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Morse-Bott Spectral Sequences and the Links of Singularities

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Morse-Bott Spectral Sequences and the Links of Singularities

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Abstract

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In this thesis, we construct spectral sequences converging to symplectic homology and equivariant symplectic homology groups. We use Morse-Bott type Hamiltonians and a natural action filtration. Those spectral sequences are called Morse-Bott spectral sequences. We apply the spectral sequences to a certain kind of symplectic manifolds with boundary, namely Milnor fibers whose boundaries are the links of singularities. In special cases, such as links of weighted homogeneous polynomials, they admit a nice symmetry along a periodic Reeb flow of a contact form. By means of those special symmetric feature, we present a systematic way of computing equivariant symplectic homology groups and its mean Euler characteristic. We obtain several applications of these machineries to exotic contact structures.

Key words: Morse-Bott spectral sequence, link of singularity, symplectic homology, equivariant symplectic homology, Milnor fiber, mean Euler characteristic

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Chapter 1

Introduction

1.1 Contact structures on the links of singularities

Links of singularities

Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial with an isolated critical point at the origin and $f(0) = 0$. The zero set $f^{-1}(0)$ then has an isolated hypersurface singularity at the origin. By the *link of singularity* f, denoted by $\Sigma(f)$, we mean the intersection $f^{-1}(0) \cap S^{2n+1}_\delta$ for a sufficiently small $\delta > 0$. It turns out that the link $\Sigma(f)$ admits a canonical smooth structure by the transversality of $f^{-1}(0) \cap S^{2n+1}_\delta$.

There are close relations between the isolated singularity of the affine variety $f^{-1}(0)$ and the smooth structure of the link $\Sigma(f)$. One remarkable result is due to Mumford [1]. He has shown that, for $n = 2$, if the link $\Sigma(f)$ is diffeomorphic to the sphere S^3 with the standard smooth structure, then the variety $f^{-1}(0)$ is in fact smooth. In this sense, the smooth structure on the link "sees" the singularity.

For higher dimensions, however, the smooth structure on the link is not enough to detect the singularity. For example, if we take $f(z) = z_0^3 + z_1^2 + z_2^2 + z_3^2$, then its link $\Sigma(f)$ is diffeomorphic to the standard sphere S^5, but the zero set $f^{-1}(0)$ is not smooth at the origin.

It turns out that an additional geometric structure, which is called *contact structure*, plays an important role in this context. Every link of isolated hypersurface singularity admits a canonical contact structure as the boundary of a Stein domain $V(f) := f^{-1}(\epsilon) \cap B^{2n+2}_\delta$. In other words, $\Sigma(f)$ has a natural Stein filling $V(f)$. A beautiful theorem by Eliashberg, Floer, and

McDuff [2] states that if the contact type boundary of a symplectically aspherical filling is contactomorphic to the sphere S^{2n-1} with the standard contact structure, then the filling must be diffeomorphic to the ball B^{2n}. This in particular implies that if the link $\Sigma(f)$ is contactomorphic to the standard sphere, then its filling $V(f)$ is diffeomorphic to the ball. In view of the Milnor number of the given singularity, it follows that the variety $f^{-1}(0)$ is smooth at the origin. In this sense, the contact structure on the link is a stronger invariant of the singularity than the smooth structure.

Symplectic homology

Keeping the above motivation in mind, it is tempting to investigate symplectic and contact topology of the filling $V(f)$ and its link $\Sigma(f)$. This thesis studies their "rigidity" in terms of symplectic invariants such as symplectic homology and equivariant symplectic homology. For example, we can show that if the isolated hypersurface singularity is singular in the sense that its Milnor number is not zero, then the symplectic homology of the Milnor fiber does not vanish.

Roughly speaking, symplectic homology $SH_*(W)$ is a Hamiltonian Floer homology of a symplectic manifold W with contact type boundary. It was first introduced by Floer-Hofer [10] for open sets in $\mathbb{C}^n$, and there has been several variants and generalizations of its definitions, [11], [12]. In this thesis, we follow the treatment of Viterbo [12], Bourgeois-Oancea [29], and for the equivariant version, [13], [14].

This invariant depends on the contact structure of the boundary in several ways. The contact condition of the boundary insures a kind of maximum principle which is technically essential in the construction of symplectic homology. Moreover, the generators with positive action correspond to periodic Reeb orbits on the boundary. If the boundary admits a property known as dynamical convexity, then symplectic homology of positive action only depends on the contact structure. Recently, for example, Uebele [15] used this as an invariant of the contact structures of the links of certain Brieskorn polynomials. See also [16].

If the contact type boundary admits a *periodic* Reeb flow, we can compute the invariants in a very explicit way. The links of weighted homogeneous polynomials provide a plenty of interesting examples in this regard.

1.2 Morse-Bott spectral sequence

Periodic flow

In the case when f is a weighted homogeneous polynomial, the contact structure on the link $\Sigma(f)$ has a nice symmetry. Namely, it admits a contact form α whose Reeb flow is *periodic*. Its Reeb flow is given by

$$Fl_t^{R_\alpha}(z) = (e^{it/w_0} z_0, e^{it/w_1} z_1, \ldots, e^{it/w_n} z_n)$$

where $(w_0, w_1, \ldots, w_n)$ is the weight of f. This flow is nothing but the (weighted) rotations on each coordinates, and gives also an S^1-action on the link. The Reeb dynamics on $(\Sigma(f), \alpha)$ is therefore quite simple and explicit. For example, we can combinatorially arrange periodic Reeb orbits according to their periods.

In the sense of Floer theory, however, the contact form α is extremely degenerate. Since the usual construction of Floer theory requires non-degenerate situation, one needs to perturb the form to make it non-degenerate. Indeed, Ustilovsky [3] gave an explicit perturbation of the contact form for certain Brieskorn polynomials, and he could compute contact homology of them with respect to the perturbed form. Nevertheless it is in general quite difficult to make an explicit perturbation and use it for computing Floer theoretical invariants.

To overcome this technical difficulty, there have been techniques developed in order to construct Floer homology using such a degenerate situation *directly*. These sorts of techniques are called *Morse-Bott techniques*. F. Bourgeois has presented a variant in his thesis [4] in terms of contact homology, and van Koert [5] applied it to compute contact homology of Brieskorn manifolds. The essential idea of the Morse-Bott technique is described for example in [18]. The philosophy of the method is that we use the perturbation only *implicitly*; once the perturbation is done in an abstract setup, we can develop Floer theory for the given degenerate Morse-Bott data without any reference of explicit perturbation.

Basically, we carry out this idea for (equivariant) symplectic homology of Liouville domains whose boundary admits a *periodic Reeb flow*. The Milnor fiber whose boundary is the link of a weighted homogeneous polynomial is a relevant example of such a domain. In particular, the idea of Morse-Bott technique leads us to construct spectral sequences which are quite useful to understand the corresponding Floer homology groups. They are called *Morse-Bott spectral sequences*.

Morse-Bott spectral sequence

We construct Morse-Bott spectral sequences for (equivariant) symplectic homology groups. A construction has been briefly sketched in literature, for example [8], and there are some variants such as [9]. We work out a version for Liouville domains with a periodic Reeb flow on the boundary, which is adapted to our purpose.

The basic idea of the construction is the following. The periodicity of the Reeb flow on the boundary implies that admissible time-independent Hamiltonians H are of *Morse-Bott type*. This means that the corresponding action functional $\mathcal{A}_H$ is Morse-Bott, i.e., the set of critical points forms a submanifold and the Hessian is non-degenerate along the normal directions. Note that Floer chain complex is naturally filtered by the action values. Applying a classical theorem in homological algebra on spectral sequences, we get a spectral sequence for (equivariant) symplectic homology.

In fact, we carefully manipulate an action filtration with respect to the standard time-dependent perturbation of H. The upshot is that E^1-page of the spectral sequence consists of *local Floer homology groups* of each Morse-Bott components Σ. The local Floer homology is isomorphic to singular homology of Σ up to a degree shifting. This was addressed in [6] for the case when $\Sigma = S^1$ using $\mathbb{Z}_2$-coefficient. We have extended their result to the general manifold Σ and the general coefficient ring R. For that, we need to construct certain local coefficient system $\mathcal{L}_\Sigma$ on each Morse-Bott component. For the links of weighted homogeneous polynomials, in particular, the local coefficient system $\mathcal{L}_\Sigma$ is turned out to be trivial. The resulting spectral sequence therefore has its E^1-page as

$$E^1_{pq} = \bigoplus_{\Sigma} H_{p+q-\mathrm{shift}(\Sigma)}(\Sigma; R).$$

The main benefit of the spectral sequence is that we do not need to perturb the degenerate Reeb flow in practice. As long as we know its Reeb dynamics sufficiently well, in particular its Morse-Bott components, we can describe E^1-page by considering their singular homology groups and their

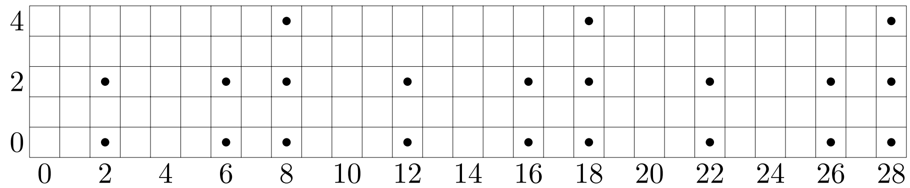


Figure 1.1: E^1-page of the spectral sequence (re-indexed)

Conely-Zehnder indices. This can be done for the link of weighted homogeneous polynomials, using classical results on their topology. We can also extend a method of van Koert in [5] computing Conley-Zehnder indices for Brieskorn case to the general weighted homogeneous case.

Another benefit of the spectral sequence, which is rather technical but very useful, is that it "visualizes" the Floer complex in an intuitive way. For example, Figure 1.1 is the E^1-page of the spectral sequence for positive equivariant symplectic homology of the Milnor fiber of the polynomial $f(z) = z_0^3 + z_1^2 + z_2^2 + z_3^2$. Each dot denotes a generator. Observe that a beauty of periodic nature of the flow is recorded as a periodic pattern of the E^1-page; the first three non-trivial "columns" are repeated in the horizontal direction. Using the pattern, we can get the full homology groups SH^{+,S^1} by just "looking" at the sequence.

Having such periodicity, the spectral sequence makes computations of the *mean Euler characteristic*, an invariant of the contact structure of the link, very efficient. This invariant will give us a series of applications in this thesis.

1.3 Applications

Mean Euler characteristic

The mean Euler characteristic $\chi_m(W)$ of equivariant symplectic homology SH^{+,S^1} is defined as follows.

$$\chi_m(W) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=-N}^{N} (-1)^i rk SH^{+,S^1}(W).$$

One can interpret the definition as an infinite dimensional analogue of the usual Euler characteristic in algebraic topology. This concept was first introduced by van Koert in his thesis [19] in terms of cylindrical contact homology. The basic motivation is that we can extract an invariant from Floer homology *without* knowing Floer trajectories which is one of the most difficult parts of the Floer theory to deal with.

By the definition, $\chi_m(W)$ is a priori an invariant of the filling. However, there are some cases where the mean Euler characteristic does not depend on the filling. For example, if the boundary admits a periodic flow, as the link of weighted homogeneous polynomials, the mean Euler characteristic provides an invariant of its contact structure. Even though it forgets the information from Floer trajectories, it is still efficient enough to distinguish contact structures for many examples.

The spectral sequence is quite useful to compute the mean Euler characteristic. Once we know the E^1-page, the computation reduces to a simple numerical counting. We just count the (signed) number of generators in E^1-page in one "periods". This gives an algorithmic way of computing the mean Euler characteristic of the link of weighted homogeneous polynomials. In this thesis, we present some applications of those computations to contact topology of the link.

Exotic contact structures

The mean Euler characteristic is sharp enough that we can re-prove the result of Ustilovsky [3]. It states that there are infinitely many contact structures on each standard spheres S^{2n-1} for odd $n \geq 3$. This follows from the following computation of the mean Euler characteristic of the links of A_k-type singularities.

Proposition. *The mean Euler characteristic of the link $\Sigma(p, 2, \ldots, 2)$ is given by*

$$\chi_m(\Sigma(p, 2, \ldots, 2)) = \frac{(p-1)(n-1)+n}{2\{(n-2)p+2\}}$$

for p, n are odd and $n \geq 3$.

Since the above formula is a one-to-one function of p, it follows that $\Sigma(p, 2, \cdots, 2)$ is not contactomorphic to $\Sigma(p', 2, \cdots, 2)$ if $p \neq p'$. A classical result due to Milnor and Kervaire shows that they are in fact all diffeomorphic to the standard sphere of the same dimension. So we get another proof of Ustilovsky's result as a corollary.

Corollary. *There are infinitely many contact structures of the standard sphere* S^{2n-1} *for odd* $n \geq 1$*, which are pairwise non-contactomorphic to each other and are in the same homotopy class.*

The mean Euler characteristic by definition takes values in the real numbers. Observe that we already have infinitely many rational numbers as mean Euler characteristics of some contact manifolds. In this sense, one can ask a natural "geography problem" of contact structures as follows.

Question. *Can we realize every rational number as a mean Euler characteristic of a contact manifold?*

We answer this question affirmatively.

Theorem. *Every rational number can be realized as the mean Euler characteristic of some contact structure on* S^5 *with its standard smooth structure.*

Note that S^5 is a simply-connected spin 5-manifold. Using a classical theorem due to S. Smale [20] on a classification of simply-connected spin 5-manifolds, we can generalize the above theorem as follows.

Theorem. *Every simply-connected spin* 5*-manifold admits infinitely many pairwise non-contactomorphic contact structures* ξ*, all satisfying* $c_1(\xi) = 0$*, and they realize all rational numbers as their mean Euler characteristic.*

The main ingredient of the proof is the fact due to Espina [17], Bourgeois-Oancea [14] on the behavior of mean Euler characteristic under the boundary connected sum, see Theorem 9.2.9.

In fact, the boundary connected sum defines a *monoid* operation on the set of certain class of the contact structures of S^5; contact structures that admit *convenient dynamics* and are convex fillable by simply connected Liouville manifolds with vanishing first Chern class. We denote the monoid by $\Xi_{nice}(S^5)$. Then we can formulate an algebraic nature of $\Xi_{nice}(S^5)$ as the following corollary.

Corollary. *The map* $\tilde{\chi}_m : (\Xi_{nice}(S^5), \#) \to (\mathbb{Q}, +)$ *defined by*

$$(\xi, W) \mapsto \chi_m(W) - \frac{1}{2}$$

is a surjective monoid homomorphism.

Chapter 2

Preliminaries

In this chapter, we present some basics on symplectic and contact topology as preliminaries. This is also for fixing notations which will be frequently used in the thesis.

2.1 Symplectic manifolds

A **symplectic manifold** W is a smooth manifold equipped with a **symplectic form** ω. To simply the notation, we usually call the pair (W, ω) a symplectic manifold. A symplectic form is by definition a *closed* differential 2-form $\omega \in \Omega^2(W)$, which is *non-degenerate*. Note that symplectic manifold is necessarily even dimensional.

One can interpret symplectic manifolds as a generalization of Hamiltonian dynamics on $\mathbb{R}^{2n}$ to smooth manifolds. Indeed, if we have a smooth function $H : W \to \mathbb{R}$ on a symplectic manifold (W, ω), then the symplectic form ω provides Hamiltonian dynamics on W due to its non-degeneracy. In this sense, a smooth function on (W, ω) is called *Hamiltonian*.

2.1.1 Hamiltonians

Let $H : W \to \mathbb{R}$ be a smooth function on a symplectic manifold (W, ω). Any such function is called **Hamiltonian**. For each hamiltonian, we can associate a canonical vector field X_H on W by the following implicit formula.

$$\iota_{X_H}\omega = dH.$$

Note here that non-degeneracy of ω uniquely determines the vector field X_H. This vector field is called a **Hamiltonian vector field** of H. We denote its flow by $Fl_t^{X_H}$ and call the **Hamiltonian flow**. Roughly speaking, the classical Hamiltonian dynamics studies the flow $Fl_t^{X_H}$ on the cotangent bundle $T^*\mathbb{R}^n$ with the canonical symplectic structure.

2.1.2 Almost complex structures

One thing that makes symplectic geometry be interesting is the existence of compatible almost complex structures. For a symplectic manifold (W, ω), an almost complex structure J on W is called **compatible** if $\omega(JX, JY) = \omega(X, Y)$ and $\omega(X, JX) > 0$ for all $X, Y \in TW$. It is well known that the set of compatible almost complex structures $\mathcal{J}(W, \omega)$ is always non-empty and forms a contractible topological space.

Let J be a compatible almost complex structure on (W, ω). There is canonically defined Riemannian metric g on W with respect to the pair (ω, J): Define a 2-tensor g on W by

$$g(X, Y) := \omega(X, JY)$$

for each $X, Y \in TW$. One can directly see that the compatible condition of (ω, J) implies that g is a metric on W. We call the triple (ω, J, g) a **compatible triple** on W.

Let $H : W \to \mathbb{R}$ be an Hamiltonian and (ω, J, g) a compatible triple on W. We can then consider the gradient vector field $\operatorname{grad} H$ with respect to the metric g. It is often useful to keep in mind the following relation between the Hamiltonian vector field X_H and the gradient vector field $\operatorname{grad} H$.

Proposition 2.1.1. *For a compatible tuple (ω, J, g) and a Hamiltonian H, we have*

$$X_H = -J \operatorname{grad} H.$$

Proof. This follows directly from the definitions. Observe that

$$\iota_{-J \operatorname{grad} H}\omega(Y) = \omega(-J \operatorname{grad} H, Y) = \omega(\operatorname{grad} H, JY) = \iota_{\operatorname{grad} H}(Y).$$

This complete the proof. $\square$

2.2 Contact manifolds

A $2n-1$ dimensional smooth manifold M is called a **(cooriented) contact manifold** if it admits a **contact 1-form** α, which means that the $(2n-2)$-form $\alpha \wedge d\alpha^{n-1}$ is nowhere vanishing. The codimension 1 distribution ξ given by $\xi := \ker \alpha$ is called a **contact structure** on M. Put differently, a contact structure ξ on M is a codimension 1 distribution which is *completely non-integrable*. Such a distribution is always given locally as a kernel of 1-form. In the case when ξ is the kernel of a globally defined 1-form α, then α is a contact form. In this thesis, by contact manifold, we always mean a cooriented contact manifold.

A contact manifold is an odd-dimensional analogue of the symplectic manifolds. There are several analogous theorems on contact manifolds and its contact structures. For example, there is a version of Darboux theorem which shows that every contact manifold looks locally the standard contact structure $\xi = \ker(dz - ydx)$ on $\mathbb{R}^{2n-1}$. One another analogy is *Gray stability* which can be proved by a kind of the *Moser trick*.

Theorem 2.2.1 (Gray stability theorem, [32])**.** *Let $\xi_t, t \in [0,1]$ be a smooth family of contact structures on a closed manifold Y. Then there is an smooth isotopy ψ_t of Y such that*

$$T\psi_t(\xi_0) = \xi_t$$

for each $t \in [0,1]$. In particular, (Y, ξ_0) is contactomorphic to (Y, ξ_1).

2.2.1 Reeb vector fields

Let $(M, \xi = \ker \alpha)$ be a contact manifold. There is a canonically defined vector field associated to the contact form α.

Definition 2.2.2. Define a vector field R_α on M by the conditions

$$\iota_{R_\alpha} d\alpha|_\xi \equiv 0, \quad R_\alpha(\alpha) \equiv 1.$$

The vector field R_α is called the **Reeb vector field** associated to the contact form α.

Note that the contact condition of α implies that $d\alpha$ restricts to a symplectic structure on the bundle $\xi \to M$ over M; in particular, $d\alpha|_\xi$ is non-degenerate. Therefore the first condition of R_α determines the vector field up to its "length". Combining with the second condition, the Reeb vector field is uniquely determined. We denote the flow of the Reeb vector field by $Fl_t^{R_\alpha} : M \to M$, which is called the **Reeb flow**. The dynamical system given by the Reeb flow is usually said to be **Reeb dynamics**.

Remind that a symplectic manifold is a framework for classical Hamiltonian dynamics on manifolds. In that regard, contact manifolds, one can say, concerns Hamiltonian dynamics on the fixed energy hypersurface of Hamiltonians. This relation can be seen rather apparently in the setting of *symplectizations*.

2.2.2 Symplectizations

Let (M, α) be a contact manifold with a contact form α. We can associate a symplectic manifold as follows. Define a 2-form on the product space $\mathbb{R} \times M$ by $\omega = d(r\alpha)$ where r is the coordinate of $\mathbb{R}$. This coordinate r is called the *cylindrical coordinate* for convenience. Then $\omega = d(r\alpha)$ is clearly closed and non-degenerate, in other words, the pair $(\mathbb{R} \times M, \omega)$ forms a symplectic manifold of dimension $2n$.

Definition 2.2.3. The symplectic manifold $(\mathbb{R} \times M, \omega)$ is called the **symplectization** of the contact manifold (M, α).

As one can surely expect, contact topology/geometry of the given contact manifold and symplectic topology/geometry of its symplectization are closely related. For example, for each Legendrian submanifold $\mathcal{L}$ of M, the product space $L := \mathbb{R} \times \mathcal{L}$ is a Lagrangian in the symplectization.

Another relation which is important in the context of this thesis is the relation between Hamiltonian flows on the symplectization $\mathbb{R} \times M$ and the Reeb flow on each hypersurface $\{r_0\} \times M$. Let $H : \mathbb{R} \times M \to \mathbb{R}$ be a Hamiltonian which is *cylindrical*, i.e, H only depends on the cylindrical coordinate, say $H(r, x) = h(r)$ for some smooth function $h : \mathbb{R} \to \mathbb{R}$. One can easily check that, on the level set $\{r_0\} \times M$,

$$X_H = -h'(r_0) R_\alpha.$$

In other words, Hamiltonian dynamics on the fixed energy level is the Reeb dynamics. This relation will play an essential role in the construction of symplectic homology groups.

2.3 Symplectic fillings of contact manifolds

Let (M, ξ) be a contact manifold of dimension $2n-1$ with a contact *structure* ξ. A **symplectic filling** or briefly a **filling** of a contact manifold (M, ξ) is a symplectic manifold (W, ω) whose boundary is the given contact manifold. There are several notions of fillings according to how much symplectic topology/geometry of the filling is related to the contact topology/geometry of the boundary.

To give a series of notions of symplectic fillings, we start with the definition of *Liouville vector field*. Let W be a symplectic manifold *with boundary* $\partial W =: M$. Let ω be a symplectic form on W.

Definition 2.3.1. A vector field X (possibly partially defined) on W is called **Liouville** if

$$\mathcal{L}_X \omega = \omega.$$

Let X be a Liouville vector field defined on a neighborhood of the boundary ∂W, and assume that X is transverse to the boundary. Then we have a canonical contact form on the boundary as follows.

Proposition 2.3.2. *For a Liouville vector field X defined on a neighborhood of M, which is transverse to M, a 1-form α defined by*

$$\alpha := \iota_X \omega|_M$$

is a contact form on M.

Proof. One observes that, using the definition of the Liouville vector field, α is contact if and only if ω is symplectic, provided that $X \pitchfork \partial W$. □

Consequently, if there is a Liouville vector field defined in a neighborhood of the boundary, then ∂W becomes a contact manifold.

Definition 2.3.3. Let (M, ξ) be a contact manifold. A **strong sympletic filling** of (M, ξ) is a symplectic manifold (W, ω) with boundary M such that

1. there is a Liouville vector field X defined on a neighborhood of the boundary and X is transverse to M pointing outward;

2. the contact structure ξ is the kernel of the contact form $\alpha := \iota_X \omega|_M$.

In this case, we say that the contact manifold (M, α) is **strongly fillable**.

If the symplectic form ω is happen to be *exact*, i.e, $\omega = d\lambda$ for some 1-form λ on W, then we have a globally defined Liouville vector field.

Proposition 2.3.4. *A Liouville vector field X is globally defined on W if and only if ω is exact.*

Proof. Let X be a Liouville vector field on W. Define a 1-form λ by

$$\lambda := \iota_X \omega.$$

By non-degeneracy λ is well-defined, and we see that

$$d\lambda = d\iota_X \omega = \mathcal{L}_X \omega = \omega.$$

Here we used Cartan's formula. By reversing this argument, the other direction also holds. This completes the proof. $\square$

Definition 2.3.5. A symplectic manifold W with boundary is called **exact** if its symplectic form ω is exact.

To step up to a next level of fillings, note that the (globally defined) contact form $\alpha = \iota_X \omega$ on ∂W gives an orientation by $\alpha \wedge (d\alpha)^{n-1}$. On the other hand ∂W has another orientation as the boundary of oriented manifold W by ω^n. If these two orientations coincide to each other, then we say that the Liouville vector field X is *positively* transverse to the boundary.

We now give a definition of Liouville domain.

Definition 2.3.6. A **Liouville domain** is an exact symplectic manifold W with boundary ∂W whose Liouville vector field is positively transverse to the boundary.

By Proposition 2.3.2, the boundary of a Liouville domain is a contact manifold with the contact form $\alpha := \lambda|_{\partial W}$. Therefore, in terms of the symplectic filling, a Liouville domain (W, ω) is an exact filling of the boundary $(\partial, \xi = \ker \alpha)$ such that the Liouville vector field is positively transverse to the boundary. In this case, we say that (M, ξ) is **Liouville fillable**, and (W, ω) is called a **Liouville filling**.

There are two more finer notions of fillings, namely, *Weinstein* and *Stein filling*. We give a definition of the Weinstein domain here, and the Stein domain shall be given in Section 7.1.

Definition 2.3.7. A **Weinstein domain** (W, ω, X, f) consists of an exact symplectic manifold (W, ω) together with a Liouville vector field and a Morse function $f : W \to \mathbb{R}$ such that

1. X is *gradient-like* with respect to the function g, i.e. $Xf > 0$;
2. the boundary ∂W is a regular level set of f.

Any Weinstein domain is clearly a Liouville domain.

Chapter 3

Symplectic Homology

Roughly speaking, symplectic homology is a Hamiltonian Floer homology for symplectic manifolds *with contact type boundary*. We have seen in Section 2.3 that there are several "levels" of symplectic manifolds with contact type boundary. In general, symplectic homology can be defined on any such symplectic manifolds, but in this thesis, we focus on the *Liouville domains*. Justifications of this choice will be discussed throughout this chapter.

We start with the notion of the completion of Liouville domains.

3.1 Completion

Let $(W, d\lambda)$ be a Liouville domain. Its boundary ∂W is then a contact manifold with a contact form $\alpha := \lambda|_{\partial W}$ The following shows that, near the boundary, W looks like the symplectization of $(\partial W, \alpha)$.

Proposition 3.1.1. *For $\epsilon > 0$ small enough, there is a symplectic embedding*

$$((-\epsilon, 0] \times \partial W, d(e^t \alpha)) \to (W, \omega)$$

such that $\{0\} \times \partial W$ maps to $\partial W \subset W$

Proof. Let X be a Liouville vector field on (W, ω). Denote its time t-flow by Fl_t^X. We have a map

$$(t, x) \mapsto Fl_t^X(x).$$

Then by the definition of Liouville vector field, Φ is a symplectic embedding. The assertion now follows. $\square$

Using Proposition 3.1.1, we can attach the positive part of the symplectization $[0,\infty)\times\partial W$ to the domain W along the boundary. The symplectic form ω on W also extends by the symplectic form $d(e^t\alpha)$ on the symplectization. More precisely we define an open symplectic manifold by

$$\widehat{W} := W \cup_{\partial W} ([0,\infty)\times\partial W)$$

with a symplectic form

$$\hat{\omega} := \begin{cases} \omega & \text{on } W, \\ d(e^r\alpha) & \text{on } [0,\infty)\times\partial W \end{cases}$$

where r denote the coordinate of $[0,\infty)$.

Definition 3.1.2. The attached open symplectic manifold $(\widehat{W},\hat{\omega})$ is called the **completion** of (W,ω).

The domain part $W_0 := W\backslash\partial W$ in the completion $\widehat{W}$ is called the *interior*, and the attached symplectization part is called the *cylindrical end* or simply *end*.

3.2 Admissible Hamiltonians

Symplectic homology of a domain (W,ω) is, roughly speaking, Hamiltonian Floer homology of its completion $(\widehat{W},\hat{\omega})$. Note that the completion is not compact, which may mean we cannot apply the standard compactness results for the closed case. To overcome this non-compactness, we choose a special kind of Hamiltonians called *admissible*.

Denote the action spectrum of the periodic Reeb orbits in the boundary by

$$Spec(\partial W,\alpha) := \{T\in\mathbb{R} \mid T \text{ is a period of a Reeb orbit of } \alpha\}.$$

Definition 3.2.1. A Hamiltonian $H : S^1\times\widehat{W}\to\mathbb{R}$ is called **admissible** if

1. H non-degenerate;
2. H is C^2-small Morse in the interior W_0;
3. H is only dependent of the cylindrical coordinate r at the end, i.e., $H(t,x) = h(e^r)$ for convex some function $h : [0,\infty)\to\mathbb{R}$.

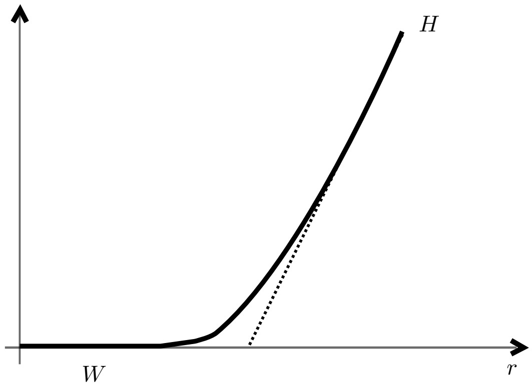


Figure 3.1: Admissible hamiltonian

4. H is linear at the end, i.e., $H(t,x) = ae^r + b$, and its slope a is not in $Spec(\partial W, \alpha)$.

The above conditions on H affects the Hamiltonian vector field of H as follows.

Lemma 3.2.2. *On a level set* $\{r\} \times M$ *of* $[0,\infty) \times M$*,*

$$X_H(r,x) = -h'(e^r)R_\alpha(x).$$

Here, R_α *denotes the Reeb vector field on the boundary* (M,α)*.*

Proof. Using the definition of the Reeb vector field, we have

$$\begin{aligned}
\iota_{-h'(e^r)R_\alpha} d(e^r\alpha)\left(e^{-r}\frac{\partial}{\partial r}\right) &= \iota_{-h'(e^r)R_\alpha}(e^r dr \wedge \alpha + e^r d\alpha)\left(e^{-r}\frac{\partial}{\partial r}\right) \\
&= e^r dr \wedge \alpha\left(-h'(e^r)R_\alpha, e^{-r}\frac{\partial}{\partial r}\right) \\
&= h'(e^r) = e^r h'(e^r) dr\left(e^{-r}\frac{\partial}{\partial r}\right) \\
&= dh\left(e^{-r}\frac{\partial}{\partial r}\right).
\end{aligned}$$

This implies the assertion. $\square$

Therefore in the region where the Hamiltonian satisfies $H(t,x) = h(t,e^r)$, its 1-periodic orbits are in one-to-one correspondence with $h'(e^r)$-periodic Reeb orbits in the boundary (M,α). In addition, by the last condition on H which is essential for *noncompact* situation, 1-periodic Hamiltonian orbits of H do not appear in $[R_0, \infty)$ for sufficiently large $R_0 > 0$. The condition that H is C^2-small and Morse in the interior is related to the behavior of Floer trajectories in the interior, and this will be clarified later.

Let $H : S^1 \times \widehat{W} \to \mathbb{R}$ be admissible. Denote the free loop space of $\widehat{W}$ by $\Lambda\widehat{W}$, i.e.,

$$\Lambda\widehat{W} := W^{1,p}(S^1, \widehat{W}) = \{\gamma : S^1 \to \widehat{W} \mid \gamma \text{ is of Sobolev class } W^{1,p}\}.$$

Define the *(Hamiltonian) action functional* by

$$\begin{aligned} \mathcal{A}_H :&\Lambda\widehat{W} \to \mathbb{R} \\ &\gamma \longmapsto -\int_{S^1} \gamma^*\hat{\lambda} - \int_0^1 H(\gamma(t), t)dt \end{aligned} \tag{3.2.1}$$

where $\hat{\lambda}$ denotes the extended primitive 1-form on $\widehat{W}$ such that $\hat{\omega} = d\hat{\lambda}$. Denote the set of critical points of the function $\mathcal{A}_H$ by $\mathcal{P}(H)$.

Lemma 3.2.3. *The set of critical points $\mathcal{P}(H)$ consists of 1-periodic orbits in $\widehat{W}$ of the Hamiltonian vector field X_H.*

Proof. We do the usual variational calculus. Let γ_s be a path in $\Lambda\widehat{W}$. Denote its derivative at $s = 0$ by $Y \in T_{\gamma_0}\Lambda\widehat{W}$. Then we compute, reminding the convention $\iota_{X_H}\omega = dH$,

$$\begin{aligned} d\mathcal{A}_H(\gamma_0)(Y) &= -\int_{S^1} \frac{d}{ds}\bigg|_{s=0} \gamma_s^*\hat{\lambda} - \int_0^1 \frac{d}{ds}\bigg|_{s=0} H(\gamma_s(t), t)dt \\ &= -\int_{S^1} \gamma_0^*\mathcal{L}_Y\hat{\lambda} - \int_0^1 dH(\gamma_0(t), t)Y dt \\ &= \int_{S^1} \hat{\omega}(\gamma_0', Y) - \int_0^1 \hat{\omega}(X_H, Y)dt \\ &= \int_{S^1} \hat{\omega}(\gamma_0' - X_H, Y)dt. \end{aligned}$$

This implies that $d\mathcal{A}_H(\gamma) \equiv 0$ if and only if $\gamma'(t) = X_H(\gamma(t))$, i.e., γ is a 1-periodic orbit of X_H. $\square$

A useful observation on the action functional is the following.

Lemma 3.2.4. *For a 1-periodic orbit γ on the level set $\{r\} \times \Sigma$, the action functional $\mathcal{A}_H$ is equal to*

$$-rh'(r) - h(r).$$

Proof. This is direct from the definition. Observe that

$$\int_{S^1} \gamma^* \hat{\alpha} = \int_{S^1} \gamma^* r\alpha = rh'(r)$$

since γ corresponds to $h'(r)$-periodic Reeb orbit of α. Moreover,

$$\int_0^1 H(\gamma(t))dt = \int_0^1 h(r)dt = h(r).$$

We are done. $\square$

3.3 Conley-Zehnder index

The chain complex for Floer homology is graded by a Maslov-type index, called *Conley-Zehnder index* or more generally *Robbin-Salamon index*. We give some detail on the definition and properties of the indices in this section.

3.3.1 For a path of symplectic matrices

Denote the group of symplectic matrices by $Sp(2n) := Sp(2n, \mathbb{R})$. Let $\Psi : [0,1] \to Sp(2n)$ be a path of symplectic matrices with $\Psi(0) = \mathbb{1}$. Such path $\Psi : [0,1] \to Sp(2n, \mathbb{R})$ is called *non-degenerate* if the end point $\Psi(1)$ has no eigenvalue equal to 1. The Conley-Zender index assign an *integer* to each non-degenerate path of symplectic matrices.

Note that the unitary group $U(n)$ is a subgroup of $Sp(2n)$ as $U(n) = Sp(2n) \cap O(2n)$, and $U(n)$ is homotopy equivalent to $Sp(2n)$. It turns out that the determinant map $\det : U(n) \to S^1$ continuously extends to $Sp(2n)$, and this is unique in the following sense.

Proposition 3.3.1 ([33])**.** *For each integer $n \leq 1$, there is a unique continuous map*

$$\rho : Sp(2n) \to S^1$$

satisfying the following:

1. *ρ is an extension of* $\det : U(n) \to S^1$*, i.e., if $A \in U(n)$, then $\rho(A) = \det(A)$;*

2. *ρ is conjugation invariant, i.e., for any $A, B \in Sp(2n)$*

$$\rho(B^{-1}AB) = \rho(A);$$

3. *If $A \in Sp(2n)$ has no eigen value on S^1, then $\rho(A) = \pm 1$;*

4. *If $A \in Sp(2n)$ is given by a direct sum of two matrices $A_1 \in Sp(2n_1)$, $A_2 \in Sp(2n_2)$, then*
$$\rho(A) = \rho(A_1) \cdot \rho(A_2).$$

Let Ψ be a non-degenerate path of symplectic matrices. Then, in view of the topology of $Sp(2n)$, the end point $\Psi(1)$ can be connected in $Sp(2n)$ either to $-\mathbb{1}$ or to the matrix

$$\mathrm{diag}(2, -1, \ldots, -1, \frac{1}{2}, -1, \ldots, -1)),$$

depending whether $\det(\Psi(1))$ is positive or negative. Denote an extended path by $\widetilde{\Psi} : [0, 2] \to Sp(2n)$. Note that the composition $\rho^2 \circ \widetilde{\Psi} : [0, 2] \to S^1$ now forms a *loop* so that its degree makes sense.

Definition 3.3.2. The **Conley-Zehnder index** of a path Ψ is defined by

$$\mu_{CZ}(\Psi) := \deg(\rho^2 \circ \widetilde{\Psi}).$$

where $\widetilde{\Psi}$ is an extended path as above.

Since an extension $\widetilde{\Psi}$ is unique up to homotopy and the degree is invariant under homotopy, the index does not depend on the choice of extensions. The Conley-Zehnder index has the following useful properties.

Proposition 3.3.3 ([33])**.** *The Conley-Zehnder index satisfies the following.*

1. *(Naturality) For any path $\Phi : [0, 1] \to Sp(2n)$,*
$$\mu_{CZ}(\Phi\Psi\Phi^{-1}) = \mu_{CZ}(\Psi);$$

2. *(Homotopy) If two paths Ψ and Φ are homotopic, then $\mu_{CZ}(\Psi) = \mu_{CZ}(\Phi)$;*

3. *(Direct sum) If a path Ψ is given as a direct sum, say $\Psi = \Psi_1 \oplus \Psi_2$, then we have*
$$\mu_{CZ}(\Psi) = \mu_{CZ}(\Psi_1) + \mu_{CZ}(\Psi_2);$$

4. *(Loop) Let $\Phi : [0, 1] \to Sp(2n)$ be a loop, i.e., $\Phi(0) = \Phi(1)$. Then we have*
$$\mu_{CZ}(\Phi\Psi) = \mu_{CZ}(\Psi) + 2\mu(\Phi).$$

5. *(Signature) Let* $\Psi(t) = \exp(J_0 St)$ *for some symmetric non-degenerate matirix* S. *If* $||S||$ *is* C^2*-small, say* $||S|| < 2\pi$, *then we have*

$$\mu_{CZ}(\Psi) = \frac{1}{2}\operatorname{sign}(S)$$

where $\operatorname{sign}(S)$ *is the signature of* S.

Remark 3.3.4. There are some remarks on the properties of μ_{CZ}.

- In the loop property, $\mu(\Phi)$ denotes the *Maslov index* of loop of symplectic matrices. It is defined by

$$\mu(\Phi) = \deg(\rho \circ \phi).$$

  For more details, see [36].

- If S is symmetric and non-degenerate, $\Psi(t) = \exp(J_0 St)$ is a path of symplectic matrices which is non-degenerate.

- The signature property gives a relation between Conley-Zehnder index and Morse index of constant periodic orbit in Floer theory. This will be explained later.

In fact, the above properties determines the Conley-Zehnder index uniquely in the following sense.

Proposition 3.3.5 ([33]). *If an assignment of integers to each non-degenerate path of symplectic matrices satisfies homotopy, loop and signature property, then it coincides with Conley-Zehnder index.*

There is another description of Conley-Zehnder index which is more useful in practice than the above "topological" definition. This will be explained in the next section.

3.3.2 Robbin-Salamon index

So far we have dealt with *non-degenerate* paths. The notion of the Conley-Zehnder index can be extended to "degenerate" cases. An extension is called *Robbin-Salamon index*. We define Robbin-Salamon index via the *crossing formula*. Denote the standard symplectic form on $\mathbb{R}^{2n}$ by $\omega_0 = \sum_j dx_j \wedge dy_j$.

Definition 3.3.6. Let $\Psi : [0, T] \to Sp(2n)$ be a path of symplectic matrices with $\Psi(0) = \mathbb{1}$. A point $t \in [0, T]$ is called a **crossing** if $\det(\Psi(t) - \mathbb{1}) = 0$. For each crossing $t \in [0, T]$, let $V_t := \ker(\Psi(t) - \mathbb{1})$. Define a quadratic form $Q_t : V_t \times V_t \to \mathbb{R}$ by

$$Q_t(v, v) := \omega_0(v, \Psi'(t)v).$$

The quadratic form Q_t is called a **crossing form**. A crossing $t \in [0, T]$ is called **non-degenerate** if Q_t is non-degenerate as a quadratic form.

Example 3.3.7. Let us consider a path $\Psi : [0, T] \to Sp(2)$ defined by

$$\Psi(t) = e^{it}.$$

Evidently, $t \in [0, T]$ is a crossing if and only if $t \in 2\pi\mathbb{Z}$. For each crossing $t = 2\pi n$, we see $\Psi(t) = \mathbb{1}$, so that $V_t = \mathbb{R}^2$ itself. The crossing form is then given by

$$\begin{aligned} Q_t(v, v) &= (dx \wedge dy)(v, ie^{it}v) \\ &= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \end{aligned}$$

where $v = (x, y)$. In particular, every crossing is non-degenerate, and its signature is 2.

Definition 3.3.8. Let $\Psi : [0, T] \to Sp(2n)$ be a path whose crossings are all non-degenerate. We define the **Robbin-Salamon index** of Ψ by

$$\mu_{RS}(\Psi) := \frac{1}{2} \operatorname{sign} Q_0 + \sum_{\text{crossings } t \text{ in } (0,T)} \operatorname{sign} Q_t + \frac{1}{2} \operatorname{sign} Q_T. \tag{3.3.1}$$

For a general path Φ, we choose a perturbation $\tilde{\Phi}$ fixing end points such that $\tilde{\Phi}$ has only non-degenerate crossings. Then define

$$\mu_{RS}(\Phi) := \mu_{RS}(\tilde{\Phi}).$$

Remark 3.3.9. This is well-defined, i.e., μ_{RS} does not depend on the choice of perturbations. For more details, see [34].

Example 3.3.10. We continue Example 3.3.7. As we have seen, the path $\Psi(t) = e^{it}$ has only non-degenerate crossings at $t = 2\pi n$, and the crossing form Q_t has signature equal to 2. Therefore, its Robbin-Salamon index is given by the following formula.

$$\mu_{RS}(e^{it}, [0, T]) = \begin{cases} \dfrac{T}{\pi} & \text{if } T \in 2\pi\mathbb{Z}, \\ 2\left\lfloor \dfrac{T}{2\pi} \right\rfloor + 1 & \text{otherwise.} \end{cases} \tag{3.3.2}$$

This example will be particularly useful in the applications.

Remark 3.3.11. The Robbin-Salamon index takes values in half integers $\frac{1}{2}\mathbb{Z}$, whereas the Conley-Zehnder index is always an integer.

As mentioned before, the Robbin-Salamon index is a generalization of the Conley-Zehnder index. Moreover, the Robbin-Salamon index satisfies the same sort of properties as the Conley-Zehnder index.

Proposition 3.3.12 ([34])**.** *The Robbin-Salamon index satisfies the following properties.*

1. *(Generalization) For a non-degenerate path $\Psi : [0, T] \to Sp(2n)$,*

$$\mu_{RS}(\Psi) = \mu_{CZ}(\Psi);$$

2. *(Naturality) For $\Psi, \Phi : [0, T] \to Sp(2n)$, we have*

$$\mu_{RS}(\Phi\Psi\Psi^{-1}) = \mu_{RS}(\Psi);$$

3. *(Homotopy) If two paths Ψ and Φ are homotopic relative to end points, then we have $\mu_{CZ}(\Psi) = \mu_{CZ}(\Phi)$;*

4. *(Product) If a path Ψ is given as a direct sum, say $\Psi = \Psi_1 \oplus \Psi_2$, then we have*

$$\mu_{CZ}(\Psi) = \mu_{CZ}(\Psi_1) + \mu_{CZ}(\Psi_2);$$

5. *(Shear) Let B be a symmetric matrix. If a path $\Psi : [0, 1] \to Sp(2n)$ is given by*

$$\Psi(t) = \begin{pmatrix} \mathbb{1} & -tB \\ O & \mathbb{1} \end{pmatrix},$$

then its index is $\mu_{RS}(\Psi) = \frac{1}{2} \operatorname{sign} B$.

3.3.3 For periodic orbits

Let $(\widehat{W}, \hat{\omega})$ be a completion of a Liouville domain $(W, \omega = d\lambda)$. Take a time dependent admissible Hamiltonian $H : S^1 \times \widehat{W} \to \mathbb{R}$. In this section, we define the Conley-Zehnder index of *contractible* 1-periodic Hamiltonian orbit γ. By 1-periodic Hamiltonian orbit, we mean that a loop

$$\gamma : [0,1] \to \widehat{W}$$

such that $\gamma'(t) = X_H(\gamma(t))$. To make the index be well-defined, we assume that

$$c_1(\widehat{W}) = 0.$$

The reason why we impose these conditions will be explained.

Since γ is contractible in $\widehat{W}$, there is a "capping" disk $u : D^2 \to \widehat{W}$ which bounds γ, i.e.,

$$u(e^{2\pi i t}) = \gamma(t).$$

In general, on any Riemann surface with *non-empty* boundary, symplectic vector bundle has a trivialization. See [35, Proposition 2.66]. Therefore we have a symplectic trivialization of $u^*T\widehat{W}$, and by restriction we get a symplectic trivialization of $\gamma^*T\widehat{W}$. Denote the trivialization by

$$\phi : S^1 \times \mathbb{R}^{2n} \to \gamma^*T\widehat{W}.$$

Note that the *linearized* Hamiltonian flow $TFl_t^{X_H} : T\widehat{W} \to T\widehat{W}$ is a symplectomorphism. Define a path of symplectic matrices $\Psi_\gamma : [0,1] \to Sp(2n)$ by taking the matrix representation of $TFl_t^{X_H}$, namely

$$\Psi_\gamma(t) := \phi(\gamma(t))^{-1} \circ T_{\gamma(0)}Fl_t^{X_H} \circ \phi(\gamma(0)).$$

We define the *Conley-Zehnder index of a 1-periodic Hamiltonian orbit* γ by

$$\mu_{CZ}(\gamma) := \mu_{CZ}(\Psi_\gamma).$$

The choice of the capping disk affects the index as follows.

Proposition 3.3.13. *For another capping disk $\tilde{u} : D^2 \to \widehat{W}$, we have*

$$\mu_{CZ}(\gamma, \tilde{u}) = \mu_{CZ}(\gamma, u) + 2c_1(\widehat{W})(A)$$

where $A \in H_2(\widehat{W})$ *a homology class of* $u\#\tilde{u}$ *(a sphere obtained by gluing along* γ*).*

Proof. Write the trivialization of $\gamma^*T\widehat{W}$ corresponds to $\tilde{u}$ by $\tilde{\phi} : S^1 \times \mathbb{R}^{2n} \to \gamma^*T\widehat{W}$. By naturality property of the index, we may assume that

$$\phi^{-1}(\gamma(0)) \circ \tilde{\phi}(\gamma(0)) = \mathrm{id}\,.$$

Then we have a loop $\tilde{\phi}^{-1}(\gamma(t)) \circ \phi(\gamma(t))$, $t \in [0, 1]$, of symplectic matrices. Using the loop property, we now see that

$$\begin{aligned}
\mu_{CZ}(\gamma, \tilde{u}) &= \mu_{CZ}(\tilde{\phi}^{-1} \circ TFl_t^{X_H} \circ \tilde{\phi}) \\
&= \mu_{CZ}(\tilde{\phi}^{-1} \circ \phi \circ \phi^{-1} \circ TFl_t^{X_H} \circ \phi \circ \phi^{-1} \circ \tilde{\phi}) \\
&= \mu_{CZ}((\tilde{\phi}^{-1} \circ \phi) \circ (\phi^{-1} \circ TFl_t^{X_H} \circ \phi)) \\
&= \mu_{CZ}(\gamma, u) + 2\mu(\tilde{\phi}^{-1}(\gamma(t)) \circ \phi(\gamma(t))).
\end{aligned}$$

Note the fact that if a loop is given as a transition map of two unitary trivializations of a complex vector bundle on a Riemann surface, then its Maslov is equal to the first Chern number of the bundle, see [35, Theorem 2.69]. Therefore the index $\mu(\tilde{\phi}^{-1}(\gamma(t)) \circ \phi(\gamma(t)))$ in the above computation is equal to $c_1(\widehat{W})(A)$, where A is the homology class in $H_2(\widehat{W})$ reprensented by $u\#\tilde{u}$. This completes the proof. ☐

As a result, since we have assumed that $c_1(\widehat{W}) = 0$, the Conley-Zehnder index of γ does not depend on the choice of a capping disk.

3.3.4 Morse index and Conley-Zehnder index

Let H be an admissible Hamiltonian on $\widehat{W}$. In particular, H is Morse and C^2-small in the interior W_0. This means that every 1-*periodic* Hamiltonian orbit x in W_0 is constant, and hence it is a non-degenerate critical point of H. Denote the Morse index of x by $\mathrm{ind}_H(x)$. Then the Conley-Zehnder index $\mu_{CZ}(x)$ of x as a periodic Hamiltonian orbit is related to $\mathrm{ind}_H(x)$ as follows.

Proposition 3.3.14. $\mu_{CZ}(x) = \mathrm{ind}_H(x) - n$.

Proof. We mainly use the signature property in Proposition 3.3.3. Let φ be a trivialization of $T_x\widehat{W}$. By the definition,

$$\mu_{CZ}(x) = \mu_{CZ}(\varphi^{-1} \circ T_x Fl_t^{X_H} \circ \varphi).$$

On the other hand, the Hessian $HessH$ at x is related to the Hamiltonian vector field X_H as

$$\nabla X_H(x) = J \cdot HessH(x).$$

Here, J is a ω-compatible almost complex structure and ∇ is taken with respect to the induced metric by (ω, J). It follows that

$$T_x Fl_t^{X_H} = \exp(J \cdot HessH(x)t).$$

Since $Hess(x)$ is a *non-degenerate* symmetric matrix, we have

$$2 \cdot (\# \text{ of nagative eigen values}) = 2 \cdot \mathrm{ind}_H(x) = 2n + \operatorname{sign} HessH$$

or equivalently,

$$\mathrm{ind}_H(x) - n = \frac{1}{2} \operatorname{sign} HessH = \mu_{CZ}(x).$$

The last equality is due to the signature property. $\square$

3.3.5 Linearized Hamiltonian flow and Reeb flow

Note that, on each level set $\{r\} \times \Sigma$, the hamiltonian vector field X_H is proportional to the Reeb vector field R_α, see Lemma 3.2.2. Accordingly, the Linearlized flow has a relation,

$$T_{\{r\}\times x} Fl_t^{X_H} = \{0\} \times T_x Fl_{h'(e^r)}^{R_\alpha}.$$

Let γ_H be a 1-periodic Hamiltonian orbit. Then γ_H is of the form (r, γ_R) for some $h'(r)$-periodic Reeb orbit γ_R in Σ. We define the Conley-Zehnder index $\mu_{CZ}(\gamma_R)$ of γ_R by

$$\mu_{CZ}(\gamma_R) = \mu_{CZ}(\phi|_\xi^{-1} \circ T_{\gamma_R(0)} Fl_t^{R_\alpha}|_\xi \circ \phi|_\xi).$$

Here we have assumed that the trivialization ϕ of $T\widehat{W}$ along γ_H respects the decomposition

$$T\widehat{W} = \xi \oplus \langle Y, R\rangle \tag{3.3.3}$$

where Y denotes the Liouville vector field. Then $\mu_{CZ}(\gamma_H)$ and $\mu_{CZ}(\gamma_R)$ are related as follows. (The notation $\mu_{CZ}(\gamma_H, \phi, [0,1])$ denotes the index of γ_H of period 1, with respect to ϕ.)

Proposition 3.3.15. $\mu_{CZ}(\gamma_H, \phi, [0,1]) = \mu_{CZ}(\gamma_R, \phi|_\xi, [0, h'(r)]) + \dfrac{1}{2}.$

Proof. Let $\phi : S^1 \times \mathbb{R}^{2n} \to \gamma_H^* T\widehat{W}$ be a trivialization which extends a trivialization which respects the decompostion (3.3.3). Then, due to the direct sum property, we have

$$\begin{aligned}
\mu_{CZ}(\gamma_H, \phi) &= \mu_{CZ}(TFl_t^{X_H}, \phi) \\
&= \mu_{CZ}(TFl_t^{X_H}|_\xi, \phi|_\xi) + \mu_{CZ}(TFl_t^{X_H}|_{\langle Y,R\rangle}, \phi|_{\langle Y,R\rangle}) \\
&= \mu_{CZ}(TFl_t^{R_\alpha}|_\xi, \phi|_\xi) + \mu_{CZ}(TFl_t^{X_H}|_{\langle Y,R\rangle}, \phi|_{\langle Y,R\rangle}) \\
&= \mu_{CZ}(\gamma_R, \phi|_\xi, [0, h'(r)]) + \mu_{CZ}(TFl_t^{X_H}|_{\langle Y,R\rangle}, \phi|_{\langle Y,R\rangle}, [0,1]).
\end{aligned}$$

One now computes

$$TFl_t^{X_H}(R) = R, \quad TFl_t^{X_H}(Y) = th''(r)R + Y.$$

This implies that

$$\mu_{CZ}(TFl_t^{X_H}|_{\langle Y,R\rangle}, \phi|_{\langle Y,R\rangle}) = \mu_{CZ}\left(\begin{pmatrix} 1 & 0 \\ th''(r) & 1 \end{pmatrix}, [0,1]\right) = \frac{1}{2}\operatorname{sign} h''(r) = \frac{1}{2},$$

where we used the signature property and the fact that $h''(r) > 0$. This completes the proof. $\square$

3.4 Moduli spaces of Floer trajectories

The differential of Floer complex counts the number of negative gradient flow lines of the action functional (3.2.1) with respect to an L^2-metric on the loop space $\Lambda\widehat{W}$. For this, we use an $\hat{\omega}$-compatible almost complex structure on $\widehat{W}$.

3.4.1 Admissible almost complex structures

Since $\widehat{W}$ is not compact, we need a special class of almost complex structures. Most of all, we would like to guarantee that all Floer trajectories with fixed asymptotics stay in a compact region. If not, we would be in trouble to apply compactness results on Moduli spaces.

Definition 3.4.1. An $\hat{\omega}$-compatible time-dependent almost complex structure $J : S^1 \times \widehat{W} \to End(T\widehat{W})$ is called **admissible** or **SFT-like** if J satisfies the following conditions on the cylindrical part $[0, \infty) \times \Sigma$:

1. J is invariant in r-direction;
2. $J(R_\alpha) = Y$;
3. J restricts to an almost complex structure on ξ.

Remark 3.4.2. One can define admissible almost complex structure in another way; choose first any almost complex structure on ξ, and extend it using the conditions (1), (2).

Let g be a metric on $\widehat{W}$ compatible to $(\hat{\omega}, J)$, that is,

$$g(X, Y) := \hat{\omega}(X, JY)$$

for $X, Y \in T\widehat{W}$. Then we have an L^2-*metric* on the free loop space $\Lambda\widehat{W}$ defined by

$$\langle X, Y \rangle := \int_0^1 g(X, Y)dt$$

for $X, Y \in T\Lambda\widehat{W}$.

Lemma 3.4.3. *With respect to the L^2-metric $\langle \cdot, \cdot \rangle$, the gradient of $\mathcal{A}_H$ at $x \in \Lambda\widehat{W}$ is given by*

$$\nabla \mathcal{A}_H = J(\gamma' - X_H).$$

Proof. As we have seen in the proof of Lemma 3.2.3, the differential of $\mathcal{A}_H$ is given by

$$d\mathcal{A}_H(\gamma)(Y) = \int_{S^1} \hat{\omega}(\gamma' - X_H, Y).$$

Accordingly, we compute

$$\begin{aligned}\langle J(\gamma' - X_H), Y\rangle &= \int_{S^1} g(J(\gamma' - X_H), Y)dt \\ &= \int_{S^1} \hat{\omega}(J(\gamma' - X_H), JY)dt \\ &= \int_{S^1} \hat{\omega}(\gamma' - X_H, Y)dt \\ &= d\mathcal{A}_H(Y).\end{aligned}$$

This proves the assertion. □

The negative gradient flow line $u : \mathbb{R} \to \Lambda\widehat{W}$, or equivalently $u : S^1 \times \mathbb{R} \to W$ is the solution of the following equation.

$$\partial_s u + J(\partial_t u - X_H) = 0. \tag{3.4.1}$$

This equation is called *Floer equation.* Let $\overline{\gamma}, \underline{\gamma} \in \mathcal{P}_H$ be 1-periodic orbits in $\widehat{W}$. Define $\widehat{\mathcal{M}}(\overline{\gamma}, \underline{\gamma}, H, J)$ is the set of solutions u of (3.4.1) such that

$$\lim_{s\to-\infty} u(s,t) = \overline{\gamma}(t), \quad \lim_{s\to\infty} u(s,t) = \underline{\gamma}(t).$$

Observe that if u is a solution of (3.4.1), then its translation $u(s+s_0,t)$ is also a solution for arbitrary $s_0 \in \mathbb{R}$. This gives a free $\mathbb{R}$-action on the set $\widehat{\mathcal{M}}(\overline{\gamma}, \underline{\gamma}, H, J)$. Define the *moduli space of Floer trajectories* $\mathcal{M}(\overline{\gamma}, \underline{\gamma}, H, J)$ by

$$\mathcal{M}(\overline{\gamma}, \underline{\gamma}, H, J) = \widehat{\mathcal{M}}(\overline{\gamma}, \underline{\gamma}, H, J)/\mathbb{R}.$$

3.4.2 Transversality

We outline the standard scheme to gent smoothness of the Moduli spaces using the implicit function theorem for Banach bundles.

Consider a Banach bundle $\mathcal{E} \to \mathcal{B}$ where

$$\mathcal{B} := \{u \in W^{1,p}(S^1 \times \mathbb{R}, \widehat{W}) \mid u(\infty, t) = \underline{\gamma}(t),\ u(-\infty,t) = \overline{\gamma}(t)\}$$

and the fiber at $u \in \mathcal{B}$ is given by

$$\mathcal{E}_u := L^p(S^1 \times \mathbb{R}, u^*T\widehat{W}).$$

Define a section $\overline{\partial}_{H,J} : \mathcal{B} \to \mathcal{E}$ by

$$u \mapsto \partial_s u + J(u)(\partial_t u + X_H) = \partial_s u + J(u)\partial_t u - \operatorname{grad} H.$$

This section is not a priori Fredholm. However, if asymptotes are *non-degenerate*, then it is Fredholm. To make this precise, we introduce:

Definition 3.4.4. An admissible time-dependent Hamiltonian $H : S^1 \times \widehat{W} \to \mathbb{R}$ is called **non-degenerate** if the linearized Hamiltonian time-1 flow $TFl_1^{X_H}$ has no eigenvalue equal to 1.

Remark 3.4.5. If H is time-*dependent* or equivalently, non-autonomous, then it is never non-degenerate; there is always S^1-degeneracy along the periodic orbits. At the best, non-autonomous Hamiltonian can be *transversely non-degenerate*, meaning that the linearized Hamiltonian time-1 flow has only one eigenvalue equal to 1.

Theorem 3.4.6 ([36])**.** *For non-degenerate Hamiltonian H, the section $\overline{\partial}_{H,J} : \mathcal{B} \to \mathcal{E}$ is Fredholm of index $\mu_{CZ}(\overline{\gamma}) - \mu_{CZ}(\underline{\gamma})$.*

One should notice that the equation $\overline{\partial}_{H,J}(u) = 0$ is exactly the same as the Floer equation (3.4.1). It follows that

$$\overline{\partial}_{H,J}^{-1}(0) = \widehat{\mathcal{M}}(\overline{\gamma}, \underline{\gamma}, H, J).$$

If we show that for all $u \in \overline{\partial}_{H,J}^{-1}(0) = \widehat{\mathcal{M}}(\overline{\gamma}, \underline{\gamma}, H, J)$, the differential of $\overline{\partial}_{H,J}$ at u,

$$T_u\overline{\partial}_{H,J} : T_u\mathcal{B} \to \mathcal{E}_u,$$

is surjective, then the moduli space $\widehat{\mathcal{M}}(\overline{\gamma}, \underline{\gamma}, H, J)$ is a smooth manifold by the implicit function theorem for Fredholm maps.

Observe that since $\overline{\partial}_{H,J}$ is a section, it is already surjective along the base direction. Therefore the only thing that matters for surjectivity of $T_u\overline{\partial}_{H,J}$ is the fiber direction. We briefly write the vertical part of $T_u\overline{\partial}_{H,J}$ by

$$D_u : W^{1,p}(S^1 \times \mathbb{R}, u^*T\widehat{W}) \to \mathcal{E}_u,$$

where its formula is

$$D_u X = \nabla_s X + J(u)\nabla_t X + \nabla_X J(u)\partial_t u - \nabla_X \operatorname{grad} H(t,u).$$

Definition 3.4.7. A pair (H, J) of admissible Hamiltonian and almost complex structure is called **regular** if D_u is surjective for all $u \in \overline{\partial}_{H,J}^{-1}(0)$.

The following standard theorem shows that we can always make (H, J) to be regular by a *generic* perturbation of J. Write the set of $\hat{\omega}$-compatible admissible almost complex structures on $\widehat{W}$ by $\mathcal{J}(\hat{\omega})$

Theorem 3.4.8. *There is a subset $\mathcal{J}_{reg}$ in $\mathcal{J}(\hat{\omega})$ of the 2nd Baire category such that for each $J \in \mathcal{J}_{reg}$, the corresponding pair (H, J) is regular.*

3.4.3 A priori energy bound

The family of Floer trajectories with *fixed asymptotes* admits a uniform energy bound, often called *a priori C^0-energy bound.* This is crucial to show compactness of moduli spaces.

Definition 3.4.9. Let $u : \Sigma \to W$ be a smooth map from a Riemann surface Σ. Its **energy** is defined by

$$E(u) := \int_{S^1\times\mathbb{R}} |\partial_s u|^2 dsdt.$$

The C^0-energy bound is then obtain by the following lemma, which estimates the energy in terms of the action of fixed asymptotes.

Lemma 3.4.10. *For a Floer trajectory $u \in \mathcal{M}(\overline{\gamma}, \underline{\gamma}, H, J)$, we have*

$$\mathcal{A}_H(\overline{\gamma}) - \mathcal{A}_H(\underline{\gamma}) = E(u)$$

Proof.

$$\begin{aligned}
E(u) &= \int_{S^1\times\mathbb{R}} g(\partial_s u, \partial_s u) dsdt \\
&= \int_{S^1\times\mathbb{R}} g(\partial_s u, -J(\partial_t u - X_H)) dsdt \\
&= \int_{S^1\times\mathbb{R}} d\hat{\lambda}(\partial_s u, \partial_t u - X_H) dsdt \\
&= \int_{S^1\times\mathbb{R}} u^* d\hat{\lambda} - \int_{S^1\times\mathbb{R}} d\hat{\lambda}(\partial_s u, X_H) dsdt \\
&= \int_{S^1} \overline{\gamma}^* \hat{\lambda} - \int_{S^1} \underline{\gamma}^* \hat{\lambda} + \int_0^1 H(\overline{\gamma}(t)) dt - \int_0^1 H(\underline{\gamma}(t)) dt \\
&= \mathcal{A}_H(\overline{\gamma}) - \mathcal{A}_H(\underline{\gamma})
\end{aligned}$$

where we used Stokes' theorem eventually. □

3.4.4 Maximum principle

A crucial difference of open manifold from closed manifold is that Floer trajectories a priori can escape to infinity. This may mean that we cannot apply the compactness results for closed manifolds. However, if the boundary of the domain is contact type, which we have imposed, then Floer trajectories must obey the *maximum principle*. This principle says that Floer trajectories in the moduli space $\mathcal{M}(\overline{\gamma}, \underline{\gamma}, H, J)$ must stay in a compact region. We can still apply the compactness results for closed case.

To illustrate a principle of the principle, we first show the case of J-holomorphic curves with fixed asymptotes. One can interpret J-holomorphic curves as solutions of (3.4.1) with $H = 0$. Let J be an SFT-like almost complex structure on a symplectization $(\mathbb{R} \times M, d(r\alpha))$. One should notice that SFT-like condition on J is quite essential for the following maximum principle.

Proposition 3.4.11. *Let (Σ, j) be a Riemann surface with complex structure j. Let $u : (\Sigma, j) \to (\mathbb{R} \times M, J)$ be a (j, J)-holomorphic curve, meaning that*

$$du \circ j = J \circ du. \tag{3.4.2}$$

Denote $u = (f, v)$. Then $f : \Sigma \to \mathbb{R}$ has no local maximum.

Proof. We just re-formulate a proof in [37]. We want to show that the function f is in fact sub-harmornic, i.e.

$$\Delta f = (\partial_s^2 + \partial_t^2) f \geq 0$$

where (s, t) denotes a local complex coordinate of Σ. This claim implies the assertion.

Note that $(du \circ j)\partial_s = \partial_t u$ and $(J \circ du)\partial_s = J(u)\partial_s u$. Therefore, in the local coordinate (s, t), the Cauchy-Riemann equation (3.4.2) leads to

$$\partial_s u + J(u)\partial_t u = 0.$$

Note that $TM \cong \xi \oplus R_\alpha$. We denote the projection from TM to the contact structure ξ by $\pi : TM \to \xi$. Since $u = (f, v)$, the above equation splits into

$$\begin{aligned} \partial_s f - \alpha(\partial_t v) &= 0, \\ \partial_t f + \alpha(\partial_s v) &= 0, \\ \pi(\partial_s v) + J\pi(\partial_t v) &= 0. \end{aligned}$$

Indeed, $\partial_s u = (\partial_s f)\partial_r + \alpha(\partial_s v)R_\alpha + \pi(\partial_s v)$ and $J\partial_t u = J[(\partial_t f)\partial_r + \alpha(\partial_t v)R_\alpha + \pi(\partial_t v)] = (\partial_t f)R_\alpha - \alpha(\partial_t v)\partial_r + J\pi(\partial_t v)$, and by comparing we get the above three identities.

Using the first and second identities we have

$$(\partial_s^2 + \partial_t^2)f = \partial_s(\alpha(\partial_t v)) - \partial_t(\alpha(\partial_s v)) = d\alpha(\partial_s v, \partial_t v).$$

Note that $d\alpha$ vanishes along the Reeb direction, and $d\alpha$ defines a symplectic structure on the bundle $\xi \to M$. Moreover J restricts to a $d\alpha$-compatible almost complex structure on ξ by the SFT-like condition. That said, using the third identity, we have

$$d\alpha(\partial_s v, \partial_t v) = d\alpha(\pi(\partial_s v), \pi(\partial_t v)) = d\alpha(\pi(\partial_s v), J\pi(\partial_s v)) \geq 0.$$

Therefore it follows. $\square$

In particular, when Σ is the cylinder $S^1 \times \mathbb{R} \cong \mathbb{C}P^1 \backslash \{0, \infty\}$, consider a family of J-holomorphic curves $u : S^1 \times \mathbb{R} \to \mathbb{R} \times M$ with fixed asymptotics. Then every such a curve must lie in a compact region, i.e., $u(S^1 \times \mathbb{R}) \subset [r_0, r_1] \times M$ for some fixed $r_0, r_1 \in \mathbb{R}$ which only depend on the asymptotics

Now we consider the case of Hamiltonian Floer trajectories with fixed asymptotes.

Proposition 3.4.12. *Let $\overline{\gamma}, \underline{\gamma}$ be 1-periodic orbits of H in $\widehat{W}$. Then every solution $u : \mathbb{R} \times S^1 \to \widehat{W}$ of (3.4.1) from $\overline{\gamma}$ to $\underline{\gamma}$ stays in a compact region $W \cup (\{r \leq r_0\} \times M)$ which depends only on the asymptotes.*

Proof. If u lies entirely in the interior W, there is nothing to prove. Suppose u touches the cylindrical part. In that region, we may write $u = (f, v) : \mathbb{R} \times S^1 \to \mathbb{R} \times M$. Then as in the proof of Proposition 3.4.11, the equation

3.4.1 splits in a local coordinate as

$$\partial_s f - \alpha(\partial_t v) - h'(t, f) = 0 \tag{3.4.3}$$

$$\partial_t f + \alpha(\partial_s v) = 0 \tag{3.4.4}$$

$$\pi(\partial_s v) + J_t \pi(\partial_t v) = 0. \tag{3.4.5}$$

Here we have used that $H(r, x) = h(r)$ for some convex function $h : \mathbb{R} \to \mathbb{R}$ in the cylindrical end.

Accordingly, we see that

$$d\alpha(\partial_s v, \partial_t v) = \partial_s(\alpha(\partial_t v)) - \partial_t(\alpha(\partial_s v)) = \Delta f - \partial_s(h'(t, f)) = \Delta f - h'' \cdot \partial_s f.$$

Just as in the proof of Proposition 3.4.11,

$$d\alpha(\partial_s v, \partial_t v) = d\alpha(\pi(\partial_s v), J_t \pi(\partial_s v)) \geq 0,$$

which implies that f is a solution of a second order partial differential inequality. It is well-known that such solutions obey the maximum principle. This completes the proof. □

Remark 3.4.13. An implicit ingredient of the principle is the *J-convexity* (or *strictly plurisubharmonic convexity*) of the contact type boundary. The boundary of a domain W in a complex manifold (V, J) is called **J-convex** if it is (locally) a regular level set of a J-convex function. If the Liouville vector field is pointing outward, then the cylindrical coordinate function $r : [1, \infty) \times \partial W$ is J-convex for SFT-like almost complex structure.

In this sense, the condition that Liouville vector field is pointing outward at the boundary of Liouville domain is quite essential in the contruction of symplectic homology.

3.4.5 Bubbling phenomenon

By a *bubble* we mean a J-holomorphic sphere $u : \mathbb{C}P^1 \to W$. The Gromov-Floer compactness theorem, [36, Corollary 3.4], tells us that a sequence of Floer trajectories with fixed asymptotes has a subsequence converging to a (possibly) broken Floer trajectories *up to bubbles*.

In Floer theory, it is very useful to exclude such bubbling phenomenon, especially in order to define a differential map by counting Floer trajectories. For closed symplectic manifolds, this is why one usually insists some assumptions on (M, ω). For example, one usually assumes the *symplectically aspherical* condition, meaning that ω vanishes on $\pi_2(M)$.

For open symplectic manifolds, such as a completion of a Liouville domain, things are even easier. Indeed, if a symplectic manifold (W, ω) is exact, then there is *no* non-constant J-holomorphic sphere. This can be seen by showing that its energy must vanish.

Remark 3.4.14. For closed manifold, symplectic form cannot be exact. This is because the top wedge of the symplectic form must serve a non-zero generator of the top cohomology, which is not the case when the symplectic form is exact.

Lemma 3.4.15. *If $u : \mathbb{C}P^1 \to W$ is J-holomorphic, then the energy of u vanishes.*

Proof. First observe that if u is J-holomorphic, then its energy $E(u)$ coincides to the *symplectic area*, that is

$$\int_{S^1 \times \mathbb{R}} |\partial_s u|^2 ds dt = \int_{S^2} u^* \omega,$$

which can be seen directly from the Cauchy-Riemann equation. Now let ω be exact, say $\omega = d\lambda$. Then we have

$$E(u) = \int_{S^2} u^* \omega = \int_{S^2} = du^* \lambda = 0$$

by Stokes' theorem. ☐

Corollary 3.4.16. *For a completed Liouville domain $\widehat{W}$, there is no non-constant J-holomorphic sphere in $\widehat{W}$.*

Proof. If u is a J-holomorphic sphere with $E(u) = 0$, then it is evident that u must be constant. So the claim follows. ☐

3.4.6 Floer trajectories of "small" Hamiltonians

In this section, we show that if a Hamiltonian is sufficiently small, then its Floer trajectories, i.e. solutions of (3.4.1) are time-independent. In other words, they are Morse trajectories. An intuition behind is that periods in

t-direction of Floer trajectories tend to zero as the given hamiltonian gets smaller.

Let $H : W \to \mathbb{R}$ be *any* Hamiltonian which is Morse. Consider a sequence of Hamiltonians $H_n : W \to \mathbb{R}$ defined by $H_n := H/n$ for each $n \in \mathbb{N}$.

Proposition 3.4.17. *Let $p, q \in W$ be critical points of H (and hence of H_n for all n) such that*

$$\mathrm{ind}_H(p) - \mathrm{ind}_H(q) \leq 2.$$

For sufficiently large n, every Floer trajectory from p to q is time-independent. Consequently, they are Morse trajectories.

Proof. We follow the proof in [38]. We only give a proof for the case when $\mathrm{ind}_H(p) - \mathrm{ind}_H(q) = 1$. The other cases follows from the same idea.

Suppose not. This means that there is a sequence $n_k \in \mathbb{N}$ which diverges to infinity and a sequence of Floer trajectories (u_{n_k}) of the Hamiltonians (H_{n_k}) from p to q such that u_{n_k} is time-*dependent*. Consider a rescaled sequence (v_{n_k}) given by

$$v_{n_k}(s,t) := u_{n_k}(n_k s, n_k t).$$

Evidently, v_{n_k}'s are Floer trajectories of H from p to q and are time-dependent. By the Gromov-Floer compactness theorem, v_{n_k} converges, after passing to a subsequence if necessary, to a possibly broken trajectory v from p to q. Since $\mathrm{ind}_H(p) - \mathrm{ind}_H(q) = 1$, it turns out that v is unbroken and hence $v \in \mathcal{M}(p, q, H, J)$. Note that $\mathcal{M}(p, q, H, J)$ is compact and is of dimension zero. Therefore v is isolated in $\mathcal{M}(p, q, H, J)$.

We now claim that v is time-*independent*. This claim in fact completes the proof; since v is isolated in $\mathcal{M}(p, q, H, J)$, v_{n_k} eventually coincides to v up to reparametrization, which contradicts to the fact that v_{n_k}'s are time-dependent.

Observe that v_{n_k} is $1/n_k$-periodic, i.e

$$v_{n_k}(s,t) = v_{n_k}(s, t + 1/n_k).$$

Fix *any* real number $r \in \mathbb{R}$. Then we see that

$$v_{n_k}(s,t) = v_{n_k}\left(s, t + \frac{\lfloor r n_k \rfloor}{n_k}\right)$$

for all k. Taking k to infinity, it follows that

$$v(s,t) = v(s,t+r).$$

Since the choice of r is arbitrary, we must have that v is time-independent. □

3.5 Definition of symplectic homology

In this section, we define symplectic homology of (W,ω) by taking a direct limit of Hamiltonian Floer homology groups.

3.5.1 Hamiltonian Floer homology

Let (H,J) be an admissible pair. Denote the set of 1-periodic Hamiltonian orbits (of course, including constant orbits) by $\mathcal{P}(H)$. We define a free $\mathbb{Z}$-module by

$$CF_k(H,J) := \bigoplus_{\substack{\gamma\in\mathcal{P}(H)\\ \mu_{CZ}(\gamma)=k}} \mathbb{Z}\langle\gamma\rangle.$$

Let $\mu_{CZ}(\overline{\gamma}) - \mu_{CZ}(\underline{\gamma}) = 1$. Then we have seen that the corresponding moduli space

$$\mathcal{M}(\overline{\gamma},\underline{\gamma},H,J)$$

is a compact smooth manifold of dimension zero. Moreover, it is equipped with a coherent orientation. It now makes sense to count the number of elements of $\mathcal{M}(\overline{\gamma},\underline{\gamma},H,J)$ with signs. Denote this algebraic number by

$$\#\mathcal{M}(\overline{\gamma},\underline{\gamma},H,J).$$

Define a linear map $\partial_k = \partial_k(H,J) : CF_k(H,J) \to CH_{k-1}(H,J)$ by

$$\partial_k(\overline{\gamma}) = \sum_{\substack{\underline{\gamma}\in\mathcal{P}(H)\\ \mu_{CZ}(\overline{\gamma})-\mu_{CZ}(\underline{\gamma})=1}} \#\mathcal{M}(\overline{\gamma},\underline{\gamma},H,J)\underline{\gamma}$$

for each generator, and extend linearly to $CF_k(H,J)$. By the maximum principle, the condition that $\partial_{k-1}\circ\partial_k = 0$ follows exactly from the same argument as the closed case. (Note that there is no sphere bubbling by exactness of ω.)

Theorem 3.5.1 ([36]). *The pair $(CF_*(H,J),\partial)$ forms a differential complex. That is $\partial_{k-1}\circ\partial_k = 0$.*

Now define its homology group by

$$HF_k(H,J) := \frac{\ker\partial_k}{\operatorname{im}\partial_{k+1}}.$$

We call this group the **Hamiltonian Floer homology of the pair** (H,J).

3.5.2 Continuation homomorphisms

In the closed case, for each homotopy of two Hamiltonians, we can define so-called *continuation homomorphism* between the corresponding Hamiltonian Floer homology groups. It counts *parametrized* Floer trajectories. We can do almost the same thing for non-closed case, but there is a major difference that the continuation homomorphism is not necessarily isomorphism in general. This is basically because a certain maximum principle is only guaranteed in one direction. We here work out some details about this.

Let (H^0,J^0) and (H^1,J^1) be admissible pairs. By a *homotopy* (H^s,J^s) between them we mean a smooth family of admissible pairs connecting them from $s=0$ to $s=1$ and (H^s,J^s) is s-independent for $s\leq 0$ or $s\geq 1$. For given such homotopy, we consider a *parametrized Floer equation*

$$\partial_s u + J^s(u)(\partial_t u - X_{H^s}) = 0 \tag{3.5.1}$$

for $u : S^1\times\mathbb{R}\to\widehat{W}$. Suppose that u is a solution of the above equation with asymptotic conditions

$$\lim_{s\to-\infty} u(t,s) = \overline{\gamma}(t)\in\mathcal{P}(H^0),\quad \lim_{s\to\infty} u(t,s) = \underline{\gamma}(t)\in\mathcal{P}(H^1). \tag{3.5.2}$$

Then we can show that u also obey a maximum principle, *provided that the slope is increasing* along the homotopy.

Proposition 3.5.2. *Let $u : S^1\times\mathbb{R}\to\widehat{W}$ be a solution of (3.5.1) with fixed asymptotic conditions as (3.5.2). Suppose that $\partial_s\partial_r H^s$ is non-negative in the cylindrical end, i.e. slope is increasing along the homotopy parameter. Then the image of u is contained in a compact region of $\widehat{W}$ which depends only on the asymptotes.*

Proof. A proof is almost similar to 3.4.12, so we just point out where the condition that $\partial_s \partial_r H^s$ is used.

Following the same computation as the proof of 3.4.12, one encounters the equation

$$\Delta f = d\alpha(\pi(\partial_s v), J_t \pi(\partial_s v)) - h'' \cdot \partial_s f + \partial_s \partial_r H^s.$$

Note that the last term of the right hand side newly appears due to t-dependence of the parametized Floer equation. However, once this term is non-negative, we still have that

$$\Delta f + h'' \cdot \partial_s f \geq 0$$

as before. That is why we have imposed that $\partial_s \partial_r H^s$ is non-negative. $\square$

By virtue of the above maximum principle, we can define a continuation homomorphism $HF_*(H^0, J^0) \to HF_*(H^1, J^1)$ by counting solutions of parametrized Floer equation (3.5.1), provided that the slope of H^1 is bigger than or equal to the slope of H^0.

More precisely, we first define a moduli space

$$\mathcal{M}(\overline{\gamma}, \underline{\gamma}, H^s, J^s)$$

to be the set of solutions $u : S^1 \times \mathbb{R} \to \widehat{W}$ with given asymptotes $\overline{\gamma} \in \mathcal{P}(H^0)$, $\underline{\gamma} \in \mathcal{P}(H^1)$. As in the closed case, we have some relavant version of transversality, compactness, and orientation results. In particular, for generic choice of homotopy, the moduli space $\mathcal{M}(\overline{\gamma}, \underline{\gamma}, H^s, J^s)$ is a smooth manifold of dimension $\mu_{CZ}(\overline{\gamma}) - \mu_{CZ}(\underline{\gamma})$.

Remark 3.5.3. Note that there is no $\mathbb{R}$-action on the moduli space anymore, so we do not mod out the moduli space.

In the case when $\mu_{CZ}(\overline{\gamma}) - \mu_{CZ}(\underline{\gamma}) = 0$, the moduli space is a compact smooth manifold of dimension zero. Therefore the algebraic number of its elements makes sense. Define a linear map $CF_k(H^0, J^0) \to CF_k(H^1, J^1)$ by

$$\overline{\gamma} \in \mathcal{P}(H^0) \longmapsto \sum_{\substack{\underline{\gamma} \in \mathcal{P}(H^1) \\ \mu_{CZ}(\overline{\gamma}) = \mu_{CZ}(\underline{\gamma})}} \# \mathcal{M}(\overline{\gamma}, \underline{\gamma}, H^s, J^s) \underline{\gamma}.$$

Then the standard compactness result implies that this map is a chain map.

Proposition 3.5.4. *The above map is a chain map.*

We now have a well-defined homomorphism at the homology level

$$\Phi_{01} : HF_*(H^0, J^0) \to HF_*(H^1, J^1)$$

which is called **continuation homomorphism** or briefly **continuation map**.

Let (H^2, J^2) is another pair which is *slope increasing* homotopic to (H^1, J^1). Denote the continuation map by

$$\Phi_{12} : HF_*(H^1, J^1) \to HF_*(H^2, J^2).$$

Note that (H^0, J^0) is also homotopic to (H^2, J^2) via slope increasing homotopy, so that the continuous map $\Phi_{02} : HF_*(H^0, J^0) \to HF_*(H^2, J^2)$ is well-defined. Furthermore, using the standard argument via *homotopy of homotopies*, see for example [36], we conclude the following.

Proposition 3.5.5. *The continuation maps are compatible with compositions, in the sense that*

$$\Phi_{12} \circ \Phi_{01} = \Phi_{02}.$$

3.5.3 A direct system of Hamiltonians and symplectic homology groups

Direct limit

We first give a brief presentation on direct limit. Let $(I, \leq)$ be a partially ordered set.

Definition 3.5.6. A partially ordered set $(I, \leq)$ is called a **directed set** if for any $i, j \in I$, there exists an element $k \in I$ such that

$$i \leq k, \quad j \leq k.$$

Let R be a commutative ring with a unit. Let $\mathcal{C}$ be a category of R-modules, i.e, its objects are R-modules and its morphisms are consists of R-module homomorphisms.

Definition 3.5.7. A **directed system** of R-modules on $(I, \leq)$ consists of a family of R-modules $(M_i)_{i\in I}$ and morphisms $f_{ij} : M_i \to M_j$ for each $i \leq j$ such that

1. $f_{ii} = \mathrm{id}_{M_i}$;
2. if $i \leq j \leq k$, then $f_{jk} \circ f_{ij} = f_{ik}$.

Let (M_i, f_{ij}) be a direct system of R-modules on a directed set $(I, \leq)$. We define its **direct limit** to be an R-module defined by

$$\varinjlim_{i\in I} M_i := (\bigsqcup_{i\in I} M_i)/\sim$$

where the equivalence relation $\sim$ on the disjoint union $\bigsqcup_{i\in I} M_i$ is given as follows; for two elements $x_i \in M_i$ and $x_j \in M_j$, we say $x_i \sim x_j$ if there exists an element $k \in I$ with $i \leq k$ and $j \leq k$ such that

$$f_{ik}(x_i) = f_{jk}(x_j).$$

Example 3.5.8. A trivial example is that we set $M_i = M$ for all $i \in I$ for some partially ordered set $(I, \leq)$ and a fixed R-module M. Assign the identity morphism to each $i \leq j$. This clearly forms a direct system. The corresponding direct limit is then M itself.

Example 3.5.9. We can define a direct limit in the category of topological spaces. Form a direct system $\{\mathbb{R}P^n, \iota_n\}_{n\in\mathbb{N}}$ with the canonical inclusion maps $\iota_n : \mathbb{R}P^n \hookrightarrow \mathbb{R}P^{n+1}$ over the direct set $(\mathbb{N}, \leq)$. Then the resulting direct limit is called the *infinite real projective space* and denoted by $\mathbb{R}P^\infty$. By the same construction, we can define, for example, the infinite dimensional sphere S^∞, as well as the infinite complex projective space $\mathbb{C}P^\infty$.

Cofinal family

In practical use of direct limit, it is often useful to consider the *cofinal family* of the given direct system. A **cofinal subset** J of a directed set $(I, \leq)$ is a subset such that for any $i \in I$, there exists an element $j \in J$ with $i \leq j$. It is evident that $(J, \leq)$ forms a directed set again.

Let (M_i, f_{ij}) be a direct system of R-modules on $(I, \leq)$. Just by restricting the index set form I to J, we have a family $(M_i, f_{ij})_{i,j\in J}$. Then it is immediate to check that the restriction still defines a direct system. Denote its corresponding direct limit by $\varinjlim_{J} M_i$. Then the standard argument using the universal property show the following.

Proposition 3.5.10. *The two direct limits are isormorphic to each other, i.e.*

$$\varinjlim_{i \in I} M_i \cong \varinjlim_{i \in J} M_i.$$

Symplectic homology

Denote the set of all pairs of admissible Hamiltonians and almost complex structures by $\mathcal{HJ}$. For two Hamiltonians H_1 and H_2 on $\widehat{W}$, we write $H_1 \leq H_2$ if $H_1(t,x) \leq H_2(t,x)$ for all $(t,x) \in S^1 \times \widehat{W}$. We give a partial order on $\mathcal{HJ}$ as follows:

$$(H_1, J_1) \preceq (H_2, J_2) \Longleftrightarrow H_1 \leq H_2.$$

Then the pair $(\mathcal{HJ}, \preceq)$ forms a *directed set*, in other words, for any two admissible pairs $(H_1, J_1), (H_2, J_2)$, there exists another pair (H, J) such that $(H_1, J_1) \preceq (H, J)$ and $(H_2, J_2) \preceq (H, J)$.

Now we define a *directed system* as follows. For each admissible pair $(H, J) \in \mathcal{HJ}$ we assign a group $HF_*(H, J)$. For any two elements $(H^1, J^1), (H^2, J^2) \in \mathcal{HJ}$ with $(H^1, J^1) \preceq (H^2, J^2)$, we assign a group homomorphism $\Phi_{12} : HF_*(H^1, J^1) \to HF_*(H^2, J^2)$. Then Proposition 3.5.5 shows that these assignments defines a direct system.

We finally define the **symplectic homology** of (W, ω) to be the direct limit

$$SH_*(W) := \varinjlim_{(H,J) \in \mathcal{HJ}} HF_*(H, J).$$

3.5.4 A natural action filtration

Hamiltonian Floer homology groups admit a natural action filtration and hence symplectic homology groups. The action filtration is a main idea of a construction of a *Morse-Bott spectral sequence*, which is a main topic of this thesis.

For a real number $a \in \mathbb{R}$, we define

$$CF_*^{(-\infty,a)}(H, J) \leq CF_*(H, J)$$

a subgroup of $CF_*(H,J)$ generated by all orbits $\gamma \in \mathcal{P}(H)$ with action $\mathcal{A}_H(\gamma) < a$. Since the action decreases along Floer trajectories, we have that $(CF_*^{(-\infty,a)}(H,J), \partial)$ forms a subcomplex. Now for $-\infty \le a < b \le \infty$, we define

$$CF_*^{[a,b)}(H,J) := CF_*^{(-\infty,b)}(H,J)/CF_*^{(-\infty,a)}(H,J).$$

Since ∂ preserves $CF_*^{(-\infty,a)}(H,J)$, it descends to $CF_*^{[a,b)}(H,J)$. Denote its homology by

$$FH_*^{[a,b)}(H,J) := H(CF_*^{[a,b)}(H,J), \partial).$$

To get the corresponding action filtration of symplectic homology, we need to examine the action difference along continuation maps. Let $u : \mathbb{R} \times S^1 \to \widehat{W}$ be a solution of the parametrized Floer equation (3.5.1) with asymptotic conditions

$$\lim_{s \to -\infty} u(t,s) = \overline{\gamma}(t) \in \mathcal{P}(H^0), \quad \lim_{s \to \infty} u(t,s) = \underline{\gamma}(t) \in \mathcal{P}(H^1).$$

The following can be shown by the same computation as Lemma 3.4.10.

Proposition 3.5.11. $\mathcal{A}_{H^0}(\overline{\gamma}) - \mathcal{A}_{H^1}(\underline{\gamma}) = E(u) + \int_{[0,1]\times S^1} \partial_s H^s(\gamma(t)) ds \wedge dt.$

Note that the additional term $\int_{[0,1]\times S^1} \partial_s H^s(\gamma(t)) ds \wedge dt$ appears due to the s-dependency. By maximum principle, this term is actually uniformly bounded for fixed asymptotes. It follows that, along parametrized Floer trajectories, which we have counted for continuation maps, the action value decreases. Therefore, the filtered Floer homology $HF_*^{[a,b)}(H,J)$ forms a direct system. Passing to its direct limit, we define

$$SH_*^{[a,b)}(W) := \varinjlim_{(H,J)\in\mathcal{HJ}} HF_*^{[a,b)}(H,J).$$

We define the *positive symplectic homology* by

$$SH_*^+(W) := SH_*^{[\epsilon,\infty)}(W)$$

where $\epsilon > 0$ is smaller than action of any non-constant periodic Reeb orbits in the boundary. Clearly $SH_*^+(W)$ does not depend on the choice of ϵ.

Likewise, we define the *negative symplectic homology* by

$$SH_*^-(W) := SH_*^{(-\infty,0)}(W).$$

3.6 Examples

We present some examples of symplectic homology groups.

3.6.1 Ball

Consider the ball $B^{2n} \in \mathbb{C}^n$ with the standard symplectic form ω_0. We can canonically identify its completion with $(\mathbb{C}^n, \omega_0)$. Choose an admissible Hamiltonian $H : \mathbb{C}^n \to \mathbb{R}$ which depends only on the cylindrical coordinate $|z|^2$. For such a suitable Hamiltonian, only 1-periodic orbit are constant orbit at the origin and two periodic Reeb orbits in the boundary at each actions level.

The point is that their Conley-Zehnder indices are dependent on the slope of Hamiltonians and even tends to infinity as the slope goes to infinity. It follows that, after passing to the direct limit, every generator disappears. See Oancea [23].

Proposition 3.6.1. $SH_*(B^{2n}) = 0$.

On the other hand the homology $H_*(B^{2n}, \partial B^{2n})$ does not vanish. Since $SH_*(B^{2n})$ is supposed to be generated by critical points in the interior and Reeb orbits in the boundary, it follows that the unit sphere with the standard contact structure satisfies *Weinstein conjecture.*

3.6.2 Annulus

By annulus, we mean a disk D^2 in $\mathbb{C}$ with one hole. Denote it by A. Then its completion is canonically isomorphic to the cotangent bundle of S^1 with the canonical Liouville form.

Consider an admissible Hamiltonian which is fiberwise convex and have exactly two critical points in the interior, which is homotopic to the zero section S^1. Then $SH_*(T^*S^1)$ is generated by these critical points, say x_0, x_1, and Reeb orbits on the unit cotangent bundle ST^*S^1. Note also that $ST^*S^1 = S^1 \sqcup S^1$ and its Reeb flow is nothing but the rotation on each components. Denote the simple periodic Reeb orbit on each component by

y_1 and y_2. By the *standard perturbation argument*, see Section 6.1.1, y_j splits into two 1-periodic orbits, say $y_j^{\pm}$.

Now we consider Floer trajectories. First of all, by topological reason, there is no trajectory between constant orbit and non-constant orbit in the boundary. Furthermore, by topological reason again, there is no Floer trajectory between non-constant orbit in different connected components of the boundary. Consequently, the only trajectories contributes are trajectories between $y_j^{\pm}$ for $j = 1, 2$, and these are nothing but the Morse trajectories on S^1 and hence the differential vanishes.

In conclusion, $SH_*(A)$ is infinitely many direct sum of homology groups of S^1, indexed by homotopy class of periodic orbits.

Proposition 3.6.2. *$SH_*(A) = \bigoplus_{\alpha \in \pi_1(A)} H_*^{\alpha}(S^1)$. In particular, if we only consider the contractible orbits, then*

$$SH_*^{cont}(A) = H_{*\pm 1}(S^1).$$

3.6.3 Cotangent bundles

For the general cotangent bundle over spin manifold Q, there is well-known result. A detailed and recent reference is Abouzaid [31]. Denote the free loop space of Q by $\mathcal{L}Q$.

Theorem 3.6.3 (Viterbo's theorem)**.** *$SH_*(T^*Q) \cong H_*(\mathcal{L}Q)$, up to a degree shift.*

As an example, in stringy topology theory, it is well-known that

$$H_*(T^*S^1) = \mathbb{Z}[y, x, y^{-1}]/\langle x^2 = 0 \rangle$$

as rings. We can see that this conincides to our computation before. Two constant orbits in the interior correspond to the unit and x, two simple non-constant orbits in a component of boundary correspond to y and xy.

3.7 Invariance

In this section, we discuss invariance of symplectic homology under Liouville isomorphisms and homotopies. To give an isormophism between symplectic homology groups in question, we construct adapted continuation maps. It therefore turns out that invariances rely on several versions of maximum principles.

3.7.1 Liouville isomorphisms

Let $(W_j, \omega_j = d\lambda_j)_{j=1,2}$ be Liouville domains, and denotes their completions by $\widehat{W_j}$. The notion of equivalence between Liouville domains can be formulated as follows.

Definition 3.7.1. A **Liouville isomorphism** $\psi : \widehat{W_1} \to \widehat{W_2}$ is a diffeomorphism such that

$$\psi^* \lambda_2 = \lambda_1 + dh$$

for some compactly supported function $h : \widehat{W_1} \to \mathbb{R}$.

Obviously, ψ is a symplectomorphism. In the cylindrical end, it looks like

$$\psi(r, x) = (r - f(x), \phi)$$

for some *contactomorphism* $\phi : \partial W_1 \to \partial W_2$ with $\phi^* \alpha_2 = e^f \alpha_1$. To see this, we denote $\psi(r, x) = (\psi_r(r, x), \phi(r, x)) \in [1, \infty] \times \partial W_1$. Then since h is compactly supported, ϕ is a contactomorphism such that

$$\phi^* \alpha_2 = e^f \alpha_1$$

for some function $f : \partial W_2 \to \mathbb{R}$. Note also that

$$\psi^*(\lambda_2) = \psi^*(e^r \alpha_2) = e^{\psi_r} \phi^* \alpha_2 = e^{\phi_r} e^f \alpha_1.$$

It follows that $\psi_r(r, x) = r - f(x)$.

In particular, a Liouville isomorphism restricts to a contactomorphism on the contact boundary. We now show that symplectic homology is invariant under Liouville isomorphisms.

Proposition 3.7.2. *If two Liouville domains* $(W_1, \omega_1), (W_2, \omega_2)$ *are Liouville isomorphic to each other, then we have*

$$SH_*(W_1) \cong SH_*(W_2).$$

Proof. This assertion actually becomes more clear if we translate the situation to a single Liouville domain with two different Liouville forms. Observe that since $\psi^*\lambda_2 = \lambda_1$ up to a compact region of W_1, the assertion amounts to claim the following statement: *For a Liouville domain (W, ω), if two Liouville 1-forms λ_1 and λ_2 coincide near the boundary, then $SH_*(W, \lambda_1)$ is isomorphic to $SH_*(W, \lambda_2)$.*

Fix an admissible Hamiltonian H on W. Since λ_1 and λ_2 coincide to each other near the boundary, H is admissible with respect to both of Liouville forms. Furthermore, admissibility imposes that H is just C^2-small Morse function in the interior W, where the form may be different to each other. It clearly follows that the generators of $CF_*(H, J, \lambda_1)$ and those of $CF_*(H, J, \lambda_2)$ are the same, and the differentials are also the same. Therefore $SH_*(W, \lambda_1)$ is isomorphic to $SH_*(W, \lambda_2)$. □

3.7.2 Liouville homotopies

Symplectic homology is also invariant under Liouville homotopies.

Definition 3.7.3. A **Liouville homotopy** is a smooth family of Liouville domain (W_s, λ_s), $s \in [0, 1]$. Since $W_s, s \in [0, 1]$ is then a diffeotopy, we usually fix the space W in the definition of Liouville homotopy.

Proposition 3.7.4. *Let $(W, \lambda_s), s \in [0, 1]$ be a Liouville homotopy. Then we have an isomorphism*

$$SH_*(W, \lambda_0) \cong SH_*(W, \lambda_1).$$

This follows from the fact that a Liouville homotopy actually gives a Liouville isomorphism in the following sense.

Proposition 3.7.5 ([30])**.** *If (W, λ_s) is a Liouville homotopy, then there is a diffeotopy $h_s : W \to W$ such that $h_0 = \mathrm{id}$ and $h_s^*\lambda_s - \lambda_0$ is exact, and $h_s^*\lambda_s - \lambda_0 = 0$ outside of a compact subset. In particular, (W, λ_0) is Liouville isomorphic to (W, λ_1).*

This is basically because one can reinterpret a Liouville homotopy as a procedure of attaching a trivial cobordism $([r_0, r_1] \times \partial W, \lambda)$ such that $\lambda = \lambda_0$ near $\{r_0\} \times \partial W$ and $\lambda = \lambda_1$ near $\{r_1\} \times \partial W$. For a detailed proof, see [30, Proposition 11.8].

Chapter 4

Equivariant Symplectic Homology

In this chapter, we define an equivariant version of symplectic homology. The main references are Bourgeois-Oancea [13], [14], and Gutt [39].

4.1 S^1-equivariant Morse homology

To illustrate a construction of S^1-equivariant symplectic homology, we first outline S^1-equivariant Morse homology.

Let M be a smooth manifold with an action of the circle S^1. Recall that the equivariant homology, denoted $H_*^{S^1}(M)$, of M is defined using so-called *Borel construction.* More precisely,

$$H_*^{S^1}(M) = H_*(M \times_{S^1} ES^1),$$

where $ES^1 = \lim\limits_{N\to\infty} S^{2N+1}$. The action of S^1 on ES^1 is given by the Hopf action, and $M \times_{S^1} ES^1$ is the quotient space of $M \times ES^1$ by the S^1-action $\tau\cdot(x,z) = (\tau\cdot x, \tau\cdot z)$ for $\tau \in S^1$, $(x,z) \in M \times ES^1$. Since $ES^1 = \lim\limits_{N\to\infty} S^{2N+1}$, one interprets equivariant homology as a direct limit as

$$H_*^{S^1}(M) = \lim_{N\to\infty} H_*(M \times_{S^1} S^{2N+1}).$$

In this section, we describe $H_*(M \times_{S^1} S^{2N+1})$ in terms of Morse homology, and consequently give a Morse homological description of equivariant homology.

4.1.1 S^1-equivariant Morse complex

Instead of doing Morse theory directly on the quotient space $M \times_{S^1} S^{2N+1}$, we work with a *Morse-Bott* function on $M \times S^{2N+1}$ which is invariant under the action. Choose a function $F : M \times S^{2N+1} \to \mathbb{R}$ such that

- F is S^1-equivariant, i.e,

$$F(\tau \cdot x, \tau \cdot z) = F(x, z)$$

  for all $\tau \in S^1$ and $(x, z) \in M \times S^{2N+1}$,

- the projected function $f : M \times_{S^1} S^{2N+1} \to \mathbb{R}$ is Morse.

Remark 4.1.1. In other words, F is Morse-Bott, and its critical submanifold consists of fibers of the projection $M \times S^{2N+1} \to M \times_{S^1} S^{2N+1}$. Each component of critical manifold forms an S^1-family.

For a critical point p of F, i.e, $p \in \mathrm{crit}(F)$, we denote the S^1-orbit space containing p by $S_p \subset M \times S^{2N+1}$. Denote the corresponding point in $M \times_{S^1} S^{2N+1}$ by $[p]$. Note that $[p]$ is then a critical point of f which is non-degenerate. Clearly from the notations, we have $S_{\tau \cdot p} = S_p$ and $[\tau \cdot p] = [p]$.

Definition 4.1.2. The **Morse-Bott index** of S_p is defined to be the Morse index of $[p]$, i.e,

$$\mathrm{ind}(S_p) := \mathrm{ind}_f([p]).$$

Definition 4.1.3. The S^1**-equivariant Morse complex** is defined by

$$C_k^{S^1}(F) := \bigoplus_{\mathrm{ind}(S_p)=k} \mathbb{Z}\langle S_p \rangle.$$

Remark 4.1.4. Observe that each generator S_p of the complex corresponds to a generator of the Morse complex for f.

4.1.2 S^1-equivariant Morse differentials

Let g be a metric on $M \times S^{2N+1}$ which is S^1-invariant. Then we have the induced metric $\underline{g}$ on $M \times_{S^1} S^{2N+1}$. Denote the negative gradient flow of F by $Fl_s^{\nabla F}$ with respect to g.

For a critical point $p \in \mathrm{crit}(F)$, we define the *unstable manifold* of S_p by

$$W^u(S_p) := \{(x, z) \in M \times S^{2N+1} \mid \lim_{s \to -\infty} Fl_s^{\nabla F}(x, z) \in S_p\}.$$

Similarly, the *stable manifold* of S_p is defined by

$$W^s(S_p) := \{(x,z) \in M \times S^{2N+1} \mid \lim_{s\to\infty} Fl_s^{\nabla F}(x,z) \in S_p\}.$$

In addition, for each critical point $[p]$ of f, we consider the unstable and stable manifolds with respect to $\underline{g}$.

Lemma 4.1.5. *For $p, q \in \mathrm{crit}(F)$, $W^u(S_p)$ intersects transversely to $W^s(S_q)$ if and only if $W^u([p])$ intersects transversely to $W^s([q])$.*

In other words, the pair $(f, \underline{g})$ is *Morse-Smale* if and only if $W^u(S_p)$ intersects transversely to $W^s(S_q)$ for all $p, q \in \mathrm{crit}(F)$. Since the Morse-Smale condition is generic, we have the following.

Lemma 4.1.6. *For generic g, $W^u(S_p)$ intersects transversely to $W^s(S_q)$.*

We assume a metric g is chosen to be generic from now on.

Definition 4.1.7. Let $\overline{p}, \underline{p} \in \mathrm{crit}(F)$. The **moduli space of gradient trajectories** $\widehat{\mathcal{M}}(S_{\overline{p}}, S_{\underline{p}}; F, g)$ from $S_{\overline{p}}$ to $S_{\underline{p}}$ consists of maps $v = (u, z) : \mathbb{R} \to M \times S^{2N+1}$ such that

- v is a negative gradient flow line of F, i.e, $v'(s) = -\nabla F(v(s))$,
- $\begin{cases} \lim\limits_{s\to-\infty} v(s) \in S_{\overline{p}} \\ \lim\limits_{s\to\infty} v(s) \in S_{\underline{p}} \end{cases}$.

Remark 4.1.8. Just put differently, the above conditions are equivalent to

- $\begin{cases} u'(s) = \nabla_x F(u(s), z(s)) \\ z'(s) = \nabla_z F(u(s), z(s)) \end{cases}$,
- $\begin{cases} \lim\limits_{s\to-\infty} (u(s), z(s)) = (\overline{x}, \overline{z}) \in S_{\overline{p}} \\ \lim\limits_{s\to\infty} (u(s), z(s)) = (\underline{x}, \underline{z}) \in S_{\underline{p}} \end{cases}$.

Here ∇_x denotes the gradient vector field of F restricted to TM and similarly for ∇_z.

Lemma 4.1.9. *The moduli space $\widehat{\mathcal{M}}(S_{\overline{p}}, S_{\underline{p}}; F, g)$ is a smooth manifold, and its dimension is*

$$\mathrm{ind}(S_{\overline{p}}) - \mathrm{ind}(S_{\underline{p}}) + 1.$$

Proof. Note that the moduli space is nothing but the intersection $W^u(S_{\overline{p}}) \cap W^s(S_{\underline{p}})$. Since g is chosen to be generic, $W^u(S_{\overline{p}})$ intersects transversely to $W^s(S_{\underline{p}})$. Therefore the moduli space is a smooth manifold. The dimension is then simply the dimension of the transverse submanifold. □

Let $\overline{p} \neq \underline{p} \in \mathrm{crit}(F)$. Then the moduli space $\widehat{\mathcal{M}}(S_{\overline{p}}, S_{\underline{p}}; F, g)$ carries a natural $\mathbb{R}$-action. Define the **moduli space of gradient trajectories** by the quotient space

$$\mathcal{M}(S_{\overline{p}}, S_{\underline{p}}; F, g) := \widehat{\mathcal{M}}(S_{\overline{p}}, S_{\underline{p}}; F, g)/\mathbb{R}.$$

Note that for each gradient trajectory, S^1-action on $M \times S^{2N+1}$ produces an S^1-family of trajectories. Therefore the moduli space $\mathcal{M}(S_{\overline{p}}, S_{\underline{p}}; F, g)$ admits a free S^1-action.

Definition 4.1.10. The **S^1-equivariant Moduli space of gradient trajectories from $S_{\overline{p}}$ to $S_{\underline{p}}$** is the quotient space

$$\mathcal{M}_{S^1}(S_{\overline{p}}, S_{\underline{p}}; F, g) := \mathcal{M}(S_{\overline{p}}, S_{\underline{p}}; F, g)/S^1.$$

Remark 4.1.11. The S^1-equivariant moduli space $\mathcal{M}_{S^1}(S_{\overline{p}}, S_{\underline{p}}; F, g) := \mathcal{M}$ corresponds to the Moduli space $\mathcal{M}[\overline{p}], [\underline{p}]; f, \underline{g})$ for the Morse homology of $(f, \underline{g})$.

The S^1-equivariant moduli space $\mathcal{M}_{S^1}(S_{\overline{p}}, S_{\underline{p}}; F, g)$ has the dimension

$$\mathrm{ind}(S_{\overline{p}}) - \mathrm{ind}(S_{\underline{p}}) - 1.$$

In particular, if $\mathrm{ind}(S_{\overline{p}}) - 1 = \mathrm{ind}(S_{\underline{p}})$, then $\mathcal{M}_{S^1}(S_{\overline{p}}, S_{\underline{p}}; F, g)$ is of zero dimensional.

Now we define the **S^1-equivariant Morse differential** $\partial^{S^1} : C_k^{S^1}(F) \to C_{k-1}^{S_1}(F)$ by counting the S^1-equivariant moduli spaces, namely

$$\partial^{S^1} S_{\overline{p}} = \sum_{\mathrm{ind}(S_{\underline{p}}) = \mathrm{ind}(S_{\overline{p}}) - 1} \sum_{[v] \in \mathcal{M}^{S^1}} \epsilon([v]) S_{\underline{p}}$$

where $\mathcal{M}^{S^1} = \mathcal{M}_{S^1}(S_{\overline{p}}, S_{\underline{p}}; F, g)$ and $\epsilon([v])$ is given by a coherent oritentation.

Remark 4.1.12. By construction it is evident that the S^1-equivariant moduli space $\mathcal{M}_{S^1}(S_{\overline{p}}, S_{\underline{p}}; F, g)$ and the *Morse moduli space* $\mathcal{M}([\overline{p}], [\underline{p}], f, \underline{g})$ are in one-to-one correspondence.

Proposition 4.1.13. *The module $C_*^{S^1}(F)$ together with the map ∂^{S^1} forms a differential complex, i.e, $\partial^{S^1} \circ \partial^{S^1} = 0$.*

Proof. It follows directly from the above remark 4.1.12. □

Since the generators, indices and differentials (even regularity as well) correspond to those of the Morse complex of $(f, \underline{g})$, we have

$$H_*(C_*^{S^1}, \partial^{S^1}) \cong HM_*(M \times_{S^1} S^{2N+1}, f, \underline{g}) \cong H_*(M \times_{S^1} S^{2N+1}).$$

Now define S^1*-equivariant Morse homology* of M by

$$HM^{S^1}(M) := \lim_N H_*(C_*^{S^1}(F, g), \partial^{S^1}).$$

Then the upshot of the above constructions is the following.

Proposition 4.1.14. *The S^1-equivariant Morse homology is equivalent to the S^1-equivariant (singular) homology of M, i.e.,*

$$HM_*^{S^1}(M) \cong M_*^{S^1}(M).$$

4.2 S^1-equivariant symplectic homology

We follow the scheme of S^1-equivariant Morse homology. Namely, the ambient space M will be replaced by the free loop space $\Lambda\widehat{W}$ of the completion $\widehat{W}$, and S^1-invariant function $F : M \times S^{2N+1} \to \mathbb{R}$ will be replaced by an S^1-invariant *Hamiltonian action functional* on $\Lambda\widehat{W} \times S^{2N+1}$ with respect to the obvious diagonal S^1-action on $\Lambda\widehat{W} \times S^{2N+1}$.

4.2.1 S^1-invariant action functional

Note that the free loop space $\Lambda\widehat{W}$ carries a natural S^1-action by

$$(g \cdot \gamma)(t) = \gamma(t - g)$$

for $g \in S^1$ and $\gamma \in \Lambda\widehat{W}$. We have the diagonal S^1-action on the product $\Lambda\widehat{W} \times S^{2N+1}$ as

$$g \cdot (\gamma, z) = (g \cdot \gamma, g \cdot z).$$

To define an S^1-invariant action functional, we consider a Hamiltonian $H : S^1 \times \widehat{W} \times S^{2N+1} \to \mathbb{R}$ invariant under this diagonal action, namley, for each $g \in S^1$

$$H(t+g, x, g \cdot z) = H(t, x, z)$$

where $(t, x, z) \in S^1 \times \widehat{W} \times S^{2N+1}$. Define the S^1-equivariant action functional

$$\begin{aligned} \mathcal{A}_H^N :& \Lambda\widehat{W} \times S^{2N+1} \to \mathbb{R}, \\ &(\gamma, z) \longmapsto -\int_{S^1} \gamma^*\lambda - \int_0^1 H(t, \gamma(t), z)dt. \end{aligned}$$

Then it is evident that $\mathcal{A}_H^N$ is S^1-invariant. Denote the set of critical points of $\mathcal{A}_H^N$ by $\mathcal{P}(H)$.

Lemma 4.2.1. *The set $\mathcal{P}(H)$ consists of pairs (γ, z_0) such that*

1. *γ is a 1-perioic orbit of H with fixed z_0;*
2. *$\int_{S^1} \frac{\partial H}{\partial z}(t, \gamma(t), z_0)dt = 0$.*

Proof. Essentially the same computation as Lemma 3.2.3 applies, except for the term

$$\int_{S^1} \frac{\partial H}{\partial z}(t, \gamma(t), z_0)dt$$

which makes the second condition of the assertion. $\square$

Since the action functional $\mathcal{A}_H^N$ is S^1-invariant, for each $p := (\gamma, z) \in \mathcal{P}(H)$, we have an S^1-orbit

$$S_p := \{(\tau \cdot \gamma, \tau \cdot z) \mid \tau \in S^1\} \subset \mathcal{P}(H).$$

As in the definition of symplectic homology, we impose the following *admissible* conditions on Hamiltonians.

Definition 4.2.2. A Hamiltonian $H : S^1 \times \widehat{W} \times S^{2N+1} \to \mathbb{R}$ is called **admissible** if

1. H is S^1-invariant, i.e., $H(t+g, x, g \cdot z) = H(t, x, z)$;
2. On the interior W_0, H is C^2-small and Morse;

3. On the symplectization part, H is cylindrical and strictly increasing that is, for $(r, y) \in [0, \infty) \times \partial W$,

$$H(t, r, y, z) = h(t, r)$$

   for sufficiently large r and a convex function h.

4. H is linear at the end, i.e., $H(t, r, y, z) = se^r + \beta(z)$ for sufficiently large r with $s \notin Spec(\partial W, \alpha)$ and $\beta \in C^\infty(S^{2N+1}, \mathbb{R})$.

5. H is *non-degenerate* in the sense that every S^1-orbit S_p is non-degenerate i.e., the Hessian $d^2\mathcal{A}_H^N(\gamma, z)$ has only 1-dimensional kernel at $(\gamma, z) \in S_p$.

Note that the third and the forth conditions are essential to guarantee the maximum principle as the non-equivariant version. We choose a family of almost complex structures $\{J_z^t\}_{z \in S^{2N+1}, t \in S^1}$ on $\widehat{W}$ such that

1. for each $z \in S^{2N+1}$, J_z^t is *admissible* in the sense of Definition 3.4.1;
2. S^1**-invariant** in the sense that $J_{\tau \cdot z}^{t+\tau} = J_z^t$ for $\tau \in S^1$.

Having such a S^1-invariant family of almost complex structures, we can formulate an L^2-metric on the free loop space $\mathcal{L}\widehat{W}$, parametrized by $z \in S^{2N+1}$, as follows;

$$\langle X, Y \rangle_z := \int_{S^1} \omega(X(t), J_z^t(\gamma(t))Y(t))dt$$

where $X, Y \in \Gamma(S^1, \gamma^* T\widehat{W})$. In addition, by choosing an S^1-invariant metric on S^{2N+1}, we obtain an S^1-invariant L^2-metric on $\mathcal{L}\widehat{W} \times S^{2N+1}$. Now, from the obvious negative L^2-gradient flow equation for the S^1-invariant action functional $\mathcal{A}_H^N$ with respect to the L^2-metric, we obtain the following *parametrized Floer equations*:

Lemma 4.2.3. *A pair $(u(s,t), z(s)) : \mathbb{R} \times S^1 \to \widehat{W} \times S^{2N+1}$ is a negative L^2-gradient solution for $\mathcal{A}_H^N$ if and only if*

$$\begin{aligned} \partial_s u + J_{z(s)}^t(u(s,t)) \Big(\partial_t u - X_{H_{z(s)}}^t(u(s,t)) \Big) &= 0, \\ \dot{z}(s) - \int_{S^1} \nabla_z H(t, u(s,t), z(s)) dt &= 0. \end{aligned}$$

Here ∇_z indicates the gradient with respect to the z-coordinates.

4.2.2 S^1-equivariant Floer complex

Let $H : S^1 \times \widehat{W} \times S^{2N+1} \longrightarrow \mathbb{R}$ be an admissible Hamiltonian. For an S^1-invariant family of almost complex structures J and a metric g on S^{2N+1}, we define S^1**-equivariant Floer chain complex** $SC_*^{S^1,N}(H,J,g)$ by

$$SC_*^{S^1,N}(H,J,g) := \bigoplus_{S_p \subset \mathcal{P}(H)} R\langle S_p \rangle.$$

Here we grade the complex by the Robbin-Salmon index $\mu(S_p) := \mu_{RS}(\gamma)$ as usual.

Let $S_{\overline{p}}$ and $S_{\underline{p}}$ be the two distinct S^1-orbits (and hence generators of the complex). We denote a moduli space of parametrized Floer solutions from $S_{\overline{p}}$ to $S_{\underline{p}}$ by $\mathcal{M}(S_{\overline{p}}, S_{\underline{p}}; H, Jg)$. In other words, the moduli space consists of solutions (u,z) of the equation in Lemma 4.2.3 with the following asymptotic conditions: $\lim_{s \to -\infty}(u(s,t), z(s)) \in S_{\overline{p}}$ and $\lim_{s \to \infty}(u(s,t), z(s)) \in S_{\underline{p}}$.

Since the action functional $\mathcal{A}_H^N$ and (J,g) are S^1-invariant, the moduli space $\mathcal{M}(S_{\overline{p}}, S_{\underline{p}}; H, Jg)$ admits an obvious S^1-action induced by the diagonal action on $C^\infty(S^1, \widehat{W}) \times S^{2N+1}$, which is free. We denote its quotient by $\mathcal{M}_{S^1}(S_{\overline{p}}, S_{\underline{p}}; H, Jg)$ and call S^1*-equivariant moduli space.*

It is shown in [52, Theorem A] that for a generic choice of S^1-invariant Floer data (H,J,g), the S^1-equivariant Moduli space is a smooth manifold of dimension $\mu(S_{\overline{p}}) - \mu(S_{\underline{p}}) - 1$. Furthermore, by taking an S^1-invariant trivializations of $\gamma^* T\widehat{W} \oplus T_z S^{2N+1}$, we have that the usual linearized Floer equation, for example [52, Formula 2.11], only depends on the orbit S_p. Therefore the usual scheme of giving a coherent orientation in [28] also works for S^1-equivariant moduli spaces.

We now define the S^1-equivariant differential ∂^{S^1} on the chain complex $SC_*^{S^1,N}(H,J,g)$ by

$$\partial^{S^1} S_{\overline{p}} := \sum_{\substack{S_{\underline{p}} \subset \mathcal{P}(H) \\ \mu(S_{\overline{p}}) - \mu(S_{\underline{p}}) = 1}} \sum_{u \in \mathcal{M}_{S^1}(S_{\overline{p}}, S_{\underline{p}}; H, Jg)} \varepsilon(u) S_{\underline{p}}.$$

Here the sign $\varepsilon(u)$ comes from the chosen coherent orientation. One can now prove that ∂^{S^1} actually defines a differential, that is, $\partial^{S^1} \circ \partial^{S^1} = 0$. We define S^1**-equivariant Hamiltonian Floer homology** by taking the homology of the S^1-equivariant Floer complex $(SC_*^{S^1,N}(H,J,g), \partial^{S^1})$:

$$HF_*^{S^1,N}(H,J,g) := H_*(SC_*^{S^1,N}(H,J,g), \partial^{S^1}).$$

As the non-equivariant version, along the slopes of Hamiltonians, admissible data (H,J,g) serves a direct system for S^1-equivariant Floer homology groups. By taking the direct limit, we define

$$SH_*^{S^1,N}(W) := \varinjlim_{(H,J,g)} FH_*^{S^1,N}(H,J,g).$$

Note that the obvious inclusion $S^{2N+1} \hookrightarrow S^{2N+3}$ also serves a direct system for the above groups. We finally define S^1**-equivariant symplectic homology group** by

$$SH_*^{S^1}(W) := \varinjlim_{N} SH_*^{S^1,N}(W).$$

Clearly, S^1-equivariant symplectic homology groups admits a natural action filtration as the non-equivariant version. This leads us to the **positive S^1-equivariant symplectic homology group** $SH_*^{+,S^1}(W)$.

Chapter 5

Morse-Bott Spectral Sequence for Morse Homology

In this chapter, we construct Morse-Bott spectral sequence for Morse homolgy. As usual, Morse theory will provide basic ideas towards Morse-Bott spectral sequences for Floer homology. The construction is based on the *Morse-Bott techniques*, described for example in the appendix of [18]; we use Morse-Bott function directly for Morse Homology, keeping in mind a standard perturbation of the function.

For a given Morse-Bott function F, we manipulate a sensible action filtration which "respects" the standard perturbation F_δ. A classical theorem on spectral sequences associated filtrations then gives a spectral sequence converging to Morse homology whose E^1-page consists of *local Morse homology* of each Morse-Bott components.

5.1 Fredholm operators

We collect some basic facts on Fredholm operators, mostly without proof. The main references for this section are a lecture note of Lee [24] and a book by Conway [25].

Let X, Y be Banach spaces, i.e. complete normed vector spaces.

Definition 5.1.1. A linear operator $F : X \to Y$ is called **bounded** if there exists a real number $M > 0$ such that

$$||F(x)|| \leq M \cdot ||x||$$

for all $x \in X$

Definition 5.1.2. A bounded linear operator $F : X \to Y$ is called **Fredholm** if its image $F(X)$ is closed in Y and $\ker F$, $\operatorname{coker} F$ are finite dimensional. The **Fredholm index** of F is defined by

$$\operatorname{ind}(F) := \dim \ker F - \dim \operatorname{coker} F.$$

Remark 5.1.3. It turns out that the condition $F(X)$ is closed is redundant since if $\dim \operatorname{coker} F$ is finite, $F(X)$ is automatically closed. The proof is not hard, but we omit this.

Remark 5.1.4. If X and Y are finite dimensional, then every bounded operator is obviously Fredholm.

A useful property of Fredholm operators is the *stability under compact perturbation.*

Definition 5.1.5. A linear operator $K : X \to Y$ is called **compact** if the image of the unit ball in X is relatively compact in Y.

Remark 5.1.6. Clearly, every compact operator is bounded.

For a Banach space X, it is well-known that the unit ball is relatively compact in X if and only if X is finite dimensional.

Proposition 5.1.7. *The identity operator $id : X \to X$ is compact if and only if X is finite dimensional.*

Theorem 5.1.8 (Index stability theorem)**.** *If $F : X \to Y$ is a Fredholm operator and $K : X \to Y$ is a compact operator, then $F + K : X \to Y$ is Fredholm, and its index is* $\operatorname{ind}(F + K) = \operatorname{ind}(F)$.

The Fredholm property is also stable under a *small* perturbation: Let $F : X \to Y$ be a Fredholm operator. Suppose that its kernel and image are complemented, i.e.

$$X = \ker F \oplus X_0, \quad Y = F(X) \oplus Y_0$$

for some *closed* subspaces $X_0 \subset X$ and $Y_0 \subset Y$. Define a new operator $\tilde{F} : X_0 \times Y_0 \to Y$ by

$$\tilde{F}(x, y) = F(x) + y.$$

Observe that $\tilde{F}$ is bijective. This is almost direct from the definition. By the open mapping theorem, it has the inverse bounded operator $\tilde{F}^{-1} : Y \to X_0 \times Y_0$.

Theorem 5.1.9 (Index continuity theorem)**.** *Let $F : X \to Y$ be a Fredholm operator and $S : X \to Y$ (not necessarily Fredholm) bounded operator. If S is small in the sense that*

$$||S|| \leq \frac{1}{||\tilde{F}^{-1}||},$$

then $F + S$ is also Fredholm and its index is

$$\mathrm{ind}(F + S) = \mathrm{ind}(F).$$

The Fredholm index is additive under the composition of Fredholm operators.

Proposition 5.1.10. *Let X, Y, Z be Banach spaces and $F : X \to Y$, $G : Y \to Z$ Fredholm operators. Then the composition $G \circ F : X \to Z$ is also a Fredholm operator and its index is*

$$\mathrm{ind}(G \circ F) = \mathrm{ind}(G) + \mathrm{ind}(F).$$

5.2 Morse-Bott functions

This section gives a definition of Morse-Bott functions. We also present a standard way of perturbing a Morse-Bott function to make it a Morse function.

5.2.1 Definition

Let M be a compact smooth manifold of dimension n.

Definition 5.2.1. A function $F : M \to \mathbb{R}$ is called **Morse-Bott** if

- the set of critical points, denoted by $\mathrm{crit}(F)$, forms a submanifold of M;

- for each connected component Σ of $\mathrm{crit}(F)$, the Hessian of F restricted to the normal bundle of Σ in M is non-degenerate.

As Morse functions, local behavior of Morse-Bott functions is completely described in the following "Morse-Bott lemma". For a proof, we refer to [26, Lemma 3.51].

Lemma 5.2.2 (Morse-Bott lemma, [26])**.** *Let $F : M \to \mathbb{R}$ be a Morse-Bott function and Σ a connected component of the critical submanifold. For any $p \in \Sigma$, there exists a coordinates chart (u, v, w) at p such that*

$$F(u, v, w) = F(\Sigma) - |v|^2 + |w|^2$$

in this chart.

Definition 5.2.3. Let $F : M \to \mathbb{R}$ be a Morse-Bott function and Σ a connected component of the critical submanifold. We define the **Morse-Bott index** $\mathrm{ind}_\Sigma F$ of Σ to be the number of negative eigenvalues of $HessF|_{\nu(\Sigma)}$.

5.2.2 Standard Perturbation

We describe a standard way of perturbing a Morse-Bott function F to a Morse function. For each connected component Σ_j, choose a Morse function $f_j : \Sigma_j \to \mathbb{R}$ such that $0 \leq f_j \leq 1$. We can extend f_j to a tubular neighborhood $\nu(\Sigma_j)$ by a suitably chosen cutoff function, depending only on the distance from Σ. More precisely, we define the extension $\overline{f}_j$ by

$$\overline{f}_j(x) = \rho_j(|x|) f_j(\pi_j(x))$$

where $x \in \nu(\Sigma_j)$ and $\pi_j : \nu(\Sigma_j) \to \Sigma_j$ is the obvious projection map. Define a perturbation $F_\delta : M \to \mathbb{R}$ of F by

$$F_\delta := F + \delta \Sigma_j \overline{f}_j$$

for $\delta > 0$.

Roughly speaking, we have perturbed F in a small neighborhood of Σ_j in a way that the perturbation does not affect the normal part, where F was already Morse along that part. The following Lemma is obvious.

Lemma 5.2.4. *The function F_δ is Morse, provided that $\delta > 0$ is small enough.*

Take a metric g on M. Denote the gradient vector field of F_δ with respect to g by $gradF_\delta$. Then it is evident from the way of perturbation that $gradF_\delta$ is "negative gradient like" to F.

Lemma 5.2.5. *For sufficiently small $\delta > 0$, we have*

$$dF(-grad_g F_\delta) \leq 0.$$

In particular, the inequality is strict on the complement of Σ_j's.

5.3 Morse Homology with local coefficient systems

We recollect definitions of Morse homology, with a special focus on the coherent orientations. We first give the story with $\mathbb{Z}$-coefficient, and with local coefficient system.

Orientation lines

Let f be a Morse function and g a metic on M. We assume that M is oriented. Let p be a critical point of f, and fix a trivialization $\mathcal{B}_p$ of T_pM, which coincides to the orientation of M. Set a "capped" real line $\overline{\mathbb{R}}$ by $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. We give a smooth structure on $\overline{\mathbb{R}}$ in a similar way with [27, Definition 2.1], namely, we impose that the function $h : \overline{\mathbb{R}} \to [-1, 1)$ given by

$$h(t) = \frac{t}{\sqrt{1+t^2}}$$

is a diffeomorphism.

With respect to the trivialization $\mathcal{B}_p$ of T_pM, we define an operator $D_p : W^{1,2}(\overline{\mathbb{R}}, \mathbb{R}^n) \to L^2(\mathbb{R}, \mathbb{R}^n)$ by

$$X \mapsto \partial_s X + B_p(s)X$$

where $B_p : \overline{\mathbb{R}} \to \mathbb{R}^{n^2}$ is a smooth function such that $B_p(s) = Hess_p f$ for sufficiently large s. One should notice that D_p is the the linearization of the Morse equation $\partial + \nabla f = 0$.

Lemma 5.3.1. *The operator D_p is Fredholm, and its Fredholm index is given by*

$$\operatorname{ind} D_p = n - \operatorname{ind}_p f.$$

Proof. It is well-known that the Fredholm property only depends on asymptotic behavior of the operator. In this case, since $B_p(s) = Hess_p f$ for $s \gg 1$ and $Hess_p f$ is non-degenerate, D_p is Fredholm.

For the index, observe that for each positive eigenvalue $\lambda > 0$ of $Hess_p f$, elements in $W^{1,2}(\overline{\mathbb{R}}, \mathbb{R}^n)$ of the form $X(s) = Ce^{-\lambda}t$ $(s \gg 1)$ forms a basis of $\ker D_p$. This implies that $\operatorname{ind} D_p = n - \operatorname{ind}_p(f)$. $\square$

Definition 5.3.2. Let $D : W \to L$ be a Fredholm operator between Banach spaces. We define its **determinant line** $\det D$ by

$$\det D := \wedge^{max} \ker D \otimes \wedge^{max} (\operatorname{coker} D)^*.$$

The **orientation line** of D, denoted by $|\det D|$, is a graded free abelian group of rank 1 defined by

$$|\det D| = \langle \sigma_1, \sigma_2 \rangle / \{\sigma_1 + \sigma_2 = 0\}$$

where σ_j's are different orientations of the determinant line $\det D$. The determinant line and orientation line are naturally graded by the index of D.

We in particular denote the orientation line of the operator D_p at p by o_p. Note that the definition of D_p is a priori dependent of the choice of a function B_p. However, we can show that its orientation line o_p is canonically determined. This follows from the following series of lemmas. In particular, the first one is [27, Lemma 2.15], and the second one is in Appendix of [28].

Lemma 5.3.3 ([27])**.** *Let $\mathcal{F}$ be the space of Fredholm operators from the Sobolev space $W^{1,2}(\mathbb{R}, \mathbb{R}^n)$ to $L^2(\mathbb{R}, \mathbb{R}^n)$. Let $\mathcal{O}$ be a subset consists of operators of the form*

$$\partial_s + B(s)$$

where B is asymtotically the same for all elements in $\mathcal{O}$. Then $\mathcal{O}$ is constractible in $\mathcal{F}$.

Lemma 5.3.4 ([28])**.** *Let $\mathcal{F} : [0,1] \to Fred(W, L)$ be a* continuous *family of Fredholm operators between Banach spaces W and L. We define*

$$\det \mathcal{F} := \bigcup_{t \in [0,1]} \det F(t).$$

Then there is a natural real line bundle structure on $\det \mathcal{F}$ *over* I *with fiber* $\det F(t)$ *at* t.

Lemma 5.3.5. *Let* $L \to B$ *be a line bundle over a connected space* B. *Let* x *be a path in* B *from* p *to* q. *Then there is an isomorphism*

$$L_p \cong L_q$$

which is canonical up to multiplication by positive constant. This isomorphism is invariant in the homotopy class of x *relative to the end points. As a result, we have a* canonical *isomorphism between orientation lines*

$$|L_p| \cong |L_q|.$$

Proof. Note that x^*L is a line bundle over the interval I which is contractible. Since the bundle x^*L is then trivial, there are only two homotopy classes of non-vanishing sections, according to "signs". On the other hand, a non-vanishing section clearly defines an isomorphism $L_p \cong L_q$. It is evident that this isomorphism is invariant in the homotopy class of the non-vanishing section, up to positive multiplication. One can moreover show that this isomorphism does not depend on the choice of homotopy classes of non-vanishing sections. This completes the proof. □

Combining the above lemmas, we conclude that the orientation line o_p does not depend on the choice of B_p if we fix its asymptotic behavior as we did. The upshot is that we have associated to each critical point p an orientation line o_p. We can now define the Morse chain group by the following graded free abelian group

$$CM_*(f) := \bigoplus_{p \in \mathrm{crit}(f)} o_p.$$

Here, the grading is defined by $\deg(o_p) := \mathrm{ind}_p(f) = n - \mathrm{ind}\, D_p$.

Glued operator

To define the differential, we use a notion of *glued operator*, which is a linear version of the gluing of Morse trajectories.

Let x be a negative gradient flow line, or briefly *Morse flow line*, from p to

q, which means that $x : \mathbb{R} \to M$ is a solution of the equation $\partial_s x + \operatorname{grad} f = 0$. Note that trivializations of T_pM and T_qM have been already fixed. We can extend them to a trivialization $\mathcal{B}_x$ of x^*TM. We then define an operator

$$D_x : W^{1,2}(\mathbb{R}, \mathbb{R}^n) \to L^2(\mathbb{R}, \mathbb{R}^n), \quad X \mapsto \partial_s X + B_x(s)X$$

where $B_x : \mathbb{R} \to \mathbb{R}^{n^2}$ is a smooth function such that

$$\begin{cases} B_x(s) = Hess_p f & s \ll 0, \\ B_x(s) = Hess_q f & s \gg 1. \end{cases}$$

The standard argument shows that D_x is Fredholm. For example, see [27, Proposition 2.16].

Lemma 5.3.6. *The operator D_x is Fredholm, and its Fredholm index is*

$$\operatorname{ind}_p f - \operatorname{ind}_q f.$$

As the case of D_p, the determinant line $\det D_x$ and the orientation line $|\det D_x|$ of the operator D_x do not depend on the choice of B_x. The determinant line $\det D_x$ do not even depend on the choice of an extension $\mathcal{B}_x$, since the resulting Fredholm operators are asymptotically the same. It follows that the orientation $|\det D_x|$ is also independent of the choice of an extension $\mathcal{B}_x$, *up to* positive scalar multiplication.

Observe that the function B_x coincide to B_p near $-\infty$ and to B_q near ∞. In particular, by gluing their domains $\overline{\mathbb{R}}$ and $\mathbb{R}$, where the glued domain is again $\overline{\mathbb{R}}$, we can obviously glue the operators D_p and D_x with some gluing parameter, say $R > 0$. Denote the glued operator by

$$D_p \#_R D_x : W^{1,2}(\overline{\mathbb{R}}, \mathbb{R}^n) \to L^2(\overline{\mathbb{R}}, \mathbb{R}^n).$$

The crucial ingredient of the coherent orientation for Morse homology is the following "Gluing lemma", [27, Theorem 6].

Lemma 5.3.7 (Gluing)**.** *There exists a canonical isomorphism up to postive multiplication*

$$\det D_p \#_R D_x \cong \det D_p \otimes \det D_x.$$

Observe that the glued operator has the same asymptotic behavior at the positive end as D_q. So we have the following an immediate corollary.

Corollary 5.3.8. *There is a canonical isomorphism up to posivite multiplication*

$$\det D_p \otimes \det D_x \cong \det D_q.$$

As a result, we have a canonical isomophism

$$o_p \otimes |\det D_x| \cong o_q.$$

Differentials

Let the metric g be *generic*. In other words, for each Morse flow line x, the corresponding operator D_x is surjective. Denote the Moduli space of Morse flow lines from p to q by $\mathcal{M}(p,q)$.

Proposition 5.3.9. *Let* $\operatorname{ind}_p f = \operatorname{ind}_q f + 1$*. For each* $x \in \mathcal{M}(p,q)$*, there is a canonical isomorphism*

$$\partial_x : o_p \to o_q$$

of orientation lines.

Proof. Since g is generic, the moduli space (before modding out by $\mathbb{R}$-action) $\tilde{\mathcal{M}}(p,q)$ is a smooth manifold of dimension $\operatorname{ind}_p f - \operatorname{ind}_q f$. Its tangent space $T_x\tilde{\mathcal{M}}$ is the same as the kernel of the operator D_x. Note also that $T_x\tilde{\mathcal{M}} = T_x\mathcal{M} \oplus \langle \partial_s \rangle$ where ∂_s denotes the $\mathbb{R}$-action direction. As a result, the determinant line $\det D_x$ is given by

$$\det D_x = \wedge^{max} \ker D_x \cong \wedge^{max} T_x\tilde{\mathcal{M}} \cong \wedge^{max} T_x\mathcal{M} \otimes \wedge^{max} \langle \partial_s \rangle.$$

Plugging this into Corollary 5.3.8, we get a canonical isomorphism

$$o_p \otimes |T_x\mathcal{M}| \otimes |\partial_s| \cong o_q.$$

The tangent space $T_x\mathcal{M}$ is of zero dimension, so we can assign its orientation in a canonical way, say "+" for example. The orientation line $|\partial_s|$ also has a natural orientation by the direction of $\mathbb{R}$-action. We therefore get the asserted canonical isomorphism $\partial_x : o_p \to o_q$. $\square$

Define the differential $\partial : CM_*(f) \to CM_*(f)$ by

$$\partial|_{o_p} = \sum_{\substack{q \in \operatorname{crit}(f) \\ \operatorname{ind}_q f = \operatorname{ind}_p f - 1}} \sum_{x \in \mathcal{M}(p,q)} \partial_x.$$

Then the standard argument shows that $\partial \circ \partial = 0$. We finally define the **Morse homology** of M, with $\mathbb{Z}$-coefficient, denoted by $HM_*(M;\mathbb{Z})$, to be the homology of the chain complex $(CM_*(f), \partial)$.

Local coefficient systems

To consider Morse homology *with local coefficient system*, we briefly describe the definition of the local coefficient system and its basic properties. Let X be a path-connected topological space.

Definition 5.3.10. A **local coefficient system $\mathcal{L}$ on X (or "local system" briefly)** consists of the following assignments:

1. we assign to each points $x \in X$ a free abelian group $\mathcal{L}_x$ of rank 1;
2. for each path $\gamma : [0,1] \to X$, there is a corresponding homomorphism
$$\mathcal{L}_\gamma : \mathcal{L}_{\gamma(0)} \to \mathcal{L}_{\gamma(1)}$$
which only depends on the homotopy class of γ;
3. this correspondence satisfies an obvious concatenation property, namely
$$\mathcal{L}_{\gamma_0 \# \gamma_1} = \mathcal{L}_{\gamma_1} \circ \mathcal{L}_{\gamma_0};$$
4. for the constant path, we assign the identity homomorphism.

Definition 5.3.11. A local system $\mathcal{L}$ on X is said to be **trivial** if all points are assigned to the same group, and all paths correspond to the identity homomorphism.

Remark 5.3.12. By the third and forth conditions, the reversed path should give the inverse homomorphism. Therefore all homomorphisms $\mathcal{L}_\gamma$ are isomorphisms automatically.

Observe that if X is simply-connected, then every local system on X is clearly trivial. There is another condition on the topological space X for triviality of local systems over X.

Proposition 5.3.13. *If $H^1(X;\mathbb{Z}_2) = 0$, then all local systems on X are trivial.*

Proof. Note that if $H^1(X;\mathbb{Z}_2)$ vanishes, then all line bundles over X are trivial. This is because there is an isomorphism between the isomorphism classes of line bundles over X and $H^1(X;\mathbb{Z}_2)$, which maps a line bundle to its first Stiefel-Whitney class. A line bundle L over X is trivial if and only if every restriction $L|_\gamma$ to a loop γ in X is trivial. This implies that the homomorphism $\mathcal{L}_\gamma$ must be the identity, otherwise it would provide a non-trivial line bundle over $S^1 = \gamma(I)/(\gamma(0) = \gamma(1)) \subset X$ by the clutching construction. Therefore all local systems on X must be trivial. □

Definition 5.3.14. For an 1-dimensional real vecter space V, an **orientation line** $o(V)$ of V is the abelian group with two generators corresponding to two orientations of V, and the relation their sum vanishes, i.e.

$$o(V) := \langle \sigma_1, \sigma_2 \rangle / \{\sigma_1 + \sigma_2 = 0\}.$$

Example 5.3.15. Let $L \to S^1$ be a (real) line bundle over the unit circle S^1. In this example, we construct local systems $\mathcal{L}$ on S^1 of rank 1 from the line bundle. Considering the clutching construction, there are only two such line bundles: the trivial line bundle and the Möbius band. In both cases, there is a natural connection, or parallel transport; just follow the parallel direction on the "bands".

Assuming we have chosen a line bundle L over S^1, we give the following assignments:

- For $x \in S^1$, $x \mapsto \mathcal{L}_x := o_x(F_x)$ where F_x is the fiber over x and $o_x(F_x)$ is its orientation line.
- let γ be a path on S^1. Using the natural connection, we get the parallel transport $\phi : F_{\gamma(0)} \to F_{\gamma(1)}$, and this induces an orientation-preserving or orientation-reversing map, so that we have an isomorphism $\mathcal{L}_\gamma : \mathcal{L}_{\gamma(0)} \to \mathcal{L}_{\gamma(1)}$.

One can easily check that this assignments satisfy all conditions of local system.

Suppose the chosen line bundle $L \to S^1$ is trivial. Then all fibers can be identified, even as sets, so that $\mathcal{L}_x$'s are the same for all $x \in S^1$. It also is clear that the natural connection gives rise to the identity homomorphisms for every $\mathcal{L}_\gamma$. They are in particular orientation-preserving. Therefore the local system induced by the trivial line bundle is trivial.

Let us assume the case when $L \to S^1$ is non-trivial, i.e., the Möbius band. In this case we get a non-trivial local system. If we take a loop γ which wounds S^1 odd number of times, then the corresponding parallel transport reverses the orientation by the construction of the Möbius band. It follows that $\mathcal{L}_\gamma = -\operatorname{id}$.

As one might expect now, the notion of local systems of rank 1 is equivalent to the line bundles, up to homotopy.

Lemma 5.3.16. *Any principal $O(1)$-bundle over X defines a local coefficients system over X in a canonical way.*

Proof. It follows by combining the proof of Proposition 5.3.13 and the construction in the above example. □

Morse homology with local coefficient systems

Let M be a compact smooth manifold, equipped with a Morse-Smale pair (f, g). Let $\mathcal{L}$ be a local system on M. This means that each point $p \in M$ is associated to a rank 1 free abelian group $\mathcal{L}_p$, and each path $\gamma : p \to q$ is associated an isomorphism $\mathcal{L}_\gamma : \mathcal{L}_p \to \mathcal{L}_q$. We define a chain group $CM_*(f; \mathcal{L})$ by

$$CM_*(f; \mathcal{L}) := \bigoplus_{p \in \operatorname{crit}(f)} o_p \otimes \mathcal{L}_p.$$

By the tensor product $\otimes$, we mean tensor over $\mathbb{Z}$.

Let x be a rigid Morse flow line from p to q. Then we canonically have an isomorphism $\partial_x : o_p \to o_q$. By tensoring the morphisms, we obtain an isomorphism

$$\partial_x \otimes \mathcal{L}_x : o_p \otimes \mathcal{L}_p \to o_q \otimes \mathcal{L}_q.$$

Define a map $\partial \otimes \mathcal{L} : CM_*(f; \mathcal{L}) \to CM_*(f; \mathcal{L})$ by

$$(\partial \otimes \mathcal{L})|_{o_p \otimes \mathcal{L}_p} = \sum_{\substack{q \in \operatorname{crit}(f) \\ \operatorname{ind}_q f = \operatorname{ind}_p f - 1}} \sum_{x \in \mathcal{M}(p,q)} \partial_x \otimes \mathcal{L}_x.$$

Proposition 5.3.17. *The pair $(CM_*(f; \mathcal{L}), \partial \otimes \mathcal{L})$ forms a chain complex, i.e., $(\partial \otimes \mathcal{L})^2 = 0$.*

Proof. The essential analytic ingredients are the same as the usual proof, using the compactness and gluing results. We need to carefully deal with local systems and the Kozul sign conventions. More details will be given. □

We define the **Morse homology with coefficient in the local system** $\mathcal{L}$, denoted by $HM_*(M;\mathcal{L})$, by taking the homology group of the complex $(CM_*(f;\mathcal{L}), \partial \otimes \mathcal{L})$. Note that if $\mathcal{L}$ is trivial, then $HM_*(M;\mathcal{L})$ is the same as $HM_*(M;\mathbb{Z})$.

Remark 5.3.18. In view of the definition, the notation "$HM_*(M; R \otimes \mathcal{L})$" looks also suitable, for a general ring R. We will use this notation later when it is better to reveal that the coefficient is *twisted* by the local system.

What local systems actually do is that they *twist* the orientations of Moduli spaces $\mathcal{M}(p,q)$. That makes us to count the flow lines with *different signs* from the usual ones. Let us clarify this by an example.

Example 5.3.19. Consier $M = S^1$ and the usual height function f witht two critical points; the maximum p and the minimum q. We have seen in Example 5.3.15 that there are only two local systems on S^1. We take the non-trivial one $\mathcal{L}$ which comes from the Möbius band.

Note that there are exactly two Morse flow lines x, y from p to q. The usual coherent orientation on $\mathcal{M}(p,q)$ would give the different signs on them, so that $\#\mathcal{M}(p,q) = 0$. In terms of the associated isomorphisms of orientation lines, the counting means that $\partial_x : o_p \to o_q$ and $\partial_y : o_p \to o_q$ are different to each other. In other words, $\partial_x \circ \partial_y^{-1} : o_p \to o_p$ is equal to $-\operatorname{id}$.

If we tensor by $\mathcal{L}$, however, things are different. By the construction of $\mathcal{L}$, we have $\mathcal{L}_x \circ \mathcal{L}_y^{-1} = \mathcal{L}_x \circ \mathcal{L}_{\overline{y}} = \mathcal{L}_{x \# \overline{y}} = -\operatorname{id}$. It follows that

$$(\partial_x \otimes \partial_x) \circ (\partial_y \otimes \partial_y)^{-1} = \operatorname{id}.$$

In terms of orientation of the Moduli space, this means that we count x and y with the *same* sign. Therefore, the resulting homology group is given by

$$HM_*(S^1;\mathcal{L}) = \begin{cases} \mathbb{Z}_2 & * = 0, \\ 0 & \text{otherwise.} \end{cases}$$

5.4 Local Morse Homology

Let $F : M \to \mathbb{R}$ be a Morse-Bott function and g a metric on M. Let Σ be a connected component of the critical submanifold. *We always assume* *that each Morse-Bott component Σ is oriented.* Choose a Morse function $f : \Sigma \to \mathbb{R}$ and a metric h on Σ. We perturb F using f in a standard way, see Section 5.2.2. Denote the perturbed Morse function F_δ.

In this section, we define the *local Morse homology* of Σ with $\mathbb{Z}$-coefficient, denoted by $HM_*^{loc}(\Sigma;\mathbb{Z})$, and we prove the following proposition.

Proposition 5.4.1. *For a specific local coefficient system $\mathcal{L}_\Sigma$ over Σ, it holds true that*

$$HM_*^{loc}(\Sigma;\mathbb{Z}) \cong HM_{*-\text{shift}(\Sigma)}(\Sigma;\mathcal{L}_\Sigma)$$

where the degree shifting is given by

$$\text{shift}(\Sigma) = \text{ind}_\Sigma(F).$$

5.4.1 Local Morse homology of Σ

Recall that for small δ, say $\delta_0 > 0$, the perturbation F_{δ_0} is Morse, and in a neighborhood $U := \nu^{\delta_0}(\Sigma)$, we have that $gradF_{\delta_0}$ is negative gradient-like to the gradient of F. A key ingredient for the definition of local Morse homology is the following lemma. A behind idea is that δ is so small that Morse trajectories do not have enough energy to escape Σ.

Lemma 5.4.2. *For any neiborhood V of U such that $V \cap \text{crit}(F) = \Sigma$, there exists $\delta_1 > 0$ with $\delta_1 < \delta_0$ such that for each $0 < \delta < \delta_1$,*

1. *$U \cap \text{crit}(F_\delta) = V \cap \text{crit}(F_\delta) = \text{crit}(f)$,*
2. *for $x \in \mathcal{M}(p,q;F_\delta,g;V)$, we have that $\text{im}(x) \subset \Sigma$, where $\mathcal{M}(p,q;F_\delta,g;V)$ denotes the moduli space of Morse flow lines* in V *from p to q with respect to the pair (F_δ, g).*

Proof. The first assertion is direct from the definition. For the second one, we prove it by a contradiction. Suppose that there is a neighborhood V of Σ and sequences $\delta_n \to \infty$ and $\{x_n\}$ of negative gradient trajectories of F_{δ_n} in U, which are *not* contained in V. Then an Arzela-Ascoli argument shows that $\{x_n\}$ has a convergent subsequence, still denoted by x_n, such that $x_n \to x$ for some negative gradient trajectory x of F. Since the asymptotics of x must be in Σ by the first assertion, we have that the energy of x is equal to zero. It follows that x is in fact constant path lying in Σ. This contradicts to the assumption that each x_n does not lie in V for all n. $\square$

Let $\delta > 0$ be as Lemma 5.4.2. Consider the Morse homolgy group $HM_*(U;\mathbb{Z})$ with respect to the pair (F_δ, g). The generators are given by critical points of F_δ in U, and the differential counts Morse flow lines in the Moduli space $\mathcal{M}(p,q;F_\delta,g;V)$. Lemma 5.4.2 tells us that the group $HM_*(U;\mathbb{Z})$ does not depend on the choice of a neighborhood U. Therefore we denote this group by $HM_*^{loc}(\Sigma;\mathbb{Z})$ (without U), and call it the **local Morse homology** of Σ.

By Lemma 5.4.2, it is immediate that the local Morse homology with $\mathbb{Z}_2$-*coefficient* is isomorphic to the Morse homology of Σ with the same coefficient. That is,

$$HM_*^{loc}(\Sigma;\mathbb{Z}_2) \cong H_{*-\mathrm{shift}(\Sigma)}(\Sigma;\mathbb{Z}_2). \tag{5.4.1}$$

With $\mathbb{Z}$-coefficients, however, this is not true in general:

Example 5.4.3. Consider the real projective plane $\mathbb{R}P^2$ and its decomposition into the Möbius band B and a disk D^2. We define a Morse-Bott function $F : \mathbb{R}P^2 \to \mathbb{R}$ such that

- on B, the function F takes the maximum value along the "center circle" of the Möbius band;
- on D^2, the function F takes the minimum value at the origin of the disk;
- there is no other critical point of F.

Such a function clearly exists. Then the critical submanifold of F consists of two Morse-Bott components, the center circle S^1 in B and the origin of D^2.

The point of this construction is that a neighborhood of the component S^1 in $\mathbb{R}P^2$ is *not* orientable. This affects the coherent orientation of moduli spaces of Morse flow lines; we would count the two trajectories from the top to the bottom of S^1 with the same sign. As a result, the local Morse homology $HM_*^{loc}(S^1,\mathbb{Z})$ is given by

$$HM_*^{loc}(S^1;\mathbb{Z}) = \begin{cases} \mathbb{Z}_2 & * = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that this group is *not* isomorphic to Morse homology $HM_*(S^1;\mathbb{Z})$ with $\mathbb{Z}$-coefficient.

In fact, we have shown that

$$HM_*^{loc}(S^1;\mathbb{Z}) \cong HM_{*-1}(S^1;\mathcal{L})$$

where $\mathcal{L}$ is the local system induced by the Möbius band, see Example 5.3.19. This isomorphism is a special case of Proposition 5.4.1. Note that the Morse-Bott index $\mathrm{ind}_{S^1}(F)$ of the component S^1 is equal to 1 since S^1 forms the minimum of F. So the degree shift is zero in this case.

5.4.2 Construction of the local system $\mathcal{L}_\Sigma$

We construct a local coefficients system $\mathcal{L}_\Sigma$ over a connected component Σ. We do this by constructing a principal $O(1)$-bundle over Σ. Roughly speaking, the fiber at $p \in \Sigma$ of this bundle will be the orientation line of the operator $\partial_s + Hess_p F$ where F is Morse-Bott. To deal with such fiber, we start with some Fredholm properties in Morse-Bott setup.

Fredholm operators in Morse-Bott setup

Let p be a point in Σ. We define a *weighted* version of Sobolev space by

$$W_\delta^{k,p}(\overline{\mathbb{R}}, T_pM) := \{X \in W^{k,p}(\overline{\mathbb{R}}, T_pM) \mid \beta_\delta X \in W^{k,p}(\overline{\mathbb{R}}, T_pM)\}$$

where $\beta_\delta : \overline{\mathbb{R}} \to \mathbb{R}$ is smooth function such that $\beta_\delta(s) = e^{\delta s}$ for $s \gg 1$, and $\delta > 0$. Also, we set

$$L_\delta^p(\overline{\mathbb{R}}, T_pM) := W_\delta^{0,p}(\overline{\mathbb{R}}, T_pM).$$

Define a map $\Phi^{k,p} : W^{k,p}(\overline{\mathbb{R}}, T_pM) \to W_\delta^{k,p}(\overline{\mathbb{R}}, T_pM)$ by

$$X \mapsto \beta_\delta X.$$

Then obviously the following lemma is true.

Lemma 5.4.4. *The map $\Phi^{k,p}$ is a Banach space isomorphism.*

Define a vector space $\mathcal{T}_p\Sigma$ by

$$\mathcal{T}_p\Sigma := span_{\mathbb{R}}(e_1, \dots e_{\dim \Sigma})$$

where $e_j : \overline{\mathbb{R}} \to T_p\Sigma$ is a C^1-function such that e_j's are asymptotically constant and linearly independent to each other. We further insist that $\{e_j\}$ fits into the orientation of $T_p\Sigma$. Note that $\mathcal{T}_p\Sigma$ is morally the tangent space of Σ at p.

We consider an operator $D_p(F) : W^{1,2}_\delta(\overline{\mathbb{R}}, T_pM) \oplus \mathcal{T}_p\Sigma \to L^2_\delta(\overline{\mathbb{R}}, T_pM)$ defined by

$$\tilde{X} = (X, e) \mapsto \partial_x \tilde{X} + B_p(F)(s)\tilde{X}$$

where $B_p(F) : \overline{\mathbb{R}} \to \mathbb{R}^{n^2}$ is such that $B_p(F)(s) = Hess_p F$ for $s \gg 1$.

Lemma 5.4.5. *The operator $D_p(F)$ is Fredholm, and its Fredholm index is* $\mathrm{ind}_\Sigma(F) + \dim \Sigma$

Proof. For notational convenience, set $W := W^{1,2}_\delta(\overline{\mathbb{R}}, T_pM)$. Note that W is a subspace of $W \oplus \mathcal{T}_pM$ of finite codimension. It follows that $D_p(F)$ is Fredholm if and only if $D_p(F)|_W$ is Fredholm, and the Fredholm index of $D_p(F)$ is given by

$$\mathrm{ind}\, D_p(F) = \mathrm{ind}\, D_p(F)|_W + \dim \Sigma.$$

We claim that $D_p(F)|_W$ is a Fredholm operator and its index is $\mathrm{ind}_\Sigma F$. By Lemma 5.4.4, it suffices to show that the conjugated operator $\tilde{D}_p(F)|_W := (\Phi^{0,2})^{-1} \circ D_p(F)|_W \circ \Phi^{1,2}$ is Fredholm of index $\mathrm{ind}_\Sigma F$. Observe that the operator $\tilde{D}_p(F)|_W$ is given by the formula

$$X \mapsto \partial_s X + (B_p(F)(s) + \delta I_n)X.$$

Since $B_p(F)(s) + \delta I_n$ is asymptotically equal to $Hess_s F + \delta I_n$, it is non-degenerate for sufficiently small δ. Therefore $\tilde{D}_p(F)|_W$ is Fredholm, and since $\delta > 0$, its index is the number of negative eigenvalues of $Hess_p F$, i.e., $\mathrm{ind}_\Sigma F$ This completes the proof. $\square$

Remark 5.4.6. Denote $\mathcal{T}_p\Sigma =: T$ for simplicity. We can decompose $D_p(F)$ as

$$D_p(F) = D_p(F)|_W \oplus D_p(F)|_T,$$

and hence there is a canonical isomorphism

$$\ker D_p(F) \cong \ker D_p(F)|_W \otimes \ker D_p(F)|_T.$$

Since e_j's are constant at the end, it follows that $D_p(F)|_T \equiv 0$ at the end. So $\ker D_p(F)|_T = \mathcal{T}_p\Sigma$. If we assume the given data (F, g) and (f, h) are regular, then we would hat the following canonical isomorphism.

$$|\det D_pF| \cong |\det D_p(F)|_W| \otimes |T_p\Sigma|. \tag{5.4.2}$$

This will be used later.

Construction of $\mathcal{L}_\Sigma$

We now construct a principal O(1)-bundle L over Σ whose fiber at p is $\det D_p(F)$. Then, as in Section 5.3, L induces a canonical local coefficients system $\mathcal{L}_\Sigma$ over Σ.

Since Σ is compact, there are only finitely many critical points of f, say $p_0, \ldots, p_k$. In addition, we can take a finite open cover $\{U_0, \ldots U_k\}$, "centered" at $p_0, \ldots, p_k$, with the following preferred choices:

- Fix paths $x_{0,j}$ in Σ from p_0 to p_j.
- For each point $p \in U_j$, fix a path $x_{j,p}$ from p_j to p.

These choices can be made in a canonical way; for example, take U_j's as normal coordinates with respect to the metric h. Now our strategy to construct an $O(1)$-bundle over Σ is as follows.

Step 1: We construct line bundles L_{U_j} on each open set U_j.

Step 2: We patch L_{U_j}'s together on each intersections by defining canonical transition functions up to multiplication by a positive scalar.

Step 3: We show that our transition functions satisfy the cocycle condition. Therefore we get an $O(1)$-bundle over Σ.

We start with a lemma which is proven in [28, Appendix].

Lemma 5.4.7 ([28])**.** *Let U be topoloical space and $\mathcal{F} : U \to Fred(W, L)$ be a continuous family of Fredholm operators from W to L. Define*

$$\det \mathcal{F} := \bigcup_{p \in U} \{p\} \times \det \mathcal{F}(p).$$

Then $\det \mathcal{F}$ admits a natural line bundle structure.

Note that a Fredholm operator

$$D_p(F) : W^{1,2}_\delta(\overline{\mathbb{R}}, T_pM) \oplus \mathcal{T}_p\Sigma \to L^2_\delta(\overline{\mathbb{R}}, T_pM)$$

is associated to each $p \in U_j$. To apply Lemma 5.4.7, we need to choose trivializations of T_pM and $T_p\Sigma$ for each $p \in U_j$ in a continuous way. We do this as follows: First, choose trivializations of $T_{p_0}M$ and $T_{p_0}\Sigma$ according to the orientations of M and Σ, respectively. Along the fixed paths $x_{0,j}$ we can push these trivializations to trivializations of $T_{p_j}M$ and $T_{p_j}\Sigma$, using Levi-Civita connections of g and h. Then for each $p \in U_j$, we give trivializations of T_pM and $T_p\Sigma$ by pushing trivializations at the center p_j, along $x_{j,p}$. In this way, we have continuous family of trivializations on each U_j's.

We now have a continuous family of Fredholm operators

$$\mathcal{F}_{U_j} : U_j \to Fred(W, L), \quad p \mapsto D_p(F)$$

where $W = W^{1,2}_\delta(\overline{\mathbb{R}}, \mathbb{R}^n) \oplus \mathbb{R}^{\dim \Sigma}$ and $L = L^2_\delta(\overline{\mathbb{R}}, \mathbb{R}^n)$. Applying Lemma 5.4.7, we get a natural line bundle L_{U_j} over U_j for each j. This completes step 1.

For step 2, we first state a lemma which is just a mixture of Lemma 5.3.4 and Lemma 5.3.5.

Lemma 5.4.8. *Let $\mathcal{F} : [0,1] \to Fred(W, L)$ be a continuous path of Fredholm operators. Then there is a canonical isomorphism up to positive multiplication*

$$\det \mathcal{F}(0) \cong \det \mathcal{F}(1).$$

Let $p \in U_i \cap U_j$. Note that we are given the two (possibly different) Fredholm operators associated to this point, say $D_p(F)_i$ and $D_p(F)_j$. Consider the path x_p from p to itself, defined by a concatenation

$$x_p := \overline{x}_{i,p} \# \overline{x}_{0,i} \# x_{0,j} \# x_{j,p}.$$

Here, $\overline{x}$ denotes the inversed path of x. Along this path, we can connect $D_p(F)_i$ with $D_p(F)_j$ through Fredholm operators. Therefore, by Lemma 5.4.8, we get a canonical isomorphism

$$\tau_{ij}(p) : L_{U_i}|_p \to L_{U_j}|_p$$

up to a positive scalar multiplication. Moreover, by construction, τ_{ij} varies continuously as p varies in $U_i \cap U_j$.

We now check the cocycle conditions for $\{\tau_{ij}\}$. A useful observation is the following.

Lemma 5.4.9. *Assume the notations in Lemma 5.4.8. Define a path $\overline{\mathcal{F}}$ by $\overline{\mathcal{F}}(t) = \mathcal{F}(1-t)$ for $t \in I$. Then the canonical isomorphism determined by $\overline{\mathcal{F}}$ is exactly the inverse.*

Proof. Note that the canonical isomorphism given in Lemma 5.4.8 is defined by choosing a non-vanishing section of the line bundle $\det \mathcal{F}$. Clearly we get a non-vanishing section of the bundle $\det \overline{\mathcal{F}}$ just by reversing. This defines a canonical isomorphism which is the inverse. $\square$

By definition, τ_{ii} is given by the canonical isomorphism along the path

$$\overline{x}_{i,p} \# \overline{x}_{o,i} \# x_{o,i} \# x_{i,p}.$$

Therefore Lemma 5.4.9 shows that $\tau_{ii} = \mathrm{id}$. Since the path for τ_{ji} is exactly the reversed path for τ_{ij}, we have $\tau_{ji} = \tau_{ij}^{-1}$. The same observation shows that $\tau_{jk} \circ \tau_{ij} = \tau_{ik}$. We conclude $\{\tau_{ij}\}$ satisfies the cocycle conditions.

Patching together the bundles L_{U_j} using transition functions $\{\tau_{ij}\}$, we get a canonically defined principal $O(1)$-bundle L over Σ. More precisely, we define a principle $O(1)$-bundle L over Σ by

$$L := \left(\coprod_j L_{U_j} \right) / \sim$$

where $(p, v) \sim (q, w)$ if and only if $p = q \in U_i \cap U_j$ and $v = \tau_{ij} w$ up to a positive scalar multiplication. By Lemma 5.3.16, we now have the induced local coefficient system $\mathcal{L}_\Sigma$ for each connected components Σ.

Glued operator in Morse-Bott setup

Fix a Morse-Bott component Σ. Denote the local system $\mathcal{L} := \mathcal{L}_\Sigma$ for simplicity. Note that the free abelian group $\mathcal{L}_p$ at $p \in \Sigma$ is by construction the orientation line of the Fredholm operator $D_p(F)$. In addition, for each path x in Σ from p to q, we have an associated isomorphism $\mathcal{L}_x : \mathcal{L}_p \to \mathcal{L}_q$.

On the other hand, by a gluing construction in Morse-Bott setup, we will show that there is a canonical isomorphism from $\mathcal{L}_p$ to $\mathcal{L}_q$, which will be denoted by $\partial_x(F) : \mathcal{L}_p \to \mathcal{L}_q$. In this section, we discuss the gluing in Morse-Bott setup and show that $\mathcal{L}_x = \partial_x(F)$.

Define a *weighted* Sobolev space $W^{k,p}_\delta(\mathbb{R}, \mathbb{R}^n)$ by

$$W^{k,p}_\delta(\mathbb{R}, \mathbb{R}^n) := \{X \in W^{k,p}(\mathbb{R}, \mathbb{R}^n) \mid \beta_\delta X \in W^{k,p}(\mathbb{R}, \mathbb{R}^n)\}$$

where $\beta_\delta : \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} \beta_\delta(s) = e^{\delta s} & s \gg 1, \\ \beta_\delta(s) = e^{-\delta s} & s \ll 0. \end{cases}$$

Then, as in Section 5.4.2, $W^{k,p}_\delta(\mathbb{R}, \mathbb{R}^n)$ is a Banach space which is isomorphic to $W^{k,p}(\mathbb{R}, \mathbb{R}^n)$.

Remind that the domain of elements in the vector space $\mathcal{T}_p\Sigma$ is the half line $\overline{\mathbb{R}}$. The corresponding vector space for the path x is defined essentially the same way but the domain should be replaced by $\mathbb{R}$. We define a vector space $\mathcal{T}^-_p\Sigma$ by

$$\mathcal{T}^-_p\Sigma := span_{\mathbb{R}}(e^-_j)$$

where $e^-_j : \mathbb{R} \to T_p\Sigma$ is a C^1-function such that $e^-_j(s) = 0$ for $s \gg 1$, and e^-_j's are constant and linearly independent for $s \ll 0$. Define $\mathcal{T}^+_p\Sigma$ similarly but $e^+_j(s) = 0$ for $s \ll 0$, and e^+_j's are constant and linearly independent for $s \gg 1$.

Let x be a Morse flow line in Σ from p to q with respect to (F_δ, g) or equivalently (f, h). According to the trivializations of Σ, we define an operator

$$D_x(F) : W^{1,2}_\delta(\mathbb{R}, \mathbb{R}^n) \oplus \mathcal{T}^-_p\Sigma \oplus \mathcal{T}^+_q\Sigma \to L^2_\delta(\mathbb{R}, \mathbb{R}^n)$$

by

$$\tilde{X} := (X, e^-, e^+) \mapsto \partial_s\tilde{X} + B_x(F)(s)\tilde{X}$$

where $B_x(F) : \mathbb{R} \to \mathbb{R}^{n^2}$ is a smooth function such that

$$\begin{cases} B_x(F)(s) = Hess_pF & s \ll 0, \\ B_x(F)(s) = Hess_qF & s \gg 1. \end{cases}$$

Then the same argument as Lemma 5.4.5 proves the following.

Lemma 5.4.10. *The operator $D_x(F)$ is Fredholm, and its index is equal to* $\dim \Sigma$. *Furthermore,* $\det D_x(F)$ *does not depend on the choice of $B_x(F)$ with fixed asymptotics.*

We now glue $\mathbb{R}$ with $\overline{\mathbb{R}}$ in an obvious way with some gluing parameter. The resulting domain is again $\overline{\mathbb{R}}$. We denote the vector space spanned by the glued maps $e_j \# e_j^-$ by $\mathcal{T}_p^0\Sigma$. Then the glued operator $D_p(F)\#D_x(F)$ has its domain as $W_\delta^{1,2}(\overline{\mathbb{R}}, \mathbb{R}^n) \oplus \mathcal{T}_p^0\Sigma \oplus \mathcal{T}_q^+\Sigma$. The following is the gluing lemma for Morse-Bott setup.

Lemma 5.4.11 (Gluing in Morse-Bott setup)**.** *For sufficiently large gluing parameter, there is a canonical isomorphism*

$$\det D_p(F)\#_R D_x(F) \cong \det D_p(F) \otimes \det D_x(F)$$

up to multiplication by a positive scalar.

Proof. This is almost a corollary of the ordinary gluing lemma. Apply the gluing lemma in Morse case to the operators modded out by finite dimensional subspaces $\mathcal{T}\Sigma$'s. To make it precise, we need to consider the conjugated operators on unweighted Sobolev spaces, and a key observation is that the determinant line of the glued operator of conjugated operators is canonically isomorphic to the determinant line of the conjugated operator of the glued operator. □

Lemma 5.4.12. *There is a canonical isomorphism up to multiplication by positive scalar*

$$\det(D_p(F)\#D_x(F)) \cong \det D_q(F) \otimes \det T_p\Sigma.$$

Proof. Note that domains of the glued operator and $D_q(F)$ only differ by $\mathcal{T}_p^0\Sigma$. So the conclusion is immediate. □

Corollary 5.4.13. *We have a canonical isomorphism*

$$\mathcal{L}_p \otimes |\det D_x(F)| \cong \mathcal{L}_q \otimes |\det T_p\Sigma|.$$

As a result, by orienting $T_p\Sigma$ by the given trivialization on Σ, Corollary 5.4.13 gives an isomorphism

$$\partial_x(F) : \mathcal{L}_p \to \mathcal{L}_q.$$

In view of the proof of the gluing lemma, we conclude that

$$\eta_x = \partial_x(F). \tag{5.4.3}$$

5.4.3 Canonical isomorphisms

In this section we give canonical identifications of generators and differentials of $HM_*^{loc}(\Sigma;\mathbb{Z})$ and $HM_{*+\mathrm{shift}(\Sigma)}(\Sigma;\mathcal{L}_\Sigma)$. A crucial observation is a canonical splitting of determinant lines of $D_p(F_\delta)$ and $D_x(F_\delta)$. We start with a general lemma.

Splitting of determinant lines

Consider an operator $D_A : W^{1,2}(\mathbb{R},\mathbb{R}^n) \to L^2(\mathbb{R},\mathbb{R}^n)$ of the form $\partial_s + A$ for some symmetric invertible matrix A.

Lemma 5.4.14. *If A is of the form*

$$A = \left[\begin{array}{c|c} B & * \\ \hline O & C \end{array}\right]$$

for some symmetic invertible matrices B and C, then there is a canonical isomorphism

$$\det D_A \cong \det D_B \otimes \det D_C$$

for sufficiently small $\delta > 0$

Proof. Since A is symmetric, there exists an invertible matrix P such that $P^{-1}AP$ is diagonal. One can choose P in the form

$$P = \left[\begin{array}{c|c} P_B & * \\ \hline O & P_C \end{array}\right]$$

where $P_B^{-1}BP_B$ and $P_C^{-1}CP_C$ are diagonal. Furthermore, by insisting on P to be of certain "length", such choice can be made in a canonical way.

Consider isomphisms $\Phi : W^{1,2}(\mathbb{R}, \mathbb{R}^n) \to W^2(\mathbb{R}, \mathbb{R}^n)$ given by

$$X \mapsto PX$$

and $\Psi : L^2(\mathbb{R}, \mathbb{R}^n) \to L^2(\mathbb{R}, \mathbb{R}^n)$ given by the same formula. Then the conjugated operator $\tilde{D}_A := \Psi^{-1} \circ D_A \circ \Phi$ is of the form $\partial_s + \mathrm{diag}(P_B^{-1} B P_B, P_C^{-1} C P_C)$. So there is a canonical isomorphism

$$\det \tilde{D}_A \cong \det \tilde{D}_B \otimes \tilde{D}_C.$$

Furthermore, since Φ and Ψ are isomorphisms, there are canonical isomorphisms between determinant lines of operators before and after conjugation. This completes the proof. $\square$

Identification of generators

We now show that there is a canonical isomorphism

$$o_p(F_\delta) \cong o_p(f) \otimes \mathcal{L}_p$$

which identifies the generators of $HM^{loc}(\Sigma; \mathbb{Z})$ and $HM(\Sigma; \mathcal{L})$.

Take a Morse-Bott coordinates chart of M at p. In this coordinates, the Hessian $Hess_p F_\delta$ decomposes as

$$Hess_p F_\delta = \left[\begin{array}{c|c} \delta Hess_p f & O \\ \hline O & Hess_p F|_\nu \end{array} \right]$$

where ν denotes the normal direction to $T_p\Sigma$ in T_pM. We have Fredholm operators $D_p(f)$ and $D_p(F)^\nu$ defined by $Hess_p f$ and $Hess_p F|_\nu$, respectively, in an obvious way. By Lemma 5.4.14, we have a canonical isomorphism

$$\det D_p(F_\delta) \cong \det D_p(f) \otimes \det D_p(F)^\nu.$$

Lemma 5.4.15. *There is a canonical isomorphism*

$$\det \ker D_p(F)^\nu \cong \det \ker D_p(F)|_W$$

where $D_p(F)|_W$ is the restriction of $D_p(F)$ to $W^{1,2}_\delta(\overline{\mathbb{R}}, \mathbb{R}^n)$.

Proof. Let $\tilde{D}_p(F)|_W$ be the conjugated operator of $D_p|_W$ as in the proof of

Lemma 5.4.5. Then

$$\ker D_p|_W \cong \ker \tilde{D}_p(F)|_W$$

canonically. By the definition, $\tilde{D}_p(F)|_W$ is given asymptotically by the matrix of the form

$$\left[\begin{array}{c|c} \delta - \delta\,\mathrm{id} & O \\ \hline O & Hess_p F|_\nu - \delta\,\mathrm{id} \end{array}\right]$$

Therefore, as Lemma 5.4.14, we have a canonical decomposition

$$\ker \tilde{D}_p(F)|_W \cong \ker K \oplus \ker D_p(F)^\nu$$

where K is an obvious operator corresponds to $-\delta\,\mathrm{id}$. Note that $\ker K$ is clearly of zero dimensional since $\delta > 0$. So we get a canonical isomorphism

$$\det\ker \tilde{D}_p(F)|_W \cong \det\ker D_p(F)^\nu$$

which completes the proof. □

Proposition 5.4.16. *Let our situation be generic in the sense of Morse-Bott homology. Then there is a canonical isomorphism*

$$o_p(F_\delta) \cong o_p(f) \otimes \mathcal{L}_p.$$

Proof. Since we are in the generic situation, we have the following sequence of canonical isomorphisms:

$$\begin{aligned} \det D_p(F_\delta) &= \det\ker D_p(F_\delta) \\ &\cong \det\ker D_p(f) \otimes \det\ker D_p(F)^\nu \\ &\cong \det D_p(f) \otimes \det D_p(F) \otimes |T_p\Sigma|^* \\ &\cong \det D_p(f) \otimes \mathcal{L}_p \otimes |T_p\Sigma|^*. \end{aligned}$$

By putting the orientation of Σ in the last one, we get the result. □

Identification of differentials

Let $p, q \in \Sigma$ be critical points of F_δ and x be a Morse flow line from p to q.

Proposition 5.4.17. *There is a commutative diagram of isomorphisms*

$$\begin{array}{ccc} o_p(F_\delta) & \xrightarrow{\cong} & o_p(f) \otimes \mathcal{L}_p \\ {\scriptstyle \partial_x(F_\delta)} \downarrow & & \downarrow {\scriptstyle \partial_x(f) \otimes \mathcal{L}_x} \\ o_q(F_\delta) & \xrightarrow{\cong} & o_q(f) \otimes \mathcal{L}_q \end{array}$$

where the horizontal arrows are as in Proposition 5.4.16.

Proof. Note that the isomorphism $\partial_x(F_\delta)$ is determined by the canonical isomorphism

$$o_p(F_\delta) \otimes |\det D_x(F_\delta)| \cong o_q(F_\delta) \tag{5.4.4}$$

obtained by gluing. Using the isomorphism in Proposition 5.4.16 we can replace $o_p(F_\delta)$ and $o_q(F_\delta)$ by $o_p(f) \otimes \mathcal{L}_p$ and $o_q(f) \otimes \mathcal{L}_q$ respectively. Moreover by a similar splitting argument as the previous section, one can show that there is a canonical isomorphism

$$|\det D_x(F_\delta)| \cong |\det D_x(f)| \otimes |\det D_x(F)|.$$

Therefore the equation (5.4.4) is equivalent to

$$o_p(f) \otimes \mathcal{L}_p \otimes |\det D_x(f)| \otimes |\det D_x(F)| \cong o_q(f) \otimes \mathcal{L}_q. \tag{5.4.5}$$

On the other hand, we have seen that the isomorphism $\mathcal{L}_x : \mathcal{L}_p \to \mathcal{L}_q$ is the same as the map $\partial_x(F)$ given by the gluing in Morse-bott setup. This implies that the equation (5.4.5) induces the same map as $\partial_x(f) \otimes \mathcal{L}_x$. This completes the proof. □

By Proposition 5.4.16 and Proposition 5.4.17, generators and differentials for $HM_*^{loc}(\Sigma; \mathbb{Z})$ and $HM_{*+\text{shift}(\Sigma)}(\Sigma; \mathcal{L}_\Sigma)$ are canonically identified. So Proposition 5.4.1 follows.

5.5 Morse-Bott spectral sequence for Morse homology

We now construct Morse-Bott spectral sequence for Morse homology groups. We do this using an action filtration on Morse chain complex and a classical theorem in homological algebra.

Action filtration

Let (C_*, d) be a differential complex. A **filtration** $\{F_p\}$ of a differential complex (C_*, d) is a family of sequences $0 \leq \cdots \leq F_p \leq F_{p+1} \leq \cdots \leq C_q$ consist of subcomplexes of C_q for each q. Then we have an associated spectral sequence to the filtration by the following classical theorem.

Theorem 5.5.1 ([7]). *For a filtration $\{F_p\}$ of (C_*, d) which is bounded from below and is exhausting, there exists a spectral sequence converging to $H(C_*, d)$ whose E^0-page is given by*

$$E^0_{p,q} = F_p C_{p+q} / F_{p-1} C_{p+q}.$$

We filter the Morse complex by the value of F_δ as

$$F_p CM_q := \{x \in CM_q(F_\delta) \mid F_\delta(x) < a(p)\},$$

where $a : \mathbb{N} \to \mathbb{R}$ is an increasing function satisfying the following condition: For any p, the interval $[a(p-1), a(p)]$ contains at most one critical value of F_δ. In particular, note that the quotient $F_p C_* / F_{p-1} C_*$ only contains critical points of $F_\delta|_\Sigma$ where Σ satisfies $F_\delta(\Sigma) \in [a(p-1), a(p)]$.

Denote $C(p)$ be the set of Morse-Bott components Σ with $F_\delta(\Sigma) \in [a(p-1), a(p)]$. Then by action reason, if δ and the action gap $[a(p-1), a(p)]$ are sufficiently small, then there is no Morse trajectory between Morse-Bott components. This implies that the quotient $F_p C_{p+q} / F_{p-1} C_{p+q}$ is the same as the complex of local Morse homology of Σ's in $C(p)$. In other words, we have manipulated an action filtration so that the complex $(\bigoplus_{p+q=*} E^0_{p+q}, \bigoplus_{p+q=*} d^0)$ is identified to the complex $(\bigoplus_{\Sigma \in C(p)} CM^{loc}_*(\Sigma), \bigoplus_{\Sigma \in C(p)} \partial)$.

Spectral sequence for Morse homology

By Theorem 5.5.1, we get a spectral sequence converging to the Morse homology $HM_*(M, F_\delta; \mathbb{Z})$. Its E^0-page is given by the chain complexes of local Morse homology of Σ, and hence E^1-page is a direct sum of $HM^{loc}_*(\Sigma; \mathbb{Z})$'s. Now Proposition 5.4.1 finishes the proof of the following theorem.

Theorem 5.5.2. *For each local coefficients systems $\mathcal{L}_\Sigma$ over Σ, there is a*

2			
0	$\mathbb{Z}$	$\mathbb{Z}_2$	
	0		2

Figure 5.1: E^1-page for $\mathbb{R}P^2$

spectral sequence converging to $HM_*(M;\mathbb{Z})$, and its E^1-page is given by

$$E^1_{p,q} = \bigoplus_{\Sigma \in C(p)} H_{p+q-\mathrm{shift}(\Sigma)}(\Sigma; \mathcal{L}_\Sigma)$$

where $\mathrm{shift}(\Sigma) := \mathrm{ind}_\Sigma(F)$*.*

Example 5.5.3. Consider the real projective plane $\mathbb{R}P^2$. Its homology groups are well-known as

$$H_*(\mathbb{R}P^2;\mathbb{Z}) = \begin{cases} \mathbb{Z}_2 & * = 1, \\ \mathbb{Z} & * = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5.5.1}$$

We can compute the homology using the spectral sequence. Consider the Morse-Bott function $F : \mathbb{R}P^2 \to \mathbb{R}$ defined in Example 5.4.3. We have two Morse-Bott components; S^1 of index $\mathrm{ind}_{S^1}(F) = 1$ and the origin O with of $(F) = 0$.

We can filter the complex $CM_*(F_\delta)$ by the action such that $C(1) = \{S^1\}$, $C(0) = \{O\}$. Then the column E^1_{0q} is given by

$$H_{0+q-0}(O;\mathbb{Z}) = \begin{cases} \mathbb{Z} & q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The column E^1_{1q} is given by

$$H_{1+q-1}(S^1;\mathcal{L}) = \begin{cases} \mathbb{Z}_2 & q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the E^1-page of the spectral sequence is as Figure 5.1. Note that there is no group homomorphism from $\mathbb{Z}_2$ to $\mathbb{Z}$. It follows that the spectral sequence is stable from the first page. We now observe that the spectral sequence converges to the same homology group of $\mathbb{R}P^2$ as (5.5.1).

Chapter 6

Morse-Bott spectral sequences for symplectic homology

The basic idea of the spectral sequence for symplectic homology is essentially the same as the Morse homology case. We need to deal with Hamiltonians instead of Morse functions. The most crucial part is again the fact that local Floer homology is isomorphic to the singular (or Morse) homology up to some degree shift.

6.1 Morse-Bott type Hamiltonians

Let $(W, \omega = d\lambda)$ be a Liouville domain, and denote its completion by $\widehat{W}$. Consider a time-independent (or equivalently autonomous) Hamiltonian $H : \widehat{W} \to \mathbb{R}$, which is admissible. Then H is in particular linear at the end and its slope does not contained in the action spectrum $Spec(\partial W, \lambda_{\partial W})$. A suitable notion for H, which corresponds to Morse-Bott function, is the following.

Definition 6.1.1. A time-independent Hamiltonian H is called **Morse-Bott** if

1. The set $C := \{x \in \widehat{W} \mid Fl_1^{X_H}(x) = x\}$ forms a compact submanifold of $\widehat{W}$ without bonndary;
2. for each connected component Σ of C, the restriction of the linearized return map $T_x Fl_1^{X_H}|_{\nu(\Sigma)}$ to a normal bundle $\nu(\Sigma)$ is non-degenerate.

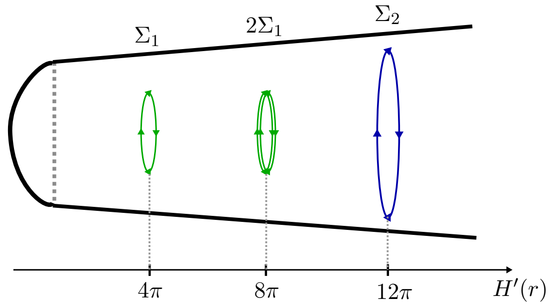


Figure 6.1: Morse-Bott components

In other words, $T_x Fl_1^{X_H}|_{\nu(\Sigma)}$ has no eigenvalue equal to 1.

The set C is called **critical manifold** or **critical submanifold** of H. Note that C is possibly disconnected. Each connected component Σ is called **Morse-Bott component** or **Morse-Bott submanifold**.

In fact, by assumptions on H, the set C consists of *finitely many* connected components: Since H is linear at infinity and its slope is not an action of periodic Reeb orbit in the boundary, every 1-periodic orbit of the Hamiltonian vector field X_H lies in a compact set of $\widehat{W}$. In addition, the second condition of the Morse-Bott Hamiltonian implies that each component is isolated. It follows that the number of Morse-Bott components are finite.

Remark 6.1.2. The Morse-Bott conditions on Hamiltonian makes the corresponding action functional $\mathcal{A}_H$ to be *Morse-Bott* in the sense of Definition 5.2.1.

Example 6.1.3. Let the boundary $(\partial W, \alpha)$ have periodic Reeb flow. Then any admissible time-independent Hamiltonians H are of Morse-Bott type. Indeed, note that the Hamiltonian vector field X_H is propotional to the Reeb vector field R_α and their ratio is exactly the derivative $-H'(r)$, see Lemma 3.2.2. Let $T_1, T_2, \ldots, T_k$ be periods of simple periodic Reeb orbits on the boundary and T_k is the minimal common period. It follows that for each level set $\{r_{T_j}\} \times \partial W$, Hamiltonian 1-periodic orbits form a Morse-Bott component Σ_{T_j} consisting of periodic Reeb orbits of period T_j. The critical submanifold C of H then consists of Σ_{T_j}'s and their iterates. Figure 6.1 is an example of the periodic flow case $\Sigma(3,2,2,2)$. Each components are located in different level set of the symplectization.

6.1.1 Standard perturbation

As in the case of Morse homology, we can perturb Morse-Bott type time-independent hamiltonians to non-degenerate *time-dependent* ones. When H has only S^1-degeneracy, in other words, H is *transversely non-degenerate*, a method of perturbation was presented in [6]. We work out here more general case. Of course, the basic idea is the same as in Section 5.2.2.

We first take a non-negative *Morse* function h on the critical manifold C. We can then extend h to an open neighborhood U of C. Consider a cut-off function ρ supported in U and ρ is constant in Σ. By choosing a metric, we assume that ρ depends only on the distance to C. In addition we can assume the product $h \cdot \rho$ satisfies $0 \leq h \cdot \rho < 1$ by choosing sufficiently small ρ.

Let Σ be a Morse-Bott component of C. Then H has a constant slope along Σ, say s. We define a time-dependent fucntion $\overline{h} : \nu(\Sigma) \times S^1 \to \mathbb{R}$ by

$$\overline{h}(p, n; t) = h(Fl_t^{sR_\alpha}(p)) \cdot \rho(n)$$

where $p \in \Sigma$ and $n \in \nu(\Sigma)_p$ is a normal vector to Σ. Up to an identification of a neighborhood of Σ and its normal bundle $\nu(\Sigma)$, we define a time-dependent perturbation of H by

$$H_\delta = H + \delta\overline{h}$$

for some $\delta > 0$. Intuitively, we have perturbed H in normal direction to each Σ, and the time parameter $t \in S^1$ is still of the Hamiltonian flow of H. It will turn out in Section 6.2.2 that each 1-periodic orbits of H splits into non-degenerate orbits of H_δ which correspond to each critical point of h.

6.1.2 A priori energy bound

The following lemma is the foundation of the definition of *local Floer homology*. It says *implicitly* that, for sufficiently small $\delta > 0$, there is an a priori energy bound needed for Floer trajectories "escaping" a neighborhood of connected component Σ. Note that this is parallel to Lemma 5.4.2. For a proof, we just imitate the proof of [6, Lemma 2.1] with minor changes to deal with $\Sigma \neq S^1$.

Lemma 6.1.4. *Let $H : \widehat{W} \to \mathbb{R}$ be a time-independent Hamiltonian which is Morse-Bott. Let Σ be a connected component of the critical submanifold C of H. Let U be an open neighborhood of Σ in $\widehat{W}$ such that $U \cap C = \Sigma$.*

Then for any open neighborhood $V \subset U$ *of* Σ*, there exists* $\delta_0 > 0$ *such that for each* $0 < \delta < \delta_0$ *the following hold.*

1. *If a* 1*-periodic orbit of the perturb Hamiltonian* H_δ *is in* U*, then in fact it is contained in* V*.*

2. *Every Floer trajectory of* (H_δ, J) *in* U *is already contained in* V*.*

Proof. We prove the first assertion. Suppose on the contrary that there is a neighborhood V of Σ in U such that $\overline{V} \subset U$ and sequences $\delta_n \to 0$ and $\{\gamma_n\}$ of 1-periodic Hamiltonian orbits of H_{δ_n} such that γ_n is not contained in V, say $\gamma_n(t_n) \notin V$ for some $t_n \in S^1$.

Then via the Sobolev embedding $W^{1,2}(S^1) \to C^0(S^1)$, there is a subsequence of $\{\gamma_n\}$, we use the same notation $\{\gamma_n\}$, such that $\gamma_n \to \gamma$ in $C^0(S^1)$ for some 1-periodic Hamiltonian orbit γ of H. Since U is an exclusive neighborhood of Σ by assumption, we must have that γ is contained in Σ. On the other hand, since $\gamma(t_n) \notin V$, it follows that $\gamma(t) \notin V$ where $t_n \to t$. In particular γ is not contained in V. This is a contradiction.

We now prove the second assertion. The idea is the same as the above. Suppose on the contrary that there is a neighborhood V of Σ in U with $\overline{V} \subset U$ and sequences $\delta_n \to 0$ and $\{u_n\}$ of Floer trajectories of H_δ in U, which is not contained in V. Denote the negative and positive asymptotics of u_n by γ_n^- and γ_n^+, respectively. Then by the first assertion, $\gamma_n^- \to \gamma^-$ and $\gamma_n^+ \to \gamma^+$ for some 1-periodic orbits $\gamma^\pm$ of H contained in Σ.

Note that the Hamiltonian action is given by

$$\begin{aligned} \mathcal{A}_{H_{\delta_n}} :& \Lambda\widehat{W} \to \mathbb{R} \\ & \gamma \longmapsto -\int_{S^1} \gamma^*\hat{\lambda} - \int_0^1 H_{\delta_n}(\gamma(t), t)dt \end{aligned}$$

Since $\delta_n \to 0$, it follows that $\mathcal{A}_{H_{\delta_n}}(\gamma_n^+) - \mathcal{A}_{H_{\delta_n}}(\gamma_n^-) \to 0$. On the other hand, the standard compactness theorem implies that u_n has a convergent subsequence, still denoted by u_n, in C^∞_{loc}, say $u_n \to u$ for some Floer trajectory of (H, J). Since the energy of u is given by the difference of actions of asymptotics, we have $e(u) = \lim_{n\to\infty} \mathcal{A}_{H_{\delta_n}}(\gamma_n^+) - \mathcal{A}_{H_{\delta_n}}(\gamma_n^-) = 0$. Therefore u is in fact constant along the cylindrical direction. As a result $u(\mathbb{R} \times S^1) \subset \Sigma$. This contradicts to the assumption that u_n is not contained in V for all n. □

From now on, we always assume that $\delta > 0$ is so small that the conclusion of Lemma 6.1.4 holds.

6.2 Local Floer homology

In this section, we give a definition of *local Floer homology* of each Morse-Bott component Σ with coefficients in a ring R. Lemma 6.1.4 is foundational for the definition. We also address that the local Floer homology is isomorphic to the Morse homology of Σ with a degree shift and a *twisted coefficients* by a local coefficient system $\mathcal{L}_\Sigma$. We finally give several conditions when $\mathcal{L}_\Sigma$ is in fact trivial.

6.2.1 Definition

Let H_δ be a perturbed time-dependent non-degenerate Hamiltonian and J a compatible almost complex structure. Fix a ring R and Σ a Morse-Bott component of H. Let U be an exclusive neighborhood of Σ, i.e, U does not contain 1-periodic orbits of H other than those of Σ. Denote 1-periodic orbits of H_δ which are contained in U by $\mathcal{P}^U_{H_\delta}$. We define a chain complex $CF^{loc}_*(\Sigma, H_\delta, J)$ by

$$CF^{loc}_k(\Sigma, H_\delta, J) = \bigoplus_{\substack{\gamma \in \mathcal{P}^U_{H_\delta} \\ \mu_{CZ}(\gamma)=k}} R\langle\gamma\rangle,$$

the free R-module generated by 1-periodic orbits of H_δ in U. We grade it by Conley-Zehnder index.

To define the differential, we introduce a moduli space $\mathcal{M}^U(\overline{\gamma}, \underline{\gamma}, H_\delta, J)$ collecting Floer trajectories of (H_δ, J) contained in U. More precisely we define

$$\mathcal{M}^U(\overline{\gamma}, \underline{\gamma}, H_\delta, J) = \{u : \mathbb{R}\times S^1 \to U \mid \partial_s u + J_t(u)(\partial_t u - X_{H_\delta}) = 0,\ \lim_{s\to\mp\infty} = \overline{\underline{\gamma}}\}.$$

As the usual Floer homology case, for generic J we achieve transversality for the moduli space so that it is a smooth manifold of dimension given by

$$\mu_{CZ}(\overline{\gamma}) - \mu_{CZ}(\underline{\gamma}) - 1.$$

Note that in the case when H_δ is C^2-small, the corresponding Floer trajectories are in fact Morse trajectories of (Σ, H_δ), see Proposition 3.4.17. We choose a coherent orientation of $\mathcal{M}^U(\overline{\gamma}, \underline{\gamma}, H_\delta, J)$ such that if H_δ is C^2-small, then the Floer trajectories are counted in the same way as the Morse trajectories with the coherent orientation of Morse homology, which makes it to be isomorphic to singular homology in $\mathbb{Z}$-coefficients. Such a orientation for Morse homology is discussed in Section 5.3.

Now we define the differential on $CF_*^{loc}(\Sigma, H_\delta, J)$ by

$$\begin{aligned}\partial_k^{loc} : CF_k^{loc}(\Sigma, &H_\delta, J) \to CF_{k-1}^{loc}(\Sigma, H_\delta, J)\\ &\overline{\gamma} \mapsto \sum_{\substack{\underline{\gamma} \in \mathcal{P}^U_{H_\delta} \\ \mu_{CZ}(\underline{\gamma}) = k-1}} \sum_{[u] \in \mathcal{M}^U(\overline{\gamma}, \underline{\gamma}, H_\delta, J)} \epsilon([u]) \underline{\gamma}\end{aligned}$$

where $\epsilon([u])$ is the sign according to the coherent orientation. By the standard recipe, the pair $(CF_*^{loc}(\Sigma, H_\delta, J), \partial^{loc})$ forms a differential complex. Its homology group

$$HF_*^{loc}(\Sigma, H_\delta, J; R) := H_*(CF_*^{loc}(\Sigma, H_\delta, J), \partial^{loc})$$

is called **local Floer homology** of Σ. Lemma 6.1.4 guarantees that local Floer homology does not depend on the choice of a neighborhood U of Σ, provided that $\delta > 0$ is sufficiently small.

6.2.2 Local Floer homology and Morse homology

In this section, we prove that the local Floer homology $HF_*^{loc}(\Sigma, H_\delta, J; R)$ is isomorphic to the singular homology $H_*(\Sigma; R \otimes \mathcal{L}_\Sigma)$, up to a degree shifing, where $\mathcal{L}_\Sigma$ is a local coefficient system over Σ. This was done for $\Sigma = S^1$ and $R = \mathbb{Z}_2$ in [6]. We here extend their proof to the general case with some technical assumption. The precise statement is as follows.

Theorem 6.2.1. *Let Σ be a Morse-Bott component of a Morse-Bott type Hamiltonian $H : \widehat{W} \to \mathbb{R}$. Let $H(\Sigma) \equiv c$ for some contant $c \in \mathbb{R}$. Assume the following.*

1. *The constant c is a regular value of H such that the Liouville vector field X is transverse to $\Sigma_c := H^{-1}(c)$.*

2. *The restriction of the Hessian of H to the Liouville direction is positive definite.*

Then we have a canonical isomorphism

$$HF_*^{loc}(\Sigma, H_\delta, J; R) \cong HM_{*-\mathrm{shift}(\Sigma)}(\Sigma; R \otimes \mathcal{L}_\Sigma)$$

where $\mathrm{shift}(\Sigma) = \mu_{RS}(\Sigma) - \frac{1}{2}\dim(\Sigma/S^1)$ *and* $\mathcal{L}_\Sigma$ *is a local coefficient system over* Σ.

Proof. A basic idea of the proof is the following: By the perturbation $H_\delta = H + \delta h$, the Morse-Bott component Σ splits into non-degenerate 1-periodic Hamiltonian orbits of X_{H_δ} which correspond to each critical point of $h : \Sigma \to \mathbb{R}$. To make this correspondence rigorous, we first "unwrap" the flow of X_{H_δ} to replace its periodic orbits by constant orbits. This step is called *spinning*, see [23].

By the first assumption, a neighborhood $\nu(\Sigma)$ of Σ in $\widehat{W}$ can be identified to $(1-\epsilon, 1+\epsilon) \times \nu_{\Sigma_c}(\Sigma)$ with the symplectic form $d\lambda_{\Sigma_c}$. Here r denotes the coordinate for the interval $(1-\epsilon, 1+\epsilon)$. Note that the Hamiltonian H depends only on r, and along Σ it has a constant slope, say $\partial_r H|_{\nu(\Sigma)} = s$. Consider the Hamiltonian $K : \nu(\Sigma) \to \mathbb{R}$ defined by $K(r, y) = -s \cdot r$, and denote its Hamiltonian flow by $\Delta_t := Fl_t^{X_K}$. We now define a "unwrapped" Hamiltonian $\tilde{K} : S^1 \times \nu(\Sigma) \to \mathbb{R}$ by

$$\tilde{K}(t, x) := H_\delta(t, \Delta_{-t}(x)) + K(x).$$

Let J be an almost complex structure for $HF^{loc}(H_\delta)$. Define another almost complex structure $J_{\tilde{K}}$ by the conjugation $J_{\tilde{K}} := T\Delta_t \circ J \circ T\Delta_t^{-1}$. Then we see that $HF^{loc}(H_\delta, J)$ is canonically isomorphic to $HF^{loc}(\tilde{K}, J_{\tilde{D}})$ as follows.

- Note that the Hamiltonian flow of $\tilde{K}$ is nothing but the composition $\Delta_t \circ Fl_t^{X_{H_\delta}}$. On chain level, we send a 1-periodic orbit x of H_δ to $\hat{x} := \Delta_t \circ x$. This map preserves the Conley-Zender index, provided that $c_1(W) = 0$ by the loop property.

- With the choice of almost complex structure $J_{\tilde{K}}$, we directly see that the every Floer trajectory for (H_δ, J) corresponds to a Floer trajectory for $(\tilde{K}, J_{\tilde{K}})$. A Floer trajectory for (H_δ, J) is regular if and only if the corresponding trajectory for $(\tilde{K}, J_{\tilde{K}})$ is so.

- For orientations of Moduli spaces, we pull back the coherent orientations for $CF^{loc}(H_{\delta,J})$ with Δ_t^{-1} to get local coefficient systems for

$CF^{loc}(\tilde{K}, J_{\tilde{K}})$.

It follows that $HF^{loc}(H_\delta, J)$ is isomorphic to $HF^{loc}(\tilde{K}, J_{\tilde{K}})$ via the above correspondence. This completes the unwrapping.

The next step is to give an isomorphism between $HF^{loc}(\tilde{K}, J_{\tilde{K}}; R)$ to Morse homology $HM(\Sigma; R \otimes \mathcal{L}_\Sigma)$ up to a degree shifting. (For more clear presentation, we prospond constructing the local coefficient system $\mathcal{L}_\Sigma$ to the next section.) Observe that $\tilde{K}$ is C^2-small along Σ. In view of Lemma 6.1.4, all 1-periodic orbits of $\tilde{K}$ in $\nu(\Sigma)$ are in fact critical points in Σ. Let J be an almost complex structure on $\nu(\Sigma)$ such that the metric $g = \omega(\cdot, J\cdot)$ is Morse-Smale. Consider the Floer equation $\partial_s u + J\partial_t u = -\operatorname{grad}_g \tilde{K}$ for the local Floer homology $HF^{loc}(\Sigma, \tilde{K}, J)$. We see that the rigid *gradient trajectories* of $\tilde{K}$ with respect to the metric g are solution of the Floer equation. In other words, they are Floer trajectories for $(\tilde{K}, J)$.

It turns out that the rigid gradient trajectories are in one-to-one correspondence to the rigid Floer trajectories, which was noted by Floer [21] as well as in [6, proof of Proposition 2.2]. The condition that the Hamiltonian is C^2-small is essential, and we have discussed such phenomenon in Section 3.4.6. For more detailed description, we refer to Poźniak [22, Proposition 3.4.6] who identifies the kernels of the linearized Floer equation and the kernel of linearized gradient flow in a Lagrangian setup. The upshot is that $HF^{loc}(\tilde{K}, J; R)$ is isomorphic to $HM(\Sigma; R \otimes \mathcal{L}_\Sigma)$ up to a degree shifting.

Finally we explain how to determine the degree shifting. Remind that $\mu_{RS}(\Sigma)$ denotes the Robbin-Salamon index of a periodic orbit γ lies in Σ, with respect to the *linearized Reeb flow restricted to the contact structure* $\xi = \ker \lambda_{\partial W}$. By direct sum ξ with $\langle R_\alpha, Y \rangle$ where Y is the Liouville vector field, we can extend the trivialization of $T\widehat{W}$. Accordingly, we have the corresponding Robbin-Salamon index of γ with respect to the *linearized Hamiltonian flow.* Denote this index by $\mu_{RS}^{Ham}(\Sigma)$. Then $\mu_{RS}^{Ham}(\Sigma)$ and $\mu_{RS}(\Sigma)$ are related by

$$\mu_{RS}^{Ham}(\Sigma) = \mu_{RS}(\Sigma) + \frac{1}{2}$$

as we have proven in Proposition 3.3.15.

On the other hand the Robbin salamon index of a *contant* Hamiltonian 1-peridic orbit x of a C^2-small Hamiltonian of $2n$-dimensional symplectic manifold is related to the Morse index by

$$\mu_{RS}^{Ham}(x) = \mathrm{ind}^{Morse}(x) - n.$$

This is proved in Proposition 3.3.14. In the proof of the proposition, $\mathrm{ind}^{Morse}(x) - n$ is actually the half of the signature of the Hessian of the Hamiltonian.

Going back to our situation, note that the degree shiftings occur twice exactly when we composite Δ_t to the Hamiltonian orbits of H_δ and when we interpret a constant periodic orbit of K as a critical point. The degree shifting for the first step amounts to $\mu_{RS}(\Sigma) - \frac{1}{2}\dim(\Sigma/S^1)$. For a detailed computation, see [23, Section 3.3]. The second step affects the degree by the relation between the indices of a contact periodic Hamiltonian orbit and a critical point. Combining all these together, we have the the desired degree shifting. $\square$

Remark 6.2.2. An easy rough way of thinking of the degree shifting is the following. By perturbing a Morse-Bott Hamiltonian to a non-degenerate one, the last term of (3.3.1) disappears by non-degeneracy. This happens for each "direction" of Morse-Bott components. This is the reason why we have the term $\frac{1}{2}\dim\Sigma$ for the degree shifting.

In particular, note that the interior W is itself a Morse-Bott component consisting constant orbits of H. Since H_δ is C^2-small and Morse, it follows from the above theorem that

$$HF_*^{loc}(W, H_\delta, J; R) \cong H_{*+n}(W, \partial W; R). \tag{6.2.1}$$

The reason why we can just take R-coefficient on the right hand side is due to the initial choice of coherent orientation of Moduli space of local Floer homology.

6.2.3 Construction of the local system $\mathcal{L}_\Sigma$

We now address how we construct the local coefficient system $\mathcal{L}_\Sigma$ in Theorem 6.2.1. We basically follow the scheme in Section 5.4.2 for Morse case. Namely, we construct an $O(1)$-bundle L whose fiber is given by the determinant line of the operator corresponding to the linearization of the Floer equation $\partial_s u + J(\partial_t u - X_H)u = 0$.

Let Σ be a Morse-Bott component of a time-independent Morse-Bott type Hamiltonian $H : \widehat{W} \to \mathbb{R}$. It is very useful in this section to think of each 1-periodic orbit γ of X_H in Σ as a point in Σ. This correspondence is obviously given by $\gamma \mapsto \gamma(0)$.

Fix a real number $R > 0$ and a cutoff function $\rho : \mathbb{C} \to \mathbb{R}$ such that $\rho(z) = 1$ for $|z| > 2R$ and $\rho(z) = 0$ for $|z| < R$. Let $\gamma \in \Sigma$. Take a capping disk $u_\gamma : \mathbb{C} \to \widehat{W}$ of γ. This means that $u_\gamma(z) = \gamma(z/|z|)$ for $|z| > R$. Then we have a symplectic trivialization $\varepsilon_\gamma : \mathbb{C} \times \mathbb{R}^{2n} \to u_\gamma^* T\widehat{W}$.

We define a vector space $\mathcal{T}_\gamma \Sigma$ for asymptotic operators as follows. Find a basis of solutions to the ODE for the vector field X along γ,

$$J(\nabla_t X - \nabla_X X_H) = 0$$

where ∇ is a symmetric connection for $T\widehat{W}$. Denote the basis by $\{e_{\gamma,i}\}_i$. Using the cutoff function ρ, we extend these functions to a map

$$\tilde{e}_{\gamma,i} : \mathbb{C} \to u_\gamma^* T\widehat{W}, \quad z \mapsto \rho(z) \cdot e_{\gamma,i}\left(\frac{z}{|z|}\right).$$

We define the vector space $\mathcal{T}_\gamma \Sigma$ by

$$\mathcal{T}_\gamma \Sigma := \mathrm{span}_{\mathbb{R}}(\tilde{e}_{\gamma,1}, \ldots, \tilde{e}_{\gamma,\dim \Sigma}).$$

Note that this vector space is morally the tangent space of the component Σ.

As in the Morse case, we also define necessary Sobolev spaces with weighted measure in Floer setup. For a cylindrical end $[R, \infty) \times S^1$ of the plane $\mathbb{C}$, we equip the measure $e^{\delta s} ds \wedge dt$, where $\delta > 0$ is chosen to be smaller then the spectral gap of the asymptotic operator. We extend this measure to the plane $\mathbb{C}$ where we use the standard measure near the origin. Denote the resulting weighted measure by $\rho_\delta ds \wedge dt$. We now define Sobolev spaces

$$\begin{aligned} W^{1,p}_\delta(\mathbb{C}, u_\gamma^* T\widehat{W}) &:= \{f : \mathbb{C} \to u_\gamma^* T\widehat{W} \mid f \in W^{1,p}(\mathbb{C}, u_\gamma^* T\widehat{W}; \rho_\delta ds \wedge dt)\}, \\ L^p_\delta(\mathbb{C}, u_\gamma^* T\widehat{W}) &:= \{f : \mathbb{C} \to u_\gamma^* T\widehat{W} \mid f \in L^p(\mathbb{C}, u_\gamma^* T\widehat{W}; \rho_\delta ds \wedge dt)\}. \end{aligned}$$

For each $\gamma \in \Sigma$ we define an operator $D_\gamma(H)$ by

$$\begin{aligned} D_\gamma(H) : & W^{1,p}_\delta(\mathbb{C}, u^*_\gamma T\widehat{W}) \oplus \mathcal{T}_\gamma \Sigma \longrightarrow L^p_\delta(\mathbb{C}, u^*_\gamma T\widehat{W}) \\ & \tilde{X} := X + e \longmapsto \nabla_s \tilde{X} + J \nabla_t \tilde{X} - B_\gamma \cdot \tilde{X} \end{aligned}$$

where $B_\gamma : \mathbb{C} \to \mathbb{R}^{2n\times 2n}$ is such that $B_\gamma(z) = Hess_\gamma H$ for $|z| > 2R$.

By the same argument as Lemma 5.4.5, the operator $D_\gamma(H)$ is Fredholm. Furthermore, the space of such Fredholm operators is contractible, see [29, Section 4.4], so that the determinant line bundle $\det D_\gamma(H)$ does not depend on the choice of B_γ, as long as its asymptotic condition is fixed.

Take a finite good covering $\{U_a\}$ of Σ. Using the technique in 5.4.2, we have a continuous family of trivializations $(u^a_\gamma, \varepsilon^a_\gamma)$ varying along $\gamma \in U_a$; we take a "center" γ_a of U_a and push a trivialization at γ_a forward to a trivialization at each $\gamma \in U_a$ along the canonical path from γ_a to γ. Then by Lemma 5.4.7, we get a line bundle L_{U_a} over U_a for each a, whose fiber at $\gamma \in U_a$ is the determinant line bundle $\det D_\gamma(H)$.

The technique in Section 5.4.2 can also be applied to show that for each $\gamma \in U_a \cap U_b$; we can identify the corresponding fibers $\det D^a_\gamma$ and $\det D^b_\gamma$ in a canonical way, up to a positive scalar multiplication. This is essentially due to Lemma 5.4.8. It follows that we have transition functions in $O(1)$, and we can check the cocycle condition holds as before. Define an $O(1)$-bundle L_Σ over Σ by attaching L_{U_a}'s using the transition functions. The corresponding local coefficient system on Σ is now the desired local system $\mathcal{L}_\Sigma$.

Local Floer homology using orientation lines

To clarify the relation between $HF^{loc}_*(\Sigma, H_\delta; R)$ and $HM_*(\Sigma, \tilde{K}, \mathcal{L}_\Sigma)$, we give a brief definition of $HF^{loc}_*(\Sigma, H_\delta; R)$ in terms of orientation lines. This approach is described in detail by Abouzaid [31], and is an analogue of Section 5.3.

Given a 1-periodic Hamiltonian orbit $\hat{\gamma}$ of H_δ, choose a capping plane $u_{\hat{\gamma}}$ and a trivialization $\varepsilon_{\hat{\gamma}}$ of $T\widehat{W}$ via $u_{\hat{\gamma}}$. With respect to the trivialization, we obtain a Fredholm operator

$$\begin{aligned} D_{\hat{\gamma}} : W^{1,p}(\mathbb{C}, \mathbb{R}^{2n}) \to L^p(\mathbb{C}, \mathbb{R}^{2n}) \\ X \longmapsto \partial_s X + J \partial_t X + B_{\hat{\gamma}} \cdot X \end{aligned}$$

where $B_{\hat{\gamma}} : \mathbb{C} \to \mathbb{R}^{2n\times 2n}$ is such that $B_{\hat{\gamma}}(z) = Hess_{\hat{\gamma}} H_\delta$ for $|z| > 2R$.

Denote its orientation line by $o_{\hat{\gamma}} := |\det D_{\hat{\gamma}}|$. Then the Floer chain complex is defined as the graded abelian group

$$CF_*^{loc}(\nu(\Sigma), H_\delta) := \bigoplus_{\hat{\gamma}\in\mathcal{P}_{H_\delta}^{\nu(\Sigma)}} o_{\hat{\gamma}}.$$

For each rigid Floer trajectory u from $\overline{\gamma}$ to $\underline{\gamma}$ induces a canonical isomorphism

$$\partial_u : o_{\overline{\gamma}} \longrightarrow o_{\underline{\gamma}}$$

by the gluing lemma [28, Proposition 9]. We now define the differential $\partial : CF_*^{loc}(\nu(\Sigma), H_\delta) \to CF_*^{loc}(\nu(\Sigma), H_\delta)$ as follows.

$$\partial|_{o_{\overline{\gamma}}} = \sum_{\underline{\gamma}} \sum_{[u]\in\mathcal{M}(\overline{\gamma},\underline{\gamma},H_\delta)} \partial_u.$$

The the local Floer homology $HF_*^{loc}(\Sigma, H_\delta; R)$ is defined by taking the homology of the complex. This is of course the same thing as the one in 6.2.1.

Recall that the Morse homology of $\tilde{K}$ with local coefficient $\mathcal{L}_\Sigma$ is “generated” by orientation lines $o_{\Delta_t\circ\gamma}(\tilde{K}) \otimes \mathcal{L}_\gamma$. The next lemma follows the same argument as Proposition 5.4.16.

Lemma 6.2.3. *The orientation line $o_{\hat{\gamma}}$ is canonically isomorphic to the orientation line $o_{\Delta_t\circ\gamma}(\tilde{K}) \otimes \mathcal{L}_\gamma$.*

We can also identify the differentials by an analogous argument of the proof of Proposition 5.4.17. We finally conclude that

$$HF_*^{loc}(\Sigma, H_\delta; R) \cong HM_{*-\mathrm{shift}(\Sigma)}(\Sigma, \tilde{K}; \mathcal{L}_\Sigma) \cong H_{*-\mathrm{shift}(\Sigma)}(\Sigma; \mathcal{L}_\Sigma).$$

There are some conditions that makes local coefficient system $\mathcal{L}_\Sigma$ to be trivial.

Lemma 6.2.4. *Suppose that one of the following conditions holds.*

1. $H^1(\Sigma; \mathbb{Z}_2) = 0$*;*
2. $\Sigma = S^1$ *is a good Reeb orbit;*
3. *the linearized Reeb flow is complex linear with respect to a unitary trivialization of the contact structure along each* $\gamma \in \Sigma$*.*

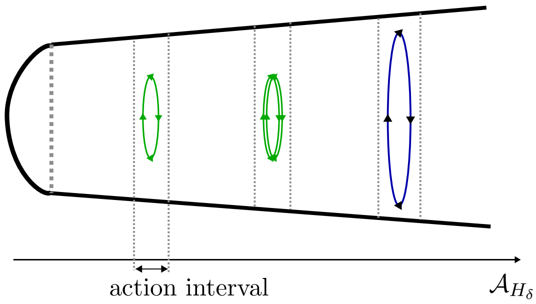


Figure 6.2: Action filtration

Then the local coefficient system $\mathcal{L}_\Sigma$ is trivial.

Proof. If the first assumption $H^1(\Sigma; \mathbb{Z}_2) = 0$ holds, then every local system on Σ is trivial, see Proposition 5.3.13. For the second condition, we refer to [29, Lemma 4.29]. It is proven there that if $\Sigma = S^1$ is a transversely non-degenerate orbit, then $\mathcal{L}_\Sigma$ is trivial if and only if γ is a good orbit. The proof of the third case is given in [51, Lemma B.7]. □

As a result, we can rewrite the isomorphism in Theorem 6.2.1 with trivial coefficients.

Corollary 6.2.5. *If the assumptions of Lemma 6.2.4 as well as the assumptions of Theorem 6.2.1 hold, then we have*

$$HF_*^{loc}(\Sigma, H_\delta, J; R) \cong HM_{*-\mathrm{shift}(\Sigma)}(\Sigma; R).$$

6.3 An action filtration of Floer chain complex

Let $H : \widehat{W} \to \mathbb{R}$ be a Morse-Bott type admissible Hamiltonian. Denote its perturbation by H_δ as before. Since the Hamiltonian action is decreasing along Floer trajectories, the chain complex of Hamiltonian Floer homology admits a natural action filtration. For a given Morse-Bott type Hamiltonian, in particular, we can give more "sensible" action filtration which is adapted to our purpose; the desired filtration distinguishes each Morse-Bott components by their actions and respects the perturbation in some sense. See Figure 6.2 for an intuitive picture of the case $\Sigma(3, 2, 2, 2)$. We assume that values of H at each constant orbits are negative and sufficiently close to zero, where as non-constant 1-periodic orbits have positive action.

Proposition 6.3.1. *There is a strictly increasing function $a_H : \mathbb{Z} \to \mathbb{R}$ satisfying the following conditions.*

1. *We set $a_H(0) = 0$, and for constant orbits x of H, we impose $\mathcal{A}_{H_\delta}(x) \in (a_H(-1), a_H(0)]$.*
2. *Let Σ be a Morse-Bott component of H with $\mathcal{A}_H \in (a_H(p-1), a_H(p)]$ for some $p \in \mathbb{Z}$, then the corresponding 1-periodic orbits of H_δ have their action in the same interval.*
3. *For $p > 0$, if Σ_1 and Σ_2 are distinct Morse-Bott components with $\mathcal{A}_H(\Sigma_j) \in (a_H(p-1), a_H(p)]$, then there are no Floer trajectories between them.*
4. *$\lim_{p\to\infty} a_H(p) = \infty$.*

Proof. The first and the forth conditions are nothing but conventions. To make sure the second condition, for a component Σ, the action difference between 1-periodic orbits corresponding to the minimum and the maximum of $h|_\Sigma$ is bounded by the the perturbation parameter δ. If we choose the difference $a_H(p) - a_H(p-1)$ sufficiently small where $\mathcal{A}_H(\Sigma) \in (a_H(p-1), a_H(p)]$, then it follows that the corresponding 1-periodic orbits of H_δ are still in the same interval.

For the third condition, observe that, a Floer trajectory of H escaping a neightborhood of Σ has energy bounded from below. Since there are only finitely many Morse-Bott component, we can take the bound $\tilde{\delta} > 0$ for the critical submanifold C of H. Note that the energy of Floer trajectories is given by the difference of actions of asymptotics. In particular, if $\Sigma_1, \Sigma_2 \in C(p)$, then a Floer trajectory between them would have energy bounded by δ. It follows that there is no such trajectory, provided that $\delta < \tilde{\delta}/2$. □

We denote the set of Morse-Bott components whose actions are contained in the interval $(a_H(p-1), a_H(p)]$ by $C(p)$, i.e.,

$$C(p) := \{\text{Morse-Bott component } \Sigma \mid \mathcal{A}_H(\Sigma) \in (a_H(p-1), a_H(p)]\}.$$

Example 6.3.2. As in Example 6.1.3, we consider the case when the boundary $(\partial W, \alpha)$ admits a periodic Reeb flow. Let H be an admissible Hamiltonian of Morse-Bott type such that $H < 0$ in the interior. Using the notations in Example 6.1.3, one can clearly choose a function a_H so that $\Sigma_{T_j} \in C(p_j)$ for some $p_j > 0$ for each j.

6.4 Morse-Bott spectral sequences for symplectic homology

In this section, we give a construction of Morse-Bott spectral sequences for symplectic homology. We first construct a spectral sequence for Hamiltonian Floer homology $HF_*(H_\delta, J)$ using the action filtration described in Section 6.3. We then pass to spectral sequence for symplectic homology.

6.4.1 For Hamiltonian Floer homology

Let $H : \widehat{W} \to \mathbb{R}$ be a Morse-Bott type time-independent admissible Hamiltonian. Let H_δ be the perturb non-degenerate admissible Hamiltonian and let us have a choice of the filtration function $a_H : \mathbb{Z} \to \mathbb{R}$. We define an action filtration on the Floer complex $CF_*(H_\delta, J)$ by

$$F_p CF_q(H_\delta, J) := \{\gamma \in CF_q(H_\delta, J) \mid \mathcal{A}_{H_\delta}(\gamma) \leq a_H(p)\}.$$

Note that H is chosen to be linear at the end and its slope is not in the action spectrum of the Reeb flow. Since $a_H(p) \to \infty$ by the construction of a_H, it follows that the filtration exhausts, in other words, we eventually have

$$F_p CF_q(H_\delta, J) = CF_q(H_\delta, J)$$

for large p. We denote the minimal value of such p by p_H.

Note also that this filtration is clearly bounded from below. By Theorem 5.5.1, we obtain the associated spectral sequence converging to Floer homology $HF_*(H_\delta, J)$, whose E^0-page is begin by

$$E^0_{pq} = F_p CF_{p+q}(H_\delta, J)/F_{p-1} CF_{p+q}(H_\delta, J),$$

and the differential d^0 of E^0-page is obviously induced by the differential of the Floer complex $CF_*(H_\delta, J)$.

Observe that, from the construction of a_H, the group E^0_{pq} is the same as the direct sum of local Floer complexes, namely

$$E^0_{pq} = \bigoplus_{\Sigma \in C(p)} CF^{loc}_{p+q}(\Sigma, H_\delta, J).$$

Moreover, the third condition of a_H in Proposition 6.3.1 guarantees that the differential of the whole complex $CF_*(H_\delta, J)$ respects the above direct sum of the local Floer complexes. It follows that the differential d^0 of E^0-page acts precisely as the differentials of $CF^{loc}_{p+q}(\Sigma, H_\delta, J)$ of each Morse-Bott component. We conclude the E^1-page of the spectral sequence is given by

$$E^1_{pq} = \bigoplus_{\Sigma \in C(p)} HF^{loc}_{p+q}(\Sigma, H_\delta, J).$$

By combining the discussions so far with Corollary 6.2.5 and (6.2.1), we have the following.

Theorem 6.4.1. *Suppose that $(W, d\lambda)$ is a Liouville domain with a time-independent Morse- Bott type Hamiltonian H satisfying the following.*

1. *The hamiltonian H is linear at the end and its slope is not the period of any periodic Reeb orbit of $\lambda_{\partial W}$.*
2. *The restriction of the Hessian of H along each Morse-Bott component Σ to the Liouville direction is positive definite.*
3. *The assumption of Lemma 6.2.4.*

Let H_δ be the perturbed time-dependent non-degenerate Hamiltonian of the same slope at the infinity. Then there exists a spectral sequence converging to $HF(H_\delta, J; R)$ whose E^1-page is given by

$$E^1_{pq}(HF(H_\delta, J; R)) = \begin{cases} \bigoplus_{\Sigma \in C(p)} H_{p+q-\mathrm{shift}(\Sigma)}(\Sigma; R) & p > 0, \\ H_{q+n}(W, \partial W; R) & p = 0, \\ 0 & p < 0. \end{cases} \tag{6.4.1}$$

Here, $\mathrm{shift}(\Sigma) = \mu_{RS}(\Sigma) - \frac{1}{2}\dim(\Sigma/S^1)$.

6.4.2 For symplectic homology

For each admissible Hamiltonian H, we have a spectral sequence (6.4.1) for Hamiltonian Floer homology group $HF_*(H_\delta, J; R)$. Note that the symplectic homology $SH_*(W;R)$ is a direct limit of $HF_*(H_\delta, J; R)$ over a direct system of admissible Hamiltonians.

When the boundary has a periodic flow, we can construct a spectral sequence for symplectic homology $SH_*(W;R)$ by passing to a direct limit of those for $HF_*(H_\delta, J; R)$. A careful choice of a cofinal family of admissible Hamiltonians is necessary to get a direct system of spectral sequences.

A cofinal family of Hamiltonians for periodic flows

Assume that the Reeb flow of the boundary $(\partial W, \lambda_{\partial W})$ is periodic with minimal periods $T_1 < T_2 < \cdots < T_k$, where T_k is the common period. We define a sequence of Hamiltonians $\{H_n : \widehat{W} \to \mathbb{R}\}_{n \in \mathbb{Z}_{\geq 0}}$ satisfying the following conditions.

1. Each Hamiltonian H_n is C^2-small in the interior W.

2. $H_n(r, x) = h_n(r)$ for an increasing function that is linear at the end with the slope $n \cdot T_k + \epsilon$. Let r_n is the infimum such that $H_n|_{[r_0,\infty)\times\partial W}$ is linear.

3. $H_n \equiv H_{n-1}$ in the region $\{r \leq r_{n-1}\}$ including the interior.

Intuitively speaking, the "limit" H_∞ of the sequence $\{H_n\}$ would be a quadratic function of the cylindrical coordinate r.

We also choose the functions $a_{H_n} : \mathbb{Z} \to \mathbb{R}$ of filtration as follows. Recall that $p_H \in \mathbb{Z}$ denotes the minimal integer such that $F_pCF_*(H, J) = CF_*(H, J)$ for all $p \geq p_H$. We define $\{a_{H_n} : \mathbb{Z} \to \mathbb{R}\}$ recursively by requiring that

1. $a_{H_n}(p) = a_{H_{n-1}}(p)$ for all $p \leq p_{H_{n-1}}$;

2. For $p > p_{H_{n-1}}$, we impose that $a_{H_n}(p)$ increases so slowly that
$$F_pCF_*(H_n, J) = CF_*(H_{n-1}, J)$$
for $p_{H_{n-1}} \leq p \leq 2p_{H_{n-1}}$, and extend a_{H_n} following conditions in Proposition 6.3.1.

In particular the second condition implies that the new column of $E^1(HF_*(H_n))$-page appears after a total degree gap lager than $2p_{H_{n-1}}$. Note that we have the function a_{H_∞} for action filtration by taking the "limit" of $\{a_{H_n}\}$ in an obvious sense. We use this function for every spectral sequences for $HF_*(H_n)$ uniformly.

Direct system of spectral sequences

Denote the spectral sequences (6.4.1) for H_n by $E(H_n)$ and its E^r-page by $E^r_{pq}(H_n)$. By the constructions of $\{H_n\}$ and the function a_{H_∞} of action filtration, the sequence $\{E(H_n)\}$ has the following property: All generators in E_n also appear in $E_{n'}$ at the same positions for $n \le n'$. Therefore, for each $n \le n'$, we can define a morphism

$$c_{nn'} : E(H_n) \to E(H_{n'})$$

between spectral sequences as obvious inclusions.

Lemma 6.4.2. *The morphism $c_{nn'}$ is a chain map on each pages E^r with respect to the differentials d^r of the spectral sequences. In other words,*

$$d^r_{n'} \circ c^r_{nn'} = c^r_{nn'} \circ d^r_n.$$

Proof. Note the second condition of a_{H_n} implies that the gap between the last non-trivial column of $E^0_{pq}(H_n)$ and the first additional column of $E^0_{pq}(H_{n'})$ is bigger than p_{H_n}. Now we divide the argument into two cases:

- Let $p \le p_{H_n}$. In this case $c_{nn'}$ is nothing but the identity map by its construction; $E(H_n)$ and $E(H_{n'})$ coincide when $p \le p_{H_n}$, and since $H_{n'}$ is linear in an interval contained in $(t_n, t_{n'q})$, there is no Floer trajectories from the new generators.
- Let $p > p_{H_n}$. Since $E^r_{pq}(H_n)$ is trivial in this case, the morphism $c^r_{nn'}$ vanishes. The second restriction on a_{H_∞} implies that d^r_n also vanishes.

This completes the proof. □

We can therefore form a direct system $(E(H_n), c_{nn'})$, over the index set $\mathbb{Z}_{\ge 0}$, of spectral sequences for Hamiltonian Floer homology $HF_*(H, J)$. By taking the direct limit we have a spectral sequence converging to the direct limit of $HF_*(H, J)$, i.e., symplectic homology $SH_*(W)$. The upshot is now the following theorem.

Theorem 6.4.3. *Let $(W, d\lambda)$ be a Liouville domain and suppose the following assumptions.*

1. *The Reeb flow of the boundary ∂W is periodic with minimal periods $T_1, T_2, \ldots, T_k$, where T_k is the common period.*
2. *There is a compatible complex structure J for the contact structure $(\xi, d(\lambda|_{\partial W}))$ of ∂W such that, for every periodic Reeb orbit γ, the linearized Reeb flow is complex linear with respect to some unitary trivialization of (ξ, J) along γ.*

Then there is a spectral sequence converging to the symplectic homology $SH(W; R)$, whose E^1-page is given by

$$E^1_{pq}(SH) = \begin{cases} \bigoplus_{\Sigma \in C(p)} H_{p+q-\mathrm{shift}(\Sigma)}(\Sigma; R) & p > 0, \\ H_{q+n}(W, \partial W; R) & p = 0, \\ 0 & p < 0. \end{cases} \tag{6.4.2}$$

In addition, the positive symplectic homology $SH^+(W; R)$ also admits a spectral sequence with E^1-page given by

$$E^1_{pq}(SH^+) = \begin{cases} \bigoplus_{\Sigma \in C(p)} H_{p+q-\mathrm{shift}(\Sigma)}(\Sigma; R) & p > 0, \\ 0 & p \leq 0. \end{cases} \tag{6.4.3}$$

6.4.3 For equivariant symplectic homology

Under the same assumptions as Theorem 6.4.3, there is also a spectral sequence converging to $SH^{+,S^1}(W; R)$, whose E^1-page is given by

$$E^1_{pq}(SH^{+,S^1}) = \begin{cases} \bigoplus_{\Sigma \in C(p)} H^{S^1}_{p+q-\mathrm{shift}(\Sigma)}(\Sigma; R) & p > 0 \\ 0 & p \leq 0. \end{cases} \tag{6.4.4}$$

The idea of the construction of equivariant the version is the same as the non-equivariant version. Recall that a key ingredient of the construction of Morse-Bott spectral sequences is that the local Floer homology of a Morse-Bott component Σ is isomorphic to Morse homology with twisted coefficient. We outline the equivariant version of this fact. Then the spectral sequence for equivariant symplectic homology (6.4.4) follows from the same argument as Section 6.4.2.

Local equivariant Floer homology

We take a time-independent Hamiltonian whose 1-periodic orbits form Morse-Bott manifolds, denoted by Σ. We also get corresponding Morse-Bott manifolds of critical points in the equivariant setup, namely $\Sigma \times S^{2N+1}$.

As the non-equivariant setup, we can again take suitable perturbations of the Hamiltonian to make it non-degenerate. Fix a component Σ and a ring R. Then local equivariant Floer homology can be constructed as follows.

- Using the same idea as Section 6.1.1, define perturbations H_δ, J_δ and g_δ, where g_δ is an S^1-equivariant metric on S^{2N+1}.
- Define $CF_*^{loc,S^1,N}(\nu(\Sigma), H_\delta,\ J_\delta, g_\delta)$ as the R-module freely generated by S^1-orbits of critical points $S^1 \cdot (\gamma_\delta, z_\delta)$ of the parametrized action functional. By choosing the perturbation parameter δ sufficiently small and applying the argument of Lemma 6.1.4, it follows that $(\gamma_\delta, z_\delta)$ converges to a point (γ, z) in the Morse-Bott manifold $\Sigma \times S^{2N+1}$ as δ converges to 0.
- Define the N-th approximation of the equivariant local "moduli space" as the set

$$\mathcal{M}_{S^1}^{\nu(\Sigma)}(S_{\overline{p}}, S_{\underline{p}}; H_\delta, J_\delta, g_\delta) := \{(u,z) \in \mathcal{M}_{S^1}(S_{\overline{p}}, S_{\underline{p}}; H_\delta, J_\delta, g_\delta) \mid \operatorname{im}(u) \subset \nu(\Sigma)\}.$$

- For the differential, we count those Floer trajectories that are completely contained in $\nu(\Sigma) \times S^{2N+1}$,

$$\partial^{loc,S^1}\overline{S}_p = \sum_{\substack{\underline{S}_p \in Crit\mathcal{A}^N \\ -\mu(\overline{S}_p)+\mu(\underline{S}_p)=1}} \sum_{u \in \mathcal{M}_{S^1}^{\nu(\Sigma)}(S_{\overline{p}}, S_{\underline{p}}; H,J,g)} \epsilon([u])\underline{S}_p.$$

- Define the N-th approximation of the local equivariant Floer homology of $\nu(\Sigma)$ by

$$HF_*^{loc,S^1,N}(\nu(\Sigma); H_\delta, J_\delta, g_\delta) := H_*(CF_*^{loc,S^1,N}(\nu(\Sigma), H_\delta, J_\delta, g_\delta)\,).$$

As Theorem 6.2.1, there is an isomorphism.

$$HF_*^{loc,S^1,N}(\nu_{\Sigma_c}(\Sigma), H_\delta, J_\delta, g_\delta; R) \cong HM_{*-\mathrm{shift}(\Sigma)}(\Sigma \times_{S^1} S^{2N+1}, \tilde{K}; \tilde{\mathcal{L}}_{\Sigma,N} \otimes_{\mathbb{Z}} R).$$

This isomorphism is a consequence of "unwrapping argument" as in the proof of Theorem 6.2.1. In particular, the local system $\tilde{\mathcal{L}}_{\Sigma,N}$ here are again obtained as an O(1)-bundle over $\Sigma \times S^{2N+1}$, whose fibers are given by the determinant lines of the linearized operator of the parametrized Floer equation of H.

Consider a critical point $(\gamma, z) \in \Sigma \times S^{2N+1}$, i.e., γ is a 1-periodic orbit in Σ and $z \in S^{2N+1}$. We choose trivializations of $\gamma^* T\widehat{W} \oplus T_z S^{2N+1}$ that are invariant under the circle action. Then one finds that the linearized operator of the parametrized Floer equation only depends on the class $[(\gamma, z)]$ with such an S^1-family of trivializations. Therefore the restriction of L to $S_{(\gamma,z)}$ is trivial. It follows that we have an induced $O(1)$-bundle over $\Sigma \times_{S^1} S^{2N+1}$, and hence a local coefficient system $\tilde{\mathcal{L}}_{\Sigma,N}$ over $\Sigma \times_{S^1} S^{2N+1}$.

Lemma 6.4.4. *An O(1) bundle $\tilde{L} \to \Sigma \times_{S^1} S^{2N+1}$ is trivial if and only if the induced bundle $L \to \Sigma \times S^{2N+1}$ is trivial.*

Proof. Suppose that σ is a nowhere vanishing section of L. Because the expression for the linearized operator for the parametrized Floer equation only depends on the orbit $S_{(\gamma,z)}$ by the above, we can assume that σ is S^1-invariant, and thus we obtain a nowhere vanishing section $\tilde{\sigma}$ of $\tilde{L}$ by putting $\tilde{\sigma}([v, z]) = \sigma(v, z)$ which is then well-defined. The converse is obtained by the same observation. □

Hence, by combining the above lemma with Lemma 6.2.4, we finally obtain the following corollary.

Corollary 6.4.5. *Suppose that the linearized Reeb flow is complex linear. Then*

$$HF_*^{loc,S^1,N}(\nu_{\Sigma_c}(\Sigma), H_\delta, J_\delta, g_\delta; R) \cong H_{*-\mathrm{shift}(\Sigma)}(\Sigma \times_{S^1} S^{2N+1}; R).$$

Chapter 7

Links of Singularities

In this chapter, we introduce the links of isolated hypersurface singularities. They admit a canonical contact structure as a boundary of Stein domain, called Milnor fiber. We in particular focus on singularities from weighted homogeneous polynomials. In this case, the contact structures admit a specific contact form whose Reeb flow is *periodic*. This implies that the contact form is of Morse-Bott type. The Reeb flow is so explicit that we can easily investigate the Reeb dynamics, for example, we can list the orbit spaces in terms of their actions. Furthermore, due to its "affine nature", we can explicitly compute Robbin-Salmon index of each orbit spaces. For these reasons, the links of weighted homogeneous polynomials give us nice examples to apply the spectral sequences we have developed in the previous chapters.

7.1 Backgrounds on Stein manifolds

We first give some backgrounds on Stein domains. The main reference for this section is Cieliebak-Eliashberg [30], as well as Forstnerič [40].

7.1.1 Definitions and examples

There are several equivalent definitions of Stein manifold. In this thesis, we only introduce the two frequently used definitions, namely *affine definition* and *J-convex definition.*

Remark 7.1.1. There are at least two more definitions of Stein manifold. One of them is a classical definition which says that a complex manifold

(V, J) is Stein if X is *holomorphically convex* and each two points can be separated by a holomorphic function on X.

The other definition uses the notion of *coherent sheaf cohomology*; a complex manifold is Stein if the first sheaf cohomology $H^1(X; \mathcal{F})$ vanishes for all coherent sheaf on X. An equivalence of this definition to other definitions can be shown for example by using so-called Cartan's theorem.

Definition 7.1.2 (Affine definition)**.** A complex manifold (V, J) is called **Stein** if there is a proper holomorphic embedding into the affine space $\mathbb{C}^N$, with the standard complex structure, for some $N \in \mathbb{N}$.

Remark 7.1.3. A continuous map is called *proper* if every preimage of compact set is again compact.

Example 7.1.4. There are several immediate examples from the affine definition.

1. The affine space $(\mathbb{C}^n, i)$ is clearly Stein.
2. Every closed complex submanifold of Stein manifold is Stein. This is because the inclusion map is *proper* by closedness.
3. As a result, a regular level set $f^{-1}(r)$ of a holomorphic function $f : V \to \mathbb{C}$ on a Stein manifold V is Stein.
4. For two Stein manifolds M and N, their product $M \times N$ with the obvious complex structure is Stein.

There is another definition of Stein manifold using the notion of J-convex functions. Let (V, J) be an almost complex manifold and $\phi : V \to \mathbb{R}$ a smooth function. Define a 1-form on V by

$$\lambda_\phi := -d\phi \circ J$$

and denote its differential by

$$\omega_\phi := d\lambda_\phi$$

Definition 7.1.5. A real valued function $\phi : V \to \mathbb{R}$ is called **J-convex** or **strictly plurisubharmonic** if

$$\omega_\phi(X, JX) > 0$$

for all $X \in TV$.

Recall that a continuous function on a topological spaces if called *exhausting* if it is proper and bounded from below.

Definition 7.1.6 (J-convex definition)**.** A complex manifold (V, J) is **Stein** if there is an exhausting J-convex function on V.

Example 7.1.7. A simplest example is of course the affine space $(\mathbb{C}^n, i)$. Consider the *radial* function $\phi : \mathbb{C}^n \to \mathbb{R}$ defined by $\phi(z) = \frac{1}{4}|z|^2 = \frac{1}{4}z\overline{z}$. Then the associated 1-form and 2-form are given as follows.

$$\lambda_\phi = -\frac{1}{4}i\overline{z}dz + \frac{1}{4}izd\overline{z}, \quad \omega_\phi = \frac{1}{2}idz \wedge d\overline{z}.$$

Since ω_ϕ is nothing but the standard symplectic form (or equivalenlty a Kähler form) on $\mathbb{C}^n$, it follow that a symmetric bilinear form $g_\phi(X, Y) := \omega_\phi(X, iY)$ defines the standard inner product. In particular, g_ϕ is positive-definite, so that ϕ is i-convex. It is evident that ϕ is proper and bounded from below.

Remark 7.1.8. A local description of ω_ϕ is as follows.

$$\begin{aligned}
d\phi &= \sum_j \left(\frac{\partial \phi}{\partial z_j} dz_j + \frac{\partial \phi}{\partial \overline{z}_j} d\overline{z}_j \right), \\
\lambda_\phi = -d\phi \circ i &= -i \sum_j \left(\frac{\partial \phi}{\partial z_j} dz_j - \frac{\partial \phi}{\partial \overline{z}_j} d\overline{z}_j \right), \\
\omega_\phi = d\lambda_\phi &= 2i \sum_{i,j} \frac{\partial^2 \phi}{\partial z_i \partial \overline{z}_j} dz_j \wedge d\overline{z}_j = 2i\partial\overline{\partial}\phi, \\
g_\phi &= 4Re \left(\sum_{i,j} \frac{\partial^2 \phi}{\partial z_i \partial \overline{z}_j} dz_j \otimes d\overline{z}_j \right).
\end{aligned}$$

If we define a Hermitian form by $H_\phi := g_\phi - i\omega_\phi$, the tripple $(H_\phi, \omega_\phi, g_\phi)$ forms a Kähler triple. In particular, for the radial function $\phi(z) = \frac{1}{4}|z|^2$, this triple is exactly the standard Kähler structure on $\mathbb{C}^n$.

Example 7.1.9. An affine algebraic subvariety A in $\mathbb{C}^N$ is Stein. Namely, we have exhausting i-convex function simply by restricting the radial function above.

There is another way to give Stein structure on A. By compactifying A in a projective space X, for example embed A into $\mathbb{C}P^N$, we have that $A = X \backslash D$ for some effective ample divisor D. Let $E \to X$ be an associated holomorphic line bundle over X and s be a holomorphic section with $s^{-1}(0) = D$. By ampleness, we have a metric g on E such that its curvature form is positive. Now define a function on $X \backslash D$ by

$$\phi : X \backslash D \to \mathbb{R}, \quad \phi = -\log ||s||_g.$$

One can easily check that the corresponding 2-form ω_ϕ is exactly the curvature form. So ϕ is J-convex.

Example 7.1.10. Another source of Stein manifold is holomorphic vector bundles over a Stein manifold. Let $E \to X$ be a holomorphic vector bundle over a Stein manifold (X, J). Let $\phi : X \to \mathbb{R}$ be an exhausting J-convex function. Define a function $\tilde{\phi} : E \to \mathbb{R}$ by

$$\tilde{\phi}(x, z) := \phi(x) + \frac{1}{4}||z||^2.$$

One can check that this is well-defined, after restricting the structure group to $SU(n)$ if necessary. It is then clear that $\tilde{\phi}$ is exhausting $J \oplus i$-convex.

It is immediate that the affine definition implies the J-convex definition: Let $i : V \to \mathbb{C}^N$ be a proper holomorphic embedding from a complex manifold (V, J). Define a smooth function $\phi : V \to \mathbb{R}$ just by pulling back the radial function on $\mathbb{C}^N$. Then it is clear that ϕ is exhausting. Since J-convexity is a local condition and the radial function is i-convex, it follows that ϕ is J-convex. Therefore (V, J) is Stein in the sense of the J-convex definition.

Conversely, J-convex definition implies the affine definition, and this is the content of the following theorem.

Theorem 7.1.11 (Grauert [41])**.** *If a complex manifold admits an exhausting J-convex function, then there is a proper holomorphic embedding into $\mathbb{C}^N$ for some N.*

7.1.2 Stein domains

Definition 7.1.12. A compact complex manifold (W, J) with boundary is called a **Stein domain** if W admits a J-convex function $\phi : W \to \mathbb{R}$ such that the boundary ∂V is a regular level set of ϕ.

A direct relation to Stein *manifold* is the following: Let (V, J) be a Stein manifold with an exhausting J-convex function $\phi : V \to \mathbb{R}$. Then Every sublevel set $V_c := \{x \in V \mid \phi(x) \leq c\}$ clearly forms a Stein domain. Conversely, if we have a Stein domain, it is clear that its completion forms a Stein manifold (of finite type).

Every Stein domain is a Weinstein domain. In particular, it is a Liouville domain. Note first that, by perturbing a little bit, we may assume that a given J-convex function is actually a Morse function. Recall that $\omega_\phi = d\lambda_\phi$ defines an exact symplectic form on W, and we also have the corresponding metric g_ϕ. Let us denote the gradient vector field of ϕ with respect to g_ϕ by $X_\phi := \operatorname{grad} \phi$.

Proposition 7.1.13. *A Stein domain (W, J, ϕ) is a Weinstein domain with a Weinstein structure $(\omega_\phi, X_\phi, \phi)$.*

Proof. The only thing that we need to prove is that X_ϕ is a Liouville vector field with respect to ω_ϕ. Since $\omega_\phi = d\lambda_\phi$ is exact, where $\lambda_\phi = -d\phi \circ J$, this is equivalent to say that $\iota_{X_\phi}\omega_\phi = \lambda_\phi$. We now compute that

$$\omega_\phi(X_\phi, \cdot) = g_\phi(JX_\phi, \cdot) = g_\phi(-X_\phi, J\cdot) = -d\phi \circ J(\cdot) = \lambda_\phi(\cdot).$$

This proves the assertion. $\square$

Remark 7.1.14. The converse statement up to homotopy is a content of the monograph, Cieliebak-Eliashberg [30].

7.1.3 Subcritical Stein domains

An important topological property of Stein domain (or manifold) comes from the following.

Proposition 7.1.15. *Let $\phi : V \to \mathbb{R}$ be a J-convex Morse function on a complex manifold (V, J) of real dimension $2n$. Then every critical point has Morse index less than or equal to n.*

Remark 7.1.16. In particular, Stein manifold is *necessarily* non-compact. This can be seen directly from the affine definition though.

Proof. We follow a proof of [30, Lemma 2.21]. Let p be a critical point of ϕ. Denote its unstable submanifold by $W^u(p)$ and the gradient vector field of ϕ by X. More precisely,

$$W^u(p) = \{x \in W \mid \lim_{t\to\infty} Fl_t^X(x) = p\}.$$

We claim that for each critical point p,

$$\lambda_\phi|_{W^u(p)} \equiv 0.$$

Let $x \in W^u(p)$ and $v \in T_x W^u(p)$. From the definition of the unstable manifold, we should have that

$$\lim_{t\to\infty} T_x Fl_t^X(v) = 0.$$

On the other hand, since X is a Liouville vector field of $(W, \omega_\phi = d\lambda_\phi)$, we know that $(Fl_t^X)^*\lambda_\phi = e^t\lambda_\phi$. Therefore we have

$$\lim_{t\to\infty} e^t\lambda_\phi(v) = \lim_{t\to\infty}(Fl_t^X)^*\lambda_\phi(v) = \lim_{t\to\infty}\lambda_\phi(T_x Fl_t^X(v)) = 0.$$

We conclude that $\lambda_\phi|_{W^u(p)} \equiv 0$. In particular, $W^u(p)$ is isotropic with respect to ω_ϕ and hence its dimension, which is equal to the Morse index of p, is less than or equal to n. $\square$

Remark 7.1.17. More directly, one can conclude the same statement by observing that the Hermitian *quadratic* form H_ϕ is related to the Hessian $Hess_p\phi : T_pW \to \mathbb{R}$ of ϕ as

$$H_\phi = Hess_p\phi + Hess_p\phi \circ i$$

at each critical point p of ϕ. Note that ϕ is i-convex if and only if H_ϕ is positive definite. It follows that the number of negative eigen values of $Hess_p\phi$ must be less than or equal to n.

An immediate corollary is the following.

Corollary 7.1.18. *Every Stein domain of (real) dimension* $2n$ *is a CW-complex consists of cells of dimension* $\leq n$.

Definition 7.1.19. A Stein domain (W, J) is called **subcritical** if it is a

CW-complex of dimension strictly less then n. Otherwise, (W, J) is called **critical**.

There are several proofs of Proposition 7.1.15. Here we give a "symplectic" proof, which says that every unstable manifold of critical point of ϕ is *isotropic* with respect to ω_ϕ. Recall that a submanifold of a symplectic manifold (M, ω) is called isotropic if ω vanishes on the submanifold.

7.2 Links of singularities

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a polynomial with an isolated critical point at the origin. Then the zero level set $f^{-1}(0)$ admits an isolated singularity at the origin. It turns out that if we intersects $f^{-1}(0)$ with sufficiently small sphere S^{2n+1}_δ, then the intersections is a smooth manifold.

Proposition 7.2.1. *For sufficiently small $\delta_0 > 0$, $f^{-1}(0)$ intersects transversely to S^{2n+1}_δ for all $0 < \delta < \delta_0$. Therefore the intersection*

$$\Sigma(f) := f^{-1}(0) \cap S^{2n+1}_\delta$$

is a smooth manifold of dimension $2n - 1$.

Proof. Denote the radial function by $r : \mathbb{C}^{n+1} \to \mathbb{C},\ z \mapsto |z|^2$. Since f is a polynomial (of finite degree), the restriction $r|_{f^{-1}(0)}$ has only finitely many critical points, including the origin. In particular, they are isolated. This proves the proposition. $\square$

Remark 7.2.2. The above proposition reveals the *local conical structure* of isolated hypersurface singularities. This means that the intersection $f^{-1}(0) \cap B^{2n+2}_\delta$ is homeomorphic to the *cone* over its boundary $f^{-1}(0) \cap S^{2n+1}_\delta$. See Figure 7.1.

Definition 7.2.3. The smooth manifold $\Sigma(f)$ is called a **link of singularity** f, or briefly, a **link** of f.

Remark 7.2.4. Since δ gives a 1-parameter family of smooth manifolds, $\Sigma(f)$ does not depend on δ.

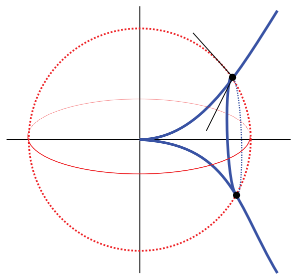


Figure 7.1: Local conical structure

Example 7.2.5. This example explains the terminology *link*. Let $p, q \in \mathbb{Z}$ be relatively prime. Consider a polynomial $f : \mathbb{C}^2 \to \mathbb{C}$ given by

$$(z, w) \mapsto z^p + w^q.$$

Note that f has critical point only at the origin, and denotes its link by

$$\Sigma(p, q) := f^{-1}(0) \cap S^3.$$

Then one can easily see that $\Sigma(p, q)$ is exactly the (p, q)-*torus knot* embedded in S^3. In the case when p, q are not relatively prime, its link may be a disjoint union of knots in S^3, in other words, they form links in S^3.

Remark 7.2.6. If 0 were regular value of f, then its link would be *trivial* in the sense that $\Sigma(f)$ is *unknotted* in S^{2n+1}.

7.3 A natural Stein filling

From now on f is an isolated singularity at the origin, as before. In fact its link $\Sigma(f)$ has a natural filling, i.e. $2n$-dimensional manifold $V(f)$ whose boundary is $\Sigma(f)$. This can be seen two equivalent ways: a smoothed variety and the Milnor fiber.

7.3.1 A smoothed variety

The variety $V(f) := f^{-1}(0)$ has an isolated singular point at the origin. By perturbing it a little bit at the origin, we can smooth the singular point as follows: Consider a smooth function $\zeta : \mathbb{R} \to \mathbb{R}$ such that $\zeta(r) = 1$ for

$r \leq 1/3$ and $\zeta(r) = 0$ for $r \geq 2/3$. Then for sufficiently small $\epsilon > 0$ we define

$$V_\epsilon(f) := \{z \in \mathbb{C}^{n+1} \mid f(z) = \zeta(|z|)\epsilon\}.$$

By the definition of the function ζ, the above smoothed one $V_\epsilon(f)$ bounds the link $\Sigma(f)$. Moreover, it coincides to the variety $V(f)$ away from the origin.

We can give a Stein structure on $V_\epsilon(f)$ as follows. Note that $V_\epsilon(f)$ is a complex submanifold of the affine space $\mathbb{C}^{n+1}$. Since $\mathbb{C}^{n+1}$ is a Stein manifold, $V_\epsilon(f)$ is also a Stein manifold with naturally inherited Stein structure. More precisely, the standard complex structure i restricts to a complex structure on $V_\epsilon(f)$, and the radial function $\rho(z) = |z|^2$ also restricts to a exhausted plurisubharmonic function on $V_\epsilon(f)$. We have an exact symplectic form $\omega_0 = d\lambda_0$ given by

$$\omega_0 = i \sum_{j=0}^{n} dz_j \wedge d\overline{z}_j \Bigg|_{V_\epsilon(f)}, \quad \lambda_0 = \frac{i}{2} \sum_{j=0}^{n} z_j d\overline{z}_j - \overline{z}_j dz_j \Bigg|_{V_\epsilon(f)}.$$

As a result, the Liouville 1-form λ_0 restricts to a contact form $\alpha_0 := \lambda_0|_{\Sigma(f)}$ on the link $\Sigma(f)$. In this way, every link of isolated hypersurface singularity admits a canonical contact structure.

7.3.2 Milnor fibers

The other description of a natural Stein filling of $\Sigma(f)$ is the *Milnor fiber*. Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be an isolated hypersurface singularity. Define a map $\phi : S^{2n+1} \backslash \Sigma(f) \to S^1$ by

$$\phi(z) = \frac{f(z)}{||f(z)||}.$$

Theorem 7.3.1 (Milnor [42])**.** *The map ϕ defines a fiber bundle over S^1.*

This fibrations is called the *Milnor fibration*, and its fiber $F := \phi^{-1}(1)$ is called the *Milnor fiber*. Note that its boundary of its closure $\partial\overline{F}$ is exactly the link $\Sigma(f)$. In [42], it is shown that $\overline{F}$ is diffeomorphic to $f^{-1}(1) \cap B^{2n+2}$, where B^{2n+1} denotes the unit ball in $\mathbb{C}^{n+1}$. Therefore the Milnor fiber has a natural Stein structure as a Stein submanifold of $\mathbb{C}^{n+1}$. In particular it serves a Stein filling of the link $\Sigma(f)$. The link $\Sigma(f)$ again inherits a natural contact form as a boundary of the Milnor fiber.

The two Stein filling of $\Sigma(f)$ is of course equivalent to each other:

Proposition 7.3.2. *The two Stein fillings of $\Sigma(f)$ are equivalent to each other as Stein manifolds.*

Proof. This is almost obvious from the definitions. In the definition of $V_\epsilon(f)$, we have used a cutoff function ζ. By homotope this function, we have a homotopy of Stein manifolds from $V_\epsilon(f)$ to the Minlor fiber $M(f)$. We then have an equivalence of Stein manifolds. □

By virture of the above equivalence, we can switch the two Stein fillings of $\Sigma(f)$ for convenience whenever we want.

7.3.3 Topology of complex hypersurface and its link

The Milnor fiber has a simple topological discription.

Proposition 7.3.3. *The Milnor F is homotopy equivalent to the wedge of n-spheres.*

$$F \simeq \bigvee_{\mu(f)} S^n.$$

This can be easily seen by perturbing f to a complex Morse function. A standard procedure is called *Morsification*. The the classical *Picard-Lefschetz theory* gives the conclusion. For more detail, we refer to [43]. There is also a proof by Milnor in [42].

The number of wedged n-spheres $\mu(f)$ in the above proposition is called the **Milnor number**. This number somehow measures how singular the singularity is. An immediate corollary is the following.

Corollary 7.3.4. *The Milnor fiber F is $(n-1)$-connected, i.e. $\pi_i(F) = 0$ for $i \leq n-1$.*

Note that $\overline{F}$ is a $2n$-dimensional compact manifold with boundary. Therefore we have its homology groups with $\mathbb{Z}$-coefficient:

$$H_*(\overline{F}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{\mu(f)} & * = n; \\ \mathbb{Z} & * = 0; \\ 0 & \text{else.} \end{cases}$$

Remark 7.3.5. Since the middle dimensional homology group $H_n(V(f);\mathbb{Z})$ is far from being trivial, $V(f)$ is a *critical* Stein filling. One can naturally ask whether the link $\Sigma(f)$ admits a *subcritical* Stein filling or not. The answer is *no*. There are several ways to think of this question. We will explain later.

Note that the link $\Sigma(f)$ is the boundary of the closure $\overline{F}$. So we also have the following corollary.

Corollary 7.3.6. *The link* $\Sigma(f)$ *is* $(n-2)$*-connected, i.e.* $\pi_i(\Sigma(f)) = 0$ *for* $i \leq n-2$.

Note that Σ is a $(2n-1)$-dimensional compact manifold without boundary. By the Poincare duality, one has, for $* \neq n, n-1$

$$H_*(\Sigma(f);\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & * = 0, 2n-1; \\ 0 & * \neq 0, n-1, n, 2n-1, \end{cases}$$

and $H_{n-1}(\Sigma(f);\mathbb{Z}) \cong H_n(\Sigma(f);\mathbb{Z})$.

7.3.4 The middle dimensional homology group of $\Sigma(f)$

We can investigate more about $H_{n-1}(\Sigma(f);\mathbb{Z})$ via Milnor fibration. Denote the total space of the fibration by $E := S^{2n+1}\backslash\Sigma(f)$. Choosing a connection of the bundle $\phi : E \to S^1$, we have a *monodromy map* along the simple loop on S^1, say $h : F \to F$, as the parallel transport. Denote the induced map on homology by $h_* : H_*(F) \to H_*(F)$. Recall the following Wang exact sequence.

Theorem 7.3.7. *(Wang) There is an exact sequence*

$$0 \longrightarrow H_{n+1}(E;\mathbb{Z}) \longrightarrow H_n(F;\mathbb{Z}) \xrightarrow{h_*-\mathrm{id}} H_n(F;\mathbb{Z}) \longrightarrow H_n(E;\mathbb{Z}) \longrightarrow 0.$$

Using Poincare and Alexander duality, we have

$$H_{n-1}(\Sigma(f);\mathbb{Z}) \cong H^n(\Sigma(f);\mathbb{Z}) \cong H_n(S^{2n+1}\backslash\Sigma(f);\mathbb{Z}).$$

Consequently, we see that

$$H_{n-1}(\Sigma(f);\mathbb{Z}) \cong \mathrm{coker}(h_* - \mathrm{id}).$$

For certain polynomials, $\operatorname{coker}(h_* - \mathrm{id})$ can be explicitly computed. This will be done in Section 7.4.1.

Remark 7.3.8. There is another description of $H_{n-1}(\Sigma(f);\mathbb{Z})$ in terms of the *intersection form*. Denote the intersection form by $S_f : H_n(F;\mathbb{Z}) \times H_n(F;\mathbb{Z}) \to \mathbb{Z}$. Then it is well-known that

$$H_{n-1}(\Sigma(f);\mathbb{Z}) \cong \operatorname{coker} S_f.$$

7.4 Weighted homogeneous polynomials

In this thesis, we are particularly interested in links of singularities of *weighted homogeneous polynomials*. Its S^1-symmetric nature plays a crucial role for our computation of homology groups as well as equivariant symplectic homology.

Definition 7.4.1. A polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is called **weighted homogeneous** if there are rational numbers $w_0, \dots, w_0$ such that

$$f(\eta^{1/w_0} z_0, \cdots, \eta^{1/w_n} z_n) = \eta f(z_0, \cdots, z_n)$$

for each $\eta \in \mathbb{C}^*$. In this case f is called a weighted homogeneous polynomial of **weights** $(w_0, \cdots, w_n)$. We briefly write **whp** instead of "weighted homogeneous polynomial."

From now on we always assume that the weights w_j's are in the irreducible form, namely $w_j = p_j/q_j$ for some relatively prime integers p_j and q_j.

Remark 7.4.2. There are two more equivalent definitions of whp's.

1. A polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a *whp1* of weight $(w_0, \cdots, w_n)$ if it consists of monomials of the form $z_0^{i_0} \cdots z_n^{i_n}$ such that

   $$i_0/w_0 + \cdots + i_n/w_0 = 1.$$

   This definition is exactly the same as the above.

2. A polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is called a *whp2* of weight $(v_0, \cdots, v_n)$ with degree d for some *integers* $v_j, d \in \mathbb{Z}$ if

   $$f(\eta^{v_0} z_0, \cdots, \eta^{v_n} z_n) = \eta^d f(z_0, \cdots, z_n)$$

for all $\eta \in \mathbb{C}^*$. Note that weights and degree are not unique, so one usually insists further that $gcd_j\ v_j = 1$. In fact f is a whp2 of weight $(v_0, \cdots, v_n)$ with degree d if and only if f is a whp of weight $(d/v_0, \cdots d/v_n)$. Reversely, Let f be a whp of weight $(w_0, \cdots, w_n)$ and $w_j = p_j/q_j$ in its irreducible form. Then f is a whp2 of weight $(\frac{lcm_j\ p_j}{w_0}, \cdots, \frac{lcm_j\ p_j}{w_n})$ of degree $d = lcm_j\ p_j$.

Example 7.4.3. Weighted homogeneous polynomials include the following examples.

1. (Homogeneous polynomials) Recall that a polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is called a homogeneous polynomial of degree d if

$$f(\eta z_0, \cdots, \eta z_n) = \eta^d f(z_0, \cdots, z_n).$$

Clearly, f is then a whp of weight $(d, \cdots, d)$.

2. (Brieskorn polynomials) A polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is called **Brieskorn** if it is of the form

$$f(z_0, \cdots, z_n) = z_0^{a_0} + \cdots z_n^{a_n}$$

for some integers $a_j \in \mathbb{Z}_{>0}$. Clearly, f is then a whp of weight $(a_0, \cdots, a_n)$. Furthermore, in the case when $a_j > 1$ for all j, f has only one critical point at the origin. So f is an isolated singularity and we denote its link by

$$\Sigma(a_0, \cdots, a_n) := \Sigma(f) = f^{-1}(0) \cap S^{2n+1}.$$

This manifold is particularly called a **Brieskorn manifold**.

7.4.1 The middle dimensional homology for weighted homogeneous polynomials

In Section 7.3.4 we have seen that

$$H_{n-1}(\Sigma(f); \mathbb{Z}) \cong \operatorname{coker}(h_* - \mathrm{id}).$$

In this subsection, we present a proof of Milnor-Orlik [44], which gives the free part of the middle dimensional homology $H_{n-1}(\Sigma(f); \mathbb{Z})$ when f is weighted homogeneous. We also give Randell's result in [45] on the torsion part, when f is in particular a Brieskorn polynomial, without proof.

Let $w_j = p_j/q_j$ be of its irreducible form and f is of weight $(w_0, \dots, w_n)$.

Theorem 7.4.4 (Milnor-Orlik [44]). *The middle betti number of $\Sigma(f)$ is given by*

$$\kappa(\Sigma(f)) = \sum_{I_t \subset I} (-1)^{n+1-t} \frac{\prod_{j \in I_t} w_j}{\mathrm{lcm}_{j \in I_t} p_j} \tag{7.4.1}$$

where $I = \{0, 1, \dots, n\}$ and I_t denotes each subset of I whose number of elements is $t \in \{0, 1, \dots, n+1\}$. As a convention, if $t = 0$, we add $(-1)^{n+1}$ to the sum.

A detailed proof can be found in [51, Section 3.6.2], and see also [44].

Example 7.4.5. Consider a Brieskorn manifold $\Sigma(2,2,2,2)$. Then $t \in \{0,1,2,3,4\}$. The list of I_t's in the sum (7.4.1) is as follows.

- $t = 0$: $\emptyset$;
- $t = 1$: $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$;
- $t = 2$: $\{0,1\}$, $\{0,2\}$, $\{0,3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$;
- $t = 3$: $\{0,1,2\}$, $\{0,1,3\}$, $\{0,2,3\}$, $\{1,2,3\}$;
- $t = 4$: $\{0,1,2,3\}$.

For example, the summand corresponding to $I_t = \{0,1\}$ is $(-1)^{4-2} \cdot \frac{2 \cdot 2}{2} = 2$, and for $I_t = \{1,2,3\}$, the summand is $(-1)^{4-3} \cdot \frac{2 \cdot 2 \cdot 2}{2} = -2^2$.

In this way, we can compute

$$\begin{aligned}
\kappa(\Sigma(f)) &= (-1)^4 + (-1)^3 \cdot (1+1+1+1) + (-1)^2 \cdot (2+2+2+2+2+2) \\
&\quad + (-1)^1 \cdot (2^2 + 2^2 + 2^2 + 2^2) + (-1)^0 \cdot (2^3) \\
&= 1 - 4 + 12 - 16 + 8 \\
&= 1.
\end{aligned}$$

As a result, we get the full homology groups of $\Sigma(2,2,2,2)$ in coefficient $\mathbb{Q}$ as follows.

$$H_*(\Sigma(2,2,2,2); \mathbb{Q}) = \begin{cases} \mathbb{Q} & * = 0, 2, 3, 5; \\ 0 & * = 1, 4. \end{cases}$$

Example 7.4.6. Consider, more generally, a homogeneous polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$ of degree d with an isolated singularity at the origin. We can compute the singular homology groups of the link $\Sigma(f)$ using the formula (7.4.1). Note that f is of weight $(d, \ldots, d)$, and weights have its irreducible form as $d/1$ so that $p_j = d$, $q_j = 1$ for all j. We can compute the middle dimensional betti number as follows.

$$\begin{aligned}
\kappa(\Sigma(f)) &= (-1)^{n+1} + (-1)^n \binom{n+1}{1} \cdot 1 + (-1)^{n-1} \binom{n+1}{2} d \\
&\quad + (-1)^{n-2} \binom{n+1}{3} d^2 + \cdots + (-1)^1 \binom{n+1}{n} d^{n-1} + (-1)^0 \binom{n+1}{n+1} d^n \\
&= (-1)^{n+1} \frac{\{(1-d)^{n+1} + d - 1\}}{d}.
\end{aligned}$$

Note in particular that the Brieskorn manifold $\Sigma(2, 2, \ldots, 2)$ is the unit cotangent bundle ST^*S^n over the sphere S^n. So the above also gives a (not efficient, but algorithmic) way to compute singular homology groups of ST^*S^n in $\mathbb{Q}$-coefficient.

Example 7.4.7. As an example of weighted homogeneous polynomial which is neither homogeneous nor Brieskorn, we consider a polynomial

$$f : \mathbb{C}^4 \to \mathbb{C}, \quad z \mapsto z_0^3 + z_0 z_1^3 + z_2^2 + z_3^3.$$

Note that f is of weights $(3, 9/2, 2, 2)$ so that $p_0 = 3, p_1 = 9, p_2 = 2, p_3 = 2$, and $n = 3$. So one can compute:

$$\begin{aligned}
\kappa(\Sigma(f)) &= 1 - (1 + 1/2 + 1 + 1) + (3/2 + 1 + 1 + 1/2 + 1/2 + 2) \\
&\quad - (3/2 + 3/2 + 2 + 1) + 3 \\
&= 1 - 7/2 + 13/2 - 6 + 3 \\
&= 1.
\end{aligned}$$

Therefore we have the homology groups in $\mathbb{Q}$-coefficient as follows.

$$H_*(\Sigma(f); \mathbb{Q}) = \begin{cases} \mathbb{Q} & * = 0, 2, 3, 5; \\ 0 & * = 1, 4. \end{cases}$$

7.4.2 Randell's algorithm: Torsion part of $H_{n-1}(\Sigma(a))$

For Brieskorn manifold in particular, we can determine torsion part of the middle dimensional homology group $H_{n-1}(\Sigma(a);\mathbb{Z})$ using Randell's algorithm.

Consider a Brieskorn manifold $\Sigma(a_0, a_1, \ldots, a_n)$. Let I_s be a subset of the index set $I = \{0, 1, \ldots, n\}$ of the form $I_s = \{i_1, \ldots, i_s\}$. We define the corresponding Brieskorn submanifolds

$$K(I_s) := \Sigma(a_{i_1}, \ldots, a_{i_n}).$$

We denote the rank of $H_{n-1}(K(I_s))$ by $\kappa(I_s)$. Define $k(I_s)$ by

$$k(I_s) := \begin{cases} \kappa(I_s) & \text{if } n+1-s \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Now we shall define a function C which sends each index subset to an integer recursively as follows.

$$C(\emptyset) = \gcd_{i \in I} a_i,$$
$$C(I_s) = \frac{\gcd_{i \in I \setminus I_s} a_i}{\prod_{I_t \not\subset I_s} C(I_t)}.$$

Then define $d_j := \displaystyle\prod_{I_s, \kappa(I_s) \geq j} C(I_s)$ and $r := \max_{I_s \subset I} k(I_s)$.

Theorem 7.4.8 (Randell [45]). *The middle dimensional homology group of $\Sigma(a)$ with $\mathbb{Z}$-coefficient is completely determined by*

$$H_{n-1}(\Sigma(a_0, \ldots, a_n);\mathbb{Z}) \cong \mathbb{Z}^{\kappa(I)} \oplus \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_r}.$$

Example 7.4.9. Consider a Brieskorn manifold $\Sigma(p, 3, 3, 3)$ where p is relatively prime to 3. By the Milnor-Orlik's formula one has $\kappa(\Sigma(p, 3, 3, 3))$=0. We compute in this example the torsion part of $H_2(\Sigma(p, 3, 3, 3);\mathbb{Z})$. Using the recursive definition of C, we can compute

$$C() = 1, C(\{0\}) = 3, C(\{1\}) = C(\{2\}) = C(\{3\}) = 1$$
$$C(\{0,1,2\}) = C(\{0,1,3\}) = C(\{0,2,3\}), C(\{1,2,3\}) = p, C(\{0,1,2,3\}) = 1.$$

It follows that $r = 2$, and we also get

$$d_1 = C(\{1,2,3\}) = p, \quad d_2 = C(\{1,2,3\}) = p.$$

We conclude that $H_2(\Sigma(p,3,3,3);\mathbb{Z}) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$.

7.4.3 Brieskorn spheres

For Brieskorn manifold, in particular, we can say more on its topology, especially when it is homeomorphic to a sphere. We call such manifolds *Brieskorn spheres.* We collect here some well-known results which we use later. The main reference is a book of Hirzebruch and Mayer [46].

Theorem 7.4.10 ([46])**.** *Let $\Sigma(a)$ be a Brieskorn sphere. If $n = 1,3,7$, then $\Sigma(a)$ is diffeomorphic to the standard sphere.*

Denote the intersection form of the singularity $f : \mathbb{C}^{n+1} \to \mathbb{C}$ by $S_f : H_n(V(f);\mathbb{Z}) \times H_n(V(f);\mathbb{Z}) \to \mathbb{Z}$. For Brieskorn polynomial, we write as $S_{V(a_0,a_1,\ldots,a_n)}$. For the case of $n \neq 1,3,7$, we have the following criterion.

Theorem 7.4.11 ([46])**.** *Let $\Sigma(a_0,\ldots,a_n)$ be a Brieskorn sphere, and assume that $n \neq 1,3,7$. If $\det S_{V(a_0,\ldots,a_n,2)} = \pm 1$ mod 8, then $\Sigma(a_0,\ldots,a_n)$ is diffeomorphic to the standard sphere.*

Note that $\Sigma(p,2,\ldots,2)$ with odd number p is homeomorphic to $(2n-1)$-sphere. Indeed, Theorem 7.4.8 tells us that $\Sigma(p,2,\ldots,2)$ is a integral homology sphere, and since it is simply connected for $n \geq 3$,it follows that $\Sigma(p,2,\ldots,2)$ is a homotopy sphere. Now the famous work of S. Smale on the generalized Poincare conjecture implies that $\Sigma(p,2,\ldots,2)$ is homeomorphic the sphere.

Having that said, if $p = \pm 1$ mod 8, then we have the following corollary.

Corollary 7.4.12. *For $p = \pm 1$ mod 8 and $n \geq 3$, the Brieskorn spheres $\Sigma(p,2,\ldots,2)$ are all diffeomorphic to the standard $(2n-1)$-sphere.*

Proof. For $n = 3,7$, it directly follows from Theorem 7.4.10. For the other dimensions, note that the determinant of the intersection form $S_{V(p,2\ldots,2,2)}$ (with one more 2) is ± 1 mod 8 when $p = \pm 1$ mod 8. This can be seen by a classical Picard-Lefschetz theory. This completes the proof. □

Remark 7.4.13. The Brieskorn spheres $\Sigma(p, 2, \ldots, 2)$ with $p = \pm 1 \bmod 8$ are in particular called *Ustilovsky spheres.* I. Ustilovsky showed that the canonical contact structure on $\Sigma(p, 2, \ldots, 2)$ are non-contactomorphic to that of $\Sigma(p', 2, \ldots, 2)$ if $p \neq p'$.

7.4.4 Equivariant singular homology of $\Sigma(f)$

Since we have an explicit formula for the homology groups of the links in $\mathbb{Q}$-coefficient, we can compute its equivariant homology groups using Gysin exact sequence for S^1-bundles.

Theorem 7.4.14 (Gysin exact sequence)**.** *Let $\pi : P \to Q$ be an S^1-bundle between smooth manifolds. Then there exists a long exact sequence*

$$\cdots \to H_*(P;\mathbb{Z}) \xrightarrow{\pi_*} H_*(Q;\mathbb{Z}) \xrightarrow{\cap e} H_{*-2}(Q;\mathbb{Z}) \xrightarrow{\partial} H_{*-1}(P;\mathbb{Z}) \to \cdots$$

where $\cap e$ denotes the cap product by the Euler class of the bundle, and ∂ denotes some boundary operator.

Note that we have an S^1-action on the link $\Sigma(f)$ of weighted homogeneous polynomial f by its Reeb flow. The equivariant singular homology of $\Sigma(f)$ is defined by the so-called Borel space, more precisely

$$H_*^{S^1}(\Sigma(f);\mathbb{Z}) := H_*(\Sigma(f) \times_{S^1} ES^1;\mathbb{Z}).$$

Note in particular that the space $\Sigma(f) \times_{S^1} ES^1$ admits an S^1-bundle structure

$$\pi : \Sigma(f) \times ES^1 \to \Sigma(f) \times_{S^1} ES^1$$

by the obvious quotient. In addition, since $ES^1 = S^\infty$ is contractible, it follows that $H_*(\Sigma(f) \times ES^1) \cong H_*(\Sigma)$. By applying the Gysin sequence with $\mathbb{Q}$-coefficient to the bundle we have

$$\to H_*(\Sigma(f);\mathbb{Q}) \xrightarrow{\pi_*} H_*^{S^1}(\Sigma(f);\mathbb{Q}) \xrightarrow{\cap e} H_{*-2}^{S^1}(\Sigma(f);\mathbb{Q}) \xrightarrow{\partial} H_{*-1}(\Sigma(f);\mathbb{Q}) \to$$

Remark 7.4.15. Since we use the rational coefficient, the situation becomes even simpler. In fact, the equivariant homology of $\Sigma(f)$ in $\mathbb{Q}$ is nothing but the homology of the quotient $\Sigma(f)/S^1$. Moreover $\Sigma(f)/S^1$ can be regarded as a hypersurface in the weighted complex projective space $\mathbb{C}P^n(w)$.

Using the sequence and the formula (7.4.1), we have the following result.

Theorem 7.4.16 (Randell)**.**

$$H_*^{S^1}(\Sigma(f);\mathbb{Q}) \cong \begin{Bmatrix} \mathbb{Q} & *\ \textit{is even and}\ 0 \le * \le 2n-2 \\ 0 & \textit{otherwise} \end{Bmatrix} \oplus \begin{Bmatrix} \mathbb{Q}^{\kappa} & * = n-1 \\ 0 & \textit{otherwise} \end{Bmatrix} \tag{7.4.2}$$

where $\kappa = \kappa(\Sigma(f))$ is given in (7.4.1).

Proof. We only describe a proof of the case of $n = 3$. The other dimensional cases are almost verbatim. Note in this case that $H_5^{S^1}(\Sigma(f)) = 0$ since $\Sigma(f)/S^1$ is of dimension 4. We have an exact sequence starting at $H_6^{S^1}(\Sigma(f))$ as follows.

$$\begin{tikzcd}
H_6^{S^1}(\Sigma(f)) \cong 0 \arrow[r, "\cap e"] & H_4^{S^1}(\Sigma(f)) \arrow[r, "\partial"] & H_5(\Sigma(f)) \cong \mathbb{Q} \arrow[dll, "\pi_*"] \\
H_5^{S^1}(\Sigma(f)) \cong 0 \arrow[r, "\cap e"] & H_3^{S^1}(\Sigma(f)) \arrow[r, "\partial"] & H_4(\Sigma(f)) \cong 0 \arrow[dll, "\pi_*"] \\
H_4^{S^1}(\Sigma(f)) \arrow[r, "\cap e"] & H_2^{S^1}(\Sigma(f)) \arrow[r, "\partial"] & H_3(\Sigma(f)) \cong \mathbb{Q}^{\kappa} \arrow[dll, "\pi_*"] \\
H_3^{S^1}(\Sigma(f)) \arrow[r, "\cap e"] & H_1^{S^1}(\Sigma(f)) \arrow[r, "\partial"] & H_2(\Sigma(f)) \cong \mathbb{Q}^{\kappa} \arrow[dll, "\pi_*"] \\
H_2^{S^1}(\Sigma(f)) \arrow[r, "\cap e"] & H_0^{S^1}(\Sigma(f)) \arrow[r, "\partial"] & H_1(\Sigma(f)) \cong 0 \arrow[dll, "\pi_*"] \\
H_1^{S^1}(\Sigma(f)) \arrow[r, "\cap e"] & H_{-1}^{S^1}(\Sigma(f)) \cong 0 \arrow[r, "\partial"] & H_0(\Sigma(f)) \cong \mathbb{Q} \arrow[dll, "\pi_*"] \\
H_0^{S^1}(\Sigma(f)) \arrow[r, "\cap e"] & H_{-2}^{S^1}(\Sigma(f)) \cong 0 \arrow[r, "\partial"] & H_{-1}(\Sigma(f)) \cong 0
\end{tikzcd}$$

From the bottom, we have $H_0^{S^1}(\Sigma(f)) \cong H_0(\Sigma(f)) \cong \mathbb{Q}$, and $H_1^{S^1}(\Sigma(f)) \cong 0$. Since we are dealing with vector spaces, the short exact sequence

$$0 \to \mathbb{Q}^{\kappa} \to H_2^{S^1}(\Sigma(f)) \to \mathbb{Q} \to 0$$

in the above long exact sequence is split exact. It follow that $H_2^{S^1}(\Sigma(f)) \cong \mathbb{Q}^\kappa \oplus \mathbb{Q}$. Since $H_5^{S^1}(\Sigma(f)) = 0$, we have $H_3^{S^1}(\Sigma(f)) = 0$. The top line now shows that $H_4^{S^1}(\Sigma(f)) \cong \mathbb{Q}$. This completes the proof. □

Example 7.4.17. Consider the Brieskorn manifold $\Sigma(2, 2, \dots, 2)$. We have computed the middle dimensional Betti number κ of $\Sigma(2, 2, \dots, 2)$ for $n = 3$ in Example 7.4.5. The same computation shows that

$$\kappa(\Sigma(2, 2, \dots, 2)) = \begin{cases} 1 & n \text{ is odd,} \\ 0 & n \text{ is even.} \end{cases}$$

It follow from the formula (7.4.2) that, for n is odd,

$$H_*^{S^1}(\Sigma(2, 2, \dots, 2); \mathbb{Q}) = \begin{cases} \mathbb{Q} & * \text{ is even and } 0 \le * \le 2n-2 \text{ and } * \ne n-1 \\ \mathbb{Q}^2 & * = n-1 \\ 0 & \text{otherwise} \end{cases}$$

and, for n is even,

$$H_*^{S^1}(\Sigma(2, 2, \dots, 2); \mathbb{Q}) = \begin{cases} \mathbb{Q} & * \text{ is even and } 0 \le * \le 2n-2 \\ 0 & \text{otherwise.} \end{cases}$$

7.5 Periodic Reeb flows on the links of weighted homogeneous polynomials

Suppose from now on that f is a weighted homogeneous polynomial of weight $(w_0, \cdots, w_n)$, with an isolated critical point at the origin. Denote its link by

$$\Sigma(f) := f^{-1}(0) \cap S^{2n+1}.$$

This link carries a nice contact form in the sense that its Reeb flow matches to the natural S^1-action on $\Sigma(f)$ as follows.

Note that the zero set of f carries an S^1-action, namely,

$$e^{it} \cdot (z_0, \cdots, z_n) = (e^{it/w_0} z_0, \cdots, e^{it/w_n} z_n)$$

for $e^{it} \in S^1$ and $z \in f^{-1}(0)$. Indeed, if $f(z) = 0$, then

$$f(e^{it} \cdot z) = f(e^{it/w_0} z_0, \cdots, e^{it/w_n} z_n) = e^{it} f(z) = 0$$

by the (weighted) homogeneity.

We consider a 1-form on $\mathbb{C}^{n+1}$

$$\tilde{\alpha} = \frac{i}{2} \sum_{j=0}^{n} w_j (z_j d\overline{z}_j - \overline{z}_j dz_j).$$

By restricting to the link $\Sigma(f)$ we have a 1-form

$$\alpha := \tilde{\alpha}|_{\Sigma(f)}.$$

Proposition 7.5.1. *The above 1-form α on $\Sigma(f)$ is a contact form.*

Proof. Define a *weighted* symplectic form on $\mathbb{C}^{n+1}$ by

$$\omega_w = i \sum w_j dz_j \wedge d\overline{z}_j.$$

Clearly, this form is exact, and its primitive 1-form is exactly

$$\lambda_w = \frac{i}{2} \sum_{j=0}^{n} w_j (z_j d\overline{z}_j - \overline{z}_j dz_j).$$

Denote $V_0 = f^{-1}(0) \backslash \{0\}$. Then its boundary (except the origin) is the link $\Sigma(f)$ by definition, and we have the restricted form $\omega_w|_{V_0} = d\lambda_w|_{V_0}$ which is exact symplectic on V_0. Consider the Liouville vector field X given by the equation

$$\iota_X \omega_w|_{V_0} = \lambda_w|_{V_0}.$$

We now claim that X is transverse to the the link $\Sigma(f)$. Recall that $\Sigma(f) = f^{-1}(0) \cap S_\delta^{2n+1}$, so we only have to check that X is transverse to the level sets of the radial function $|z|^2$ *on* V_0. Define a vector field R on $\mathbb{C}^{n+1}$ by

$$R = i \sum \frac{1}{w_j} \left(z_j \frac{\partial}{\partial z_j} - \overline{z}_j \frac{\partial}{\partial \overline{z}_j} \right).$$

Then one can check directly that

$$\iota_R \omega_w = -\sum (z_j d\overline{z}_j - \overline{z}_j dz_j) = -d(|z|^2),$$

so that $d(|z|^2)(X) = -\omega_w(X, R)$. What we can also observe is that $\omega_w(X, R) = \lambda_w(R) \neq 0$ on V_0 by a simple calculation. It follows that X is transverse to $\Sigma(f)$ as we claimed. Now Proposition 2.3.2 shows that $\alpha = \lambda_w|_{\Sigma(f)}$ is a contact form on the link $\Sigma(f)$. □

One can also compute that the Reeb flow of $\tilde{\alpha}$ is given by

$$Fl_t(z) = (e^{it/w_0} z_0, e^{it/w_1} z_1, \dots, e^{it/w_n} z_n),$$

which is exactly the same as the S^1-action. This coincidence will help in several way for computations later.

In fact, the nice contact form induces the same contact structure on $\Sigma(f)$ as the standard one.

Proposition 7.5.2. *The contact structure $\xi = \ker \alpha$ is contactomorphic to the canonical contact structure on $\Sigma(f)$ as a boundary of the Stein domain $V_\epsilon(f)$.*

Proof. Denote the canonical contact form from the filling by α_0 and the nice contact form by α_1, temporarily in this proof. We find an isotopy between α_0 and α_1 and apply Gray stability theorem. Our technical strategy to find an isotopy is that we first isotope via varying complex structures and then Liouville vector fields.

We may assume that $w_j \geq 2$ for all j by rescaling α_1. Then define a family of almost complex structures J_s, $s \in [0, 1]$, on C^{n+1} by

$$J_s := \sum_{j=0}^{n} \left\{ c_j(s) \frac{\partial}{\partial y_j} \otimes dx_j - \frac{1}{c_j(s)} \frac{\partial}{\partial x_j} \otimes dy_j \right\},$$

where $c_j(s)$'s are non-zero real numbers such that $c_j(0) = 1$, $c_j(1) + 1/c_j(1) = w_j$ for each j. Note that such choices are possible when $w_j \geq 2$ as we have assumed. Now by a direct computation, we have that the function $|z|^2$ on $\mathbb{C}^{n+1}$ is J_s-convex for all $s \in [0, 1]$, and hence we get a family of contact forms

$$\tilde{\alpha}_s := (-d(|z|^2) \circ J_s)|_{\Sigma(f)}$$

on the link $\Sigma(f)$. Observe that $\tilde{\alpha}_0 = \alpha_0$ and $d\tilde{\alpha}_1 = \omega_w$.

Now we isotope α_1 and $\tilde{\alpha}_1$, which completes the proof. Define a Liouville vector field X_1 by

$$\iota_{X_1}\omega_w|_{V_0} = \tilde{\alpha}_1|_{V_0}.$$

Then clearly X_1 is *positively* transverse to $\Sigma(f)$, i.e, $X_1(|z|^2) > 0$. Denote the ω_w-dual of α_1 by X. Note that X is also a Liouville vector field and is *positively* transverse to $\Sigma(f)$. Since both of X and X_1 are positively transverse Liouville vector fields, their convex sum

$$X_t := tX_1 + (1-t)X, \quad t \in [0,1]$$

defines a family of Liouville vector fields which are positively transverse to $\Sigma(f)$. The family of contact forms $\iota_{X_t}\omega_w$ then gives a desired isotopy between α_1 and $\tilde{\alpha}_1$. □

From now on, we always use the contact form α for every link $\Sigma(f)$ of weighted homogeneous polynomials f.

7.6 Reeb dynamics of $(\Sigma(f), \alpha)$

Let $\alpha = \sum w_j(z_j d\overline{z}_j - \overline{z}_j dz_j)$ be as in Section 7.5. Its Reeb flow is particularly simple and extremely degenerate. In this section, we investigate the Reeb dynamics of $(\Sigma(f), \alpha)$ in terms of its orbit spaces and the corresponding actions.

Its Reeb flow is given by the formula

$$Fl_t^{R_\alpha}(z) = (e^{it/w_0}z_0, e^{it/w_1}z_1, \cdots, e^{it/w_n}z_n).$$

This formula coincides to the canonical S^1-action on $\mathbb{C}^{n+1}$ associated to the weighted homogeneous polynomial.

We observe that *every* Reeb orbit is periodic. For example, take a least common multiple $T := \mathrm{lcm}_j\, p_j$, where $w_j = p_j/q_j$ is of irreducible form. We have $Fl_{t+2\pi T}^{R_\alpha}(z) = Fl_t^{R_\alpha}(z)$ for every $z \in \mathbb{C}^{n+1}$. In this sense, we say the Reeb flow is *periodic*.

Example 7.6.1. The link of a homogeneous polynomial has a very simple Reeb dynamics. Let f be a homogeneous polynomial of degree k. Its weights is $(k, \ldots, k)$. The corresponding Reeb flow is the Hopf circle action on $\mathbb{C}^{n+1}$

$$Fl_t^{R_\alpha}(z) = (e^{it/k}z_0, e^{it/k}z_1, \cdots, e^{it/k}z_n).$$

The minimal common period is $2\pi k = 2\pi \cdot \mathrm{lcm}_j\, k$. Since every periodic orbit has period $2\pi k$, there is no other possible period.

7.6.1 Brieskorn case

We first investigate the Brieskorn case, which is easier to deal with and illuminates the general weighted homogeneous case. A Brieskorn polynomial f is a weighted homogeneous polynomial of the form

$$f(z) = z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n}$$

where a_j's are integers lager than 1. Observe that the weights are exactly its exponents $a_0, \cdots, a_n$. We briefly denote its link by $\Sigma(a)$, where $a = (a_0, \ldots, a_n)$. The smallest common period of Reeb flow is $2\pi \cdot \mathrm{lcm}_j\, a_j$. All the other periods of Reeb orbits are 2π multiplied by a divisor of $\mathrm{lcm}_j\, a_j$.

For such divisors T of $\mathrm{lcm}_j\, a_j$, we define a notation

$$\Sigma(a)_T := \{z \in \Sigma(a) \mid Fl^{R_\alpha}_{2\pi T}(z) = z\}$$

for an "orbit space" consists of (the images of) Reeb orbits which come back after the time $2\pi T$ (not necessarily minimal). We call $\Sigma(a)_T$ a **Morse-Bott submanifold** or **Morse-Bott component** of $\Sigma(a)$ with period/action $2\pi T$.

Remark 7.6.2. Such orbit spaces are supposed to form Morse-Bott components of Morse-Bott type Hamiltonian for symplectic homology. This is why we call them "Morse-Bott components".

Example 7.6.3. Consider an A_2-singularity

$$f(z) = z_0^3 + z_1^2 + \cdots + z_n^2.$$

This polynomial is Brieskorn of weights $(3, 2, \ldots, 2)$, and its link is denoted by $\Sigma(3, 2, \ldots, 2)$. As we have seen, its Reeb flow is given by the formula

$$Fl^{R_\alpha}_t(z) = (e^{it/3} z_0, e^{it/2} z_1, \cdots, e^{it/2} z_n).$$

The minimal common period is $2\pi \cdot 6$ where 6 is the least common multiple of the weights. The divisors of 6 are $1, 2, 3, 6$, and they form possible periods (after multiplying by 2π).

Observe that there is no periodic Reeb orbit of period $2\pi\cdot 1$, i.e $\Sigma(a)_1 = \emptyset$. This can be seen by considering the definition of the link. We can also check that $\Sigma(a)_3$ is empty by the definition of the link.

One the other hand, two remaining divisors, 2 and 6 correspond to non-empty Morse-Bott components. Clearly, $\Sigma(a)_6$ is the ambient manifold $\Sigma(a)$ itself. Since the flow is the coordinate-wise rotations, the component $\Sigma(a)_2$ is the subset $\{z_0 = 0\} \subset \Sigma(a)$. Observe that, by the definition of the link, the subset $\{z_0 = 0\} \subset \Sigma(a)$ forms again a Brieskorn manifold $\Sigma(2, \cdots, 2) \subset \mathbb{C}^n$.

In the above example, every Morse-Bott component of a Brieskorn manifold forms a Brieskorn manifold. This is a general phenomenon.

Proposition 7.6.4. *Each Morse-Bott submanifold $\Sigma(a)_T$ is a (possibly empty) Brieskorn manifold.*

Proof. A key obsevation is that if there is a *periodic* Reeb orbit γ of period T such that $\gamma([0,T]) \cap \{z_j \neq 0\} \subset \Sigma(a)$, then the Morse-Bott submanifold $\Sigma(a)_T$ contains *every* points z with $z_j \neq 0$ in $\Sigma(a)$. Consequently, it follows that $\Sigma(a)_T$, for a divisor T of $\mathrm{lcm}_j\, a_j$, is embedded in $\Sigma(a)$ as a subset $\{z_j = 0\} \cap \{z_k \neq 0\}$ for $j \notin I(T)$ and $k \in I(T)$ where $I(T) \subset \{0, 1, \ldots, n\}$ is the maximal index subset such that $\mathrm{lcm}_{j \in I(T)}\, a_j = T$. By the definition of the link, one can easily see that this embedded subset is nothing but $\Sigma(a_{i_1}, \cdots, a_{i_l})$ for $i_j \in I(T)$. $\square$

For later convenience, we introduce the following definitions. Let (Σ, α) be a compact contact manifold with a contact form α, whose Reeb flow is periodic. Then periods of simple orbits are only finitely many, say $T_1, T_2, \ldots, T_k$, where T_k is the common period.

Definition 7.6.5. The common period T_k is called a **principal period** and corresponding orbits are called **principal orbits.** The other periods are called **exceptional periods** and corresponding orbits are called **exceptional orbits**.

Example 7.6.6. For A_2 singularity, as we have see, exeptional periods are $2\pi \cdot 2, 2\pi \cdot 6$, and $2\pi \cdot 6$ is the principle period. Exceptional orbits forms the component $\Sigma(a)_2 = \{0\} \times \Sigma(2, \cdots, 2) \subset \Sigma(a)$, and the principal orbits form the ambient space.

In short, on Brieskorn manifold $(\Sigma(a), \alpha)$, principle orbits have period $2\pi \operatorname{lcm}_j a_j$, and for each divisor T of $\operatorname{lcm}_j a_j$ we have a (possibly empty) Morse-Bott component $\Sigma(a)_T$ consists of exceptional orbits of period $2\pi T$. Each Morse-Bott submanifold is again a Brieskorn manifold $\Sigma(I_T)$ where I_T is the maximal subset of I such that $\operatorname{lcm}_{j\in I_T} a_j = T$.

7.6.2 General weighted homogeneous case

Least common multiples and divisors of rational numbers.

To describe Reeb dynamics of $(\Sigma(f), \alpha)$ for general weighted homogeneous polynomial f in terms of Morse-Bott submanifolds, it is convenient to extend the notion of the least common multiple and divisors of integers to rational numbers.

Let $r \in \mathbb{Q}$ be an irreducible rational number. If we allow *negative* powers, the number r has the *unique* factorization by prime numbers. For example, $9/2 = 2^{-1}3^2$.

Definition 7.6.7. Let $r, r' \in \mathbb{Q}$ be irreducible rational numbers whose the unique prime factorizations are given by

$$r = \prod_p p^{a_p}, \quad r' = \prod_p p^{b_p}$$

where p's are prime numbers and $a_p, b_p \in \mathbb{Z}$. Then the **least common multiple** of r and r', denoted by $\operatorname{lcm}(r, r')$, is defined by

$$\operatorname{lcm}(r, r') := \prod_p p^{\max(a_p, b_p)}.$$

Put differently, $\operatorname{lcm}(r, r') = \frac{\operatorname{lcm}(p,p')}{\gcd(q,q')}$ where $r = p/q$ and $r' = p'/q'$ are of irreducible forms. One should notice that this definition coincides to the usual least common multiple of two integers. Moreover, observe that $\operatorname{lcm}(r, r')/r$ and $\operatorname{lcm}(r, r')/r'$ are both *integers*.

Example 7.6.8. Let us compute $\operatorname{lcm}(9/2, 3/4)$. Their unique factorizations are given by

$$9/2 = 2^{-1}3^2, \quad 3/4 = 2^{-2}3.$$

Therefore we have

$$\operatorname{lcm}(9/2, 3/4) = 2^{-1}3^2 = 9/2.$$

We also extend the notion of divisors of integers to rational numbers as follows.

Definition 7.6.9. Let $r \in \mathbb{Q}$ be an irreducible rational number with the prime factorization

$$r = \prod_p p^{a_p}.$$

An irreducible rational number r' is called a **divisor** of r if it has the form of

$$r' = \prod_p p^{b_p}$$

where $0 \leq b_p \leq a_p$ if $a_p \geq 0$ and $a_p \leq b_p \leq 0$ if $a_p \leq 0$.

For example, consider $9/2 = 2^{-1}3^2$. Then the list of all divisors of $9/2$ is the following.

$$\begin{aligned} &2^{-1}3^2 = 9/2, \quad 2^{-1}3^1 = 3/2, \quad 2^{-1}3^0 = 1/2 \\ &2^0 3^2 = 9, \quad 2^0 3^1 = 3, \quad 2^0 3^0 = 1. \end{aligned}$$

Note also that the above notion coincide to the usual divisor of integers.

General weighted homogeneous case

Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a weighted homogeneous polynomial of weights $(w_0, w_1, \ldots, w_n)$. Then the Reeb flow on the link $(\Sigma(f), \alpha)$ is given by

$$Fl_t^{R_\alpha}(z) = (e^{it/w_0} z_0, e^{it/w_1} z_1, \cdots, e^{it/w_n} z_n).$$

Consider the least common multiple $\operatorname{lcm}_j w_j$ of weights (possibly rational numbers). Since T/w_j is integer for all j, we have that $2\pi \operatorname{lcm}_j w_j$ is the common period of the Reeb flow. Moreover, it is evident that $2\pi \operatorname{lcm}_j w_j$ is the smallest common period of the Reeb flow, in other words.

As in the Brieskorn case, Periods of exceptional orbits can be obtained by divisors of $\operatorname{lcm}_j w_j$. Exceptional orbit spaces are (possibly empty) Morse-Bott submanifolds

$$\Sigma(f)_T = \{z \in \Sigma(f) \mid Fl_{2\pi T}^{R_\alpha}(z) = z\}$$

for each divisors T of $\operatorname{lcm}_j w_j$. Moreover, each orbit space $\Sigma(f)_T$ itself is a link of some weighted homogeneous polynomial. To see this, denote by I_T the maximal subset of $\{0, 1, \ldots, n\}$ such that $\mathrm{lcm}_{j \in I_T} w_j = T$. We define a polynomial $f_T : \mathbb{C}^{\#(I_T)} \to \mathbb{C}$ by $f_T := f \circ \iota_T$ where ι_T is a natural inclusion $\mathbb{C}^{\#(I_T)} \to \mathbb{C}^{n+1}$, preserving the indices. For example, if $I_T = \{0, 2\} \in \{0, 1, 2\}$, then $\iota_T(z_0, z_2) = (z_0, 0, z_2)$.

Lemma 7.6.10. *For each I_T, the polynomial $f_T : \mathbb{C}^{\#(I_T)} \to \mathbb{C}$ is a weighted homogeneous polynomial of weight $(w_{i_1}, \ldots, w_{i_k})$ for $i_j \in I_T$.*

Proof. This is clear from the definition of weighted homogeneous polynomials. $\square$

Proposition 7.6.11. *Each Morse-Bott submanifold $\Sigma(f)_T$ is a (possible empty) link of a weighted homogeneous polynomial.*

Proof. As the Brieskorn case, for each divisor T, the Morse-Bott submanifold $\Sigma(f)_T$ is the same as $\Sigma(f_T)$ up to the inclusion $\iota_T : \mathbb{C}^{\#(I_T)} \to \mathbb{C}^{n+1}$. $\square$

Example 7.6.12. Consider a polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$ given by

$$f(z) = z_0^3 + z_0 z_1^3 + z_2^2 + \cdots + z_n^2.$$

Then f has a unique critical point at the origin. This singularity is one of the *simple* singularity, which is called E_7-type. From the definition of weighted homogeneous polynomials, we find that f is a weighted homogeneous polynomial of weights $(3, 9/2, 2, \ldots, 2)$. So its link $(\Sigma(f), \alpha)$ admits the Reeb flow

$$Fl_t^{R_\alpha}(z) = (e^{it/3} z_0, e^{2it/9} z_1, e^{it/2} z_2, \ldots, e^{it/2} z_n).$$

Note that $3 = 3^1$, $9/2 = 2^{-1}3^2$, and $2 = 2^1$. The principal period is then $2\pi\, \mathrm{lcm}_j\, w_j = 2\pi 2^1 3^2 = 2\pi 18$. The divisors of 18 are $1, 2, 3, 6, 9$, and 18. So we have the following full list of (simple) Morse-Bott components:

- (Exceptional orbits) $\Sigma(f)_1 = \emptyset$, $\Sigma(f)_2 = \Sigma(f_2)$ where $f_2(z) = z_2^2 + \cdots + z_n^2$, $\Sigma(f)_3 = \emptyset$, $\Sigma(f)_6 = \Sigma(f_6)$ where $f_6(z) = z_0^3 + z_2^2 + \cdots + z_n^2$, $\Sigma(f)_9 = \Sigma(f_9)$ where $f_9(z) = z_0^3 + z_0 z_1^3$.
- (Principle orbits) $\Sigma(f)_{18} = \Sigma(f)$.

7.7 Robbin-Salamon indices

Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a weighted homogeneous polynomial with an isolated critical point at the origin. Denote its weights by $(w_0, \dots, w_n)$. Let γ be a periodic Reeb orbit in $(\Sigma(f), \alpha)$ of period $2\pi T$. Note that T is a divisor of $\mathrm{lcm}_j\, w_j$. In this section, we give a computation of Robbin-Salmon index of γ. An essential trick is to use the embedding $\xi \hookrightarrow T\mathbb{C}^{n+1}$ of the contact structure and the direct sum property of the Robbin-Salamon index.

This trick was first used by Ustilovsky [3] for certain Brieskorn manifolds, and extended to general Briekorn manifolds by van Koert in [5]. Our computation now extends the trick to general weighted homogeneous polynomials.

Note that $\Sigma(f)$ is simply-connected and $c_1(\xi) = 0$, provided that n is hight enough. So the index does not depend on the choice of capping disk. Take any capping disk of $\gamma(t)$, say $\beta : D^2 \to \Sigma(f)$. This gives us a symplectic trivialization $\beta : S^1 \times \mathbb{R}^{2n-2} \to \gamma^*(\xi, d\alpha)$ (we abuse the notation β). With respect to the trivialization, we get a path of symplectic matrices $\Phi(t) : [0, T] \to Sp(2n)$ by

$$\Phi(t) := \beta^{-1}(\gamma(t)) \circ T_{\gamma(t)} Fl_t^{R_\alpha}|_\xi \circ \beta(\gamma(0)).$$

The Robbin-Salamon index of γ is then defined by the Robbin-Salamon index of the path Φ.

Equip $T\mathbb{C}^{n+1}$ with the weighted symplectic form $\omega_w = i \sum w_j dz_j \wedge d\bar{z}_j = d\alpha$. Then the symplectic vector bundle $(\xi, d\alpha)$ is a subbundle of $(T\mathbb{C}^{n+1}|_{\Sigma(f)}, \omega_w)$. We now can consider a decomposition of $T\mathbb{C}^{n+1}|_{\Sigma(f)}$,

$$T\mathbb{C}^{n+1}|_{\Sigma(f)} \cong \xi \oplus \xi^{\omega_w}$$

as symplectic vector bundles, where ξ^{ω_w} denotes the symplectic complement of ξ. Note that $T\mathbb{C}^{n+1}$ has a natural symplectic basis, namely

$$\epsilon := \left\{ \frac{1}{\sqrt{w_j}} \left(\frac{\partial}{\partial z_j} + \frac{\partial}{\partial \bar{z}_j} \right) \right\}_{0 \le j \le n}.$$

We also have a symplectic basis β of $\gamma^*\xi$. Therefore, to apply direct sum property, we need to specify a symplectic basis of ξ^{ω_w}.

A quite natural candidate for a (not necessarily symplectic) basis of ξ^{ω_w}

is the (complex) gradient vector field $\operatorname{grad}^{\mathbb{C}} f$, its i-multiplication $i \cdot \operatorname{grad}^{\mathbb{C}} f$, the Reeb vector field R_α, and the Liouville vector field. More precisely, their explicit formulas are as follows.

$$X_1 = \sum_j \overline{z}_j^{w_j-1} \frac{\partial}{\partial z_j} + z_j^{w_j-1} \frac{\partial}{\partial \overline{z}_j}, \quad Y_1 = i \sum z_j^{w_j-1} \frac{\partial}{\partial z_j} - z_j^{w_j-1} \frac{\partial}{\partial \overline{z}_j},$$
$$X_2 = -i \sum_j \frac{z_j}{w_j} \frac{\partial}{\partial z_j} - \frac{\overline{z}_j}{w_j} \frac{\partial}{\partial \overline{z}_j}, \quad Y_2 = \sum_j z_j \frac{\partial}{\partial z_j} + \overline{z}_j \frac{\partial}{\partial \overline{z}_j}.$$

In view of their origin, it is clear that they are linearly independent to each other. In other words,

Lemma 7.7.1. *The set $\{X_1, Y_1, X_2, Y_2\}$ forms a basis of ξ^{ω_w}.*

However $\{X_1, Y_1, X_2, Y_2\}$ is not a *symplectic* basis. For example, one can check directly that $\omega_w(X_1, Y_1) \neq \pm 1$. Instead, we can perform a kind of *Gram-Schmidts process.* We formulate

$$\widetilde{X_1} = \frac{X_1}{\sqrt{\omega_w(X_1, Y_1)}}, \quad \widetilde{Y_1} = \frac{Y_1}{\sqrt{\omega_w(X_1, Y_1)}}$$
$$\widetilde{X_2} = X_2, \quad \widetilde{Y_2} = Y_2 - \frac{\omega_w(X_1, Y_2) Y_1 - \omega_w(Y_1, Y_2) X_1}{\omega_w(X_1, Y_1)}.$$

Lemma 7.7.2. *The set $\beta^{\omega_w} := \{\widetilde{X}_1, \widetilde{Y}_1, \widetilde{X}_2, \widetilde{Y}_2\}$ forms a symplectic basis of ξ^{ω_w}.*

Proof. For example we check $\omega(\widetilde{X_2}, \widetilde{Y_2}) = 1$. For that we first compute $\omega(X_1, X_2)$.

$$\begin{aligned} \omega(X_1, Y_2) &= i \sum_j w_j \left(\frac{1}{w_j} \overline{\frac{\partial f}{\partial z_j}} \right) \left(\frac{-i \overline{z}_j}{w_j} \right) - i \sum_j w_j \left(\frac{1}{w_j} \frac{\partial f}{\partial z_j} \right) \left(\frac{i z_j}{w_j} \right) \\ &= \sum_j \frac{1}{w_j} \overline{\frac{\partial f}{\partial z_j}} \overline{z}_j + \sum_j \frac{1}{w_j} \frac{\partial f}{\partial z_j} z_j. \end{aligned}$$

On the other hand, note that $f(z) = 0$ on $\Sigma(f)$, so that we have

$$f(e^{it/w_0} z_0, e^{it/w_1} z_1, \ldots, e^{it/w_n} z_n) = 0$$

by the definition of weighted homogeneous polynomials. Taking the derivative $\frac{d}{dt}|_{t=0}$ to this and using the (complex) chain rule, we get

$$\sum_j \frac{1}{w_j}\frac{\partial f}{\partial z_j} z_j = 0.$$

It follow that $\omega(X_1, X_2) = 0$. Similarly, one can check $\omega(Y_1, X_2) = 0$. From these formulas, we now see

$$\omega(\widetilde{X_2}, \widetilde{Y_2}) = \omega(X_2, Y_2).$$

Note also that

$$\begin{aligned}
\omega(X_2, Y_2) &= -\frac{1}{2}\left(i\sum_j w_j \frac{iz_j}{w_j}\overline{z}_j - i\sum_j w_j \frac{-i\overline{z}_j}{w_j} z_j\right) \\
&= -\frac{1}{2}\left(-\sum_j |z_j|^2 - \sum_j |z_j|^2\right) \\
&= 1
\end{aligned}$$

where the last equality holds because $z \in \Sigma(f) \subset S^{2n+1}$. Therefore we can conclude $\omega(\widetilde{X_2}, \widetilde{Y_2}) = 1$ $\square$

The linearized Reeb flow $TFl_t^{R_\alpha}$ restricted to the complement ξ^{ω_w} is very simple with respect to the trivialization β^{ω_w}.

Lemma 7.7.3. *With respect to the basis β^{ω_w}, the matrix representation of the linearized Reeb flow is* $\mathrm{diag}(e^{it}, 1) \in Sp(4)$.

Proof. This can be seen by a direct computation. As an example, we compute $TFl_t^{R_\alpha}(\widetilde{X}_1)$. First of all, note that

$$TFl_{t,z}^{R_\alpha}(X_1) = \sum_j e^{it/w_j}\overline{\frac{\partial f}{\partial z_j}}(z)\frac{\partial}{\partial z_j}(e^{it}\cdot z) + cc,$$

where $e^{it}\cdot z = (e^{it/w_0}z_0, e^{it/w_1}z_1, \ldots, e^{it/w_n}z_n)$, and "$cc$" means the "complex conjugate". Taking the derivative $\frac{\partial}{\partial z_j}$ on both sides of

$$f(\eta^{1/w_0}z_0, \cdots, \eta^{1/w_n}z_n) = \eta f(z_0, \cdots, z_n),$$

we have that

$$e^{it/w_j}\frac{\partial f}{\partial z_j}(e^{it}\cdot z) = e^{it}\frac{\partial f}{\partial z_j}(z)$$

for each j, and by conjugation we get

$$e^{it}\overline{\frac{\partial f}{\partial z_j}}(e^{it}\cdot z) = e^{it/w_j}\overline{\frac{\partial f}{\partial z_j}}(z).$$

Applying this to the first equation, we have

$$TFl_{t,z}^{R_\alpha}(X_1) = \sum_j e^{it}\overline{\frac{\partial f}{\partial z_j}}(e^{it}\cdot z)\frac{\partial}{\partial z_j}(e^{it}\cdot z) + cc.$$

It follows that $TFl_{t,z}^{R_\alpha}(\widetilde{X}_1) = e^{it}\widetilde{X}_1$, as we asserted. $\square$

We denote the trivialization of $\gamma^* T\mathbb{C}^{n+1}|_{\Sigma(f)}$ obtained by the union of β and β^{ω_w} by $\beta \oplus \beta^{\omega_w}$. Due to the decomposition $T\mathbb{C}^{n+1}|_{\Sigma(f)} \cong \xi \oplus \xi^{\omega_w}$ and the direct sum property of the Robbin-Salamon index, we have

$$\mu_{RS}(\gamma; \beta \oplus \beta^{\omega_w}) = \mu_{RS}(\gamma; \beta) + \mu_{RS}(\gamma; \beta^{\omega_w}). \tag{7.7.1}$$

Note that $T\mathbb{C}^{n+1}$ has another natural symplectic basis, namely

$$\epsilon := \left\{ \frac{1}{\sqrt{w_j}}\left(\frac{\partial}{\partial z_j} + \frac{\partial}{\partial \overline{z}_j}\right)\right\}_{0\le j\le n}.$$

Two bases, $\beta \oplus \beta^{\omega_w}$ and ϵ, only differ by a linear symplectomorphism on $T\mathbb{C}^{n+1}$. It follow from the naturality property that

$$\mu_{RS}(\gamma; \beta \oplus \beta^{\omega_w}) = \mu_{RS}(\gamma; \epsilon).$$

With respect to the basis ϵ, the linearized Reeb flow is simply the diagonal matrix

$$[TFl_t^{R_\alpha}]_\epsilon = \operatorname{diag}(e^{it/w_0}, \dots, e^{it/w_n}).$$

Therefore, using the formula (3.3.2) and the direct sum property of the index, we have

$$\mu_{RS}(\gamma; \beta \oplus \beta^{\omega_w}) = \mu_{RS}(\gamma; \epsilon) = 2\sum_{j\in I(T)}\frac{T}{w_j} + 2\sum_{j\in I\setminus I(T)}\left\lfloor\frac{T}{w_j}\right\rfloor + \#(I\setminus I(T))$$

where $I(T) \subset I = \{0, 1, \dots, n\}$ be the maximal subset such that $\mathrm{lcm}_{j \in I(T)} p_j = T$.

We can also compute $\mu_{RS}(\gamma; \beta^{\omega_w})$ using the same formula (3.3.2), and the result is

$$\mu_{RS}(\gamma; \beta^{\omega_w}) = 2T.$$

Plugging these into (7.7.1), we now have

$$\begin{aligned} \mu_{RS}(\gamma) &= \mu_{RS}(\gamma; \beta) \\ &= \mu_{RS}(\gamma; \epsilon) - \mu_{RS}(\gamma; \beta^{\omega_w}) \\ &= 2 \sum_{j \in I(T)} \frac{T}{w_j} + 2 \sum_{j \in I \setminus I(T)} \left\lfloor \frac{T}{w_j} \right\rfloor + \#(I \setminus I(T)) - 2T. \end{aligned} \tag{7.7.2}$$

Since any periodic orbit in the same Morse-Bott component has the same index, it makes sense to use the notation $\mu_{RS}(\Sigma(f)_T)$ and $\mu_{RS}(N \cdot \Sigma(f)_T)$ for its N-th iterate. By the same computation as the simple case above, we can finally formulate the index of iterates as follows.

Proposition 7.7.4. *The index of the iterate $N \cdot \Sigma(f)_T$ is given by*

$$\mu_{RS}(N \cdot \Sigma(f)_T) = 2 \sum_{j \in I(T)} \frac{NT}{w_j} + 2 \sum_{j \in I \setminus I(T)} \left\lfloor \frac{NT}{w_j} \right\rfloor + \#(I \setminus I(T)) - 2NT.$$

In particular, the index of principle orbit is given by

$$\mu_{RS}(N \cdot \Sigma(f)) = 2 \cdot (\mathrm{lcm}_j\, w_j) \left(\sum_{j=0}^{n} \frac{1}{w_j} - 1 \right) N.$$

Proof. We have already shown the simple orbit case. The index of N-multiply covered orbit is clearly obtained by just put the period $2\pi NT$ instead of $2\pi T$ in the equation (7.7.2). This completes the proof. $\square$

Example 7.7.5. Consider the Brieskorn polynomial

$$f(z) = z_0^3 + z_1^2 + \cdot + z_n^2.$$

As we have seen in Example 7.7.5 there are only two simple Morse-Bott submanifolds: $\Sigma(a)_2 = \Sigma(2, \dots, 2)$ (exceptional) and $\Sigma(a)_6 = \Sigma(3, 2, \dots, 2)$ (principal). Using the above formula, we can compute their indices. Note that the corresponding index subsets are $I(2) = \{1, 2, \ldots, n\}$ and $I(6) = I$, respectively. Therefore we have

$$\mu_{RS}(N \cdot \Sigma(a)_2) = 2nN + 2\left\lfloor \frac{2N}{3} \right\rfloor + 1 - 4N = (2n-4)N + 2\left\lfloor \frac{2N}{3} \right\rfloor + 1,$$
$$\mu_{RS}(N \cdot \Sigma(a)_6) = (4N + 6nN) + 0 + 0 - 12N = 2(3n-4)N.$$

Example 7.7.6. Consider the E_7-singularity

$$f(z) = z_0^3 + z_0 z_1^3 + z_2^2 + \cdots + z_n^2.$$

In Example 7.6.12 we listed Morse-Bott submanifolds in $\Sigma(f)$. Using the formula of the index we can compute their Robbin-Salamon indices easily. For example,

$$\begin{aligned}\mu_{RS}(N \cdot \Sigma(f)_2) &= 2\left((n-1)\frac{2N}{2}\right) + 2(\left\lfloor \frac{2N}{3} \right\rfloor + \left\lfloor \frac{2N}{9/2} \right\rfloor) + 2 - 4N \\ &= 2(n-3)N + 2(\left\lfloor \frac{2N}{3} \right\rfloor + \left\lfloor \frac{4N}{9} \right\rfloor) - 2.\end{aligned}$$

Similarly, we can compute

$$\begin{aligned}\mu_{RS}(N \cdot \Sigma(f)_6) &= 2(3n-7)N + 2\left\lfloor \frac{4N}{3} \right\rfloor + 1, \\ \mu_{RS}(N \cdot \Sigma(f)_9) &= -8N + (n-1)\left(2\left\lfloor \frac{9N}{2} \right\rfloor + 1\right), \\ \mu_{RS}(N \cdot \Sigma(f)_{18}) &= 2(9n-17)N.\end{aligned}$$

7.7.1 Maslov index and the first Chern number: homogeneous case

The index of the principle orbit is in fact closely related to the first (orbifold) Chern number of a suitable symplectic manifold.

This relation is rather apparent for homogeneous case. Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a homogeneous polynomial of degree d. Suppose that f has an isolated singularity at the origin. A typical example of such polynomial is $f(z) = z_0^d + z_1^d + \cdots z_n^d$, which is Brieskorn. Note that f is then a weighted homogeneous polynomial of weights $(d, \ldots, d)$. One can easily see that the simple principle Morse-Bott submanifold has period $2\pi d$ and there is no proper exceptional orbit space. Using Proposition 7.7.4, we have

$$\mu_{RS}(N \cdot \Sigma(f)) = 2(n+1-d)N.$$

On the other hand, note that the link is by definition

$$\Sigma(f) = f^{-1}(0) \cap S^{2n+1}.$$

Note also that the complex projective space $\mathbb{C}P^n$ is defined by the quotient

$$\mathbb{C}P^n = S^{2n+1}/S^1$$

where S^1-action on S^{2n+1} is the same as the Reeb flow on $\Sigma(f)$. In particular, $\Sigma(f)$ descends to the quotient as the hypersurface $X_d \subset \mathbb{C}P^n$ of degree d. It is well-known that the first Chern number $c_1 = c_1(X_k)$ of X_d is exactly $n+1-d$.

An upshot is that

$$\mu_{RS}(N \cdot \Sigma(f)) = 2c_1 N.$$

Note that $\Sigma(f)$ is the prequantization bundle over an integral symplectic manifold X_d. The following proposition show that the above relation is not an accident.

Proposition 7.7.7. *Let (Q, ω) be a simply-connected integral symplectic manifold. Let P be an associated prequantization bundle over $(Q, k\omega)$. Assume that Q is monotone in the sense that $c_1^{TQ} = c[\omega]$ in $H^2(Q;\mathbb{Z})$ for some $c > 0$. Then the Maslov index of the fiber is given by*

$$\mu(\gamma^k) = 2c$$

where $\pi_1(P) = \mathbb{Z}_k$ so that γ^k is contractible periodic Reeb orbit in P.

Proof. We follow the argument in [47]. For more "topological proof", see [48]. Denote the principal S^1-bundle by $\pi : (P, \alpha) \to Q$ with the connection form α such that $d\alpha = 2\pi \cdot \pi^* k\omega$, and consider the associated complex line bundle $\pi : E \to Q$. In particular, the first Chern class of the line bundle c_1^E satisfies

$$c_1^E = -k[\omega].$$

Since TE splits into the horizontal part and the vertical part, it follows that

$$\begin{aligned} c_1^{TE} &= c_1^{TB} + c_1^E \\ &= c[\omega] - k[\omega] = (c-k)[w]. \end{aligned}$$

Take any capping disk $u : D^2 \to P$ of γ. Note that γ has another capping disk *in* E canonically, namely, the obvious *fiber disk* $v : D^2 \to E$ in the fiber $\pi^{-1}(b)$ of E where $\pi(\gamma) = b \in Q$. Denote the *sphere* in E obtained by attaching u and v^k along γ by $u\#v^k$. Then by the loop axiom of the index, we have

$$\begin{aligned} \mu(\gamma^k; u) &= \mu(\gamma^k; v^k) + 2\mu(\gamma^k; u\# - v^k) \\ &= \mu(e^{it}; t \in [0, 2\pi k]) + 2c_1^{TE}([u\# - v^k]) \\ &= 2k + 2(c-k)[w]([u\# - v^k]) \\ &= 2k + 2(c-k)\left(\int_{D^2} u^*\omega - \int_{D^2} v^*\omega\right) \\ &= 2k + 2(c-k)\int_{D^2} u^*\omega \\ &= 2k + 2(c-k)\frac{1}{2k\pi}\int_{\gamma^k} \alpha \\ &= 2c. \end{aligned}$$

This completes the proof. □

7.7.2 Maslov index and the first Chern number: the weighted homogeneous case

Now we consider general weighted homogeneous polynomial. Let f be a weighted homogeneous polynomial of weight $(w_0, \ldots, w_n)$ with an isolated singularity at the origin. Then the Maslov index of the principle orbit is given by Proposition 7.7.4, namely

$$\mu(N \cdot \gamma) = 2 \cdot (\mathrm{lcm}_j\, w_j)\left(\sum_{j=0}^{n} \frac{1}{w_j} - 1\right) N.$$

As in the case of homogeneous polynomials, the link of f can be interpreted as a prequantization bundle. However, for general weighted homogeneous polynomial, the base is possibly a symplectic *orbifold*. Indeed, the S^1-action or equivalently the Reeb flow on $\Sigma(f)$ is only *locally free* rather than free. As a result, the quotient space Σ/S^1 is now an orbifold in general. More precisely, the quotient is exactly the weighted projective variety $\mathcal{X} := f^{-1}(0)$ in the weighted complex projective space $\mathbb{C}P^n(w)$. It is well-known that the first orbifold Chern number of $\mathcal{X}$ is $c_1^{orb} = (\operatorname{lcm}_j w_j)\left(\sum_{j=0}^n \frac{1}{w_j} - 1\right)$. Therefore we still have the same relation

$$\mu(N \cdot \gamma) = 2c_1^{orb} N.$$

Of course, this relation between the Maslov index and the first orbifold Chern number is not an accident.

Proposition 7.7.8. *Let P be a prequantization orbibundle over an integral symplectic orbifold $(Q, k\omega)$. Assume that Q is monotone, i.e. $c_1^{orb}(TQ) = c[\omega]$ in $H^2_{orb}(Q;\mathbb{Z})$ for some $c > 0$. Then the Maslov index of the fiber is given by*

$$\mu(\gamma^k) = 2c.$$

For a proof, we refter to Pati [49].

Chapter 8

Equivariant symplectic homology of the links of weighted homogenous polynomials

In this chapter, we give some computational results on (equivariant) symplectic homology groups of Milnor fibers and its links of weighted homogeneous polynomials. The Main tool for computations is the Morse-Bott spectral sequence we have developed in Chapter 6.

8.1 First examples

This section gives instructive examples which shows how to compute spectral sequences. For this purpose, we provide rather detailed explanation of computations for some simple examples. First recall the E^1-page of the spectral sequence looks like:

$$E^1_{pq}(SH^{+,S^1}) = \begin{cases} \bigoplus_{\Sigma \in C(p)} H^{S^1}_{p+q-\mathrm{shift}(\Sigma)}(\Sigma; R) & p > 0 \\ 0 & p \leq 0. \end{cases}$$

To compute the above E^1-page, we need the following ingredients:

- Understand Reeb dynamics, and find full list of Morse-Bott submanifolds sorted by its action.

- Compute Robbin-Salamon index of each Morse-Bott components Σ and its iterates.
- Compute (equivariant) singular homology groups of each Morse-Bott components.
- Plugging the above computations into the E^1-page of the spectral sequence and play around.

Conditions for applying the spectral sequences

Before applying the spectral sequence of Theorem 6.4.3 to the links of singularities, let us point out that the Stein manifold $(V_\epsilon(f), \Sigma(f))$ satisfies all the conditions of the theorem 6.4.3. We have already seen that the boundary $\Sigma(f)$ admits a periodic flow, discussed in Section 7.5. The second condition for triviality of the local system, we show the following.

Proposition 8.1.1. *There is a compatible complex structure J for $(\xi, d\alpha)$ such that, for every periodic Reeb orbit γ, the linearized Reeb flow is complex linear with respect to some unitary trivialization of (ξ, J) along γ.*

Proof. We have seen that the contact structure $(\xi, d\alpha)$ fits into a direct sum of bundles over $\Sigma(f)$ as

$$T\mathbb{C}^{n+1} \cong \xi \oplus \xi^{\omega_w}.$$

It is clear from the explicit formula of the linearized Reeb flow that the linearized flow is complex linear with respect to the standard complex structure on $\mathbb{C}^{n+1}$. Furthermore, we have shown in Lemma 7.7.3 that the linearized flow is also complex linear on ξ^{ω_w}. It follows that it is complex linear on ξ. □

8.1.1 Example: $\Sigma(3, 2, 2, 2)$

Let us compute the E^1-page of the spectral sequence for positive equivariant symplectic homology $SH^{+,S^1}(V(3, 2, 2, 2); \mathbb{Q})$. As we have mentioned, we need to list the Morse-Bott submanifolds and compute their Robbin Salmon indices, equivariant singular homology groups in $\mathbb{Q}$-coefficient.

Note that we have already investigated Reeb dynamics for $\Sigma(3,2,2,2)$ in Example 7.7.5. Copying the result there, a list of first six Morse-Bott components sorted by its action is the following.

Morse-Bott component	period
$\Sigma(2,2,2)$	$2\pi \cdot 2$
$2 \cdot \Sigma(2,2,2)$	$2\pi \cdot 4$
$\Sigma(3,2,2,2)$	$2\pi \cdot 6$
$4 \cdot \Sigma(2,2,2)$	$2\pi \cdot 8$
$5 \cdot \Sigma(2,2,2)$	$2\pi \cdot 10$
$2 \cdot \Sigma(3,2,2,2)$	$2\pi \cdot 12$

(8.1.1)

A pattern is that the first three components form a "period" of the list. In view of the action pattern above, the set $C(p)$ of Morse-Bott submanifolds of return time $2\pi p$ consists of at most one Morse-Bott submanifold for each $p \in \mathbb{N}$. For examples, $C(1) = \emptyset$, $C(2) = \{\Sigma(2,2,2)\}$, $C(3) = \emptyset$, $C(4) = \{2 \cdot \Sigma(2,2,2)\}$, and so on.

We also need to compute the S^1-equivariant singular homology groups of Morse-Bott submanifolds, namley, in this case, $H_*^{S^1}(\Sigma(2,2,2);\mathbb{Q})$ and $H_*^{S^1}(\Sigma(3,2,2,2);\mathbb{Q})$. Note that $H_*^{S^1}(\Sigma(2,2,2);\mathbb{Q})$ is already computed in Example (7.4.17). For $H_*^{S^1}(\Sigma(3,2,2,2);\mathbb{Q})$, we again use Theorem 7.4.2, and it is enough to compute $\kappa(\Sigma(3,2,2,2))$. Using the formula (7.4.1), we can check that $\kappa(\Sigma(3,2,2,2)) = 0$. Therefore the equivariant homology of $\Sigma(3,2,2,2)$ with $\mathbb{Q}$-coefficient is given by

$$H_*^{S^1}(\Sigma(3,2,2,2);\mathbb{Q}) = \begin{cases} \mathbb{Q} & * = 0,2,4; \\ 0 & \text{otherwise.} \end{cases}$$

In Example 7.7.5, we have computed the Robbin-Salamon indices of each Morse-Bott submanifolds. Accordingly, we get the index shifting $\operatorname{shift}(\Sigma) = \mu_{RS}(\Sigma) - \dim \Sigma/S^1$ for E^1-page. The result is the following.

$$\begin{aligned} &\text{(Exceptional)} \quad \operatorname{shift}(N \cdot \Sigma(2,2,2)) = \left(2\left\lfloor \tfrac{2N}{3} \right\rfloor + 1\right) - 1 = 2\left\lfloor \tfrac{2N}{3} \right\rfloor, \\ &\text{(Principal)} \quad \operatorname{shift}(N \cdot \Sigma(3,2,2,2)) = 10N - 2. \end{aligned}$$

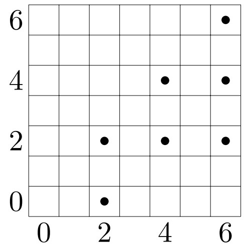


Figure 8.1: E^1-page for $V(3,2,2,2)$

$$\text{(Exceptional orbits)} \quad \mathrm{shift}(N\cdot\Sigma(2,2,2)) = \left(2\left\lfloor\frac{2N}{3}\right\rfloor+1\right)-1 = 2\left\lfloor\frac{2N}{3}\right\rfloor,$$
$$\text{(Principal orbits)} \quad \mathrm{shift}(N\cdot\Sigma(3,2,2,2)) = 10N-2.$$

For computational convenience, it is useful to compute the q-coordinate of E^1-page of the bottom generator for each Morse-Bott submanifold. For example, the equivariant singular homology of $\Sigma(2,2,2)$ will occupy the column $p=2$ and the q-coordinate of the bottom generator of this homology can be computed as follows.

$$q_{bot}(\Sigma(2,2,2)) = \mathrm{shift}(\Sigma(2,2,2)) - p = 2+0-2 = 0.$$

In general, we have

$$q_{bot}(N\cdot\Sigma(2,2,2)) = 2\left\lfloor\frac{2N}{3}\right\rfloor,$$
$$q_{bot}(N\cdot\Sigma(3,2,2,2)) = 4N-2.$$

In particular, the first column $p=2$ has the bottom generator at $q=0$, the second column $p=4$ has the bottom generator at $q=2$, and the third column $p=6$ has the bottom generator at $q=2$. Using these data, we can now fill out E^1-page completely. See Figure 8.1, which shows the E^1-page up to $p=6$, in other words, up to the first three non-trivial columns.

The Reeb flow is periodic, and Morse-Bott submanifolds appear in a repeated pattern as in (8.1.1). Accordingly, the E^1-page is also "periodic" in the horizontal direction. In order to present this pattern more clearly, one can re-index the E^1-page such that the bottom dots of each columns have

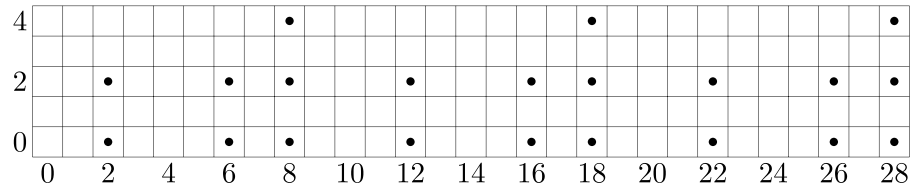


Figure 8.2: E^1-page for $V(3,2,2,2)$ after re-indexing

q-coordinates equal to zero, preserving the total degree $p+q$. A part the the re-indexed E^1-page is Figure 8.2. We can now see the dots in the columns $p \in [0,8]$ appear repeatedly.

Observe that there is no non-trivial differential by the degree reason. Indeed, every generator in E^1-page has even total degree. This can be rigorously checked as follows: The total degree of the bottom generator corresponds to the Morse-Bott submanifold Σ is $p+q = \text{shift}(\Sigma) = \mu_{RS}(\Sigma) - \dim \Sigma/S^1$. We see this numbers are all even. Since the equivariant homology groups are also have generators only in even degree, it follow that every generator in E^1-page has even total degree.

Since the differential d^r vanishes for every $r \geq 1$, the spectral sequence stabilizes from the E^1-page. Therefore the positive S^1-equivariant symplectic homology of $V(3,2,2,2)$ is given by

$$SH_*^{+,S^1}(V(3,2,2,2);\mathbb{Q}) = \bigoplus_{p+q=*} E^1_{pq}.$$

By considering the pattern of E^1-page more carefully, we have

$$SH_*^{+,S^1}(V(3,2,2,2);\mathbb{Q}) = \begin{cases} 0 & * \text{ is odd or } * < 2, \\ \mathbb{Q}^2 & * = 2\left\lfloor \frac{2N}{3} \right\rfloor + 2(N+1) \text{ with } N \in \mathbb{Z}_{\geq 1},\ 2N+1 \notin 3\mathbb{Z}, \\ \mathbb{Q} & \text{otherwise.} \end{cases}$$

The same computational principle works for $V(k,2,2,2)$ with k odd. The result is

$$SH_*^{+,S^1}(V(k,2,2,2);\mathbb{Q})$$
$$= \begin{cases} 0 & * \text{ is odd or } * < 2, \\ \mathbb{Q}^2 & * = 2\left\lfloor \frac{2N}{k} \right\rfloor + 2(N+1) \text{ with } N \in \mathbb{Z}_{\geq 1},\ 2N+1 \notin k\mathbb{Z}, \\ \mathbb{Q} & \text{otherwise.} \end{cases}$$

Remark 8.1.2. A technical reason why the differential vanishes by the degree reason is as follows: Robbin-Salmon indices and non-trivial degrees of equivariant homology groups of Morse-Bott components are all even. In particular, the equivariant homology (in $\mathbb{Q}$-coefficient) of a component is the non-equivariant homology of a projective hypersurface in weighted projective space $\mathbb{C}P^n(w)$. The "even degree" of equivariant homology groups essentially comes from there.

8.1.2 Example: E_7-singularity with $n = 3$

As an example of weighted homogeneous polynomial which is neither Brieskorn nor homogeneous, we consider the E_7-singularity with $n = 3$, namely

$$f(z) = z_0^3 + z_0 z_1^3 + z_2^2 + z_3^2.$$

We have been investigated its Reeb flow in Example 7.6.12. The smallest common period is $2\pi 18$. The list of Morse-Bott components within this periods is as follows.

Morse-Bott component	Polynomial	period
$\Sigma(f)_2$	$z_2^2 + z_3^2$	$2\pi \cdot 2$
$2 \cdot \Sigma(f)_2$	$z_2^2 + z_3^2$	$2\pi \cdot 4$
$\Sigma(f)_6$	$z_0^3 + z_2^2 + z_3^2$	$2\pi \cdot 6$
$4 \cdot \Sigma(f)_2$	$z_2^2 + z_3^2$	$2\pi \cdot 8$
$\Sigma(f)_9$	$z_0^3 + z_0 z_1^3$	$2\pi \cdot 9$
$5 \cdot \Sigma(f)_2$	$z_2^2 + z_3^2$	$2\pi \cdot 10$
$2 \cdot \Sigma(f)_6$	$z_0^3 + z_2^2 + z_3^2$	$2\pi \cdot 12$
$7 \cdot \Sigma(f)_2$	$z_2^2 + z_3^2$	$2\pi \cdot 14$
$8 \cdot \Sigma(f)_2$	$z_2^2 + z_3^2$	$2\pi \cdot 16$
$\Sigma(f)_{18}$	$z_0^3 + z_0 z_1^3 + z_2^2 + z_3^2$	$2\pi \cdot 18$

(8.1.2)

According to the corresponding polynomials, the components are given by $\Sigma(f)_2 = \Sigma(2,2)$, $\Sigma(f)_6 = \Sigma(3,2,2)$, $\Sigma(f)_9 = \Sigma(z_0^3 + z_0 z_1^3)$, and $\Sigma(f)_{18} = \Sigma(f)$. We have computed the middle dimensional Betti number κ for $\Sigma(2,2)$, $\Sigma(3,2,2)$, and $\Sigma(f)$, for instance in Example 7.4.7. Their equivariant homology groups are

$$H_*^{S^1}(\Sigma(2,2);\mathbb{Q}) = \begin{cases} \mathbb{Q}^2 & * = 0; \\ 0 & \text{otherwise.} \end{cases}$$

$$H_*^{S^1}(\Sigma(3,2,2);\mathbb{Q}) = \begin{cases} \mathbb{Q} & * = 0,2; \\ 0 & \text{otherwise.} \end{cases}$$

$$H_*^{S^1}(\Sigma(f);\mathbb{Q}) = \begin{cases} \mathbb{Q} & * = 0,4; \\ \mathbb{Q}^2 & * = 2; \\ 0 & \text{otherwise.} \end{cases}$$

We can also compute $\kappa(\Sigma(z_0^3 + z_0 z_1^3))$ using the formula (7.4.1). The result is $\kappa((\Sigma(z_0^3 + z_0 z_1^3))) = 1$. Therefore

$$H_*^{S^1}(\Sigma(z_0^3 + z_0 z_1^3);\mathbb{Q}) = \begin{cases} \mathbb{Q}^2 & * = 0; \\ 0 & \text{otherwise.} \end{cases}$$

The Robbin-Salamon indices of each Morse-Bott submanifolds are computed in Example 7.7.6. We have the degree shifts as follows.

$$\begin{aligned}
&\mathrm{shift}(N \cdot \Sigma(f)_2) = -4N + 2(\lfloor \tfrac{2N}{3} \rfloor + \lfloor \tfrac{4N}{9} \rfloor) - 2, \\
&\mathrm{shift}(N \cdot \Sigma(f)_6) = -2N + 2\lfloor \tfrac{4N}{3} \rfloor, \\
&\mathrm{shift}(N \cdot \Sigma(f)_9) = -8N + 4\lfloor \tfrac{9N}{2} \rfloor + 2, \\
&\mathrm{shift}(N \cdot \Sigma(f)_{18}) = 20N - 2.
\end{aligned}$$

As the previous section, it is convenient to compute $q_{bot}(\Sigma) = \mathrm{shift}(\Sigma) - p$ of each Morse-Bott submanifold:

$$\begin{aligned}
&q_{bot}(N \cdot \Sigma(f)_2) = -6N + 2(\lfloor \tfrac{2N}{3} \rfloor + \lfloor \tfrac{4N}{9} \rfloor) - 2, \\
&q_{bot}(N \cdot \Sigma(f)_6) = -8N + 2\lfloor \tfrac{4N}{3} \rfloor, \\
&q_{bot}(N \cdot \Sigma(f)_9) = -17N + 4\lfloor \tfrac{9N}{2} \rfloor + 2, \\
&q_{bot}(N \cdot \Sigma(f)_{18}) = 2N - 2.
\end{aligned}$$

Now one can give the E^1-page for E_7-singularity using the above data.

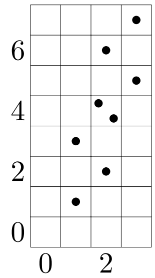


Figure 8.3: E^1-page for $V(4, 2, 2, 2)$

8.2 Simple singularities

Using the same technique as the previous section, we can compute the positive S^1-equivariant symplectic homology of the Milnor fibers of simples singularities (or ADE-type singularities). We provide here a result for A_k-type simple singularties.

A_k-type

The positive S^1-equivariant symplectic homology groups for A_k-singularities with $n \geq 3$ are as follows. For odd n, we have

$$SH_*^{+,S^1}(V(2k, 2, \ldots, 2)) = \begin{cases} \mathbb{Q}^2 & * = 2\lfloor \frac{N}{k} \rfloor + (2n-4)N + n - 1; \text{ or} \\ & * \in 2\{(n-2)k+1\}\mathbb{N} \\ 0 & * \text{ is odd or } * < n-1 \\ \mathbb{Q} & \text{otherwise} \end{cases}$$

and

$$SH_*^{+,S^1}(V(2k+1, 2, \ldots, 2)) = \begin{cases} \mathbb{Q}^2 & * = 2\lfloor \frac{2N}{2k+1} \rfloor + (2n-4)N + n - 1 \\ & \quad \text{for } 2N+1 \notin (2k+1)\mathbb{N} \\ 0 & * \text{ is odd or } * < n-1 \\ \mathbb{Q} & \text{otherwise.} \end{cases}$$

And for n even we have

$$SH_*^{+,S^1}(V(2k,2,\dots,2)) = \begin{cases} \mathbb{Q}^2 & * = 2\lfloor \frac{N}{k} \rfloor + (2n-4)N + n - 1;\ \text{or} \\ & * = \ 2\lfloor \frac{N}{k} \rfloor + (2n-4)N + 1 \text{ for } N \notin k\mathbb{N} \\ 0 & * \text{ is even or } * < n-1 \\ \mathbb{Q} & \text{otherwise} \end{cases}$$

and

$$\begin{aligned} & SH_*^{+,S^1}(V(2k+1,2,\dots,2)) \\ & \qquad = \begin{cases} \mathbb{Q}^2 & * = 2\lfloor \frac{2N}{2k+1} \rfloor + (2n-4)N + n - 1 \\ & \qquad \text{for } 2N+1 \notin (2k+1)\mathbb{N};\ \text{or} \\ & * = 2\lfloor \frac{2N}{2k+1} \rfloor + (2n-4)N + 1 \text{ for } N \notin (2k+1)\mathbb{N} \\ 0 & * \text{ is even or } * < n-1 \\ \mathbb{Q} & \text{otherwise.} \end{cases} \end{aligned}$$

8.3 Homogeneous polynomials

The E^1-page of the case when f is homogeneous is simpler than the above. We collect some computational results including the cotangent bundles over spheres.

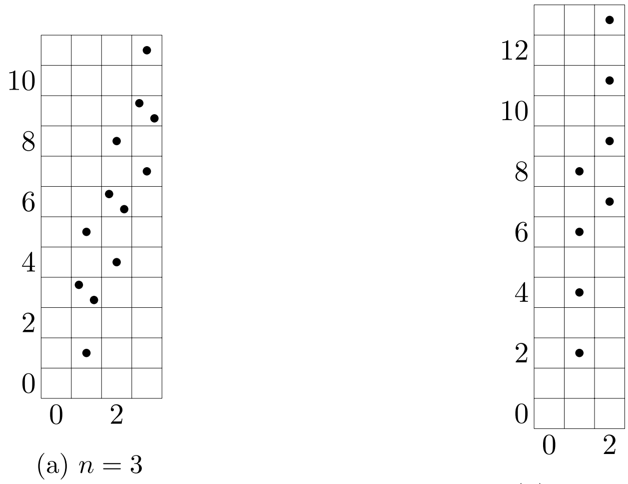
The cotangent bundle over sphere T^*S^n

The cotangent bundle over spheres are Milnor fibers of the quadratic polynomials $z_0^2 + \cdots + z_n^2$. On the one hand, they are homogeneous polynomials of degree 2 and, on the other hand, they are A_1-type Brieskorn polynomials.

Figure 8.4 shows E^1-pages for cotangent bundles over spheres in low dimensions. In general, the positive equivariant symplectic homology of T^*S^n is given as follows.

Proposition 8.3.1. *If n is odd, then*

$$SH_k^{+,S^1}(T^*S^n, \lambda_{can}) \cong \begin{cases} \mathbb{Q}^2 & \text{if } k = n-1+d(n-1) \text{ with } d \in \mathbb{Z}_{\geq 1} \\ 0 & \text{if } k \text{ is odd or } k < n-1 \\ \mathbb{Q} & \text{otherwise.} \end{cases}$$



(a) $n = 3$

(b) $n = 4$

Figure 8.4: E^1-pages of Morse-Bott spectral sequence for ST^*S^n

If n is even, then

$$SH_k^{+,S^1}(T^*S^n, \lambda_{can}) \cong \begin{cases} \mathbb{Q}^2 & \text{if } k = n-1+2d(n-1) \text{ with } d \in \mathbb{Z}_{\geq 1} \\ 0 & \text{if } k \text{ is even or } k < n-1 \\ \mathbb{Q} & \text{otherwise.} \end{cases}$$

General homogeneous polynomials

For general homogeneous polynomials, (equivariant) symplectic homology groups are given as a direct sum of (equivariant) singular homology of the link. This can be easily seen in the spectral sequence. Let f be a homogeneous polynomial of degree k with an isolated singularity at the origin. Denote $Q_k = \Sigma(f)/S^1$ be the corresponding projective hypersurface of degree k in $\mathbb{C}P^n$.

For equivariant symplectic homology, by the degree reason, the differential vanishes. So

$$\begin{aligned} SH_*^{+,S^1}(V(f);\mathbb{Q}) &= \bigoplus H^{S^1}_{*-\mathrm{shift}(\Sigma(f))}(\Sigma(f);\mathbb{Q}) \\ &= \bigoplus H_{*-\mathrm{shift}(\Sigma(f))}(Q_k;\mathbb{Q}) \end{aligned}$$

where $\mathrm{shift}(\Sigma(f))$ is completely determined by $c_1(Q_k)$ as Proposition 7.7.7.

For non-equivariant symplectic homology, the differential does not vanish in general. However, using spectral sequence, we can show that if the degree is sufficiently high relative to n, then the differential vanishes. So $SH_*(V(f);\mathbb{Q})$ is a direct sum of singular homology of $H(\Sigma(f);\mathbb{Q})$.

Chapter 9

Invariants of contact structures from symplectic homology

9.1 Towards invariants of contact structures

We have discussed symplectic homology and its equivariant version as an invariant of symplectic manifolds with contact boundary. In some cases, however, symplectic homology of domains can serve an invariant of contact structure of the boundary. In this thesis, we discuss a couple of ways to get an invariant of contact structure on the boundary from (equivariant) symplectic homology:

- *Positive symplectic homology*: If the index of periodic Reeb orbits is sufficiently large, then the positive part of symplectic homology SH^+ only depend on the contact structure of the boundary.

  The main ingredient of showing the invariance is a neck-stretching argument. The index condition then implies that flow lines in the moduli space that we count for the differential map lie in the cylindrical end of the completion. Such an argument is addressed by Uebele [15].

- *The mean Euler characteristic*: The mean Euler characteristic χ_m of the positive S^1-equivariant symplectic homology SH^{+,S^1} does not depend on the filling, in some cases.

A rough idea of the invariance is the following. The positive part is only generated by periodic Reeb orbits on the boundary (no critical points in the filling). Therefore only the differentials need the filling to be defined. However, by the same reason as the ordinary Euler characteristic of singular homology, the mean Euler characteristic does not depend on the differential. Furthermore, as we have checked, the (equivariant) symplectic homology does not depend on the choice of contact *form*. It follows that the mean Euler characteristic only depends the contact structure on the boundary.

Remark 9.1.1. Here are some remarks.

1. To ignore the effect of the filling, it is apparent that we necessarily deal with the *positive* part rather than the whole symplectic homology.

2. One may also wonder why we do not consider the mean Euler characteristic of *non-equivariant* symplectic homology. The reason is that, conjecturally, the mean Euler characteristic of symplectic homology is equal to 0. This conjecture seems true in view of the usual perturbation argument of transversely non-degenerate Hamiltonians; each S^1-family of periodic orbits breaks into two generators of index difference by 1. This conjecture is true for the case when the boundary admits a periodic flow.

An example of invariance of SH^+

Before we focus on the mean Euler characteristics, we present here an example of a case when positive symplectic homology gives an invariant of the contact structure. We use the spectral sequence to show the invariance.

Proposition 9.1.2. *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a homogeneous polynomial of degree k with an isolated singularity at the origin, for $k > 3n$. Assume that W is a simply-connected Liouville filling with $c_1(W) = 0$. Then $SH^+(W)$ does not depend on the choice of Liouville filling W with the same assumptions. In particular, $SH^+(W)$ is then an invariant of the contact structure.*

Proof. Consider the Morse-Bott spectral sequence (6.4.3) for $SH^+(W)$. By Proposition 7.7.7, we see that the index of the principal orbit is given by $n+1-k$. We compare two adjacent columns on the E^1-page. The maximum on the $p+1$-st column (after re-indexing) has total degree $(p+1)\cdot(n+1-k)+2n-1$ and the minimum on the p-th column has total degree $p\cdot(n+1-k)+0$. As $k>3n$, the differential vanishes, and the spectral sequence terminates from the E^1-page. This identically happens for every filling, so all fillings have the same E^1-page. Since the E^1-pages are isomorphic, so are the symplectic homology groups SH^+. □

9.2 The mean Euler characteristic

In this section, we define the mean Euler characteristic of the positive S^1-equivariant symplectic homology. The mean Euler characteristic was first introduced by van Koert in this thesis [19] in the context of contact homology. The main motivation was to give an invariant of contact structure from contact homology, which does not depend on the differential. Note that understanding the differential of Floer-theoretic homology is usually difficult.

Even if we loose essential information about the homology, the mean Euler characteristic still gives a useful numerical invariant. For example, it can be used to re-prove Ustilovsky's result, states that there are infinitely many contact structures on the spheres. We shall give several applications of the mean Euler characteristic on such a existence/classification problem of contact structures.

Our main tool for computing this invariant is again the Morse-Bott spectral sequence. The spectral sequences makes the computation quite efficient, by "visualizing" the chain complex. This gives rise to an easy and simple principle for mean Euler characteristic.

9.2.1 Definition

From now on we use $\mathbb{Q}$-coefficient for any symplectic homology groups. Let (W,ω) be a Liouville domain which (equivariant) symplectic homology is well-defined. We denote the i-th Betti number of the positive S^1-equivariant symplectic homology group of (W,ω) by

$$sb_i(W) := rkSH_i^{+,S^1}(W).$$

Definition 9.2.1. We define the **mean Euler characteristic** of the domain (W, ω) by

$$\chi_m(W) = \frac{1}{2}\left(\liminf_{N\to\infty} \frac{1}{N}\sum_{i=-N}^{N}(-1)^i sb_i(W) + \limsup_{N\to\infty}\frac{1}{N}\sum_{i=-N}^{N}(-1)^i sb_i(W)\right)$$

if the limits exist.

The above limit of course does not always exist. There are even some examples that the Betti number sb_i is infinite for some i. For example, $rkSH_0^{+,S^1}(V(4,4,4,4)) = \infty$. However in the case when we have a *uniform* bound of the Betti numbers, the $\liminf$ and $\limsup$ converge. Morevoer, for all examples here, the periodicity of the flow will imply that $\liminf$ and $\limsup$ coincide to each other. Therefore one could take the definition of χ_m in such cases as the following simple form.

$$\chi_m(W) = \lim_{N\to\infty}\frac{1}{N}\sum_{i=-N}^{N}(-1)^i sb_i(W).$$

9.2.2 Invariance

In this section, we introdunce some conditions that makes the mean Euler characteristic be an invariant of the contact structure on the boundary.

Theorem 9.2.2. *Let $(W, d\lambda)$ be a Liouville domain with $\pi_1(W) = 0$ and $c_1(W) = 0$ (so that the absolute grading of symplectic homology is well-defined). Suppose that*

1. *the boundary $(\partial W, \xi = \ker(\alpha = \lambda|_{\partial W}))$ admits a contact form whose Reeb flow is periodic;*
2. *the mean index of of a principle orbit is not equal to 0.*

Then $\chi_m(W, d\lambda)$ is an invariant of the contact manifold $(\partial W, \xi)$.

For a proof we first prove the following lemma, which says that the mean Euler characteristic is the same as the "mean Euler characteristic of E^1-page".

Lemma 9.2.3. *Under the same assumption as Theorem 9.2.2, the mean Euler characteristic of $SH^{+,S^1}(W;\mathbb{Q})$ is equal to*

$$\chi_m(E^1) := \lim_{N\to\infty} \frac{1}{N} \sum_{-N}^{N} (-1)^i rk \left(\bigoplus_{p+q=i} E^1_{pq} \right).$$

Proof. We define the mean Euler characteristic of "N-degree window" of the E^r-page of the spectral sequence for SH^{+,S^1} by

$$\chi^r_N := \sum_{-N}^{N} (-1)^i rk \left(\bigoplus_{p+q=i} E^r_{pq} \right).$$

We define the mean Euler characteristic of the E^r-page by

$$\chi^r_m := \lim_{N\to\infty} \frac{1}{N} \chi^r_N.$$

Note that if $d^r \equiv 0$ for all $r \geq r_0$ for some $r_0 \geq 1$, then $\bigoplus_{p+q=i} E^r_{pq} \cong SH^{+,S^1}_i(W;\mathbb{Q})$ so that $\chi_m(W) = \lim_{N\to\infty} \frac{1}{N}\chi^r_N$ for all $r \geq r_0$.

Since the Reeb flow of the boundary is periodic, the rank of $\bigoplus_{p+q=i} E^r_{pq}$ is *uniformly* bounded. In this case, as a lemma of the lemma, we can show that the mean Euler characteristic of E^r-page is actually identical for all r.

Lemma 9.2.4. *For all $r \in \mathbb{N}$,*

$$\lim_{N\to\infty} \frac{1}{N}\chi^r_N = \lim_{N\to\infty} \frac{1}{N}\chi^{r+1}_N.$$

Proof. Denote for each $i \in \mathbb{Z}$,

$$d^r_i := \bigoplus_{p+q=i} d^r_{pq} : \bigoplus_{p+q=i} E^r_{pq} \to \bigoplus_{p+q=i-1} E^r_{pq},$$

and

$$Z^r_i := \ker d^r_i, \quad B^r_{i+1} := \operatorname{im} d^r_{i+1}.$$

Then we see that

$$\bigoplus_{p+q=i} E^r_{pq} = Z^r_i \oplus B^r_i, \quad \bigoplus_{p+q=i} E^{r+1}_{pq} = \frac{Z^r_i}{B^r_{i+1}},$$

so that

$$rk\left(\bigoplus_{p+q=i} E^r_{pq}\right) = rkZ^r_i + rkB^r_i, \quad rk\left(\bigoplus_{p+q=i} E^{r+1}_{pq}\right) = rkZ^r_i - rkB^r_{i+1}.$$

Then we see that

$$\begin{aligned}
\left|\chi^r_N - \chi^{r+1}_N\right| &= \left|\sum_{-N}^{N}(-1)^i\left(rkZ^r_i - rkB^r_{i+1}\right) - \sum_{-N}^{N}(-1)^i\left(rkZ^r_i + rkB^r_i\right)\right| \\
&\leq \left|rkB^r_{N+1}\right| + \left|rkB^r_{-N}\right| \\
&\leq C
\end{aligned}$$

for some constant $C \in \mathbb{Z}$ which is *not* dependent of N. (It may depend on r though.) This is because we have assumed that $rkH^{S^1}(\Sigma;\mathbb{Q})$ is uniformly bounded.

Therefore

$$\lim_{N\to\infty}\frac{1}{N}\left|\chi^r_N - \chi^{r+1}_N\right| = 0$$

and hence the lemma follows. □

Applying the above lemma until the spectral sequence becomes stable, it follows that the mean Euler characteristic is in fact the same as the the mean Euler characteristic at E^1-page. This completes the proof of the lemma. □

Proof of Theorem 9.2.2. Note that the E^1-page is generated by periodic Reeb orbits and its index does not depend on the filling since $c_1(W) = 0$. Furthermore, the rank of the direct sum $\bigoplus_{p+q=i} E^1_{pq}$ is uniformly bounded due to periodicity of the flow. It follows that $\chi_m(E^1)$ does not depend on the differential, as the usual Euler characteristic. We conclude that $\chi_m(E^1)$ is completely determined by the boundary and its contact form, and so is $\chi_m(W)$ by Lemma 9.2.3. Finally, the invariance of the (equivariant) symplectic homology under the Liouville homotopy gives the conclusion. □

Note that every link of weighted homogeneous polynomials $\Sigma(f)$ admits periodic Reeb flow. So if the Maslov index of the principle orbit is not zero, then the mean Euler characteristic $\chi_m(V(f))$ is an invariant of the contact structure of the link $\Sigma(f)$. In such cases, it makes sense to use the notation $\chi_m(\Sigma(f))$ for the mean Euler characteristic, instead of $\chi_m(V(f))$.

For later use, it is useful to consider a bit more general condition for the invariance of the mean Euler characteristic.

Definition 9.2.5. Let (M^{2n-1}, ξ) be a contact manifold. We say (M^{2n-1}, ξ) has **convenient dynamics** if there exists a smooth family of contact forms $\{\alpha_T = f_T \alpha\}_{T \in [T_0, \infty)}$ with $\xi = \ker \alpha_T$ and $T_0 > 0$, a positive constant Δ_m and a positive integer k with the following properties.

1. all periodic orbits of α_T with period less than T are non-degenerate, and satisfy the inequality for the mean index,
$$|\Delta(\gamma)| > \Delta_m;$$
2. there are simple periodic Reeb orbits of α_T, denoted by $\gamma_1^T, \ldots, \gamma_k^T$, with $\mathcal{A}_{\alpha_T}(\gamma_i^T) < T_0$ such that every periodic Reeb orbit of α_T with period at most T is a cover of one of those;
3. the following inequality holds for all $x \in M$;
$$T f_T(x) > (T-1) f_{T-1}(x).$$

For example, if the Reeb flow is periodic, then the contact manifold has convenient dynamics.

Lemma 9.2.6 ([51])**.** *Suppose that (M, α) is a compact, simply-connected, cooriented contact manifold with the following properties:*

1. $c_1(\xi = \ker \alpha) = 0$*;*
2. *the Reeb flow of α is periodic and the mean index of a principle orbit is not equal to* 0*.*

Then $(M, \xi = \ker \alpha)$ has convenient dynamics.

The mean Euler characteristic is an invariant of contact structures which have convenient dynamics. For a proof, we refer to [51, Lemma 5.15].

Theorem 9.2.7 ([51])**.** *Suppose $(P, \alpha = \lambda|_{\partial W})$ is a compact, simply-connected contact manifold admitting a simply-connected Liouville filling $(W, d\lambda)$ with $c_1(W) = 0$, so that grading in symplectic homology is well-defined. Assume that the boundary $(\partial W, \ker \alpha)$ has convenient dynamics. Then $\chi_m(W, d\lambda)$ is an invariant of the contact manifold $(P, \xi = \ker \alpha)$.*

9.2.3 The mean Euler characteristic and the subcritical handle attachment

The mean Euler characteristic satisfies a simple formula under the subcritical handle attachment. This was shown by J. Espina [17] in the setting of contact homology, and a recent version can be deduced by the following surgery exact sequence of positive S^1-equivariant symplectic homology [14]. Denote the positive S^1-equivariant symplectic homology of the 1-handle by $SH_*^{+,S^1}(tube)$.

Theorem 9.2.8 ([14])**.** *Let W^{2n} be a Liouville domain with $c_1(W) = 0$ and W' a Liouville domain obtained from W by 1-handle attachment. Then there exists a long exact sequence:*

$$\cdots \to SH_*^{+,S^1}(tube) \to SH_*^{+,S^1}(W') \to SH_*^{+,S^1}(W) \to SH_{*+1}^{+,S^1}(tube) \to \cdots .$$

It is well-known that the symplectic homology of 1-handle is given by

$$SH_*^{+,S^1}(tube) = SH_*^{+,S^1}(D^{2n}, \omega_0) = \begin{cases} \mathbb{Q} & * = n, n+2, n+4, \dots, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\chi_m(tube) = (-1)^n \frac{1}{2}$. Let (W_1^{2n}, ω_1) and (W_2^{2n}, ω_2) be Liouville manifolds for which the mean Euler characteristic is defined. Their boundary connected sum $(W_1, \omega_1)\natural(W_2, \omega_2)$ is obtained from the disjoint union $W_1 \sqcup W_2$ by 1-handle attachment. The mean Euler characteristic of the disjoint union is nothing but the sum of the two. Therefore, by applying the surgery exact sequence to $W = W_1 \sqcup W_2$ and $W' = (W_1, \omega_1)\natural(W_2, \omega_2)$, we immediately have the following very useful formula.

Theorem 9.2.9 (Espina, Bourgeois-Oancea)**.** *Let (W_1^{2n}, ω_1) and (W_2^{2n}, ω_2) be Liouville manifolds for which the mean Euler characteristic is defined. Then the boundary connected sum of W_1 and W_2 satisfies*

$$\chi_m((W_1, \omega_1)\natural(W_2, \omega_2)) = \chi_m(W_1, \omega_1) + \chi_m(W_2, \omega_2) + (-1)^n \frac{1}{2}. \tag{9.2.1}$$

Recall from Theorem 9.2.7 that the mean Euler characteristic is an invariant of the contact structure of the boundary. In the case when the contact structures of the each boundaries ∂W_1 and ∂W_2 have convenient dynamics, then the mean Euler characteristic is again an invariant of the contact structure of its boundary.

Lemma 9.2.10. *Let $(W, d\lambda)$ be a Liouville domain with convenient dynamics on the boudnary. Let W' be a Liouville domain from W by a subcritical surgery. Then W' also admits convenient dynamics on the boundary.*

Proof. For a proof, we refer to [51, Lemma 5.18]. □

We directly have the following corollary.

Corollary 9.2.11. *Let (W_1^{2n}, ω_1) and (W_2^{2n}, ω_2) be Liouville manifolds for which the mean Euler characteristic is defined. Suppose that both admit convenient dynamics on the boundary. Then the mean Euler characteristic*

$$\chi_m((W_1, \omega_1)\natural(W_2, \omega_2))$$

of their boundary connected sum is an invariant of the its boundary.

9.2.4 The case when $SH = 0$

The mean Euler characteristic is particulary simple in the case when the symplectic homology $SH_*(W)$ vanishes.

Proposition 9.2.12. *Suppose that $(W, d\lambda)$ be a Liouville domain with $\pi_1(W) = 0 = c_1(W)$. If $SH_*(W) = 0$, then its mean Euler characteristic is given by*

$$\chi_m(W) = (-1)^n \frac{\chi^{S^1}(W, \partial W)}{2}.$$

In particular, $\chi_m(W)$ is an half interger.

In other words, the mean Euler characteristic serves as an obstruction to vanishing of symplectic homology; if $\chi_m(W)$ is not an half integer, then $SH \neq 0$.

Proof. If $SH(W) = 0$, then by [14, Theorem 4.1] we have $SH^{S^1}(W) = 0$. By the Viterbo exact sequence for S^1-equivariant symplectic homology,

$$\cdots \to SH_{k+1}^{+,S^1}(W) \to H_{n+k}^{S^1}(W, \partial W) \to SH_k^{S^1}(W) \to SH_k^{+,S^1}(W) \to \cdots$$

it follows that $SH^{+,S^1}_{k+1}(W) \cong H^{S^1}_{n+k}(W, \partial W)$. Then the assertion

$$\chi_m(W) = (-1)^n \frac{\chi^{S^1}(W, \partial W)}{2}$$

is now a simple algebraic consequence. □

Corollary 9.2.13. *Assume the conditions of Proposition 9.2.12. If W is subcritical Stein or flexible Weinstein, then its mean Euler characteristic is a half integer.*

Therefore the mean Euler characteristic can be used as an obstruction of subcritical or flexible Weinstein fillability of contact manifolds. For example, Jung [50] and see also Remark 9.3.7.

Example 9.2.14. Note that $SH_*(B^{2n}) = 0$, see Section 3.6.1. It follows that the mean Euler characteristic of the sphere S^{2n-1} with its standard contact structure is given by

$$\chi_m(S^{2n-1}, \xi_{std}) = (-1)^n \frac{\chi^{S^1}(B^{2n}, S^{2n-1})}{2} = (-1)^n \frac{1}{2}.$$

9.2.5 Computations via spectral sequences

Let (W, ω) be a simply connected Liouville domain such that $c_1(W) = 0$ and its boundary is also simply connected and $c_1(\xi) = 0$. In Section 9.2.2, the mean Euler characteristic is an invariant of the contact structure of the boundary if the Reeb flow on the boundary is periodic and the Maslov index of the principal orbit is not equal to 0. In this section, under the same section, we give a numerical principle for computing the mean Euler characteristic of the boundary in terms of the E^1-page of the spectral sequence.

From Lemma 9.2.3 we know that the mean Euler characteristic of $(\partial W, \xi)$ is the same as the corresponding mean Euler characteristic of the E^1-page. If the flow is periodic, then the E^1-page is also "periodic" in the horizontal direction. This leads to the following computational principle of the mean Euler characteristic.

In the following proposition, let $\Sigma_{T_1}, \ldots, \Sigma_{T_k}$, $T_1 < T_2 < \cdots < T_k$, be simple Morse-Bott submaifolds of the periodic flow, and Σ_{T_k} is the principal one.

Proposition 9.2.15. *Let (W,ω) be a simply connected Liouville domain such that $c_1(W)=0$ and its boundary is also simply connected and $c_1(\xi)=0$. Suppose that*

1. *the Reeb flow on ∂W is periodic;*

2. *the Maslov index of the principal orbit μ_P is not zero.*

Then the mean Eular characteristic is given by

$$\chi_m(W) = \frac{\sum_{i=1}^{k}(-1)^{\Sigma_{T_i}-\frac{1}{2}\dim(\Sigma_{T_i}/S^1)}\varphi_{T_i;T_{i+1},\cdots,T_k}\chi^{S^1}(\Sigma_{T_i})}{|\mu_P|} \tag{9.2.2}$$

where $\varphi_{T_i;T_{i+1},\cdots,T_k} = \#\{a\in\mathbb{N} \mid aT_i < T_k$ and $aT_i \notin T_j\mathbb{N}$ for $j=i+1,\dots,k\}$.

Proof. By the discussion so far, $\chi_m(W)$ is equal to the mean Euler characteristic of the E^1-page

$$\chi_m^1 = \lim_{N\to\infty}\frac{1}{N}\sum_{-N}^{N}(-1)^i rk\left(\bigoplus_{p+q=i}E^1_{pq}\right)$$

Note that the denominator of the asserted equation (9.2.2) is nothing but the signed number of generators in E^1-page, which correspond to simple orbit spaces Σ_{T_i}. Note also that the total degree of the top generator $q_{top}(N\cdot\Sigma_{T_k})$ of $N\cdot\Sigma_{T_k}$ is $2n-2+\text{shift}(N\cdot\Sigma_{T_k})-N\cdot p(\Sigma_{T_k})$ where $p(\Sigma_{T_k})$ is the p-coordinate where the column of $H_*(N\cdot\Sigma_{T_k})$ is located in E^1-page. It follows that the difference

$$|q_{top}((N+1)\cdot\Sigma_{T_k})-q_{top}(N\cdot\Sigma_{T_k})| = |\mu_{RS}((N+1)\cdot\Sigma_{T_k})-\mu_{RS}(N\cdot\Sigma_{T_k})+o(N)|$$

is the same as the absolute value of the Maslov index of the principal orbit $|\mu_P|$ up to an error in $o(N)$. This implies that the finite sum

$$\sum_{-N}^{N}(-1)^i rk\left(\bigoplus_{p+q=i}E^1_{pq}\right)$$

is a linear function of N with slope equal to the signed number of generators in E^1-page, which correspond to simple orbit spaces devided by $|\mu_P|$.

Taking the limit $\lim_{N\to\infty}\frac{1}{N}$ to the finite sum, the mean Euler characteristic is then the same as the slope. This completes the proof. $\square$

It is useful to keep in mind the following informal principle for χ_m, which is used in the above proof:

$$\chi_m(W) = \frac{\text{The signed number of generators in one period of } E^1\text{-page}}{|\mu_P|}.$$

Example 9.2.16. We consider a Brieskorn manifold $\Sigma(2,2,p,q)$ where p and q are odd integers and they are relatively prime. Then the simple Morse-Bott spaces are

$$\Sigma(2,2), \Sigma(2,2,p), \Sigma(2,2,q), \Sigma(p,q), \Sigma(2,2,p,q).$$

The Malsov index of the principal orbit $\mu_P = \mu_{RS}(\Sigma(2,2,p,q))$ can be computed using Proposition 7.7.4, and the S^1-equivariant Euler characteristic χ^{S^1} of each subspaces can also be obtain from the Randell's formula. Note that the period of the principal orbit is $2\pi \cdot 2pq$, and the periods of exceptional orbits are $2, 2p, 2q, pq$, respectively. The value of the "frequency" function φ is then easily computed. Here is a result:

Orbit space	period	χ^{S^1}	frequency (in one period of E^1)
$\Sigma(2,2,p,q)$	$2pq$	3	1
$\Sigma(2,2,p)$	$2p$	2	$q-1$
$\Sigma(2,2,q)$	$2q$	2	$p-1$
$\Sigma(2,2)$	2	2	$pq-q-p+1$
$\Sigma(p,q)$	pq	1	1

We finally have

$$\begin{aligned}\chi_m(V(2,2,p,q)) &= \frac{1+2(pq-p-q+1)+2(q-1)+2(p-1)+3}{4(p+q)} \\ &= \frac{1+pq}{2(p+q)}.\end{aligned}$$

9.2.6 Some explicit formulas

Note that every Milnor fiber of weighted homogeneous polynomials with $n > 3$ satisfies the conditions of Proposition 9.2.15, except for the condition

(2). Therefore, in principle, we can compute the mean Euler characteristic of them unless $\mu_P = 0$. We now deal with some cases when explicit formula can be given without much effort.

If the exponents of a Brieskorn manifold are relatively prime to each other, then the frequencies of each Morse-Bott submanifold can be written rather simply. For example we have:

Proposition 9.2.17. *Let $a_0, a_1, \ldots, a_n$ be pairwise relatively prime numbers greater than 1. Then the mean Euler characteristic of $\Sigma(a_0, \ldots, a_n)$ is given by*

$$(-1)^{n+1}\frac{n + (n-1)\sum_{a_0}(a_{i_0}-1) + \cdots + 1 \cdot \sum_{i_0 < \cdots < i_{n-2]}}(a_{i_0}-1)\cdots(a_{i_{n-2}}-1)}{2|(\sum_j a_0 \cdots \hat{a}_j \cdots a_n) - a_0 \cdots a_n|}.$$

Proof. This is almost directly follows from the formula in Proposition 9.2.15 where the frequency functions φ can be explicitly obtained by the assumption that the coefficients are pairwise relatively prime. $\square$

Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a homogeneous polynomial of degree d with an isolated singularity at the origin. Suppose that $n+1 \neq d$. Then by the computation in Section 7.7.1, the Maslov index of the principal orbit $\mu_P = 2(n+1-d)$ is nonzero. Since there no exceptional orbits in this case, the E^1-page consists of the repetition of $H_*^{S^1}(\Sigma(f); \mathbb{Q})$ with the degree shift. Applying Proposition 9.2.15, we have the following formula.

Proposition 9.2.18. *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a homogeneous polynomial of degree d with an isolated singularity at the origin. If $n+1 \neq d$ and $n > 3$, then we have*

$$\chi_m(\Sigma(f)) = \frac{(-1)^{n+1}(n + (-1)^{n-1}\kappa(f))}{|2(n+1-d)|}.$$

where $\kappa(f)$ is explicitly given as in Example 7.4.6.

Note that the numerator of the above formula is given by the equivariant Euler characteristic of the link $\Sigma(f)$. Since the link of homogeneous polynomial of degree d is a principal S^1-bundle over the projective hypersurface Q_d of degree d in $\mathbb{C}P^n$, it follows that $\chi^{S^1}(\Sigma(f)) = \chi(Q_d)$. The denominator is the Maslov index of the principle orbit, which is equal to $|2c_1(Q_d)|$ by the proposition 7.7.7. Therefore we can rewrite the mean Euler characteristic of

$\Sigma(f)$ purely in terms of the projective hypersurface Q_d as follows.

$$\chi_m(\Sigma(f)) = \frac{(-1)^{n+1}\chi(Q_d)}{|2c_1(Q_d)|}.$$

9.3 Applications of the mean Euler characteristic

We can apply computational results of the mean Euler characteristic to the existence and classification problems on contact structures. We first reprove the result of I. Ustilovsky, [3], which asserts that there are infinitely many contact structures on S^{4m+1}, $m \geq 1$, that are non-contactomorphic to each other and are in the same homotopy class. He used the cylindrical contact homology as an invariant of contact structures. We give an another proof by the mean Euler characteristic of positive S^1-equivariant symplectic homology.

Consider the Brieskorn manifold $\Sigma(p, 2, \ldots, 2)$ where p is odd and $n \geq 3$ is also odd. We denote its canonical contact structure by ξ_p for each p.

Proposition 9.3.1. *The mean Euler characteristic of* $\Sigma(p, 2, \ldots, 2)$ *is given by*

$$\chi_m(\Sigma(p, 2, \ldots, 2)) = \frac{(p-1)(n-1)+n}{2\{(n-2)p+2\}}$$

for p, n *are odd and* $n \geq 3$.

In particular, we observe that $\chi_m(\Sigma(p, 2, \ldots, 2))$ is a one-to-one function of p for each n. Therefore we immediately have the following corollary.

Corollary 9.3.2. *For each odd* $n \geq 3$*, the contact structure* ξ_p *on* $\Sigma(p, 2, \ldots, 2)$ *is not contactomorphic to the contact structure* $\xi_{p'}$ *on* $\Sigma(p', 2, \ldots, 2)$ *if* $p \neq p'$ *for two odd numbers* p, p'.

Proof of Proposition 9.3.1. There are two simple Morse-Bott submanifolds; $\Sigma(2, \ldots, 2)$ (exceptional), $\Sigma(p, 2, \ldots, 2)$ (principle). Note that the frequencies are $p-1$ and 1, respectively. By the formula (9.2.2), we see that

$$\chi_m(\Sigma(p, 2, \ldots, 2)) = \frac{(p-1)\chi^{S^1}(\Sigma(2, \ldots, 2)) + \chi^{S^1}(\Sigma(p, 2, \ldots, 2))}{|\mu_P|}$$

where μ_P is the Maslov index of the principal orbit. The index μ_P can be computed as

$$\mu_P = \mu_{RS}(\Sigma(p,2,\ldots,2)) = 2\cdot 2p\cdot\left(\frac{1}{p}+\frac{n}{2}-1\right) = 2\{(n-2)p+2\}$$

by Proposition 7.7.4.

Since n is odd, we have

$$H_*^{S^1}(\Sigma(2,2,\ldots,2);\mathbb{Q}) = \begin{cases}\mathbb{Q} & * \text{ is even and } 0\le * \le 2n-4\\ 0 & \text{otherwise.}\end{cases}$$

so that $\chi^{S^1}(\Sigma(2,\ldots,2)) = n-1$, and for p is add we also have

$$H_*^{S^1}(\Sigma(p,2,2,\ldots,2);\mathbb{Q}) = \begin{cases}\mathbb{Q} & * \text{ is even and } 0\le * \le 2n-2\\ 0 & \text{otherwise.}\end{cases}$$

so that $\chi^{S^1}(\Sigma(p,2,\ldots,2)) = n$.

By plugging all of these in the formula, we completes the proof. □

Recall from Corollary 7.4.12 that $\Sigma(p,2,\ldots,2)$ is diffeomorphic to the standard $(2n-1)$-sphere if $p=\pm1 \bmod 8$ and $n\ge 3$.

Corollary 9.3.3 (Ustilovsky)**.** *There are infinitely many contact structures of the standard $(4m+1)$-sphere, $m\ge 1$, which are pairwise non-contactomorphic to each other and are in the same homotopy class.*

Proof. Since $\Sigma(p,2,\ldots,2)$ is diffeomorphic to S^{4m+1}, where $n=2m+1$, for $p=\pm1 \bmod 8$, the first assertion follows from Corollary 9.3.2. Furthermore, since $c_1(\xi)=0$ for all Brieskorn manifold of dimension ≥ 5, it follows that the contact structures on S^{4m+1} have the same homotopy class in $\pi_{2n-1}(SO(2n)/U(n))$. □

We now focus on the case of 5-sphere. Note that every contact structure from the Ustilovsky spheres $\Sigma(p,2,2,2)$ has the mean Euler characteristic in $\mathbb{Q}$. We can prove a kind of converse statement in this regard as follows.

Theorem 9.3.4. *Every rational number can be realized as the mean Euler characteristic of some contact structures on S^5 with its standard smooth structure.*

Proof. The basic idea is to produce all rational numbers using the sum formula (9.2.1) and Brieskorn manifolds.

Consider Brieskorn manifolds of the form $\Sigma(k,2,2,2)$, $\Sigma(2,2,p,q)$, and $\Sigma(p,q,r,s)$, where k is odd and p,q,r,s are pairwise relatively prime. They all satisfies the following properties:

- The mean index of the principal orbit is non-zero;
- They are diffeomorphic to S^5.

Therefore, their connected sum is again S^5 and its mean Euler characteristic is still an invariant of the contact structure.

Using (9.2.2), we find

$$\chi_m(\Sigma(k,2,2,2)) = \frac{2k+1}{2(k+2)}, \quad \chi_m(\Sigma(p,q,r,s)) < \frac{1}{4}, \quad \chi_m(\Sigma(2,2,p,q)) = \frac{1+pq}{2(p+q)}$$

where p,q,r,s are sufficiently large. In particular, since $\chi_m(\Sigma(2,2,3,5)) = 1$, we have

$$\chi_m(\Sigma_1 \# \Sigma_2 \# \Sigma(2,2,3,5)) = \chi_m(\Sigma_1) + \chi_m(\Sigma_2) \tag{9.3.1}$$

by the boundary connected sum formula. Note here that by Corollary 9.2.11 the notation $\chi_m(\Sigma_1 \# \Sigma_2)$ makes sense and χ_m is still an invariant of the contact structure of the connected sum.

Observe that if p,q,r,s are sufficiently large, then

$$\chi_m(\Sigma(p,q,r,s) \# \Sigma(p,q,r,s)) < \frac{1}{4} + \frac{1}{4} - \frac{1}{2} = 0.$$

Therefore, in view of (9.3.1), it suffices to produce rational numbers of the form $\frac{1}{p^l}$ for each prime p. For an odd prime p, we take $k = 3 \cdot p^l - 2$ and we that

$$\chi_m(\Sigma(k,2,2,2) \# \Sigma(k,2,2,2) \# \Sigma(2,2,3,5)) = \frac{2k+1}{k+2} = 2 - \frac{1}{p^l}.$$

For the prime $p = 2$, we compute

$$\chi_m(\Sigma(2,2,2^l-3,2^l+3)) = 2^{l-2} - \frac{1}{2^{l-1}}.$$

This completes the proof. $\square$

Remark 9.3.5. One can try to formulate the corresponding statement for higher dimensions. As the above proof, it is a matter of finding enough examples.

Note that 5-sphere is simply-connected and spin because the second Stiefel-Whitney class vanishes. We can generalize Theorem 9.3.4 to the general 5-manifolds which are simply connected and spin.

Theorem 9.3.6. *Every simply-connected spin* 5*-manifold admits infinitely many pairwise non-contactomorphic contact structures* ξ*, all satisfying* $c_1(\xi) = 0$*, and they realize all rational numbers as their mean Euler characteristic.*

Remark 9.3.7. U. Lararev [16] has recently found infinitely many contact structures on simply-connected spin 5-manifolds, which are all *flexible Weinstein fillable*. Note that the mean Euler characteristic of flexible Weinstein fillable contact manifold is always an half integer, see Corollary 9.2.13. Therefore his contact structures are different from those in Theorem 9.3.6.

We first address the following lemma since its proof contains the main ingredient of a proof of the above theorem.

Lemma 9.3.8. *Every simply connected spin* 5*-manifold is diffeomorphic to a connected sum of Brieskorn manifolds.*

Proof. The main ingredient is Smale's theorem on classification of 5-manifolds, which says that simply-connected spin 5-manifolds M admit a prime decomposition of the form

$$M \cong \#_m S^2 \times S^3 \# M_{q_1} \# \cdots \# M_{q_l},$$

where q_i's are powers of a prime number, and we here use the following conventions.

- The empty connected sum is S^5;
- The manifold M_k is a spin manifold with $H_2(M_k; \mathbb{Z}) \cong \mathbb{Z}_k \oplus \mathbb{Z}_k$.

We now claim that every such a prime manifold M_k can be given by a Brieskorn manifold. Note first that Brieskorn manifolds of dimension 5 are simply-connected, as well as spin. This is because $w_2(\Sigma) = c_1(\xi_\Sigma)$ mod 2, and $c_1(\xi_\Sigma)$ vanishes for Brieskorn manifolds. So Brieskorn manifolds can serve as prime manifolds if they have the proper second homology group.

The unit sphere S^5 is diffeomorphic to the Brieskorn manifold $\Sigma(1,2,2,2)$, and $S^2 \times S^3 \cong ST^*S^3 \cong \Sigma(2,2,2,2)$. For "$M_k$-part", consider Brieskorn manifolds $\Sigma(p,3,3,3)$ and $\Sigma(q,4,4,2)$ for p relatively prime to 3 and q relatively prime to 2. Using the Randell's formula, we can compute its second homology group in $\mathbb{Z}$-coefficient. The result is that

$$H_2(\Sigma(p,3,3,3);\mathbb{Z}) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p, \quad H_2(\Sigma(q,4,4,2);\mathbb{Z}) \cong \mathbb{Z}_q \oplus \mathbb{Z}_q.$$

For example, the first one was computed in Example 7.4.9. This completes the proof. □

We now prove the theorem.

Proof of Theorem 9.3.6. Let Σ be a simply-conneted spin 5-manifold. Then the proof of Lemma 9.3.8 shows that Σ is the boundary of a boundary connected sum of Brieskorn varieties corresponding to

$$S^5,\ \Sigma(2,2,2,2),\ \Sigma(p^k,3,3,3),\ \Sigma(p^k,4,4,2).$$

Note that all of these have non-zero Maslov index of principle orbit as we can directly check using the formula in Proposition 7.7.4. Therefore the mean Euler characteristic of the boundary connected sum of these Brieskorn varieties is an invariant of the contact structure of the boundary. In particular, by connected sum with S^5 if necessary, every rational number can be realized as a mean Euler characteristic of a contact structure of Σ by Theorem 9.3.4. Note further that all contact structures of Σ obtained in this way have zero first Chern number. This completes the proof. □

Remark 9.3.9. A natural open question on the mean Euler characteristic in this context would be whether there is a contact manifold whose mean Euler characteristic is *irrational*. This question is open. The methods for computation of mean Euler characteristic in this thesis suggest that if the boundary of the domain has a periodic flow, then its mean Euler characteristic is necessarily rational. Therefore, to find an irrational number as the mean Euler characteristic, it seems we need a *new* construction of contact manifold or symplectic manifold.

Note that contact structures on S^{2n-1} forms a monoid with respect to the connected sum operation. We denote this monoid by $\Xi(S^{2n-1})$. The identity element is the standard contact sphere (S^{2n-1}, ξ_{st}). We define a submonoid $\Xi_{nice}(S^{2n-1})$ by collecting contact structures on S^{2n-1}, which is convex fillable by Liouville domains such that

- simply connected with vanishing first Chern class,
- admit convenient dynamics.

Note that those conditions are sufficient for well-definedness of their mean Euler characteristics and invariance with respect to the contact structures on S^{2n-1}. Define a map $\tilde{\chi}_m : (\Xi_{nice}(S^5), \#) \longrightarrow (\mathbb{Q}, +)$ by

$$\xi \longmapsto \tilde{\chi}_m(\xi) - \frac{1}{2}.$$

Corollary 9.3.10. *The map* $\tilde{\chi}_m : (\Xi_{nice}(S^5), \#) \longrightarrow (\mathbb{Q}, +)$ *is a monoid homomorphism which is surjective.*

Proof. The assertion that the map $\tilde{\chi}_m$ is a monoid homomorphism is a direct consequence of the boundary connected sum formula, Theorem 9.2.9, and the surjectivity follows from Theorem 9.3.4. $\square$

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국문초록

본 학위 논문에서는 사교 호몰로지(symplectic homology)로 수렴하는 몇 가지 스펙트럴 수열(spectral sequence)을 건설한다. 이 스펙트럴 수열은 모스-보트(Morse-Bott) 테크닉과 그에 따른 자연스러운 액션 여과(action filtration)를 반영한다는 의미에서 모스-보트 스펙트럴 수열이라 부른다. 이는 특히 밀너 다발(Milnor fiber)이라는 특정한 종류의 사교 다양체와 그의 경계로 주어지는 특이점의 고리(link of singularity)에 적용할 수 있다.

특이점의 고리가 특별히 가중된 동형 다항식(weighted homogeneous polynomial)으로 주어지는 경우에는, 대응하는 립 흐름(Reeb flow)이 주기적으로 나타나서 다루기 좋은 대칭성을 갖게 된다. 이러한 주기적 성질을 이용하면, 등변(equivariant) 사교 호몰로지와 그의 평균 오일러 표수(mean Euler characteristic)를 구하는 체계적인 방법을 구현할 수 있다. 본 학위 논문에서는 이러한 기법을 사용해서 특이 접촉 구조(exotic contact structure)들과 관련한 여러 응용을 논한다.

주요어휘: 모스-보스 스펙트럴 수열, 특이점의 고리, 사교 호몰로지, 등변 사교 호몰로지, 밀너 파이버, 평균 오일러 표수

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