Once a person gets married, he/she is said to exit the marriage market for the period that his/her marriage contract is effective.

The two-sided matching theory has been used to study the marriage market since the seminal work of Gale and Shapley (1962). A more comprehensive treatment of the theory is provided by Roth and Sotomayor (1990). This paper characterizes and distinguishes the equilibrium of a simple one-period-two-sided matching game with complete information under different forms of societies and sharing rules. The games being considered are: a) A democratic society where marriage requires mutual consents, and people of the same sex pay wages to the opposite sex and claim whatever leftovers; b) A dictatorial society where the dictator arranges marriage in order to maximize the total output of society.

The Coase Theorem in Section IV states that, regardless of which party gets the right to claim the residual, the equilibrium marital status maximizes the total output, and is identical to the optimal marital status in a central planner dictatorial economy.

II. The Model

Let $H=\{m_1, m_2, \cdots, m_n\}$ be a set of men and $F=\{f_1, f_2, \cdots, f_n\}$ be a set of women. Each person is totally characterized by a vector of standardized traits, namely, $m_h=(t^1_{m_h}, t^2_{m_h}, \cdots, t^K_{m_h})\in B\subseteq R^K$ and $f_j=(t^1_{fj}, t^2_{fj}, \cdots, t^K_{fj})\in B'\subseteq R^K$. We assume $|H|=|F|=n<\infty$ and $H\cap F=\phi$. The traits of each person are generated by a multivariate process $\zeta(t^1, t^2, \cdots, t^K)$. Assume the traits of the same sort are additive and the production technology is given by a real-value continuous function $\varphi: R^K\rightarrow R_+$. The output of a single man h is $\varphi_{h0}=\varphi(t^1_{m_h}, t^2_{m_h}, \cdots, t^K_{m_h})$ and the output of a marriage between man h and woman j is $\varphi_{hj}=\varphi(t^1_{m_h}+t^1_{fj}, t^2_{m_h}+t^2_{fj}, \cdots, t^K_{m_h}+t^K_{fj})$. We assume there is a $k\in\{1, 2, \cdots, K\}$ such that $\varphi'_{t^k}>0$. Let E be the set of all conceivable outcomes. For instance, everybody remaining single is one of the elements in E. Given that we have n men and n women in the economy, with heterosexual one to one marriage and being single is allowed, the number of elements in E is

$$|E| = \sum_{s=0}^{n} \binom{n}{s}^2 (n-s)!.$$

One thing deserves our attention is that, marriage based merely on production considerations could not explain why it needs to be one-to-one and occurs only between the opposite sex. Thus we should also assume that the output of a marriage from at least two people of the same sex is less than the sum of output of those people when they remain singles, so that there exists some coalitions that block marriages consisting of the same sex. Without assuming this, marriages are not necessarily occurring between two disjoint sets, and the matching problem becomes one-sided instead of two-sided. In this paper, we only discuss the case of monogamy where marriage takes place between the opposite sex and is one to one. Before proceeding to the next section, let us define the followings:

***Definition* 1**

A matching μ is a one-to-one correspondence from the set $H \cup F$ onto itself of order two (i.e., $\mu^2(x) = x$) such that if $\mu(m_h) \neq m_h$, $h=1, 2, \cdots, n$, then $\mu(m_h) \in F$ and if $\mu(f_j) \neq f_j$, $j=1, 2, \cdots, n$, then $\mu(f_j) \in H$. We refer to $\mu(x)$ as the *mate* of x.

To achieve a notational economy, we write $\mu(m_h) = f_j$ and $\mu(h) = j$. Let $\lambda_{h\mu(h)}$ be the share of output of m_h when he marries $\mu(m_h)$. When $\mu(m_h) = m_h$, we define $j=0$ and $\lambda_{h0}=1$.

III. Democratic Society, Residual Claimer Sharing Rule

Consider a democratic society where marriage requires mutual consents. Each person tries to find a mate to maximize his/her utility. Since the production function φ has already incorporated all the traits of men and women, the utility of a man can simply be represented by $\lambda_{h\mu(h)} \varphi_{h\mu(h)}$, so that his utility is linear to output. Throughout this paper, we assume the preference of each individual is determined entirely by the production function φ. An individual chooses a mate according to endowments of traits of himself/herself and of the partner.

Consider the output table (Table 1) generated by some underlying endowments of traits and a special production function:

TABLE 1

	S	f_1	f_2
S		9	6
m_1	11	18	20
m_2	7	17	13

Suppose the sharing rule is that people of the same sex pay wages to the opposite sex and claim whatever leftovers. Suppose also that a wage earner is willing to get married as long as the wage offered is no less than his/her reservation wage (which is the output of being single). Suppose the right to claim the residual is assigned to men. If a woman is wanted by many men, then those men have to bid for this woman and she will get some economic rents. In the first trial, men should not offer wages higher than women's reservation wages unless there are competitors. If a man offers the woman he prefers the most her reservation wage, then the preference structure under Table 1 will be:

$$P(m_1) = f_2,\ S,\ f_1.$$

$$P(m_2) = f_1,\ [S,\ f_2].$$

$$P(f_1) = [S,\ m_2],\ m_1.$$

$$P(f_2) = [S,\ m_1],\ m_2.$$

where $P(x)$ represents the ordered list of preferences of x on the set $\{x\} \cup F$ if $x \in H$, and it represents the ordered list of preferences of x on the set $\{x\} \cup H$ if $x \in F$.

For example, the above preference indicates that m_1's first choice is to marry f_2, his second choice is to remain single, and his third choice is to marry f_1. For m_2, his first choice is to marry f_1, his second choice is to remain single or to marry f_2.

There is no conflict between men, m_1 will marry f_2 in order to get a residual of 14(20−6), and m_2 will marry f_1 in order to get a residual of 8(17−9). The market is cleared and the total output yielded is $\varphi_{12} + \varphi_{21} = 20 + 17 = 37$.

If women are residual claimers, then f_1 will choose m_2 in order to capture a residual of 10(17−7), and f_2 will choose m_1 in order to capture a residual of 9(20−11). Again, the total output is maximized and there is no conflict.

We will show that the total output under residual claimer sharing rule is always the maximum possible output and this does not

TABLE 2

	S	f_1	f_2	f_3
S		4	5	7
m_1	2	6	9	12
m_2	3	8	10	14
m_3	5	10	12	11

TABLE 3

	S	f_1	f_2	f_3
S				
m_1				5, 7
m_2				7, 7
m_3			7, 5	

depend on which party has the right to claim the residual. We argue that, no matter men or women are the residual claimers, the equilibrium marital status will still be the same and should also be equal to that under a dictatorial society. The only difference is which party gets the rent.

Let us consider an example where conflict exists:

In Table 2, m_1 will get 2 if he is single. If he marries f_1, they will split an output of 6 units, and so on. If men claim the residual, m_1 will find f_3 gives him the highest residual. Thus, he will offer f_3 her reservation wage 7. m_2 will also give an offer to f_3 and m_3 will give an offer to f_2. The situation is depicted in Table 3.

If a woman is desired by many men, then those men have to bid for her. The bidding process follows assumption 1.

***Assumption* 1:** Deferred Acceptance Mechanism

(i) Each man offers the woman he prefers the most her reservation wage. If a woman gets only one offer, then she is engaged to the man who offers her.

(ii) If there are more than one man pursuing the same woman, then those men decide whether to bid for that woman. These men follow a rule that, if there are other alternatives which give them the same return but need no bidding, then they will choose those alternatives and will not take part in this bid. Otherwise, they participate in the bid. The bidding takes the form of increasing that woman's wage by 1 unit at a time (we assume output is discrete to simplify our analysis). A man decides whether to keep increasing

TABLE 4

	S	f_1	f_2	f_3
S				
m_1			4, 5	4, 8
m_2				6, 8
m_3			7, 5	

TABLE 5

	S	f_1	f_2	f_3
S				
m_1			3, 6	3, 9
m_2				5, 9
m_3		6, 4	6, 6	

the wage to this woman in the third round by comparing the new residual from this woman to the return from his second best alternative.

If his new residual from this first best woman is the same as his return from the second best alternatives, and if his second best woman is neither engaged nor offered by other men in this round, then he leaves his first best for this second best woman. He will remain single if his second best alternative is to remain single.

If his second best woman has been engaged, then he has to raise wages to both the first and the second best women, and compares his new returns from these two women to his third best alternatives. This process goes on until everybody is engaged or remains single.

In Table 3, f_3 is wanted by m_1 and m_2. Under assumption 1, they have to bid for her by raising her wage to 8 in the second trial. However, if m_1 gives 8 to f_3, he can only get 4. As a result, f_3 is no longer superior to f_2 from m_1 point of view. The situation is depicted at Table 4 below.

By the deferred acceptance mechanism, the preference of men is lexicographic: i.e., each man chooses a woman who gives him the highest residual, but if he gets the same residual from different women, he prefers the one without competition. Now m_1 faces competitors no matter he chooses f_2 or f_3, thus he is indifferent between f_2 and f_3. He will bid for both women in the third trial as shown below in Table 5.

TABLE 6

	S	f_1	f_2	f_3
S				
m_1			3, 6	
m_2				5, 9
m_3		6, 4		

TABLE 7

	S	f_1	f_2	f_3
S				
m_1			2, 7	
m_2		3, 5	3, 7	3, 11
m_3		5, 5	5, 7	

TABLE 8

	S	f_1	f_2	f_3
S				
m_1			2, 7	
m_2				3, 11
m_3		5, 5		

Given that the wage of f_2 is 6 now, m_3 will begin to offer f_1 who gives him the same residual. m_3 will prefer f_1 since there is no competitor. Given m_3 is not competing with m_1 for f_2, m_1 will choose f_2 instead of competing with m_2 for f_3. Therefore, m_2 has no competitor and the game is over, with m_1 marrying f_2, m_2 marrying f_3, and m_3 marrying f_1. The take-home output of each player is shown in Table 6 below.

If women are residual claimers, the game is played in a similar manner, but will be finished earlier.

In the first trial depicted in Table 7, f_1 offers to m_2 and m_3, f_2 offers to all men, f_3 offers to m_2 only. By the lexicographic preference structure, the game is over in the second trial as shown in Table 8.

The outcome under the lexicographic tie-breaking rule will be in the core. Note that we end up with the same marital assignment as the case where men are residual claimers.

It should be mentioned that the bidding process under assumption 1 may not generate a well-defined outcome if the core is not unique. Consider the following example[1]:

[1]I would like to thank an anonymous referee for pointing out this problem.

TABLE 9

	S	f_1	f_2	f_3
S		0	0	0
m_1	0	10	10	
m_2	0		10	10
m_3	0	10		10

In Table 9, as long as all people get married, the total output is identical. Thus if we do not assume uniqueness of outcome, we can only conclude that the core are the same regardless which party get the right to claim residuals. To ensure a well-defined outcome, we have to assume that the core only consists of a single element.

***Assumption* 2**
The total societal output has a unique maximum.

IV. Dictatorial Society and the Coase Theorem

In a dictatorial society, the dictator decides the marital status of everybody according to assumption 3.

***Assumption* 3**
The dictator is a residual claimer who offers reservation wages to each individual, so that the individual rationality condition still preserves.

Since the sum of all individuals' reservation wages is independent of the matching outcomes, the objective of the dictator under assumption 2 is just to maximize the total output by choosing some marital assignments. The dictator is facing the following optimal assignment problem:

$$\max_{x_{hj}\in\{0,\ 1\}} \sum_{h=0}^{n} \sum_{j=0}^{n} \varphi_{hj}\, x_{hj}$$

s.t.

$$\sum_{j=0}^{n} x_{hj} = 1,$$

$$\sum_{h=0}^{n} x_{hj} = 1.$$

Define $x_{00}=\varphi_{00}=0$. For $j=0$ and $h=1, 2, \cdots, N$, $x_{h0}=1$ means m_h is assigned to remain single. For $h=0$ and $j=1, 2, \cdots, N$, $x_{0j}=1$ means f_j is assigned to remain single. For $h=1, 2, \cdots, N$ and $j=1, 2, \cdots, N$, $x_{hj}=1$ means m_h is assigned to f_j. It is obvious that the solution to this linear programming problem for Table 1 is $x_{01}=x_{02}=x_{10}=x_{20}=x_{11}=x_{22}=0$ and $x_{12}=x_{21}=1$.

Therefore the dictator will assign m_1 to f_2, and m_2 to f_1, achieving a total output of 37, and he makes 2 units of profits.

Theorem 1 below is a version of Coase Theorem (Coase 1960) which states the relationship between the marriage market equilibria under a central planning economy and a democratic economy with residual claimer sharing rule:

***Theorem* 1 (Coase 1960)**

Under assumptions 1 to 3, the equilibrium marital status in a residual claimer sharing a rule economy is the same regardless of which party gets the right to claim the residual. The equilibrium marital status maximizes the total output and is identical to the optimal marital status in a central planner dictatorial economy.

***Proof*:** Consider the case where men are residual claimers. For each man h, he chooses a woman j to maximize his residual $\varphi_{hj}-\varphi_{0j}$, $j=1, 2, \cdots, n$. If $\varphi_{hj}-\varphi_{0j}<\varphi_{h0}$ for all j, then he prefers to remain single. We consider two cases.

Case 1: If $\mu(m_h)\neq\mu(m_{h'})$ $\forall h\neq h'$, there is no conflict of interest between men and every man maximizing his residual. This implies the total residual of society is maximized. The problem is reduced to

$$\max_{x_{hj}\in\{0,1\}} \sum_{h=1}^{n}\sum_{j=0}^{n}(\varphi_{hj}-\varphi_{0j})x_{hj}$$

s.t.

$$\sum_{j=0}^{n} x_{hj}=1,$$

$$\sum_{h=0}^{n} x_{hj}=1.$$

Note that

$$\sum_{h=1}^{n}\sum_{j=0}^{n}(\varphi_{hj}-\varphi_{0j})x_{hj}$$

$$=\sum_{h=1}^{n}\sum_{j=0}^{n}\varphi_{hj}x_{hj}-\sum_{j=0}^{n}\varphi_{0j}\sum_{h=1}^{n}x_{hj}$$

$$= \sum_{h=1}^{n}\sum_{j=0}^{n} \varphi_{hj} x_{hj} - \sum_{j=0}^{n} \varphi_{0j}(1 - x_{0j})$$

$$= \sum_{h=0}^{n}\sum_{j=0}^{n} \varphi_{hj} x_{hj} - \sum_{j=0}^{n} \varphi_{0j}.$$

Since $\sum_{j=0}^{n} \varphi_{0j}$ does not depend on x_{hj}, our problem is further reduced to maximizing the social output, which is equivalent to the social planner problem.

If women are residual claimers, we get a similar objective function:

$$\max_{x'_{hj}\in\{0,\ 1\}} \sum_{j=1}^{n}\sum_{h=0}^{n} (\varphi_{hj} - \varphi_{h0})\, x'_{hj}$$

s.t.

$$\sum_{j=0}^{n} x'_{hj} = 1,$$

$$\sum_{h=0}^{n} x'_{hj} = 1.$$

Note that

$$\sum_{j=1}^{n}\sum_{h=0}^{n} (\varphi_{hj} - \varphi_{h0})\, x'_{hj}$$

$$= \sum_{j=1}^{n}\sum_{h=0}^{n} \varphi_{hj} x'_{hj} - \sum_{h=0}^{n} \varphi_{h0} \sum_{j=1}^{n} x'_{hj}$$

$$= \sum_{j=1}^{n}\sum_{h=0}^{n} \varphi_{hj} x'_{hj} - \sum_{h=0}^{n} \varphi_{h0}(1 - x'_{h0})$$

$$= \sum_{j=0}^{n}\sum_{h=0}^{n} \varphi_{hj} x'_{hj} - \sum_{h=0}^{n} \varphi_{h0}.$$

By a similar argument, the societal output is also maximized. Hence,

$$\sum_{h=0}^{n}\sum_{j=0}^{n} \varphi_{hj} x_{hj} = \sum_{j=0}^{n}\sum_{h=0}^{n} \varphi_{hj} x'_{hj}.$$

Since we assume an unique maximum, we have $x_{hj} = x'_{hj}\ \forall h, j$.

Therefore, the matching are the same irrespective of which party being the residual claimer.

Case 2: If a woman is desired by many men, then those men have to bid for that woman. We assume the bidding process follows the deferred acceptance mechanism. Since the residuals of men are decreasing with the number of bids, when the bidding process is completed, no man can find other women who give him a strictly higher residual than the one he is engaged to. Similarly, no woman can find other men who are willing to give her a higher wage than the one she has already been engaged to, otherwise the bidding process has to go on. The outcome is unblocked by any individual or any pair of agents and by definition it is in the core. The case where women being residual claimers can be analyzed analogously. Since the core is also the social planner outcomes, we conclude that the residual- claimer rule generates outcomes identical to the social planner problem irrespective of which party has the right to claim the residual.

Q.E.D.

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