이학박사 학위논문

Regularity for nonlocal operators with kernels of functional order

(함수적 차수의 커널을 갖는 비국소 작용소의 정칙성)

2016년 2월

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Abstract

In this thesis, we investigate the regularity results for nonlocal operators with kernels of functional order. We consider a nonlocal operator

$$Lu(x) = \int_{\mathbb{R}^d} (u(x+h) - u(x) - \nabla u(x) \cdot h \mathbf{1}_{\{|h| \le 1\}}) J(x,h) dh$$

for a given jumping kernel $J(x,h)$ comparable to $|h|^{-d} \varphi(|h|)^{-1}$ near zero. Assuming φ has a weak scaling condition one can generalize the fractional Laplace operator. Firstly, we obtain the Hölder continuity of solutions to fully nonlinear nonlocal equations with respect to a certain class of linear nonlocal operators in analytic method. Secondly, we prove the Schauder estimates for nonlocal equation $Lu = f$ by introducing the generalized Hölder spaces.

Key words: Nonlocal operator, integro-differential operator, Hölder continuity, Schauder estimate, Lévy process
Student Number: 2009-22886

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Chapter 1

Introduction

Regularity theory for nonlocal operators has been investigated a lot in both probability theory and analysis. A nonlocal operator is defined as an integral over a given region. This means that it needs values of a function everywhere in the region. It is different from a local operator which is a combination of derivatives because derivatives just need values in a neighbourhood of point. In the probabilistic view point the Laplacian, which is one of local operators, is the infinitesimal generator of a Brownian motion. The path continuity of the Brownian motion implies the localness of the Laplacian. Intuitively, the distribution of Brownian motion when it exits a ball is supported on its boundary. It gives the mean value property of a harmonic function of the Laplace equation. Similarly, a nonlocal operator corresponds to a jump process. When a jump process exits a ball, it can reach everywhere outside the ball. This phenomenon gives a relation between nonlocal operators and jump processes.

To study Markov processes with discontinuous sample paths, it is important to estimate transition probabilities and jump measures. Such estimates are closely related to the Harnack inequalities and Hölder estimates for harmonic functions with respect to the processes, which have been active research areas over last decade and more. For given stochastic processes, harmonic functions are represented by an expectation of values with respect to its distribution at the exit time of the domain.

Recently a potential theory for jump processes was developed rapidly.

Bass and Levin starts a study on Harnack inequality of harmonic functions for α-stable(like) processes in [5]. Song and Vondraček extends the results of [5] to a certain type of Markov processes in [38]. In [3, 4] the Harnack inequality and the Hölder estimate of nonnegative bounded harmonic functions were extended to the operators of variable order. Two-sided heat kernel estimate and a scale-invariant parabolic Harnack inequality for symmetric α-stable like operators on d-sets are obtained in [13]. Chen and Kumagai developed in [14] the heat kernel estimates of Hunt processes for given jumping kernel. Chen, Kim and Kumagai obtained in [11, 12] various heat kernel estimates for jump processes which leads to regularity estimates of the corresponding harmonic functions. By investigating independent subordinators, the authors in [27, 28, 29] and references therein obtained results on subordinate Brownian motions. At the same time an α-stable like process, which is a process having stable like kernels depending on spatial variables, was dealt with in those literatures.

Above results with respect to the stable like operators have been studied in analytic method. Silvestre obtained the Hölder estimates for nonlocal operators using analytic techniques in [37]. Caffarelli and Silvestre established the regularity theory for nonlocal operators with symmetric kernels by extending the method for second order differential equations in [9]. They obtained uniform estimates so that the constants in the results are invariant as the order of kernel approaches to 2. These results were extended to nonlocal operators with non symmetric kernels in [10, 32, 33].

One of the other type regularity results for nonlocal operators is the Schauder estimate. For differential operators of second order, the main assertion in this field is an estimate of the form

$$\|u\|_{C^{2+\beta}} \leq c\,(\|u\|_{C^0} + \|f\|_{C^\beta})$$

for all solutions u to elliptic equations of second order $Lu = f$ [17, Theorem 6.2] with some positive generic constant c, depending on the ellipticity of L, the dimension d and the number $\beta \in (0,1)$. The order 2 and the value β are independent quantities in this estimate. The estimate holds for different values of β but, as shown in [2, 15, 20, 35], it holds analogously for solutions to integro-differential equations where the driving operator is an integro-differential operator with fractional order of differentiability $\alpha \in (0,2)$.

Nonlocal operators with respect to stable processes have the scaling property that makes a large jump intensities to be controlled by small jump intensities. Weakening the scaling property allows to extend the theory to more general processes. In this thesis, we mostly consider a nonlocal operator with kernels of functional order which is defined by

$$Lu(x) = \int_{\mathbb{R}^d} (u(x+h) - u(x) - \nabla u(x) \cdot h \mathbf{1}_{\{|h|\le 1\}}) \frac{a(x,h)}{|h|^d \varphi(|h|)} dh$$

for a function $u \in C_b^2(\mathbb{R}^d)$. Here $a(x,h)$ is a measurable function and we assume that $\varphi : (0,\infty) \to (0,\infty)$ has a weak scaling condition, i.e. there exists constants $a_1, a_2 \in (0,\infty)$, $\delta_1, \delta_2 \in (0,2)$ and $R_0 \in (0,\infty]$ such that

$$a_1 \left(\frac{R}{r}\right)^{\delta_1} \le \frac{\varphi(R)}{\varphi(r)} \le a_2 \left(\frac{R}{r}\right)^{\delta_2}, \quad \text{for} \quad 0 < r \le R \le R_0.$$

We will discuss the following two topics on the regularity of solutions to nonlocal equations in analytic methods with probabilistic point of view.

Firstly, we discuss the Hölder continuity of solutions for fully nonlinear equations driven by nonlocal operators with kernels of functional order. Silvestre obtained in [37] the Hölder estimates for solutions to integro-differential equation $Lu = 0$ when $\varphi(r) = r^\alpha$. He introduced a bump function to obtain a growth lemma for nonlocal operators. We will use this method to prove the Hölder continuity of solutions to the fully nonlinear nonlocal equations.

Secondly, we prove the Schauder estimates for nonlocal operators with kernels of functional order in generalized Hölder spaces. In [2] Bass extended the classical Schauder estimates for differential equations to the fractional Laplace operators. He estimated the derivatives of semigroups with respect to the fractional Laplacian to deal with the constant coefficient case. The perturbation argument was useful to obtain the result for the variable coefficient case. We define generalized Hölder spaces because the differentiability of our operator is not represented by a number. We also use the derivative estimates of transition densities for subordinate Brownian motions to prove the results for translation invariant case. Finally we will obtain the Schauder estimates for variable coefficient case from the perturbation argument.

This thesis is organized as follows. Hölder continuity is proved in Chapter 2. The setting and main result is given in Section 2.1. In Section 2.2 we recall the notion of viscosity solution introduced in [9]. In Section 2.3 we prove the main result by modifying the method of [37] properly to our consideration. In Section 2.4 we give examples covered in this chapter. We recall recent results in [8] on the estimates for densities of isotropic unimodal Lévy processes, which serve as main examples. The Schauder estimates in generalized Hölder spaces is shown in Chapter 3. We set up the problem and give the main result in Section 3.1. In Section 3.2 we define and study the generalized Hölder spaces $C^{\psi}(\mathbb{R}^d)$. The proof of Theorem 3.1.2 relies on a perturbation technique. We firstly study the operator L_0 which is obtained by "freezing" the coefficients for some arbitrary but fixed point $x_0 \in \mathbb{R}^d$. In Section 3.3 we derive estimates on the transition density for the semigroup generated by L_0. Section 3.3 also contains the proof of Theorem 3.1.1. In Section 3.4 we prove our main result, Theorem 3.1.2. The proof of Theorem 3.1.3 is given in Section 3.5.

Throughout this thesis an integer $d \geq 1$ denotes the dimension of Euclidean space $\mathbb{R}^d$ and the constants for weak scaling condition R_0, a_1, a_2, δ_1 and δ_2 is regarded as being fixed. We use ":=" to denote a definition, which is read as "is defined to be"; we denote $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. We use dx to denote the Lebesgue measure in $\mathbb{R}^d$; for a Borel set $A \subset \mathbb{R}^d$, we use $|A|$ to denote its Lebesgue measure and $\mathbf{1}_A$ to denote its indicator function; we denote by $B(x, r)$ the open ball centered at $x \in \mathbb{R}^d$ of radius $r > 0$. In the statements of results and the proofs, the constants $c_1, c_2, \ldots$, and C denote generic constants and they are given anew in each statement and each proof.

Chapter 2

Hölder estimates for fully nonlinear equations

Hölder estimates for integro-differential operators have been studied in many literatures. Analytically, Silvestre obtained in [37] the Hölder estimates for solutions in proper sense of integro-differential equations with a kernel comparable to that of the fractional Laplacian. Kassmann obtained in [22] the same result for weak solution by developing a nonlocal version of De Giorgi-Nash-Moser theorem. Caffarelli and Silvestre established in [9] the regularity theory for fully nonlinear integro-differential equations extending the results for elliptic partial differential equations. Overcoming the difficulty of non-symmetry of the kernels, the authors in [10, 32, 33] obtained Hölder and $C^{1,\alpha}$ estimates for the case of non-symmetric kernels. All of them basically considered families of kernels comparable to that of the fractional Laplacian. In this chapter we will consider kernels more general than the kernel of the fractional Laplacian.

Throughout this chapter the constants M_0, λ, Λ will be fixed. For every function f, let $f_+ := f \vee 0$. We use C_1, C_2, C_3 to denote the constants which are obtained in the proofs of theorems and depends only the aforementioned constants. We define $\mathrm{Osc}_E u = \sup_{x\in E} u(x) - \inf_{x\in E} u(x)$ for a subset E in $\mathbb{R}^d$. For any Borel subset $E \subset \mathbb{R}^d$, $\overline{E}$ stands for the closure of E. We denote by $\omega_d = 2\pi^{d/2}/\Gamma(d/2)$ the surface measure of the unit sphere in $\mathbb{R}^d$.

2.1 Setting and main results

We consider a measurable function $J : \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \to \mathbb{R}$ satisfying that there exists a positive constant R_0 such that

$$M_0 := \sup_{x \in \mathbb{R}^d} \int_{|h| \geq R_0} J(x, h) dh < \infty, \tag{2.1.1}$$

and

$$J(x, h) = \frac{1}{|h|^d \varphi(|h|)} \qquad \text{for} \quad 0 < |h| < R_0, \tag{2.1.2}$$

where φ is a non-decreasing function from $(0, \infty)$ to $(0, \infty)$ having the following upper and lower scaling conditions at infinity;

$$a_1 \left(\frac{R}{r}\right)^{\delta_1} \leq \frac{\varphi(R)}{\varphi(r)} \leq a_2 \left(\frac{R}{r}\right)^{\delta_2} \qquad \text{for} \quad r \leq R \leq R_0, \tag{2.1.3}$$

with some constants $a_1, a_2 > 0$ and $\delta_1, \delta_2 \in (0, 2)$.

If (2.1.2) holds for all $x \in \mathbb{R}^d$ and $h \in \mathbb{R}^d \setminus \{0\}$, then J is the jump density of an isotropic unimodal Lévy process. An isotropic unimodal Lévy process is an important example of Markov process having jump density comparable to J. It contains subordinate Brownian motions which have been dealt with in a lot of literatures in probability and potential theory. Recently, it is shown in [8] that (2.1.3) holds if the characteristic exponent of the corresponding isotropic unimodal Lévy process has the above upper and lower scaling conditions at infinity.

In this chapter we fix constants $\lambda, \Lambda > 0$ and a function J satisfying (2.1.1) and (2.1.2). We define integro-differential operators comparable to J as follows. Let $K : \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ be a function comparable to J uniformly as

$$\lambda J(x, h) \leq K(x, h) \leq \Lambda J(x, h), \quad x \in \mathbb{R}^d, h \in \mathbb{R}^d \setminus \{0\} \tag{2.1.4}$$

and define an integro-differential operator with the kernel K as

$$\begin{aligned} Lu(x) &= L_K u(x) \\ &= \int_{\mathbb{R}^d \setminus \{0\}} \left(u(x+h) - u(x) - \nabla u(x) \cdot h \mathbf{1}_{\{|h|<R_0\}} \mathbf{1}_{\{\delta_2 \geq 1\}} \right) K(x,h) dy. \end{aligned} \tag{2.1.5}$$

Note that L may not be a symmetric operator.

We will investigate the Hölder regularity of the solution to fully nonlinear nonlocal equation $\mathcal{I}u = g$ in an open set D which is uniformly elliptic with respect to a family $\mathcal{L}$ of integro-differential operators L, i.e.,

$$\mathcal{M}^-_{\mathcal{L}}(u-v)(x) \leq \mathcal{I}u(x) - \mathcal{I}v(x) \leq \mathcal{M}^+_{\mathcal{L}}(u-v)(x) \tag{2.1.6}$$

for $x \in D$ where maximal and minimal operators $\mathcal{M}^+_{\mathcal{L}}$, $\mathcal{M}^-_{\mathcal{L}}$ are defined by

$$\mathcal{M}^+_{\mathcal{L}} u(x) = \sup_{L \in \mathcal{L}} Lu(x) \quad \text{and} \quad \mathcal{M}^-_{\mathcal{L}} u(x) = \inf_{L \in \mathcal{L}} Lu(x).$$

In [9] Caffarelli and Silvetre established the theory which can be used to prove the Harnack inequality and the Hölder continuity of the solution u with respect to the family of integro-differential operators whose kernels $K(x,h)$ are satisfying

$$\frac{(2-\alpha)\lambda}{|h|^{d+\alpha}} \leq K(x,h) \leq \frac{(2-\alpha)\Lambda}{|h|^{d+\alpha}}, \quad x \in \mathbb{R}^d, h \in \mathbb{R}^d \setminus \{0\}, \tag{2.1.7}$$

and

$$K(x,h) = K(0,h) = K(0,-h), \quad x \in \mathbb{R}^d, h \in \mathbb{R}^d \setminus \{0\}.$$

The purpose of this chapter is to obtain the Hölder continuity of solutions to uniformly elliptic fully nonlinear nonlocal equations with respect to more general class of operators. Two classes of integro-differential operators in our consideration are

$$\mathcal{L} := \{ L_K : K \text{ satisfies } (2.1.4) \}, \tag{2.1.8}$$

and

$$\mathcal{L}_{sym} := \left\{ L_K \in \mathcal{L} : K(x,h) = K(x,-h) \text{ for } x \in \mathbb{R}^d, h \in \mathbb{R}^d \setminus \{0\} \right\}. \quad (2.1.9)$$

Uniformly ellipticity of the operator $\mathcal{I}$ in (2.1.6) allows to regard the solution to $\mathcal{I}u = g$ as the sub and super solution to maximal and minimal operators respectively.

The main result in this chapter is the following. See Section 2.2 for the definition of solutions in the viscosity sense.

Theorem 2.1.1. *Assume that* (2.1.1), (2.1.2) *and* (2.1.3) *hold for the function* $J(x,h)$ *with* $\delta_1 \in (1,2)$ *or* $\delta_2 \in (0,1)$. *Let* $z_0 \in \mathbb{R}^d$, $r > 0$, $C_0 > 0$ *and* u *be a bounded function so that*

$$\begin{aligned} \mathcal{M}^+_{\mathcal{L}} u(x) &\geq -C_0 \quad \text{for} \quad x \in B(z_0, r) \\ \mathcal{M}^-_{\mathcal{L}} u(x) &\leq C_0 \quad \text{for} \quad x \in B(z_0, r) \end{aligned}$$

in the viscosity sense. Then there exist two constants $\alpha > 0$ *and* $C > 0$ *such that*

$$\sup_{x,y \in B(z_0, r/2)} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq C \left(r \wedge R_0\right)^{-\alpha} \left(\|u\|_\infty + C_0 \varphi(r)\right)$$

where the values of α *and* C *depend only on* $d, R_0, \lambda, \Lambda, a_1, a_2, \delta_1, \delta_2$ *and* $\varphi(R_0)$. *In the case* $\delta_1 \leq 1 \leq \delta_2$, *the above assertion still holds for* $\mathcal{L}_{sym}$ *in place of* $\mathcal{L}$.

In [37] Silvestre developed an analytic method to obtain this result for integro-differential operator L satisfying an assumption that given a constant $\delta > 0$ and the auxiliary function $b(x) = (1 - |x|^2)^2_+$, there are positive constants κ and η satisfying

$$\begin{aligned} &\kappa L b(x) + 2 \int_{|h| \geq 1/4} (|8h|^\eta - 1) K(x,h) dh \\ &\qquad < \frac{1}{2} \inf_{A \subset B(0,2), |A| > \delta} \int_A K(x,h) dh \end{aligned} \quad (2.1.10)$$

for every $x \in \mathbb{R}^d$. The kernels of operators in consideration here are assumed to have similar behaviour at every scale. Considering an operator L_{r,x_0} obtained by a change of variables, Silvestre proved the Hölder continuity of the solution u to $Lu = 0$ in $B(0,1)$ using the fact that (2.1.10) holds for every $L_{r,x_0},\ r > 0, x_0 \in \mathbb{R}^d$ by the scaling. The argument using scaling property of operator was employed to prove the same result for fully nonlinear equation with respect to the family of operators having kernels comparable to that of fractional Laplacian.

In this chapter we will deal with the kernels comparable to $J(x,h)$ which is equal to $|h|^{-d}\varphi(|h|)^{-1}$ only on the near $h = 0$. The function $J(x,h)$ has no scaling property and it is not comparable to fractional Laplacian near $h = 0$. Instead, we assume the weak scaling condition (2.1.3) of φ instead of the stability of the fractional Laplacian. We extend the condition (2.1.10) to the function J and for all small scales.

Caffarelli and Silvestre established in [9] the regularity results for fully nonlinear nonlocal equations extending the method for differential equations. By this work they could obtain the result uniformly as α in (2.1.7) goes to 2. However they made the best use of the symmetry of the kernels which imply no effect of the gradient term in integro-differential operators. In our consideration we just impose the symmetry condition in the case $\delta_1 \le 1 \le \delta_2$. There are some results for the non symmetric case when the symmetric part of operators are comparable to the fractional Laplacian (see [10, 25, 32, 33]).

The authors in [23] used probabilistic method to prove Hölder regularity of harmonic function with respect to a linear operator with kernels comparable to $|h|^{-d}l(|h|)$ near zero for some regularly varying function $l(r)$ at zero of order $\alpha \in (-2,0]$. When $\alpha = 0$ they could obtain the result for the kernels comparable to $|h|^{-d}$ near zero. It was also established in [24] and [31] the Hölder regularity results with respect to the regularly varying kernels. The authors in [24] also proved analytically that bounded solutions to (non-homogeneous) linear integro-differential equations with respect to the same kernels. Although the results in [23] and [24] are more general than this work in the way that they cover the case of slowly varying functions, it is still in consideration that the kernels are symmetric and regularly varying functions satisfy assumptions in this thesis. The results in this thesis contain partially the above results in this sense. It is proved in [31] that the uniform regularity results for fully nonlinear intogro-differential operators with respect to kernels $|h|^{-d}l(|h|)$ for some regularly varying function $l(r)$ at zero of order $\alpha \in (-2, -\alpha_0)$.

2.2 Viscosity solution and elliptic operator

In this section we recall the definitions for viscosity solutions and elliptic operators.

Definition 2.2.1. A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be $C^{1,1}$ at x if there exists a vector $v \in \mathbb{R}^d$ and constants $r > 0$ and $M > 0$ such that

$$|f(y) - f(x) - v \cdot (y - x)| \leq M|y - x|^2 \quad \text{for} \quad |y - x| < r.$$

When D is an open subset of $\mathbb{R}^d$, we say that f is $C^{1,1}$ in D and write $f \in C^{1,1}(D)$ if the constant M is independent of $x \in D$.

Note that the integro-differential operator $Lu(x)$ in (2.1.5) is well-defined for $u \in C^{1,1}$ at x.

Every solution of equations in this chapter is regarded as the viscosity solution. We follow the definition of viscosity solutions described in [9].

Definition 2.2.2. Let $\mathcal{I}$ be a nonlocal operator and $g : D \to \mathbb{R}$ be a continuous function. A function $u : \mathbb{R}^d \to \mathbb{R}$, upper(resp. lower) semicontinuous in $\overline{D}$, is said to be a viscosity subsolution(resp. supersolution) to $\mathcal{I}u = g$, and we write $\mathcal{I}u \geq g$(resp. $\mathcal{I}u \leq g$) if the following holds : if we have a function w defined by

$$w(y) = \begin{cases} f(y) & \text{if } y \in N_x \\ u(y) & \text{if } y \in \mathbb{R}^d \setminus N_x \end{cases}$$

where N_x is a neighborhood of x in D and f is a $C^{1,1}$ function touching u from above (resp. below) at x, i.e. $f(x) = u(x)$ and $f(y) > u(y)$ (resp. $f(y) < u(y)$) for $y \in N_x \setminus \{x\}$, then $\mathcal{I}w(x)$ is well-defined and $\mathcal{I}w(x) \geq g(x)$ (resp. $\mathcal{I}w(x) \leq g(x)$). A viscosity solution is a function u that is both a viscosity subsolution and a viscosity supersolution.

Let $\mathcal{L}$ be a set of integro-differential operators given in (2.1.8) or (2.1.9).

We consider all of nonlocal operators which are so called elliptic with respect to the class $\mathcal{L}$.

Definition 2.2.3. An operator $\mathcal{I}$ is called elliptic with respect to $\mathcal{L}$ if the following properties hold:

(i) If u is bounded and $C^{1,1}$ at x, $\mathcal{I}u(x)$ is well-defined.

(ii) If u is C^2 in some open set D, then $\mathcal{I}u(x)$ is a continuous function in D.

(iii) If u and v are bounded and $C^{1,1}$ at x, then

$$\mathcal{M}^-_{\mathcal{L}}(u-v)(x) \le \mathcal{I}u(x) - \mathcal{I}v(x) \le \mathcal{M}^+_{\mathcal{L}}(u-v)(x).$$

If $\mathcal{I}0 = 0$ then $\mathcal{I}u = g$ in the viscosity sense implies that u satisfies both $\mathcal{M}^+_{\mathcal{L}}u(x) \ge g(x)$ and $\mathcal{M}^-_{\mathcal{L}}u(x) \le g(x)$ in the viscosity sense. We refer the reader to [31] for the stability property and comparison principle of uniformly elliptic operators with respect to the family $\mathcal{L}$.

2.3 The proof of Theorem 2.1.1

We mainly follow the method developed by Silvestre in [37]. Recall that $a_1, a_2, \delta_1, \delta_2, R_0, \lambda, \Lambda, M_0$ are fixed constants in (2.1.1), (2.1.2), (2.1.3) and (2.1.4).

We start from simple calculations for integrals of a function satisfying the weak scaling conditions.

Lemma 2.3.1. *Let φ be a function from $(0,\infty)$ to $(0,\infty)$ satisfying* (2.1.3). *Then we have the following: for $0<r<R_0$,*

$$\int_0^r \frac{s}{\varphi(s)}ds \le \frac{a_2}{2-\delta_2}\cdot\frac{r^2}{\varphi(r)}, \tag{2.3.1}$$

$$\int_r^{R_0} \frac{1}{s\varphi(s)}ds \le \frac{1}{a_1\delta_1}\cdot\frac{1}{\varphi(r)}, \tag{2.3.2}$$

$$\int_0^r \frac{1}{\varphi(s)}ds \le \frac{a_2}{1-\delta_2}\frac{r}{\varphi(r)}, \qquad \text{if } \delta_2<1, \tag{2.3.3}$$

$$\int_r^{R_0} \frac{1}{\varphi(s)}ds \le \frac{1}{a_1(\delta_1-1)}\frac{r}{\varphi(r)}, \qquad \text{if } \delta_1>1. \tag{2.3.4}$$

Proof. Since we have $s<r<R_0$, we get $\varphi(s)^{-1}\le a_2 s^{-\delta_2}r^{\delta_2}\varphi(r)^{-1}$ by the upper scaling condition in (2.1.3). Therefore we obtain

$$\int_0^r \frac{s}{\varphi(s)}ds \le a_2\frac{r^{\delta_2}}{\varphi(r)}\int_0^r s^{1-\delta_2}ds = \frac{a_2}{2-\delta_2}\cdot\frac{r^2}{\varphi(r)},$$

and

$$\int_0^r \frac{1}{\varphi(s)}ds \le a_2\frac{r^{\delta_2}}{\varphi(r)}\int_0^r s^{-\delta_2}ds = \frac{a_2}{1-\delta_2}\cdot\frac{r}{\varphi(r)}, \qquad \text{if } \delta_2<1.$$

We have proved (2.3.1) and (2.3.3). Using the lower scaling condition in (2.1.3), the proofs of inequalities (2.3.2) and (2.3.4) are similar and we omit the proof. □

Lemma 2.3.2. *Suppose $J(x,h)$ satisfies* (2.1.1), (2.1.2) *and* (2.1.3). *Then we have*

$$\sup_{x\in\mathbb{R}^d\setminus\{0\}}\int_{\mathbb{R}^d}\left(1\wedge\left(\frac{|h|}{r}\right)^2\right)J(x,h)\,dh \le C_1\varphi(r)^{-1} \tag{2.3.5}$$

for $r\le R_0$, where $C_1=\omega_d\left(\frac{a_2}{2-\delta_2}+\frac{1}{a_1\delta_1}\right)+M_0\varphi(R_0)$.

Proof. We first decompose the integral in the left side of (2.3.5) into three parts and use (2.1.1) and (2.1.2) so that

$$\begin{aligned}
&\int_{\mathbb{R}^d\setminus\{0\}} \left(1 \wedge \frac{|h|^2}{r^2}\right) J(x,h)\,dh \\
&= \int_{|h|<r} \frac{|h|^2}{r^2} \frac{\varphi(|h|)^{-1}}{|h|^d}\,dh + \int_{r\le|h|<R_0} \frac{\varphi(|h|)^{-1}}{|h|^d}\,dh + \int_{|h|\ge R_0} J(x,h)\,dh \\
&\le \omega_d \int_0^r r^{-2}\frac{s}{\varphi(s)}\,ds + \omega_d \int_r^{R_0} \frac{1}{s\varphi(s)}\,ds + M_0.
\end{aligned}$$

By (2.3.1), (2.3.2) and the monotonicity of φ, we have (2.3.5). □

When the integro-differential operator L is the fractional Laplacian, the effect on Lu of scaling to enlarge u is transferred to the the scaling of kernel. We show in the next lemma that the effect of magnifying support outside of the origin can be controlled by the growth conditions of the kernel near $h = 0$.

Lemma 2.3.3. *For any $\varepsilon > 0$ there are constants $r = r(\varepsilon) \in (0, R_0)$ and $\eta = \eta(\varepsilon) \in (0, \delta_1)$ such that for all $0 < s < r$,*

$$\sup_{x\in\mathbb{R}^d} \int_{|h|>\frac{s}{4}} \left(\left(2\frac{|4h|\wedge R_0}{s}\right)^\eta - 1\right) J(x,h)dh < \frac{\varepsilon}{\varphi(s)}. \tag{2.3.6}$$

Proof. We decompose the integral in the left side of (2.3.6) into two parts according to the absolute value of h as follows

$$\begin{aligned}
&\int_{|h|>s/4} \left(\left(2\frac{|4h|\wedge R_0}{s}\right)^\eta - 1\right) J(x,h)\,dh \\
&= \int_{|h|>R_0/4} (2^\eta (R_0/s)^\eta - 1) J(x,h)\,dh \\
&\quad + \int_{s/4<|h|\le R_0/4} \left(2^\eta\left(\frac{4|h|}{s}\right)^\eta - 1\right) J(x,h)\,dh \\
&=: I_1 + I_2.
\end{aligned}$$

From the obvious inequality $\mathbf{1}_{\{|h|\ge R_0/4\}} \le 1 \wedge (4|h|/R_0)^2 \le 16(1 \wedge (|h|/R_0))^2$, and (2.3.5) we obtain the bound for I_1 as

$$\begin{aligned} I_1 &\le 16(2^\eta (R_0/s)^\eta - 1) \int_{\mathbb{R}^d \setminus \{0\}} \left(1 \wedge \left(\frac{|h|}{R_0} \right)^2 \right) J(x,h) dh \\ &\le 2^{4+\eta} (R_0/s)^\eta C_1 \varphi(R_0)^{-1}. \end{aligned}$$

Since $\varphi(s)^{-1} \ge a_1 (R_0/s)^{\delta_1} \varphi(R_0)^{-1}$ by the lower scaling condition of (2.1.3), if η is less than δ_1 then

$$I_1 \le 2^{4+\delta_1} a_1^{-1} C_1 (s/R_0)^{\delta_1 - \eta} \varphi(s)^{-1}.$$

On the other hand, by (2.1.2)

$$\begin{aligned} I_2 &= \int_{s/4 < |h| \le R_0/4} \left(2^\eta \left(\frac{4|h|}{s} \right)^\eta - 1 \right) \frac{\varphi(|h|)^{-1}}{|h|^d} dh \\ &= \omega_d \int_s^{R_0} \left(2^\eta (t/s)^\eta - 1 \right) \frac{\varphi(t/4)^{-1}}{t} dt. \end{aligned}$$

Since $\varphi(t/4)^{-1} \le a_2 4^{\delta_2} \varphi(t)^{-1} \le a_2 4^{\delta_2} a_1^{-1} (t/s)^{-\delta_1} \varphi(s)^{-1}$ by (2.1.3), we have

$$\begin{aligned} I_2 &\le \omega_d a_2 4^{\delta_2} a_1^{-1} \varphi(s)^{-1} \int_s^{r_0} (2^\eta (t/s)^\eta - 1)(t/s)^{-\delta_1} t^{-1} dt \\ &= \omega_d a_2 4^{\delta_2} a_1^{-1} \varphi(s)^{-1} \int_1^{r_0/s} ((2t)^\eta - 1) t^{-\delta_1 - 1} dt \\ &\le \omega_d a_2 4^{\delta_2} a_1^{-1} \varphi(s)^{-1} \int_1^{\infty} ((2t)^\eta - 1) t^{-\delta_1 - 1} dt \end{aligned}$$

By the dominated convergence theorem, we can choose $\eta < \delta_1$ such that

$$\omega_d a_2 4^{\delta_2} a_1^{-1} \int_1^\infty ((2t)^\eta - 1) t^{-\delta_1 - 1} dt < \varepsilon/2,$$

and then find r such that $2^{4+\delta_1} a_1^{-1} C_1 (r/r_0)^{\delta_1 - \eta} < \varepsilon/2$. $\square$

Define test functions

$$\beta(t) = (1 - t^2)_+^2, \quad t \ge 0, \quad \text{and} \quad b_{z,r}(x) := \beta(|x - z|/r), \quad x, z \in \mathbb{R}^d, r > 0.$$

In the following lemma we prove that the minimal operator applied to the test function $b_{z,r}$ is bounded by $c\varphi(r)^{-1}$ for all $r < R_0$. In [37] Silvestre obtained this result for the operators like fractional Laplacian for every $r > 0$ using the stability.

Lemma 2.3.4. *For any $0 < r \le R_0$ and $x, z \in \mathbb{R}^d$, if $\delta_1 \in (1,2)$ or $\delta_2 \in (0,1)$ then we have*

$$\left|\mathcal{M}_{\mathcal{L}}^{-} b_{z,r}(x)\right| \le C_2 \varphi(r)^{-1}$$

where the constant C_2 is depending only on $d, a_1, a_2, \delta_1, \delta_2, \Lambda, M_0$ and $\varphi(R_0)$. In the case $\delta_1 \le 1 \le \delta_2$ we have the same bound for $\mathcal{M}_{\mathcal{L}_{sym}}^{-} b_{z,r}(x)$ with $12d\Lambda C_1$ in place of C_2.

Proof. We consider the following three integrals;

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^d \setminus \{0\}} \left(b_{z,r}(x+h) - b_{z,r}(x) - \nabla b_{z,r}(x) \cdot h \mathbf{1}_{\{|h|<r\}}\right) K(x,h) dh, \\
I_2 &= \int_{|h|<r} \left(\nabla b_{z,r}(x) \cdot h\right) K(x,h) dh, \\
I_3 &= \int_{r \le |h| < R_0} \left(\nabla b_{z,r}(x) \cdot h\right) K(x,h) dh.
\end{aligned}$$

Since $Lb_{z,r}(x) = L_K b_{z,r}(x) = I_1 + I_2 \mathbf{1}_{\{\delta_2 < 1\}} - I_3 \mathbf{1}_{\{\delta_2 \ge 1\}}$, it is enough to estimate I_1, I_2 and I_3.

First, from the definition of $b_{z,r}$ we have that for $|h| < r$

$$\begin{aligned}
&|b_{z,r}(x+h) - b_{z,r}(x) - \nabla b_{z,r}(x) \cdot h \mathbf{1}_{\{|h|<r\}}| \\
\le& \sup_{w \in \mathbb{R}^d, 1 \le i,j \le d} \left| \frac{\partial^2 b_{z,r}}{\partial x_i \partial x_j}(w) \right| d|h|^2 \\
\le& \frac{d|h|^2}{r^2} \sup_{0 \le s \le 1} \left\{ |\beta''(s)| + |\beta'(s)/s| \right\} \\
\le& 12d \frac{|h|^2}{r^2},
\end{aligned}$$

and clearly the integrand in I_1 is bounded by $K(x,h)$ for $|h| \ge r$. Thus by

(2.1.4) and (2.3.5)

$$|I_1| \leq 12d\Lambda \int_{\mathbb{R}^d\setminus\{0\}} \left(1 \wedge \frac{|h|^2}{r^2}\right) J(x,h)\,dh \leq 12d\Lambda C_1 \varphi(r)^{-1}. \tag{2.3.7}$$

The gradient term in the integrand of I_2 and I_3 is bounded by $\sup_{0\leq s\leq 1} |\beta'(s)| \leq 4$. Thus, by (2.1.2) and (2.3.3), we have the bound for I_2 as

$$\begin{aligned}
|I_2| &\leq 4\Lambda \int_{|h|<r} \frac{|h|}{r} J(x,h)\,dh \\
&= 4\Lambda \int_{|h|<r} \frac{|h|}{r} \frac{\varphi(|h|)^{-1}}{|h|^d}\,dh \\
&= 4\omega_d \Lambda r^{-1} \int_0^r \varphi(s)^{-1}\,ds \\
&\leq 4\omega_d \Lambda \frac{a_2}{1-\delta_2} \varphi(r)^{-1}, \qquad \text{if } \delta_2 < 1.
\end{aligned} \tag{2.3.8}$$

By using (2.3.4) instead of (2.3.3) we have the bound for I_3 as

$$\begin{aligned}
|I_3| &\leq 4\Lambda \int_{r\leq|h|<R_0} \frac{|h|}{r} J(x,h)dh \\
&= 4\Lambda \int_{r\leq|h|<R_0} \frac{|h|}{r} \frac{\varphi(|h|)^{-1}}{|h|^d} dh \\
&= 4\omega_d \Lambda \int_r^{r_0} r^{-1} \varphi(s)^{-1} ds \\
&\leq 4\omega_d \Lambda \frac{1}{a_1(\delta_1 - 1)} \varphi(r)^{-1}, \qquad \text{if } \delta_1 > 1.
\end{aligned} \tag{2.3.9}$$

Combining (2.3.7)–(2.3.9) and then taking $\inf_{L\in\mathcal{L}}$, we get that for $r \leq R_0$ if $\delta_2 < 1$ or $\delta_1 > 1$, $\left|\mathcal{M}^-_{\mathcal{L}} b_{z,r}(x)\right| \leq C_2 \varphi(r)^{-1}$ where $C_2 = 12d\Lambda\big(C_1 + \frac{a_2\omega_d}{1-\delta_2} + \frac{\omega_d}{a_1(\delta_1-1)}\big)$.

In the case $\delta_1 \leq 1 \leq \delta_2$ we can get $Lb_{z,r}(x) = I_1$ for $L \in \mathcal{L}_{sym}$ because $I_3 = 0$ by the symmetry of $K(x,\cdot)$. Therefore we obtain the bound $|\mathcal{M}^-_{\mathcal{L}_{sym}} b_{z,r}| \leq 12d\Lambda C_1 \varphi(r)^{-1}$. □

In the proof of the following theorem we will consider a test function w, in the definition of viscosity solution, touching u from above at a maximum point.

Theorem 2.3.5. *Suppose $\delta_1 \in (1, 2)$ or $\delta_2 \in (0, 1)$. Then there exist constants $r_1 \in (0, R_0)$, $\eta_1 > 0$ and $\theta > 0$ such that if u is a function that satisfies the following assumptions for $z \in \mathbb{R}^d$, $0 < r < r_1$*

$$\begin{aligned} \mathcal{M}^+_{\mathcal{L}} u(x) &\geq -\theta\varphi(r)^{-1} && \text{for } x \in B(z, r), \\ u(x) &\leq \frac{1}{2} && \text{for } x \in B(z, r), \\ u(x) &\leq \left(2\frac{|x - z| \wedge R_0}{r}\right)^{\eta_1} - \frac{1}{2} && \text{for } x \in \mathbb{R}^d \setminus B(z, r), \\ \frac{1}{2}|B(z, r)| &< |\{x \in B(z, r) : u(x) \leq 0\}|, \end{aligned} \tag{2.3.10}$$

then $u \leq 1/2 - \gamma$ in $B(z, r/2)$ where $\gamma = \theta(\beta(1/2) - \beta(3/4)) \in (0, 1 - 2^{-\eta_1})$ depending on r_1 and η_1.

If we suppose $\delta_1 \leq 1 \leq \delta_2$ then the above assertion holds for $\mathcal{L}_{sym}$ instead of $\mathcal{L}$.

Proof. The proofs are the same for both the case $\delta_1 \in (1, 2)$ or $\delta_2 \in (0, 1)$ and the case $\delta_1 \leq 1 \leq \delta_2$. Thus we give the proof for the first case only.

We first have to choose $r_1 \in (0, R_0/2)$ and $\eta_1 > 0$. By using Lemma 2.3.3 with $\varepsilon = \frac{\omega_d\lambda}{\Lambda 2^{d+5+\delta_2} a_2 d}$ there exist $r_1 \in (0, R_0)$ and $\eta_1 > 0$ such that for any $r < r_1$

$$\begin{aligned} &\int_{|h| \geq \frac{r}{4}} \left(\left(2\frac{|4h| \wedge R_0}{r}\right)^{\eta_1} - 1\right) J(x_1, h)\, dh \\ &\qquad\qquad \leq \frac{\omega_d\lambda}{\Lambda 2^{d+5+\delta_2} a_2 d}\varphi(r)^{-1} \end{aligned} \tag{2.3.11}$$

Let θ be a small positive constant which will be chosen later depending on r_1 and η_1. Define $\gamma = \theta(\beta(1/2) - \beta(3/4))$.

Suppose there is a point $x_0 \in B(z, r/2)$ such that $u(x_0) > 1/2 - \gamma = 1/2 - \theta\beta(1/2) + \theta\beta(3/4)$. Then we have

$$u(x_0) + \theta b_{z,r}(x_0) \geq u(x_0) + \theta\beta(1/2) > 1/2 + \theta\beta(3/4) \geq u(x) + \theta b_{z,r}(x)$$

for $x \in B(z,r) \setminus B(z,3r/4)$. This means that the supremum of $u + \theta b_{z,r}$ in $B(z,r)$ is greater than $1/2$ and is taken at an interior point x_1 of $B(z,3r/4)$. Since $u + \theta b_{z,r}$ has a maximum at x_1, we have a test function w touching $u + \theta b_{z,r}$ from above at x_1. For $\tilde{\varepsilon} > 0$ and $0 < s < r/4$ define a function w by

$$w(y) = \begin{cases} u(x_1) + \theta b_{z,r}(x_1) + \tilde{\varepsilon}|y - x_1|^2 & \text{if } |y - x_1| < s, \\ u(y) + \theta b_{z,r}(y) & \text{otherwise.} \end{cases}$$

Now we evaluate $\mathcal{M}^+_{\mathcal{L}} w(x_1)$. On the one hand, the fact that $w - \theta b_{z,r}$ is a test function touching u from above at x_1 and the ellipticity of $\mathcal{M}^+_{\mathcal{L}}$ imply

$$\begin{aligned} \mathcal{M}^+_{\mathcal{L}} w(x_1) &\geq \mathcal{M}^+_{\mathcal{L}}(w - \theta b_{z,r})(x_1) + \theta \mathcal{M}^-_{\mathcal{L}} b_{z,r}(x_1) \\ &\geq -\theta \varphi(r)^{-1} + \theta \mathcal{M}^-_{\mathcal{L}} b_{z,r}(x_1). \end{aligned} \tag{2.3.12}$$

On the other hand, from $\nabla w(x_1) = 0$ we can divide $Lw(x_1)$ into two parts as follows

$$\begin{aligned} Lw(x_1) &= \int_{|x_1+h-z| \geq r} \left(w(x_1+h) - w(x_1) \right) K(x_1,h)\, dh \\ &\quad + \int_{|x_1+h-z| < r} \left(w(x_1+h) - w(x_1) \right) K(x_1,h)\, dh \\ &=: I_1 + I_2. \end{aligned} \tag{2.3.13}$$

Since the support of $b_{z,r}$ is in $B(z,r)$ and $w(x_1) > 1/2$, by (2.3.10) and (2.1.4)

$$\begin{aligned} I_1 &= \int_{|x_1+h-z| \geq r} (u(x_1+h) - w(x_1)) K(x_1,h)\, dh \\ &\leq \int_{|x_1+h-z| \geq r} \left(\left(2 \frac{|x_1+h-z| \wedge R_0}{r} \right)^{\eta_1} - \frac{1}{2} - \frac{1}{2} \right) \Lambda J(x_1,h)\, dh. \end{aligned}$$

Since $|h| \geq |x_1 + h - z| - |z - x_1| > r/4$ and $|x_1 - z| \leq 3|h|$ if $|x_1 - z| < 3r/4$ and $|x_1 + h - z| \geq r$, we have

$$I_1 \leq \Lambda \int_{|h| \geq \frac{r}{4}} \left(\left(2 \frac{|4h| \wedge R_0}{r} \right)^{\eta_1} - 1 \right) J(x_1,h)\, dh. \tag{2.3.14}$$

To estimate I_2 we decompose the region $\{h : |x_1 + h - z| < r\}$ in the integral into $\{h : |x_1 + h - z| < r\} \cap B(0, s)$ and $\{h : |x_1 + h - z| < r\} \cap B(0, s)^c$ and control the integrand as

$$\begin{aligned}
I_2 &= \int_{\{h:|x_1+h-z|<r\}\cap B(0,s)} (w(x_1 + h) - w(x_1))K(x_1, h)dh \\
&\quad + \int_{\{h:|x_1+h-z|<r\}\cap B(0,s)^c} (w(x_1 + h) - w(x_1))K(x_1, h)dh \\
&\le \int_{B(0,s)} \tilde{\varepsilon}|h|^2 \Lambda J(x_1, h)dh \\
&\quad + \int_{B(z-x_1,r)\cap B(0,s)^c} (u(x_1 + h) + \theta b_{z,r}(x_1 + h) - u(x_1) - \theta b_{z,r}(x_1)) \\
&\qquad\qquad \times K(x_1, h)dh.
\end{aligned}$$

The first term is bounded by $\tilde{\varepsilon}\Lambda R_0^2 C_1 \varphi(R_0)^{-1}$ for $s < R_0$ by (2.3.5). The integrand in the second term is nonpositive so that the value of the integral will be greater if we restrict the region in the integral to the set $A_s := \{h \in B(0, s)^c : |x_1+h-z| < r \text{ and } u(x_1+h) \le 0\}$. If s approaches 0 then $|A_s|$ goes to $|\{h \in B(z - x_1, r) : u(x_1 + h) \le 0\}| = |\{y \in B(z, r) : u(y) \le 0\}|$ which is greater than $|B(z, r)|/2$. So we can find small $s > 0$ satisfying $A_s \subset B(0, 2r)$ and $|A_s| > |B(z, r)|/2$. Thus we have the following estimate for $\theta < 1/4$

$$\begin{aligned}
I_2 &\le \tilde{\varepsilon}\Lambda R_0^2 C_1 \varphi(R_0)^{-1} + (\theta - \frac{1}{2})\lambda \int_{\substack{|x_1+h-z|<r, |h|\ge s \\ u(x_1+h)\le 0}} J(x_1, h)dh \\
&\le \tilde{\varepsilon}\Lambda R_0^2 C_1 \varphi(r)^{-1} - \frac{\lambda}{4} \inf_{\substack{A\subset B(0,2r), \\ |A|>|B(z,r)|/2}} \int_A J(x_1, h)dh\,.
\end{aligned}$$

For the last term above we observe that, by (2.1.2), the monotonicity of φ and (2.1.3), we have for $r < R_0/2$ and $A \subset B(0, 2r)$ with $|A| > |B(z, r)|/2$,

$$\begin{aligned}
\frac{\lambda}{4}\int_A J(x_1, h)dh &= \frac{\lambda}{4}\int_A \frac{\varphi(|h|)^{-1}}{|h|^d}dh \ge \frac{\lambda}{4}\int_A \frac{\varphi(2r)^{-1}}{2^d r^d}dh \\
&\ge \frac{\lambda}{2^{d+2}} a_2^{-1} 2^{-\delta_2} \varphi(r)^{-1} r^{-d}|A| \ge C_3\varphi(r)^{-1}
\end{aligned}$$

where $C_3 = \dfrac{\omega_d \lambda}{2^{d+3+\delta_2} a_2 d}$. Thus we have $I_2 \le -C_3 \varphi(r)^{-1}/2$ for $\theta < 1/4$ and $\tilde{\varepsilon} < C_3/(2\Lambda R_0^2 C_1)$. Hence from (2.3.13), (2.3.14) and this, we obtain the following

$$\mathcal{M}_{\mathcal{L}}^{+} w(x_1) \le \Lambda \int_{|h| \ge \frac{r}{4}} \left(\left(2 \frac{|4h| \wedge R_0}{r} \right)^{\eta_1} - 1 \right) J(x_1, h)\, dh - \frac{C_3}{2} \varphi(r)^{-1}$$

for $\theta < 1/4$ and $r < R_0/2$. Therefore Lemma 2.3.4, (2.1.3), (2.3.12) and (2.3.11) yields that for any $r < r_1$

$$-\theta(C_2 + 1)\varphi(r)^{-1} \le \theta \mathcal{M}_{\mathcal{L}}^{-} b_{z,r}(x_1) - \theta\varphi(r)^{-1} \le \mathcal{M}_{\mathcal{L}}^{+} w(x_1) \le -\frac{C_3}{4}\varphi(r)^{-1}.$$

We now choose $\theta = \frac{C_3}{8(C_2+1)} \wedge \frac{1-2^{-\eta_1}}{2(\beta(1/2)-\beta(3/4))}$ so that it yields a contradiction. □

Remark 2.3.6. We can prove the above Theorem 2.3.5 for $r^d \delta$ for some constant $\delta > 0$ instead of $|B(z,r)|/2$.

When one deals with fractional Laplacian $\Delta^{\alpha/2}$, for example, the equation $\Delta^{\alpha/2} u = 0$ in a ball $B(0,r)$, one may assume $r = 1$ in the equation $Lu = 0$ in a ball $B(0,r)$ by the scaling invariant property of the equation, i.e. $\Delta^{\alpha/2}\tilde{u} = 0$ in a ball $B(0,1)$ where $\tilde{u}(x) = u(rx)$. But in our case, we don't have such scaling property so we prove Theorem 2.1.1 directly without using any scaling.

Proof of Theorem 2.1.1. Let r_1, η_1, θ and γ be the constants obtained in Theorem 2.3.5. Without loss of generality we assume that $z_0 = 0$. By normalization we may assume $\sup_{x \in \mathbb{R}^d} |u(x)| = 1/2$ and $C_0 \le a_1 \theta \varphi(r)^{-1}$. Let $\alpha := -\log_2(1-\gamma)$, which is less than η_1 and x_0 be a point in $B(0, r/2)$ and s be the minimum of $r/2$ and $r_1/2$.

We will show by induction that

$$\mathrm{Osc}_{B(x_0, 2^{-k}s)} u \le (1-\gamma)^k \qquad \text{for all } k = 0, 1, 2, \cdots. \tag{2.3.15}$$

First, the case $k = 0$ is true obviously. Suppose $\mathrm{Osc}_{B(x_0, 2^{-k}s)} u \le (1-\gamma)^k$ for some nonnegative integer $k \geq 0$. Define $v(x) = (1-\gamma)^{-k}(u(x) - a_k)$ where $a_k = \min_{B(x_0,2^{-k}s)} u + (1-\gamma)^k/2$. We have two cases

$$\text{(i) } |\{x \in B(x_0, 2^{-k}s) : v(x) \leq 0\}| \geq |B(x_0, 2^{-k}s)|/2;$$
$$\text{(ii) } |\{x \in B(x_0, 2^{-k}s) : v(x) \geq 0\}| \geq |B(x_0, 2^{-k}s)|/2.$$

Without loss of generality we assume (i) holds since we may apply the same argument on $-v$ for the case (ii).

Let us check the other conditions in Theorem 2.3.5. By (2.1.3) we have

$$a_1\theta\varphi(r)^{-1} \leq a_1\theta a_1^{-1}2^{-k\delta_1}(s/r)^{\delta_1}\varphi(2^{-k-1}s)^{-1} \leq 2^{-k\delta_1}\theta\varphi(2^{-k}s)^{-1}.$$

It follows from $\gamma \in (0, 1-2^{-\delta_1})$ and $\eta < \delta_1$ that

$$\begin{aligned} \mathcal{M}_{\mathcal{L}}^+ v(x) &= (1-\gamma)^{-k}\mathcal{M}_{\mathcal{L}}^+ u(x) \\ &\geq -(1-\gamma)^{-k}2^{-k\delta_1}\theta\varphi(2^{-k}s)^{-1} \\ &\geq -\theta\varphi(2^{-k}s)^{-1} \end{aligned}$$

for $x \in B(x_0, 2^{-k}s)$ and

$$\begin{aligned} v(x) &\leq (1-\gamma)^{-k}(\max_{B(x_0,2^{-k}s)} u - a_k) \\ &\leq (1-\gamma)^{-k}(\mathrm{Osc}_{B(x_0,2^{-k}s)} u - (1-\gamma)^k/2) \\ &\leq 1/2. \end{aligned}$$

We now check the third condition in Theorem 2.3.5. When $2^{-k+j}s \leq |x-x_0| < 2^{-k+j+1}s$, $j = 0, \cdots, k-1$, we have

$$\begin{aligned} v(x) &\leq (1-\gamma)^{-k}\left(\max_{B(x_0,2^{-k+j+1}s)} u + \left(-\min_{B(x_0,2^{-k+j+1}s)} u + \min_{B(x_0,2^{-k}s)} u\right) - a_k\right) \\ &\leq (1-\gamma)^{-k}((1-\gamma)^{k-j-1} - (1-\gamma)^k/2) \\ &\leq (1-\gamma)^{-\log_2(2|x-x_0|/(2^{-k}s))} - \frac{1}{2} \\ &\leq \left(\frac{2|x-x_0|}{2^{-k}s}\right)^{\eta_1} - \frac{1}{2}. \end{aligned}$$

When $|x - x_0| \geq s$, we simply have

$$v(x) \leq (1-\gamma)^{-k} - \frac{1}{2} \leq \left(2\frac{R_0}{2^{-k}s}\right)^{\eta_1} - \frac{1}{2}.$$

Thus

$$v(x) \leq \left(2\frac{|x-x_0| \wedge R_0}{2^{-k}s}\right)^{\eta_1} - \frac{1}{2} \quad \text{for} \quad |x-x_0| \geq 2^{-k}s.$$

We have checked that all conditions in Theorem 2.3.5 holds. Therefore we obtain $v(x) \leq 1/2 - \gamma$ for $|x - x_0| \leq 2^{-k-1}s$, that is,

$$\mathrm{Osc}_{B(x_0,2^{-k-1}s)}u = (1-\gamma)^k \mathrm{Osc}_{B(x_0,2^{-k-1}s)}v \leq (1-\gamma)^{k+1}.$$

We have proved (2.3.15), which implies that

$$|u(x) - u(x_0)| \leq (2/s)^\alpha \, |x-x_0|^\alpha \leq C(r \wedge R_0)^{-\alpha}|x-x_0|^\alpha \quad \text{for all } x \in \mathbb{R}^d$$

where $C = (4R_0/r_1)^\alpha$. □

2.4 Example: Isotropic unimodal Lévy process

Let $(X_t, t \geq 0)$ be a pure-jump isotropic Lévy process in $\mathbb{R}^d$. Its characteristic function is

$$\mathbb{E}[\exp(i\xi \cdot X_t)] = e^{-t\Phi(\xi)}$$

where $\xi \to \Phi(\xi)$ is called the characteristic exponent of X. Since X is isotropic, Φ is a radial function and we use itself for its radial part for the convenience, i.e. $\Phi(\xi) = \Phi(|\xi|)$. It is well known that Φ has the representation

$$\Phi(|\xi|) = \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot x))\nu(dx).$$

The measure ν is called the Lévy measure of X and satisfies $\int(1 \wedge |x|^2)\nu(dx) < \infty$. $(X_t, t \geq 0)$ is called isotropic unimodal Lévy process if the transition probability $\mathbb{P}(X_t \in dx)$ has non-increasing density $p_t(x)$ with respect to

Lebesgue measure. It is well known that $(X_t, t \geq 0)$ is an isotropic unimodal Lévy process if and only if the Lévy measure $\nu(dx)$ of X has non-increasing density, say, $\nu(x)$ (see [40]). Note that $p_t(x)/t$ converges vaguely to $\nu(x)$.

If we denote $\sup_{s\leq t}\Phi(s)$ by $\Phi^*(t)$, by [8, Proposition 2] and [19, Proposition 1], Φ is almost increasing;

$$\Phi^*(t) \leq \pi^2\Phi(t) \quad \text{for all } t > 0. \tag{2.4.1}$$

The following upper bound holds for ν without any extra condition (see [8, Corollary 6] and [26, Theorem 2.2]).

Theorem 2.4.1. *For an isotropic unimodal Lévy process X in $\mathbb{R}^d$, there is $C = C(d)$ such that*

$$\nu(x) \leq C\frac{\Phi^*(|x|^{-1})}{|x|^d}, \quad x \in \mathbb{R}^d \setminus \{0\}. \tag{2.4.2}$$

To obtain the estimates of density $p_t(x)$ and $\nu(x)$ we need assumptions on scaling of Φ near infinity (see [8, Section 3], [30, Section 2] and [41, (2.7) and (2.20)]): there exist constants $a_1, a_2, r_0 > 0$ and $\delta_1, \delta_2 \in (0, 2)$ such that

$$a_1 s^{\delta_1} \leq \frac{\Phi(st)}{\Phi(t)} \leq a_2 s^{\delta_2} \qquad \text{for all } s \geq 1 \text{ and } t \geq 1/r_0. \tag{\textbf{H}}$$

Recently, in [8] Bogdan, Grzywny and Ryznar obtained an interesting equivalence on the upper and lower bounds of the densities. We state a partial result relevant to our setting.

Theorem 2.4.2 ([8, Theorem 26])**.** *For an isotropic unimodal Lévy process X in $\mathbb{R}^d$, the following are equivalent:*

(i) (**H**) *holds for the characteristic function Φ of X.*

(ii) The transition density $p_t(x)$ of X has following lower bound; for some $r_0 \in (0, \infty)$ and a constant c,

$$p_t(x) \geq c\frac{t\Phi^*(|x|^{-1})}{|x|^d}, \quad 0 < |x| < r_0, 0 < t\Phi^*(|x|^{-1}) < 1.$$

(iii) The Lévy density $\nu(x)$ of X has the following lower bound; for some $r_0 \in (0,\infty)$ and a constant c,

$$\nu(x) \geq c\frac{\Phi^*(|x|^{-1})}{|x|^d}, \quad 0 < |x| < r_0.$$

From (2.4.1), (2.4.2) and Theorem 2.4.2, we conclude that our result cover isotropic unimodal Lévy process satisfying ($\mathbf{H}$) with $\varphi(r) = \Phi(r^{-1})^{-1}$. Recently the Harnack inequality and the Hölder estimate for harmonic functions with respect to isotropic unimodal Lévy process was proved in [19].

A typical example of isotropic unimodal Lévy process is a subordinate Brownian motion. If the characteristic exponent $\Phi(r)$ is of the form $\phi(r^2)$ for some Bernstein function

$$\phi(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda t})\mu(dt),$$

where $b \geq 0$ and μ is a measure on $(0,\infty)$ satisfying $\int_{(0,\infty)}(1 \wedge t)\mu(dt) < \infty$, then the associated process $(X_t, t \geq 0)$ is a subordinate Brownian motion with Lévy density

$$\nu(x) = \int_0^\infty (4\pi t)^{-d/2} e^{-|x|^2/(4t)}\mu(dt)$$

and the infinitesimal generator of X is $\phi(\Delta) := -\phi(-\Delta)$.

One can find an extensive list of explicit Bernstein functions satisfying our assumption in [36]. Here are a few of them.

(1) $\phi(\lambda) = \lambda^{\alpha/2}$, $\alpha \in (0,2)$ (symmetric α-stable process);

(2) $\phi(\lambda) = (\lambda + m^{2/\alpha})^{\alpha/2} - m$, $\alpha \in (0,2)$ and $m > 0$ (relativistic α-stable process);

(3) $\phi(\lambda) = \lambda^{\alpha/2} + \lambda^{\beta/2}$, $0 < \beta < \alpha < 2$ (mixed symmetric α- and β-stable processes);

(4) $\phi(\lambda) = \lambda^{\alpha/2}(\log(1+\lambda))^p$, $\alpha \in (0,2)$, $p \in [-\alpha/2, (2-\alpha)/2]$.

Chapter 3

Schauder estimates in generalized Hölder spaces

Schauder estimates are a central tool in the study of classical solutions to differential equations with Hölder continuous coefficients. In short the idea of this approach is to view these equations on a small scale as perturbations to equations with constant coefficients. This approach allows to use ideas from potential theory when treating equations with variable coefficients. An exposition of this method can be found in any serious textbook on partial differential equations.

The aim of this chapter is to prove Schauder estimates for a finer scale of function spaces and of operators at the same time. Let us explain this idea step by step. The Hölder space $C^\alpha(\mathbb{R}^d)$ is characterized by the number $\alpha \in (0,1)$ which appears in the bound of the modulus of continuity:

$$v \in C^\alpha(\mathbb{R}^d) \quad \Leftrightarrow \quad \sup_{x\in\mathbb{R}^d} |v(x)| + \sup_{x,y\in\mathbb{R}^d} \frac{|v(x)-v(y)|}{|x-y|^\alpha} < +\infty$$

We will study estimates in more general space $C^\psi(\mathbb{R}^d)$ where a function $\psi : (0,1] \to (0,\infty)$ is used to replace $|x-y|^\alpha$ in the above expression by $\psi(|x-y|)$. In this sense, we study a much finer scale of function spaces. This scale is of particular interest when studying mapping properties of integrodifferential operators because for them such scales turn out to be natural. Note that generalized Hölder spaces have been studied in different contexts and for very long. We mention several articles in Section 3.2 when we define and discuss these spaces.

Schauder estimates for integrodifferential operators are proved in [2, 15, 35, 20] in different contexts. Our approach is inspired by the straightforward approach in [2]. We also make use of recent developments in potential theory obtained in [26] when studying the translation invariant case. The new contribution of this work is twofold. On the one hand, we allow the integration kernels to have general singularities at $h = 0$. On the other hand, we study the resulting a priori estimates in a much finer scale of function spaces. Although these developments could be approached separately, the main new finding of this work is that they naturally belong together. This phenomenon does not exist in the study of differential equations of second order.

3.1 Setting and main results

Let us first discuss the function spaces $C^\psi(\mathbb{R}^d)$. We assume $\lim_{r\to 0+}\psi(r) = 0$. In order to describe the order of differentiability induced by a particular function ψ, we need to introduce two indices. We define indices M_ψ and m_ψ by

$$M_\psi = \inf\left\{\alpha\in\mathbb{R} | r\to \tfrac{\psi(r)}{r^\alpha} \text{ is almost decreasing in } (0,1]\right\}, \tag{3.1.1}$$
$$m_\psi = \sup\left\{\alpha\in\mathbb{R} | r\to \tfrac{\psi(r)}{r^\alpha} \text{ is almost increasing in } (0,1]\right\}. \tag{3.1.2}$$

See the definition for the almost monotonicity in Section 3.2. Note that if ψ is a regularly varying function of order $\alpha\in(0,1)$ at zero like $\psi(r) = r^\alpha$ or $\psi(r) = r^\alpha|\ln(\frac{2}{r})|$ we find $M_\psi = m_\psi = \alpha$. We denote the closed interval $[m_\psi, M_\psi]$ by I_ψ. This interval describes the range of the functional order of differentiability induced by ψ. For example, the condition $I_\psi\subset(0,1)$ implies that $C^{M_\psi}(\mathbb{R}^d)\subset C^\psi(\mathbb{R}^d)\subset C^{m_\psi}(\mathbb{R}^d)$. On the other hand, cases $I_\psi\cap\mathbb{N}\neq\emptyset$ lead to well-known technical difficulties which we want to avoid. See Section 3.2 for a more detailed discussion of the spaces $C^\psi(\mathbb{R}^d)$ including appropriate norms.

Our ultimate goal is to study integrodifferential operators which are not translation invariant, i.e., which have state dependent kernels. As it is usually done in the theory of Schauder estimates, we first study the translation invariant case, i.e., we study integrodifferential operators with constant coefficients. Since, in this case, our assumptions imply that these operators satisfy the maximum principle and generate Lévy processes, we can employ techniques from potential theory.

Let us define the integrodifferential operators under consideration. Let $\varphi : (0, \infty) \to (0, \infty)$ be a smooth function with $\varphi(1) = 1$. We assume that the function ϕ defined by $\phi(r) = \varphi(r^{-1/2})^{-1}$ is a Bernstein function, i.e., satisfies $(-1)^n \phi^{(n)}(r) \le 0$ for every $n \in \mathbb{N}$. Furthermore, we assume the scaling condition

$$a_1 \lambda^{\delta_1} \phi(r) \le \phi(\lambda r) \le a_2 \lambda^{\delta_2} \phi(r) \qquad (\lambda \ge 1, r \in (0, \infty)) \tag{3.1.3}$$

or, equivalently,

$$a_1 \lambda^{2\delta_1} \varphi(r) \le \varphi(\lambda r) \le a_2 \lambda^{2\delta_2} \varphi(r) \qquad (\lambda \ge 1, r \in (0, \infty)) \tag{3.1.4}$$

for some constants $0 < \delta_1 \le \delta_2 < 1$, $a_1 \in (0, 1]$, and $a_2 \in [1, \infty)$. Typical examples are given by $\varphi(s) = s^\alpha$ or $\varphi(s) = s^\alpha \ln(1+s^\beta)$ for $\alpha, \beta, \alpha+\beta \in (0, 2)$. In particular, we point out $I_\varphi \subset [2\delta_1, 2\delta_2]$.

Let a_0 be a measurable function on $\mathbb{R}^d \setminus \{0\}$ into $[\Lambda_1, \Lambda_2]$ for some positive constants Λ_1, Λ_2. In the case $M_\varphi \in [1, 2)$ we define L_0 by

$$L_0 u(x) = \int_{\mathbb{R}^d} (u(x+h) - u(x) - \nabla u(x) \cdot h \mathbf{1}_{\{|h| \le 1\}}) \frac{a_0(h)}{|h|^d \varphi(|h|)} \, \mathrm{d}h, \tag{3.1.5}$$

and in the case $M_\varphi \in (0, 1)$ by

$$L_0 u(x) = \int_{\mathbb{R}^d} (u(x+h) - u(x)) \frac{a_0(h)}{|h|^d \varphi(|h|)} \, \mathrm{d}h \tag{3.1.6}$$

for all continuous functions $u : \mathbb{R}^d \to \mathbb{R}$ for which the integral is well defined for every point $x \in \mathbb{R}^d$. Note that its domain $\mathcal{D}(L_0)$ includes functions $u \in C^2(\mathbb{R}^d)$ which we assume to be bounded. The class of operators L_0 is significantly larger than those of fractional Laplace operators. The main difference is that the order of differentiability of L_0 is represented by the function φ and cannot be represented by a single number. Note that φ may be chosen as not regularly varying at zero. As we will see, our scale of function spaces C^{ψ} is well suited to formulate mapping properties of such operators.

Our Schauder estimate for translation invariant operators reads as follows.

Theorem 3.1.1. *Let φ and ψ be functions described above. Suppose $I_{\psi} \subset (0,1)$ and $I_{\varphi\psi} \subset (0,1) \cup (1,2) \cup (2,3)$. There exists C_1 such that for $u \in C^{\varphi\psi}(\mathbb{R}^d) \cap C_{\infty}(\mathbb{R}^d)$ and $f \in C^{\psi}(\mathbb{R}^d)$ satisfying $L_0 u = f$, the following estimate holds:*

$$\|u\|_{C^{\varphi\psi}} \leq C_1(\|u\|_{C^0} + \|f\|_{C^{\psi}}).$$

We prove this result in Section 3.3 using a semigroup approach. In our proofs we benefit from ideas in [2] and [26].

Once Theorem 3.1.1 is established, we can use a perturbation argument to treat integrodifferential operators with variable coefficients $a : \mathbb{R}^d \times \mathbb{R}^d \to [\Lambda_1, \Lambda_2]$. Let L be defined by

$$Lu(x) = \int_{\mathbb{R}^d} \big(u(x+h) - u(x) - \nabla u(x) \cdot h \mathbf{1}_{\{|h| \leq 1\}}\big) \frac{a(x,h)}{|h|^d \varphi(|h|)} \, \mathrm{d}h \tag{3.1.7}$$

when $M_{\varphi} \in [1,2)$ and by

$$Lu(x) = \int_{\mathbb{R}^d} (u(x+h) - u(x)) \frac{a(x,h)}{|h|^d \varphi(|h|)} \, \mathrm{d}h \tag{3.1.8}$$

when $M_{\varphi} \in (0,1)$ for all continuous functions $u : \mathbb{R}^d \to \mathbb{R}$ for which the integral is well defined for every point $x \in \mathbb{R}^d$. Note that this domain equals $\mathcal{D}(L_0)$. The coefficient function $a : \mathbb{R}^d \times \mathbb{R}^d \to [\Lambda_1, \Lambda_2]$ is assumed to satisfy

$$\sup_{x \in \mathbb{R}^d} \sup_{|h|>0} |a(x+z,h) - a(x,h)| \leq \Lambda_3 \psi(|z|) \qquad (|z| \leq 1) \tag{3.1.9}$$

for some positive constant $\Lambda_3 \geq 1$. This condition requires the function $x \mapsto a(x,h)$ to be ψ-continuous uniformly in h.

We have already mentioned that the definition of Hölder and Hölder-Zygmund spaces is delicate when the order of differentiability is an integer. In order to formulate our main result we need to exclude this case. We assume further:

$$[m_{\varphi\psi}, M_{\varphi\psi}] \cap \mathbb{N} = \emptyset \,, \qquad M_\varphi \vee M_\psi < m_{\varphi\psi} \,. \tag{3.1.10}$$

Let us formulate the main result of this work.

Theorem 3.1.2. *Assume that, in addition to the assumptions of Theorem 3.1.1, condition* (3.1.10) *is satisfied. In the case* $1 \in I_\varphi$*, we assume* $a(x,h) = a(x,-h)$ *for all* $x, h \in \mathbb{R}^d$*. Then there exists a positive constant* C_2 *such that for every* $f \in C^\psi(\mathbb{R}^d)$ *and every solution* $u \in C^{\varphi\psi}(\mathbb{R}^d)$ *to the equation* $Lu = f$ *the following estimate holds:*

$$\|u\|_{C^{\varphi\psi}} \le C_2\big(\|u\|_{C^0} + \|f\|_{C^\psi}\big) \,.$$

Let us make a few comments. Note that Theorem 3.1.2 implies Theorem 3.1.1. The assumption $a(x,h) = a(x,-h)$ for all $x, h \in \mathbb{R}^d$ in the case $1 \in I_\varphi$ is natural due to the appearance of the gradient term in the definition of L. Note that this assumption needs to be added in [2] in order for [2, Corollary 5.2] to be correct. The first part of (3.1.10) is natural and resembles the fact that Lipschitz function space is not equal to the space $C^1(\mathbb{R}^d)$. The other parts of (3.1.10) would vanish if we restricted ourselves to the (large) class of regularly varying functions ψ and φ.

It is important to note that the a priori estimate provided by Theorem 3.1.2 is the best possible. This follows from the mapping properties of L, which are given by the following result Theorem 3.1.3. We defer the proof of this result to Section 3.5.

Theorem 3.1.3. *Suppose that* $M_\varphi \vee M_\psi < m_\varphi + m_\psi$*. Assume* L *and* $a(\cdot,\cdot)$ *satisfy* (3.1.7) *resp.* (3.1.8)*, and* (3.1.9)*. Furthermore, if* $1 \in I_\varphi$ *we assume that* $a(x,h) = a(x,-h)$ *for every* $x, h \in \mathbb{R}^d$*. Then the operator* L *is a continuous operator from* $C^{\varphi\psi}(\mathbb{R}^d)$ *to* $C^\psi(\mathbb{R}^d)$*.*

3.2 Generalized Hölder spaces

In this section we define the function spaces $C^{\psi}(\mathbb{R}^d)$ and discuss several of their properties. Unfortunately, we are not able to use results from the literature despite an intensive search. Since generalized smoothness of functions and related function spaces have been studied for several decades, it is likely that the results of this section have been proved somewhere else. Let us mention only a very few expositions which might be valuable for the interested reader. A very early source is [1]. Several cases and results are established in [39, 18, 21]. Some more recent works include [16, 34] where many more references can be found.

We denote by $C^0(\mathbb{R}^d)$ the Banach space of real-valued, bounded, and continuous functions on $\mathbb{R}^d$ equipped with the norm $\|f\|_{C^0(\mathbb{R}^d)} = \|f\|_{C^0} = \sup_{x\in\mathbb{R}^d} |f(x)| < \infty$. For $m \in \mathbb{N}$ we denote by $C^m(\mathbb{R}^d)$ the Banach space of functions $f \in C^0(\mathbb{R}^d)$ with all derivatives $D^\gamma f \in C^0(\mathbb{R}^d)$ for $|\gamma| \leq m$. Here, we denote by $D^\gamma f$ the partial derivative $\partial_{x_1}^{\gamma_1} \cdots \partial_{x_d}^{\gamma_d} f$ and $|\gamma| = \sum_{i=1}^d \gamma_i$ for the multi index $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathbb{N}^d$. By $C(\mathbb{R}^d)$ we denote the Fréchet space of real-valued continuous functions on $\mathbb{R}^d$.

Let ψ be a positive real valued function on $(0, 1]$ with $\psi(1) = 1$ and $\lim_{r\to 0+} \psi(r) = 0$. For $j \in \mathbb{N}_0$ we define a seminorm $[f]_{C^{-j;\psi}(\mathbb{R}^d)}$ by

$$[f]_{C^{-j;\psi}(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d} \sup_{0<|h|\leq 1} \frac{|f(x+h) - f(x)|}{\psi(|h|)|h|^{-j}},$$

and a vector space of functions $C^{-j;\psi}(\mathbb{R}^d)$ by

$$C^{-j;\psi}(\mathbb{R}^d) = \left\{ f \in C(\mathbb{R}^d) \middle| [f]_{C^{-j;\psi}(\mathbb{R}^d)} < \infty \right\} .$$

We abuse the notation $[u]_{C^{0;\psi}} = [u]_{C^\psi}$ for the convenience. Following [7], for a subinterval I on $(0, \infty)$ we call a function $\psi : I \to (0, \infty)$ *almost increasing* if there is a constant $c \in (0, 1]$ such that $c\psi(r) \leq \psi(R)$ for $r, R \in I, r \leq R$. On the other hand, we call such ψ *almost decreasing* if there is $C \in [1, \infty)$ such that $\psi(R) \leq C\psi(r)$ for $r, R \in I$ and $r \leq R$. Recall the definition of the indices M_ψ and m_ψ from (3.1.1) and (3.1.2). Now we can finally define the function spaces $C^\psi(\mathbb{R}^d)$.

Definition 3.2.1. In the case of $m_\psi \in (0,1]$, let $C^\psi(\mathbb{R}^d)$ be defined by

$$C^\psi(\mathbb{R}^d) = \{f \in C(\mathbb{R}^d) | f \in C^0(\mathbb{R}^d) \text{ and } [f]_{C^{0;\psi}} < \infty\}.$$

In the case of $m_\psi \in (k, k+1]$ for some $k \in \mathbb{N}$, let $C^\psi(\mathbb{R}^d)$ be defined by

$$\begin{aligned} C^\psi(\mathbb{R}^d) = \{f \in C(\mathbb{R}^d) \,|\, & D^\gamma f \in C^0(\mathbb{R}^d) \text{ for all } 0 \le |\gamma| \le k \\ & \text{and } D^\gamma f \in C^{-k;\psi}(\mathbb{R}^d) \text{ for all } |\gamma| = k\}. \end{aligned}$$

In the case of $m_\psi \in (k, k+1]$ for some $k \in \mathbb{N}_0$, the ψ-Hölder norm $\|\cdot\|_{C^\psi}$ is defined by

$$\|f\|_{C^\psi} = \sum_{j=0}^{k} \left\|D^j f\right\|_{C^0} + [D^k f]_{C^{-k;\psi}(\mathbb{R}^d)}.$$

Here, we use the notation $D^0 f = f$ and denote the maximum of C^0-norms and $C^{-k;\psi}$-seminorms of all k-th derivatives of f by $\|D^k f\|_{C^0}$ and $[D^k f]_{C^{-k;\psi}}$ respectively. In particular, when $k = 1$, we omit the exponent for the sake of brevity.

If there is no ambiguity, then we write C^ψ instead of $C^\psi(\mathbb{R}^d)$. Let us start with some observations. Note that for $\alpha \in (0,1)$ the two seminorms

$$\begin{aligned} [f]^{(1)}_{C^\alpha} &= \sup_{x\in\mathbb{R}^d} \sup_{|h|>0} \frac{|f(x+h) - f(x)|}{|h|^\alpha} \qquad \text{and} \\ [f]^{(2)}_{C^\alpha} &= \sup_{x\in\mathbb{R}^d} \sup_{|h|>0} \frac{|f(x+h) - 2f(x) + f(x-h)|}{|h|^\alpha} \end{aligned}$$

are equivalent. We prove an analogous property in our more general function spaces. The condition $\alpha \in (0,1)$ translates to $I_\psi \subset (0,1)$ in our setting. To shorten notation, let us write first-order and second-order differences as follows:

$$\begin{aligned} \Delta_h f(x) &= f(x+h) - f(x) \qquad \text{and} \\ \Delta_h^2 f(x) &= f(x+h) - 2f(x) + f(x-h). \end{aligned}$$

For the sake of brevity we use the notation

$$[[f]]_{C^\psi} = \sup_{x\in\mathbb{R}^d} \sup_{0<|h|\le 1} \frac{|f(x+h)-2f(x)+f(x-h)|}{\psi(|h|)}.$$

Triangle inequality gives the trivial inequality

$$[[f]]_{C^\psi} \le 2[f]_{C^\psi}.$$

We will show in the following lemma that the seminorm $[f]_{C^\psi}$ is bounded above by the sum of $\|f\|_{C^0}$ and a seminorm $[[f]]_{C^\psi}$. Summing up we get the equivalence between the two norms $\|f\|_{C^\psi}$ and $\|f\|_{C^0} + [[f]]_{C^\psi}$.

Lemma 3.2.2. *Let $I_\psi \subset (0,1)$ and $f \in C^\psi$. There exists a constant $C = C(\psi)$ such that*

$$[f]_{C^\psi} \le C\left(\|f\|_{C^0} + [[f]]_{C^\psi}\right).$$

Proof. From the definition of m_ψ and M_ψ we choose constants $c_1 \in (0,1]$ and $c_2 \in [1,\infty)$ such that

$$c_1\left(\frac{R}{r}\right)^{m_\psi/2} \le \frac{\psi(R)}{\psi(r)} \le c_2\left(\frac{R}{r}\right)^{(M_\psi+1)/2} \qquad \text{for } r \le R \le 1. \tag{3.2.1}$$

Let n be an integer greater than $(2c_2)^{2/(1-M_\psi)}$. For every $0 < |h| \le 1$, we have

$$n\Delta_{h/n} f(x) = \Delta_h f(x) - \sum_{k=1}^{n-1}(n-k)\Delta^2_{h/n} f(x+\frac{k}{n}h).$$

Dividing by $\psi(|h|)$, we obtain

$$\frac{n\psi(|h|/n)}{\psi(|h|)}\frac{|\Delta_{h/n}f(x)|}{\psi(|h|/n)} \le \frac{|\Delta_h f(x)|}{\psi(|h|)} + \sum_{k=1}^{n-1}(n-k)\frac{|\Delta^2_{h/n}f(x+\frac{k}{n}h)|}{\psi(|h|/n)}\frac{\psi(|h|/n)}{\psi(|h|)}.$$

Using (3.2.1) with $R=|h|$ and $r=|h|/n$ and taking supremum over $x\in\mathbb{R}^d$

and $0 < |h| \le 1$, we get

$$c_2^{-1} n^{(1-M_\psi)/2} \sup_{x\in\mathbb{R}^d} \sup_{0<|h|<1/n} \frac{|\Delta_h f(x)|}{\psi(|h|)} \le [f]_{C^\psi} + \frac{(n-1)n^{1-m_\psi/2}}{2c_1}[[f]]_{C^\psi}.$$

Since $\psi(|h|) \ge c_2^{-1}|h|^{(M_\psi+1)/2} \ge c_2^{-1} n^{-(M_\psi+1)/2}$ for $1/n \le |h| \le 1$, we also have

$$c_2^{-1} n^{(1-M_\psi)/2} \sup_{x\in\mathbb{R}^d} \sup_{1/n\le|h|\le 1} \frac{|\Delta_h f(x)|}{\psi(|h|)} \le 2n\|f\|_{C^0}.$$

Therefore, from the choice of n we obtain

$$[f]_{C^\psi} \le 2n\|f\|_{C^0} + (2c_1)^{-1}(n-1)n^{1-m_\psi/2}[[f]]_{C^\psi},$$

which implies the result. □

This equivalence is allowed for the case $I_\psi \subset (1,2)$ by the following lemmas.

Lemma 3.2.3. *If $I_\psi \subset (1,2)$, then for small $\varepsilon > 0$ there exists a constant $C = C(d,\psi,\varepsilon) > 0$ such that*

$$\|Df\|_{C^0} \le C\|f\|_{C^0} + \varepsilon[Df]_{C^{-1;\psi}}. \tag{3.2.2}$$

If $I_\psi \subset (2,3)$, then for small $\varepsilon > 0$ there exists a constant $C = C(d,\psi,\varepsilon) > 0$ such that

$$\|Df\|_{C^0} + \|D^2 f\|_{C^0} \le C\|f\|_{C^0} + \varepsilon[D^2 f]_{C^{-2;\psi}}. \tag{3.2.3}$$

Proof. First consider the case $I_\psi \subset (1,2)$. Fix $1 \le i \le d$ and x be a point in $\mathbb{R}^d$. Let $c_1 \in (0,1]$ such that

$$c_1 \left(\frac{R}{r}\right)^{(m_\psi+1)/2} \le \frac{\psi(R)}{\psi(r)}, \quad \text{for } r \le R \le 1.$$

The case when $[D_i f]_{C^{-1;\psi}} = 0$ is trivial, so we suppose not. Define $N = (\|f\|_{C^0}/[D_i f]_{C^{-1;\psi}})^{2/(m_\psi+1)}$ if $\|f\|_{C^0} \le [D_i f]_{C^{-1;\psi}}$ and

$N = (\|f\|_{C^0}/[D_i f]_{C^{-1;\psi}})^{2/(M_\psi+2)}$ otherwise. We may only consider the first case because the proof for the other case is the same. By the mean value theorem, there exists x' on the line segment between x and $x+Ne_i$ such that

$$|D_i f(x')| = \frac{|f(x+Ne_i) - f(x)|}{N} \leq \frac{2\|f\|_{C^0}}{N}.$$

Thus

$$\begin{aligned} |D_i f(x)| &\leq |D_i f(x')| + |D_i f(x') - D_i f(x)| \\ &\leq \frac{2\|f\|_{C^0}}{N} + c_1^{-1}[D_i f]_{C^{-1;\psi}}\psi(N)N^{-1}. \end{aligned}$$

With the fact $\psi(N) \leq c_1^{-1} N^{(m_\psi+1)/2}$ and the choice of N,

$$|D_i f(x)| \leq (2 + c_1^{-2})\|f\|_{C^0}^{1-2/(m_\psi+1)}[D_i f]_{C^{-1;\psi}}^{2/(m_\psi+1)}. \tag{3.2.4}$$

Taking the supremum over $x \in \mathbb{R}^d$ and then applying the inequality

$$r^\theta s^{1-\theta} \leq r + s, \quad r, s > 0, \theta \in (0,1),$$

we obtain

$$\|D_i f\|_{C^0} \leq (2 + c_1^{-2})(\|f\|_{C^0} + [D_i f]_{C^{-1;\psi}}).$$

By the scaling argument we get (3.2.2).

Now, we assume $I_\psi \subset (2,3)$. Let c_2 be a constant such that

$$c_2 \left(\frac{R}{r}\right)^{(m_\psi+2)/2} \leq \frac{\psi(R)}{\psi(r)}, \quad \text{for } r \leq R \leq 1.$$

By the above argument used to obtain (3.2.4), we have

$$\|D_i f\|_{C^0} \leq 3\|f\|_{C^0}^{1/2}\|D_{ii} f\|_{C^0}^{1/2}$$

Thus it suffices to show the second result for the left hand side replaced by

$\|D^2 f\|_{C^0}$. The same argument for (3.2.4) we get

$$\begin{aligned}|D_{ij}f(x)| &\leq (2+c_2^{-2})\|D_j f\|_{C^0}^{1-2/m_\psi}[D_{ij}f]_{C^{-2;\psi}}^{2/m_\psi} \\ &\leq 3(2+c_2^{-2})\|f\|_{C^0}^{1/2-1/m_\psi}\|D^2 f\|_{C^0}^{1/2-1/m_\psi}[D_{ij}f]_{C^{-2;\psi}}^{2/m_\psi},\end{aligned}$$

which implies

$$\begin{aligned}\|D^2 f\|_{C^0} &\leq c_3\|f\|_{C^0}^{(m_\psi-2)/(m_\psi+2)}[D^2 f]_{C^{-2;\psi}}^{4/(m_\psi+2)} \\ &\leq c_3(\|f\|_{C^0}+[D^2 f]_{C^{-2;\psi}}).\end{aligned}$$

By the scaling argument we get (3.2.3). □

Lemma 3.2.4. *Assume $I_\psi \subset (1,2)$ and $f \in C^\psi(\mathbb{R}^d)$. Then there exists a constant $C = C(\psi)$ such that*

$$[D_i f]_{C^{-1;\psi}} \leq C\left(\|f\|_{C^0} + [[f]]_{C^\psi}\right), \tag{3.2.5}$$

for every $i = 1, \ldots, d$.

Proof. Define $\bar{\psi}(r) := r^{-1}\psi(r)$. Then $I_{\bar{\psi}} \subset (0,1)$. First note that it is shown in Lemma 3.2.2 that the seminorm $[\,\cdot\,]_{C^{\bar{\psi}}} (= [\,\cdot\,]_{C^{-1;\psi}})$ is bounded by $\|\cdot\|_{C^0} + [[\,\cdot\,]]_{C^{\bar{\psi}}}$. Choose $c_1 \geq 1$ such that for every $g \in C^{-1;\psi}$

$$[g]_{C^{-1;\psi}} \leq c_1\left(\|g\|_{C^0} + [[g]]_{C^{\bar{\psi}}}\right). \tag{3.2.6}$$

Since $D_i f \in C^{-1;\psi}$ for any $f \in C^\psi$,

$$[D_i f]_{C^{-1;\psi}} \leq c_1\left(\|D_i f\|_{C^0} + [[D_i f]]_{C^{\bar{\psi}}}\right).$$

We claim that

$$[[D_i f]]_{C^{\bar{\psi}}} \leq c_2(\|D_i f\|_{C^0} + [[f]]_{C^\psi}), \quad i = 1, \ldots, d \tag{3.2.7}$$

for some constant c_2 not depending on f. If we prove the claim, then the following estimate from Lemma 3.2.3 below

$$\|D_i f\|_{C^0} \leq (2c_1(1+c_2))^{-1}[D_i f]_{C^{-1;\psi}} + c_3\|f\|_{C^0}$$

implies (3.2.5) with $C = 2c_1(1+c_2)(1+c_3)$.

In order to prove (3.2.7) we only consider the case $i = 1$. The remaining cases can be dealt with analogously. For $k, h \in \mathbb{R}^d$ we obtain

$$\begin{aligned}
&|k||\Delta_h^2(D_1 f)(x)| \\
&= \Big| |k| D_1 f(x+h) - \Delta_{|k|e_1} f(x+h) + \Delta_{|k|e_1} f(x+h) \\
&\quad - 2\big(|k| D_1 f(x) - \Delta_{|k|e_1} f(x)\big) - 2\Delta_{|k|e_1} f(x) \\
&\quad + |k| D_1 f(x-h) - \Delta_{|k|e_1} f(x-h) + \Delta_{|k|e_1} f(x-h) \Big| \\
&\leq |k| \big| D_1 f(x+h) - D_1 f(x+h+\theta_1 |k| e_1) \big| \\
&\quad + 2|k| \big| D_1 f(x) - D_1 f(x+\theta_2 |k| e_1) \big| \\
&\quad + |k| \big| D_1 f(x-h) - D_1 f(x-h+\theta_3 |k| e_1) \big| \\
&\quad + |\Delta_h^2 f(x+|k|e_1)| + |\Delta_h^2 f(x)|
\end{aligned}$$

for some real numbers $\theta_1, \theta_2, \theta_3 \in [0,1]$. Let $c_4 \in (0,1]$ be a constant such that

$$c_4 \left(\frac{R}{r}\right)^{(1+m_\psi)/2} \leq \frac{\psi(R)}{\psi(r)} \quad \text{for } r \leq R \leq 1. \tag{3.2.8}$$

If $|k| \leq |h| \leq 1$ then the sum of the first three terms is bounded by

$$4c_4^{-1} [D_1 f]_{C^{-1;\psi}} \psi(|k|), \tag{3.2.9}$$

and the sum of the last two terms is bounded by

$$2[[f]]_{C^\psi} \psi(|h|). \tag{3.2.10}$$

Using (3.2.6), (3.2.9) and (3.2.10) we have

$$|k||\Delta_h^2(D_1 f)(x)| \leq 4c_1 c_4^{-1}([[D_1 f]]_{C^{\bar\psi}} + \|D_1 f\|_{C^0})\psi(|k|) + 2[[f]]_{C^\psi}\psi(|h|)$$

for $|k| \le |h| \le 1$. Taking $k = \varepsilon h$ with $\varepsilon = (8c_1 c_4^{-2})^{2/(1-m_\psi)}$ gives us

$$\varepsilon|h||\Delta_h^2(D_1 f)(x)| \le 4c_1 c_4^{-1}([[D_1 f]]_{C^{\bar{\psi}}} + \|D_1 f\|_{C^0})\psi(\varepsilon|h|) + 2[[f]]_{C^\psi}\psi(|h|).$$

Dividing both sides by $\varepsilon\psi(|h|)$ and using (3.2.8), we obtain

$$\begin{aligned} &\frac{|\Delta_h^2(D_1 f)(x)|}{\psi(|h|)|h|^{-1}} \\ &\le 4c_1 c_4^{-2}\varepsilon^{(m_\psi-1)/2}([[D_1 f]]_{C^{\bar{\psi}}} + \|D_1 f\|_{C^0}) + 2\varepsilon^{-1}[[f]]_{C^\psi} \end{aligned} \tag{3.2.11}$$

for $|h| \le 1$. Taking supremum to (3.2.11) over $x \in \mathbb{R}^d$ and $0 < |h| \le 1$ we have an inequality

$$[[D_1 f]]_{C^{\bar{\psi}}} \le 4c_1 c_4^{-2}\varepsilon^{(m_\psi-1)/2}\left([[D_1 f]]_{C^{\bar{\psi}}} + \|D_1 f\|_{C^0}\right) + 2\varepsilon^{-1}[[f]]_{C^\psi}$$

which implies (3.2.7) with $c_2 = 1 + 4(8c_1 c_4^{-2})^{2/(m_\psi-1)}$. □

By summing up the results in Lemma 3.2.2, Lemma 3.2.3 and Lemma 3.2.4 we get the following equivalence.

Proposition 3.2.5. *Let $I_\psi \subset (0,1) \cup (1,2)$. For $f \in C^\psi$ the norm $\|f\|_{C^\psi}$ is equivalent to the norm*

$$\|f\|_{C^0} + [[f]]_{C^\psi}.$$

Proposition 3.2.6. *Assume $I_{\psi_1}, I_{\psi_2} \subset (0,1)\cup(1,2)\cup(2,3)$ and $M_{\psi_1} < m_{\psi_2}$. For any $0 < \varepsilon < 1$, there exists $C = C(d, \psi_1, \psi_2, \varepsilon) > 0$ such that*

$$\|f\|_{C^{\psi_1}} \le C\,\|f\|_{C^0} + \varepsilon\,\|f\|_{C^{\psi_2}}$$

Proof. We first consider the case $I_{\psi_2} \subset (0,1)$. Let c_1 and c_2 be the constants such that $\psi_1(|h|) \ge c_1|h|^{(2M_{\psi_1}+m_{\psi_2})/3}$ and $\psi_2(|h|) \le c_2|h|^{(M_{\psi_1}+2m_{\psi_2})/3}$. Let $h_0 = (c_1 c_2^{-1}\varepsilon)^{3/(m_{\psi_2}-M_{\psi_1})}$. If $|h| \le h_0$, then

$$\frac{|\Delta_h f(x)|}{\psi_1(|h|)} \le [f]_{C^{0;\psi_2}}\frac{\psi_2(|h|)}{\psi_1(|h|)} \le \varepsilon[f]_{C^{0;\psi_2}}.$$

If $h_0 < |h| \le 1$, then

$$\frac{|\Delta_h f(x)|}{\psi_1(|h|)} \le \frac{2}{\psi_1(|h|)} \|f\|_{C^0} \le c_3 \psi_1(h_0)^{-1} \|f\|_{C^0}.$$

Combining the above two inequalities and taking supremum, we have

$$\sup_{x \in \mathbb{R}^d} \sup_{0<|h|\le 1} \frac{|\Delta_h f(x)|}{\psi_1(|h|)} \le c_4 \|f\|_{C^0} + \varepsilon [f]_{C^{0;\psi_2}}. \tag{3.2.12}$$

Now we consider the case $I_{\psi_2} \subset (1,2)$. When $I_{\psi_1} \subset (0,1)$, it follows from (3.2.2) that

$$\|f\|_{C^{\psi_1}} \le c_5 (\|f\|_{C^0} + \|Df\|_{C^0}) \le c_6 \|f\|_{C^0} + \varepsilon [Df]_{C^{-1;\psi_2}}.$$

When $I_{\psi_1} \subset (1,2)$, it also follows from (3.2.2) that

$$\|f\|_{C^0} + \|Df\|_{C^0} \le c_8 \|f\|_{C^0} + \varepsilon [Df]_{C^{-1;\psi_2}}.$$

Since $[Df]_{C^{-1;\psi}} = [Df]_{C^{0;\bar\psi}}$, where $\bar\psi(r) = r^{-1}\psi(r)$, applying (3.2.12) to Df with $\bar\psi_1$ and $\bar\psi_2$ follows that

$$[Df]_{C^{-1;\psi_1}} \le c_9 \|f\|_{C^0} + \varepsilon [Df]_{C^{-1;\psi_2}}.$$

The remaining case $I_{\psi_2} \subset (2,3)$ is also proved by the same argument combined with (3.2.3). $\square$

The product rule of derivatives gives us the following Lemma.

Lemma 3.2.7. *Assume $I_\psi \subset (k, k+1)$ for $k \in \mathbb{N}$. Then, there exists a constant $c_1 = c_1(d,k,\psi) > 0$ such that*

$$\|fg\|_{C^\psi} \le c_1 \|f\|_{C^\psi} \|g\|_{C^\psi}.$$

Proof. By the product rule and the fact that $[D^j f]_{C^{-k;\psi}} \le c_2 \|D^{j+1} f\|_{C^0}$ for $j \le k$ we can obviously obtain the result. $\square$

3.3 The translation invariant case

The aim of this section is to prove Theorem 3.1.1. Let us recall the main assumptions from the introduction. As explained in (3.1.5), (3.1.6) we study operators of the form

$$L_0 u(x) = \int_{\mathbb{R}^d} (u(x+h) - u(x) - \nabla u(x) \cdot h \mathbf{1}_{\{|h| \leq 1\}}) \frac{a_0(h)}{|h|^d \varphi(|h|)} \, \mathrm{d}h, \quad (3.3.1)$$

where $a_0 : \mathbb{R}^d \backslash \{0\} \to [\Lambda_1, \Lambda_2]$ is a measurable function and Λ_1, Λ_2 are positive numbers. The domain of this operator L_0 contains bounded and smooth functions, e.g., $u \in C^2(\mathbb{R}^d)$. Recall that we assume that $\varphi : (0, \infty) \to (0, \infty)$ is a smooth function with $\varphi(1) = 1$ and the function ϕ defined by $\phi(r) = \varphi(r^{-1/2})^{-1}$ is a Bernstein function, i.e., satisfies $(-1)^n \phi^{(n)}(r) \leq 0$ for every $n \in \mathbb{N}$. Furthermore, we assume the scaling condition (3.1.3) or, equivalently, (3.1.4).

Our main idea is to apply methods from potential theory. Note that

$$\nu(\mathrm{d}h) = \frac{a_0(h)}{|h|^d \varphi(|h|)} \, \mathrm{d}h$$

defines a Lévy measure with respect to a centering function $\mathbf{1}_{\{|h| \leq 1\}}$. This measure ν induces a strongly continuous contraction semigroup (P_t) on the Banach space $C_\infty(\mathbb{R}^d)$ of continuous functions from $\mathbb{R}^d$ to $\mathbb{R}$ that vanish at infinity. We write C_∞ instead of $C_\infty(\mathbb{R}^d)$. In fact, (P_t) is also a semigroup on C^0 but not strongly continuous in general. Denote by $C^2_\infty = C^2_\infty(\mathbb{R}^d)$ the space of functions from C_∞ with the property that all derivatives up to order 2 are elements from C_∞. The infinitesimal generator $(A, \mathcal{D}(A))$ of the semigroup (P_t) satisfies $Au = L_0 u$ for every $u \in C^2_\infty$.

Our aim is to study the semigroup (P_t). To do this, we first consider a subordinate Brownian motion X with subordinator whose Laplace exponent is ϕ, see Section 3.3.1. If we denote by (Q_t) the semigroup of X, i.e., if

$$Q_t f(x) = \int_{\mathbb{R}^d} q_d(t, x - y) f(y) dy,$$

for $f \in C_\infty(\mathbb{R}^d)$, then its infinitesimal generator $(L_Q, \mathcal{D}(L_Q))$ acts on functions $f \in C^2_\infty(\mathbb{R}^d)$ in the following form:

$$L_Q f(x) = \int_{\mathbb{R}^d} (f(x+h) - f(x) - \nabla f(x) \cdot h \mathbf{1}_{\{|h| \le 1\}}) J(h) \, \mathrm{d}h. \tag{3.3.2}$$

The values of the so-called jumping function $J(h)$ are known to be comparable to $|h|^{-d}\phi(|h|^{-2})$. Since $\varphi(r) = \phi(r^{-2})^{-1}$, these values are also comparable to $\frac{a_0(h)}{|h|^d \varphi(|h|)}$ appearing in the definition of L_0 in (3.3.1). That is why estimates of the semigroup (Q_t) and its derivatives imply estimates of the semigroup (P_t).

3.3.1 Semigroup of subordinate Brownian motion

Let $S = (S_t, t \ge 0)$ be a subordinator that is a nonnegative valued increasing Lévy process starting at zero. It is characterized by its Laplace exponent ϕ via

$$\mathbb{E}[\exp(-\lambda S_t)] = e^{-t\phi(\lambda)}, \quad t \ge 0, \lambda > 0.$$

The Laplace exponent ϕ can be written in the form

$$\phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t}) \mu(dt),$$

where $b \ge 0$ and μ is a measure on $(0,\infty)$ satisfying $\int_{(0,\infty)} (1 \wedge t) \mu(dt) < \infty$, called the Lévy measure. Here, b and $\mu(A)$ describes the drift of S_t and the intensity of its jumps of size A. In this chapter we assume that $b = 0, \phi(1) = 1$ and $\lim_{\lambda \to \infty} \phi(\lambda) = \infty$. Thus

$$\phi(\lambda) = \int_{(0,\infty)} (1 - e^{-\lambda t}) \mu(dt).$$

Let $W = (W_t : t \ge 0)$ be the d-dimensional Brownian motion with the transition density $(4\pi t)^{-d/2} \exp(-|x|^2/(4t))$ independent of S. Define a subordinate Brownian motion $X = (X_t : t \ge 0)$ by $X_t = W_{S_t}$. The characteristic function of X is given by

$$\mathbb{E}\big[e^{i\xi\cdot X_t}\big] = e^{-t\phi(|\xi|^2)}$$

and X has the transition density

$$q_d(t,x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\xi\cdot x}e^{-t\phi(|\xi|^2)}d\xi.$$

Furthermore, if we denote the distribution of S_t by $\eta_t(dr) = \mathbb{P}(S_t \in dr)$ then $q_d(t,x)$ is the same as

$$\int_{(0,\infty)} (4\pi s)^{-d/2}\exp\left(-\frac{|x|^2}{4s}\right)\eta_t(ds).$$

Thus $q_d(t,x)$ is smooth in x. Moreover, its Lévy measure has a rotationally symmetric density $J(x) = j(|x|)$ with respect to the Lebesgue measure given by

$$j(r) = \int_0^\infty (4\pi t)^{-d/2}e^{-r^2/(4t)}\mu(dt)\,.$$

Note that J is the same function as in (3.3.2).

In order to obtain the necessary estimates on the semigroup of subordinate Brownian motion we make use of estimates on the transition density and its derivatives. In [26] the authors obtain upper bounds of spatial derivatives of $q_d(t,x)$ when ϕ has a certain scaling condition. For our purposes a weaker version than [26, Lemma 4.1] is sufficient. We formulate this result without a proof.

Theorem 3.3.1. *Suppose ϕ satisfies condition* (3.1.3). *There exists a constant $C \geq 1$ such that the following inequalities hold:*

$$q_d(t,x) \asymp \left(\phi^{-1}(t^{-1})^{d/2} \wedge \frac{t\phi(|x|^{-2})}{|x|^d}\right), \tag{3.3.3}$$

$$\sum_{|\gamma|=k} |D^\gamma q_d(t,x)| \leq C\phi^{-1}(t^{-1})^{k/2}\left(\phi^{-1}(t^{-1})^{d/2} \wedge \frac{t\phi(|x|^{-2})}{|x|^d}\right), \tag{3.3.4}$$

for every $k \in \mathbb{N}$ and for all $(t,x) \in (0,\infty)\times\mathbb{R}^d$.

Corollary 3.3.2. *Suppose ϕ satisfies condition (3.1.3) and $k \in \mathbb{N}$. There exists a constant C depending only on $k, a_1, a_2, \delta_1, \delta_2$ and d such that for every multi-index γ with $|\gamma| = k$, and every bounded function f*

$$|D^\gamma Q_t f(x)| \le C\phi^{-1}(t^{-1})^{k/2}\|f\|_{C^0}, \quad t \in (0,\infty), x \in \mathbb{R}^d .$$

Proof. Comparability of the heat kernel $q_d(t,x)$ and $\phi^{-1}(t^{-1})^{d/2} \wedge \big(t\phi(|x|^{-2})|x|^{-d}\big)$ from (3.3.3) and estimate (3.3.4) imply

$$|D^\gamma Q_t f(x)| \le \int_{\mathbb{R}^d} c_1 \phi^{-1}(t^{-1})^{k/2} q_d(t, x-y) f(y) dy \le c_1\phi^{-1}(t^{-1})^{k/2}\|f\|_{C^0}$$

for every multi-index γ with $|\gamma| = k$. □

3.3.2 Proof of Theorem 3.1.1

The aim of this subsection is to prove Theorem 3.1.1. Recall that (P_t) is the semigroup corresponding to the infinitesimal generator A. Let $C_0 \ge 1$ be the constant that ensures

$$C_0^{-1}|h|^{-d}\varphi(|h|)^{-1} \le J(h) \le C_0|h|^{-d}\varphi(|h|)^{-1} \qquad (h \in \mathbb{R}^d \setminus \{0\}) .$$

An immediate consequence is that

$$(\Lambda_2 C_0)^{-1}\, \frac{a_0(h)}{|h|^d\varphi(|h|)} \le J(h) \le \Lambda_1^{-1} C_0\, \frac{a_0(h)}{|h|^d\varphi(|h|)} \qquad (h \in \mathbb{R}^d \setminus \{0\}) .$$

The derivative estimates of (Q_t) from Corollary 3.3.2 imply estimates of (P_t) as the next result shows.

Theorem 3.3.3. *If $f \in C^0$, then $P_t f$ is $C^\infty(\mathbb{R}^d)$ for $t > 0$ and for each multi-index γ with $|\gamma| = k$, there exists $C > 0$ (depending on k) such that*

$$|D^\gamma P_t f(x)| \le C\varphi^{-1}(t)^{-k}\|f\|_{C^0}.$$

Proof. We define

$$L_1 f(x) = \int_{\mathbb{R}^d} (f(x+h) - f(x) - \nabla f(x) \cdot h \mathbf{1}_{|h|\leq 1})(\Lambda_2 C_0)^{-1} J(h)\,\mathrm{d}h$$

and $L_2 f(x) = L_0 f(x) - L_1 f(x)$ for every $f \in C_b^2(\mathbb{R}^d)$. Let Q_t^1 and Q_t^2 be the semigroups whose infinitesimal generators are L_1 and L_2 respectively. Then $P_t = Q_t^1 Q_t^2$. Since Q_t^1 is the semigroup of the deterministic time changed process considered in Theorem 3.3.1, we can apply it to Q_t^1. Using the contraction property of Q_t^2, we get

$$|D^\gamma P_t f(x)| \leq c_1 \varphi^{-1}(t)^{-k} \|Q_t^2 f\|_{C^0} \leq c_2 \varphi^{-1}(t)^{-k} \|f\|_{C^0}.$$

□

Recall that we denote the interval of scaling orders of ψ by $I_\psi = [m_\psi, M_\psi]$. We define the potential operator as

$$Rf(x) = \int_0^\infty P_t f(x) dt$$

when the function $t \mapsto P_t f(x)$ is integrable. We want to prove that R takes functions in C^ψ into functions in $C^{\varphi\psi}$, provided that both I_ψ and $I_{\varphi\psi}$ contain no integer and Rf is bounded.

Lemma 3.3.4. *Let ρ be a non-negative C^∞ symmetric function with its support in $B(0,1)$ such that $\int_{\mathbb{R}^d} \rho(x)dx = 1$, and let $\rho_\varepsilon(x) = \varepsilon^{-d}\rho(x/\varepsilon)$. Define $f_\varepsilon = f * \rho_\varepsilon$. Then for a function ψ with $I_\psi \subset (0,1)$ there exists a constant $C > 0$ such that*

$$\|f - f_\varepsilon\|_{C^0} \leq C \|f\|_{C^\psi} \psi(\varepsilon), \tag{3.3.5}$$

$$\|D^k f_\varepsilon\|_{C^0} \leq C \|f\|_{C^\psi} \psi(\varepsilon) \varepsilon^{-k}, \quad k \geq 1. \tag{3.3.6}$$

Proof. (3.3.5) follows from

$$|f(x) - f_\varepsilon(x)| \leq \|f\|_{C^\psi} \int_{\mathbb{R}^d} \psi(\varepsilon|y|)\rho(y)dy \leq c_1 \|f\|_{C^\psi} \psi(\varepsilon).$$

In the last inequality we used the fact that $r^{-m_\psi/2}\psi(r)$ is almost increasing. Using $\int_{\mathbb{R}^d} D^\gamma \rho(y)dy = 0$ for $|\gamma| \geq 1$, we can get (3.3.6) from

$$\begin{aligned}
|D^\gamma f_\varepsilon(x)| &= \left| \int_{\mathbb{R}^d} (f(x-y) - f(x)) D^\gamma \rho_\varepsilon(y)\, \mathrm{d}y \right| \\
&\leq \varepsilon^{-k} \int_{\mathbb{R}^d} |f(x-\varepsilon y) - f(x)||D^\gamma \rho(y)|\, \mathrm{d}y \\
&\leq c_2 \|f\|_{C^\psi} \frac{\psi(\varepsilon)}{\varepsilon^k} \int_{\mathbb{R}^d} |D^\gamma \rho(y)|\, \mathrm{d}y
\end{aligned}$$

for every $x \in \mathbb{R}^d$ and $|\gamma| = k$. □

Proposition 3.3.5. *Suppose $I_\psi \subset (0,1)$ and $I_{\varphi\psi} \subset (0,1) \cup (1,2)$. If $f \in C^\psi$ and $\|Rf\|_{C^0} < \infty$, then $Rf \in C^{\varphi\psi}$ and there exists C not depending on f such that*

$$\|Rf\|_{C^{\varphi\psi}} \leq C(\|f\|_{C^\psi} + \|Rf\|_{C^0}).$$

Proof. By Proposition 3.2.5 it is enough to show that $[[Rf]]_{C^{\varphi\psi}}$ is bounded by $\|f\|_{C^\psi} + \|Rf\|_{C^0}$. Since $|\Delta_h^2(Rf)(x)| \leq 4\|Rf\|_{C^0}$ for any $x \in \mathbb{R}^d$, we may assume $|h| \leq 1$. First we show

$$|\Delta_h^2(P_s f)(x)| \leq c_1 |h|^2 \|f\|_{C^\psi} \frac{\psi(\varphi^{-1}(s))}{\varphi^{-1}(s)^2}. \tag{3.3.7}$$

By the Taylor's theorem, Theorem 3.3.3, and (3.3.5),

$$\begin{aligned}
|\Delta_h^2(P_s(f - f_\varepsilon))(x)| &\leq |h|^2 \|D^2 P_s(f - f_\varepsilon)\|_{C^0} \\
&\leq \frac{c_2}{\varphi^{-1}(s)^2} |h|^2 \|f - f_\varepsilon\|_{C^0} \\
&\leq \frac{c_3}{\varphi^{-1}(s)^2} |h|^2 \psi(\varepsilon) \|f\|_{C^\psi}.
\end{aligned} \tag{3.3.8}$$

Since Δ_h^2 and P_s commute and P_s is a contraction semigroup, (3.3.6) implies

$$|\Delta_h^2(P_s f_\varepsilon)(x)| = |P_s(\Delta_h^2 f_\varepsilon)(x)| \leq \|\Delta_h^2 f_\varepsilon\|_{C^0} \leq c_4 |h|^2 \frac{\psi(\varepsilon)}{\varepsilon^2} \|f\|_{C^\psi}. \tag{3.3.9}$$

Letting $\varepsilon = \varphi^{-1}(s)$ and combining with (3.3.8), we obtain (3.3.7).

Let σ be a small number such that $M_\varphi + M_\psi + 2\sigma < 2$. Using (3.3.7) and noting $M_\varphi + M_\psi < 2$, we have that for $|h| < 1$,

$$\begin{aligned}
\int_{\varphi(|h|)}^{1} |\Delta_h^2(P_s f)(x)| ds &\le c_4 |h|^2 \|f\|_{C^\psi} \int_{\varphi(|h|)}^{1} \frac{\psi(\varphi^{-1}(s))}{\varphi^{-1}(s)^2} ds \\
&\le c_5 |h|^2 \|f\|_{C^\psi} \frac{\psi(|h|)}{|h|^2} \int_{\varphi(|h|)}^{1} \left(\frac{|h|}{\varphi^{-1}(s)}\right)^{2-M_\psi-\sigma} ds \\
&\le c_6 \|f\|_{C^\psi} \psi(|h|) \int_{\varphi(|h|)}^{1} \left(\frac{\varphi(|h|)}{s}\right)^{(2-M_\psi-\sigma)/(M_\varphi+\sigma)} ds \\
&\le c_7 \varphi(|h|)\psi(|h|)\|f\|_{C^\psi} \qquad (3.3.10)
\end{aligned}$$

Also the Hölder continuity of f gives

$$|\Delta_h^2(P_s f)(x)| = |P_s(\Delta_h^2 f)(x)| \le \|\Delta_h^2 f\|_{C^0} \le c_8 \|f\|_{C^\psi} \psi(|h|),$$

and thus

$$\int_0^{\varphi(|h|)} |\Delta_h^2(P_s f)(x)| ds \le c_8 \|f\|_{C^\psi} \varphi(|h|)\psi(|h|). \qquad (3.3.11)$$

Since $|\Delta_h^2(P_s f)(x)| \le \|D^2 P_s f\|_{C^0} |h|^2 \le c_9 \varphi^{-1}(s)^{-2} \|f\|_{C^0}$ and $\int_1^\infty \varphi^{-1}(s)^{-2} ds < \infty$,

$$\begin{aligned}
\int_1^\infty \left|\Delta_h^2(P_s f)(x)\right| ds &\le c_9 |h|^2 \|f\|_{C^0} \int_1^\infty \varphi^{-1}(s)^{-2} ds \\
&\le c_{10} \varphi(|h|)\psi(|h|)\|f\|_{C^0}. \qquad (3.3.12)
\end{aligned}$$

Adding (3.3.10), (3.3.11), and (3.3.12) we conclude

$$|\Delta_h^2(Rf)(x)| \le c_{11} \|f\|_{C^\psi} \varphi(|h|)\psi(|h|).$$

□

Finally we consider the case when $I_{\varphi\psi} \subset (2,3)$.

Proposition 3.3.6. *Suppose $I_\psi \subset (0,1)$, and $I_{\varphi\psi} \subset (2,3)$. If $f \in C^\psi$ and $\|Rf\|_{C^0} < \infty$, then $Rf \in C^{\varphi\psi}$ and there exists C not depending on f such* *that*

$$\|Rf\|_{C^{\varphi\psi}} \leq C(\|f\|_{C^\psi} + \|Rf\|_{C^0}).$$

Proof. Since $I_\psi = [m_\psi, M_\psi] \subset (0,1)$ and $I_{\varphi\psi} = [m_\varphi + m_\psi, M_\varphi + M_\psi] \subset (2,3)$, necessarily $m_\varphi > 1$. Define $\bar\varphi(r) = r^{-1}\varphi(r)$ then $I_{\bar\varphi\psi} \subset (1,2)$. In view of Proposition 3.2.5 it suffices to show

$$[[DRf]]_{C^{\bar\varphi\psi}} \leq c_1\|f\|_{C^\psi}. \tag{3.3.13}$$

Fix i and let $Q_s f(x) = D_i(P_s f)(x)$. From Theorem 3.3.3 we have

$$\|D^2 Q_s f\|_{C^0} \leq c_2 \varphi^{-1}(s)^{-3}\|f\|_{C^0}.$$

Note that Q_s is translation invariant. As the proof of Proposition 3.3.5 we assume $|h| \leq 1$. Analogously to (3.3.8) and (3.3.9),

$$|\Delta_h^2(Q_s(f - f_\varepsilon))(x)| \leq \frac{c_4}{\varphi^{-1}(s)^3}|h|^2\psi(\varepsilon)\|f\|_{C^\psi}$$

and

$$|\Delta_h^2(Q_s f_\varepsilon)(x)| \leq |h|^2\|D^2 Q_s f_\varepsilon\|_{C^0} \leq c_5|h|^2\|D^3 f_\varepsilon\|_{C^0} \leq c_6|h|^2\frac{\psi(\varepsilon)}{\varepsilon^3}\|f\|_{C^\psi}.$$

Taking $\varepsilon = \varphi^{-1}(s)$ we obtain

$$|\Delta_h^2(Q_s f)(x)| \leq c_6|h|^2\frac{\psi(\varphi^{-1}(s))}{\varphi^{-1}(s)^3}\|f\|_{C^\psi}.$$

Integrating the right hand side with respect to s over the interval $[\varphi(|h|), 1)$ yields $c_7\bar\varphi(|h|)\psi(|h|)\|f\|_{C^\psi}$.

On the other hand,

$$|\Delta_h^2(Q_s f)(x)| \leq c_8\varphi^{-1}(s)^{-1}\|\Delta_h^2 f\|_{C^0} \leq 2c_8\varphi^{-1}(s)^{-1}\psi(|h|)\|f\|_{C^\psi}.$$

and integrating this bound over $(0, \varphi(|h|))$ yields $c_9\bar\varphi(|h|)\psi(|h|)\|f\|_{C^\psi}$; we use $m_\varphi > 1$ here. Since $|\Delta_h^2 Q_s f(x)| \leq c_{10}|h|^2\|f\|_{C^0}\varphi^{-1}(s)^{-3}$ and

$\int_1^\infty \varphi^{-1}(s)^{-3} ds < \infty$,

$$\int_1^\infty |\Delta_h^2(Q_s f)(x)| ds \leq c_{10} |h|^2 \|f\|_{C^0} \int_1^\infty \varphi^{-1}(s)^{-3} ds \leq c_{11} \bar{\varphi}(|h|) \psi(|h|) \|f\|_{C^0}.$$

Therefore

$$|\Delta_h^2(D_i Rf)(x)| \leq c_{12} \bar{\varphi}(|h|) \psi(|h|) \|f\|_{C^\psi},$$

which yields (3.3.13). □

Now, the proof of Theorem 3.1.1 follows by the preceding propositions.
Proof of Theorem 3.1.1. According to the assumptions of Theorem 3.1.1 the function u is an element of $C^{\varphi\psi}(\mathbb{R}^d) \cap C_\infty(\mathbb{R}^d)$. Without loss of generality we may assume that u belongs to $C_\infty^2(\mathbb{R}^d) \cap C^{\varphi\psi}(\mathbb{R}^d)$ and thus to the domain $\mathcal{D}(A)$ of the infinitesimal generator $(A, \mathcal{D}(A))$ of the semigroup (P_t). Because we may convolve u with a mollifier ρ_ε like in Lemma 3.3.4. Then $u_\varepsilon = u * \rho_\varepsilon$ is a smooth function vanishing at infinity and satisfies the equation $L_0 u_\varepsilon = f * \rho_\varepsilon$. We would then obtain the estimate claimed in Theorem 3.1.1 for u_ε. Since the three norms in this estimate converge for $\varepsilon \to 0$, the desired estimate for u would follow.

Recall that the infinitesimal generator $(A, \mathcal{D}(A))$ of the semigroup (P_t) satisfies $Av = L_0 v$ for every $v \in C_\infty^2(\mathbb{R}^d)$ and L_0 as in (3.3.1). Denote by $(R, \mathcal{D}(R))$ be the potential operator of (P_t), i.e.,

$$Rf = \lim_{t \to \infty} \int_0^t P_s f \, \mathrm{d}s \, .$$

Note that in general the potential operator is not identical with the zero-resolvent operator $(R_0, \mathcal{D}(R_0))$. However, the property that

$$\|P_t v\|_{C^0} \to 0 \text{ as } t \to \infty \quad \text{ for every } v \in C_\infty$$

from the translation invariance implies that $(R, \mathcal{D}(R))$ is densely defined and $R = R_0 = -A^{-1}$ [6, Proposition 11.9]. Hence $u = -Rf$ and we can apply Proposition 3.3.5 and Proposition 3.3.6 from above. □

3.4 Proof of Theorem 3.1.2

The aim of this section is to prove Theorem 3.1.2 using Theorem 3.1.1 and a well-known perturbation technique. Let us first establish some auxiliary results.

We show that (3.1.4) implies

$$\int_r^\infty \frac{\mathrm{d}s}{s\varphi(s)} \le \frac{C}{\varphi(r)} \qquad (r>0)\,, \tag{3.4.1}$$

where C is some positive constant. The second inequality in (3.1.4) with $\lambda = s/r$ implies

$$\varphi(s) \ge a_1 (s/r)^{2\delta_1}\varphi(r)\,.$$

The above observation (3.4.1) now follows from

$$\int_r^\infty \frac{\mathrm{d}s}{s\varphi(s)} \le a_1^{-1}\frac{r^{2\delta_1}}{\varphi(r)}\int_r^\infty \frac{\mathrm{d}s}{s^{1+2\delta_1}} = \frac{1}{2a_1\delta_1\varphi(r)}\,.$$

Let $B(x,r)$ denote the ball of radius r centered at x. Let $\bar\eta \in C_c^\infty(\mathbb{R}^d)$ be a cut-off function which equals 1 on $B(0,1)$, equals 0 on $B(0,2)^c$ and satisfies $\bar\eta \in [0,1]$. Let $\eta_{r,x_0}(x) = \bar\eta((x-x_0)/r)$. When there is no ambiguity we write η instead of η_{r,x_0}.

Proposition 3.4.1. *Assume $I_\psi \subset (0,1)$, $I_{\varphi\psi} \subset (0,1)\cup(1,2)\cup(2,3)$, $u \in C^{\varphi\psi}(\mathbb{R}^d)$ and $f \in C^\psi(\mathbb{R}^d)$. Suppose that for each $\varepsilon > 0$ there exist $r > 0$ and $c_1 \ge 1$ depending on ε such that*

$$\|u\eta_{r,x_0}\|_{C^{\varphi\psi}} \le c_1(\|f\|_{C^\psi} + \|u\|_{C^0}) + \varepsilon\|u\|_{C^{\varphi\psi}} \tag{3.4.2}$$

for all $x_0 \in \mathbb{R}^d$. Then there exists a constant C such that

$$\|u\|_{C^{\varphi\psi}} \le C(\|f\|_{C^\psi} + \|u\|_{C^0}).$$

Proof. First, we consider the case $I_{\varphi\psi} \subset (0,1)\cup(1,2)$. Set $\varepsilon = 1/2$ and choose r and c_1 satisfying (3.4.2). For any $x_0 \in \mathbb{R}^d$, if $|h| < r$ then $\Delta_h^2 u(x_0) =$

$\Delta_h^2(u\eta_{r,x_0})(x_0)$, and (3.4.2) yields

$$\begin{aligned}|\Delta_h^2 u(x_0)| &\le \|u\eta_{r,x_0}\|_{C^{\varphi\psi}}\varphi(|h|)\psi(|h|)\\ &\le \left(c_1\|f\|_{C^\psi} + c_1\|u\|_{C^0} + \frac{1}{2}\|u\|_{C^{\varphi\psi}}\right)\varphi(|h|)\psi(|h|)\end{aligned}$$

On the other hand, if $r \le |h| \le 1$ then the fact that $\varphi(s)\psi(s) \ge c_2$ for any $s \in [r, 1]$ yields

$$|\Delta_h^2 u(x_0)| \le 4c_2^{-1}\|u\|_{C^0}\varphi(|h|)\psi(|h|).$$

Combining the above two inequalities we obtain

$$\sup_{x\in\mathbb{R}^d}\sup_{0<|h|\le 1}\frac{|\Delta_h^2 u(x)|}{\varphi(|h|)\psi(|h|)} \le c_1\|f\|_{C^\psi} + c_1(1+4c_2^{-1})\|u\|_{C^0} + \frac{1}{2}\|u\|_{C^{\varphi\psi}}$$

Therefore we obtain

$$\|u\|_{C^{\varphi\psi}} \le c_3\left(\|f\|_{C^\psi} + \|u\|_{C^0}\right).$$

Now we consider the case when $I_{\varphi\psi} \subset (2, 3)$. Let $\bar\varphi(r) = r^{-1}\varphi(r)$. By the definition of $C^{\varphi\psi}$ and Proposition 3.2.5 it is enough to show that

$$\|Du\|_{C^0} + [[Du]]_{C^{\bar\varphi\psi}} \le c_4(\|f\|_{C^\psi} + \|u\|_{C^0}).$$

For $\varepsilon = 1/4$ choose r satisfying (3.4.2). We use the same argument above to obtain that if $|h| < r$ then

$$|\Delta_h^2(Du)(x_0)| \le \left(c_5\|f\|_{C^\psi} + c_5\|u\|_{C^0} + \frac{1}{4}\|u\|_{C^{\varphi\psi}}\right)\bar\varphi(|h|)\psi(|h|)$$

for any $x_0 \in \mathbb{R}^d$. On the other hand, if $r \le |h| \le 1$ then the fact that $\bar\varphi(s)\psi(s) \ge c_6$ for any $s \in [r, 1]$ yields

$$|\Delta_h^2(Du)(x_0)| \le 4c_6^{-1}\|Du\|_{C^0}\bar\varphi(|h|)\psi(|h|).$$

Combining above two inequalities and we get

$$\sup_{x\in\mathbb{R}^d}\sup_{0<|h|\le 1}\frac{|\Delta_h^2(Du)(x)|}{\bar{\varphi}(|h|)\psi(|h|)}\le c_7\left(\|f\|_{C^\psi}+\|u\|_{C^0}+\|Du\|_{C^0}\right)+\frac{1}{4}\|u\|_{C^{\varphi\psi}}$$

By Proposition 3.2.6 we have

$$\|Du\|_{C^0}\le c_8\|u\|_{C^0}+(4(1+c_7))^{-1}\|u\|_{C^{\varphi\psi}},$$

which implies

$$\|Du\|_{C^0}+\sup_{x\in\mathbb{R}^d}\sup_{0<|h|\le 1}\frac{|\Delta_h^2(Du)(x)|}{\bar{\varphi}(|h|)\psi(|h|)}\le c_9\left(\|f\|_{C^\psi}+\|u\|_{C^0}\right)+\frac{1}{2}\|u\|_{C^{\varphi\psi}}.$$

Therefore we obtain the desired estimate

$$\|u\|_{C^{\varphi\psi}}\le c_{10}(\|f\|_{C^\psi}+\|u\|_{C^0}).$$

□

Before proving the main theorem, we give an auxiliary inequality, which we will often apply.

Lemma 3.4.2. *Let $\Psi:(0,1]\to(0,\infty)$ be a function with $M_\varphi<m_\Psi$. There exists a constant $C>0$ such that for any $0<r\le 1$,*

$$\int_{\mathbb{R}^d}\frac{\Psi(|h|\wedge r)}{|h|^d\varphi(|h|)}\,\mathrm{d}h\le C\frac{\Psi(r)}{\varphi(r)}. \tag{3.4.3}$$

Proof. Let $\sigma=(m_\Psi-M_\varphi)/3>0$. By the definition of I_φ and I_Ψ, there exists a constant $c_1>0$ such that for $0<|h|<r$,

$$\frac{\Psi(|h|)}{\varphi(|h|)}\le c_1\frac{(|h|/r)^{m_\Psi-\sigma}\Psi(r)}{(|h|/r)^{M_\varphi+\sigma}\varphi(r)}\le c_1(|h|/r)^\sigma\frac{\Psi(r)}{\varphi(r)}.$$

It follows that

$$\int_{|h|<r}\frac{\Psi(|h|)}{|h|^d\varphi(|h|)}\,\mathrm{d}h\le c_1\frac{\Psi(r)}{\varphi(r)}\int_{|h|<r}\frac{(|h|/r)^\sigma}{|h|^d}\,\mathrm{d}h\le c_2\frac{\Psi(r)}{\varphi(r)}.$$

Combining this with (3.4.1), we get the result. □

Let H be a function defined by

$$H(x) = \int_{\mathbb{R}^d} (u(x+h) - u(x))(\eta(x+h) - \eta(x)) \frac{a(x,h)}{|h|^d \varphi(|h|)} \, \mathrm{d}h.$$

Lemma 3.4.3. *Let $\varepsilon > 0$ be a small constant. Assume that condition (3.1.10) is satisfied. If $u \in C^{\varphi\psi}$, then there exists a constant $C = C(r, \varepsilon) > 0$ such that*

$$\|H\|_{C^{\psi}} \le C\|u\|_{C^0} + \varepsilon \|u\|_{C^{\varphi\psi}}.$$

Proof. Observe that

$$\begin{aligned} &|\Delta_h u(x)| \\ &\le \begin{cases} 2\|u\|_{C^0}, & \text{if } I_{\varphi\psi} \subset (0,1), \\ 2(\|u\|_{C^0} + \|Du\|_{C^0})(|h| \wedge 1), & \text{if } I_{\varphi\psi} \subset (1,2) \cup (2,3), \end{cases} \end{aligned} \tag{3.4.4}$$

and

$$\begin{aligned} &|\Delta_h \eta(x)| \\ &\le \begin{cases} 2\|\bar{\eta}\|_{C^{\varphi\psi}} (\varphi\psi) \left(\frac{|h|}{r} \wedge 1 \right), & \text{if } I_{\varphi\psi} \subset (0,1), \\ 2(\|\bar{\eta}\|_{C^0} + \|D\bar{\eta}\|_{C^0}) \left(\frac{|h|}{r} \wedge 1 \right), & \text{if } I_{\varphi\psi} \subset (1,2) \cup (2,3). \end{cases} \end{aligned} \tag{3.4.5}$$

When $I_{\varphi\psi} \subset (0,1)$, using (3.4.4) and (3.4.5) we get

$$\begin{aligned} |H(x)| &\le \int_{\mathbb{R}^d} |\Delta_h u(x)| |\Delta_h \eta(x)| \frac{|a(x,h)|}{|h|^d \varphi(|h|)} \, \mathrm{d}h \\ &\le 4\|u\|_{C^0} \|\bar{\eta}\|_{C^{\varphi\psi}} \int_{\mathbb{R}^d} (\varphi\psi) \left(\frac{|h|}{r} \wedge 1 \right) \frac{\Lambda_2}{|h|^d \varphi(|h|)} \, \mathrm{d}h \\ &\le \frac{c_1}{\varphi(r)} \|u\|_{C^0}. \end{aligned}$$

We used (3.4.3) with $\Psi(t) = (\varphi\psi)(t/r)$ in the last inequality. When $I_{\varphi\psi} \subset$

$(1,2) \cup (2,3)$, by (3.4.4), (3.4.5), and (3.4.3) with $\Psi(t) = (t/r)^2$, we get

$$|H(x)| \le \frac{c_2}{\varphi(r)}(\|u\|_{C^0} + \|Du\|_{C^0}).$$

By Proposition 3.2.6, we can find a constant $c_3 = c_3(r, \varepsilon) > 0$ such that

$$\|H\|_{C^0} \le c_3(r,\varepsilon)\|u\|_{C^0} + \varepsilon\|u\|_{C^{\varphi\psi}},$$

whenever $I_{\varphi\psi} \subset (0,1) \cup (1,2) \cup (2,3)$.

Now, we consider $|\Delta_k H(x)|$ in order to estimate the ψ-Hölder seminorm of H. We may assume $|k| < r$ because $|\Delta_k H(x)|/\psi(|k|) \le c_4\|H\|_{C^0}/\psi(r)$ otherwise. Observe that

$$\begin{aligned}
\Delta_k H(x) &= \int_{\mathbb{R}^d} \Delta_k\Delta_h u(x)\, \Delta_h\eta(x)\, a(x+k,h)\, \frac{\mathrm{d}h}{|h|^d\varphi(|h|)} \\
&\quad + \int_{\mathbb{R}^d} \Delta_h u(x)\, \Delta_k\Delta_h\eta(x)\, a(x+k,h)\, \frac{\mathrm{d}h}{|h|^d\varphi(|h|)} \\
&\quad + \int_{\mathbb{R}^d} \Delta_h u(x)\, \Delta_h\eta(x)\, \Delta_k(a(\cdot,h))(x)\, \frac{\mathrm{d}h}{|h|^d\varphi(|h|)} \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

Thus it suffices to show that

$$|I_1| + |I_2| + |I_3| \le \psi(|k|)(c_5\|u\|_{C^0} + \varepsilon\|u\|_{C^{\varphi\psi}}) \tag{3.4.6}$$

for some constant $c_5 = c_5(r,\varepsilon) > 0$.

For $|I_1|$, it follows from the facts $u \in C^{\varphi\psi}$ and $\eta \in C^\infty$ that

$$\begin{aligned}
&|\Delta_k\Delta_h u(x)\, \Delta_h\eta(x)| \\
&\le c_6 \cdot \begin{cases}
\|u\|_{C^\psi}\psi(|k|)\left(\frac{|h|}{r} \wedge 1\right), & \text{if } I_{\varphi\psi} \subset (0,1), \\
\|u\|_{C^\sigma}|k|\,\frac{\sigma(|h|\wedge 1)}{|h|\wedge 1}\left(\frac{|h|}{r} \wedge 1\right), & \text{if } I_{\varphi\psi} \subset (1,2), \\
(\|Du\|_{C^0} + \|D^2u\|_{C^0})|k|(|h|\wedge 1)\left(\frac{|h|}{r} \wedge 1\right), & \text{if } I_{\varphi\psi} \subset (2,3),
\end{cases}
\end{aligned}$$

for some constant $c_6 > 0$, where σ is a function on $(0,1]$ defined by

$$\sigma(t) = t^{(1 \vee M_\varphi + m_\varphi + m_\psi)/2}.$$

Note that the exponent of σ is greater than both 1 and M_φ. Since r is less than one, we can apply Lemma 3.4.2 with $\Psi_1(t) = t/r$, $\Psi_2(t) = \sigma(t/r)$, and $\Psi_3(t) = (t/r)^2$ for each cases. Then we have

$$|I_1| \le \frac{c_7 \Lambda}{\varphi(r)} \cdot \begin{cases} \|u\|_{C^\psi} \psi(|k|), & \text{if } I_{\varphi\psi} \subset (0,1), \\ \|u\|_{C^\sigma} |k|, & \text{if } I_{\varphi\psi} \subset (1,2), \\ (\|Du\|_{C^0} + \|D^2 u\|_{C^0}) |k|, & \text{if } I_{\varphi\psi} \subset (2,3), \end{cases}$$

for some constant $c_7 > 0$. By the fact $|k| \le c_8 \psi(|k|)$ and Proposition 3.2.6, there exists a constant $c_9 = c_9(r, \varepsilon) > 0$ such that

$$|I_1| \le \psi(|k|)(c_9(r,\varepsilon)\|u\|_{C^0} + \varepsilon \|u\|_{C^{\varphi\psi}})$$

whenever $I_{\varphi\psi} \subset (0,1) \cup (1,2) \cup (2,3)$.

For $|I_2| + |I_3|$, we first consider the case $I_{\varphi\psi} \subset (0,1)$. Since $|\Delta_k \Delta_h \eta(x)| \le c_{10} \left(\frac{|h|}{r} \wedge 1 \right) \psi \left(\frac{|k|}{r} \right)$, we have

$$\begin{aligned} &|\Delta_h u(x) \Delta_k \Delta_h \eta(x) a(x+k, h)| \\ &\le 2 c_{10} \Lambda_2 \|u\|_{C^0} \left(\frac{|h|}{r} \wedge 1 \right) \psi \left(\frac{|k|}{r} \right). \end{aligned} \tag{3.4.7}$$

Similarly, the facts that $|\Delta_h \eta(x)| \le c_{11} \left(\frac{|h|}{r} \wedge 1 \right)$ and (3.1.9) implies

$$\begin{aligned} &|\Delta_h u(x) \Delta_h \eta(x) \Delta_k (a(\cdot, h))(x)| \\ &\le 2 c_{11} \Lambda_3 \|u\|_{C^0} \left(\frac{|h|}{r} \wedge 1 \right) \psi(|k|). \end{aligned} \tag{3.4.8}$$

By integrating (3.4.7), (3.4.8), and then applying Lemma 3.4.2 with $\Psi(t) =$

t/r, we can obtain

$$|I_2| + |I_3| \le \frac{c_{12}}{\varphi(r)} \|u\|_{C^0} \psi(|k|)$$

for some constant $c_{12} > 0$. When $I_{\varphi\psi} \subset (1,2) \cup (2,3)$, we just use

$$|\Delta_h u(x)| \le 2(\|u\|_{C^0} + \|Du\|_{C^0})(|h| \wedge 1),$$

and apply Lemma 3.4.2 with $\Psi(t) = (t/r)^2$. Finally, we use Proposition 3.2.6 to obtain (3.4.6) whenever $I_{\varphi\psi} \subset (0,1) \cup (1,2) \cup (2,3)$. □

Let x_0 be a fixed point in $\mathbb{R}^d$ and $b(x,h) = a(x,h) - a(x_0,h)$. When $M_\varphi < 1$, we define $\mathcal{B}v(x)$ by

$$\mathcal{B}v(x) = \int_{\mathbb{R}^d} (v(x+h) - v(x)) \frac{b(x,h)}{|h|^d \varphi(|h|)} \,\mathrm{d}h.$$

When $M_\varphi \ge 1$, we define $\widetilde{\mathcal{B}}v(x)$ by adding a gradient term as

$$\widetilde{\mathcal{B}}v(x) = \int_{\mathbb{R}^d} (v(x+h) - v(x) - \nabla v(x) \cdot h \mathbf{1}_{\{|h| \le 1\}}) \frac{b(x,h)}{|h|^d \varphi(|h|)} \,\mathrm{d}h.$$

Lemma 3.4.4. *Let $\varepsilon > 0$ be given. Assume $M_\varphi < 1$, $I_{\varphi\psi} \subset (0,1) \cup (1,2)$, and that condition* (3.1.10) *is satisfied. Then there exists $r = r(\varepsilon) \in (0, 1/4)$ such that for every $v \in C^{\varphi\psi}$ with its support in $B(x_0, 2r)$,*

$$\|\mathcal{B}v\|_{C^\psi} \le C\|v\|_{C^0} + \varepsilon \|v\|_{C^{\varphi\psi}}$$

for $C = C(\varepsilon) > 0$

Proof. Let v be a function with its support in $B(x_0, 2r)$ for $0 < r < 1/4$. We first obtain a bound of the C^0-norm of $\mathcal{B}v$. If $x \notin B(x_0, 3r)$, then $v(x) = 0$

and $v(x+h) = 0$ for $|h| \le r$. Thus (3.4.1) yields

$$\begin{aligned} |\mathcal{B}v(x)| &= \left| \int_{|h|>r} v(x+h) \frac{b(x,h)}{|h|^d \varphi(|h|)} \,\mathrm{d}h \right| \\ &\le \|v\|_{C^0} \int_{|h|>r} \frac{2\Lambda_2 \,\mathrm{d}h}{|h|^d \varphi(|h|)} \\ &\le \frac{c_1}{\varphi(r)} \|v\|_{C^0}. \end{aligned} \tag{3.4.9}$$

Let us look at the case $x \in B(x_0, 3r)$. Note that, in this case, $|b(x,h)| \le c_2 \Lambda_3 \psi(r)$ for every $h \in \mathbb{R}^d$. Observe that

$$\begin{aligned} &|\Delta_h v(x)| \\ &\le \begin{cases} 2\|v\|_{C^\sigma} \sigma(|h| \wedge 1), & \text{if } I_{\varphi\psi} \subset (0,1), \\ 2(\|v\|_{C^0} + \|Dv\|_{C^0})(|h| \wedge 1), & \text{if } I_{\varphi\psi} \subset (1,2), \end{cases} \end{aligned} \tag{3.4.10}$$

where σ is a function on $(0,1]$ defined by

$$\sigma(t) = t^{(M_\varphi + m_\varphi + m_\psi)/2}.$$

Note that $M_\varphi < m_\sigma = (M_\varphi + m_\varphi + m_\psi)/2$. Applying Lemma 3.4.2 to its integration with $\Psi(t) = \sigma(t)$ when $I_{\varphi\psi} \subset (0,1)$, and with $\Psi(t) = t$ when $I_{\varphi\psi} \subset (1,2)$, we get

$$|\mathcal{B}v(x)| \le c_3 \psi(r) \cdot \begin{cases} \|v\|_{C^\sigma}, & \text{if } I_{\varphi\psi} \subset (0,1), \\ \|v\|_{C^0} + \|Dv\|_{C^0}, & \text{if } I_{\varphi\psi} \subset (1,2). \end{cases} \tag{3.4.11}$$

It follows from (3.4.9) and Proposition 3.2.6 that

$$\|\mathcal{B}v\|_{C^0} \le c_4 \|v\|_{C^0} + (\varepsilon/5) \|v\|_{C^{\varphi\psi}} \tag{3.4.12}$$

for some constant $c_4 = c_4(r, \varepsilon) > 0$.

In the next step we estimate the ψ-Hölder seminorm of $\mathcal{B}v$. If $r/2 < |k| \le$

1, then (3.4.9) and (3.4.11) yield

$$
\begin{aligned}
&\frac{|\Delta_k(\mathcal{B}v)(x)|}{\psi(|k|)}\\
&\le \frac{c_5\|v\|_{C^0}}{\varphi(r)\psi(r/2)} + \frac{c_5\psi(r)}{\psi(r/2)} \cdot \begin{cases} \|v\|_{C^\sigma}, & \text{if } I_{\varphi\psi} \subset (0,1), \\ (\|v\|_{C^0} + \|Dv\|_{C^0}), & \text{if } I_{\varphi\psi} \subset (1,2), \end{cases}
\end{aligned}
$$

By the fact $\psi(r) \le c_6\psi(r/2)$ and Proposition 3.2.6, the quotient $\frac{|\Delta_k \mathcal{B}(x)|}{\psi(|k|)}$ for $r/2 < |k| \le 1$ is bounded by

$$
c_7(r,\varepsilon)\|v\|_{C^0} + (\varepsilon/5)\|v\|_{C^{\varphi\psi}}. \tag{3.4.13}
$$

Now consider the case $|k| \le r/2$. First suppose $x \notin B(x_0, 3r)$. Then $v(x+k) = v(x) = 0$ and $v(x+k+h) = v(x+h) = 0$ for $|h| \le r/2$. Thus, the inequality $|\Delta_k v(x+h)| \le \|v\|_{C^\psi}\psi(|k|)$, (3.4.1) and (3.1.9) yield

$$
\begin{aligned}
&|\Delta_k(\mathcal{B}v)(x)|\\
&= \left| \int_{|h|>r/2} (v(x+k+h)b(x+k,h) - v(x+h)b(x,h)) \frac{\mathrm{d}h}{|h|^d\varphi(|h|)} \right|\\
&\le \int_{|h|>r/2} |\Delta_k v(x+h)| \frac{2\Lambda_2 \,\mathrm{d}h}{|h|^d\varphi(|h|)} + \int_{|h|>r/2} |v(x+h)| \frac{|\Delta_k(b(\cdot,h))(x)|}{|h|^d\varphi(|h|)}\,\mathrm{d}h\\
&\le 2(\Lambda_2+\Lambda_3)(\|v\|_{C^\psi} + \|v\|_{C^0})\psi(|k|) \int_{|h|>r/2} \frac{\mathrm{d}h}{|h|^d\varphi(|h|)}\\
&\le \frac{c_8}{\varphi(r/2)}\|v\|_{C^\psi}\psi(|k|)
\end{aligned}
$$

By Proposition 3.2.6 we get

$$
|\Delta_k(\mathcal{B}v)(x)| \le (c_9(r,\varepsilon)\|v\|_{C^0} + (\varepsilon/5)\|v\|_{C^{\varphi\psi}})\psi(|k|). \tag{3.4.14}
$$

Now suppose $x \in B(x_0, 3r)$. We decompose the integral into two parts as follows

$$\begin{aligned}\Delta_k(\mathcal{B}v)(x) &= \int_{\mathbb{R}^d} \Delta_k\Delta_h v(x)\frac{b(x+k,h)}{|h|^d\varphi(|h|)}\,\mathrm{d}h + \int_{\mathbb{R}^d} \Delta_h v(x)\frac{\Delta_k(b(\cdot,h))(x)}{|h|^d\varphi(|h|)}\,\mathrm{d}h \\ &=: I_4 + I_5\,.\end{aligned}$$

For I_4, we observe that $x+k \in B(x_0, 4r)$ and $|b(x+k,h)| \le c_{10}\psi(r)$ since $|k| \le r/2$. When $I_{\varphi\psi} \subset (0,1)$, by the inequality

$$|\Delta_k\Delta_h v(x)| \le 2\|v\|_{C^{\varphi\psi}}(\varphi\psi)(|h| \wedge |k|), \tag{3.4.15}$$

and (3.4.3) with $\Psi(t) = (\varphi\psi)(t)$, we obtain

$$\begin{aligned}|I_4| &\le 2c_{10}\psi(r)\|v\|_{C^{\varphi\psi}} \int_{\mathbb{R}^d} \frac{(\varphi\psi)(|h| \wedge |k|)}{|h|^d\varphi(|h|)}\,\mathrm{d}h \\ &\le c_{11}\psi(r)\|v\|_{C^{\varphi\psi}}\psi(|k|).\end{aligned}$$

If $I_{\varphi\psi} \subset (1,2)$, the following inequality is used instead of (3.4.15),

$$|\Delta_k\Delta_h v(x)| \le 2 \cdot \begin{cases} \|v\|_{C^{\varphi\psi}}|h|\frac{(\varphi\psi)(|k|)}{|k|}, & \text{if } |h| \le |k|, \\ \|v\|_{C^{\varphi\psi}}|k|\frac{(\varphi\psi)(|h|)}{|h|}, & \text{if } |k| < |h| \le 1, \\ \|v\|_{C^{\psi}}\psi(|k|), & \text{if } |h| > 1. \end{cases}$$

By the integration and Proposition 3.2.6 we obtain

$$|I_4| \le c_{12}\psi(r)(\|v\|_{C^0} + \|v\|_{C^{\varphi\psi}})\psi(|k|), \tag{3.4.16}$$

whenever $I_{\varphi\psi} \subset (0,1) \cup (1,2)$.

Since $|\Delta_k(b(\cdot,h))(x)| \le 2\Lambda\psi(|k|)$ for every $h \in \mathbb{R}^d$, (3.4.10) and the continued argument yields

$$|I_5| \le (c_{13}\|v\|_{C^0} + (\varepsilon/5)\|v\|_{C^{\varphi\psi}})\psi(|k|), \tag{3.4.17}$$

whenever $I_{\varphi\psi} \subset (0,1)\cup(1,2)$. Combining (3.4.12), (3.4.13), (3.4.14), (3.4.16) and (3.4.17), we have

$$\|\mathcal{B}v\|_{C^\psi} \le c_{14}(r,\varepsilon)\|v\|_{C^0} + c_{12}\psi(r)\|v\|_{C^{\varphi\psi}} + (4\varepsilon/5)\|v\|_{C^{\varphi\psi}}$$

Choosing r such that $c_{12}\psi(r) \le \varepsilon/5$, we can obtain the result. $\square$

Lemma 3.4.5. *Let $\varepsilon > 0$ be given. Assume $M_\varphi \ge 1$, $I_{\varphi\psi} \subset (1,2) \cup (2,3)$, and the condition (3.1.10) is satisfied. In the case $1 \in I_\varphi$ we further assume $a(x,h) = a(x,-h)$ for all $x, h \in \mathbb{R}^d$. Then there exists $r = r(\varepsilon) \in (0,1/4)$ such that for every $v \in C^{\varphi\psi}$ with its support in $B(x_0, 2r)$,*

$$\|\widetilde{\mathcal{B}}v\|_{C^\psi} \le C\|v\|_{C^0} + \varepsilon\|v\|_{C^{\varphi\psi}}$$

for $C = C(\varepsilon) > 0$.

Proof. For the C^0-norm of $\widetilde{\mathcal{B}}v$, the only difference from the proof of Lemma 3.4.4 is that we replace (3.4.10) with

$$\begin{aligned} &|\Delta_h v(x) - \nabla v(x)\cdot h\mathbf{1}_{\{|h|\le 1\}}| \\ &\le \begin{cases} 2\|v\|_{C^{\tilde\sigma}}\tilde\sigma(|h|\wedge 1), & \text{if } I_{\varphi\psi} \subset (1,2), \\ 2(\|v\|_{C^0} + \|D^2 v\|_{C^0})(|h|^2 \wedge 1), & \text{if } I_{\varphi\psi} \subset (2,3), \end{cases} \end{aligned} \tag{3.4.18}$$

where $\tilde\sigma$ is a function on $(0,1]$ defined by

$$\tilde\sigma(t) = t^{(M_\varphi + m_\varphi + v_\psi)/2}.$$

Then we have from (3.4.2) and Proposition 3.2.6 that

$$\|\mathcal{B}v\|_{C^0} \le c_1\|v\|_{C^0} + (\varepsilon/5)\|v\|_{C^{\varphi\psi}}$$

for some constant $c_1 = c_1(r,\varepsilon) > 0$.

For the ψ-Hölder seminorm of $\widetilde{\mathcal{B}}v$, firstly we can use the inequality (3.4.13) without any change. We can also use (3.4.14) as it is, because $\nabla v(x+k) = \nabla v(x) = 0$ for $|k| \le r/2$ and $x \notin B(x_0, 3r)$. Now suppose $x \in B(x_0, 3r)$.

This implies $|b(x+k,h)| \le c_2\psi(r)$ for any $h \in \mathbb{R}^d$ and $|k| \le r/2$. We have

$$\begin{aligned}
&\Delta_k(\widetilde{\mathcal{B}}v)(x)\\
&= \int_{\mathbb{R}^d} (\Delta_k\Delta_h v(x) - \Delta_k(\nabla v)(x)\cdot h\mathbf{1}_{\{|h|\le 1\}})b(x+k,h)\frac{\mathrm{d}h}{|h|^d\varphi(|h|)}\\
&\quad + \int_{\mathbb{R}^d} (\Delta_h v(x) - \nabla v(x)\cdot h\mathbf{1}_{\{|h|\le 1\}})\Delta_k(b(\cdot,h))(x)\frac{\mathrm{d}h}{|h|^d\varphi(|h|)}\\
&= I_6 + I_7.
\end{aligned}$$

If $I_{\varphi\psi} \subset (1,2)$, then we have

$$\begin{aligned}
&|\Delta_k\Delta_h v(x) - \Delta_k(\nabla v)(x)\cdot h\mathbf{1}_{\{|h|\le 1\}}|\\
&\le 2\cdot\begin{cases} \|v\|_{C^{\varphi\psi}}(\varphi\psi)(|h|), & \text{if } |h| \le |k|,\\ \|v\|_{C^{\varphi\psi}}|h|\frac{(\varphi\psi)(|k|)}{|k|}, & \text{if } |k| < |h| \le 1,\\ \|v\|_{C^{\psi}}\psi(|k|), & \text{if } |h| > 1. \end{cases}
\end{aligned} \tag{3.4.19}$$

Taking the integration, we get

$$\begin{aligned}
|I_6| \le& \int_{|h|\le|k|} 2\|v\|_{C^{\varphi\psi}}(\varphi\psi)(|h|)\frac{c_2\psi(r)}{|h|^d\varphi(|h|)}\,\mathrm{d}h\\
&+ \int_{|k|<|h|\le 1} 2\|v\|_{C^{\varphi\psi}}\frac{|h|(\varphi\psi)(|k|)}{|k|}\cdot\frac{c_2\psi(r)}{|h|^d\varphi(|h|)}\,\mathrm{d}h\\
&+ \int_{|h|>1} 2\|v\|_{C^{\psi}}\psi(|k|)\frac{c_2\psi(r)}{|h|^d\varphi(|h|)}\,\mathrm{d}h\\
\le& c_3\psi(r)(\|v\|_{C^{\psi}} + \|v\|_{C^{\varphi\psi}})\psi(|k|).
\end{aligned}$$

For the case $1 \in I_\psi$, the additional assumption that $a(x,h) = a(x,-h)$ for $x,h \in \mathbb{R}^d$ allows for the integrand over the region $\{h \in \mathbb{R}^d : |k| < |h| \le 1\}$ to be reduced to

$$|\Delta_k\Delta_h v(x)| \le 2\|v\|_{C^{\varphi\psi}}\frac{|k|(\varphi\psi)(|h|)}{|h|}.$$

If $I_{\varphi\psi} \subset (2,3)$, then we just replace (3.4.19) with

$$\begin{aligned}&|\Delta_k\Delta_h v(x) - \Delta_k(\nabla v)(x)\cdot h\mathbf{1}_{\{|h|\le 1\}}|\\ &\le 2\cdot\begin{cases}\|v\|_{C^{\varphi\psi}}\frac{|h|^2(\varphi\psi)(|k|)}{|k|^2}, & \text{if } |h|\le|k|,\\ \|v\|_{C^{\varphi\psi}}\frac{|k|(\varphi\psi)(|h|)}{|h|}, & \text{if } |k|<|h|\le 1,\\ \|v\|_{C^{\psi}}\psi(|k|), & \text{if } |h|>1.\end{cases}\end{aligned}$$

Hence applying Proposition 3.2.6 to $\|\cdot\|_{C^\psi}$ and $\|\cdot\|_{C^{\varphi\psi}}$ implies

$$|I_6| \le c_4\psi(r)(\|v\|_{C^0} + \|v\|_{C^{\varphi\psi}})\psi(|k|),$$

whenever $I_{\varphi\psi} \subset (1,2)\cup(2,3)$.

For I_7, we use $|\Delta_k(b(\cdot,h))(x)| \le 2\Lambda_3\psi(|k|)$ and (3.4.18) to get

$$|I_7| \le (c_5\|v\|_{C^0} + (\varepsilon/5)\|v\|_{C^{\varphi\psi}})\psi(|k|),$$

for some constant $c_5 = c_5(r,\varepsilon) > 0$. By summing up the above result and choosing r such that $c_4\psi(r) \le \varepsilon/5$, we get the result. □

Remark 3.4.6. In the proof of Lemma 3.4.4 and Lemma 3.4.5, the fact that $b(x,h) = a(x,h) - a(x_0,h)$ is only used to estimate I_4 and I_6. It allows us to find r depending on ε. If we fix ε as a number, then we can replace $b(x,h)$ with $a(x,h)$ and obtain

$$\|L\eta\|_{C^\psi} \le C. \tag{3.4.20}$$

for a constant C depending on r.

We are finally ready to prove our main result.

Proof. [Proof of Theorem 3.1.2] By Proposition 3.4.1 it is enough to show that for any $\varepsilon > 0$ there exist positive constants r and c_1 such that for all $x_0 \in \mathbb{R}^d$

$$\|u\eta_{r,x_0}\|_{C^{\varphi\psi}} \le c_1\|f\|_{C^\psi} + c_1\|u\|_{C^0} + \varepsilon\|u\|_{C^{\varphi\psi}}\,. \tag{3.4.21}$$

First we consider the case $M_\varphi < 1$. We define a freezing operator

$$L_0 u(x) = \int_{\mathbb{R}^d} \Delta_h u(x) \frac{a(x_0,h)}{|h|^d \varphi(|h|)} \,\mathrm{d}h\,,$$

and $\mathcal{B} = L - L_0$. Let $v(x) = u(x)\eta_{r,x_0}(x)$. As we mentioned at the beginning of this section we write η instead of η_{r,x_0}. Observe that the identity

$$\Delta_h(u\eta)(x) = \eta(x)\Delta_h u(x) + u(x)\Delta_h \eta(x) + \Delta_h u(x)\Delta_h \eta(x),$$

yields

$$Lv(x) = \eta(x)Lu(x) + u(x)L\eta(x) + H(x),$$

where H is defined by

$$H(x) = \int_{\mathbb{R}^d} \Delta_h u(x)\Delta_h \eta(x) \frac{a(x,h)}{|h|^d \varphi(|h|)} \,\mathrm{d}h.$$

Then we have

$$L_0 v(x) = \eta(x)f(x) + u(x)L\eta(x) + H(x) - \mathcal{B}v(x).$$

Theorem 3.1.1 now implies that with some constant $c_2 \geq 1$

$$\|v\|_{C^{\varphi\psi}} \leq c_2(\|\eta f + uL\eta + H - \mathcal{B}v\|_{C^\psi} + \|v\|_{C^0}).$$

Choose $r > 0$ from Lemma 3.4.4 such that

$$\|\mathcal{B}v\|_{C^\psi} \leq c_3\|v\|_{C^0} + (2c_2)^{-1}\|v\|_{C^{\varphi\psi}}.$$

It is obvious that $\|\eta f\|_{C^\psi} \leq c_4\|f\|_{C^\psi}$. It follows from (3.4.20) and Proposition 3.2.6 that

$$\|uL\eta\|_{C^\psi} \leq c_6\|u\|_{C^0} + (4c_2)^{-1}\varepsilon\|u\|_{C^{\varphi\psi}}.$$

Finally, (3.4.3) implies that for a given $\varepsilon > 0$, there exists a constant $c_7 =$

$c_7(r,\varepsilon) > 0$ such that

$$\|H\|_{C^\psi} \le c_7\|u\|_{C^0} + (4c_2)^{-1}\varepsilon\|u\|_{C^{\varphi\psi}}.$$

Hence (3.4.21) holds true. For the case $M_\varphi \ge 1$, we denote a freezing operator by

$$\widetilde{L}_0 u(x) = \int_{\mathbb{R}^d} (\Delta_h u(x) - \nabla u(x)\cdot h\mathbf{1}_{\{|h|\le 1\}})\frac{a(x_0,h)}{|h|^d\varphi(|h|)}\,\mathrm{d}h,$$

and $\widetilde{\mathcal{B}} = L - \widetilde{L}_0$. Then we have

$$\widetilde{L}_0 v(x) = \eta(x)f(x) + u(x)L\eta(x) + H(x) - \widetilde{\mathcal{B}}v(x).$$

Using Lemma 3.4.5 instead of Lemma 3.4.4, we get the result from the same argument above. □

3.5 Continuity of L

In this section we give a proof of Theorem 3.1.3.

Proof of Theorem 3.1.3. We first consider the case $M_\varphi < 1$ that L is defined by

$$Lu(x) = \int_{\mathbb{R}^d} (u(x+h) - u(x))\frac{a(x,h)}{|h|^d\varphi(|h|)}\,\mathrm{d}h.$$

We observe that

$$|\Delta_h u(x)| \le 2\|u\|_{C^{\varphi\psi}} \begin{cases} (\varphi\psi)(|h|\wedge 1), & I_{\varphi\psi} \subset (0,1), \\ |h|\wedge 1, & I_{\varphi\psi} \subset (1,2). \end{cases}$$

It follows from Lemma 3.4.2 that $|Lu(x)| \le c_1\|u\|_{C^{\varphi\psi}}$ for all $x \in \mathbb{R}^d$. For the seminorm of Lu, we know that

$$\Delta_k(Lu)(x) = \int_{\mathbb{R}^d} \Delta_k\Delta_h u(x)\frac{a(x+k,h)}{|h|^d\varphi(|h|)}\,\mathrm{d}h + \int_{\mathbb{R}^d} \Delta_h u(x)\frac{\Delta_k(a(\cdot,h))(x)}{|h|^d\varphi(|h|)}\,\mathrm{d}h.$$

For the case $I_{\varphi\psi} \subset (0,1)$, we have

$$|\Delta_k \Delta_h u(x)| \leq 2\|u\|_{C^{\varphi\psi}} (\varphi\psi)(|h| \wedge |k|),$$

and, if $I_{\varphi\psi} \subset (1,2)$, then

$$|\Delta_k \Delta_h u(x)| \leq 4\|u\|_{C^{\varphi\psi}} \cdot \begin{cases} \frac{(\varphi\psi)(|k|)}{|k|}|h|, & |h| \leq |k|, \\ \frac{(\varphi\psi)(|h|)}{|h|}|k|, & |k| < |h| \leq 1, \\ \psi(|k|), & |h| > 1. \end{cases}$$

We used Proposition 3.2.6 with the assumption in the last case. By integrating the above terms with respect to h and using assumptions (3.1.9) and Lemma 3.4.2, we get

$$|\Delta_k (Lu)(x)| \leq c_2 \|u\|_{C^{\varphi\psi}} \psi(|k|)$$

for every $x \in \mathbb{R}^d$ and $|k| \leq 1$, which implies $[Lu]_{C^{0;\psi}} \leq c_2 \|u\|_{C^{\varphi\psi}}$.

For the case $M_\varphi \geq 1$, the operator that we consider is given by

$$Lu(x) = \int_{\mathbb{R}^d} (u(x+h) - u(x) - \nabla u(x) \cdot h \mathbf{1}_{\{|h| \leq 1\}}) \frac{a(x,h)}{|h|^d \varphi(|h|)} \, \mathrm{d}h.$$

When $1 \notin I_\varphi$, similarly to the above case, we only need to observe that

$$\begin{aligned} &|\Delta_h u(x) - \nabla u(x) \cdot h \mathbf{1}_{\{|h| \leq 1\}}| \\ &\leq 2\|u\|_{C^{\varphi\psi}} \cdot \begin{cases} (\varphi\psi)(|h| \wedge 1), & I_{\varphi\psi} \subset (1,2), \\ (|h| \wedge 1)^2, & I_{\varphi\psi} \subset (2,3), \end{cases} \end{aligned}$$

and if $I_{\varphi\psi} \subset (1,2)$,

$$\begin{aligned} &|\Delta_k \Delta_h u(x) - \Delta_k \nabla u(x) \cdot h \mathbf{1}_{\{|h| \leq 1\}}| \\ &\leq c_3 \|u\|_{C^{\varphi\psi}} \cdot \begin{cases} \frac{(\varphi\psi)(|h| \wedge |k|)}{|h| \wedge |k|}|h|, & |h| \leq 1, \\ \psi(|k|), & |h| > 1, \end{cases} \end{aligned} \tag{3.5.1}$$

and if $I_{\varphi\psi} \subset (2,3)$,

$$\begin{aligned}
&|\Delta_k \Delta_h u(x) - \Delta_k \nabla u(x) \cdot h \mathbf{1}_{\{|h| \le 1\}}| \\
&\le c_4 \|u\|_{C^{\varphi\psi}} \cdot \begin{cases} \frac{(\varphi\psi)(|k|)}{|k|^2} |h|^2, & |h| \le |k|, \\ \frac{(\varphi\psi)(|h|)}{|h|^2} |h||k|, & |k| < |h| \le 1, \\ \psi(|k|), & |h| > 1. \end{cases}
\end{aligned}$$

When $1 \in I_\varphi$, it is easily shown $I_{\varphi\psi} \subset (1,2)$, and we can replace the indicator function $\mathbf{1}_{\{|h| \le 1\}}$ in (3.5.1) with $\mathbf{1}_{\{|h| \le |k|\}}$ from the symmetry of $h \mapsto a(x,h)$. Thus we replace (3.5.1) with

$$\begin{aligned}
&|\Delta_k \Delta_h u(x) - \Delta_k \nabla u(x) \cdot h \mathbf{1}_{\{|h| \le 1\}}| \\
&\le c_3 \|u\|_{C^{\varphi\psi}} \cdot \begin{cases} (\varphi\psi)(|h|), & |h| \le |k|, \\ \frac{(\varphi\psi)(|h|)}{|h|} |k|, & |k| < |h| \le 1, \\ \psi(|k|), & |h| > 1. \end{cases}
\end{aligned}$$

Calculations similar to the case $M_\varphi < 1$ give rise to the result. $\square$

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국문초록

이 논문에서는 함수적 차수의 커널함수를 갖는 비국소 작용소에 관한 정칙성 결과를 도출한다. 이 논문에서 다루는 작용소 L 은

$$Lu(x) = \int_{\mathbb{R}^d} (u(x+h) - u(x) - \nabla u(x) \cdot h \mathbf{1}_{\{|h| \leq 1\}}) J(x,h) dh$$

로 정의된다. 이 때, 주어진 점프 커널함수 $J(x,h)$ 와 함수 $|h|^{-d}\varphi(|h|)^{-1}$ 의 비율이 $h = 0$ 근방에서 위, 아래로 유계임을 가정한다. 함수 φ에 약한 스케일링 조건을 가정하여 프랙셔널 라플라스 작용소를 일반화시킬 수 있다. 먼저 선형 비국소 작용소들의 적당한 클래스에 대하여 완전 비선형 비국소 방정식의 해의 횔더 연속성을 구하고, 일반화된 횔더 공간을 도입하여 비국소 방정식 $Lu = f$ 에 대한 샤우더 추정을 증명한다.

주요어휘: 비국소 작용소, 적미분 작용소, 횔더 연속, 샤우더 추정, 레비 과정
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