Supplementary Material for "Adaptive Regulation to Nominal Response for Uncertain Mechanical Systems and Its Application to Optical Disk Drive"

Gyunghoon Park and Hyungjong Kim

This supplementary material aims at providing the detailed proof of the technical lemmas in the authors' work [SM1].

SM.I. Proof of Lemma 1

The proof begins by computing the time derivative of η using [SM1, Eq. (18a)] as follows:

$$\begin{aligned}\dot\eta &= \frac{1}{\tau}\Upsilon\dot p + \frac{1}{\tau}\Upsilon\Gamma\Phi_{\mathsf n}\frac{1}{g_{\mathsf n}}\dot x + \frac{1}{\tau}\Upsilon\dot\Gamma\Phi_{\mathsf n}\frac{1}{g_{\mathsf n}}x\\ &= \frac{1}{\tau}\Upsilon\left(A_4 - \Upsilon^{-1}\alpha C_4\right)p - \frac{1}{\tau}\gamma u + \frac{1}{\tau}\beta\frac{1}{g_{\mathsf n}}C_2x + \frac{1}{\tau}\Upsilon\Gamma\Phi_{\mathsf n}\frac{1}{g_{\mathsf n}}\dot x + \frac{1}{\tau}\Upsilon\dot\Gamma\Phi_{\mathsf n}\frac{1}{g_{\mathsf n}}x\\ &= \Upsilon\left(A_4 - \Upsilon^{-1}\alpha C_4\right)\Upsilon^{-1}\eta - \frac{1}{\tau}\gamma u + \frac{1}{\tau}\beta\frac{1}{g_{\mathsf n}}C_2x - \frac{1}{\tau}\Upsilon(A_4 - \Upsilon^{-1}\alpha C_4)\Gamma\Phi_{\mathsf n}\frac{1}{g_{\mathsf n}}x + \frac{1}{\tau}\Upsilon\Gamma\Phi_{\mathsf n}\frac{1}{g_{\mathsf n}}\dot x + \frac{1}{\tau}\Upsilon\dot\Gamma\Phi_{\mathsf n}\frac{1}{g_{\mathsf n}}x\end{aligned} \tag{SM1}$$

where the last equality is derived from

$$p = \tau\Upsilon^{-1}(\tau)\eta - \Gamma(\hat\theta;\tau)\Phi_{\mathsf n}\frac{1}{g_{\mathsf n}}x. \tag{SM2}$$

It is pointed out that $u = u_{\mathsf c} - \bar s(C_4\eta)$,

$$\Upsilon\left(A_4 - \Upsilon^{-1}\alpha C_4\right)\Upsilon^{-1} = \Upsilon A_4\Upsilon^{-1} - \alpha C_4\Upsilon^{-1} = \frac{1}{\tau}\left(A_4 - \alpha C_4\right), \quad \text{and} \quad \Upsilon\Gamma\Phi_{\mathsf n}B_2 = \gamma.$$

On the other hand, using the nominal components $\phi_{\mathsf n}$ and $g_{\mathsf n}$, the x-dynamics [SM1, Eq. (2)] can be rewritten as

$$\dot x = \big(A_2 + B_2\phi_{\mathsf n}^\top\big)x + B_2g_{\mathsf n}u_{\mathsf c} - B_2g\big(\bar s(C_4\eta) - \lambda\big). \tag{SM3}$$

Then with (SM3) and an additional symbol

$$\Pi := \beta C_2 - \Upsilon\left(A_4 - \Upsilon^{-1}\alpha C_4\right)\Gamma\Phi_{\mathsf n} + \Upsilon\Gamma\Phi_{\mathsf n}(A_2 + B_2\phi_{\mathsf n}^\top),$$

the η-dynamics (SM1) turns out to be a simpler form

$$\begin{aligned}\tau\dot\eta &= \tau\left(\Upsilon\left(A_4 - \Upsilon^{-1}\alpha C_4\right)\Upsilon^{-1}\right)\eta - \gamma u + \beta\frac{1}{g_{\mathsf n}}C_2x - \Upsilon(A_4 - \Upsilon^{-1}\alpha C_4)\Gamma\Phi_{\mathsf n}\frac{1}{g_{\mathsf n}}x\\ &\qquad + \Upsilon\Gamma\Phi_{\mathsf n}\frac{1}{g_{\mathsf n}}\Big(\big(A_2 + B_2\phi_{\mathsf n}^\top\big)x + B_2g_{\mathsf n}u_{\mathsf c} - B_2g\left(\bar s(C_4\eta) - \lambda\right)\Big) + \Upsilon\dot\Gamma\Phi_{\mathsf n}\frac{1}{g_{\mathsf n}}x\\ &= \left(A_4 - \alpha C_4\right)\eta - \gamma\big(u_{\mathsf c} - \bar s(C_4\eta)\big) + \Pi\frac{1}{g_{\mathsf n}}x + \Upsilon\Gamma\Phi_{\mathsf n}B_2\left(u_{\mathsf c} - \frac{g}{g_{\mathsf n}}\left(\bar s(C_4\eta) - \lambda\right)\right) + \Upsilon\dot\Gamma\Phi_{\mathsf n}\frac{1}{g_{\mathsf n}}x\\ &= (A_4 - \alpha C_4)\eta + \left(1 - \frac{g}{g_{\mathsf n}}\right)\bar s(C_4\eta) + \Pi\frac{1}{g_{\mathsf n}}x + \frac{g}{g_{\mathsf n}}\gamma\lambda + \Upsilon\frac{\partial\Gamma}{\partial\hat\theta}\dot{\hat\theta}\Phi_{\mathsf n}\frac{1}{g_{\mathsf n}}x.\end{aligned}$$

From now on, we claim that $\Pi = 0$ regardless of τ, which concludes the proof. To see this, compute $\Gamma\Phi_{\mathsf n}$ as

$$\Gamma\Phi_{\mathsf n} = \begin{bmatrix}(c_2/\tau^2) & 0\\ (c_1/\tau^3) - (c_2/\tau^2)\phi_{\mathsf n,2} & (c_2/\tau^2)\\ (c_0/\tau^4) - (c_1/\tau^3)\phi_{\mathsf n,2} & (c_1/\tau^3)\\ 0 & (c_0/\tau^4)\end{bmatrix}.$$

After additional calculations, each of the last two components of Π is expressed as

$$-\Upsilon(A_4 - \Upsilon^{-1}\alpha C_4)\Gamma\Phi_{\mathsf{n}} = \begin{bmatrix} (c_2/\tau^2)a_3 - c_1/\tau^2 + (c_2/\tau)\phi_{\mathsf{n},2} & -c_2/\tau \\ (c_2/\tau^2)a_2 - c_0/\tau^2 + (c_1/\tau)\phi_{\mathsf{n},2} & -c_1/\tau \\ (c_2/\tau^2)a_1 + (c_0/\tau)\phi_{\mathsf{n},2} & -c_0/\tau \\ (c_2/\tau^2)a_0 & 0 \end{bmatrix},$$

$$\Upsilon\Gamma\Phi_{\mathsf{n}}(A_2 + B_2\phi_{\mathsf{n}}^\top) = \begin{bmatrix} 0 & c_2/\tau \\ (\phi_{\mathsf{n},1} - \phi_{\mathsf{n},2})c_2 & c_1/\tau \\ (\phi_{\mathsf{n},1} - \phi_{\mathsf{n},2})c_1 & c_0/\tau \\ (\phi_{\mathsf{n},1} - \phi_{\mathsf{n},2})c_0 & 0 \end{bmatrix}.$$

This directly implies that $\Pi = 0$, by the definition of β.

SM.II. Proof of Lemma 2

Firstly, we differentiate $\tilde{\theta}$ in [SM1, Eq. (26)] as

$$\begin{aligned} \dot{\tilde{\theta}} = \dot{\hat{\theta}} - \dot{\theta} &= -\kappa\Xi^\top C_3^\top C_3\hat{\xi} + \kappa\Xi^\top C_3^\top u - \dot{\theta} \\ &= -\kappa\Xi^\top C_3^\top C_3\big(\Xi\tilde{\theta} + \tilde{\chi} + \xi\big) + \kappa\Xi^\top C_3^\top u - \dot{\theta} \\ &= -\kappa\Xi^\top C_3^\top C_3\Xi\tilde{\theta} + \kappa\Xi^\top\left(-C_3^\top C_3\tilde{\chi} + C_3^\top(u - u^\star)\right) \end{aligned} \tag{SM4}$$

in which $\hat{\xi} = \tilde{\xi} + \xi = \Xi\tilde{\theta} + \tilde{\chi} + \xi$ and $u^\star = C_3\xi$ are used. It is further noted that

$$\lambda = \mathbf{C}_\lambda \begin{bmatrix} x \\ c \end{bmatrix} + \mathbf{D}_\lambda \begin{bmatrix} r \\ d \end{bmatrix} = \mathbf{C}_\lambda \begin{bmatrix} \tilde{x} \\ \tilde{c} \end{bmatrix} + \lambda_{\mathsf{n}} = \mathbf{C}_\lambda \begin{bmatrix} \tilde{x} \\ \tilde{c} \end{bmatrix} + (\lambda_{\mathsf{n}} - \lambda_{\mathsf{n}}^\star) + \lambda_{\mathsf{n}}^\star = \mathbf{C}_\lambda \begin{bmatrix} \tilde{x} \\ \tilde{c} \end{bmatrix} + \mathbf{C}_\lambda \begin{bmatrix} \tilde{x}_{\mathsf{n}} \\ \tilde{c}_{\mathsf{n}} \end{bmatrix} + \lambda_{\mathsf{n}}^\star, \tag{SM5}$$

$$u_{\mathsf{c}} = \mathbf{C}_{\mathsf{c}} \begin{bmatrix} x \\ c \end{bmatrix} + \mathbf{D}_{\mathsf{c}} \begin{bmatrix} r \\ d \end{bmatrix} = \mathbf{C}_{\mathsf{c}} \begin{bmatrix} \tilde{x} \\ \tilde{c} \end{bmatrix} + u_{\mathsf{n}} = \mathbf{C}_{\mathsf{c}} \begin{bmatrix} \tilde{x} \\ \tilde{c} \end{bmatrix} + (u_{\mathsf{n}} - u_{\mathsf{n}}^\star) + u_{\mathsf{n}}^\star = \mathbf{C}_{\mathsf{c}} \begin{bmatrix} \tilde{x} \\ \tilde{c} \end{bmatrix} + \mathbf{C}_{\mathsf{c}} \begin{bmatrix} \tilde{x}_{\mathsf{n}} \\ \tilde{c}_{\mathsf{n}} \end{bmatrix} + u_{\mathsf{n}}^\star, \tag{SM6}$$

$$w = -\bar{s}\left(C_4 p + \frac{c_2}{\tau^2}\frac{1}{g_{\mathsf{n}}}C_2 x\right) = -\bar{s}(C_4\eta) = -\bar{s}(C_4\tilde{\eta} + \lambda) = -\bar{s}\left(C_4\tilde{\eta} + \mathbf{C}_\lambda \begin{bmatrix} \tilde{x} \\ \tilde{c} \end{bmatrix} + \lambda_{\mathsf{n}}\right) \tag{SM7}$$

where the matrices $\mathbf{C}_\lambda$, $\mathbf{D}_\lambda$, $\mathbf{C}_{\mathsf{c}}$, and $\mathbf{D}_{\mathsf{c}}$ are defined below [SM1, Eqs. (16) and (23)]. By the definition of Ω in [SM1, Eq. (30)], the term $u - u^\star$ in (SM4) is computed as follows:

$$\begin{aligned} u - u^\star = u_{\mathsf{c}} + w - u_{\mathsf{n}}^\star + \lambda_{\mathsf{n}}^\star &= (u_{\mathsf{c}} - u_{\mathsf{n}}^\star) - (\lambda - \lambda_{\mathsf{n}}^\star) + (w + \lambda) \\ &= (\mathbf{C}_{\mathsf{c}} - \mathbf{C}_\lambda)\begin{bmatrix} \tilde{x} \\ \tilde{c} \end{bmatrix} + (\mathbf{C}_{\mathsf{c}} - \mathbf{C}_\lambda)\begin{bmatrix} \tilde{x}_{\mathsf{n}} \\ \tilde{c}_{\mathsf{n}} \end{bmatrix} - \Omega. \end{aligned} \tag{SM8}$$

With the matrices

$$\mathbf{M}_1 := -C_3^\top C_3, \quad \mathbf{M}_2 = \mathbf{M}_3 := C_3^\top(\mathbf{C}_{\mathsf{c}} - \mathbf{C}_\lambda), \quad \mathbf{N}_1 := -C_3^\top,$$

one can derive the $\tilde{\theta}$-dynamics as [SM1, Eq. (27)].

We notice that using the x-dynamics (SM3), it is easy to obtain the $[\tilde{x}; \tilde{c}]$- and $[\tilde{x}_{\mathsf{n}}; \tilde{c}_{\mathsf{n}}]$-dynamics as in [SM1, Eq. (28)]. To carry out the $\tilde{\chi}$-dynamics, we substitute [SM1, Eq. (15b)] into [SM1, Eq. (25)], from which it follows:

$$\begin{aligned} \dot{\tilde{\xi}} = \dot{\hat{\xi}} - \dot{\xi} &= \left(A_3\hat{\xi} + \Psi_u\hat{\theta} + L(u - C_3\hat{\xi}) + \Xi\dot{\hat{\theta}}\right) - \left(A_3\xi + \Psi_{u^\star}\theta\right) \\ &= A_3(\hat{\xi} - \xi) + L(u^\star - C_3\hat{\xi}) + \Xi\dot{\hat{\theta}} + \Psi_u(\hat{\theta} - \theta) + L(u - u^\star) - (\Psi_u - \Psi_{u^\star})\theta \\ &= (A_3 - LC_3)\tilde{\xi} + \Xi\dot{\tilde{\theta}} + \Psi_u\tilde{\theta} + (L - \Psi_\theta)(u - u^\star). \end{aligned}$$

(Here, we use the equalities $(\Psi_u - \Psi_{u^\star})\theta = \Psi_{(u-u^\star)}\theta = \Psi_\theta(u - u^\star)$.) The time derivative of $\tilde{\chi}$ is then derived as

$$\begin{aligned} \dot{\tilde{\chi}} = \dot{\tilde{\xi}} - \dot{\Xi}\tilde{\theta} - \Xi\dot{\tilde{\theta}} &= (A_3 - LC_3)\tilde{\xi} + \Xi\dot{\tilde{\theta}} + \Psi_u\tilde{\theta} + (L - \Psi_\theta)(u - u^\star) - \Xi\dot{\tilde{\theta}} - \dot{\Xi}\tilde{\theta} \\ &= (A_3 - LC_3)\tilde{\xi} + \Psi_u\tilde{\theta} + (L - \Psi_\theta)(u - u^\star) - ((A_3 - LC_3)\Xi + \Psi_u)\,\tilde{\theta} \\ &= (A_3 - LC_3)(\tilde{\xi} - \Xi\tilde{\theta}) + (L - \Psi_\theta)(u - u^\star). \end{aligned} \tag{SM9}$$

Applying (SM8) to the above result, we finally have the $\tilde{\chi}$-dynamics as

$$\begin{aligned}\dot{\tilde{\chi}} &= (A_3 - LC_3)\tilde{\chi} + (L - \Psi_\theta)\left((\mathbf{C}_\mathsf{c} - \mathbf{C}_\lambda)\begin{bmatrix}\tilde{x}\\ \tilde{c}\end{bmatrix} + (\mathbf{C}_\mathsf{c} - \mathbf{C}_\lambda)\begin{bmatrix}\tilde{x}_\mathsf{n}\\ \tilde{c}_\mathsf{n}\end{bmatrix} - \Omega\right)\\ &= (A_3 - LC_3)\tilde{\chi} + \mathbf{M}_4\begin{bmatrix}\tilde{x}\\ \tilde{c}\end{bmatrix} + \mathbf{M}_5\begin{bmatrix}\tilde{x}_\mathsf{n}\\ \tilde{c}_\mathsf{n}\end{bmatrix} + \mathbf{N}_2\Omega\end{aligned}$$

where

$$\mathbf{M}_4 = \mathbf{M}_5 := (L - \Psi_\theta)(\mathbf{C}_\mathsf{c} - \mathbf{C}_\lambda) \quad \text{and} \quad \mathbf{N}_2 := -(L - \Psi_\theta).$$

In computing the $\tilde{\eta}$-dynamics in [SM1, Eq. (29)], we differentiate $\tau\tilde{\eta}$ using the η-dynamics [SM1, Eq. (21)] as follows:

$$\begin{aligned}\tau\dot{\tilde{\eta}} &= \tau\dot{\eta} - \tau\dot{\eta}^\star\\ &= (A_4 - \alpha C_4)\eta + \left(1 - \frac{g}{g_\mathsf{n}}\right)\gamma(\hat{\theta};\tau)\bar{s}(C_4\eta) + \frac{g}{g_\mathsf{n}}\gamma(\hat{\theta};\tau)\lambda + \Upsilon(\tau)\frac{\partial\Gamma}{\partial\hat{\theta}}\dot{\hat{\theta}}\Phi_\mathsf{n}\frac{1}{g_\mathsf{n}}x - \tau\dot{\eta}^\star\\ &= (A_4 - \alpha C_4)(\tilde{\eta} + \eta^\star) + \left(1 - \frac{g}{g_\mathsf{n}}\right)\gamma(\hat{\theta};\tau)\bar{s}(C_4\eta) + \frac{g}{g_\mathsf{n}}\gamma(\hat{\theta};\tau)C_4\eta^\star + \Upsilon(\tau)\frac{\partial\Gamma}{\partial\hat{\theta}}\dot{\hat{\theta}}\Phi_\mathsf{n}\frac{1}{g_\mathsf{n}}x - \tau\dot{\eta}^\star\\ &= (A_4 - \alpha C_4)\tilde{\eta} + \left(1 - \frac{g}{g_\mathsf{n}}\right)\gamma(\theta;0)\Big(\bar{s}(C_4\tilde{\eta} + C_4\eta^\star) - C_4\eta^\star\Big) + \tau\Delta\\ &= (A_4 - \alpha C_4)\tilde{\eta} + \left(1 - \frac{g}{g_\mathsf{n}}\right)\gamma(\theta;0)\Omega + \tau\Delta \end{aligned}\tag{SM10}$$

where the perturbation term Δ is given as follows:

$$\begin{aligned}\Delta := &\frac{1}{\tau}(A_4 - \alpha C_4)\eta^\star + \frac{1}{\tau}\left(1 - \frac{g}{g_\mathsf{n}}\right)\Big(\gamma(\hat{\theta};\tau) - \gamma(\theta;0)\Big)\bar{s}(C_4\eta) + \frac{1}{\tau}\gamma(\theta;0)C_4\eta^\star\\ &+ \frac{1}{\tau}\frac{g}{g_\mathsf{n}}\Big(\gamma(\hat{\theta};\tau) - \gamma(\theta;0)\Big)C_4\eta^\star + \frac{1}{\tau}\Upsilon(\tau)\frac{\partial\Gamma}{\partial\hat{\theta}}\dot{\hat{\theta}}\Phi_\mathsf{n}\frac{1}{g_\mathsf{n}}x - \dot{\eta}^\star.\end{aligned}\tag{SM11}$$

For ease of explanation, we rewrite Δ as $\Delta = \Delta_1^\star + \Delta_2^\star + \Delta_3^\star + \Delta_4^\star$ with the 4 sub-components

$$\begin{aligned}\Delta_1^\star &:= \frac{1}{\tau}\left(1 - \frac{g}{g_\mathsf{n}}\right)\Big(\gamma(\hat{\theta};\tau) - \gamma(\theta;\tau)\Big)\bar{s}(C_4\eta) + \frac{1}{\tau}\frac{g}{g_\mathsf{n}}\Big(\gamma(\hat{\theta};\tau) - \gamma(\theta;\tau)\Big)C_4\eta^\star,\\ \Delta_2^\star &:= \frac{1}{\tau}\left(1 - \frac{g}{g_\mathsf{n}}\right)\Big(\gamma(\theta;\tau) - \gamma(\theta;0)\Big)\bar{s}(C_4\eta) + \frac{1}{\tau}\frac{g}{g_\mathsf{n}}\Big(\gamma(\theta;\tau) - \gamma(\theta;0)\Big)C_4\eta^\star + \frac{1}{\tau}\gamma(\theta;0)C_4\eta^\star - \frac{1}{\tau}\gamma(\theta;\tau)C_4\eta^\star,\\ \Delta_3^\star &:= \frac{1}{\tau}\Upsilon(\tau)\frac{\partial\Gamma}{\partial\hat{\theta}}\dot{\hat{\theta}}\Phi_\mathsf{n}\frac{1}{g_\mathsf{n}}x,\\ \Delta_4^\star &:= \frac{1}{\tau}\gamma(\theta;\tau)C_4\eta^\star + \frac{1}{\tau}(A_4 - \alpha C_4)\eta^\star - \dot{\eta}^\star.\end{aligned}$$

We complete the proof of the lemma by showing that each sub-component $\Delta_i^\star$, $i = 1, \ldots, 4$, is a continuous function of the state variables that vanishes at the origin; thus, their linear combination Δ also does. Some useful equalities are

$$\frac{1}{\tau}\Big(\gamma(\hat{\theta};\tau) - \gamma(\theta;\tau)\Big) = -\tau\begin{bmatrix}0\\1\\a_3\\0\end{bmatrix}\tilde{\theta},\quad \frac{1}{\tau}\Big(\gamma(\theta;\tau) - \gamma(\theta;0)\Big) = -\tau\begin{bmatrix}0\\1\\a_3\\0\end{bmatrix}\theta,\quad \frac{\partial\Gamma}{\partial\hat{\theta}} = \begin{bmatrix}-1 & 0\\ -a_3/\tau & -1\\ 0 & -a_3/\tau\\ 0 & 0\end{bmatrix}.$$

Then one can compute $\Delta_1^\star$, $\Delta_2^\star$, and $\Delta_3^\star$ as follows:

$$\begin{aligned}\Delta_1^\star &= \frac{1}{\tau}\Big(\gamma(\hat{\theta};\tau) - \gamma(\theta;\tau)\Big)\left(\left(1 - \frac{g}{g_\mathsf{n}}\right)\bar{s}(C_4\eta) + \frac{g}{g_\mathsf{n}}C_4\eta^\star\right)\\ &= -\tau\begin{bmatrix}0\\1\\a_3\\0\end{bmatrix}\tilde{\theta}\left(\left(1 - \frac{g}{g_\mathsf{n}}\right)\bar{s}\left(C_4\tilde{\eta} + \mathbf{C}_\lambda\begin{bmatrix}\tilde{x}\\ \tilde{c}\end{bmatrix} + \lambda_\mathsf{n}\right) + \frac{g}{g_\mathsf{n}}\left(\mathbf{C}_\lambda\begin{bmatrix}\tilde{x}\\ \tilde{c}\end{bmatrix} + \lambda_\mathsf{n}\right)\right),\end{aligned}\tag{SM12}$$

$$\Delta_2^\star = \frac{1}{\tau}\Big(\gamma(\theta;\tau) - \gamma(\theta;0)\Big)\left(1 - \frac{g}{g_\mathsf{n}}\right)\left(\bar{s}(C_4\eta) - C_4\eta^\star\right)$$

$$= -\tau \begin{bmatrix} 0 \\ 1 \\ a_3 \\ 0 \end{bmatrix} \theta \left(1 - \frac{g}{g_\mathsf{n}}\right) \left(\bar{s} \left(C_4 \tilde{\eta} + \mathbf{C}_\lambda \begin{bmatrix} \tilde{x} \\ \tilde{c} \end{bmatrix} + \lambda_\mathsf{n} \right) - \left(\mathbf{C}_\lambda \begin{bmatrix} \tilde{x} \\ \tilde{c} \end{bmatrix} + \lambda_\mathsf{n} \right) \right) = -\tau \begin{bmatrix} 0 \\ 1 \\ a_3 \\ 0 \end{bmatrix} \theta \left(1 - \frac{g}{g_\mathsf{n}}\right) \Omega, \tag{SM13}$$

$$\Delta_3^\star = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \tau & 0 & 0 \\ 0 & 0 & \tau^2 & 0 \\ 0 & 0 & 0 & \tau^3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -a_3/\tau & -1 \\ 0 & -a_3/\tau \\ 0 & 0 \end{bmatrix} \dot{\tilde{\theta}} \Phi_\mathsf{n} \frac{1}{g_\mathsf{n}} (\tilde{x} + x_\mathsf{n})$$

$$= - \begin{bmatrix} 1 & 0 \\ a_3 & \tau \\ 0 & a_3\tau \\ 0 & 0 \end{bmatrix} \kappa \left(-\Xi^\top C_3^\top C_3 \Xi \tilde{\theta} + \Xi^\top \left(\mathbf{M}_1 \tilde{\chi} + \mathbf{M}_2 \begin{bmatrix} \tilde{x} \\ \tilde{c} \end{bmatrix} + \mathbf{M}_3 \begin{bmatrix} \tilde{x}_\mathsf{n} \\ \tilde{c}_\mathsf{n} \end{bmatrix} + \mathbf{N}_1 \Omega \right) \right) \Phi_\mathsf{n} \frac{1}{g_\mathsf{n}} (\tilde{x} + x_\mathsf{n}), \tag{SM14}$$

from which the arguments on the first three sub-components $\Delta_1^\star$, $\Delta_2^\star$, and $\Delta_3^\star$ are obtained. For the last term, we represent $\Delta_4^\star$ as follows:

$$\Delta_4^\star = \frac{1}{\tau} \begin{bmatrix} 0 \\ a_2 - \tau^2\theta \\ a_1 - \tau^2 a_3 \theta \\ a_0 \end{bmatrix} C_4 \eta^\star + \frac{1}{\tau} A_4 \eta^\star - \frac{1}{\tau} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} C_4 \eta^\star - \dot{\eta}^\star$$

$$= \frac{1}{\tau} \begin{bmatrix} 0 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} \lambda - \frac{1}{\tau} \begin{bmatrix} 0 \\ \tau^2\theta \\ \tau^2 a_3 \theta \\ 0 \end{bmatrix} \lambda + \frac{1}{\tau} \begin{bmatrix} a_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \lambda + \frac{1}{\tau} \begin{bmatrix} \tau \lambda_\mathsf{n}^{\star(1)} \\ \tau^2 \lambda_\mathsf{n}^{\star(2)} + \tau a_3 \lambda_\mathsf{n}^{\star(1)} + \tau^2 \theta \lambda_\mathsf{n}^\star \\ \tau^3 \lambda_\mathsf{n}^{\star(3)} + \tau^2 a_3 \lambda_\mathsf{n}^{\star(2)} + \tau^3 \theta \lambda_\mathsf{n}^{\star(1)} + \tau^2 a_3 \theta \lambda_\mathsf{n}^\star \\ 0 \end{bmatrix} - \frac{1}{\tau} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} \lambda$$

$$- \begin{bmatrix} 1 \\ a_3 \\ 0 \\ 0 \end{bmatrix} \dot{\lambda} - \begin{bmatrix} 0 \\ \tau \lambda_\mathsf{n}^{\star(2)} \\ \tau^2 \lambda_\mathsf{n}^{\star(3)} + \tau a_3 \lambda_\mathsf{n}^{\star(2)} + \tau^2 \theta \lambda_\mathsf{n}^{\star(1)} \\ \tau^3 \lambda_\mathsf{n}^{\star(4)} + \tau^2 a_3 \lambda_\mathsf{n}^{\star(3)} + \tau^3 \theta \lambda_\mathsf{n}^{\star(2)} + \tau^2 a_3 \theta \lambda_\mathsf{n}^{\star(1)} \end{bmatrix}$$

$$= - \begin{bmatrix} 0 \\ \tau\theta \\ \tau a_3 \theta \\ 0 \end{bmatrix} (\lambda - \lambda_\mathsf{n}^\star) - \begin{bmatrix} 1 \\ a_3 \\ 0 \\ 0 \end{bmatrix} (\dot{\lambda} - \dot{\lambda}_\mathsf{n}) - \begin{bmatrix} 0 \\ 0 \\ 0 \\ \tau^3 \lambda_\mathsf{n}^{\star(4)} + \tau^2 a_3 \lambda_\mathsf{n}^{\star(3)} + \tau^3 \theta \lambda_\mathsf{n}^{\star(2)} + \tau^2 a_3 \theta \lambda_\mathsf{n}^{\star(1)} \end{bmatrix}$$

$$= - \begin{bmatrix} 0 \\ \tau\theta \\ \tau a_3 \theta \\ 0 \end{bmatrix} \left(\mathbf{C}_\lambda \begin{bmatrix} \tilde{x} \\ \tilde{c} \end{bmatrix} + \mathbf{C}_\lambda \begin{bmatrix} \tilde{x}_\mathsf{n} \\ \tilde{c}_\mathsf{n} \end{bmatrix} \right) - \begin{bmatrix} 1 \\ a_3 \\ 0 \\ 0 \end{bmatrix} \left(\mathbf{C}_\lambda \mathbf{A}_\mathsf{n} \begin{bmatrix} \tilde{x} \\ \tilde{c} \end{bmatrix} + \mathbf{C}_\lambda \mathbf{N}_2 \Omega + \mathbf{C}_\lambda \mathbf{A}_\mathsf{n} \begin{bmatrix} \tilde{x}_\mathsf{n} \\ \tilde{c}_\mathsf{n} \end{bmatrix} \right)$$

$$- \begin{bmatrix} 0 \\ 0 \\ 0 \\ \tau^3 \lambda_\mathsf{n}^{\star(4)} + \tau^2 a_3 \lambda_\mathsf{n}^{\star(3)} + \tau^3 \theta \lambda_\mathsf{n}^{\star(2)} + \tau^2 a_3 \theta \lambda_\mathsf{n}^{\star(1)} \end{bmatrix}$$

where the last equality comes from

$$\lambda - \lambda_\mathsf{n}^\star = \mathbf{C}_\lambda \begin{bmatrix} \tilde{x} \\ \tilde{c} \end{bmatrix} + \mathbf{C}_\lambda \begin{bmatrix} \tilde{x}_\mathsf{n} \\ \tilde{c}_\mathsf{n} \end{bmatrix} \quad \text{and} \quad \dot{\lambda} - \dot{\lambda}_\mathsf{n}^\star = \mathbf{C}_\lambda \begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{c}} \end{bmatrix} + \mathbf{C}_\lambda \begin{bmatrix} \dot{\tilde{x}}_\mathsf{n} \\ \dot{\tilde{c}}_\mathsf{n} \end{bmatrix} = \mathbf{C}_\lambda \mathbf{A}_\mathsf{n} \begin{bmatrix} \tilde{x} \\ \tilde{c} \end{bmatrix} + \mathbf{C}_\lambda \mathbf{N}_2 \Omega + \mathbf{C}_\lambda \mathbf{A}_\mathsf{n} \begin{bmatrix} \tilde{x}_\mathsf{n} \\ \tilde{c}_\mathsf{n} \end{bmatrix}.$$

It should be emphasized that, since $\lambda_\mathsf{n}^\star(t)$ in fact has the form of a biased sinusoid

$$\lambda_\mathsf{n}^\star(t) = M_0^\star + M_1^\star \sin(\sigma t + \rho_1^\star),$$

with some constants $M_0^\star$, $M_1^\star$, and $\rho_1^\star$, one can obtain

$$\tau^3\lambda_{\mathsf{n}}^{\star(4)}(t)+\tau^2 a_3\lambda_{\mathsf{n}}^{\star(3)}(t)+\tau^3\theta\lambda_{\mathsf{n}}^{\star(2)}(t)+\tau^2 a_3\theta\lambda_{\mathsf{n}}^{\star(1)}(t)\equiv 0. \tag{SM15}$$

This completes the claim on $\Delta_4^\star$ and also concludes the lemma.

SM.III. Proof of Lemma 3

The proof of the lemma is mainly based on the singular perturbation theory, especially on the Tikhonov's theorem [SM2, Thm. 11.2]. To this end, we regard the transformed system [SM1, Eqs. (27)–(29)] as a standard (or two-time scaled) singular perturbation form with respect to the perturbation parameter τ. More specifically, the former two subsystems [SM1, Eqs. (27) and (28)] are viewed as the *slow* subsystem in the standard singular perturbation theory.

Before going on further, we emphasize again that the initial value $\tilde{\eta}(0)$ of the fast variable is a polynomial of $1/\tau$, as seen in [SM1, Eq. (32)]. This implies that the requirements of the Tikhonov's theorem [SM2, Thm. 11.2] are not fulfilled yet. Thus as in an alternative way, we choose a small time instant $T^\star>0$ (independent on τ) such that

$$\|[\tilde{x}(t);\tilde{c}(t)]\|<\sqrt{\frac{\min(\mathcal{L}(\mathbf{P}_{\mathsf{n}}))}{\max(\mathcal{L}(\mathbf{P}_{\mathsf{n}}))}}\frac{\epsilon}{2},\quad \forall t\in[0,T^\star] \tag{SM16}$$

where $\mathbf{P}_{\mathsf{n}}$ is the positive definite matrix such that $\mathbf{A}_{\mathsf{n}}^\top\mathbf{P}_{\mathsf{n}}+\mathbf{P}_{\mathsf{n}}\mathbf{A}_{\mathsf{n}}=-I$, and $\mathcal{L}(\mathbf{P}_{\mathsf{n}})$ stands for the set of all the eigenvalues of $\mathbf{P}_{\mathsf{n}}$. This selection of $T^\star$ is always possible, because the velocity of $[\tilde{x},\tilde{c}]$ is bounded around its starting point $[\tilde{x},\tilde{c}]=0$ due to the saturation function $\overline{s}$. With $T^\star$ selected as above, we will analyze the overall system [SM1, Eqs. (27)–(29)] sequentially for the transient period $[0,T^\star)$ and the steady-state period $[T^\star,\infty)$.

The following lemma indicates that, during the transient period $t\in[0,T^\star)$, the fast variable $\tilde{\eta}(t)$ converges around its *quasi-steady-state* $\tilde{\eta}=0$ with sufficiently small τ.

Lemma SM.III.1. *There exists $\overline{\tau}_1>0$ such that $\tau\in(0,\overline{\tau}_1)$, the solution $\tilde{\eta}(t)$ of [SM1, Eq. (27)] satisfies*

$$\|\tilde{\eta}(T^\star)\|\le k_1 e^{-\varphi_1(T^\star/\tau)}\|\tilde{\eta}(0)\|+\tau k_2 \tag{SM17}$$

where k_1, k_2, and φ_1 are independent on τ. □

Proof of Lemma SM.III.1. With the time scale $\varrho:=t/\tau$, the $\tilde{\eta}$-dynamics [SM1, Eq. (29)] is rewritten as a Lur'e-type nonlinear system

$$\begin{aligned}
\frac{d\tilde{\eta}}{d\varrho}&=(A_4-\alpha C_4)\tilde{\eta}+\gamma(\theta;0)u_\eta+\tau\Delta,\quad y_\eta:=C_4\tilde{\eta}, \\
u_\eta&=-\left(\frac{g}{g_{\mathsf{n}}}-1\right)\Omega=-\left(\frac{g}{g_{\mathsf{n}}}-1\right)\left(\overline{s}\Big(y_\eta+\mathbf{C}_\lambda\begin{bmatrix}\tilde{x}(t)\\ \tilde{c}(t)\end{bmatrix}+\lambda_{\mathsf{n}}(t)\Big)-\Big(\mathbf{C}_\lambda\begin{bmatrix}\tilde{x}(t)\\ \tilde{c}(t)\end{bmatrix}+\lambda_{\mathsf{n}}(t)\Big)\right)=:-\Omega_\eta(t,y_\eta)
\end{aligned} \tag{SM18}$$

where $\tau\Delta$ is the perturbation term which vanishes when $\tau=0$. First, we claim that if $\tau=0$, then the origin of the nonlinear system (SM18) is globally exponentially stable. Indeed, the transfer function

$$\begin{aligned}
\frac{1+(\overline{g}/g_{\mathsf{n}}-1)C_4(sI-A_4+\alpha C_4)^{-1}\gamma(\theta;0)}{1+(\underline{g}/g_{\mathsf{n}}-1)C_4(sI-A_4+\alpha C_4)^{-1}\gamma(\theta;0)}&=\frac{1+\left(\dfrac{\overline{g}}{g_{\mathsf{n}}}-1\right)\dfrac{a_2s^2+a_1s+a_0}{s^4+a_3s^3+a_2s^2+a_1s+a_0}}{1+\left(\dfrac{\underline{g}}{g_{\mathsf{n}}}-1\right)\dfrac{a_2s^2+a_1s+a_0}{s^4+a_3s^3+a_2s^2+a_1s+a_0}}\\
&=\frac{s^4+a_3s^3+(\overline{g}/g_{\mathsf{n}})a_2s^2+(\overline{g}/g_{\mathsf{n}})a_1s+(\overline{g}/g_{\mathsf{n}})a_0}{s^4+a_3s^3+(\underline{g}/g_{\mathsf{n}})a_2s^2+(\underline{g}/g_{\mathsf{n}})a_1s+(\underline{g}/g_{\mathsf{n}})a_0}
\end{aligned} \tag{SM19}$$

is strictly positive real [SM2] due to the construction of a_i's. In addition, for all $0\le t\le T^\star$,

$$\begin{aligned}
&\frac{\underline{g}}{g_{\mathsf{n}}}-1\le\frac{\partial\Omega_\eta}{\partial y_\eta}=\left(\frac{g}{g_{\mathsf{n}}}-1\right)\frac{\partial\overline{s}}{\partial y_\eta}\le\frac{\overline{g}}{g_{\mathsf{n}}}-1,\quad\text{and}\\
&\Omega_\eta(t,0)=\left(\frac{g}{g_{\mathsf{n}}}-1\right)\left(\overline{s}\Big(\mathbf{C}_\lambda\begin{bmatrix}\tilde{x}(t)\\ \tilde{c}(t)\end{bmatrix}+\lambda_{\mathsf{n}}(t)\Big)-\Big(\mathbf{C}_\lambda\begin{bmatrix}\tilde{x}(t)\\ \tilde{c}(t)\end{bmatrix}+\lambda_{\mathsf{n}}(t)\Big)\right)=0
\end{aligned}$$

(where the latter equality follows from (SM16)). These facts imply that the nonlinearity $\Omega_\eta(t, y_\eta)$ is contained in the sector $[\underline{g}/g_{\mathsf{n}} - 1, \overline{g}/g_{\mathsf{n}} - 1]$ (with respect to the input y_η) for $0 \le t \le T^\star$. The circle criterion [SM2, Thm. 7.1] completes the claim on the case $\tau = 0$.

For the remainder of the proof, we remind that $\tau\Delta$ is a continuous function of the state variable satisfying $\lim_{\tau\to 0} \tau\Delta = 0$, and the slower variables $\tilde{\theta}(t)$ and $[\tilde{x}(t); \tilde{c}(t); \tilde{x}_{\mathsf{n}}(t); \tilde{c}_{\mathsf{n}}(t); \tilde{\chi}(t)]$ are bounded for $0 \le t \le T^\star$. Finally, the lemma follows from the vanishing and non-vanishing perturbation theory [SM2, Lems. 9.1 and 9.2]. ■

It is important to note that, by Lemma SM.III.1 and the inequality [SM1, Eq. (32)], $\|\tilde{\eta}(T^\star)\|$ has an upper bound as

$$\|\tilde{\eta}(T^\star)\| \le k_1 e^{-\varphi_1(T^\star/\tau)} \left\| \frac{1}{\tau}\Upsilon(\tau)p(0) + \frac{1}{\tau}\Upsilon(\tau)\Gamma(\hat{\theta}(0);\tau)\Phi_{\mathsf{n}}\frac{1}{g_{\mathsf{n}}}x(0) - \eta^\star(0) \right\| + \tau k_2. \tag{SM20}$$

Here as τ approaches zero, the right hand-side of the inequality converges to zero, and so $\tilde{\eta}(T^\star)$ also does. (Notice that this does not take place with $\tilde{\eta}(0)$.) Keeping this in mind, we now concentrate on the steady-state period $t \in [T^\star, \infty)$.

In what follows, we call

$$\frac{d\tilde{\eta}}{d\sigma} = (A_4 - \alpha C_4)\tilde{\eta} - \gamma(\theta; 0)\Omega_\eta(T^\star, C_4\tilde{\eta}) \tag{SM21}$$

(which is the same as the $\tilde{\eta}$-dynamics (SM18) with $\tau = 0$ and with the slow variables frozen at $t = T^\star$) as the *boundary-layer system*. In a similar manner, one can consider the following *reduced system* in the sense of the singular perturbation theory (to avoid confusion on the terminology, we denote the state variable of the reduced system as $[\tilde{\theta}^\circ; \tilde{x}^\circ; \tilde{c}^\circ; \tilde{x}^\circ_{\mathsf{n}}; \tilde{c}^\circ_{\mathsf{n}}; \tilde{\chi}^\circ]$ rather than $[\tilde{\theta}; \tilde{x}; \tilde{c}; \tilde{x}_{\mathsf{n}}; \tilde{c}_{\mathsf{n}}; \tilde{\chi}]$):

$$\frac{1}{\kappa}\dot{\tilde{\theta}}^\circ = -\Xi^{\circ\top} C_3^\top C_3 \Xi^\circ \tilde{\theta} + \Xi^{\circ\top}\left(\mathbf{M}_1\tilde{\chi}^\circ + \mathbf{M}_2\begin{bmatrix}\tilde{x}^\circ\\ \tilde{c}^\circ\end{bmatrix} + \mathbf{M}_3\begin{bmatrix}\tilde{x}^\circ_{\mathsf{n}}\\ \tilde{c}^\circ_{\mathsf{n}}\end{bmatrix} + \mathbf{N}_1\Omega^\circ\right), \tag{SM22a}$$

$$\begin{bmatrix}\dot{\tilde{x}}^\circ\\ \dot{\tilde{c}}^\circ\\ \dot{\tilde{x}}^\circ_{\mathsf{n}}\\ \dot{\tilde{c}}^\circ_{\mathsf{n}}\\ \dot{\tilde{\chi}}^\circ\end{bmatrix} = \begin{bmatrix}\mathbf{A}_{\mathsf{n}} & 0 & 0\\ 0 & \mathbf{A}_{\mathsf{n}} & 0\\ \mathbf{M}_4 & \mathbf{M}_5 & A_3 - LC_3\end{bmatrix}\begin{bmatrix}\tilde{x}^\circ\\ \tilde{c}^\circ\\ \tilde{x}^\circ_{\mathsf{n}}\\ \tilde{c}^\circ_{\mathsf{n}}\\ \tilde{\chi}^\circ\end{bmatrix} + \begin{bmatrix}\mathbf{N}_2\\ 0\\ \mathbf{N}_3\end{bmatrix}\Omega^\circ, \tag{SM22b}$$

where

$$\Omega^\circ = \bar{s}\left(\mathbf{C}_\lambda\begin{bmatrix}\tilde{x}^\circ\\ \tilde{c}^\circ\end{bmatrix} + \lambda_{\mathsf{n}}\right) - \mathbf{C}_\lambda\begin{bmatrix}\tilde{x}^\circ\\ \tilde{c}^\circ\end{bmatrix} - \lambda_{\mathsf{n}}, \quad \text{and} \tag{SM22c}$$

$$\dot{\Xi}^\circ = (A_3 - LC_3)\Xi^\circ + \begin{bmatrix}0\\ -1\\ 0\end{bmatrix}\left(u^\star + (\mathbf{C}_{\mathsf{c}} - \mathbf{C}_\lambda)\begin{bmatrix}\tilde{x}^\circ\\ \tilde{c}^\circ\end{bmatrix} + (\mathbf{C}_{\mathsf{c}} - \mathbf{C}_\lambda)\begin{bmatrix}\tilde{x}^\circ_{\mathsf{n}}\\ \tilde{c}^\circ_{\mathsf{n}}\end{bmatrix} - \Omega^\circ\right). \tag{SM22d}$$

Among them, the stability of the boundary-layer system is a natural consequence of Lemma SM.III.1.

Corollary SM.III.1. *The origin of the boundary-layer system* (SM21) *is globally exponentially stable.* □

From now on, we take a closer look at the reduced system (SM22). Let us define a set

$$\mathcal{R}_0^\circ := \Big\{[\tilde{\theta}(T^\star); \tilde{x}(T^\star); \tilde{c}(T^\star); \tilde{x}_{\mathsf{n}}(T^\star); \tilde{c}_{\mathsf{n}}(T^\star); \tilde{\chi}(T^\star)] \\ : [\tilde{\theta}(t); \tilde{x}(t); \tilde{c}(t); \tilde{x}_{\mathsf{n}}(t); \tilde{c}_{\mathsf{n}}(t); \tilde{\chi}(t)] \text{ is generated by [SM1, Eqs. (27)–(29)]}\Big\}.$$

Then we are now ready to derive the stability of the reduced system.

Lemma SM.III.2. *Suppose that all the assumptions hold. Then there exists $\overline{\kappa} > 0$ such that for $\kappa \in (0, \overline{\kappa})$,*

(a) *the state trajectory of the reduced system* (SM22) *initiated in $\mathcal{R}_0^\circ$ satisfies*

$$\|[\tilde{x}^\circ(t); \tilde{c}^\circ(t)]\| < \frac{\epsilon}{2}, \quad \forall t \ge T^\star; \tag{SM23}$$

(b) *the origin of the reduced system* (SM22) *is exponentially stable with the region of attraction containing $\mathcal{R}_0^\circ$.* □

Proof of Lemma SM.III.2. To prove the item (a), consider the $[\tilde{x}^\circ; \tilde{c}^\circ]$-dynamics

$$\begin{bmatrix} \dot{\tilde{x}}^\circ \\ \dot{\tilde{c}}^\circ \end{bmatrix} = \mathbf{A}_\mathsf{n} \begin{bmatrix} \tilde{x}^\circ \\ \tilde{c}^\circ \end{bmatrix} + \mathbf{N}_2 \Omega^\circ \tag{SM24}$$

with the initial condition $[\tilde{x}^\circ(T^\star); \tilde{c}^\circ(T^\star)]$ satisfying

$$\|[\tilde{x}^\circ(T^\star); \tilde{c}^\circ(T^\star)]\| < \sqrt{\frac{\min(\mathcal{L}(\mathbf{P}_\mathsf{n}))}{\max(\mathcal{L}(\mathbf{P}_\mathsf{n}))}} \frac{\epsilon}{2}$$

For the analysis, we now employ a Lyapunov function $V(\tilde{x}^\circ, \tilde{c}^\circ) := [\tilde{x}^\circ; \tilde{c}^\circ]^\top \mathbf{P}_\mathsf{n} [\tilde{x}^\circ; \tilde{c}^\circ]$. It is noted that, by definition of $\bar{s}$ and $T^\star$,

$$V(\tilde{x}^\circ, \tilde{c}^\circ) < \min(\mathcal{L}(\mathbf{P}_\mathsf{n})) \frac{\epsilon^2}{4} \quad \text{implies} \quad \mathbf{C}_\lambda \begin{bmatrix} \tilde{x}^\circ \\ \tilde{c}^\circ \end{bmatrix} + \lambda_\mathsf{n} \in \hat{\Lambda}_\mathsf{n} \quad \text{and thus}$$
$$\Omega^\circ = \bar{s}\left(\mathbf{C}_\lambda \begin{bmatrix} \tilde{x}^\circ \\ \tilde{c}^\circ \end{bmatrix} + \lambda_\mathsf{n} \right) - \left(\mathbf{C}_\lambda \begin{bmatrix} \tilde{x}^\circ \\ \tilde{c}^\circ \end{bmatrix} + \lambda_\mathsf{n} \right) = 0.$$

It means that the set

$$\mathcal{V} := \left\{ [\tilde{x}^\circ; \tilde{c}^\circ] : \quad V(\tilde{x}^\circ, \tilde{c}^\circ) < \min(\mathcal{L}(\mathbf{P}_\mathsf{n})) \frac{\epsilon^2}{4} \right\}$$

is positive invariant. Notice that $[\tilde{x}^\circ(T^\star); \tilde{c}^\circ(T^\star)]$ is already included in $\mathcal{V}$, because

$$V\big(\tilde{x}^\circ(T^\star), \tilde{c}^\circ(T^\star)\big) \le \max(\mathcal{L}(\mathbf{P}_\mathsf{n})) \, \|[\tilde{x}^\circ(T^\star); \tilde{c}^\circ(T^\star)]\|^2 < \min(\mathcal{L}(\mathbf{P}_\mathsf{n})) \frac{\epsilon^2}{4}.$$

This proves the item (a). We further point out that $\Omega^\circ = \Omega^\circ(t) = 0$ holds for all $t \ge T^\star$.

Next, putting $\Omega^\circ = 0$ into the reduced system (SM22a) and (SM22b), one has

$$\frac{1}{\kappa} \dot{\tilde{\theta}}^\circ = -\Xi^{\circ\top} C_3^\top C_3 \Xi^\circ \tilde{\theta}^\circ + \Xi^{\circ\top} \left(\mathbf{M}_1 \tilde{\chi}^\circ + \mathbf{M}_2 \begin{bmatrix} \tilde{x}^\circ \\ \tilde{c}^\circ \end{bmatrix} + \mathbf{M}_3 \begin{bmatrix} \tilde{x}_\mathsf{n}^\circ \\ \tilde{c}_\mathsf{n}^\circ \end{bmatrix} \right) \tag{SM25}$$

$$\begin{bmatrix} \dot{\tilde{x}}^\circ \\ \dot{\tilde{c}}^\circ \\ \dot{\tilde{x}}_\mathsf{n}^\circ \\ \dot{\tilde{c}}_\mathsf{n}^\circ \\ \dot{\tilde{\chi}}^\circ \end{bmatrix} = \begin{bmatrix} \mathbf{A}_\mathsf{n} & 0 & 0 \\ 0 & \mathbf{A}_\mathsf{n} & 0 \\ \mathbf{M}_4 & \mathbf{M}_5 & A_3 - LC_3 \end{bmatrix} \begin{bmatrix} \tilde{x}^\circ \\ \tilde{c}^\circ \\ \tilde{x}_\mathsf{n}^\circ \\ \tilde{c}_\mathsf{n}^\circ \\ \tilde{\chi}^\circ \end{bmatrix}. \tag{SM26}$$

We emphasize that the reduced system (SM25) and (SM26) has the standard singular perturbation form again, with respect to the perturbation parameter κ. It is readily obtained from the fact that $\mathbf{A}_\mathsf{n}$ and $A_3 - LC_3$ in (SM26) are Hurwitz that the faster subsystem (SM26) is globally exponentially stable. Therefore, in accordance with the singular perturbation theory [SM2, Thm. 11.4], it is enough for the item (b) to show that the following reduced dynamics, which is computed by putting the quasi-steady-state $[\tilde{\theta}^\circ; \tilde{x}^\circ; \tilde{c}^\circ; \tilde{x}_\mathsf{n}^\circ; \tilde{c}_\mathsf{n}^\circ; \tilde{\chi}^\circ] = 0$ into (SM25), is stable:

$$\frac{d\tilde{\theta}^\circ}{d\sigma_\kappa} = -\Xi^{\circ\top} C_3^\top C_3 \Xi^\circ \tilde{\theta}^\circ \tag{SM27}$$

where $\sigma_\kappa := \kappa t$, and $\Xi^\circ(t)$ is the solution of

$$\dot{\Xi}^\circ = (A_3 - LC_3)\Xi^\circ + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} u^\star.$$

Note that the Ξ°-dynamics above is input-to-state stable (ISS) and its input $u^\star = u_\mathsf{n}^\star - \lambda_\mathsf{n}^\star$ is a bounded sinusoidal signal of the frequency σ. This implies that $C_3 \Xi^\circ(t)$ is *persistently exciting* [SM3, Thm. 5.2.1] and thus, the origin of (SM27) is exponentially stable. In summary, by the singular perturbation theory [SM2, Thm. 11.4], there exists $\overline{\kappa} > 0$ such that the origin of (SM22) is exponentially stable for all $\kappa \in (0, \overline{\kappa})$. ■

We have observed so far in Corollary SM.III.1 and Lemma SM.III.2 that both the boundary-layer system (SM21) and the reduced system (SM22) (with fixed $\kappa \in (0, \overline{\kappa})$) are exponentially stable. Thus the singular perturbation theory [SM2, Thm.

11.4] says that there exists $\overline{\tau}_2 > 0$ such that the overall singularly perturbed system [SM1, Eqs. (27)–(29)] is exponentially stable for all $\tau \in (0, \overline{\tau}_2)$. On the other hand, by employing the Tikhonov's theorem [SM2, Thm. 11.2] for the steady-state period $t \in [T^\star, \infty)$, it follows that there exists $\overline{\tau}_3 > 0$ such that for all $\tau \in (0, \overline{\tau}_3)$, the actual state $[\tilde{x}; \tilde{c}]$ remains close to its counterpart $[\tilde{x}^\circ; \tilde{c}^\circ]$ in the reduced system (SM22); more precisely,

$$\|[\tilde{x}(t); \tilde{c}(t)] - [\tilde{x}^\circ(t); \tilde{c}^\circ(t)]\| < \frac{\epsilon}{2}, \quad \forall t \geq T^\star \tag{SM28}$$

where $[\tilde{x}(t); \tilde{c}(t)]$ is the solution of (SM22) initiated at $[\tilde{x}(T^\star); \tilde{c}(T^\star)] = [\tilde{x}^\circ(T^\star); \tilde{c}^\circ(T^\star)]$. Combining (SM28) with the item (a) of Lemma SM.III.2 implies that

$$\|[\tilde{x}(t); \tilde{c}(t)]\| \leq \|[\tilde{x}(t); \tilde{c}(t)] - [\tilde{x}^\circ(t); \tilde{c}^\circ(t)]\| + \|[\tilde{x}^\circ(t); \tilde{c}^\circ(t)]\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad \forall t \geq T^\star.$$

Finally, with $\overline{\tau}$ set as $\overline{\tau} = \min(\overline{\tau}_1, \overline{\tau}_2, \overline{\tau}_3)$, the proof of the lemma is completed.

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