이학박사 학위논문

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(스테이블 사상의 모듈라이 공간의 기하와 변형)

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Geometry of moduli space of stable maps and degeneration

A dissertation
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Abstract

Geometry of moduli space of stable maps and degeneration

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We study geometry of moduli space of holomorphic maps from curves to projective scheme through various compactifications. Most famous one is moduli space of stable maps introduced by Kontsevich. When genus is one and target space is projective space, main component of moduli space of stable maps is nonsingular. Vakil and Zinger found some desingularization via modular blow ups. Kim introduced log stable maps with target expansions which gives another desingularization of moduli space of stable maps. We compare theses two desingularization. Also, Gross-Seibert and Abramovich-Chen defined logarithmic stable maps without target expansions. Using these moduli space, one can define log Gromov-Witten invariants. We prove the degeneration formula of log Gromov-Witten invariants.

Key words: stable map, logarithmic structure, Gromov-Witten invariant
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Contents

Chapter 1

Introduction

In physics, the number of rational curves in calabi-yau manifold, which is called Gromov-Witten invariant, is very important. In mathematics, this invariant can be defined by intersection theory on the moduli space of stable maps introduced by Kontsevich. More precisely, we can define the virtual fundamental class of expected dimension through natural perfect obstruction theory coming from deformation theory of the moduli space of stable maps.

When X is quintic Calabi-Yau threefold in $\mathbb{P}^4$, virtual fundamental class of the moduli space of stable maps with target X is same as Euler class of certain complex of vector bundles in the moduli space of stable maps with target $\mathbb{P}^4$. In genus 0 case, the complex of vector bundles is actually vector bundle. But in higher genus case, this is no longer true. This is due to some higher genus components contracted to point. When genus is one, there are two ways to resolve this problems. First way done by Vakil and Zinger[27] is to desingularize the moduli space of stable maps with target $\mathbb{P}^4$ so that the complex of vector bundle become a vector bundle. Second ways done by Kim[15] is to define the new moduli space containing more information which resolve the problem of contraction of higher genus component. Each method gives new Gromov-Witten type invariants. These two spaces are desingularization of the moduli space of stable maps. In chapter 2, we define these two space and study how they are related.

Even if we defined Gromov-Witten invariants, It is hard to actually compute them in general. There are several techniques to reduce to more simple one. One method is degeneration method. Roughly, we degenerate target X into X' which consists of two components X_1 and X_2, and prove the formula relating Gromov-Witten invariant of X and Gromov-Witten invariant of X_1 and X_2. In chapter 3, we prove the degeneration formula for log Gromov-Witten invariants introduced by Gross-Seibert [10] and Abramovich-Chen[1].

Chapter 2

Comparison of two desingularizations of moduli space of stable maps

2.1 Introduction

The Kontsevich's moduli space of stable maps $\overline{\mathbf{M}}_{g,k}(X,d)$ is a moduli space which parametrizes maps from k-marked nodal curve of arithmetic genus g to projective variety X satisfying stability conditions. See [9] for precise definitions and properties. In this paper we only consider Kontsevich's moduli space of elliptic stable maps $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n,d)$. $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n,d)$ is known to have several components. We call the component parametrizing elliptic stable maps whose domain curve have non-contracted elliptic subcurves the main component. We denote the main component of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n,d)$ as $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n,d)_0$. It is known that $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n,d)_0$ is singular.

Recently many birational model of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n,d)_0$ have been introduced by many authors. In [27], Vakil and Zinger found a canonical desingularization $\widetilde{\mathbf{M}}_{1,0}(\mathbb{P}^n,d)_0$ of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n,d)_0$ by blowing-up $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n,d)_0$. In [15], Kim introduced another desingularization of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n,d)_0$ called the moduli space of logarithmic stable maps by using log structures. We denote this space as $\overline{\mathbf{M}}_{1,0}^{log,ch}(\mathbb{P}^n,d)$. In [20], Marian, Oprea and Pandharipande constructed moduli space of stable quotients denoted by $\mathbf{Q}_g(\mathbb{P}^n,d)$. They defined a moduli space of stable quotients of the rank n trivial sheaf on nodal curves. They also proved that when the genus is 1, $\mathbf{Q}_1(\mathbb{P}^n,d)$ is a smooth Delign-Mumford stack. So this gives another smooth birational model. In [29], Viscardi constructed a moduli space of (m)-stable maps denoted by $\overline{\mathbf{M}}^{(m)}_{1,k}(\mathbb{P}^n, d)$. He defined a moduli space using (m)-stable curves which was introduced by Smyth [26]. He also proved $\overline{\mathbf{M}}^{(m)}_{1,k}(\mathbb{P}^n, d)$ is smooth if $d + k \leq m \leq 5$.

In general, it is not known how these birational moduli spaces are related to each others. In this paper, we compare Vakil-Zinger's desingularization and the moduli space of logarithmic stable mpas. We show that $\overline{\mathbf{M}}^{log,ch}_{1,0}(\mathbb{P}^n, 3)$ can be obtained by blowing up $\widetilde{\mathbf{M}}_{1,0}(\mathbf{P}^n, 3)_0$ along the locus $\sum_2$, Γ_2, $\sum_1$, Γ_1.

$\overline{\sum_1}$ is the closure of the locus of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)_0$ parametrizing stable maps such that their domain curves consist of a elliptic component of the degree 0 and a rational component of the degree 3 and the morphism restricted to the rational component has ramification order 3 at the nodal point.

$\overline{\sum_2}$ is the closure of the locus of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)_0$ parametrizing stable maps such that their domain curves consist of a elliptic component of the degree 0, and two rational components with the degree 1,2, each meeting the elliptic component at one point and the morphism restricted to degree 2 rational component has ramification order 2 at the nodal point.

$\overline{\Gamma_1}$ is the closure of the locus of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)_0$ parametrizing stable maps such that their domain curves consist of a elliptic component of the degree 0 and a rational component of the degree three, and there exists a smooth point q on the rational component such that p, q go to same point, where p is the node point.

$\overline{\Gamma_2}$ is the closure of the locus of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)_0$ parametrizing stable maps such that their domain curves consist of a elliptic component of the degree 0 and two rational components with the degree 1, 2, each meeting the elliptic component at one point and there exists smooth point q on degree 2 rational component such that p, q go to same point where p is nodal point on degree 2 rational component.

$\sum_1$,$\sum_2$,Γ_1,Γ_2 are proper transforms of $\overline{\sum_1}$,$\overline{\sum_2}$,$\overline{\Gamma_1}$,$\overline{\Gamma_2}$.

The outline of this paper is as follows. In section 2, we give some preliminaries. In section 3, we present an example of a degeneration where a nontrivial elliptic logarithmic stable map occurs. In section 4, we calculate the fiber of the natural morphism from the moduli space of admissible stable maps to the Kontsevich's moduli space of stable maps. In section 5, we prove two moduli spaces are equal if the degree is 2. In section 6, we describe etale charts of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, 3)_0$ explicitly and by blowing up suitable subschemes, we obtain

Theorem 2.1.1. $\overline{\mathbf{M}}_{1,0}^{log,ch}(\mathbb{P}^n, 3)$ *can be obtained by blowing-up* $\widetilde{\mathbf{M}}_{1,0}(\mathbb{P}^n, 3)_0$ *along the locus* $\sum_2$, Γ_2, $\sum_1$, Γ_1.

2.2 Preliminaries

In this section we introduce some notations. We also briefly recall some definitions and properties of Vakil-Zinger's desingularization and the moduli space of logarithmic stable maps.

2.2.1 Notations

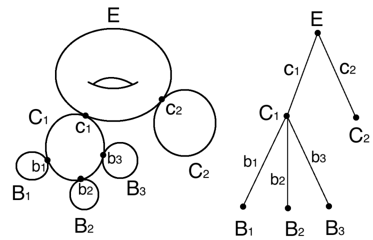
A dual graph of domain curves

In this paper we only consider connected curves of arithmetic genus 1. Note that every connected curve of arithmetic genus 1 has the unique minimal subcurve of arithmetic genus 1. we give names to this subcurve.

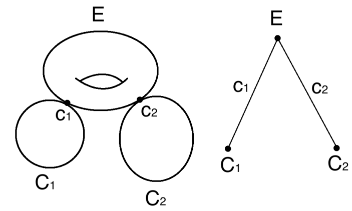
Definition 2.2.1. *Let C be connected curve of arithmetic genus 1. Let C' be the minimal subcurve of arithmetic genus 1 of C. We call C' the* essential part *of C.*

For every nodal curve, we can associate a graph called the dual graph. Irreducible components of the nodal curve correspond to vertices of the graph. And nodal points of nodal curve correspond to edges of graph.

If curve C is connected curve of arithmetic genus 1 whose essential part is irreducible curve, we can represent its dual graph as following. Suppose C has 6 irreducible components E, C_1, C_2, B_1, B_2, B_3. E is a smooth curve of arithmetic genus 1. Two smooth rational components C_1, C_2 are connected to E. And three smooth rational components B_1, B_2, B_3 are connected to C_1. Then we can represent the dual graph of C as $E[C_1[B_1, B_2, B_3], C_2]$. In this case, we say C is of the type $E[C_1[B_1, B_2, B_3], C_2]$. We denote the intersection point of E and C_1 as c_1. And we denote intersection point of C_1 and B_1 as b_1 and so on.



The real picture and the dual graph of the curve of the type E[C1[B1,B2,B3],C2]



The real picture and the dual graph of the curve of the type E[C1,C2]

Furthermore, if a curve C is the domain curve of the Kontsevich's moduli space of elliptic stable maps, we record information of the degree in the parenthesis. For example, if we say that C is of the type $E(0)[B_1(0)[C_1(1), C_2(2)]]$, then the dual graph of C is represented as $E[B_1[C_1, C_2]]$ and the degrees of maps restricted to E, B_1, C_1, C_2 are 0, 0, 1, 2, respectively.

The expanded target

Let $\mathbb{P}^n$ be a n-dimensional projective space. We define $\mathbb{P}^n(1)$ to be $(Bl_{c(0)}\mathbb{P}^n) \bigcup \mathbb{P}^n$. Here $c(0)$ is a point in $\mathbb{P}^n$. And $Bl_{c(0)}\mathbb{P}^n$ and $\mathbb{P}^n$ are glued along $D(1)$. In $Bl_{c(0)}\mathbb{P}^n$, $D(1)$ is the exceptional divisor. And in $\mathbb{P}^n$, $D(1)$ is a hyperplane. We can give the linear order to the set of irreducible components of $\mathbb{P}^n(1)$ such that component corresponding to $\mathbb{P}^n$ is the largest one. We denote the irreducible components of $\mathbb{P}^n(1)$ by $\mathbb{P}^n_1$, $\mathbb{P}^n_2$ according to this order. i.e. $\mathbb{P}^n_2$ is the largest one.

We define $\mathbb{P}^n(2)$ to be $(Bl_{c(1)}\mathbb{P}^n(1)) \bigcup \mathbb{P}^n$. Here $c(1)$ is a point in $\mathbb{P}^n_2$ not contained in D(1). And $Bl_{c(1)}\mathbb{P}^n(1)$ and $\mathbb{P}^n$ are glued along $D(2)$. In $Bl_{c(0)}\mathbb{P}^n(1)$, $D(2)$ is the exceptional divisor. And in $\mathbb{P}^n$, $D(2)$ is a hyperplane. We can give the linear order to the set of irreducible components of $\mathbb{P}^n(2)$ such that component corresponding to $\mathbb{P}^n$ is the largest one. We denote the irreducible components of $\mathbb{P}^n(2)$ by $\mathbb{P}^n_1$, $\mathbb{P}^n_2$, $\mathbb{P}^n_3$ according to this order. i.e. $\mathbb{P}^n_3$ is the largest one.

In this way, we define $\mathbb{P}^n(k)$, $\mathbb{P}^n_1, \mathbb{P}^n_2, \cdots, \mathbb{P}^n_{k+1}$, $D(1), D(2), \cdots, D(k)$ inductively.

The sequence of blow up

Let X be an algebraic scheme. Let V_1, V_2, $\cdots$, V_n be subschemes of X. When we say that we blow up X along V_1, V_2, $\cdots$, V_n, we mean that we first blow up V_1, blow up the proper transform of $V_2, \cdots$, and blow up the proper transform of V_n.

By abuse of notation, we identify an ideal J with the subscheme V_J defined by J.

2.2.2 Vakil-Zinger Desingularization

In [28], Vakil and Zinger defined the m-tail locus of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)_0$ to be the locus parametrizing maps such that in the domain the contracted elliptic curve meets the rest of the curve a total of precisely m points. Desingularization is described as following way; blow up the 1-tail locus, then the proper transform of the 2-tail locus, etc. This process stop at finite steps, and resulting space $\widetilde{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)_0$ is smooth Delign-Mumford stack.

In [11], Hu and Li described local equations of $\overline{\mathbf{M}}_{1.0}(\mathbb{P}^n, d)$. They first defined the terminally weighted tree γ. To each γ, they associated the variety Z_γ called local model and the subvariety $Z^0_\gamma \subset Z_\gamma$ called the type γ loci in Z_γ. They defined DM-stack S to have singularity type γ at a closed point $s \in S$ if there is a scheme Y, a point $y \in Y$ and two smooth morphisms $q_1 : Y \to S$, $q_2 : Y \to Z_\gamma$ such that $q_1(y) = s$ and $q_2(y) \in Z^0_\gamma$. To each element $[u] \subset \overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)$, they associated terminally weighted rooted tree. They defined the substack $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)_\gamma \subset \overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)$ to be the subset of all $[u] \subset \overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)$ whose associated terminally weighted rooted trees is γ. Finally they showed that the stack $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)$ has singularity type γ along $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)_\gamma$

We do not present full details here since our case is quite simple. When $d = 3$, 1-tail locus $D_1 \subset \overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, 3)_0$ and 2-tail locus $D_2 \subset \overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, 3)_0$ are smooth divisors. 3-tail locus $D_3 \subset \overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, 3)_0$ has description as follows. Let Z be $\{(a_1, a_2, \cdots, a_{n-1}, b_1, b_2, \cdots, b_{n-1}, z_1, z_2) \in \mathbb{A}^{2n} : a_1z_1 - b_1z_2 = a_2z_1 - b_2z_2 = \cdots = a_{n-1}z_1 - b_{n-1}z_2 = 0\}$, where $\mathbb{A}^n$ is n-dimensional affine space. $Z^0 \subset Z$ is $\{(a_1, a_2, \cdots, a_{n-1}, b_1, b_2, \cdots, b_{n-1}, z_1, z_2) \in Z : z_1 = z_2 = 0\}$. To each element $[u] \subset D_3$, there is a scheme Y, a point $y \in Y$ and two smooth morphisms $q_1 : Y \to \overline{\mathbf{M}}_{1.0}(\mathbb{P}^n, d)_0$, $q_2 : Y \to Z$ such that $q_1(y) = [u]$ and $q_2(y) \in Z^0$. Therefore $\widetilde{\mathbf{M}}_{1,0}(\mathbb{P}^n, 3)_0 =$

$Bl_{D_3}\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, 3)_0$.

2.2.3 Logarithmic stable maps

We briefly introduce logarithmic stable maps following [15]. There is standard reference for definition and some properties of the log structures ([13]). We do not give full details about the log structures since the log structures are not used extensively in this paper.

Definition 2.2.2. *An algebraic space W over S is called a* Fulton-Macpherson (FM) type space *if*

1. *$W \to S$ is a proper, flat morphism;*

2. *for every closed point $s \in S$, etale locally there is an etale morphism*

$$W_{\bar{s}} \to Spec(k(\bar{s})[x, y, z_1, z_2, \cdots, z_{k-1}]/(xy))$$

where x, y and z_i are indeterminates.

Definition 2.2.3 ([15],5.1.1)**.** *A triple $((C/S, \mathbf{p}), W/S, f : C \longrightarrow W)$ is called a n-pointed, genus g,* admissible map *to a FM type space W/S if*

1. *$(C/S, \mathbf{p} = (p_1, ..., p_n))$ is a n-pointed, genus g, prestable curve over S.*

2. *W/S is a FM type space.*

3. *$f : C \longrightarrow W$ is a map over S.*

4. *(*Admissibility*) If a point $p \in C$ is mapped into the relatively singular locus $(W/S)^{\mathrm{sing}}$ of W/S, then étale locally at $\bar{p}$, f is factorized as*

$$\begin{tikzcd}
C \arrow[dr] \arrow[dd, "f"'] & & U \arrow[ll] \arrow[rr] \arrow[dd] \arrow[dr] & & \mathrm{Spec}(A[u,v]/(uv-t)) \arrow[dl] \arrow[dd] \\
& S & & \mathrm{Spec}A \arrow[ll] & \\
W \arrow[ur] & & V \arrow[ll] \arrow[rr] \arrow[ur] & & \mathrm{Spec}A[x,y,z_1,...,z_{r-1}]/(xy-\tau) \arrow[ul]
\end{tikzcd}$$

where all 5 horizontal maps are formally étale; u, v, x, y, z_i are indeterminates; $x = u^l$, $y = v^l$ under the far right vertical map for some positive integer l; t, τ are elements in the maximal ideal $\mathfrak{m}_A$ of the local ring A; and $\bar{p}$ is mapped to the point defined by the ideal $(u, v, \mathfrak{m}_A)$.

A log morphism $(W, M_W)/(S, N)$ is called an extended log twisted FM type space if $W \to S$ is FM type space and M_W, N are log structures on W, S satisfying some conditions.

Definition 2.2.4 ([15],5.2.2)**.** *A log morphism $(f : (C, M_C, \mathbf{p}) \longrightarrow (W, M_W))\,/(S, N)$ is called a (g, n)* logarithmic prestable map *over (S, N) if*

1. *$((C, M)/(S, N), \mathbf{p})$ is a n-pointed, genus g, minimal log prestable curve.*
2. *$(W, M_W)/(S, N)$ is an extended log twisted FM type space.*
3. *(*Corank = # Nondistinguished Nodes Condition*) For every $s \in S$, the rank of* $\mathrm{Coker}(N_{\bar{s}}^{W/S} \longrightarrow N_{\bar{s}})$ *coincides with the number of nondistinguished nodes on $C_{\bar{s}}$.*
4. *$f : (C, M_C) \longrightarrow (W, M_W)$ is a log morphism over (S, N).*
5. *(*Log Admissibility*) either of the following conditions, equivalent under the above four conditions, holds:*
   - *$\underline{f}$ is admissible.*
   - *$f^b : f^*M_W \longrightarrow M_C$ is simple at every distinguished node.*

Definition 2.2.5 ([15],8.1)**.** *Let $\overline{\mathbf{M}}_{1,0}^{log,ch}(X, d)$ be the moduli stack of $(g = 1, n = 0, d \neq 0)$ logarithmic stable maps (f, C, W) satisfying the following conditions additional to those in Definition 3.0.2. For every $s \in S$,*

1. *Every end component of $W_{\bar{s}}$ contains the entire image of the essential part of $C_{\bar{s}}$ under $f_{\bar{s}}$.*
2. *The image of the essential part of $C_{\bar{s}}$ is nonconstant.*

Here, it is possible that some of irreducible components in the essential part are mapped to points. Note that the dual graph of the target W_s must be a chain. Such a log stable map is called an elliptic log stable map to a chain type FM space W of the smooth projective variety X.

Theorem 2.2.6 ([15],Main Theorem B)**.** *The moduli stack $\overline{\mathbf{M}}_{1,0}^{log,ch}(X,d)$ of elliptic logarithmic stable maps to chain type FM spaces of X is a proper Delign-Mumford stack. When X is a projective space $\mathbb{P}^n$, the stack is smooth.*

We define the moduli space $\overline{\mathbf{M}}_{1,0}^{ch}(X,d)$ of admissible stable maps to chain type FM spaces of X to be same as $\overline{\mathbf{M}}_{1,0}^{log,ch}(X,d)$ without log structures. That is, an element of $\overline{\mathbf{M}}_{1,0}^{ch}(X,d)$ is an element of $\overline{\mathbf{M}}_{1,0}^{log,ch}(X,d)$ without log structures.

2.3 An example of degeneration

First we construct a family of elliptic stable maps over $S = \mathbb{A}^2$. Let $R = k[t,a]$ be a coordinate ring of S, where k is a algebraically closed field and t,a are indeterminates. Let $C' = Proj(R[x,y,z]/zy^2 - x^3 - z^2x - z^3)$. Let $f' : C' \dashrightarrow \mathbb{P}^2$ be given by $[t^3y, at^2x, z]$. It is a well defined family of elliptic stable maps except at $\{(t,a) : t = 0\} \subset S$. If we blow up an ideal (t,x,z), it extends to family of elliptic stable maps on whole S. That is, if we let $C = Bl_{(t,x,z)}C'$, the rational morphism $f' : C' \dashrightarrow \mathbb{P}^2$ extends to $f : C \longrightarrow \mathbb{P}^2$ and f gives a family of elliptic stable maps over S. At $t \neq 0$, its domain curve is smooth. At $t = 0$ its domain curve consists of an elliptic component whose degree is 0 and one rational component whose morphism is given by $[s^3, as^2, 1]$ where s is the local coordinate of the rational component such that $\{s = 0\}$ is the intersection point with elliptic component.

Now we construct a family of elliptic stable admissible maps over $\widetilde{S} = Bl_{(t,a)}S$ in following way.

Proposition 2.3.1. *Let R, C', C, f be as above. Let $\widetilde{S}$ be the blow up of S at the origin and let E be the exceptional divisor. Let D be the proper transform of subscheme defined by $(t) \subset R$. Let C'' be a pullback of C' along $\widetilde{S} \longrightarrow S$ and $\widetilde{C}$ be the blow up of C'' along ideals (D, x, z), (E, x, z); Here we mean that first blow up along (D, x, z) and next blow up along the proper transform of (E, x, z). Let $\widetilde{W}$ be the blow up of $\widetilde{S} \times \mathbb{P}^2$ along ideals $(E^3, x_0, x_1), (D^2, x_0, x_1)$, where x_0, x_1, x_2 are coordinates of $\mathbb{P}^2$. Then $f : C \longrightarrow \mathbb{P}^2$ extends to map $\widetilde{f} : \widetilde{C} \longrightarrow \widetilde{W}$ and $(\widetilde{f} : \widetilde{C} \longrightarrow \widetilde{W})$ is a family of admissible stable maps.*

Proof. we choose one local coordinate of $\widetilde{S}$ as $\{(t, a)\} \simeq \mathbb{A}^2$ such that $\widetilde{S} \longrightarrow S$ is given by $(t, a) \mapsto (ta, a)$. Then the induced morphism is given by $[t^3a^3y : t^2a_3x, z]$. Since we only need to consider a neighborhood of $\{[x : y : z] = [0, 1, 0]\}$ which is smooth point, the problem is reduced to the following lemma.

Lemma 2.3.2. *Suppose $(f : C = \mathbb{A}^1 \times \mathbb{A}^2 = Spec(k[x, t, a]) \longrightarrow W = \mathbb{A}^2 \times \mathbb{A}^2 = Spec(k[X, Y, t, a]))$ is given by $(x, t, a) \mapsto (\frac{t^3a^3}{z}, \frac{t^2a^3x}{z}, t, a)$, where z is a function of x such that the vanishing order of z at $x = 0$ is 3. If we let $\widetilde{C}$ be the blow up of C along ideals (x, t), (x, a) and $\widetilde{W}$ be the blow up of W along ideals (X, Y, a^3), (X, Y, t^2), then $(f : C \longrightarrow W)$ extends to morphism $(\widetilde{f} : \widetilde{C} \longrightarrow \widetilde{W})$*

Proof. Using universal property of blow ups, we need to check that inverse image sheaves of (X, Y, a^3) and (X, Y, t^3) are invertible. For example, at an open set $U \subset C$ given by $\{(x, t, a) : x \neq 0)\}$, inverse image sheaves of (X, Y, a^3) and (X, Y, t^3) are $(\frac{t^3a^3}{z}, \frac{t^2a^3x}{z}, a_3) = (a^3)$ and $(\frac{t^3a^3}{z}, \frac{t^2a^3x}{z}, t^2) = (t^2)$ respectively which are invertible sheaves. Other cases are left to the reader. □

We can easily check that $\widetilde{f}$ satisfies admissible conditions. In the same way we can prove the case of the other open sets of $\widetilde{S}$. This proves theorem. □

Remark 2.3.3. the origin in S parameterizes stable map whose domain curve consist of an elliptic component of degree 0 and one rational component whose morphism has ramification order 3 at the intersection point with the elliptic component. i.e. it is an element of $\sum_1$.

Remark 2.3.4. In the proof, we can describe an element of admissible stable map explicitly. For example over $\{(t, a) : a = 0, t \neq 0\} \subset \widetilde{S}$, $\widetilde{C}$ is of the type $E[C_1]$ and

$\widetilde{W} = \mathbb{P}^2(1)$ and $\widetilde{f}|_E : E \longrightarrow \mathbb{P}^2_1$ is given by $[X_0, X_1, X_2] = [t^3y : t^2x : z] = [ty : x : z]$ where X_0, X_1, X_2 are coordinates of $\mathbb{P}^2_1$ such that $D(1)$ is given by $\{[X_1, X_2, X_3] : X_2 = 0\}$. The last equality is due to the existence of an automorphism of $\mathbb{P}^2_1$ fixing $D(1)$.

2.4 The description of fiber in the moduli space of elliptic admissible stable maps

By the definition of admissible stable map we get the following proposition.

Proposition 2.4.1. *There is a natural morphism ϕ from the moduli space of elliptic admissible stable maps to the Kontsevich's moduli space of stable maps. Proof. A family of admissible maps over $\tilde{S}$ consist of $((\tilde{C}/\tilde{S}, \mathbf{p}), W/\tilde{S}, \tilde{f} : \tilde{C} \longrightarrow W)$, where $\tilde{C}$ is a family of pre-stable curves over $\tilde{S}$ and W is FM type space of $\mathbb{P}^2$. By just forgetting W, we obtain Kontsevich's pre-stable maps and after stabilization we get Kontsevich's stable maps.* □

Now we describe set theoretic fibers of ϕ, when $d = 3$. Note that if the essential part is not contracted to a point, the fiber is just one point because W is trivial. i.e. $W = \mathbb{P}^n$. Let's consider the fiber of element where the essential part is contracted to a point.

Lemma 2.4.2. *Let $(C, f : C \longrightarrow \mathbb{P}^n)$ be an element of the main component of the Kontsevich's moduli space of elliptic stable maps satisfying following condition. C is of the type $E(0)[C_1(3)]$.*

1. *if f has ramification order 2 at c_1 and there is no smooth point $q_1 \in C_1$ such that $f(c_1) = f(q_1)$, then the fiber of ϕ is equal to a point set theoretically.*

2. *if f has ramification order 2 at c_1 and there is a smooth point $q_1 \in C_1$ such that $f(c_1) = f(q_1)$, then the fiber of ϕ is equal to $\mathbb{P}^{n-1}$ set theoretically.*

3. *if f has ramification order 3 at c_1, then the fiber of ϕ is equal to $Bl_{pt}\mathbb{P}^n$ set theoretically.*

Proof.

1. • **point :** The domain curve $\tilde{C}$ is of the type $E[C_1]$ and $W = \mathbb{P}^n(1)$. $\tilde{f} : C_1 \longrightarrow \mathbb{P}^n_0$ is already given. $\tilde{f} : E \longrightarrow \mathbb{P}^n_1$ is given by $[X_0 : X_1 : \cdots : X_n] = [x : 0 : \cdots : 0 : z]$, where E are given by $\{[x, y, z] : zy^2 = x^3 + z^2x + Az^3\}$ and $X_0, X_1, \cdots, X_n$ are coordinates of $\mathbb{P}^n$. c_1 is given by $\{[x : y : z] : z = x = 0\}$ and $D(1)$ is given by $\{[X_1, X_2, \cdots, X_n] : X_n = 0\}$.

2. • $\mathbb{A}^{n-1}$ **with parameter $\{\,[\alpha_0 : \alpha_1 : \cdots : \alpha_{n-1}]\,, \alpha_{n-1} \neq 0\,\}$:** The domain curve $\tilde{C}$ is of the type $E[C_1[A_1]]$ and $W = \mathbb{P}^n(1)$. $\tilde{f}|_{C_1} : C_1 \longrightarrow \mathbb{P}^n_0$ is already given. $\tilde{f}|_E : E \longrightarrow \mathbb{P}^n_1$ is given by $[X_0 : X_1 : \cdots : X_n] = [x : 0 : \cdots : 0 : z]$. $\tilde{f}|_{A_1} : A_1 \longrightarrow \mathbb{P}^n(1)$ is given by $[1 : \alpha_0 t : \alpha_1 t : \cdots : \alpha_{n-1} t]$, where t are a local parameter of A_1 such that a_1 is given by $\{t = 0\}$ and $D(1)$ is given by $\{[X_1, X_2, \cdots, X_n] : X_n = 0\}$.

   • $\mathbb{P}^{n-2}$ **with parameter $\{\,[\alpha_0 : \alpha_1 : \cdots : \alpha_{n-2}]\,\}$:** The domain curve $\tilde{C}$ is of the type $E[C_1'[C_1[A_1]]]$ and $W = \mathbb{P}^n(2)$. $\tilde{f}|_{C_1} : C_1 \longrightarrow \mathbb{P}^n_0$ already given. $\tilde{f}|_{C_1'} : C_1' \longrightarrow \mathbb{P}^n_1$ is given by $[t^2 : 0 : \cdots : 0 : 1]$ where t is a local parameter of C_1' such that c_1 are given by $\{t = 0\}$ and $D(1)$ is given by $\{[X_1, X_2, \cdots, X_n] : X_n = 0\}$. $\tilde{f}|_E : E \longrightarrow \mathbb{P}^n_2$ is given by $[x : 0 : \cdots : 0 : z]$ where $D(2)$ are given by $\{[X_1, X_2, \cdots, X_n] : X_n = 0\}$. $\tilde{f}|_{A_1} : A_1 \longrightarrow \mathbb{P}^2_1$ is given by $[1 : \alpha_0 s : \alpha_1 s : \cdots : \alpha_{n-2} s : s]$ where s is a local parameter of A_1 such that a_1 is given by $\{s = 0\}$.

3. • $\mathbb{A}^n$ **with parameter $\{[\alpha_0 : \alpha_1 : \cdots : \alpha_n], \alpha_n \neq 0\}$:** The domain curve $\tilde{C}$ is of the type $E[C_1]$ and $W = \mathbb{P}^n(1)$. $\tilde{f}|_{C_1} : C_1 \longrightarrow \mathbb{P}^n_0$ already given. $\tilde{f}|_E : E \longrightarrow \mathbb{P}^n_1$ is given by $[X_0 : X_1 : \cdots : X_n] = [\alpha_0 x + \alpha_n y : \alpha_1 x : \alpha_2 : \cdots : \alpha_{n-1} : z]$, where $D(1)$ is given by $\{[X_1, X_2, \cdots, X_n] : X_n = 0\}$.

   • $\mathbb{P}^{n-1} \setminus pt$ **with parameter $\{[\alpha_0 : \alpha_1 : \cdots : \alpha_{n-1}]$, not all α_k are 0 for $1 \leqslant k \leqslant n-1\}$:** The domain curve $\tilde{C}$ is of the type $E[C_1'[C_1]]$ and $W = \mathbb{P}^n(2)$. $\tilde{f}|_{C_1} : C_1 \longrightarrow \mathbb{P}^n_0$ is already given. $\tilde{f}|_{C_1'} : C_1' \longrightarrow \mathbb{P}^n_1$

is given by $[1-\alpha_0 t : \alpha_1 t : \alpha_2 t : \cdots : \alpha_{n-1} t : t^3]$ where t is a local parameter of C_1' such that c_1 is given by $\{t=0\}$ and $D(1)$ is given by $\{[X_1, X_2, \cdots, X_n] : X_n = 0\}$. $\tilde{f}|_E : E \longrightarrow \mathbb{P}^n_2$ is given by $[x : 0 : \cdots : 0 : z]$ where $D(2)$ are given by $\{[X_1, X_2, \cdots, X_n] : X_n = 0\}$.

- $\mathbb{A}^{n-1}$ **with parameter** $\{\, [\alpha_0 : \alpha_1 : \cdots : \alpha_{n-1}]\, , \alpha_{n-1} \neq 0 \,\}$ **:** The domain curve $\tilde{C}$ is of the type $E[C_1'[C_1, A_1]]$ and $W = \mathbb{P}^n_2$. $\tilde{f}|_{C_1} : C_1 \longrightarrow \mathbb{P}^n_0$ is already given. $\tilde{f}|_{C_1'} : C_1' \longrightarrow \mathbb{P}^n_1$ is given by $[1-t : 0 : \cdots : 0 : t^3]$ where t is a local parameter of C_1' such that c_1 is given by $\{t=0\}$ and a_1 is given by $\{t=1\}$ and $D(1)$ is given by $\{[X_1, X_2, \cdots, X_n] : X_n = 0\}$. $\tilde{f}|_E : E \longrightarrow \mathbb{P}^n_2$ is given by $[x : 0 : \cdots : 0 : z]$. $\tilde{f}|_{A_1} : A_1 \longrightarrow \mathbb{P}^n_2$ is given by $[1 : \alpha_0 : \alpha_1 : \cdots : \alpha_{n-1} s]$, where s is a local parameter of A_1 such that a_1 is given by $\{s=0\}$ and $D(2)$ is given by $\{[X_1, X_2, \cdots, X_n] : X_n = 0\}$.

- $\mathbb{P}^{n-2}$ **with parameter** $\{\, [\alpha_0 : \alpha_1 : \cdots : \alpha_{n-2}] \,\}$ **:** The domain curve $\tilde{C}$ is of the type $E[C_1''[C_1'[C_1, A_1]]]$ and $W = \mathbb{P}^n(3)$. $\tilde{f}|_{C_1} : C_1 \longrightarrow \mathbb{P}^n_0$ is already given. $\tilde{f}|_{C_1'} : C_1' \longrightarrow \mathbb{P}^n_1$ is given by $[1-t : 0 : \cdots : 0 : t^3]$ where t is a local parameter of C_1' such that c_1 is given by $\{t=0\}$ and a_1 is given by $\{t=1\}$ and $D(1)$ is given by $\{[X_1, X_2, \cdots, X_n] : X_n = 0\}$. $\tilde{f}|_{C_1''} : C_1'' \longrightarrow \mathbb{P}^n_2$ is given by $[1 : 0 : \cdots : 0 : s^2]$ where s is parameter of C_1'' such that c_1' is given by $\{s=0\}$. $\tilde{f}|_{A_1} : A_1 \longrightarrow \mathbb{P}^n_2$ is given by $[1 : \alpha_0 u : \alpha_1 u : \cdots : \alpha_{n-2} u : u]$ where u is a local parameter of A_1 such that a_1 is given by $\{u=0\}$ and $D(2)$ is given by $\{[X_1, X_2, \cdots, X_n] : X_n = 0\}$. $\tilde{f}|_E : E \longrightarrow \mathbb{P}^n_3$ is given by $[x : 0 : \cdots : 0 : z]$ where $D(3)$ is given by $\{[X_1, X_2, \cdots, X_n] : X_n = 0\}$.

$\square$

Note that the case where essential part is singular curves can be stated and proved in the same way. Note that we actually know every element of the fiber explicitly. By similar way we can prove the following lemmas whose proof will be omitted.

Lemma 2.4.3. *Let $(C, f : C \longrightarrow \mathbb{P}^n)$ be an element of the main component of the Kontsevich's moduli space of stable maps satisfying the following conditions. C is of the type $E(0)[C_1(1), C_2(2)]$.*

1. *if f has ramification order 1 at c_2 and there is no smooth point $q_2 \in C_2$ such that $f(c_2) = f(q_2)$, then the fiber of ϕ is equal to a point set theoretically.*
2. *if f has ramification order 1 at c_2 and there is a smooth point $q_2 \in C_2$ such that $f(c_2) = f(q_2)$, then the fiber of ϕ is equal to $\mathbb{P}^n$ set theoretically.*
3. *if f has ramification order 2 at c_2.and images of C_1 and C_2 are different lines, then fiber of ϕ is equal to $\mathbb{P}^1$ set theoretically.*
4. *if f has ramification order 2 at c_2, images of C_1 and C_2 are same lines, then the fiber of ϕ is equal to $\mathbb{P}^{n-1} \bigcup \mathbb{P}^1$ glued at one point, set theoretically.*

Lemma 2.4.4. *Let $(C, f : C \longrightarrow \mathbb{P}^n)$ be an element of the main component of the Kontsevich's moduli space of stable maps satisfying the following conditions. C is of the type $E(0)[C_1(1), C_2(1), C_3(1)]$.*

1. *if images of C_1 and C_2 and C_3 are distinct lines, then the fiber of ϕ is equal to a point set theoretically.*
2. *if images of C_1 and C_2 are same lines and the image of C_3 is the distinct line, then the fiber of ϕ is equal to a point set theoretically.*
3. *if images of C_1 and C_2 and C_3 are all same lines, then the fiber of ϕ is equal to $\mathbb{P}^1$ set theoretically.*

Lemma 2.4.5. *Let $(C, f : C \longrightarrow \mathbb{P}^n)$ be an element of the main component of the Kontsevich's moduli space of stable maps satisfying the following conditions. C is of the type $E(0)[B_1(0)[C_1(1), C_2(2)]]$.*

1. *if f has ramification order 1 at c_2 and there is no smooth point $q_2 \in C_2$ such that $f(c_2) = f(q_2)$, then the fiber of ϕ is equal to a point, set theoretically.*
2. *if f has ramification order 1 at c_2 and there is a smooth point $q_2 \in C_2$ such that $f(c_2) = f(q_2)$, then the fiber of ϕ is equal to $\mathbb{P}^{n-1} \bigcup \mathbb{P}^{n-1}$ glued along $\mathbb{P}^{n-2}$, set theoretically.*

3. *if f has ramification order 2 at c_2 and the tangent lines of images of C_1 and C_2 are independent, then the fiber of ϕ is equal to $\mathbb{P}^1$, set theoretically.*

4. *if the tangent lines of images of C_1 and C_2 are dependent, then the fiber of ϕ is equal to $Bl_{pt}\mathbb{P}^n \bigcup(\mathbb{P}^1 \times \mathbb{P}^{n-1}) \bigcup Bl_{pt}\mathbb{P}^n$ glued along $\mathbb{P}^{n-1}$, $\mathbb{P}^{n-2}$, set theoretically.*

Lemma 2.4.6. *Let $(C, f : C \longrightarrow \mathbb{P}^n)$ be an element of the main component of the Kontsevich's moduli space of stable maps satisfying the following conditions. C is of the type $E(0)[B_1(0)[C_1(1), C_2(1), C_3(1)]]$.*

1. *if the images of C_1 and C_2 and C_3 are distinct lines, then the fiber of ϕ is equal to a point set theoretically.*

2. *if the images of C_1 and C_2 are same line and the image of C_3 is distinct line, then the fiber of ϕ is equal to a point set theoretically.*

3. *if the images of C_1 and C_2 and C_3 are all same lines, then the fiber of ϕ is equal to $(\mathbb{P}^1 \times \mathbb{P}^{n-1}) \bigcup Bl_{pt}\mathbb{P}^n$ glued along $\mathbb{P}^{n-1}$set theoretically.*

Lemma 2.4.7. *Let $(C, f : C \longrightarrow \mathbb{P}^n)$ be an element of the main component of the Kontsevich's moduli space of stable maps satisfying the following conditions. C is of the type $E(0)[B_1(0)[C_1(1), C_2(1)], C_3(1)]]$.*

1. *if the images of C_1 and C_2 are distinct lines, then fiber of ϕ is equal to a point, set theoretically.*

2. *if the images of C_1 and C_2 are same lines and the image of C_3 is distinct line, then the fiber of ϕ is equal to $\mathbb{P}^{n-1}$, set theoretically.*

3. *if the images of C_1 and C_2 and C_3 are all same lines, then the fiber of ϕ is equal to $(\mathbb{P}^1 \times \mathbb{P}^{n-1}) \bigcup \mathbb{P}^{n-1}$ glued along $\mathbb{P}^{n-2}$, set theoretically.*

Lemma 2.4.8. *Let $(C, f : C \longrightarrow \mathbb{P}^n)$ be an element of the main component of the Kontsevich's moduli space of stable maps satisfying the following conditions. C is of the type $E(0)[B_1(0)[B_2(0)[C_1(1), C_2(1)], C_3(1)]]$.*

1. *if the images of C_1 and C_2 are distinct lines, then the fiber of ϕ is equal to point, set theoretically.*

2. *if the images of C_1 and C_2 are same lines and the image of C_3 is the distinct line, then the fiber of ϕ is equal to $\mathbb{P}^{n-1}$, set theoretically.*

3. *if the images of C_1 and C_2 and C_3 are all same lines, then the fiber of ϕ is equal to $Bl_{pt}\mathbb{P}^n \bigcup Bl_{pt}(\mathbb{P}^{n-1}\times\mathbb{P}^1)\bigcup(\mathbb{P}^{n-1}\times\mathbb{P}^1)\bigcup(\mathbb{P}^{n-1}\times\mathbb{P}^1)$, set theoretically.*

Remark 2.4.9. What we showed is that the fiber of ϕ is ,at least set theoretically, same as the fiber of corresponding blow-ups which we will describe later. Actually it is same scheme theoretically.

2.5 The case of the degree 2

In this section, we show that when $d = 2$, two moduli spaces are same. i.e. $\widetilde{\mathbf{M}}_{1,0}(\mathbb{P}^n, 2)_0 = \overline{\mathbf{M}}_{1,0}^{log,ch}(\mathbb{P}^n, 2)$. Note that if the degree is 2, $\widetilde{\mathbf{M}}_{1,0}(\mathbb{P}^n, 2)_0 = \overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, 2)_0$. As in the previous section, we can calculate the fiber of $\phi : \overline{\mathbf{M}}_{1,0}^{ch}(\mathbb{P}^n, 2) \longrightarrow \widetilde{\mathbf{M}}_{1,0}(\mathbb{P}^n, 2)_0$ and it is easy to see that every fiber is just one point. This actually suffices to conclude that $\widetilde{\mathbf{M}}_{1,0}(\mathbb{P}^n, 2)_0 = \overline{\mathbf{M}}_{1,0}^{log,ch}(\mathbb{P}^n, 2)$ by the Zariski's main theorem. Still we construct an actual morphism for the completeness. We only do the case $n = 1$ for simplicity.

Note that when the essential part is not contracted to a point, two moduli spaces are naturally isomorphic. So we only need to consider neighborhoods of points where the essential part is contracted to point.

We describe an etale atlas of stack $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^1, 2)_0$. Because of stackyness of the moduli space of elliptic curves, we need to separate the case according to j-invariant of the essential part of the domain curve.

when essential part is smooth elliptic curve with $\mathbf{j} \neq 0$

Let k be an algebraically closed field and t, α, γ, c, A be indeterminates. Let $R = k[t, \alpha, \gamma, c, A]/(\gamma - \alpha^3 - \gamma^2\alpha - A\gamma^3)$. Let D_1,D_2 be subschemes defined by ideals (α, γ),(t). Let $S = Spec(R)\backslash V$ where $V \subset Spec(R)$ is a subscheme defined by an ideal $(4+27A^2)$ and $C' = Proj(R[x,y,z]/zy^2 - x^3 - z^2x - Az^3)$.

Then the rational map $f' : C' \dashrightarrow \mathbb{P}^1$ defined by $[t\gamma(x+\alpha y)+c(\gamma x-\alpha z), \gamma x-\alpha z]$ gives the family of elliptic stable maps except at D_1 and D_2. But if we let C be the blow up of C' along (D_1, x, z),$(D_2, x-\alpha y, z-\gamma y)$,$(D_2, x, z)$, we easily see that $f' : C' \dashrightarrow \mathbb{P}^1$ extends to $f : C \longrightarrow \mathbb{P}^1$ and f gives a family of elliptic stable maps over whole S.

Moreover we know every element of family over S explicitly as follows.
Over $\{\gamma = 0, t \neq 0\}$, the domain curves are of the type $E[C_1]$ and $f|_{C_1} : C_1 \to \mathbb{P}^1$ is given by $[ts^2 + c(s-1) : s-1]$, where s is a local parameter of C_1 such that c_1 is given by $\{s = 0\}$.
Over $\{t = 0, \gamma \neq 0\}$, the domain curves are of the type $E[C_1, C_2]$ and c_1 and c_2 are given by $(z = 0)$, $(x = \alpha y, z = \gamma y)$ in $E = \{[x;y;z] : zy^2 = x^3 + z^2x + Az^3\}$ and $f|_{C_i} : C_i \to \mathbb{P}^1$ is given by $[s_i + c, 1]$ where c_i is a local parameter of C_i such that c_i is given by $(s_i = 0)$ for $i = 1, 2$.
Over $\{\gamma = 0, t = 0\}$, the domain curves are of the type $E[B_1[C_1, C_2]]$ and $f|_{C_i} : C_i \to \mathbb{P}^1$ is given by $[s_i + c, 1]$ where s_i local parameter of C_i such that c_i is given by $(s_i = 0)$ for $i = 1, 2$.
By looking at a local deformation, we can check that S is an etale atlas of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^1, 2)_0$.

Proposition 2.5.1. *Let us assume above. If we let W be the blow up of $S \times \mathbb{P}^1$ along $(D_2, x_0 - cx_1)$, $(D_1^2, x_0 - cx_1)$ where x_0, x_1 are coordinates of $\mathbb{P}^1$, then $f : C \longrightarrow \mathbb{P}^1$ extends to $\widetilde{f} : C \longrightarrow W$ and $\widetilde{f}$ gives a family of elliptic admissible maps over S.*

Remark 2.5.2. We know every element over S explicitly. Here we only describe the type of domain curves and target. Over $\{t \neq 0, \gamma \neq= 0\}$, the domain curve C is of type E and the target $W = \mathbb{P}^1$. Over $\{t = 0, \gamma \neq 0\}$, the domain curve C is of the type $E[C_1, C_2]$ and the target $W = \mathbb{P}^1(1)$. Over $\{t \neq 0, \gamma = 0\}$, the domain curve C is of the type $E[C_1]$ and the target $W = \mathbb{P}^1(1)$. Over $\{t = 0, \gamma = 0\}$, domain $C = E[B_1[C_1, C_2]]$ and the target $W = \mathbb{P}^1(2)$.

when essential part is smooth elliptic curve with j $\neq$ 1728

Everything is same if we change the equation $\gamma - \alpha^3 - \gamma^2\alpha - A\gamma^3$ above to $\gamma - \alpha^3 - A\gamma^2\alpha - \gamma^3$.

when essential part is singular curve

Let k be an algebraically closed field and t, α, β, c, A be indeterminates. Let $R = k[t,\alpha,\beta,c,A]/(\beta^2-\alpha^3-\alpha^2-A)$. Let D_2 be a subscheme defined by an ideal (t). Let $S = Spec(R)$ and $C' = Proj(R[x,y,z]/(zy^2-x^3-zx^2-Az^3))$.

Then a rational map $f' : C' \dashrightarrow \mathbb{P}^1$ defined by $[t(y+\beta z)+c(x-\alpha z), x-\alpha z]$ gives a family of elliptic stable maps except at D_2 and $\{\alpha=\beta=0\}$. But if we let C be the blow up of C' along ideals $(y-x-\frac{x^2+\alpha x+\alpha^2}{2}, \beta-\alpha-\frac{x^2+\alpha x+\alpha^2}{2})$, (D_2,x,z), $(D_2, x-\alpha z, y-\beta z)$, we can see that $f' : C' \dashrightarrow \mathbb{P}^1$ extends to $f : C \longrightarrow \mathbb{P}^1$ and f gives a family of elliptic stable maps over whole S. Note that the effect of blowing up along ideal $(y-x-\frac{x^2+\alpha x+\alpha^2}{2}, \beta-\alpha-\frac{x^2+\alpha x+\alpha^2}{2})$ is inserting a rational component at singular point in the rational nodal curve at $\{\alpha=\beta=0\}$.

Proposition 2.5.3. *Let us assume above. If we let W be the blow up of $S\times\mathbb{P}^1$ along ideal (D_2, x_0-cx_1) where x_0,x_1 are coordinates of $\mathbb{P}^1$, then $f : C \longrightarrow \mathbb{P}^1$ extends to $\widetilde{f} : C \longrightarrow W$ and $\widetilde{f}$ gives a family of elliptic admissible maps over S.*

Summing up previous results we get a morphism from $\widetilde{\mathbf{M}}_{1,0}(\mathbb{P}^1,2)_0$ to $\overline{\mathbf{M}}^{ch}_{1,0}(\mathbb{P}^1,2)$ where $\overline{\mathbf{M}}^{ch}_{1,0}(\mathbb{P}^1,2)$ is the moduli space of elliptic admissible stable maps without log structures. Actually it is one to one morphism. On the other hand we also have one to one morphism from $\overline{\mathbf{M}}^{ch,log}_{1,0}(\mathbb{P}^1,2)$ to $\overline{\mathbf{M}}^{ch}_{1,0}(\mathbb{P}^1,2)$, which is just forgetting log structures([**?**]). By the uniqueness of the normalization, we see that $\widetilde{\mathbf{M}}_{1,0}(\mathbb{P}^1,2)_0 = \overline{\mathbf{M}}^{ch,log}_{1,0}(\mathbb{P}^1,2)$.

2.6 The case of the degree 3

In this section we describe a local chart of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n,3)_0$ and $\overline{\mathbf{M}}^{log,ch}_{1,0}(\mathbb{P}^n,3)$. Throughout the section, we will denote the proper transforms of subscheme as the same notations as original subschemes.

Because of the stackyness of the moduli space of elliptic curves, we need to separate the case according to j-invariant of the essential part of the domain curve.

2.6.1 Etale chart of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, 3)_0$

(a)when essential part is smooth elliptic curve with j $\neq 0$.
Let k be an algebraically closed field and $a_1, a_2, \cdots, a_{n-1}, b_1, b_2, \cdots, b_{n-1}, c_1, c_2, \cdots, c_{n-1}, d_1, d_2, \cdots, d_n, z_1, z_2, A, \alpha, \gamma, \alpha', \gamma'$ be indeterminates. Let $R = \mathrm{k}[a_1,a_2,\cdots,a_{n-1},b_1,b_2,\cdots,b_{n-1}, c_1,c_2,\cdots,c_{n-1},d_1,d_2,\cdots,d_n,z_1,z_2,A,\alpha,\gamma,\alpha',\gamma']/(a_1z_1 - b_1z_2,a_2z_1 - b_2z_2,\cdots,a_{n-1}z_1 - b_{n-1}z_2,\gamma - \alpha^3 - \gamma^2\alpha - A\gamma^3,\gamma' - \alpha'^3 - \gamma'^2\alpha' - A\gamma'^3)$.

Let $D_{2,\alpha}$, $D_{2,\alpha'}$, $D_{2,\alpha-\alpha'}$, D_3, F_α, $F_{\alpha'}$, $F_{\alpha-\alpha'}$, G be subschemes defined by ideals (α, γ), (α', γ'), $(\alpha - \alpha', \gamma - \gamma')$, (z_1, z_2), $(z_1, b_1, b_2, \cdots, b_{n-1})$, $(z_2, a_1, a_2, \cdots, a_{n-1})$, $(z_1 - z_2, a_1 - b_1, a_2 - b_2, \cdots, a_{n-1} - b_{n-1})$, $(\frac{\alpha'\gamma\frac{z_1}{z_2}(x+\alpha y)(\gamma' x-\alpha' z)-\alpha\gamma'(\gamma x-\alpha z)(x+\alpha' y)}{\alpha\alpha'(\alpha-\alpha')(\gamma x-\alpha z)(\gamma' x-\alpha' z)})$.

Let $\widehat{S} = Spec(R) \setminus V$, $S = Bl_{(\alpha,\alpha',\gamma,\gamma')}\widehat{S}$, where V is the subscheme defined by an ideal $(4 + 27A^2)(\alpha - \alpha', \gamma - \gamma')$. Let D_1 be the exceptional divisor. Let $\widehat{C} = Proj(R[x, y, z]/zy^2 - x^3 - z^2x - Az^3)$, C' be a pull-back of $\widehat{C}$, and C be the blow up of C' along ideals

(D_1, x, z),
$(D_{2,\alpha}, x - \alpha' y, z - \gamma' y)$,$(D_{2,\alpha}, x, z)$,$(D_{2,\alpha'}, x - \alpha y, z - \gamma y)$,$(D_{2,\alpha'}, x, z)$,
(D_3, x, z),$(D_3, x - \alpha y, z - \gamma y)$,$(D_3, x - \alpha' y, z - \gamma' y)$,
$(F_\alpha, x - \alpha y, z - \gamma y)$,$(F_{\alpha'}, x - \alpha' y, z - \gamma' y)$,$(F_{\alpha-\alpha'}, x, z)$.

Then rational map $\widehat{f} : \widehat{C} \dashrightarrow \mathbb{P}^n$ given by

$$[\alpha'\gamma(a_1+c_1)z_1(x+\alpha y)(\gamma' x-\alpha' z)-\alpha\gamma'(b_1+c_1)z_2(x+\alpha' y)(\gamma x-\alpha z)+d_1(\alpha-\alpha')(\gamma x-\alpha z)(\gamma' z-\alpha' x), \alpha'\gamma(a_2+c_2)z_1(x+\alpha y)(\gamma' x-\alpha' z)-\alpha\gamma'(b_2+c_2)z_2(x+\alpha' y)(\gamma x-\alpha z)+d_2(\alpha-\alpha')(\gamma x-\alpha z)(\gamma' z-\alpha' x), \cdots, \alpha'\gamma(a_{n-1}+c_{n-1})z_1(x+\alpha y)(\gamma' x-\alpha' z)-\alpha\gamma'(b_{n-1}+c_{n-1})z_2(x+\alpha' y)(\gamma x-\alpha z)+d_{n-1}(\alpha-\alpha')(\gamma x-\alpha z)(\gamma' z-\alpha' x), \alpha'\gamma z_1(x+\alpha y)(\gamma' x-\alpha' z)-\alpha\gamma' z_2(x+\alpha' y)(\gamma x-\alpha z)+d_n(\alpha-\alpha')(\gamma x-\alpha z)(\gamma' z-\alpha' x), (\alpha-\alpha')(\gamma x-\alpha x)(\gamma' x-\alpha' z)]$$

extends to a morphism $f : C \longrightarrow \mathbb{P}^n$. This gives a family of semi-stable maps of elliptic curves and after the stabilization we get a family of stable maps of elliptic curves over S.

Note that as in the case of degree 2, we can actually describe every element parametrized by S and check it is an etale atlas.

(b)when essential part is smooth elliptic curve with j $\neq$ 1728.
Everything is same if we change the equation $\gamma - \alpha^3 - \gamma^2\alpha - A\gamma^3$ above to $\gamma - \alpha^3 - A\gamma^2\alpha - \gamma^3$.

(c)when essential part is singular curve.
Let k be an algebraically closed field and $a_1, a_2, \cdots, a_{n-1}, b_1, b_2, \cdots, b_{n-1}, c_1, c_2, \cdots, c_{n-1}, d_1, d_2, \cdots, d_n, z_1, z_2, A, \alpha, \beta, \alpha', \beta'$ be indeterminates. Let R= k$[a_1,a_2,\cdots,a_{n-1},b_1,b_2,\cdots,b_{n-1},c_1,c_2,\cdots,c_{n-1},d_1,d_2,\cdots,d_n,z_1,z_2,A,\alpha,\beta,\alpha',\beta']/(a_1z_1-b_1z_2 ,a_2z_1 - b_2z_2,\cdots,a_{n-1}z_1 - b_{n-1}z_2,\beta^2 - \alpha^3 - \alpha - A,\beta' - \alpha'^3 - \alpha' - A)$

Let $D_2, D_3, F_\alpha, F_{\alpha'}, F_{\alpha-\alpha'}, G$ be ideals defined by $(\alpha - \alpha',\beta - \beta')$, (z_1, z_2), $(z_1, b_1, b_2, \cdots, b_{n-1})$, $(z_2, a_1, a_2, \cdots, a_{n-1})$, $(z_1 - z_2, a_1 - b_1, a_2 - b_2, \cdots, a_{n-1} - b_{n-1})$, $(\frac{z_1}{z_2}(y + \beta)(x - \alpha') - (y + \beta')(x - \alpha))$.

Let $\widehat{S} = Spec(R)$, $S = Bl_{(\beta-\alpha-\frac{\alpha^2+\alpha\alpha'+\alpha'^2}{2},\beta'-\alpha'-\frac{\alpha^2+\alpha\alpha'+\alpha'^2}{2})}\widehat{S}$.
Let $\widehat{C} = Proj(R[x, y, z]/zy^2 - x^3 - x^2z - Az^3)$, let C' be a pull back of $\widehat{C}$, and C be the blow up of C' along ideals

$(y - x - \frac{x^2+\alpha x+\alpha^2}{2},\beta - \alpha - \frac{x^2+\alpha x+\alpha^2}{2})$,$(y - x - \frac{x^2+\alpha' x+\alpha'^2}{2},\beta' - \alpha' - \frac{x^2+\alpha' x+\alpha'^2}{2})$,
$(D_2, x - \alpha z, y - \beta z)$,$(D_2, x, z)$,
(D_3, x, z),$(D_3, x - \alpha z, y - \beta z)$,$(D_3, x - \alpha' z, y - \beta')$
$(F_\alpha, x - \alpha z, y - \beta z)$,$(F_{\alpha'}, x - \alpha' z, y - \beta' z)$,$(F_{\alpha-\alpha'}, x, z)$.

Then rational map $\widehat{f} : \widehat{C} \dashrightarrow \mathbb{P}^n$ given by

$[(\alpha-\alpha')(a_1+c_1)z_1(y+\beta z)(x-\alpha' z)-(\alpha-\alpha')(b_1+c_1)z_2(y+\beta' z)(x-\alpha z)+d_1(\beta-\beta')(x-\alpha z)(x-\alpha' z), (\alpha-\alpha')(a_2+c_2)z_1(y+\beta z)(x-\alpha' z)-(\alpha-\alpha')(b_2+c_2)z_2(y+\beta' z)(x-\alpha z)+d_1(\beta-\beta')(x-\alpha z)(x-\alpha' z), \cdots, (\alpha-\alpha')(a_{n-1}+c_{n-1})z_1(y+\beta z)(x-\alpha' z)-(\alpha-\alpha')(b_{n-1}+c_{n-1})z_2(y+\beta' z)(x-\alpha z)+d_{n-1}(\beta-\beta')(x-\alpha z)(x-\alpha' z), (\alpha-\alpha')z_1(y+\beta z)(x-\alpha' z)-(\alpha-\alpha')z_2(y+\beta' z)(x-\alpha z)+d_n(\beta-\beta')(x-\alpha z)(x-\alpha' z), (\beta-\beta')(x-\alpha z)(x-\alpha' z)]$

extends to a morphism $f : C \longrightarrow \mathbb{P}^n$. This gives the family of semi-stable maps of elliptic curves and after the stabilization we get a family of stable map of elliptic curves over S.

2.6.2 Local chart of $\overline{\mathbf{M}}_{1,0}^{log,ch}(\mathbb{P}^n, 3)$

(a)when essential part is smooth elliptic curve with j $\neq 0$.
In previous local chart S of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^2, 3)_0$, the blow-up center of Vakil-Zinger desingularization is given by D_3. And $\sum_1$, $\sum_2$, Γ_1, Γ_2 are given by proper transforms of

$$(D_1, a_1, a_2, \cdots, a_{n-1}, b_1, b_2, \cdots, b_{n-1}, \alpha z_1 + \alpha' z_2),$$
$$(D_{2,\alpha}, z_1, b_1, b_2, \cdots, b_{n-1})(D_{2,\alpha'}, z_2, a_1, a_2, \cdots, a_{n-1}),$$
$$(D_1, a_1, a_2, \cdots, a_{n-1}, b_1, b_2, \cdots, b_{n-1}),$$
$$(D_{2,\alpha}, a_1, a_2, \cdots, a_{n-1}, b_1, b_2, \cdots, b_{n-1})(D_{2,\alpha'}, a_1, a_2, \cdots, a_{n-1}, b_1, b_2, \cdots, b_{n-1}).$$

Let $\widetilde{S}$ be the blow up of S along D_3, $\sum_2$, Γ_2, $\sum_1$, Γ_1 and let E_1, $E_{2,\alpha} \bigcup E_{2,\alpha'}$, L_1, $L_{2,\alpha} \bigcup L_{2,\alpha'}$ be the exceptional divisors corresponding to $\sum_1$, $\sum_2$, Γ_1, Γ_2. Note that after blowing up along $\sum_2$, $\sum_1$ and Γ_2 are separated. Now let C'' be the pull back of $\widehat{C}$ along $\widetilde{S}$ and let $\widetilde{C}$ be the blow up of C" along ideals

(D_1, x, z), (L_1, x, z), (E_1, x, z),
$(D_{2,\alpha}, x - \alpha' y, z - \gamma' y)$,$(L_{2,\alpha}, x - \alpha' y, z - \gamma' y)$,$(E_{2,\alpha}, x - \alpha' y, z - \gamma' y)$,$(D_{2,\alpha}, x, z)$, $(L_{2,\alpha}, x, z)$,$(E_{2,\alpha}, x, z)$,
$(D_{2,\alpha'}, x - \alpha y, z - \gamma y)$,$(L_{2,\alpha'}, x - \alpha y, z - \gamma y)$,$(E_{2,\alpha'}, x - \alpha y, z - \gamma y)$,$(D_{2,\alpha'}, x, z)$, $(L_{2,\alpha'}, x, z)$,$(E_{2,\alpha'}, x, z)$,
(D_3, x, z),$(D_3, x - \alpha y, z - \gamma y)$,$(D_3, x - \alpha' y, z - \gamma' y)$,
$(F_\alpha, x - \alpha y, z - \gamma y)$,$(F_{\alpha'}, x - \alpha' y, z - \gamma' y)$,$(F_{\alpha-\alpha'}, x, z)$,
$(\widetilde{L}_1^2, G)$, $(\widetilde{L}_{2,\alpha}, G)$,$(\widetilde{L}_{2,\alpha'}, G)$,
where $\widetilde{L}_1$,$\widetilde{L}_{2,\alpha}$,$\widetilde{L}_{2,\alpha}$ are exceptional divisor of (L_1, x, z),$(L_{2,\alpha}, x, z)$,$(L_{2,\alpha'}, x, z)$.

Let $\widetilde{W}$ be the blow-up of $\widetilde{S} \times \mathbb{P}^2$ along ideals

$$(D_3, x_0 - d_1 x_n, x_1 - d_2 x_n, \cdots, x_{n-1} - d_n x_n),$$

$(E_2^2, x_0 - d_1x_n, x_1 - d_2x_n, \cdots, x_{n-1} - d_nx_n)$, $(L_2, x_0 - d_1x_n, x_1 - d_2x_n, \cdots, x_{n-1} - d_nx_n)$, $(D_2, x_0 - d_1x_n, x_1 - d_2x_n, \cdots, x_{n-1} - d_nx_n)$,
$(E_1^3, x_0 - d_1x_n, x_1 - d_2x_n, \cdots, x_{n-1} - d_nx_n)$, $(L_1^2, x_0 - d_1x_n, x_1 - d_2x_n, \cdots, x_{n-1} - d_nx_n)$, $(D_1^2, x_0 - d_1x_n, x_1 - d_2x_n, \cdots, x_{n-1} - d_nx_n)$,
where x_0,x_1,$\cdots$,x_n are coordinates of $\mathbb{P}^n$.

Then $\widehat{f} : \widehat{C} \dashrightarrow \mathbb{P}^n$ extends to $\widetilde{f} : \widetilde{C} \longrightarrow \widetilde{W}$ and we get a family of admissible maps over $\widetilde{S}$.

(b)when the essential part is smooth elliptic curve with j $\neq$ 1728.
Everything is same if we change the equation $\gamma - \alpha^3 - \gamma^2\alpha - A\gamma^3$ above to $\gamma - \alpha^3 - A\gamma^2\alpha - \gamma^3$.

(c)when the essential part is singular curve.
In previous local chart S of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^2, 3)_0$, the blow up center of Vakil-Zinger desingularization is given by D_3. And $\sum_2, \Gamma_2$ are given by proper transforms of $(D_2, z_1 - z_2, a_1 - b_1)$, (D_2, a_1, b_1).

Let $\widetilde{S}$ be the blow up of S along D_3, $\sum_2, \Gamma_2$ and let E_2, L_2 be the exceptional divisors corresponding to $\sum_2, \Gamma_2$. Now let C'' be the pull back of C' along $\widetilde{S}$ and let $\widetilde{C}$ be the blow up of C'' along ideals

$(D_2, x-\alpha z, y-\beta z)$,$(D_2, x, z)$,$(L_2, x-\alpha z, y-\beta z)$,$(L_2, x, z)$,$(E_2, x-\alpha z, y-\beta z)$,$(E_2, x, z)$,
(D_3, x, z),$(D_3, x - \alpha z, y - \beta z)$,$(D_3, x - \alpha' z, y - \beta' z)$,
$(F_\alpha, x - \alpha z, y - \beta z)$,$(F_{\alpha'}, x - \alpha' z, y - \beta' z)$,$(F_{\alpha-\alpha'}, x, z)$.
$(\widetilde{L_2}, G)$,
where $\widetilde{L_2}$ is exceptional divisor of (L_2, x, z).

Let $\widetilde{W}$ be blow-up of $\widetilde{S} \times \mathbb{P}^2$ along ideal

$(D_3, x_0 - d_1x_n, x_1 - d_2x_n, \cdots, x_{n-1} - d_nx_n)$,
$(E_2^2, x_0 - d_1x_n, x_1 - d_2x_n, \cdots, x_{n-1} - d_nx_n)$, $(L_2, x_0 - d_1x_n, x_1 - d_2x_n, \cdots, x_{n-1} - d_nx_n)$, $(D_2, x_0 - d_1x_n, x_1 - d_2x_n, \cdots, x_{n-1} - d_nx_n)$.

Then $\widehat{f} : \widehat{C} \dashrightarrow \mathbb{P}^n$ extends to $\widetilde{f} : \widetilde{C} \longrightarrow \widetilde{W}$ and we get the family of admissible maps over $\widetilde{S}$.

2.6.3 Main result

Let $\widehat{\mathbf{M}}$ be the blow up of $\widetilde{\mathbf{M}}_{1,0}(\mathbf{P}^n, 3)_0$ along $\sum_2$, Γ_2, $\sum_1$, Γ_1. By previous subsections, we can find a morphism from $\widehat{\mathbf{M}}$ to $\overline{\mathbf{M}}_{1,0}^{ch}(\mathbb{P}^n, 3)$. Here $\overline{\mathbf{M}}_{1,0}^{ch}(\mathbb{P}^n, 3)$ is moduli space of admissible stable maps of chain type without log structures. One can check that this morphism is finite surjective by using the result of section 5. Actually it is one to one morphism. On the other hand, We also have a finite surjective map from $\overline{\mathbf{M}}_{1,0}^{ch,log}(\mathbb{P}^n, 3)$ to $\overline{\mathbf{M}}_{1,0}^{ch}(\mathbb{P}^n, 3)$, which is just forgetting log structures([**?**]). By the uniqueness of the normalization, we get following theorem.

Theorem 1.0.1. $\overline{\mathbf{M}}_{1,0}^{log,ch}(\mathbb{P}^n, 3)$ *can be obtained by blowing-up* $\widetilde{\mathbf{M}}_{1,0}(\mathbf{P}^n, 3)_0$ *along the locus* $\sum_2$, Γ_2, $\sum_1$, Γ_1.

Remark 2.6.1. Note that we only used the fact that forgetting morphism ψ : $\overline{\mathbf{M}}_{1,0}^{log,ch}(\mathbb{P}^n, 3) \longrightarrow \overline{\mathbf{M}}_{1,0}^{ch}(\mathbb{P}^n, 3)$ is a finite morphism. It follow from above that it is actually one to one morphism in our cases. We can also get this fact by calculating possible log structures. i.e. when $d \leqslant 3$, there exists unique log structure on each admissible stable map. If $d \geqslant 4$, there could be more than one log structures on one admissible stable map.

Chapter 3

Degeneration of log stable maps

3.1 Introduction

3.1.1 The main result

Consider a projective morphism $W \to B$ from a nonsingular complex variety W to a nonsingular curve B with a distinguished closed point $0 \in B$ such that the central fiber W_0 consists of two irreducible nonsingular components intersecting transversely. In this paper we prove the degeneration formula (Theorem 3.1.1) for the degeneration W/B in the framework of minimal/basic stable log maps of D. Abramovich and Q. Chen [1]; Q. Chen [6]; M. Gross and B. Siebert [10]. In papers [10] (resp. [1, 6]), without expanding targets they have constructed the virtually smooth proper DM stacks of basic (resp. minimal) stable log maps to a Zariski-globally generated (resp. Deligne-Faltings) log smooth target with a fixed numerical class. The Deligne-Faltings log structures are special cases of Zariski-globally generated log structures. Associated to the divisors W_0 of W and 0 of B, the schemes W, B are equipped with the natural Deligne-Faltings log structures making $W \to B$ log smooth (see [13]).

The degeneration formulas for W/B in the framework of the "expanded" (relative) stable maps were already discovered: in symplectic geometry set-up by A.-M. Li and Y. Ruan [17], by E. Ionel and T. Parker [12]; in algebraic geometry set-up by J. Li [18]. The degeneration formula is also proven by D. Abramovich and B. Fantechi [2] using stable twisted maps and by Q. Chen [5] using stable log maps in the sense of [5, 15]. All of these methods commonly use target expansions.

Without target expansions it is under development to obtain a degeneration formula for general degenerations in the realm of minimal/basic stable log maps (however see also [25] in a symplectic set-up). We are contented to prove the degeneration formula for the simple case W/B, expecting that the proof will serve to establish a general degeneration formula. We mention that the splitting method in this paper works for the general case, too.

3.1.2 The precise statement

For the precise statement of the degeneration formula we need some preparations. Below for a log scheme S, denote by $\underline{S}$ the underlying scheme of S. Let $\underline{W}_0$ be $\underline{X}_1 \sqcup_D \underline{X}_2$, where D is the singular locus of $\underline{W}_0$ and $\underline{X}_i$, $i = 1, 2$, are the irreducible components of $\underline{W}_0$. Let $\Bbbk$ denote the field $\mathbb{C}$ of complex numbers. The point $0 = \mathrm{Spec}(\Bbbk)$ of $\underline{B}$ has the induced log structure, the so-called standard log point, from B, denoted by $\mathrm{Spec}(\Bbbk^\dagger)$. We consider the target $W_0/\mathrm{Spec}(\Bbbk^\dagger)$ as the log scheme over $\mathrm{Spec}(\Bbbk^\dagger)$, whose log structure is defined to be the inverse image of the log structures of W under $\underline{W}_0 \subset \underline{W}$.

For an effective curve class $\beta \in H_2(\underline{W}_0, \mathbb{Z})$, denote by

$$\overline{M}_{g,n}(W_0/\mathrm{Spec}(\Bbbk^\dagger), \beta)$$

the moduli stack of n-pointed, genus g, class β, minimal/basic stable log maps to $W_0/\mathrm{Spec}(\Bbbk^\dagger)$ (see [1, 6, 10]). The moduli stack is a proper DM stack over $\Bbbk$, with the canonical virtual fundamental class and evaluation maps at markings, denoted by

$$[\overline{M}_{g,n}(W_0/\mathrm{Spec}(\Bbbk^\dagger), \beta)]^{\mathrm{vir}} \quad \text{and}$$
$$ev_i : \overline{M}_{g,n}(W_0/\mathrm{Spec}(\Bbbk^\dagger), \beta) \to \underline{W}_0, \quad i = 1, ..., n,$$

respectively.

Let Γ be a decorated connected graph satisfying the following. The set $V(\Gamma)$ of vertices is partitioned into two sets $V_1(\Gamma)$, $V_2(\Gamma)$. Every edge connects a vertex from $V_1(\Gamma)$ and a vertex from $V_2(\Gamma)$. To $v \in V_i(\Gamma)$, a decoration (g_v, β_v, N_v) is given with $g_v \in \mathbb{Z}_{\geq 0}$, β_v an effective curve class of X_i, $N_v \subset [n]$. Here, $[n] := \{1, ..., n\}$. To each edge e, a strictly positive integer c_e is given. Finally, a label on $V(\Gamma)$ is chosen. Let $\iota_{\underline{X}_a}$ denote the closed embedding $\underline{X}_a \to \underline{W}_0$. We require that

$$\sum_{v \in V_1(\Gamma)} \iota_{X_1,*}\beta_v + \sum_{v \in V_1(\Gamma)} \iota_{X_2,*}\beta_v = \beta;$$

for $v \in V_i(\Gamma)$

$$\beta_v \cdot D = \sum_{v \in e} c_e;$$

the "stability" condition $\beta_v \neq 0$ whenever $2g_v + |N_v| + \mathrm{val}(v) < 3$;

$$1 - \chi(\Gamma) + \sum_v g_v = g;$$

and

$$\coprod_v N_v = \{1, ..., n\}.$$

Let $r(v) = i$ if $v \in V_i(\Gamma)$ and let E_v be the set of edges adjacent to v. Let

$$\overline{M}_{g_v, N_v|E_v}(X_{r(v)}, \beta)$$

be the moduli stack of N_v-pointed, genus g_v, class β_v, minimal/basic stable log maps to $X_{r(v)}$ with relative markings $\{e \mid e \in E_v\}$ whose contact orders are $\{c_{e_v} \mid e \in E_v\}$ with respect to the divisor D. Here, X_i is considered the log scheme, obtained from the divisor $D \subset \underline{X}_i$, with the log smooth morphism $X_i \to \mathrm{Spec}(\Bbbk)$. For $e \in E_v$, denote by ev_e be the "relative" evaluation map

$$\overline{M}_{g, N_v|E_v}(X_{r(v)}, \beta) \to D$$

and for $i \in N_v$, denote by ev_i be the "absolute" evaluation map

$$\overline{M}_{g, N_v|E_v}(X_{r(v)}, \beta) \to \underline{X}_{r(v)}.$$

Let $\Omega(g, n, \beta)$ be the set of all such graphs Γ. Note that $\Omega(g, n, \beta)$ is a finite set. Let $\{\delta_j^1\}$ be a homogeneous basis of $H^*(D, \mathbb{Q})$ and let $\{\delta_j^2\}$ be the dual basis in the sense that

$$\int_D \delta_i^1 \delta_j^2 = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

Now we are ready to state the degeneration formula.

Theorem 3.1.1. *For $\gamma_i \in H^*(\underline{W}_0, \mathbb{Q})$ and the Psi classes ψ_i, $i = 1, ..., n$,*

$$\int_{[\overline{M}_{g,n}(W_0,\beta)]^{\mathrm{vir}}} \prod_{i\in[n]} \psi_i^{m_i} ev_i^*(\gamma_i) = \sum_{\Gamma\in\Omega(g,N,\beta)} \sum_{(j_e)_e\in\{1,\dots,\mathrm{rk}H^*(D)\}^{E(\Gamma)}} (-1)^{\epsilon}$$

$$\frac{\prod_e c_e}{|V(\Gamma)|!} \prod_v \int_{[\overline{M}_{g,N_v|E_v}(X_v,\beta)]^{\mathrm{vir}}} \prod_{i\in N_v} \psi_i^{m_i} ev_i^* \iota_{\underline{X}_a}^*(\gamma_i) \prod_{e\in E_v} ev_e^*(\delta_{j_e}^{r(v)}),$$

where $(-1)^{\epsilon}$ is given by the equality

$$\prod_{i=1}^{n} \gamma_i \prod_{j\in M} \delta_j^1 \delta_j^2 = (-1)^{\epsilon} \prod_{i\in N_1} \gamma_i \prod_{j\in M} \delta_j^1 \prod_{i\in N_2} \gamma_i \prod_{j\in M} \delta_j^2.$$

3.1.3 Conventions

Throughout the paper unless otherwise specified: every scheme is a scheme locally of finite type over the field $\mathbb{k} := \mathbb{C}$ of complex numbers and every log structure $\alpha : \mathcal{M}_S \to \mathcal{O}_S$ on a scheme S is a fine and saturated (fs for short) log structure on the étale site of S. The underlying scheme of a log scheme S will be denoted by $\underline{S}$. A log morphism f from a log scheme S_1 to a log scheme S_2 consists of a morphism $\underline{f} : \underline{S}_1 \to \underline{S}_2$ of schemes and a homomorphism $f^\flat : f^{-1}\mathcal{M}_{S_2} \to \mathcal{M}_{S_1}$ compatible with log structure $\alpha_i : \mathcal{M}_{S_i} \to \mathcal{O}_{S_i}$.

The standard log point will be denoted by $\mathrm{Spec}(\mathbb{k}^\dagger)$. The sum $\mathcal{M}_1 \oplus \mathcal{M}_2$ of log structures $\mathcal{M}_i$ on a scheme $\underline{S}$ always means the push-out $\mathcal{M}_1 \oplus_{\mathcal{O}_{\underline{S}}^\times} \mathcal{M}_2$.

For a toric monoid Q, meaning that Q is a fs monoid with no nontrivial invertibles, $Q^\vee$ denotes $\mathrm{Hom}(Q, \mathbb{N})$ and $Q^\vee[1]$ denotes the set of minimal integral elements in the extremal rays in $Q^\vee \otimes \mathbb{Q}_{\geq 0}$.

For $s \in \underline{S}$, $\bar{s} \to \underline{S}$ denotes the associated geometric point by taking the separable closure of s. Let P be a monoid. Then a monoid homomorphism $\alpha : P \to \mathcal{O}_{S,\bar{s}}$ will be called a log structure on $\mathcal{O}_{S,\bar{s}}$ if α induces an isomorphism from $\alpha^{-1}(\mathcal{O}_{S,\bar{s}}^\times)$ to $\mathcal{O}_{S,\bar{s}}^\times$.

By an algebraic stack $\mathfrak{X}$ over a scheme, we mean a stack $\mathfrak{X}$ over the scheme such that the diagonal is representable and of finite presentation; and it allows a surjective smooth morphism $U \to \mathfrak{X}$ from a scheme U. This is the convention following [23].

For a curve C over an algebraically closed field, denote by C^{nd} the set of nodal points, denote by $\mathrm{Irr}(C)$ the set of generic points of the irreducible components of C, and denote by $\mathrm{Con}(C)$ the set of connected components of C.

3.2 Basic/Minimal stable log maps

3.2.1 Log structures on $\mathfrak{M}_{g,n}$ and $\mathfrak{C}_{g,n}$

Let $\mathfrak{M}_g$ be the algebraic stack of genus g prestable curves. The boundary divisor parameterizing singular curves gives rise to the divisorial log structure on the lisse-étale site of $\mathfrak{M}_g$. Define the log structure also on the algebraic stack $\mathfrak{M}_{g,n}$ of n-pointed genus g prestable curves by taking the inverse image of the log structure on $\mathfrak{M}_g$ under the natural projection $\mathfrak{M}_{g,n} \to \mathfrak{M}_g$. Denote by $\mathfrak{C}_{g,n}$ the universal curve with the log structure attached to the normal crossing divisors of boundaries viewed as $\mathfrak{M}_{g,n+1}$.

3.2.2 Prestable log curves

Let $\underline{S}$ be a $\Bbbk$-scheme. A n-pointed prestable curve $(\underline{C}/\underline{S}, \mathbf{p} := (p_1, ..., p_n))$ amounts to a map $\underline{S} \to \underline{\mathfrak{M}}_{g,n}$. We denote by

$$\mathcal{M}_{\underline{S}}^{\underline{C}/\underline{S}}$$

the associated pullback log structure of $\mathcal{M}_{\mathfrak{M}_{g,n}}$ under the map. Strictly speaking, the pullback log structure is on the lisse-étale site of $\underline{S}$. However, by [23, Theorem A.1], it is induced from a unique log structure $\mathcal{M}_{\underline{S}}^{\underline{C}/\underline{S}}$ on the étale site of $\underline{S}$.

Similarly, on $\underline{C}$ there is the log structure

$$\mathcal{M}_{\underline{C}}^{\underline{C}/\underline{S},\mathbf{p}}$$

inherited from the log structure of $\mathfrak{C}_{g,n}$ by the pullback. These $\mathcal{M}_{\underline{S}}^{\underline{C}/\underline{S}}$ (resp. $\mathcal{M}_{\underline{C}}^{\underline{C}/\underline{S},\mathbf{p}}$) will be called the *canonical log structure* on $\underline{S}$ (resp. on $\underline{C}$) attached to the n-pointed prestable curve $(\underline{C}/\underline{S}, \mathbf{p})$.

A log morphism from a fine saturated log scheme S to $(\underline{S}, \mathcal{M}_{\underline{S}}^{\underline{C}/\underline{S}})$ will be called a *n-pointed prestable log curve*. Denote by

$$\varphi : \mathcal{M}_{\underline{S}}^{\underline{C}/\underline{S}} \to \mathcal{M}_S \tag{3.2.1}$$

the homomorphism attached to the log morphism. Note that this is equivalent to an object over a scheme $\underline{S}$ of the log stack $\mathcal{L}og_{\mathfrak{M}_{g,n}}$, which is, by definition, a log 1-morphism from a fine saturated log scheme S to the log stack $\mathfrak{M}_{g,n}$ (see [23, Proposition 5.9]).

There is another interchangeable description of the prestable log curve. From

$$S \to \mathfrak{M}_{g,n}, \tag{3.2.2}$$

we obtain C by taking the fiber product of log stacks

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\pi"'] & \mathfrak{C}_{g,n} \arrow[d] \\ S \arrow[r] & \mathfrak{M}_{g,n} \end{tikzcd}$$

and hence a log smooth, integral morphism

$$\pi : C \to S \tag{3.2.3}$$

with n-pointed prestable curves $(\underline{C}/\underline{S}, \mathbf{p} = p_1, ..., p_n)$ satisfying the following. The log structure $\mathcal{M}_C|_{C^{\mathrm{sm}}}$ on the π-smooth locus C^{sm} is isomorphic to the log structure $\pi^*(\mathcal{M}_S) \oplus \bigoplus_i \mathcal{M}_{D_i}$ where $\mathcal{M}_{D_i}$ denotes the log structure standardly defined by the smooth divisor p_i.

The converse direction (3.2.3) $\Rightarrow$ (3.2.2) is also true by [14, 22].

An isomorphism between two n-pointed prestable log curve over $\underline{S}$ in the stack $\mathcal{L}og_{\mathfrak{M}_{g,n}}$ can be reinterpreted exactly as a pair of isomorphisms $h : C \to C'$, $h_S : S \to S'$ of log schemes making a fiber product of log spaces

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d] & C' \arrow[d] \\ S \arrow[r, "h_S"] & S' \end{tikzcd}$$

and satisfying $\underline{h_S} = \mathrm{Id}_{\underline{S}}$.

From now on, C/S always means a pointed prestable log curves unless otherwise stated.

3.2.3 Generization maps

Let q be a node point of $\underline{C}_{\bar{s}}$ for $s \in S$. There is a corresponding component $\mathbb{N}_{\bar{q}} = \mathbb{N}$ in the free monoid $\overline{\mathcal{M}}^{\underline{C}/\underline{S}}_{\underline{S},\bar{s}}$.

According to F. Kato [14],

$$\overline{\mathcal{M}}_{C,\bar{q}} = \overline{\mathcal{M}}_{S,\bar{s}} \oplus_{\mathbb{N}_{\bar{q}}} \mathbb{N}^2. \tag{3.2.4}$$

Here RHS of (3.2.4) is the push-out of a diagram

$$\begin{array}{ccc} \mathbb{N}_{\bar{q}} & \longrightarrow & \mathbb{N}^2 \\ \downarrow & & \\ \overline{\mathcal{M}}_{S,\bar{s}}, & & \end{array}$$

where $\mathbb{N}_{\bar{q}} \to \mathbb{N}^2$ is the diagonal map and $\mathbb{N}_q \to \overline{\mathcal{M}}_{S,\bar{s}}$ is the induced from φ in (3.2.1). The isomorphism (3.2.4) preserves the natural homomorphisms from $\overline{\mathcal{M}}_{S,\bar{s}}$ to $\overline{\mathcal{M}}_{C,\bar{q}}$ and $\overline{\mathcal{M}}_{S,\bar{s}} \oplus_{\mathbb{N}_q} \mathbb{N}^2$.

In what follows, λ_q will denote the image of $1_q := 1 \in \mathbb{N}_q$ under the homomorphism $\mathbb{N}_q \to \overline{\mathcal{M}}_{S,\bar{s}}$. Note that

$$\lambda_q \neq 0$$

since $\bar{\varphi}^{-1}(0) = 0$.

Adjacent to the node q there are two generic points η_i, $i = 1, 2$ and so we obtain two generization maps $\overline{\mathcal{M}}_{C,\bar{q}} \to \overline{\mathcal{M}}_{C,\bar{\eta}_i}$, which will be denoted by $\chi^S_{q,i}$ and have explicit expressions:

$$\begin{aligned} \chi^S_{q,i} : \overline{\mathcal{M}}_{S,\bar{s}} \oplus_{\mathbb{N}_{\bar{q}}} \mathbb{N}^2 &\to \overline{\mathcal{M}}_{S,\bar{s}} \\ (m, (a_1, a_2)) &\mapsto m + a_i \lambda_q. \end{aligned} \tag{3.2.5}$$

Since $\lambda_q \neq 0$ and $\overline{\mathcal{M}}_{S,\bar{s}}$ is a toric monoid, observe that

$$\chi^S_{q,1} \times \chi^S_{q,2} : \overline{\mathcal{M}}_{S,\bar{s}} \oplus_{\mathbb{N}_q} \mathbb{N}^2 \to \overline{\mathcal{M}}_{S,\bar{s}} \times \overline{\mathcal{M}}_{S,\bar{s}}$$

is injective. Therefore we may identify the monoid $\overline{\mathcal{M}}_{S,\bar{s}} \oplus_{\mathbb{N}_q} \mathbb{N}^2$ with the image

$$\{(m_1, m_2) \in \overline{\mathcal{M}}_{S,\bar{s}} \times \overline{\mathcal{M}}_{S,\bar{s}} \mid m_2 - m_1 \in \mathbb{Z}\lambda_q \text{ in } \overline{\mathcal{M}}^{\mathrm{gp}}_{S,\bar{s}}\}$$

(see [10]). This point of view indicates that a homomorphim to $\overline{\mathcal{M}}_{C_{\bar{s}}}$ from a sheaf of monoids on $\underline{C}_{\bar{s}}$ is determined by the restriction to the set of all generic or marked points.

3.2.4 Stable log maps

We recall stable log maps introduced by D. Abramovich, Q. Chen [1]; Q. Chen [6]; and M. Gross, B. Siebert [10]. Our presentation will closely follow [10].

Let W, B be fine saturated log schemes and $\pi_W : W \to B$ be a log smooth, projective morphism. We assume that the log structure on W is induced from a log structure on the *Zariski* site W.

Definition 3.2.1. *A* stable log map *to W/B is a triple*

$$((C/S, \mathbf{p}), f, f_S) \tag{3.2.6}$$

as follows:

- $(C/S, \mathbf{p})$ *is a n-pointed prestable log curve.*
- *f (resp. f_S) is a log morphism from C (resp. S) to W (resp. B) fitting into a commutative diagram of log morphisms*

$$\begin{array}{ccc} C & \xrightarrow{f} & W \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_W} \\ S & \xrightarrow{f_S} & B \end{array} \tag{3.2.7}$$

whose underlying pair $((\underline{C}/\underline{S}, \mathbf{p}), \underline{f}, \underline{f_S})$ is a $\underline{S}$-family of n-pointed stable maps to $\underline{W}/\underline{B}$.

An isomorphism between two stable log maps

$$(C/S, \mathbf{p}, f),\ (C'/S', \mathbf{p}', f')$$

over $\underline{S} \to \underline{B}$ is defined to be an isomorphism (h, h_S) between $(C/S, \mathbf{p})$ and $(C'/S', \mathbf{p}')$ in $\mathcal{L}og_{\mathfrak{M}_{g,n}}$ satisfying the compatibility

$$f' \circ h = f, \quad f'_{S'} \circ h_S = f_S$$

as in diagram

$$\begin{tikzcd}
& W \arrow[dd] & \\
C \arrow[ur, "f"] \arrow[rr, "h"] \arrow[dd] & & C' \arrow[ul, "f'"'] \arrow[dd] \\
& B & \\
S \arrow[ur, "f_S"] \arrow[rr, "h_S"'] & & S' \arrow[ul, "f'_{S'}"']
\end{tikzcd}$$

Therefore we may define a $\underline{B}$-stack of stable log maps to W/B, which is shown to be an algebraic stack locally of finite type over $\underline{B}$ (see [1, 6, 10]).

3.2.5 Class

Let $NE_1(\underline{W})$ be the submonoid of $H_2(\underline{W}, \mathbb{Z})$ generated by effective curve classes. Let R be a subset of $\{1, ..., n\}$. Fix $Z_i \subset W, i \in R$ which are strict closed log schemes and fix global sections $\sigma_i : \overline{\mathcal{M}}^{\mathrm{gp}}_{Z_i} \to \mathbb{Z}_{Z_i}$ on $\underline{Z}_i$. We say that a stable log map (3.2.6) is of class $(g, \beta, \{\sigma_i\}_{i\in R})$ if the following conditions are satisfied. The genus of $\underline{C}/\underline{S}$ is g; the stable map $\underline{f}$ has the curve class $\beta \in NE_1(\underline{W})$; $\underline{f}(p_i) \subset \underline{Z}_i$; and finally the equality

$$\mathrm{pr}_2 \circ \overline{f \circ p_i}^{\flat} = (f \circ p_i)^*(\sigma_i)$$

of homomorphisms

$$(f \circ p_i)^{-1}\overline{\mathcal{M}}^{\mathrm{gp}}_{Z_i} \to \mathbb{Z}_S,\ i \in R.$$

3.2.6 Types

Consider a stable log map $(C/S, \mathbf{p}, f)$ and let $\bar{s} \to S$ be a geometric point. For a marked point p of $C_{\bar{s}}$ (i.e., $p = p_i|_{\bar{s}}$ for some i), we define u_p to be the composite

$$\mathrm{pr}_2 \circ \overline{f}^{\flat}_p : \overline{\mathcal{M}}_{W,\underline{f}(p)} \to \overline{\mathcal{M}}_{S,\bar{s}} \oplus \mathbb{N} \to \mathbb{N}.$$

At a node q, we define a homomorphism

$$u_q : \overline{\mathcal{M}}_{W,\underline{f}(q)} \to \mathbb{Z}$$

by the equation

$$((\overline{\pi}^\flat)^{-1}_{\bar{\eta}_2} \circ \overline{f^\flat}_{\bar{\eta}_2} \circ \chi_{q,2})(m) - ((\overline{\pi}^\flat)^{-1}_{\bar{\eta}_1} \circ \overline{f^\flat}_{\bar{\eta}_1} \circ \chi_{q,1})(m) = u_q(m)\lambda_q \tag{3.2.8}$$

where η_1, η_2 are generic points of two components of $C_{\bar{s}}$ adjacent to q and $\chi_{q,i}$: $\overline{\mathcal{M}}_{W,\underline{f}(q)} \to \overline{\mathcal{M}}_{W,\underline{f}(\bar{\eta}_i)}$ are the generization maps. Note that u_q is determined up to sign depending on the orderings of the two components of C.

Let $((C'/S', p'_1, ..., p'_n), f')$ be obtained from a base change $S' \to S$ and let $\bar{s} \to S' \to S$. Then for $p' \mapsto p$, $q' \mapsto q$, note that $\lambda_q \mapsto \lambda_{q'}$ under $\overline{\mathcal{M}}_{S,\bar{s}} \to \overline{\mathcal{M}}_{S',\bar{s}}$ and the types $u_{p'}, u_{q'}$ of f' coincide with the types u_p, u_q of f, respectively.

Remark 3.2.2. Note that, generally, for a sheaf P of monoids on $\underline{C}$ and a homomorphim from $P \to \overline{\mathcal{M}}_C$, we can define u_p and u_q.

Remark 3.2.3. If q is a self-intersection point, then $u_q = 0$ since $\overline{\mathcal{M}}_W$ is a log structure on a Zariski site. Also, if $\overline{\mathcal{M}}_{W,\underline{f}(q)} \cong \overline{\mathcal{M}}_{B,\underline{\pi}_W(\underline{f}(q))}$ under $\bar{\pi}^\flat_W$, then $u_q = 0$ by the commutativity of (3.2.7).

3.2.7 Universality

Given a stable log map (3.2.6) we construct a universal one following [1, 6, 10]. We proceed in three steps.

Characteristic level

There is a natural complex of abelian groups

$$\prod_{q \in C^{\mathrm{nd}}_{\bar{s}}} \overline{\mathcal{M}}^{\mathrm{gp}}_{W,\underline{f}(q)} \xrightarrow{g_1} \prod_{\eta \in \mathrm{Irr}(C)_{\bar{s}}} \overline{\mathcal{M}}^{\mathrm{gp}}_{W,\underline{f}(\bar{\eta})} \times \prod_{q \in C^{\mathrm{nd}}_{\bar{s}}} \mathbb{N}^{\mathrm{gp}}_q \xrightarrow{g_2} \overline{\mathcal{M}}^{\mathrm{gp}}_{S,\bar{s}} \to 0$$

where:

- The first homomorphism g_1 is defined by

$$g_1 = \sum_q (\iota_{\bar{\eta}_1(q)} \circ \chi_{q,1} - \iota_{\bar{\eta}_2(q)} \circ \chi_{q,2} + \iota_q \circ u_q) \circ \mathrm{pr}_q$$

for the homomorphism $u_q : \overline{\mathcal{M}}_{W,\underline{f}(q)} \to \mathbb{Z}$ defined in (3.2.8). Here pr_q is the projection to the component $\overline{\mathcal{M}}^{\mathrm{gp}}_{W,\underline{f}(q)}$ and $\iota_\star$ denotes the injection associated to $\star$-component.

- The second homomorphism g_2 is induced from the log map $f : C/S \to W/B$ and the homomorphism $\mathbb{N}_q = \mathbb{N} \to \overline{\mathcal{M}}_{S,\bar{s}}$ sending $1_q = 1$ to λ_q. This means that

$$g_2 = \sum_\eta \bar{f}^\flat_{\bar{\eta}} \circ \mathrm{pr}_{\bar{\eta}} + \sum_q \lambda_q \mathrm{pr}_q.$$

Let $Q^{\mathrm{gp}}_{\bar{s}}$ be the cokernel of g_1 and let $Q_{\bar{s}}$ be the saturation of the quotient image of $\prod_\eta \overline{\mathcal{M}}_{W,\underline{f}(\bar{\eta})} \times \prod_q \mathbb{N}_q$ in $Q^{\mathrm{gp}}_{\bar{s}}$. Let

$$[g_2] : Q_{\bar{s}} \to \overline{\mathcal{M}}_{S,\bar{s}}$$

be the induced homomorphism.

Definition 3.2.4. *This $Q_{\bar{s}}$ is called* basic *in [10] and* minimal *in [6, 1]. We call $\mathcal{M}_{S,\bar{s}}$ and $\overline{\mathcal{M}}_{S,\bar{s}}$* basic *or* minimal *if* $[g_2]$ *is an isomorphism (i.e. the complex above becomes exact).*

We will use the latter terminology 'minimal' following [8] for more general context. We remark that in [15] non-degenerate case was treated (see §3.3.2 below for details).

Remark 3.2.5. In [10], by the existence of the stable log map $(C/S, \mathbf{p}, f)$, it was shown that $Q_{\bar{s}}$ is in fact a toric monoid. Therefore $Q_{\bar{s}}$ can be recovered from its dual

$$\begin{aligned} &Q^\vee_{\bar{s}} = \{(V_\eta, l_q) \in \textstyle\prod_\eta \overline{\mathcal{M}}^\vee_{W,\underline{f}(\bar{\eta})} \times \prod_q \mathbb{N}^\vee_q \mid \forall q \in C^{\mathrm{nd}}_{\bar{s}}, \\ &V_{\eta_2(q)} \circ \chi_{q,2} - V_{\eta_1(q)} \circ \chi_{q,1} = l_q u_q \text{ as elements of } \mathrm{Hom}(\overline{\mathcal{M}}_{W,\underline{f}(q)}, \mathbb{Z})\} \end{aligned} \tag{3.2.9}$$

which is often easier to be computed than $Q_{\bar{s}}$.

Remark 3.2.6. When $\overline{\mathcal{M}}_{W,\underline{f}(q)}$ is free for every node q, there is another equivalent description for Q. Choose an isomorphism $\overline{\mathcal{M}}_{W,\underline{f}(q)} \cong \mathbb{N}^{r_q}$ (unique up to orderings of the basis elements) for some nonnegative integer r_q and then express $u_q \in \mathrm{Hom}(\overline{\mathcal{M}}_{W,\underline{f}(q)}, \mathbb{Z})$ uniquely as

$$u_q = u_{q,1} - u_{q,2}, \quad u_{q,i} \in \mathrm{Hom}(\overline{\mathcal{M}}_{W,\underline{f}(q)}, \mathbb{N}).$$

These components $u_{q,i}$ do not depend upon the choices of isomorphisms. Now we may describe Q as the saturated co-equalizer of

$$\prod_{q\in C^{\mathrm{nd}}_{\bar{s}}} \overline{\mathcal{M}}_{W,\underline{f}(q)} \rightrightarrows \prod_{\eta\in \mathrm{Irr}(C)_{\bar{s}}} \overline{\mathcal{M}}_{W,\underline{f}(\bar{\eta})} \times \prod_{q\in C^{\mathrm{nd}}_{\bar{s}}} \mathbb{N}_q, \tag{3.2.10}$$

where two homomorphisms are given by

$$\sum_q (\iota_{\bar{\eta}_i(q)} \circ \chi_{q,i} + \iota_q \circ u_{q,i}) \circ \mathrm{pr}_q, \quad i = 1, 2.$$

It is straightforward to check that this definition agrees with the first one.

Construction of $\overline{\mathcal{M}}^{\min}_S$

We will define a canonical sheaf $\overline{\mathcal{M}}^{\min}_S$ of monoids on the étale site of S such that $\overline{\mathcal{M}}^{\min}_{S,\bar{s}} \cong Q_{\bar{s}}$ for every $\bar{s} \to S$. First note that for every $\sigma \in Q_{\bar{s}}$, there are pairs (U, σ_U) of open étale neighborhoods U of $\bar{s}$ and functions $\sigma_U : U \to \bigsqcup_{u\in U} Q_{\bar{u}}$, satisfying that $\sigma_U|_{U'} = \sigma_{U'}$ whenever $(U', \sigma_{U'})$ is such a pair with an S-étale morphism $U' \to U$. Now for any étale morphism $V \to S$ define $\overline{\mathcal{M}}^{\min}_S(V)$ to be the set of functions $t : V \to \bigsqcup_{v\in V} Q_{\bar{v}}$ such that t allows a collection $\{(U_i, \sigma_{U_i})\}$ with a cover $\{U_i\}$ of V and $\sigma_{U_i,\bar{v}} = t(v)$ for $\bar{v} \to U_i$.

It is clear that $\overline{\mathcal{M}}^{\min}_S$ is a sheaf with a canonical isomorphism $\overline{\mathcal{M}}^{\min}_{S,\bar{s}} \cong Q_{\bar{s}}$ for every $\bar{s}$. Note also that there is a canonical homomorphism

$$\begin{tikzcd} \overline{\mathcal{M}}^{C/\underline{S}}_{\underline{S}} \arrow[r] \arrow[dr] & \overline{\mathcal{M}}^{\min}_S \arrow[d] \\ & \overline{\mathcal{M}}_S \end{tikzcd}$$

sending $1_q \in \mathbb{N}_q$ to $[\iota_q(1)]$ in $\overline{\mathcal{M}}^{\min}_S$.

Let

$$\overline{\mathcal{M}}^{\min}_C := \pi^{-1}\overline{\mathcal{M}}^{\min}_S \oplus_{\pi^{-1}\overline{\mathcal{M}}^{C/S}_{\underline{S}}} \overline{\mathcal{M}}^{C/\underline{S}}_{\underline{C}},$$

then we obtain a natural commutative diagram of sheaf homomorphisms

$$
\begin{tikzcd}
& \overline{\mathcal{M}}_{\underline{C}}^{C/\underline{S}} \arrow[dl] \arrow[dr] & \\
\overline{\mathcal{M}}_C^{\min} \arrow[rr, dashed] & & \overline{\mathcal{M}}_C \\
& \pi^{-1}\overline{\mathcal{M}}_{\underline{S}}^{C/\underline{S}} \arrow[uu] \arrow[dl] \arrow[dr] & \\
\pi^{-1}\overline{\mathcal{M}}_S^{\min} \arrow[uu] \arrow[rr] & & \pi^{-1}\overline{\mathcal{M}}_S .
\end{tikzcd}
$$

Homomorphisms between characteristics

By the connectedness of $C_{\bar{s}}$ and the fact that $u_q|_{\mathrm{Im}\overline{\mathcal{M}}_{B,\underline{f}_{\underline{S}}(\bar{s})}} = 0$, the composite of natural maps $\overline{\mathcal{M}}_{B,\underline{f}_{\underline{S}}(\bar{s})} \to \overline{\mathcal{M}}_{W,\underline{f}(\bar{\eta})} \to \overline{\mathcal{M}}_{S,\bar{s}}^{\min}$ is independent of the choices of η. Therefore we have a natural commuting diagram of homomorphisms

$$
\begin{tikzcd}
& \overline{\mathcal{M}}_{S,\bar{s}}^{\min} \arrow[dr, "{[g_2]}"] & \\
\overline{\mathcal{M}}_{B,\underline{f}_{\underline{S}}(\bar{s})} \arrow[ur, "\text{via } \overline{\mathcal{M}}_{W,\underline{f}(\bar{\eta})}"] \arrow[rr, "\overline{f_{\underline{S}}^{\flat}}"'] & & \overline{\mathcal{M}}_{S,\bar{s}}.
\end{tikzcd}
\tag{3.2.11}
$$

We define a homomorphism

$$\mathfrak{gen} : f^{-1}\overline{\mathcal{M}}_W \to \overline{\mathcal{M}}_C^{\min}$$

as follows. At a generic point η of a component of $C_{\bar{s}}$, we take the natural one $\overline{\mathcal{M}}_{W,\underline{f}(\bar{\eta})} \to \overline{\mathcal{M}}_{C,\bar{\eta}}^{\min}$. At a node q, we use the generization method to define a homomorphism

$$
\begin{aligned}
\overline{\mathcal{M}}_{W,\underline{f}(q)} \to \overline{\mathcal{M}}_{S,\bar{s}}^{\min} \oplus_{\mathbb{N}_q} \mathbb{N}^2 \subset \overline{\mathcal{M}}_{S,\bar{s}}^{\min} \times \overline{\mathcal{M}}_{S,\bar{s}}^{\min} \\
m \mapsto ([\iota_{\bar{\eta}_1(q)} \circ \chi_{q,1}(m)], [\iota_{\bar{\eta}_2(q)} \circ \chi_{q,2}(m)])
\end{aligned}
$$

Recall that $\overline{\mathcal{M}}^{\min}_{S,\bar{s}}$ is a toric monoid so that $\overline{\mathcal{M}}^{\min}_{S,\bar{s}} \oplus_{\mathbb{N}_q} \mathbb{N}^2$ can be viewed as a submonoid of $\overline{\mathcal{M}}^{\min}_{S,\bar{s}} \times \overline{\mathcal{M}}^{\min}_{S,\bar{s}}$. At a marking p, take

$$(\chi_p, \sigma_p) : \overline{\mathcal{M}}_{W,\underline{f}(p)} \to \overline{\mathcal{M}}^{\min}_{S,\bar{s}} \oplus \mathbb{N}.$$

At other closed points p' of $C_{\bar{s}}$, take the composite

$$\overline{\mathcal{M}}_{W,\underline{f}(p')} \xrightarrow{\chi_{p'}} \overline{\mathcal{M}}_{W,\underline{f}(\bar{\eta})} \to \overline{\mathcal{M}}^{\min}_{S,\bar{s}}.$$

In fact, the generization map $\chi_{p'}$ is an isomorphism since $\overline{\mathcal{M}}_{W,\underline{f}(p')} \to \overline{\mathcal{M}}_{C_{\bar{s}},\bar{p}'} \cong \overline{\mathcal{M}}_{S,\bar{s}}$ is injective.

These maps can be glued since they are compatible with generization maps. Therefore we obtain a commutative diagram of monoid-valued sheaves

$$\begin{tikzcd} & \overline{\mathcal{M}}^{\min}_C \arrow[dr, "\widetilde{[g_2]}"] & \\ f^{-1}\overline{\mathcal{M}}_W \arrow[ur, "\mathrm{gen}"] \arrow[rr, "\overline{f^\flat}"] & & \overline{\mathcal{M}}_C \end{tikzcd} \tag{3.2.12}$$

The diagram (3.2.12) is compatible with (3.2.11) under the natural homomorphisms.

This generization method is systematically written as a Proposition below. In what follows, $\overline{\mathcal{M}}_C$ is also said to be minimal if its associated $\overline{\mathcal{M}}_S$ is minimal.

Proposition 3.2.7. ([10, Proposition 1.18]) *Let $(\underline{C}/\underline{S}, p_1, ..., p_n)$ be a family of n-pointed prestable curves and let P be a Zariski-site fs sheaf of toric monoids on $\underline{C}$ with $u_{p_i} : P_{p_i} \to \mathbb{N}$. For every $s \in S$ and every node q of $C_{\bar{s}}$, let $u_q : P_q \to \mathbb{Z}$ be a homomorphism. Assume that u_{p_i}, u_q are compatible with generizations, respectively. Suppose that $\overline{\mathcal{M}}_S$, $\overline{\mathcal{M}}_C$ be fs sheaves of toric monoids on $\underline{S}$, $\underline{C}$ such that*

$$\overline{\mathcal{M}}_C \cong \overline{\mathcal{M}}_S \oplus_{\overline{\mathcal{M}}^{C/\underline{S},\mathbf{p}}_{\underline{S}}} \overline{\mathcal{M}}^{C/\underline{S},\mathbf{p}}_{\underline{C}}$$

for some sharp homomorphism $\overline{\mathcal{M}}^{C/\underline{S},\mathbf{p}}_{\underline{S}} \to \overline{\mathcal{M}}'_S$, meaning that only invertible elements are mapped to invertible elements. If there is a sharp homomorphism $P \to \overline{\mathcal{M}}_C$ with type $\mathbf{u}$, then there is a unique minimal fs sheaves of toric monoids

$\overline{\mathcal{M}}_C^{\min}$, $\overline{\mathcal{M}}_S^{\min}$ on $(\underline{C}/\underline{S}, p_1, ..., p_n)$ with a sharp homomorphism $P \to \overline{\mathcal{M}}_C^{\min}$ with type $\mathbf{u}$. *Moreover there are unique sharp homomorphisms $\overline{\mathcal{M}}_S^{\min} \to \overline{\mathcal{M}}_S$, $\overline{\mathcal{M}}_C^{\min} \to \overline{\mathcal{M}}_C$ making the commutative diagram*

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[dr] & \\ \overline{\mathcal{M}}_C^{\min} \arrow[rr, dashed] & & \overline{\mathcal{M}}_C \\ \pi^{-1}\overline{\mathcal{M}}_S^{\min} \arrow[u] \arrow[rr, dashed] & & \pi^{-1}\overline{\mathcal{M}}_S \arrow[u] \end{tikzcd}$$

Log morphisms

Let
$$\mathcal{M}_S^{\min} = \overline{\mathcal{M}}_S^{\min} \times_{\overline{\mathcal{M}}_S} \mathcal{M}_S$$
with the homomorphism $\mathcal{M}_S^{\min} \to \mathcal{O}_S$ induced from the structure map $\mathcal{M}_S \to \mathcal{O}_S$. Since $[g_2]^{-1}(0) = 0$, this pre-log structure is in fact a log structure, whose characteristic is $\overline{\mathcal{M}}_S^{\min}$. It is obvious that the projection $\mathcal{M}_S^{\min} \to \mathcal{M}_S$ is a log homomorphism. Similarly, on $\underline{C}$ we define a log structure and a log homomorphism:

$$\mathcal{M}_C^{\min} := \overline{\mathcal{M}}_C^{\min} \times_{\overline{\mathcal{M}}_C} \mathcal{M}_C \to \mathcal{M}_C.$$

By the very definitions of the above, there is a natural log morphism

$$C^{\min} := (\underline{C}, \mathcal{M}_C^{\min})) \to S^{\min} := (\underline{S}, \mathcal{M}_S^{\min}).$$

We construct the natural log lifts of diagrams (3.2.11), (3.2.12) by the following Lemma 3.2.8.

Lemma 3.2.8. *For given two homomorphisms of log structures $(\mathcal{M}_1, \mathcal{O}_1) \to (\mathcal{M}_3, \mathcal{O}_3)$, $(\mathcal{M}_2, \mathcal{O}_2) \to (\mathcal{M}_3, \mathcal{O}_1 = \mathcal{O}_3)$ with a compatible $\overline{\mathcal{M}}_2 \to \overline{\mathcal{M}}_1$, there is a unique lift homomorphism $(\mathcal{M}_2, \mathcal{O}_2) \to (\mathcal{M}_1, \mathcal{O}_1)$ making a commuting diagram*

$$\begin{tikzcd} & (\mathcal{M}_1, \mathcal{O}_1) \arrow[dr] & \\ (\mathcal{M}_2, \mathcal{O}_2) \arrow[ur, dashed] \arrow[rr] & & (\mathcal{M}_3, \mathcal{O}_1) \end{tikzcd}$$

Proof. This follows from the fact $\mathcal{M}_1 = \overline{\mathcal{M}}_1 \times_{\overline{\mathcal{M}}_3} \mathcal{M}_3$. □

Now for $c \in C, w \in W, b \in B$ with $\underline{f}(c) = w$, $\pi(c) = s$, $\pi_W(w) = b$, we obtain a commuting diagram of homomorphisms of log structures

$$\begin{tikzcd}
 & (\mathcal{M}^{\min}_{C,\bar{c}}, \mathcal{O}_{C,\bar{c}}) \arrow[dr] & \\
(\mathcal{M}_{W,\bar{w}}, \mathcal{O}_{W,\bar{w}}) \arrow[rr] \arrow[ur, dashed] & & (\mathcal{M}_{C,\bar{c}}, \mathcal{O}_{C,\bar{c}}) \\
 & (\mathcal{M}^{\min}_{S,\bar{s}}, \mathcal{O}_{S,\bar{s}}) \arrow[uu] \arrow[dr] & \\
(\mathcal{M}_{B,\bar{b}}, \mathcal{O}_{B,\bar{b}}) \arrow[rr] \arrow[ur, dashed] \arrow[uu] & & (\mathcal{M}_{S,\bar{s}}, \mathcal{O}_{S,\bar{s}}). \arrow[uu]
\end{tikzcd}$$

Here the commutativity of the left back side square can be deduced from Lemma (3.2.8).

Therefore we have constructed a minimal/basic stable log map $f^{\min} : (C^{\min}/S^{\min}, \mathbf{p}') \to W/B$ with the same underlying stable map $\underline{f}$ and a log morphism $h : C/S \to C^{\min}/S^{\min}$ such that $f = f^{\min} \circ h$ and $f_S = f_S^{\min} \circ h_S$. The minimal/basic stable log map and the morphism are unique up to unique isomorphism (see [10, Proposition 1.24]).

3.2.8 The works of Abramovich, Chen, Gross, and Siebert

The universal stable log maps are called basic in [10] and minimal in [6]. From now on, all stable log maps are assumed to be minimal unless otherwise stated.

Theorem 3.2.9. *[10, 6, 1] Suppose that the characteristic sheaf $\overline{\mathcal{M}}_W$ is globally generated. The moduli stack $\overline{M}_{g,\sigma}(W/S,\beta)$ of stable log maps of type (g,β,σ) is a DM-stack, proper over $\underline{B}$, carrying a canonical virtual fundamental class.*

Remark 3.2.10. We refer readers to [1, 6, 10] for weaker conditions of the log structure $\mathcal{M}_W$ under which the above properness theorem were proven.

Let $\pi : \mathfrak{C} \to \mathcal{K}$ be the universal curve and let $T_{W/B}$ denote the log tangent sheaf of W relative to B. There is a canonical perfect obstruction theory relative to the stack $\mathfrak{M} := \mathcal{L}og_{\mathfrak{M}_{g,n}} \times_{\mathcal{L}og_{\Bbbk}} \mathcal{L}og_B$

$$(\mathbb{R}\pi_* f^* T_{W/B})^\vee \to \mathbb{L}_{\mathcal{K}/\mathfrak{M}}$$

(for this particular form see [10, 15]).

3.3 Simple degenerations

In this section W/B is the degeneration as in §3.1.

3.3.1 Basic facts

We let $X_1 = (x_1)$ and $X_2 = (x_2)$ in the local sense in what follows.

Lemma 3.3.1. *Let $(C/S, p, f)$ be a stable log map to W/B and let $s \in S$.*

1. *If $\overline{\mathcal{M}}_{W,\underline{f}(q)} = \mathbb{N}$, then $u_q = 0$.*
2. *If $\overline{\mathcal{M}}_{W,\underline{f}(q)} \cong \mathbb{N}^2$, then $u_q((a_1, a_2)) = c_q(a_2 - a_1)$ for some $c_q \in \mathbb{Z}$.*
3. *If $\overline{\mathcal{M}}_{W,\underline{f}(q)} \cong \mathbb{N}^2$ with $\overline{\mathcal{M}}_{W,\underline{f}(\bar{\eta}_i)} = \mathbb{N}$ for some $i = 1, 2$, then $u_q \neq 0$.*

Proof. (1) Both $\overline{f^\flat_{\bar{\eta}_i}} \circ \chi_{q,i} : \overline{\mathcal{M}}_{W,\underline{f}(q)} \to \overline{\mathcal{M}}_s$, $i = 1, 2$, coincide with the homomorphism $\mathbb{N} \to \overline{\mathcal{M}}_s$ inherited from $s \to \mathrm{Spec}(\Bbbk^\dagger)$.

(2) This is immediate from $u_q((1, 1)) = 0$.

(3) Suppose $\underline{f}(\bar{\eta}_1) \in X_1$ and let x_1 be a local regular function at $\underline{f}(q)$ defining the divisor X_1. Then $f^\sharp(x_1)$ is a local equation defining the component corresponding to η_1 with some positive multiplicity. Hence $\overline{f^\flat}((1, 0)) = [(0, c, 0)] \in \overline{\mathcal{M}}_s \oplus_{\mathbb{N}} \mathbb{N}^2$ (up to the isomorphisms of $\mathbb{N}^2$), which is $\pm c\lambda_q$ as an element in $\overline{\mathcal{M}}_{S,\bar{s}} \times \overline{\mathcal{M}}_{S,\bar{s}}$ by (3.2.5)*. Thus $u_q((1, 0)) = \pm c\lambda_q \neq 0$.* □

In the case of Lemma 3.3.1 (2), we call $|c_q|$ the *contact order* of the node.

3.3.2 Non-degenerate case

In this subsection we consider the case $\underline{S} = \underline{s} = \operatorname{Spec}\Bbbk$. Let $(C/S, f) \in \overline{M}_{g,0}(W_0, \beta_0)$ is non-degenerate, i.e., $\underline{f}^{-1}(D) \subset C^{\mathrm{nd}}$. Let $\underline{f}^{-1}(D) = \{q_i \mid i = 1, ..., m\}$ and let $c_{,}..., c_m$ be the contact orders of $\underline{f}$ at $q_1, ..., q_m$ (see Lemma 3.3.1 (2)). They are nonzero by Lemma 3.3.1 (3).

We claim that $\overline{\mathcal{M}}_s \cong \mathbb{N} \oplus \mathbb{N}^{m'}$ where $m' := |C^{\mathrm{nd}}| - m$. Let

$$l = \mathrm{LCM}(c_1, ..., c_m)$$

and let $[1_\eta]$ denote $\iota_{\bar{\eta}}(1)$ in $\iota_{\bar{\eta}} : \mathbb{N} = \overline{\mathcal{M}}_{C,\bar{\eta}} \to \overline{\mathcal{M}}_s$. If η and η' are connected in $\underline{C} \setminus \underline{f}^{-1}(D)$, note $[1_\eta] = [1_{\eta'}]$ by Lemma 3.3.1 (1) and the definition of the minimal $\overline{\mathcal{M}}_s$. Also note that

$$[1_\eta] = c_j \lambda_{q_j} \tag{3.3.1}$$

whenever q_j is contained in the component associated to η. Therefore combined with the connectedness of C,

$$[1_\eta] = [1_{\eta'}] \tag{3.3.2}$$

for every components η, η' of C. Thus, (3.3.1) and (3.3.2) together with Lemma 3.3.1 (1) & (2) implies that $\overline{\mathcal{M}}_s^{\mathrm{gp}} \cong R \times \mathbb{Z}^{m'}$ for the quotient R of $\mathbb{Z}^m$ divided by relations $c_i 1_{q_i} = c_j 1_{q_j}$. We note that R^{gp} is isomorphic to $\mathbb{Z}$ using the exact sequence

$$0 \longrightarrow \mathbb{Z}^{m-1} \xrightarrow{\phi_1} \mathbb{Z}^m \xrightarrow{\phi_2} \tfrac{1}{l}\mathbb{Z} \longrightarrow 0$$

in which ϕ_i are defined by

$$\phi_1(1_{q_i}) = c_i 1_{q_i} - c_{i+1} 1_{q_{i+1}} \text{ and } \phi_2(1_{q_i}) = \frac{1_{q_i}}{c_i}.$$

By the saturation of $\overline{\mathcal{M}}_s$ we conclude the following description:

$$\overline{\mathcal{M}}_{\underline{s}}^{C/\underline{s}} = \textstyle\prod_{i=1}^m \mathbb{N}_{q_i} \oplus \mathbb{N}^{m'} \xrightarrow{(l_1,...,l_m)\oplus \mathrm{Id}} \overline{\mathcal{M}}_s = \mathbb{N} \oplus \mathbb{N}^{m'} \xleftarrow{(l,0)} \overline{\mathcal{M}}_{B,0} = \mathbb{N}$$

where $l_i = l/c_i$.

This shows that for the non-degenerate f, the minimality/basicness of f coincides with the minimality in the sense of [15, §5.2].

3.3.3 Splitting

For each $i = 1, 2$, let $\mathcal{M}_{X_i}$ denote the divisorial log structure associated to the divisor $\underline{X}_1 \cap \underline{X}_2$ in $\underline{X}_i$. There is a canonical homomorphism $\iota_{X_i}^{\flat} : \mathcal{M}_{X_i} \to \iota_{X_i}^{*}\mathcal{M}_{W_0}$. Under this homomorphism

$$\overline{\mathcal{M}}_{X_i} = \mathbb{N}_{\underline{X}_1 \cap \underline{X}_2} \hookrightarrow \iota_{X_i}^{-1}\overline{\mathcal{M}}_{W_0} = \mathbb{N}_{\underline{X}_1} \oplus \mathbb{N}_{\underline{X}_1 \cap \underline{X}_2}.$$

Consider a stable log map $(C/S, \mathbf{p}, f)$ whose underlying stable map is a join of two maps $\underline{f}_i$, $i = 1, 2$, that is:

- $(\underline{C}, \mathbf{p}) = (\underline{C}_1, \{p_j\}_{j \in N_1}, \{p_j^1\}_{j \in M}) \coprod_{q_j, j \in M} (\underline{C}_2, \{p_j\}_{j \in N_2}, \{p_j^2\}_{j \in M})$ and
- $\underline{f}(\underline{C}_i) \subset \underline{X}_i$, $\underline{f}_i = \underline{f}_{|\underline{C}_i}$.

We include the case when $\underline{C}_1 = \emptyset$ or $\underline{C}_2 = \emptyset$. For $i = 1, 2$, we will construct a log structure C_i/S_i on the pointed possibly disconnected prestable curve $(\underline{C}_i/\underline{S}, \{p_j\}_{j \in N_i}, \{p_j^i\}_{j \in M})$ over $\underline{S}$ and a minimal/basic stable log map f_i from C_i/S_i to X_i whose underlying stable map is exactly $\underline{f}_i$.

Let $\bar{s} \to \underline{S}$ be a geometric point and let $\bar{s}_i$ be two copies of $\bar{s}$. For a node $q \in \underline{C}_{i|\bar{s}_i}$, we denote

$$u_q^i := u_q \circ \overline{\iota_{X_i}^{\flat}} : \overline{\mathcal{M}}_{X_i, \underline{f}_i(q)} \to \overline{\mathcal{M}}_{W_0, \underline{f}(q)} \to \mathbb{Z}. \tag{3.3.3}$$

By Proposition 3.2.7 we obtain the monoid $\overline{\mathcal{M}}_{\bar{s}_i}$ minimal with respect to data

$$(\underline{f}_i^{-1}\overline{\mathcal{M}}_{X_i}, u_q^i; q \in (\underline{C}_i)^{\mathrm{nd}}_{|\bar{s}_i}). \tag{3.3.4}$$

Note that $\overline{\mathcal{M}}_{\bar{s}_i}$ is a toric monoid since it is a finitely generated saturated submonoid of $\overline{\mathcal{M}}_{S,\bar{s}}$. Note also that there is a natural homomorphism $\overline{\mathcal{M}}^{\underline{C}_i/\underline{S}_i}_{\underline{S}_i, \bar{s}_i} \to \overline{\mathcal{M}}_{\bar{s}_i}$ from the construction, hence we can define a sheaf of monoids on $\underline{C}_{i|\bar{s}_i}$

$$\overline{\mathcal{M}}_{C_i} := (\overline{\mathcal{M}}^{\underline{C}_i/\underline{S}_i, \{p_j\}_{j \in N_i}}_{\underline{C}_{i|\bar{s}_i}} \oplus_{\pi_{i|\bar{s}_i}^{-1}\overline{\mathcal{M}}^{\underline{C}_i/\underline{S}_i}_{\underline{S}_i, \bar{s}_i}} \pi_{i|\bar{s}_i}^{-1}\overline{\mathcal{M}}_{\bar{s}_i}) \oplus \bigoplus_{j \in M} \mathbb{N}_{p_j^i}$$

where $\mathbb{N}_{p_j^i}$ is the constant sheaf $\mathbb{N}$ supported on p_j^i and $\pi_i : \underline{C}_i \to \underline{S}_i$ is the projection.

By informally considering local equations $x_1 = 0$ for X_1 and $x_2 = 0$ for X_2 at $\underline{f}(p_j) \in \underline{W}$, it is convenient to write

$$\overline{\mathcal{M}}_{W_0,\underline{f}(p_j)} = \mathbb{N}_{x_1} \oplus \mathbb{N}_{x_2} \quad \text{and} \quad \overline{\mathcal{M}}_{X_i,\underline{f}_i(p^i_j)} = \mathbb{N}_{x_{i'}},$$

where $i' \in \{1,2\}$ with $i' \neq i$. Define

$$u_{p^i_j} : \overline{\mathcal{M}}_{X_i,\underline{f}_i(p^i_j)} \to \mathbb{N}$$

by the equation

$$u_{q_j} = u_{p^1_j} \circ \mathrm{pr}_{x_2} - u_{p^2_j} \circ \mathrm{pr}_{x_1}. \tag{3.3.5}$$

where pr_{x_i} is the projection $\mathbb{N}_{x_1} \oplus \mathbb{N}_{x_2} \to \mathbb{N}_{x_i}$. *We assume that with respect to the order selecting the second branch from $\underline{C}_2$, the equation* (3.3.5) *is fulfilled.*

Define a natural homomorphism $\overline{\mathcal{M}}_{C_i} \to \iota^{-1}_{\underline{C}_i}\overline{\mathcal{M}}_C$ by the generization method of

$$\begin{tikzcd}
\prod_{q\in C_i^{\mathrm{nd}}} \mathbb{N}_q \arrow[d] \arrow[dr] & \\
\overline{\mathcal{M}}_{\bar{s}_i} \arrow[r, hook] & \overline{\mathcal{M}}_{S,\bar{s}}.
\end{tikzcd}$$

together with the composite

$$\overline{\mathcal{M}}_{C_i}|_{p^i_j} = \mathbb{N} \oplus \overline{\mathcal{M}}_{\bar{s}_i} \xhookrightarrow{\text{facet}} \mathbb{N}^2 \oplus \overline{\mathcal{M}}_{\bar{s}_i} \to \mathbb{N}^2 \oplus_{\mathbb{N}_{p^1_j}} \overline{\mathcal{M}}_{S,\bar{s}} = \iota^{-1}_{\underline{C}_i}\overline{\mathcal{M}}_C|_{p^i_j}.$$

By Proposition 3.2.7, we derive a homomorphism $\underline{f}_i^\flat$ making a commutative diagram of sharp sheaf homomorphisms

$$\begin{tikzcd}
\underline{f}_i^{-1}\overline{\mathcal{M}}_{X_i} \arrow[r, dashed, "\underline{f}_i^\flat"] \arrow[d] & \overline{\mathcal{M}}_{C_i} \arrow[d] \\
\underline{f}_i^{-1}\iota^{-1}_{\underline{X}_i}\overline{\mathcal{M}}_W \arrow[r] & \iota^{-1}_{\underline{C}_i}\overline{\mathcal{M}}_C
\end{tikzcd} \quad . \tag{3.3.6}$$

In fact we need an extra treatment at p^i_j. We simply define $(\underline{f}_i^\flat)_{p^i_j} : (\underline{f}_i^{-1}\overline{\mathcal{M}}_{X_i})_{p^i_j} \to (\overline{\mathcal{M}}_{C_i})_{p^i_j} = \mathbb{N} \oplus \overline{\mathcal{M}}_{s_i}$ to be the sum of $u_{p^i_j}$ and the composite

$$(\underline{f}_i^{-1}\overline{\mathcal{M}}_{X_i})_{p^i_j} \to (\underline{f}_i^{-1}\overline{\mathcal{M}}_{X_i})_{\bar{\eta}} \to \overline{\mathcal{M}}_{s_i}$$

of the generization followed by the natural one in the construction of $\overline{\mathcal{M}}_{s_i}$ where η is the generic point of the curve component containing p^i_j. We check the commutativity at such points by the following diagram:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{X_i,\underline{f}_1(p^i_j)}, a & \longrightarrow & \mathbb{N}_{z_{i'}} \oplus \overline{\mathcal{M}}_{\bar{s}_i}, (u_{p^i_j}(a), [\chi(a)]) \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{W_0,\underline{f}(q_j)}, (0,a) & \longrightarrow & \mathbb{N}^2_{z_i,z_{i'}} \oplus_{\mathbb{N}} \overline{\mathcal{M}}_{\bar{s}}, (0, u_{p^i_j}(a), [\chi(a)]). \end{array}$$

Now we define the log structures on $\underline{s}_i$ and $\underline{C}_i$ by fiber products as follows:

$$\mathcal{M}_{S_i} := \overline{\mathcal{M}}_{S_i} \times_{\overline{\mathcal{M}}_S} \mathcal{M}_S, \quad \mathcal{M}_{C_i} := \overline{\mathcal{M}}_{C_i} \times_{\iota^{-1}_{\underline{C}_i}\overline{\mathcal{M}}_C} \iota^*_{\underline{C}_i}\mathcal{M}_C$$

By (3.3.6) and Lemma 3.2.8, we obtain $f_i^\flat$ making the commutative diagram, for every $c \in \underline{C}_{i|_{\bar{s}_i}}$,

$$\begin{array}{ccc} (\mathcal{M}_{X_i,\underline{f}_i(\bar{c})}, \mathcal{O}_{X_i,\underline{f}_i(\bar{c})}) & \overset{f_i^\flat}{\dashrightarrow} & (\mathcal{M}_{C_i}, \mathcal{O}_{C_i}) \\ \downarrow & & \downarrow \\ (\mathcal{M}_{W_0,\underline{f}_i(\bar{c})}, \mathcal{O}_{X_i,\underline{f}_i(\bar{c})}) & \xrightarrow{f^\flat} & (\mathcal{M}_{C,\bar{c}}, \mathcal{O}_{C_i,\bar{c}}). \end{array} \tag{3.3.7}$$

Therefore, we get two universal stable log maps

$$(C_i/S_i, p_j, j \in A_i, p^i_j, j \in J, f_i) \tag{3.3.8}$$

to X_i with contact orders c_j at p^i_j.

3.3.4 Gluing

Conversely, starting from given two stable log maps (3.3.8) we want to glue them in order to get a stable log map to W_0. To do so, we will need some additional data which will be specified later.

Let $\mathrm{pr}_i : \overline{\mathcal{M}}_{W_0,q} \to \overline{\mathcal{M}}_{X_i,q} \cong \mathbb{N}_{x_{i'}}$ and recall that $X_i = (x_i)$. First note that (3.3.3) and Lemma (3.3.1) determine type u_q for $q \in (C_i)^{\mathrm{nd}}_{\bar{s}}$, that is,

$$u_q := u^i_q \circ \mathrm{pr}_{x_{i'}} - u^i_q(1)\mathrm{pr}_{x_i}.$$

For a gluing node $q_j \in \underline{C}_1 \cap \underline{C}_2$, define

$$u_q := u_{p_j^1} \circ \mathrm{pr}_{x_2} - u_{p_j^2} \circ \mathrm{pr}_{x_1}.$$

Now on $C_{\bar{s}}$, we have data $(\underline{f}^{-1}\overline{\mathcal{M}}_{W_0}, u_q; q \in \underline{C}_{\bar{s}}^{\mathrm{nd}})$, hence obtain $\overline{\mathcal{M}}_{\bar{s}}$ minimal with respect to the data.

Lemma 3.3.2. *The monoid $\overline{\mathcal{M}}_{\bar{s}}$ is a toric monoid with a facet $\overline{\mathcal{M}}_{\bar{s}_1} \times \overline{\mathcal{M}}_{\bar{s}_2}$.*

Proof. Let $l_q = \mathrm{LCM}(c_i, i \in M)/c_q$. It is straightforward to see that $\overline{\mathcal{M}}_s$ is the saturation of a quotient of

$$\widetilde{\overline{\mathcal{M}}_s} := \overline{\mathcal{M}}_{\bar{s}_1} \times \prod_{\eta_1 \in \mathrm{Con}(\underline{C}_{1,\bar{s}})} \mathbb{N}_{x_1} \times \overline{\mathcal{M}}_{\bar{s}_2} \times \prod_{\eta_1 \in \mathrm{Con}(\underline{C}_{2,\bar{s}})} \mathbb{N}_{x_2} \times \prod_{q \in \underline{C}_{1,\bar{s}} \cap \underline{C}_{2,\bar{s}}} \mathbb{N}_q$$

Consider a homomorphism

$$\sigma : \widetilde{\overline{\mathcal{M}}_s} \to \mathbb{N}$$

determined by that the kernel of σ is $\overline{\mathcal{M}}_{\bar{s}_1} \times \overline{\mathcal{M}}_{\bar{s}_2}$ and $\sigma|_{\log x_i} = \times l : \mathbb{N}_{\log x_i} \to \mathbb{N}$, $\sigma|_{\mathbb{N}_q} = \times l_q : \mathbb{N}_q \to \mathbb{N}$. Note that σ factors through a homomorphism $\overline{\mathcal{M}}_{\bar{s}} \to \mathbb{N}$, which is also denoted by σ. Since the kernel of σ is a toric monoid $\overline{\mathcal{M}}_{\bar{s}_1} \times \overline{\mathcal{M}}_{\bar{s}_2}$, $\overline{\mathcal{M}}_{\bar{s}}^{\times}$ must be trivial and hence $\overline{\mathcal{M}}_{\bar{s}}$ is also a toric monoid. Since $\overline{\mathcal{M}}_{\bar{s}_1} \times \overline{\mathcal{M}}_{\bar{s}_2}$ is the kernel of $\sigma \in \overline{\mathcal{M}}_{\bar{s}}^{\vee}$, it is a face. It is easy to check that σ is a 1-dimensional extremal ray of $\overline{\mathcal{M}}_{\bar{s}}^{\vee}$ so that $\overline{\mathcal{M}}_{\bar{s}_1} \times \overline{\mathcal{M}}_{\bar{s}_2}$ is a facet. □

Note that there are natural commuting homomorphisms

$$\begin{tikzcd}
\overline{\mathcal{M}}_{\underline{S}}^{C/\underline{S}} \arrow[r] & \overline{\mathcal{M}}_S & \overline{\mathcal{M}}_b \arrow[l] \\
\overline{\mathcal{M}}_{\underline{S}_i}^{\underline{C}_i/\underline{S}_i} \arrow[u] \arrow[r] & \overline{\mathcal{M}}_{S_i} \arrow[u] &
\end{tikzcd}$$

Here $\overline{\mathcal{M}}_b \to \overline{\mathcal{M}}_S$ is well-defined via $\overline{\mathcal{M}}_{\bar{\eta}_i} \to \overline{\mathcal{M}}_s$ with any choice of a component of $\underline{C}_s$ since $u_q = 0$ on $\overline{\mathcal{M}}_b$. Therefore by the generization method we obtain a commuting diagram of homomorphisms

$$\begin{tikzcd}
\overline{\mathcal{M}}_C & \overline{\mathcal{M}}_{W_0} \arrow[l] \\
\overline{\mathcal{M}}_S \arrow[u] & \overline{\mathcal{M}}_b \arrow[l] \arrow[u]
\end{tikzcd}$$

Lemma 3.3.3. *The homomorphism $\overline{\mathcal{M}}_{W_0,\underline{f}(q_j)} \to \overline{\mathcal{M}}_{C,q_j}$, defined by the generization, coincides with the one through $\overline{\mathcal{M}}_{s_i} \oplus \mathbb{N}$.*

Proof.

$$\begin{aligned}
&\overline{\mathcal{M}}_{W_0,\underline{f}(q)} = \overline{\mathcal{M}}_{X_2,\underline{f}_2(q)} \oplus \overline{\mathcal{M}}_{X_1,\underline{f}_1(q)} = \mathbb{N}_{x_1} \oplus \mathbb{N}_{x_2} \ni (a,b) \\
&\longrightarrow (\overline{\mathcal{M}}_{s_2} \oplus \mathbb{N}_x) \oplus (\overline{\mathcal{M}}_{s_1} \oplus \mathbb{N}_y) \ni (\overline{f}_2^{\flat} \circ \chi(a), c_q a, \overline{f}_1^{\flat} \circ \chi(b), c_q b) \\
&\longrightarrow \overline{\mathcal{M}}_s \oplus_{\mathbb{N}} (\mathbb{N}_x \oplus \mathbb{N}_y) \ni (\overline{f}_1^{\flat} \circ \chi(b) + \overline{f}_2^{\flat} \circ \chi(a), (c_q a, c_q b)) - - - (*)
\end{aligned}$$

There are three possible cases.

Case 1. When q is non-degenerate, then $(*) = (0, (c_q a, c_q b))$. On the other hand, $(0,b)1_{\eta_2} - (a,0)1_{\eta_1} = c_q(b-a)\lambda_q$ in $\overline{\mathcal{M}}_s$. Hence

$$(*) = ((a,0)1_{\eta_1}, (0,b)1_{\eta_2})$$

in $\overline{\mathcal{M}}_s \times \overline{\mathcal{M}}_s$.

Case 2. When η_1, η_2 are mapped into $\underline{X}_1 \cap \underline{X}_2$, $(*) = ((0,b)1_{\eta_1} + (a,0)1_{\eta_2} + c_q a\lambda_q, (0,b)1_{\eta_1} + (a,0)1_{\eta_2} + c_q b\lambda_q)$. Since $(a,b)1_{\eta_2} - (a,b)1_{\eta_1} = c_q(b-a)\lambda_q$,

$$(*) = ((a,b)1_{\eta_1}, (a,b)1_{\eta_2})$$

Case 3. When η_2 is mapped into $\underline{X}_1 \cap \underline{X}_2$, $(*) = ((a,0)1_{\eta_2} + c_q a\lambda_q, (a,0)1_{\eta_2} + c_q b\lambda_q)$. Since $(a,b)1_{\eta_2} - (a,0)1_{\eta_1} = c_q(b-a)\lambda_q$,

$$(*) = ((a,0)1_{\eta_1}, (a,b)1_{\eta_2})$$

□

Suppose that we are given the following additional data: a log structure $\mathcal{M}_S$ with the characteristic $\overline{\mathcal{M}}_S$ and a commutative diagram of log homomorphisms (denoted by dotted arrows)

$$\begin{tikzcd}
\mathcal{M}^{C/S}_{\underline{S},q_j} \arrow[r] & \mathcal{M}^{C/S}_{\underline{S}} \arrow[r, dashed] & \mathcal{M}_S & \mathcal{M}_b \arrow[l, dashed] \\
& \mathcal{M}^{C_i/S_i}_{\underline{S}_i} \arrow[u] \arrow[r] & \mathcal{M}_{S_i} \arrow[u, dashed] &
\end{tikzcd}$$

where solid arrows are already determined. This in turn defines a log structure $\mathcal{M}_C$. We will construct a natural log homomorphism $\underline{f}^*\mathcal{M}_{W_0} \to \mathcal{M}_C$. On the way we will adjust the homomorphism $\mathcal{M}_{\underline{S},q_j}^{\underline{C}/\underline{S}} \to \mathcal{M}_S$ as in Proposition 3.3.4.

Note that there is a natural inclusion $\mathcal{M}_{\underline{C}_i}^{\underline{C}_i/\underline{S}_i,\{p_j\}_{j\in A_i},\{p^i_j\}_{j\in J}} \to \iota_{\underline{C}_i}^{-1}\mathcal{M}_C^{C/S}$ which in turn induces a homomorphism $\mathcal{M}_{C_i} \to \iota_{C_i}^*\mathcal{M}_C$. This will be used in the commutative diagram below. Since

$$\iota_{\underline{X}_i}^*\mathcal{M}_{W_0}^{\mathrm{gp}} \cong \mathcal{M}_{X_1}^{\mathrm{gp}} \oplus_{\mathcal{O}_{\underline{X}_i}^\times} \iota_{\underline{X}_i}^*\underline{\pi}_{W_0}^*\mathcal{M}_b^{\mathrm{gp}},$$

we obtain a commutative diagram

$$\begin{tikzcd}
\iota_{C_i}^*\mathcal{M}_C^{\mathrm{gp}} & \mathcal{M}_{C_i}^{\mathrm{gp}} \arrow[l] & \\
\iota_{C_i}^*\pi^*\mathcal{M}_S^{\mathrm{gp}} \arrow[u] & \underline{f}_i^*\iota_{\underline{X}_i}^*\mathcal{M}_{W_0}^{\mathrm{gp}} \arrow[ul, dotted] & \underline{f}_i^*\mathcal{M}_{X_i}^{gp} \arrow[l] \arrow[ul, bend right] \\
 & \underline{f}_i^*\iota_{\underline{X}_i}^*\underline{\pi}_{W_0}^*\mathcal{M}_b^{\mathrm{gp}} \arrow[u] \arrow[ul, bend left] & \mathcal{O}_{C_i}^\times \arrow[l] \arrow[u]
\end{tikzcd}$$

and hence a natural homomorphism from

$$\iota_{C_i}^*\underline{f}^*\mathcal{M}_{W_0} \to \iota_{C_i}^*\mathcal{M}_C$$

Therefore a homomorphism $\underline{f}^*\mathcal{M}_{W_0} \to \mathcal{M}_C$ is well-defined by the above *except at the gluing nodes q_j, $j \in J$*. It is straightforward to check that this is compatible with the one in the characteristic level given by the generization method.

Proposition 3.3.4. *Assume that S = Spec$A$ for a local noetherian henselian ring A. At the gluing node q_j, there exist log homomorphisms $\mathcal{M}_{\underline{S},q_j}^{\underline{C}/\underline{S}} \to \mathcal{M}_S$, $j \in J$ and $h_{q_j} : \mathcal{M}_{W_0,\underline{f}(q_j)} \to \mathcal{M}_{C,q_j}$ satisfying the followings.*

1. They make the commutative diagram

$$\begin{tikzcd}
\mathcal{M}_{W_0,\underline{f}(q_j)} \arrow[r, dashed, "h_{q_j}"] & \mathcal{M}_{C,q_j} \\
\mathcal{M}_{B,0} \arrow[r] \arrow[u] & \mathcal{M}_{S,\bar{s}} \arrow[u]
\end{tikzcd}$$

where $\bar{s}$ is the unique closed point in S.

2. They induce two given homomorphisms $\mathcal{M}_{X_i,\underline{f}_i(p^i_j)} \to \mathcal{M}_{C_i,p^i_j}$ as in diagram (3.3.7)

Furthermore, the such homomorphisms $h_{q_j} : \mathcal{M}_{W_0,\underline{f}(q_j)} \to \mathcal{M}_{C,q_j}$ and the restriction of $\mathcal{M}^{C/S}_{\underline{S},q_j} \to \mathcal{M}_S$ to $\mathcal{M}^{C/\underline{S},c_{q_j}}_{\underline{S},q_j}$ are uniquely determined, where $\mathcal{M}^{C/\underline{S},c_{q_j}}_{\underline{S},q_j}$ is the subsheaf of $\mathcal{M}^{C/S}_{S,q_j}$ consisting of sections whose c_{q_j}-th roots exist at least locally.

Proof. Let R be the henselization of $A[x,y]/(xy)$. We need to define a log homomorphism h in the diagram

$$\begin{tikzcd}
R^\times \oplus \mathbb{N}^2 \cong f^*\mathcal{M}_{W_0,\underline{f}(q_j)} \arrow[r, dashed, "h"] & \mathcal{M}^{C/S}_{\underline{C},q_j} \oplus_{\mathcal{M}^{C/S}_{\underline{S},q_j}} \mathcal{M}_{S,\bar{s}} \cong R^\times \oplus \mathbb{N}^2 \oplus_{A^\times \oplus \mathbb{N}} (A^\times \oplus \overline{\mathcal{M}}_{S,\bar{s}}) \\
A^\times \oplus \mathbb{N} \cong f^*\mathcal{M}_{B,0} \arrow[u, "(\mathrm{id}_A,\Delta)"] \arrow[r, "h_b"'] & \mathcal{M}_{S,\bar{s}} \cong A^\times \oplus \overline{\mathcal{M}}_{S,\bar{s}} \arrow[u, "(0,\mathrm{id}_{\mathcal{M}_{\bar{s}}})"']
\end{tikzcd}$$

The homomorphism h restricted to each branch is already given so that

$$\begin{aligned}
h(1_{x_1}) &= ((v_1, c_q 1_x), (1, \overline{f}_2^\flat \circ \chi(1_{x_1}))), \\
h(e_{x_2}) &= ((v_2, c_q 1_y), (1, \overline{f}_1^\flat \circ \chi(1_{x_2}))
\end{aligned}$$

for some $v_1 \in R^\times/(y)$, $v_2 \in R^\times/(x)$. Therefore h is determined by lifts $\tilde{v}_i$ of v_i in $R^\times$. However, the commutativity of the diagram requires that the product $\tilde{v_1}\tilde{v_2}$ must be in $A^\times$. We show that such lifts are unique as follows. Let $x' = x\tilde{v}_1$, $y' = y\tilde{v}_2$ in R. Then according to [21, §3.B] and [14, Lemma 2.1], there are unique $u_1, u_2 \in R^\times$ such that $u_1u_2 \in A^\times$ and $x' = xu_1, y' = yu_2 \in R$. This analysis shows that h can be uniquely determined. Note also that making the square diagram above commutes for a given h_b, the homomorphism $\mathcal{M}^{C/S}_{\underline{S},q_j} \to \mathcal{M}_S$ can be constructed and is, moreover, uniquely determined when it is restricted to $\mathcal{M}^{C/\underline{S},c_{q_j}}_{\underline{S},q_j}$. It is straightforward to check that h preserves the structure maps of log structures. $\square$

3.4 Proof of Theorem 3.1.1

3.4.1 The splitting stack

For a fine log scheme S, denote by $\mathcal{L}og_S$ the stack over $\underline{S}$ whose fiber over $\underline{T} \to \underline{S}$ is the groupoid of log morphisms $T \to S$ over $\underline{T} \to \underline{S}$ and whose homomorphisms between $T \to S$ and $T' \to S$ are isomorphisms $h : T \to T'$ over S such that $\underline{h} = \mathrm{id}_{\underline{T}}$. The fibered category $\mathcal{L}og_S$ is an algebraic stack locally of finite presentation over $\underline{S}$ (see [23, Theorem 1.1]). Let $\mathcal{T}or_S$ be the open substack of $\mathcal{L}og_S$ classifying fs log schemes over S.

Definition 3.4.1. *Denote by $\mathcal{L}og^{\mathrm{spl}}_{\Bbbk^\dagger}$ the category fibered in groupoids over* $(\mathrm{Sch}/\mathrm{Spec}(\Bbbk))$ *whose fiber over $\underline{T} \to$* $\mathrm{Spec}(\Bbbk)$ *is the groupoid of triples $(T, h, \mathcal{F}_T)$ where $(T, h : \mathbb{N}_T \oplus \mathcal{O}^*_T \to \mathcal{M}_T)$ is an object in $\mathcal{T}or_{\Bbbk^\dagger}$ and $\mathcal{F}_T$ is a subsheaf of $\mathcal{M}_T$ satisfying that:*

1. *For every $t \in \underline{T}$, $\overline{\mathcal{F}}_{T,\bar{t}} \subset \overline{\mathcal{M}}_{T,\bar{t}}$ is a facet.*
2. *For the log structure $\alpha : \mathcal{M}_T \to \mathcal{O}_T$, $\alpha_{|_{\mathcal{M}_T \setminus \mathcal{F}_T}} = 0$.*
3. *For every $t \in \underline{T}$, $\langle \mathcal{F}_{T,\bar{t}}, p\rangle \otimes_{\mathbb{Z}} \mathbb{Q} = \overline{\mathcal{M}}_{T,\bar{t}} \otimes \mathbb{Q}$ where p is the image of* 1 *under the induced homomorphism $\bar{h}_{\bar{t}} : \mathbb{N}_{T,\bar{t}} \to \overline{\mathcal{M}}_{T,\bar{t}}$ and $\langle \mathcal{F}_{T,\bar{t}}, p\rangle$ is the monoid generated by $\mathcal{F}_{T,\bar{t}}$ and p.*

Note that by Condition (2), $(\mathcal{F}_T, \alpha_{|_{\mathcal{F}_T}})$ is also a log structure on $\underline{T}$.

Lemma 3.4.2. 1. *The fibered category $\mathcal{L}og^{\mathrm{spl}}_{\Bbbk^\dagger}$ is a zero pure-dimensional algebraic stack over* $\mathrm{Spec}(\Bbbk)$.

2. *The forgetful morphism $\mathcal{L}og^{\mathrm{spl}}_{\Bbbk^\dagger} \to \mathcal{L}og_{\Bbbk^\dagger}$ is representable and proper.*

Proof. It is straightforward to check that $\mathcal{L}og^{\mathrm{spl}}_{\Bbbk^\dagger}$ is a stack over $\mathrm{Spec}(\Bbbk)$. We apply [3, Lemma C.5] to prove (1) and (2). Let $\underline{T} \to \mathcal{T}or_{\Bbbk^\dagger}$ be the morphism obtained from a log morphims $T \to \mathrm{Spec}(\Bbbk^\dagger)$ together with its toric chart

$$\begin{tikzcd} \mathbb{N}_T \arrow[r] \arrow[d] & Q_T \arrow[d] \\ \mathbb{N}_T \oplus \Bbbk^*_T \arrow[r] & \mathcal{M}_T \end{tikzcd}$$

(in particular, Q is a toric monoid).

For $\rho \in Q^\vee[1]$, let $Q_\rho := \{q \in Q \mid \rho(q) = 0\}$ be the facet of Q associated to ρ and let p be the image of 1 under the homomorphism $\mathbb{N} \to Q$.

Lemma 3.4.3. *The fiber product $\underline{T} \times_{\mathcal{T}or_{\Bbbk^\dagger}} \mathcal{L}og^{\mathrm{spl}}_{\Bbbk^\dagger}$ is representable by a $\underline{T}$ - scheme $\coprod_{\rho \in Q^\vee[1], \rho(p) \neq 0} T_\rho$, where T_ρ is the closed subscheme of $\underline{T}$ defined by the ideal generated by Q/Q_ρ under the composition $Q_T \to \mathcal{M}_T \to \mathcal{O}_T$.*

Proof. We first show that there is a natural $\underline{T}$-morphism from $\coprod T_\rho$ to $\underline{T} \times_{\mathcal{T}or_{\Bbbk^\dagger}} \mathcal{L}og^{\mathrm{spl}}_{\Bbbk^\dagger}$. Consider pre-log structure $Q_T \to h_\rho^{-1}\mathcal{O}_T \to \mathcal{O}_{T_\rho}$ on T_ρ, where $h_\rho : T_\rho \to \underline{T}$. There is the natural morphism f of sheaves of monoids on T_ρ.

$$f : Q_T \to Q_T^a \to \overline{Q_T^a}$$

We can check $f(Q_\rho) \subset \overline{Q_T^a}$ is a facet. We define subsheaf $\mathcal{F} \subset Q_T^a$ as the fibered product

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & Q_T^a \arrow[d] \\ f(Q_\rho) \arrow[r] & \overline{Q_T^a} \end{tikzcd}$$

Then $(\mathcal{F} \to Q_T^a \leftarrow \Bbbk_T \oplus \mathbb{N})$ is an object of $\underline{T} \times_{\mathcal{T}or_{\Bbbk^\dagger}} \mathcal{L}og^{\mathrm{spl}}_{\Bbbk^\dagger}(T_\rho)$

Conversely, we show that there is a natural $\underline{T}$-morphism from $\underline{T} \times_{\mathcal{T}or_{\Bbbk^\dagger}} \mathcal{L}og^{\mathrm{spl}}_{\Bbbk^\dagger}$ to $\coprod_\rho T_\rho$. Suppose that we are given $h : \underline{S} \to \underline{T}$, $\mathbb{N}_S \oplus \Bbbk_S^* \to \mathcal{M}_S \leftarrow \mathcal{F}_S$ such that $\mathbb{N}_S \oplus \mathcal{O}_S^* \to \mathcal{M}_S$ is the pull-back of $\mathbb{N}_T \oplus \Bbbk_T^* \to \mathcal{M}_T$ by h. Consider following morphism of monoids by choosing any section $h^{-1}\overline{\mathcal{M}}_T \to h^{-1}\mathcal{M}_T$,

$$g : h^{-1}\overline{\mathcal{M}}_T \to h^{-1}\mathcal{M}_T \to (h^{-1}\mathcal{M}_T)^a \cong \mathcal{M}_S \to \overline{\mathcal{M}}_S$$

We can check $g^{-1}(\overline{\mathcal{F}}_S)$ is a facet not containing p. Hence $g^{-1}(\overline{\mathcal{F}}_S)$ is equal to T_ρ for some ρ. Therefore, $h : \underline{S} \to \underline{T}$ factors though $\underline{S} \to T_\rho \to \underline{T}$.

It's straightforward to check that above two natural morphisms are inverse to each other.

□

This shows the proof of Lemma 3.4.2.

□

3.4.2 Graphs

Consider a morphism $\underline{S} \to \mathcal{T}or_{\mathfrak{M}_{g,n}} \times_{\mathcal{T}or_{\Bbbk}} \mathcal{L}og^{\mathrm{spl}}_{\Bbbk^\dagger}$ and suppose that $\underline{S}$ is a connected scheme. Let $\rho_s \in \overline{\mathcal{M}}^{\vee}_s[1]$ such that $\mathrm{Ker}\rho_s = \overline{\mathcal{F}}_s$ and let s be a specialization of $\xi \in \underline{T}$. Then by Lemma **??** (2) we have the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\overline{\mathcal{F}}_s)^{gp} & \longrightarrow & \overline{\mathcal{M}}^{gp}_s & \xrightarrow{\rho^{gp}_t} & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & (\overline{\mathcal{F}}_\xi)^{gp} & \longrightarrow & \overline{\mathcal{M}}^{gp}_\xi & \xrightarrow{\rho^{gp}_\xi} & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

so that there is a canonical isomorphism $\mathcal{M}_s/\mathcal{F}_s \cong \mathcal{M}_\xi/\mathcal{F}_\xi$. Therefore for $s, s' \in \underline{S}$, via cospecialization, canonically

$$\mathbb{N} \cong \mathcal{M}_s/\mathcal{F}_s \cong \mathcal{M}_{s'}/\mathcal{F}_{s'}.$$

We can associate a graph Γ for a geometric point $\bar{s} \to \underline{S}$ as follows. Let $C_{\bar{s}}$ be the curve over the point $\bar{s}$ induced from the morphism $\underline{S} \to \mathcal{T}or_{\mathfrak{M}_{g,n}} \times_{\mathcal{T}or_{\Bbbk}} \mathcal{L}og^{\mathrm{spl}}_{\Bbbk^\dagger}$. We call a node e of $C_{\bar{s}}$ a *splitting node* if the induced homomorphism

$$\times l_e : \mathbb{N}_e \to \overline{\mathcal{M}}^{C_{\bar{s}}/\bar{s}}_{\bar{s}} \to \overline{\mathcal{M}}_{\bar{s}} \to \overline{\mathcal{M}}_{\bar{s}}/\overline{\mathcal{F}}_{\bar{s}} = \mathbb{N}$$

is nonzero (i.e., $l_e \neq 0$). Using a local chart, we see that the integer l_e is well-defined, independent of the choices of $\bar{s}$. Now define Γ be the dual graph associated to the curve obtained smoothing all non-splitting nodes of $C_{\bar{s}}$. This graph Γ is independent of the choices of a geometric point $\bar{s} \to \underline{S}$. We say it is $(\pm)$-orientable if there exists an assignment $r : V(\Gamma) \to \{\pm\}$ such that $\{r(v), r(v')\} = \{\pm\}$ whenever $(v, v') \in E(\Gamma)$. In particular, there is no loops in $E(\Gamma)$ if Γ is orientable. Let $l_\Gamma : \mathbb{N}_b \to \overline{\mathcal{M}}_{\bar{s}}/\overline{\mathcal{F}}_{\bar{s}}$ and $c_e = l_\Gamma/l_e$. We say that if $c_e \in \mathbb{N}$ for every $e \in E(\Gamma)$, then Γ with l_Γ, l_e is divisible. We also give a genus and marking decorations, g_e, n_e.

Assume that the decorated graph Γ with $\{l_\Gamma, l_e, g_e, n_e : e\}$ is orientable and divisible. These decorations l_Γ, l_e, g_e, n_e are well-defined, independent of the choices of $\bar{s}$. Denote by $\mathfrak{B}_\Gamma$ the open and closed substack of $\mathcal{T}or_{\mathfrak{M}_{g,n}} \times_{\mathcal{T}or_{\Bbbk}} \mathcal{L}og^{\mathrm{spl}}_{\Bbbk^\dagger}$ whose associated graph is the decorated graph Γ.

Let

$$\mathfrak{B} := \coprod_{\Gamma} \mathfrak{B}_\Gamma.$$

There is uniquely an open and closed substack $\mathfrak{M}_0$ of $\mathcal{T}or_{\mathfrak{M}_{g,n}} \times_{\mathcal{T}or_k} \mathcal{T}or_{k^\dagger}$ such that $\mathfrak{B} \to \mathcal{T}or_{\mathfrak{M}_{g,n}} \times_{\mathcal{T}or_k} \mathcal{T}or_{k^\dagger}$ factored through a surjective morphism $\mathfrak{B} \to \mathfrak{M}_0$. Define

$$\mathcal{K}_\Gamma^{\mathrm{spl}} := \mathfrak{B}_\Gamma \times_{\mathfrak{M}_0} \mathcal{K}_0, \qquad \mathcal{K}^{\mathrm{spl}} := \coprod_\Gamma \mathcal{K}_\Gamma^{\mathrm{spl}}.$$

Then canonically

$$\mathcal{K}^{\mathrm{spl}} \cong (\mathcal{T}or_{\mathfrak{M}_{g,n}} \times_{\mathcal{T}or_{\Bbbk}} \mathcal{L}og_{\Bbbk^\dagger}^{\mathrm{spl}}) \times_{\mathfrak{M}_0} \mathcal{K}_0$$

since we can check that if (α, β) is an object of $(\mathcal{T}or_{\mathfrak{M}_{g,n}} \times_{\mathcal{T}or_{\Bbbk}} \mathcal{L}og_{\Bbbk^\dagger}^{\mathrm{spl}}) \times_{\mathfrak{M}_0} \mathcal{K}_0$, α is an object of $\mathfrak{B}_\Gamma \subset \mathcal{T}or_{\mathfrak{M}_{g,n}} \times_{\mathcal{T}or_{\Bbbk}} \mathcal{L}og_{\Bbbk^\dagger}^{\mathrm{spl}}$.

Define an assignment $r : V \to \{1, 2\}$ by the rule $r(v) = i$ if the composite

$$\mathbb{N}_{x_i} \xrightarrow[\mathrm{inc}]{} \mathbb{N}_{x_1} \oplus \mathbb{N}_{x_2} \xrightarrow[\mathrm{pr}]{} \overline{\mathcal{M}}_{W_0, f(v)} \longrightarrow \overline{\mathcal{M}}_{\bar{s}} \xrightarrow[\rho]{} \mathbb{N}$$

is nonzero. The following Lemma shows that the map r is well-defined. Denote by $\langle \rho, \log x_i \otimes e_\eta \rangle$ the value of 1 under the composite.

Lemma 3.4.4. *For each component η of $C_{\bar{s}}$, $\langle \rho, \log x_i \otimes e_\eta \rangle \neq 0$ if and only if $\langle \rho, \log x_{t(i)} \otimes e_\eta \rangle = 0$ for $i = 1, 2$. Furthermore, these imply $\underline{f}(\bar{\eta}) \in \underline{X}_i$.*

Proof. Since $\overline{\mathcal{M}}_{\bar{s}}^\vee \otimes \mathbb{R}_{\geq 0}$ is a strictly convex rational cone, there is no small plane passing through ρ. Since $\rho(p) \neq 0$, $\langle \rho, \log x_i \otimes e_\eta + \log x_{t(i)} \otimes e_\eta \rangle > 0$. Represent ρ by $\prod_\eta (\alpha_\eta, \beta_\eta) \times \prod \gamma_q \in \prod \mathbb{N}$ with certain relation at nodes q. Suppose that $\langle \rho, \log x_i \otimes e_\eta \rangle > 0$ for both $i = 1, 2$. For such η, the relation is that $\alpha_\eta + \beta_\eta = \alpha_{\eta'} + \beta_{\eta'}$ so that we can vary α_η, β_η on the line $x + y =$constant. This proves the first statement. For the second statement, note that if $\underline{f}(\bar{\eta}) \notin \underline{X}_i$, then $\log x_i \otimes e_\eta = 0$ in $\overline{\mathcal{M}}_{\bar{s}}$. □

By using the fiber product

$$\begin{tikzcd} \mathcal{K}_\Gamma^{\mathrm{spl}} \arrow[r, "\sigma_\Gamma"] \arrow[d] & \mathcal{K}_0 \arrow[d] \\ \mathfrak{B}_\Gamma \arrow[r, "\sigma_{\mathfrak{B}_\Gamma}"'] & \mathfrak{M}_0 \end{tikzcd}$$

let

$$E_{\mathcal{K}_\Gamma/\mathfrak{B}_\Gamma} := \sigma_\Gamma^* E_{\mathcal{K}_0/\mathfrak{M}_0}.$$

This defines a perfect obstruction theory for $\mathcal{K}_\Gamma$ relative to $\mathfrak{B}_\Gamma$ (see [4, Theorem 4.5 & Proposition 7.1]).

Lemma 3.4.5. *Under the projective morphisms $\sigma_{\mathfrak{M}_\Gamma}$,*

$$\sum_\Gamma l_\Gamma (\sigma_{\mathfrak{B}_\Gamma})_* [\mathfrak{B}_\Gamma] = [\mathfrak{M}_0].$$

Proof. Consider the following fibered diagram

$$\begin{tikzcd} \mathcal{T}or_{\mathfrak{M}_{g,n}} \times_{\mathcal{T}or_{\Bbbk}} \mathcal{L}og^{\mathrm{spl}}_{\Bbbk^\dagger} \arrow[r] \arrow[d] & \mathcal{T}or_{\mathfrak{M}_{g,n}} \times_{\mathcal{T}or_{\Bbbk}} \mathcal{L}og_{\Bbbk^\dagger} \arrow[r] \arrow[d] & \mathcal{T}or_{\mathfrak{M}_{g,n}} \arrow[d, "h"] \\ \mathcal{L}og^{spl}_{\Bbbk^\dagger} \arrow[r, "f"'] & \mathcal{L}og_{\Bbbk^\dagger} \arrow[r] & \mathcal{T}or_{\Bbbk} \end{tikzcd}$$

Since $\mathfrak{M}_{g,n}$ is log smooth over Spec($\Bbbk$), h is smooth by [23, Theorem 4.6 (ii)]. Hence, it's enough to prove the statement for $f : \mathcal{L}og^{spl}_{\Bbbk^\dagger} \to \mathcal{L}og_{\Bbbk^\dagger}$.

First note that locally we may assume that any morphism $\underline{T} \to \mathcal{T}or_{\Bbbk^\dagger}$ is of form $\underline{T} = \mathrm{Spec}(\Bbbk[Q]/(p))$ with chart $\mathbb{N} \to Q$, $1 \mapsto p$, where (p) is the ideal of $\Bbbk[Q]$ generated by $p \neq 0$. The associated reduced scheme is the union of toric divisors T_ρ of Spec($\Bbbk[Q]$) with $\rho(p) \neq 0$. For some positive integer m, mp is in the sum of Q_ρ, $\forall \rho$. Note that the hypersurface associated to mp of Spec($\Bbbk[Q]$) has the multiplicity $m[p]$ with respect to the irreducible component Spec($\Bbbk[Q_\rho]$) of the hypersurface, where $[p]$ is the integer in $Q/Q_\rho = \mathbb{N}$ associated to p. ◻

Consider $\mathfrak{B}'_\Gamma := \mathfrak{B}_\Gamma \times_\Bbbk \mathrm{Spec}(\Bbbk[x]/(x^{l_\Gamma}))$ in order to have the degree-1 induced morphism $\coprod_\Gamma \mathfrak{B}'_\Gamma \to \mathfrak{M}_0$. Now by Theorem 5.0.1 of [7] (see also Proposition 5.29 of [19]), under the projective morphisms σ_Γ,

$$\sum_\Gamma l_\Gamma (\sigma_\Gamma)_* [\mathcal{K}_\Gamma, E_{\mathcal{K}_\Gamma/\mathfrak{B}_\Gamma}] = [\mathcal{K}_0, E_{\mathcal{K}_0/\mathfrak{M}_0}]. \tag{3.4.1}$$

3.4.3 Gluing of underlying maps

Fix a graph Γ and an element $\tau \in \mathfrak{S}_{E(\Gamma)}$, let

$$\mathcal{K}_v := \overline{M}_{g_v, n_v, \cup_e \tau_e}(X_v, \beta_v)$$

where e runs for $e = (v, v')$ for some $v' \in V(\Gamma)$. Let ev_{v,τ_e} be the evaluation map $\mathcal{K}_v \to D$ at the relative marking τ_e. Define a stack $\bigodot_v \mathcal{K}_v$ as the fiber product

$$\begin{tikzcd} \bigodot_v \mathcal{K}_v \arrow[r, "u"] \arrow[d, "\prod_e ev_e"'] & \prod_v \mathcal{K}_v \arrow[d, "\prod_{e=(v,v')}(ev_{v,\tau_e} \times ev_{v',\tau_e})"] \\ D^{E(\Gamma)} \arrow[r, "\Delta"'] & D^{E(\Gamma)} \times D^{E(\Gamma)} \end{tikzcd} \tag{3.4.2}$$

We define a natural perfect obstruction theory on $\bigodot_v \mathcal{K}_v$ relative to $\prod_v \mathcal{L}og_{\prod \mathfrak{M}_{g_v,n_v+1}}$ as follows. Over $\bigodot_v \mathcal{K}_v$, there is a universal curve $\underline{\mathfrak{C}}$ joined by two universal curves $\underline{\mathfrak{C}_v}$ associated to the universal curve over $\mathcal{K}_v$. Let f be the universal map $\underline{\mathfrak{C}} \to \underline{W}$, let $\iota_{\underline{\mathfrak{C}}_v} : \underline{\mathfrak{C}}_v \to \underline{\mathfrak{C}}$ the inclusion, let $\iota_{q_e} : \bigodot_v K_v \to \underline{\mathfrak{C}}$ be the section associated to the nodes corresponding to e. There is a natural sheaf epimorphism

$$\begin{array}{ccc} \oplus_v (\iota_{\underline{\mathfrak{C}}_v})_* f_v^* T^\dagger_{X_v} & \to & \oplus_e (\iota_{q_e})_* ev_e^* T_D \\ (\xi_v)_v & \mapsto & (D(\xi_v)_{v \mapsto e})_e \end{array}$$

where $D(\xi_v)$ is the part of ξ_v tangent along D. If we denote $\mathcal{E}$ the Kernel of the above epimorphism, we obtain a homomorphism of distinguished triangles

$$\begin{tikzcd} ev^* \mathbb{L}_{D^{E(\Gamma)}/D^{2E(\Gamma)}}[-1] \arrow[r] \arrow[d] & u^* E_{\prod K_v/\mathcal{L}og_{\prod \mathfrak{M}_{g_v,n_v+1}}} \arrow[r] \arrow[d] & (\mathbb{R}\pi_* \mathcal{E})^\vee \arrow[d] \\ \mathbb{L}_{\bigodot K_v/\prod K_v}[-1] \arrow[r] & u^* \mathbb{L}_{\prod K_v/\prod \mathcal{L}og_{\mathfrak{M}_{g_v,n_v+1}}} \arrow[r] & \mathbb{L}_{\bigodot K_v/\prod \mathcal{L}og_{\mathfrak{M}_{g_v,n_v+1}}} \end{tikzcd}$$

where

$$E_{\prod K_v/\mathcal{L}og_{\prod \mathfrak{M}_{g_v,n_v+1}}} = \boxtimes_v (\mathbb{R}(\pi_v)_* f_v^* T^\dagger_{X_v})^\vee.$$

By the diagram chasing, it is straightforward to check that $(\mathbb{R}\pi_* \mathcal{E})^\vee \to \mathbb{L}_{\bigodot K_v/\prod \mathcal{L}og_{\mathfrak{M}_{g_v,n_v+1}}}$ is a perfect obstruction theory. By the functoriality of [4, Proposition 5.10] we conclude that

$$[\bigodot K_v, (\mathbb{R}\pi_* \mathcal{E})^\vee] = \Delta^! \prod_v [K_v, (\mathbb{R}(\pi_v)_* f_v^* T^\dagger_{X_v})^\vee]. \tag{3.4.3}$$

3.4.4 Gluing of log structures

Define $\tilde{\mathcal{K}}^{spl}_\Gamma \to \mathcal{K}^{spl}_\Gamma$ by choosing an order on the set of the splitting nodes. Note that it's principal $\mathfrak{S}_{E(\Gamma)}$-bundle.

By forgetting some data, we have the commuting diagram of natural morphisms

$$\begin{tikzcd}
\tilde{\mathcal{K}}_{\Gamma}^{spl} \arrow[r, "\phi"] \arrow[d, "\nu"'] & \bigodot K_v \arrow[d] \\
\mathfrak{B}_{\Gamma} \arrow[dr, "\psi"'] & \prod \mathcal{L}og_{\mathfrak{M}_{g_v,n_v+1}} \arrow[d, "\nu'"] \\
& \mathcal{L}og_{\prod \mathfrak{M}_{g_v,n_v+1}}
\end{tikzcd}$$

Lemma 3.4.6. *1. The morphism ν' is étale.*

2. The morphism ψ is DM-type and smooth.

3.

$$\nu^* \mathbb{L}_{\mathfrak{B}_\Gamma / \mathcal{L}og_{\prod \mathfrak{M}_{g_v,n_v+1}}} \cong \phi^* \prod_e ev_e^* N_{D/X_e}^{\dagger \boxtimes E(\Gamma)}$$

4. The morphism ϕ is DM-type and étale of degree $\frac{\prod_e c_e}{l_\Gamma}$.

Proof. For (1): This is clear by considering the lifting criterion for formally étale morphisms.

For (2): First note that ψ is DM-type since there is no infinitesimal automorphisms σ of a geometric point of $\mathfrak{B}_\Gamma$ with $\psi(\sigma) = \mathrm{id}$. This implies that ψ is DM-type. Now to prove ψ is smooth, it is enough to show ψ is formally smooth since it is locally of finite presentation. The corresponding lifting property of ψ can be checked by considering charts of log morphisms. Let I be a nilpotent ideal of a finitely generated ring Λ over k and let $S = \mathrm{Spec}(\Lambda/I)$. We may assume that there is a chart

$$\begin{tikzcd}
\overline{M}_S^{C/S} \arrow[r] \arrow[dr] & Q \arrow[d] & \mathbb{N}_b \arrow[l] \arrow[dl] \\
& O_S &
\end{tikzcd}$$

of $\mathcal{M}_S^{C/S} \to \mathcal{M}_S \leftarrow \mathbb{N} \oplus O_S^\times$. By Definition 3.4.1, the liftings of log structure on $\mathcal{M}_S$ uniquely exists. It is also obvious that $\mathcal{M}_S^{C/S} \to \mathcal{M}_S$ and $\mathcal{M}_S \leftarrow \mathcal{M}_b$ have lifts (may not be unique).

For (3): Since $N^{\dagger}_{D/X_e}$ is trivial bundle, it's enough to show that $\mathbb{L}_{\prod \mathfrak{B}/\mathcal{L}og\mathfrak{M}_{g_v,n_v+1}}$ is also trivial bundle. Let $(C/S, \mathbb{N}_S \oplus O_S^* \xrightarrow{h} \mathcal{M}_S \xleftarrow{j} \mathcal{M}_S^{C/S})$ be an object of $\mathfrak{B}_\Gamma$ over $\underline{S}$. The isomorphism set of the lifting to $S[\epsilon] := Spec(O_S[\epsilon]/\epsilon^2)$ is canonically isomorphic to the free O_S-module whose basis $\{\beta_e\}$ is described as follows:

$$(C[\epsilon]/S[\epsilon], \mathbb{N}_{S[\epsilon]} \oplus O_S^*[\epsilon] \xrightarrow{h} \mathcal{M}_S[\epsilon] \xleftarrow{j_e} \mathcal{M}_S^{C/S}[\epsilon])$$

where h is the trivial extension and j_e is the homomorphism determined by following condition. For each splitting node q, there is the canonical subsheaf $\mathcal{N}_q \subset \mathcal{M}_S^{C/S}$ such that $\mathcal{N}_q \cong \mathbb{N} \oplus O_S^*$. Let $1_q \in \mathcal{N}_q[\epsilon]$ be a primitive element. Then j_e is the homomorphism which satisfy that: $j_e(1_q) = j(1_q)$ for $q \neq e$, $j_e(1_q) = j(1_q)(1+\epsilon)$ for $q = e$. We can check that $\{\beta_e\}$ is well-defined. This show the proof of (3)

For (4): The proof of (1) and Proposition 3.3.4 shows that ϕ is formally étale and hence it is étale. Now we count the degree of ϕ. Let $\underline{S} = \mathrm{Spec}(\mathbb{k})$. By considering charts, we note that the homomorphisms $\mathcal{M}^{C/S} \to \mathcal{M}_S \leftarrow \mathcal{M}_b$ modulo $\mathcal{M}_{S_v}^{C_v/S_v} \to \mathcal{M}_S$ are determined by homomorphism $\mathbb{N}_e \to \mathbb{k}^\times$, $1 \mapsto \zeta_e$, $\mathbb{N}_b \to \mathbb{k}^\times$, $1 \mapsto \zeta$ with $\zeta_e^{c_e} = \zeta$. By an isomorphism of $\mathcal{M}_S$, we may let $\zeta = 1$. There are still remained isomorphisms of $\mathcal{M}_S$ which form a multiplicative group $\{t \in \mathbb{k}^\times : t^l = 1\}$. This group acts on the set of homomorphisms $(\zeta_e)_e$ by $\zeta_e \mapsto t\zeta_e$. This shows that the degree of ϕ is $(\prod_e c_e)/l_\Gamma$. □

Let

$$E_{\tilde{\mathcal{K}}_\Gamma^{spl}/\prod \mathcal{L}og\mathfrak{M}_{g_v,n_v+1}} := \phi^*(\mathbb{R}\pi_*\mathcal{E})^\vee$$

By (4) of Lemma 3.4.6, $E_{\tilde{\mathcal{K}}_\Gamma^{spl}/\prod \mathcal{L}og\mathfrak{M}_{g_v,n_v+1}}$ is a perfect obstruction theory relative to $\prod \mathcal{L}og\mathfrak{M}_{g_v,n_v+1}$ and hence by the very definition of virtual fundamental classes,

$$\phi^*[\bigodot K_v, (\mathbb{R}\pi_*\mathcal{E})^\vee] = [\tilde{\mathcal{K}}_\Gamma^{spl}, E_{\tilde{\mathcal{K}}_\Gamma^{spl}/\prod \mathcal{L}og\mathfrak{M}_{g_v,n_v+1}}]. \tag{3.4.4}$$

By (2) & (3) of Lemma 3.4.6, we get the exact triangle

$$E_{\tilde{\mathcal{K}}_\Gamma^{spl}/\prod \mathcal{L}og\mathfrak{M}_{g_v,n_v+1}} \to E_{\tilde{\mathcal{K}}_\Gamma^{spl}/\mathfrak{B}_\Gamma} \to \nu^*\mathbb{L}_{\mathfrak{B}_\Gamma/\prod \mathcal{L}og\mathfrak{M}_{g_v,n_v+1}}[1]$$

Therefore,

$$[\tilde{\mathcal{K}}_\Gamma^{spl}, E_{\tilde{\mathcal{K}}_\Gamma^{spl}/\prod \mathcal{L}og_{\mathfrak{M}_{g_v,n_v+1}}}] = [\tilde{\mathcal{K}}_\Gamma^{spl}, E_{\tilde{\mathcal{K}}_\Gamma^{spl}/\mathfrak{B}_\Gamma}]. \tag{3.4.5}$$

3.4.5 The conclusion

Now combining (3.4.1), (3.4.5), (3.4.4), (3.4.3), we obtain Theorem 3.1.1. This part is standard (see [2],[5]).

$$\int_{[\overline{M}_{g,n}(W_0,\beta)]^{\mathrm{vir}}} \prod_{i\in[n]} \psi_i^{m_i} ev_i^*(\gamma_i) = \sum_{\Gamma\in\Omega(g,N,\beta)} \frac{l_\Gamma}{|V(\Gamma)|!} deg(\prod_{i\in[n]} \psi_i^{m_i} ev_i^*(\gamma_i) \cap [\tilde{\mathcal{K}}_\Gamma^{spl}])$$

$$= \sum_{\Gamma\in\Omega(g,N,\beta)} \frac{\prod_e c_e}{|V(\Gamma)|!} deg(\prod_{i\in[n]} \psi_i^{m_i} ev_i^*(\gamma_i) \cap [\bigodot_v K_v])$$

$$= \sum_{\Gamma\in\Omega(g,N,\beta)} \frac{\prod_e c_e}{|V(\Gamma)|!} deg(\prod_{i\in[n]} \psi_i^{m_i} ev_i^*(\gamma_i) \prod_{e\in E(\Gamma)} (\sum_j ev_{1,e}^*\delta_{j_e}^1 \times ev_{2,e}^*\delta_{j_e}^2) \cap \prod_v [K_v])$$

$$= \sum_{\Gamma\in\Omega(g,N,\beta)} \sum_{(j_e)_e\in\{1,\dots,\mathrm{rk}H^*(D)\}^{E(\Gamma)}} (-1)^\epsilon \frac{\prod_e c_e}{|V(\Gamma)|!} \prod_v \int_{[\overline{M}_{g,N_v|E_v}(X_v,\beta)]^{\mathrm{vir}}} \prod_{i\in N_v} \psi_i^{m_i} ev_i^* \iota_{\underline{X}_a}^*(\gamma_i) \prod_{e\in E_v} ev_e^*(\delta_{j_e}^{r(v)}).$$

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국문초록

대수적 곡선에서 프로젝티브 다양체로의 홀로모픽 사상의 모듈라이 공간의 옹골화들을 통해서 그 기하를 공부한다. 가장 유명한 옹골화는 컨세비치에 의해 소개된 스테이블 사상의 모듈라이 공간이다. 종수가 1이고 타겟이 프로젝티브 다양체인 경우에, 스테이블 사상의 모듈라이공간의 메인 컴포넌트는 논싱귤러한 공간이다. 바킬과 징어에 의해 모듈러 블로우 업에 통한 특이점 해소화가 발견되었다. 김에 의해 로그 스테이블 공간이 소개되었데, 이 공간 역시 스테이블 사상의 모듈라이 공간의 특이점 해소화 공간이다. 이 두가지의 특이점 해소화 공간이 어떻게 연관이 되어 있는지를 밝히겠다. 또한 그로스-지버트 와 아브라모비치-첸에 의해 로가리드믹 스테이블맵이 소개되었다. 이 공간을 이용하아, 로그 그로모브 위튼 불변량을 정의 할 수 잇다. 로가리드믹 스테이블 맵의 변형에 대하여 공부하고, 이를 통해 로그 그로모브 위튼 불변량의 디제너레이션 공식을 증명하겠다.

주요어휘: 스테이블 사상, 로가리드믹 구조, 그로모브 위튼 불변량

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