이학박사 학위논문

Signature of Surface bundles over Surfaces and Mapping Class Group

(곡면 위의 곡면 다발의 부호수와 사상류 군)

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수리과학부

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Abstract

Signature of Surface bundles over Surfaces and Mapping Class Group

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In this thesis, we study the topological constraints on the signature and the Euler characteristic $(\sigma(X), e(X))$ for smooth 4-manifolds X (or complex surfaces X) which are surface bundles over surfaces with nonzero signature. The first main result is about the improved upper bounds for the minimal base genus function $b(f, n)$ for a fixed fiber genus f and a fixed signature $4n$. In particular, we construct new smooth 4-manifolds with a fixed signature 4 and small Euler characteristic which are surface bundles over surfaces by subtraction of Lefschetz fibrations. They include an example with the smallest Euler characteristic among known examples with non-zero signature. Secondly, we explore possibilities to construct Kodaira fibrations with small signature which are smooth surface bundles over surfaces as ramified coverings of products of two complex curves. To obtain the minimal base genus and the smallest possible signature, we investigate the action of the monodromy of the fibration of pointed curves. Throughout the paper we'll see that the surface mapping class group plays an important role in both constructions and the control of topological invariants.

Key words: signature, surface bundles over surfaces, mapping class group, Lefschetz fibration, Kodaira fibration, Birman exact sequence
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Chapter 1

Introduction

It is a longstanding problem of 4-dimensional manifold theory to determine which simply connected topological 4-manifolds carry smooth structures. The following theorems tell us that there are some topological constraints for the existence of smooth structures.

Theorem 1.0.1 (Rohlin). *If* X *is a simply connected, closed, oriented, spin, smooth* 4*-manifold, then the signature* $\sigma(X)$ *is divisible by* 16*.*

Theorem 1.0.2 (Donaldson). *If* X *is a simply connected, closed, oriented, smooth* 4*-manifold with the definite intersection form* Q_X*, then* Q_X *is diagonalisable, that is,* Q_X *equivalent to* $\pm \oplus (+1)$*.*

Theorem 1.0.3 (Furuta). *If* X *is a simply connected, closed, oriented, smooth* 4*-manifold with the intersection form* $Q_X \cong 2kE_8 \oplus lH$*, then* $l \geq 2|k| + 1$*, equivalently,* $\frac{b_2}{|\sigma|} \geq \frac{10}{8}$*.*

Now the next natural question to ask is the following: what are the topological constraints a smooth, closed, oriented 4-manifold must satisfy in order to support an additional structure such as complex structure, symplectic structure, Lefschetz fibration, or surface bundle over surface. In this thesis, we'll focus on this problem for the case of surface bundle over surface (with nonzero signature).

The constraints on the topology of the underlying smooth 4-manifold X are often measured by the signature $\sigma(X)$ and the Euler characteristic $e(X)$. The Euler characteristic is comparatively easy because it's multiplicative in fiber bundles. Under the assumption that the fundamental group of the base acts trivially on the cohomology of the fiber, the signature is also known to be multiplicative[11]. However, this is not true in general. Atiyah and Kodaira [1, 35], independently found examples of surface bundles over surfaces with non-zero signature as the first counter-example.

First we introduce what we know about the restrictions on $\sigma(X)$ and $e(X)$ of a smooth 4-manifold X which is a surface bundle over a surface.

Theorem 1.0.4 (Kotschick'98)**.** *[37] Let X be an aspherical surface bundle over a surface. Then $2|\sigma(X)| \leq e(X)$. If in addition, X admits a complex structure, then $3|\sigma(X)| \leq e(X)$.*

Kotschick proved this theorem using the results about Seiberg-Witten invariants of symplectic 4-manifolds. Recently, Hamenstadt improved this result as follows.

Theorem 1.0.5 (Hamenstadt'12)**.** *[25] $3|\sigma(X)| \leq e(X)$ for every aspherical surface bundle X over a surface.*

In Example 5.9 in [12], Catanese and Rollenske constructed an example with the largest known ratio between the signature and the Euler characteristic. Precisely, it realises $\frac{3\sigma}{e} = \frac{2}{3}$, equivalently, the Chern slope $\frac{c_1^2}{c_2} = \frac{8}{3}$.

It is easily seen that the signature of any surface bundle is divisible by 4 because it admits an almost complex structure[52]. Furthermore, every multiple of 4 is equal to the signature of some surface bundle X as stated in the following theorem.

Theorem 1.0.6 (Meyer'73)**.** *[43]*

1. *Every oriented surface bundle* $X \to \Sigma_b$ *with fiber genus* $f \leq 2$ *or base genus* $b \leq 1$ *has zero signature.*

2. *For every* $f \geq 3$ *and* $4n \in 4\mathbb{Z}$*, there exists a genus* f *surface bundle* $X \to \Sigma_b$ *with signature* $4n$ *for some* b*.*

In order to prove this theorem, Meyer introduced a 2-cocycle τ_g on the symplectic group $Sp(2g;\mathbb{Z})$ and gave the signature formula for surface bundles over surfaces. We'll revisit Meyer's formula in Section 2. Based on this Meyer's result, Endo[16] started the systemetic study of the following question:

Problem 1.0.7. *For each* $f \geq 3$ *and each* $n \in \mathbb{Z}$*,*

$$b(f,n) := \min\{b \mid \exists\ \Sigma_f\text{-}bundle\ X \to \Sigma_b \ with\ \sigma(X) = 4n\}$$

Determine the value $b(f,n)$*.*

This problem was studied by many 4-manifold topologists and algebraic geometers [16, 51, 14, 5, 7].

Our first main result (Chapter 4 based on [38]) is about the improved upper bounds for this minimal base genus fuction $b(f,n)$.

Theorem 1.0.8 (Lee'15)**.** *[38]*

(a) For every $f \geq 3$ *and* $n \neq 0$*,* $b(f,n) \leq 7|n| + 1$*. In particular, there exists a smooth* 4*-manifold with signature* 4 *which is a* Σ_3*-bundle over* Σ_8*.*
(b) For every $f \geq 5$ *and* $n \neq 0$*,* $b(f,n) \leq 6|n| + 1$*. In particular, there exists a smooth* 4*-manifold with signature* 4 *which is a* Σ_5*-bundle over* Σ_7*.*
(c) For every $f \geq 6$ *and* $n \neq 0$*,* $b(f,n) \leq 5|n| + 1$*. In particular, there exists a smooth* 4*-manifold with signature* 4 *which is a* Σ_6*-bundle over* Σ_6*.*

In particular, a genus 3 surface bundle over a surface of genus 8 with signature 4 we found is the example with the smallest Euler characteristic among known examples with non-zero signature.

The main technique for the construction is the subtraction of Lefschetz fibrations introduced in [14]. This is a kind of surgery operation on Lefschetz fibrations. In fact, both the construction of building blocks and the operations on them can be done by computations in mapping class groups. Before we discuss about this result in detail in Section 4, we give some necessary backgrounds on mapping class groups in Section 3.

Next, we would like to require further that our smooth 4-manifold which is a surface bundle over a surface admits a complex structure. Eventhough it is not immediate to answer if the above examples admit any complex structure or not because they all have even b_1, historically we have a tautological construction, due to Kodaira[35] and Atiyah[1], which allows us to construct complex surfaces with positive signature which are smooth surface bundles over surfaces. We usually call such a surface a Kodaira fibration.

Our second main result(Chapter 5 based on [39]) is about the possibilities to construct Kodaira fibrations with signature 4 as ramified coverings of products of two Riemann surfaces.

Proposition 1.0.9 (Lee-Lönne-Rollenske '16)**.** *[39] The possible numerical invariants of a virtual double étale Kodaira fibration of virtual signature* 4 *are as follows: for each component* D_i *of* D *the ramification order* $r_i = 2$ *and the other invariants can be (up to reordering the* D_i*) given as in Table5.1, where we also collect partial information on realisability.*

In order to classify double étale Galois double Kodaira fibrations with signature 4*, we first make a list of all possible candidates of double étale virtual Kodaira fibrations with virtual signature* 4*, and then check the realisability of each case in the list. The core part is to develop the method to compute the signature of the Kodaira fibration. In the earlier constructions[1, 35, 27, 5, 7], they start with the product* $B \times F$ *of two Riemann surfaces and the divisor* D *inside it and then take an unramified pull-back of the base, which is explicitly chosen but has sufficiently large degree to guarantee the divisibility of the divi-*

Table 1.1: virtual Kodaira fibraions with virtual signature 4

type	b	f	$\|G\|$	$g(D_i)$	(d_i, e_i)	realisability	$((\tilde{f}, \tilde{b}), (\tilde{f}_2, \tilde{b}_2), \tilde{\sigma})$
G_1	2	2	8	2	$(1,1)$	no	((11,9),(81,2),32)
G_2	2	2	4	$(2,2)$	$(1,1),(1,1)$	no	((7,17),(97,2),64)
							((4,17),(49,2),32) $G = \mathbb{Z}/2$)
G_3	3	3	2	$(3,3)$	$(1,1),(1,1)$	no	((6,9),(21,3),16),
							((6,33),(81,3),64)
G_4	3	2	2	$(3,3)$	$(1,2),(1,2)$	no	((4,17),(49,2),32)
C_1	2	2	2	5	$(4,4)$	?	
C_2	3	2	2	5	$(2,4)$	?	
C_3	2	3	2	5	$(4,2)$	?	
C_4	3	3	2	5	$(2,2)$	no	((6,9),(21,3),16)
							((6,17),(41,3),32),
							((6,65),(161,3),128)
C_5	2	5	2	5	$(4,1)$	no	((11,9),(21,5),32)
C_6	3	5	2	5	$(2,1)$	no	((10,65),(145,5),128)
C_7	2	2	4	3	$(2,2)$	no	((7,9),(49,2),32)
							((4,9),(25,2),16) $G = \mathbb{Z}/2$)
C_8	2	3	4	3	$(2,1)$	no	((11,9),(41,3),32)
C_9	2	2	2	$(3,3)$	$(2,2),(2,2)$	?	
C_{10}	2	2	2	$(4,2)$	$(3,3),(1,1)$	?	
C_{11}	2	3	2	$(3,3)$	$(2,1),(2,1)$	no	((7,9),(25,3),32)
C_{12}	2	2	2	$(2,2,3)$	$(1,1),(1,1),(2,2)$	?	

The double Kodaira fibration $S \to \tilde{B} \times F$ coming from each virtual Kodaira fibration in the above list has two different fiberings: one is given by the composition with $pr_1\colon \tilde{B} \times F \to \tilde{B}$, and another is given by the composition with $pr_2\colon \tilde{B} \times F \to F$. We denote the corresponding fiber and base genera by $(\tilde{f}, \tilde{b})$ and $(\tilde{f}_2, \tilde{b}_2)$.

sor. On the other hand, Catanese and Rollenske[12] proved that every virtual Kodaira fibration $(\mathsf{B} \times \mathsf{F}, \mathsf{D}, \theta \colon \pi_1(\mathsf{F} \setminus \mathsf{F} \cap \mathsf{D}) \to \mathsf{G})$ *is realisable after a suitable finite étale pull-back. However, since we want to obtain the minimal base genus and the smallest possible signature, we investigate the minimal degree pull-back among those which grarantee the existence of ramified covering (Proposition 5.2.3). The study of the monodromy action of the fibration of pointed curves* $(\mathsf{F}, \mathsf{F} \cap \mathsf{D})$ *(Theorem 5.3.5) allows us to compute the minimal degree of the required pull-back and hence the signature of Kodaira fibration.*

Before we discuss about this result in detail in Section5, we give some backgrounds necessary to study the monodromy action in Section3. One more thing to remark is that we found some interesting examples by generalising D *to multisections eventhough all the previous examples in the literature come from the pure sections.*

Chapter 2

Signature of fibered 4-manifolds

2.1 Signature of 4-manifolds

For any compact oriented manifold X *of dimension* $4k$*, the cup-product defines a non-degenerate symmetric bilinear form on the middle dimensional cohomology* $H^{2k}(X;\mathbb{R})$*. Let* $b_{2k}^{+}(b_{2k}^{-})$ *denote the maximal dimension of a subspace on which this form is positive (negative) definite. Then the difference* $b_{2k}^{+} - b_{2k}^{-}$ *is defined to be the signature of the (topological) manifold* X*, and usually denoted by* $\sigma(X)$*.*

In this section, we review the proof due to Atiyah and Singer[2] that this topological definition of the signature coincides with the differential geometric definition. In other words, the claim is that the signature $\sigma(X)$ *is the index of a certain elliptic operator on* X *associated to the* $SO(4k)$*-structure.*

Let X *be a* $4k$ *dimensional oriented Riemannian manifold. Then the metric on* TX *induces metrics on the bundles of* p*- forms and hence, by integration over X, inner products on the space* Ω^p*. Hence, we have a Hodge star operator* $* : \Omega^p \to \Omega^{4k-p}$ *satisfying* $*^2 = (-1)^p$*. The bundle* $\bigwedge^*(X)$ *has a canonical first order differential operator* d *which is usually called the exterior derivative and its formal adjoint* d^* *is given by* $d^*(\alpha) = - * d * \alpha$*. We will consider the first* *order operator* $D = d + d^*$. *Since it is self-adjoint,* $D^2 = D^*D = \Delta$ *where* $\Delta = dd^* + d^*d$ *is the Laplacian. From this, it follows that* $\mathrm{Ker} D$ *coincides with the space of harmonic forms.*

We now introduce a map τ *on differential forms defined by*

$$\tau(\alpha) = i^{p(p-1)+2k} * \alpha, \quad \alpha \in \Omega^p.$$

Then, $\tau^2(\alpha) = \alpha$, *i.e.* τ *is an involution, and we can therefore decompose the space* $\Omega = \sum \Omega^p$ *into* $\pm$ *eigenspaces* $\Omega^{\pm}$ *of* τ. *Then we consider the following restriction maps of D*

$$D^+ : \Omega^+ \to \Omega^-, \quad D^- : \Omega^- \to \Omega^+$$

Each is elliptic and they are formal adjoints of each other. If we denote by $H^{\pm}$ *the space of harmonic forms in* $\Omega^{\pm}$, *respectively, then we have*

$$\mathrm{index} D^+ = \dim H^+ - \dim H^-$$

Observe that for each $0 \leq j < 2k$, *the dimensions of the* $+1$ *and* -1 *eigenspaces of* τ *in* $V_j = H^j \bigoplus H^{4k-j}$ *are equal. Now, we get*

$$\mathrm{index} D^+ = \dim H^{2k}_+ - \dim H^{2k}_-$$

Recall that $\sigma(X) = b^+_{2k} - b^-_{2k}$. *By the classical Hodge theorem, we can consider each* $2k$ *dimensional homology class* α *of* X *as the element of either* H^{2k}_+ *or* H^{2k}_-.
For nonzero $\alpha \in H^{2k}_+$,

$$Q_X(\alpha, \alpha) = \int_X \alpha \wedge \alpha = \int_X \alpha \wedge *\alpha = (\alpha, \alpha) > 0$$

while for nonzero $\alpha \in H^{2k}_-$,

$$Q_X(\alpha, \alpha) = \int_X \alpha \wedge \alpha = \int_X \alpha \wedge (- * \alpha) = -(\alpha, \alpha) < 0$$

Therefore, $\mathrm{dim}H^{2k}_{+} = b^{+}_{2k}$ *and* $\mathrm{dim}H^{2k}_{-} = b^{-}_{2k}$.

2.2 Surface bundles over surfaces and Lefschetz fibrations

Surface bundles over surfaces and Lefschetz fibrations constitute a rich source of examples of smooth, symplectic, and complex 4-manifolds.

Lefschetz fibrations (including surface bundles Section 4.1.3) can be described combinatorially by means of their monodromy factorizations.

Theorem 2.2.1 ([30], [40]). *For* $g \geq 2$*, genus* g *Lefschetz fibrations* $f\colon X \to B$ *over connected surfaces are completely determined, up to isomorphism, by their monodromy representations* $\pi_1(B \setminus f(C), b_0) \to \mathrm{Mod}(\Sigma_g)$ *up to equivalence relations.*

In particular, to every genus g *Lefschetz fibration over* S^2*, one can associate a factorization of identity into a product of right-handed Dehn twists in the mapping class group* $\mathrm{Mod}(\Sigma_g)$*. Conversely, given a positive relation* $t_1 t_2 \cdots t_n = 1$ *in* $\mathrm{Mod}(\Sigma_g)$*, one can construct a smooth* 4*- manifold* X *and the corresponding Lefschetz fibration* $f\colon X \to S^2$*. For the detailed explanation, refer to [30, 23].*

Example 2.2.2. The most typical example of a Lefschetz fibration over S^2 is an elliptic fibration with 12 Fishtail fibers on $E(1)$ diffeomorphic to $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}}$. This Lefschetz fibration with torus regular fibers can be constructed from a generic pencil $\{t_0 p_0 + t_1 p_1 \mid [t_0 : t_1] \in \mathbb{CP}^1\}$ of cubic curves in $\mathbb{CP}^2$ by blowing up 9 base points where all the cubic curves intersect. The fishtail fiber is an immersed 2-sphere with one transverse positive double point. This singular fiber is obtained by collapsing a nonseparating simple closed curve, called the vanishing cycle, on the nearby smooth fiber. We must have 12 fishtail fibers since the Euler characteristic of the total space is 12, and moreover the monodromy factorization is equivalent to $(t_a t_b)^6 = 1$ in $\mathsf{Mod}(T^2)$, where $t_a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $t_b = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ if we identify $\mathsf{Mod}(T^2)$ with $SL(2, \mathbb{Z})$ [44]. This positive relation is usually called the 2-chain relation or the one-holed torus relation.

Various Lefschetz fibrations over a surface of positive genus will be introduced in Proposition 4.2.3, 4.2.4 where we obtain from a product of lantern relations, and in Proposition 4.2.5, 4.2.6, 4.2.7 from a product of n*-holed torus relation and lantern relations.*

2.3 Various signature formulas for surface bundles

In this section, we introduce various signature formulas for surface bundles over surfaces from the literature and then we can take the benefit from each formula.

(1) Signature formula derived from the Atiyah-Singer Index theorem [1]

Theorem 2.3.1. *Let* $Z \to X$ *be a differentiable fiber bundle over* X *with a fiber* Y*. Then by Hirzebruch's signature theorem,*

$$\begin{aligned} \mathsf{Sign}(Z) &= \widetilde{\mathcal{L}}(TZ)[Z] \\ &= \{\widetilde{\mathcal{L}}(T_\pi) \cdot \pi^* \widetilde{\mathcal{L}}(TX)\}[Z] \\ &= \{\pi_* \widetilde{\mathcal{L}}(T_\pi) \cdot \widetilde{\mathcal{L}}(TX)\}[X] \end{aligned}$$

The Atiyah-Singer Index theorem for families of elliptic operators applied to the signature operators $D+$ *on the fibers gives*

$$\mathsf{ch}(\mathsf{index} D+) = \pi_*(\widetilde{\mathcal{L}}(T_\pi)) \in H^*(X; \mathbb{Q})$$

where $\mathrm{ch}\colon K(X) \to H^*(X;\mathbb{Q})$ *is the Chern character and* $\pi_* : H^*(Z;\mathbb{Q}) \to H^*(X;\mathbb{Q})$ *is the Gysin homomorphism.*

Therefore,

$$\mathrm{Sign}(Z) = \{\mathrm{ch}(\mathrm{index}D^+) \cdot \widetilde{\mathcal{L}}(X)\}[X].$$

This formula explains why the $\mathrm{Sign}(Z)$ *vanishes under the assumption that* $\rho : \pi_1(X) \to Sp(2g;\mathbb{R})$ *is trivial.* $\mathrm{ch}(\mathrm{index}D^+)$ *is the pullback of the universal characteristic class by the classifying map* $\phi : X \to BSp(2g)$ *and* $\rho = \phi_*$.

(2) Hirzebruch's signature formula for n*-fold cyclic branched covering [27]*

Theorem 2.3.2 (Atiyah, Hirzebruch)**.** *Let* M *be a compact complex manifold and* D *be a divisor on* M *of the form* $D = D_1 - D_2$ *where* D_1, D_2 *are smooth disjoint curves. Suppose for the holomorphic line bundle* $\{D\}$ *defined by* D *that the first Chern class* $c_1(\{D\})$ *is divisible by* n*. Then there exists a compact complex manifold* $\tilde{M}$ *which is an* n*-fold cyclic cover of* M *branched along* D*. Moreover, its signature is given by*

$$\sigma(\tilde{M}) = n\sigma(M) - \frac{n^2-1}{3n}D^2 \tag{2.3.1}$$

This construction and the signature formula made it possible to construct the first example and then many more examples of complex surfaces which are surface bundles over surfaces with non-zero signature.[35, 1, 27, 5, 7] It was done by starting with the product of two Riemann surfaces $B \times F$ *and* $D = \Gamma_1 - \Gamma_2$ *two disjoint graphs. In section 5, we'll introduce the signature formula generalising* (2.3.1) *for more general construction.*

(3) As an evaluation of the first Chern class of the Hodge bundle [29, 5]

If we fix a Riemannian metric on a surface bundle $f\colon X \to B$ *with fiber* Σ_g*, a surface of genus* g*, then the fibers become complex curves and thus there's an induced map* $\phi_f\colon B \to \mathcal{M}_g$*, where* $\mathcal{M}_g$ *is the moduli space of genus* g *complex curves. Non-torsion part of* $H^2(\mathcal{M}_g;\mathbb{Z})$ *has rank one and is generated* *by the first Chern class of the Hodge bundle. Since* $\mathsf{det}(\mathbb{E})$ *is ample on* $\mathcal{M}_g$ *for every* $g \geq 3$, $\phi_f^*(c_1(\mathbb{E}))$ *will evaluate non-trivially on* B *for any non-constant holomorphic orbi-map* ϕ_f.

For such surfaces the signature is indeed positive, for example, because of the formula

$$\sigma(S) = 4\int_B \phi_f^* c_1(\mathbb{E})$$

.

(4) Meyer's signature formula using an explicit 2*-cocycle [43]*

Theorem 2.3.3. *[42, 43] Let* $\pi : E \to \Sigma_h$ *be an oriented surface bundle over an oriented closed surface* Σ_h *with genus* $h \geq 1$, *and with fiber* Σ_g *an oriented closed surface of genus* $g \geq 2$. *We denote its monodromy homomorphism by* $\chi \colon \pi_1(\Sigma_h) \to \mathsf{Mod}(\Sigma_g)$. *Then the signature of the total space is given by evaluating the "signature class" on the fundamental class of the base:*

$$\sigma(E) = -\langle (\psi \circ \chi)^*[\tau_g], [\Sigma_h]\rangle. \tag{2.3.2}$$

where $\psi \colon \mathsf{Mod}(\Sigma_g) \to \mathsf{Sp}(2g;\mathbb{Z})$ *is a canonical symplectic representation, and the cohomology class* $[\tau_g] \in H^2(\mathsf{Sp}(2g;\mathbb{Z});\mathbb{Z})$ *is represented by the signature* 2*-cocyle on* $\mathsf{Sp}(2g;\mathbb{Z})$.

Alternatively, we may interpret this formula 2.3.2 as the second cohomology class of $\mathsf{Mod}(\Sigma_g)$ *evaluated on the second homology class of* $\mathsf{Mod}(\Sigma_g)$ *as stated in Theorem 3.2.8. By proving this signature formula, Meyer proved that the signature of any surface bundle over a surface is divisible by* 4 *and conversely, every multiple of* 4 *can be realised as the signature of some surface bundle over a surface.*

The details on the signature 2*-cocyle* τ_g *will be discussed in Section 4.3 and those on* $H_2(\mathsf{Mod}(\Sigma_g);\mathbb{Z})$ *in Section 3.2.*

Chapter 3

Mapping class group

Let $\Sigma_{g,p}^{b}$ be a compact oriented surface of genus g with b boundary components and p marked points in its interior. When the surface has no boundary components or no marked points, we often use the notation Σ_g omitting the 0 for b or p. The mapping class group $\mathrm{Mod}(\Sigma_{g,p}^{b})$ of a surface $\Sigma_{g,p}^{b}$ is the group of isotopy classes of orientation-preserving self-homeomorphisms of $\Sigma_{g,p}^{b}$ that restrict to the identity on the boundary and leave the set of marked points invariant. The isotopies are required to fix the boundary pointwisely and preserve the marked points setwisely.

Definition 3.0.4 (Dehn twist)**.** Let α be a simple closed curve on an oriented surface Σ_g. The (right-handed) Dehn twist, denoted by t_α, about a simple closed curve α is a self-homeomorphism of Σ_g, supported on a regular neighborhood A of α, defined by

$$(r, \theta) \mapsto (r, \theta + 2\pi r) \quad \text{on } A,$$

where A is parametrized by $(r, \theta) \quad \text{with} \quad 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi$.

Remark 3.0.5. This is a diffeomorphism obtained by cutting the surface Σ_g along α, twisting $360°$ to the right and then regluing.

The isotopy class of t_α only depends on the isotopy class a of α, and hence t_a is well-defined element of the mapping class group $\mathrm{Mod}(\Sigma_g)$.

The monodromy around a Lefschetz critical value is known to be the right-handed Dehn twist about the vanishing cycle on the general fiber. (See Section4.1.3)

Theorem 3.0.6 (Dehn, Lickorish, Humphries). $\mathrm{Mod}(\Sigma_g)$ *is generated by finitely many Dehn twists.*

3.1 Presentation of Mapping class group

The mapping class groups are finitely presented. In this section, we introduce Wajnryb's presentation for $\mathrm{Mod}(\Sigma_g)$.

Theorem 3.1.1 (Wajnryb's presentation of mapping class group[53]). *Let* $g \geq 2$*. The mapping class group* $\mathrm{Mod}(\Sigma_g)$ *admits a presentation with generators* $a_0, a_1, \cdots, a_{2g}$*, the usual Humphries generators, and with defining relations*

A. *(Commutativity relations)* $[a_i, a_j] = 1$ *if* $i(a_i, a_j) = 0$

B. *(Braid relations)* $a_i a_j a_i = a_j a_i a_j$ *if* $i(a_i, a_j) = 1$

C. *(3-Chain relation)* $(a_1 a_2 a_3)^4 = a_0 (a_4 a_3 a_2 a_1^2 a_2 a_3 a_4) a_0 (a_4 a_3 a_2 a_1^2 a_2 a_3 a_4)^{-1}$

D. *(Lantern relation)* $a_0 (t_2 t_1)^{-1} a_0 (t_2 t_1) t_2^{-1} a_0 t_2 = a_1 a_3 a_5 w a_0 w^{-1}$ *where*

   (a) $t_2 = a_4 a_3 a_5 a_4$

   (b) $t_1 = a_2 a_1 a_3 a_2$

   (c) $w = a_6 a_5 a_4 a_3 a_2 (t_2 a_6 a_5)^{-1} a_0 (t_2 a_6 a_5)(a_4 a_3 a_2 a_1)^{-1}$

E. *(Hyperelliptic relation)* $[a_{2g+1}, a_{2g} a_{2g-1} \cdots a_3 a_2 a_1^2 a_2 a_3 \cdots a_{2g-1} a_{2g}] = 1$

We'll briefly explain why each of the relations holds in Section 4.1.2.

3.2 Homology of Mapping class group

(Co)Homology of a group is defined to be the (co)homology of a classifying space of a group. We can associate to each group G*, a classifying space* BG*, a connected topological space which classifies the principal* G *bundles over any compact Hausdorff space* X *by the one-to-one correspondence* $\mathrm{Prin}_G(X) \cong [X, BG]$*. In particular, the oriented surface bundles with the fiber* F *of genus* g *are classified by the homotopy classes of classifying maps from the base to the classifying space of the structure group* $\mathrm{Homeo}^+(F)$*. On the other hand,* $\mathrm{Homeo}^+(F)$ *is known to be homotopy equivalent to the discrete topological group* $\mathrm{Mod}(F)$ *for any* $g \geq 2$*. Therefore, we have the isomorphism* $H^*(B\mathrm{Homeo}^+(F); \mathbb{Z}) \cong H^*(\mathrm{Mod}(F); \mathbb{Z})$ *which allows us to consider the element of* $H^*(\mathrm{Mod}(F); \mathbb{Z})$ *as the characteristic class of the surface bundle. In this section, we review the definition of group homology, the known results on the low dimensional (co)homology of mapping class groups, and its relation to the Meyer's signature formula.*

Definition 3.2.1. For any (discrete) group G, a classifying space BG is defined to be a topological space such that
(1) $\pi_1(BG) = G$
(2) universal cover of BG is contractible.

Definition 3.2.2 (Group Homology)**.** $H_*(G) := H_*(BG)$

Remark 3.2.3. The above definition is well-defined because the classifying space BG is unique up to homotopy equivalence.

Since a classifying space BG *can be chosen as a CW-complex, we can get the cellular chain complex* $(C_n(BG), \partial_n)$ *of* BG *from the cellular chain complex* $(C_n(EG), \partial_n)$ *of its universal cover* EG *by dividing out the free* G*-action. Therefore we have the following equivalent definition for group homology.*

Definition 3.2.4 (algebraic definition of Group Homology)**.** Let $F : \ldots F_k \to \ldots \to F_0 \to \mathbb{Z} \to 0$ be a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$. We denote the complex obtained from F by tensoring with the trivial $\mathbb{Z}\mathsf{G}$-module $\mathbb{Z}$ by $\mathsf{F}_\mathsf{G} : \cdots \to \mathbb{Z} \otimes_{\mathbb{Z}\mathsf{G}} \mathsf{F}_k \to \cdots \to \mathbb{Z} \otimes_{\mathbb{Z}\mathsf{G}} \mathsf{F}_0 \to 0$. Then we define $\mathsf{H}_*(\mathsf{G}) := \mathsf{H}_*(\mathsf{F}_\mathsf{G})$

Theorem 3.2.5. *For any $g \geq 3$, the group $\mathsf{H}_1(\mathsf{Mod}(\Sigma_g); \mathbb{Z})$ is trivial. In other words, $\mathsf{Mod}(\Sigma_g)$ is perfect, that is $\mathsf{Mod}(\Sigma_g) = [\mathsf{Mod}(\Sigma_g), \mathsf{Mod}(\Sigma_g)]$.*

It is easy to prove this theorem. Since $\mathsf{H}_1(\mathsf{Mod}(\Sigma_g); \mathbb{Z})$ is the abelianization of $\mathsf{Mod}(\Sigma_g)$, it is generated by the class τ of any Dehn twist about a nonseparating simple closed curve and for $g \geq 3$ this class τ is trivial by the lantern relation. Alternatively, we can explicitly write every Dehn twist as the product of two commutators, and then use the Theorem 3.0.6. In Section 4.2, this Theorem 3.2.5 makes it possible to change arbitrary part of the monodromy factorization of a Lefschetz fibration into the monodromy of the bundle part over the positive genus.

Theorem 3.2.6 ([26, 36, 19])**.** *For any $g \geq 4$, the group $\mathsf{H}_2(\mathsf{Mod}(\Sigma_g); \mathbb{Z}) \cong \mathbb{Z}$.*

This theorem was first proved by Harer but his proof relies on the modification of the Hatcher-Thurston complex and it is very complicated. Instead, we have a simpler proof, proposed by Pitsch and then extended by Korkmaz and Stipsicz, based on the presentation of the mapping class group and the following Hopf theorem.

Theorem 3.2.7 (Hopf,[10])**.** *Suppose that a group G is given as a quotient F/R, where F is free. Then $\mathsf{H}_2(\mathsf{G}) \cong \mathsf{R} \cap [\mathsf{F}, \mathsf{F}]/[\mathsf{R}, \mathsf{F}]$.*

By Theorem3.2.5, for every $g \geq 3$, $\mathsf{H}^2(\mathsf{Mod}(\Sigma_g); \mathbb{Z}) \cong \mathsf{Hom}(\mathsf{H}_2(\mathsf{Mod}(\Sigma_g); \mathbb{Z}), \mathbb{Z})$. On the other hand, the Meyer 2-cocycle τ_g on the symplectic group $\mathsf{Sp}(2g; \mathbb{Z})$ represents the second cohomology class $[\tau_g] \in \mathsf{H}^2(\mathsf{Sp}(2g; \mathbb{Z}); \mathbb{Z})$ (Section 4), and hence after taking the pull-back by the symplectic representation $\psi \colon \mathsf{Mod}(\Sigma_g) \to \mathsf{Sp}(2g; \mathbb{Z})$, we get the cohomology class $\psi^*[\tau_g] \in \mathsf{H}^2(\mathsf{Mod}(\Sigma_g); \mathbb{Z})$. Now we are ready to introduce the Meyer's signature formula for a surface bundle over surface.

Theorem 3.2.8 (Meyer'73,[43]). *Let $E \to \Sigma_h$ be an oriented surface bundle over a surface Σ_h with fiber Σ_g whose monodromy homomorphism is denoted by $\chi\colon \pi_1(\Sigma_h) \to \mathsf{Mod}(\Sigma_g)$. Then the signature of the total space is given by*

$$\sigma(E) = -\langle \psi^*[\tau_g], \chi_*[\Sigma_h] \rangle.$$

3.3 Birman exact sequence

There are two important exact sequences of mapping class groups. One is the boundary capping sequence and the other is the Birman exact sequence. Here we focus on the latter which explains the relation between two mapping class groups $\mathsf{Mod}(\Sigma_{g,p})$ and $\mathsf{Mod}(\Sigma_g)$. First, we give some necessary definitions.

Definition 3.3.1 (Configuration spaces).

1. The configuration space $C^{ord}(\Sigma_g, n)$ of ordered n-tuples of Σ_g is the subspace of the n-fold product space $\prod_1^n \Sigma_g$ defined by

$$C^{ord}(\Sigma_g, n) = \{(z_1, \cdots, z_n) \in \prod_1^n \Sigma_g \mid z_i \neq z_j, i \neq j\}.$$

2. The configuration space $C(\Sigma_g, n)$ of unordered n distinct points in Σ_g is the quotient space of $C^{ord}(\Sigma_g, n)$ under the free action by the symmetric group S_n

$$C(\Sigma_g, n) = C^{ord}(\Sigma_g, n)/\Sigma_n.$$

Definition 3.3.2 (Braid groups).

1. $\mathsf{Br}_n(\Sigma_g) := \pi_1(C(\Sigma_g, n))$ is called the n-stranded surface braid group of Σ_g.

2. $\mathsf{PBr}_n(\Sigma_g) := \pi_1(C^{ord}(\Sigma_g, n))$ is called the n-stranded pure surface braid group of Σ_g.

It is a generalised notion of usual braid group $B_n = \mathsf{Br}_n(D^2)$ and the pure braid group $P_n = \mathsf{PBr}_n(D^2)$.

Theorem 3.3.3 (Birman exact sequence, [4, 19]). *Let S be a closed surface with genus $g(S) \geq 2$ and S^* be the surface obtained from S by marking a point x in the interior of S. Then we have the following short exact sequence.*

$$1 \to \pi_1(S, x) \xrightarrow{\mathsf{Push}} \mathsf{Mod}(S^*) \xrightarrow{\mathsf{Forget}} \mathsf{Mod}(S) \to 1$$

Theorem 3.3.4 (Birman exact sequence, generalised, [4, 19]). *Let S be a closed surface with genus $g(S) \geq 2$ and S^* be the surface obtained from S by marking n points in the interior of S. Then we have the following short exact sequence.*

$$1 \to \pi_1(C(S, n)) \xrightarrow{\mathsf{Push}} \mathsf{Mod}(S^*) \xrightarrow{\mathsf{Forget}} \mathsf{Mod}(S) \to 1.$$

Now we describe the push map (Birman called it a spin map [21]) which appeared in the above Birman exact sequence. A push map is a special type of self-homeomorphisms of a surface with marked points, which lives in the kernal of the Forget map in the Birman exact sequence.

Let $\alpha : [0, 1] \to S$ be a simple loop in S based at the point x. Then there exists an isotopy $f_t : S \to S$ supported in a small neighborhood $N(\alpha)$ of the loop $\alpha([0, 1])$ such that $f_0 = \mathrm{id}$ and $f_t(x) = \alpha(t)$. If we parametrize $N(\alpha)$ by cylindrical coordinates $(\theta, y), 0 \leq \theta \leq 2\pi, -1 \leq y \leq 1$, we can take a specific isotopy of the annulus $N(\alpha)$ given by

$$f_t(\theta, y) = (\theta + 2\pi t(1 - y^2), y).$$

Since this map is identity on the two boundary components of the annulus for each t, each f_t may be extended using the identity map on $S - N(\alpha)$ to a self-homeomorphism of the whole surface S.

The homeomorphism of S obtained at the end of this isotopy is defined to be the push map $\mathsf{Push}(\alpha)$ along α. We usually abuse the same notation $\mathsf{Push}(\alpha)$ for the isotopy class of this push map in the marked mapping class group $\mathsf{Mod}(S, x)$. The reason why we named it as "Push map" is that the isotopy f_t pushes the marked point x around the core of the annulus. In fact,

$\mathsf{Push}(\alpha)$ *only depends on the homotopy class* $[\alpha]$ *of the loop* α*. In other words, we have a well-defined push map* $\mathsf{Push}\colon \pi_1(S, x) \to \mathsf{Mod}(S^*)$.

We can observe the following fact from the definition:

Fact 3.3.5. *Let* α *be a simple loop in a surface* S *representing an element of* $\pi_1(S)$*. Then*

$$\mathsf{Push}([\alpha]) = t_a t_b^{-1}$$

where a *and* b *are the isotopy classes of the simple closed curves in* $\hat{S} = S - \{x\}$ *obtained by pushing* α *off itself to the left and right, respectively.*

Chapter 4

Surface bundles over surfaces with a fixed signature

In this chapter, we prove the following two main theorems.

Theorem 4.0.6. *(a) For every* $f \geq 3$ *and* $n \neq 0$, $b(f, n) \leq 7|n| + 1$. *In particular, there exists a smooth* 4*-manifold with signature* 4 *which is a* Σ_3*-bundle over* Σ_8.
(b) For every $f \geq 5$ *and* $n \neq 0$, $b(f, n) \leq 6|n| + 1$. *In particular, there exists a smooth* 4*-manifold with signature* 4 *which is a* Σ_5*-bundle over* Σ_7.
(c) For every $f \geq 6$ *and* $n \neq 0$, $b(f, n) \leq 5|n| + 1$. *In particular, there exists a smooth* 4*-manifold with signature* 4 *which is a* Σ_6*-bundle over* Σ_6.

Remark 4.0.7. [37] We may think of $b(f, n)$ as the minimal genus of the surfaces representing the n times generator of $H_2(\mathsf{Mod}(\Sigma_f); \mathbb{Z})/\mathsf{Tor}$ for fixed $f \geq 3$ and n.

On the other hand, the lower bound for $b(f, n)$ *was also investigated. Kotschick [37] proved* $b(f, n) \geq \frac{2|n|}{f-1} + 1$, *and Hamenstadt[25] proved* $b(f, n) \geq \frac{3|n|}{f-1} + 1$. *Combining the latter with our result, we have* $3 \leq b(3, 1) \leq 8$, $2 \leq b(5, 1) \leq 7$, *and* $2 \leq b(6, 1) \leq 6$.

It is not hard to see that $\frac{b(f,n)}{n}$ *converges. Now we define* $G_f := \lim_{n\to\infty} \frac{b(f,n)}{n}$ *and improve a priori the upper bound for* G_f *that appeared in [14].*

Theorem 4.0.8. *For every odd* $f \geq 3$*,* $G_f \leq \frac{14}{f-1}$.

Remark 4.0.9. As far as I know, this is the best known upper bound for $f = 3$ or every odd f of the form $3k+1, 3k+2$. In fact, for some other f's, better upper bounds are known : for even $f \geq 4$, $G_f \leq \frac{6}{f-2}$[5], and for $f = 3k \geq 6$, $G_f \leq \frac{9}{f-2}$[7].

4.1 Preliminaries

4.1.1 Signature

Let M *be a compact oriented topological manifold of dimension* $4m$*. Since* M *is oriented, it admits the fundamental class* $[M] \in H_{4m}(M, \partial M)$*.*

Definition 4.1.1. The symmetric bilinear form $Q_M : H^{2m}(M, \partial M) \times H^{2m}(M, \partial M) \to \mathbb{Z}$ defined by $Q_M(a, b) := \langle a \cup b, [M] \rangle$ is called the intersection form of M.

Remark 4.1.2. In the smooth case, we can understand Q_M above as the algebraic intersection number of smoothly embedded oriented submanifolds in M representing the Poincaré duals of a and b.

If a *or* b *is a torsion element, then* Q_M *vanishes, and hence* Q_M *descends to the cohomology modulo torsion.*

Definition 4.1.3. The signature of M, denoted by $\sigma(M)$, is defined to be the signature of the symmetric bilinear form Q_M on $H^{2m}(M, \partial M)/\mathrm{Tor}$. If the dimension of M is not divisible by 4, $\sigma(M)$ is defined to be zero.

4.1.2 Mapping class group

Let Σ_g^r *be an oriented surface of genus* g *with* r *boundary components and let* Σ_g *be a closed oriented surface of genus* g*. The mapping class group* $\mathrm{Mod}(\Sigma_g^r)$

of Σ_g^r *is defined to be the group of isotopy classes of orientation preserving self-homeomorphisms which are identity on each boundary component. Based on the theorem of Dehn, we have a surjective homomorphism* $\pi : \mathsf{F}(\mathsf{S}) \to \mathsf{Mod}(\Sigma_g)$, *where* $\mathsf{F}(\mathsf{S})$ *is the free group generated by the generating set* S *consisting of all the Dehn twists over all isotopy classes of simple closed curves on* Σ_g. *Set* $\mathsf{R} := \mathsf{Ker}\pi$ *and call each word* w *in the generators* S *of* $\mathsf{Mod}(\Sigma_g)$ *a relation of* $\mathsf{Mod}(\Sigma_g)$ *if* $w \in \mathsf{R}$. *Now, let us review some famous relations of mapping class groups.*

Let a *and* b *be two simple closed curves on* Σ_g. *If* a *and* b *are disjoint, then the supports of the Dehn twists* t_a *and* t_b *can be chosen to be disjoint. Hence, there exist commutativity relations* $t_a t_b t_a^{-1} t_b^{-1}$ *for any disjoint simple closed curves* a *and* b. *If* a *intersects* b *transversely at one point, then there exists a braid relation* $t_a t_b t_a t_b^{-1} t_a^{-1} t_b^{-1}$. *It can be derived from more general fact that* $f t_a f^{-1} = t_{f(a)}$ *in* $\mathsf{Mod}(\Sigma_g)$ *for any simple closed curve* a *on* Σ_g *and any orientation preserving homeomorphism* f *of* Σ_g. *For braid relations, we will take the latter general form* $f t_a f^{-1} t_{f(a)}^{-1}$. *Consider the planar surface* Σ_0^4 *with boundary components* a, b, c, *and* d. *On the left hand side of Figure* 1, *the boundary curves* a, b, c, *and* d *are in black and the interior curves* x, y, *and* z *are in different colors. One can easily check that* $t_a t_b t_c t_d = t_z t_y t_x$ *holds in* $\mathsf{Mod}(\Sigma_0^4)$ *by applying the Alexander method, and we call* $t_d^{-1} t_c^{-1} t_b^{-1} t_a^{-1} t_z t_y t_x$ *the lantern relations for all embedded subsurfaces* $\Sigma_0^4 \hookrightarrow \Sigma_g$. *For the* k*-chain relations and any other details for mapping class groups, refer to [19]. One can also deduce the star relations* $t_{\delta_3}^{-1} t_{\delta_2}^{-1} t_{\delta_1}^{-1} (t_{\alpha_1} t_{\alpha_2} t_{\alpha_3} t_\beta)^3$ *supported on any embedded subsurfaces* $\Sigma_1^3 \hookrightarrow \Sigma_g$. *See Figure* 4 *as an example.*

We say that two simple closed curves a *and* b *on* Σ_g *are topologically equivalent if there exists a homeomorphism of* Σ_g *sending* a *to* b. *Similarly, the two collections* $\{a_1, \cdots, a_n\}$ *and* $\{b_1, \cdots, b_n\}$ *of simple closed curves on* Σ_g *are called topologically equivalent if there exists a homeomorphism of* Σ_g *sending* a_i *to* b_i *simultaneously for all* $1 \le i \le n$. *To simplify the notation in the rest of this paper, we will use the notation* $w_1^{w_2}$ *for the conjugation* $w_2^{-1} w_1 w_2$.

4.1.3 Lefschetz fibrations and surface bundles

Definition 4.1.4. Let X be a compact oriented 4-manifold, and B a compact oriented 2-manifold. A smooth surjective map $f : X \to B$ is called a Lefschetz fibration if for each critical point $p \in X$ there are local complex coordinates (z_1, z_2) on X around p and z on B around $f(p)$ compatible with the orientations and such that $f(z_1, z_2) = z_1^2 + z_2^2$.

It follows that f has only finitely many critical points, and we may assume that f is injective on the critical set $C = \{p_1, \cdots, p_k\}$. A fiber of f containing a critical point is called a singular fiber, and the genus of f is defined to be the genus of the regular fiber. Notice that if $\nu(f(C))$ denotes an open tubular neighborhood of the set of critical values $f(C)$, then the restriction of f to $f^{-1}(B - \nu(f(C)))$ is a smooth surface bundle over $B - \nu(f(C))$.

For a smooth surface bundle $f : E \to B$ with a fixed identification ϕ of the fiber over the base point p of B with a standard genus g surface Σ_g, the monodromy representation of f is defined to be an antihomomorphism $\chi : \pi_1(B, p) \to \mathsf{Mod}(\Sigma_g)$ defined as follows. For each loop $l : [0, 1] \to B$, $l^*(E) \to [0, 1]$ is trivial and hence there exists a parametrization $\Phi : [0, 1] \times \Sigma_g \to f^{-1}(l[0, 1]))$ with $\Phi|_{0\times\Sigma_g} = \phi^{-1}$. Now define $\chi([l]) := [\Phi|^{-1}_{0\times\Sigma_g} \circ \Phi|_{1\times\Sigma_g}]$. For the genus g Lefschetz fibration $f : X \to B$ with a fixed identification of the fiber with Σ_g, we define the monodromy representation of f to be the monodromy representation of the surface bundle $f : X - f^{-1}(f(C)) \to B - f(C)$.

A Lefschetz singular fiber can be described by its monodromy. By looking at the local model of the Lefschetz critical point, one can see that the singular fiber is obtained from the regular fiber by collapsing a simple closed curve, called the vanishing cycle. One can also observe that the monodromy along the loop going around one Lefschetz critical value is given by the right-handed Dehn twist along the vanishing cycle. Hence, from the monodromy *representation* χ *of a Lefschetz fibration, after fixing the generating system* $\{a_1, b_1, \cdots, a_h, b_h, l_1, \cdots, l_k\}$ *of* $\pi_1(B - f(C), p)$*, we get the global monodromy* $\prod_{i=1}^{h}[\chi(a_i), \chi(b_i)] \prod_{j=1}^{k} t_{\gamma_j}$ *since we have* $\chi(l_j) = t_{\gamma_j}$ *for each* $j = 1, \cdots, k$*; and when* B *is closed,* $\prod_{i=1}^{h}[\chi(a_i), \chi(b_i)] \prod_{j=1}^{k} t_{\gamma_j} = 1$ *in* $\mathsf{Mod}(\Sigma_g)$*, and this is called the monodromy factorization of a Lefschetz fibration. Conversely, a factorization* $\prod_{i=1}^{h}[\alpha_i, \beta_i] \prod_{j=1}^{k} t_{\gamma_j} = 1$ *of identity in* $\mathsf{Mod}(\Sigma_g)$ *gives rise to a genus* g *Lefschetz fibration over* Σ_h*. For this, first observe that a product* $\prod_{i=1}^{h}[\alpha_i, \beta_i]$ *of* h *commutators in* $\mathsf{Mod}(\Sigma_g)$ *gives a* Σ_g *bundle over* Σ_h^1*. Also, a product* $\prod_{j=1}^{k} t_j$ *of right-handed Dehn twists* t_j *in* $\mathsf{Mod}(\Sigma_g)$ *gives a genus* g *Lefshetz fibration over* D^2*. By combining these two constructions, a word* $w = \prod_{i=1}^{h}[\alpha_i, \beta_i] \prod_{j=1}^{k} t_j \in \mathsf{Mod}(\Sigma_g)$ *gives the genus* g *Lefschetz fibration over* Σ_h^1*, and if* $w = 1$ *in* $\mathsf{Mod}(\Sigma_g)$ *we can close up to a Lefschetz fibration over* Σ_h*.*

Two Lefschetz fibrations $f_1 : X_1 \to B_1, f_2 : X_2 \to B_2$ *are called isomorphic if there exist orientation preserving diffeomorphisms* $H : X_1 \to X_2$ *and* $h : B_1 \to B_2$ *such that* $f_2 \circ H = h \circ f_1$*. The isomorphism class of a Lefschetz fibration is determined by an equivalence class of its monodromy representation. Oriented genus* g *surface bundles over surfaces of genus* h *are classified, up to isomorphism, by homotopy classes of the classifying map* $\Sigma_h \to \mathsf{BDiff}^+\Sigma_g$ *since the structure group is* $\mathsf{Diff}^+\Sigma_g$*. If* $g \geq 2$*, then according to the Earle-Eells theorem and the* $K(\pi, 1)$ *theory, they are classified by the conjugacy classes of the induced homomorphisms* $\pi_1(\Sigma_h) \to \mathsf{Mod}(\Sigma_g)$*. Therefore,* $\prod_{i=1}^{h}[\alpha_i, \beta_i] = 1$ *in* $\mathsf{Mod}(\Sigma_g)$ *, up to global conjugations, determines the genus* g *surface bundle over a surface of genus* h*.*

4.2 Subtraction of Lefschetz fibrations

In the study of manifold theory, a common way to construct a new manifold from a given manifold is a cut-and-paste operation. To construct a new 4-manifold which is a surface bundle over a surface, H. Endo, M. Korkmaz, *D. Kotschick, B. Ozbagci and A. Stipsicz introduced an operation, called the "subtraction of Lefschetz fibrations", in [14]. Let us first explain it here in a generalized version.*

Let $f : X \to B_1$ *be a Lefschetz fibration with* m *critical values* $q_1^{(1)}, \cdots, q_m^{(1)}$ *and let* $g : Y \to B_2$ *be another Lefschetz fibration with* $k \leq m$ *critical values* $q_1^{(2)}, \cdots, q_k^{(2)}$*. Suppose that* $f : f^{-1}(D_1) \to D_1$ *and* $g : g^{-1}(D_2) \to D_2$ *are isomorphic where* $D_1 \subset B_1$ *is a disk containing some critical values* $q_1^{(1)}, \cdots, q_k^{(1)}$ *and* $D_2 \subset B_2$ *is a disk containing* $q_1^{(2)}, \cdots, q_k^{(2)}$*. Then, the manifolds* $X \backslash f^{-1}(D_1)$ *and* $Y \backslash g^{-1}(D_2)$ *have a diffeomorphic boundary, and after reversing the orientation of one of them, this diffeomorphism can be chosen to be fiber-preserving and orientation-reversing. Fix such a diffeomorphism* ϕ *and then glue* $\overline{Y \backslash g^{-1}(D_2)}$*, the manifold* $Y \backslash g^{-1}(D_2)$ *with the reversed orientation, to* $X \backslash f^{-1}(D_1)$ *using this diffeomorphism* ϕ*. Note that the resulting manifold, denoted by* $X - Y$*, inherits a natural orientation and admits a smooth fibration* $f \cup g : X \backslash f^{-1}(D_1) \cup \overline{Y \backslash g^{-1}(D_2)} \to B_1 \# B_2$*. This is a Lefschetz fibration with* $m - k$ *singular fibers. In particular, for* $k = m$*, we get a surface bundle over a surface. In general, after repeatedly subtracting Lefschetz fibrations, we get* $X - Y_1 - Y_2 - \cdots - Y_n$*, a surface bundle over a surface, under the following assumptions. Let* $f : X \to B_0$ *be a Lefschetz fibration with* m *critical values* $\{q_{1,1}^{(0)}, \cdots, q_{1,k_1}^{(0)}, q_{2,1}^{(0)}, \cdots, q_{2,k_2}^{(0)}, \cdots, q_{n,1}^{(0)}, \cdots, q_{n,k_n}^{(0)}\}$ *and* $g_1 : Y_1 \to B_1$, $\cdots$ *,* $g_n : Y_n \to B_n$ *be Lefschetz fibrations with critical values* $\{q_1^{(1)}, \cdots, q_{k_1}^{(1)}\}$, $\cdots$ *,* $\{q_1^{(n)}, \cdots, q_{k_n}^{(n)}\}$*, respectively. We assume that* $k_1 + \cdots + k_n = m$ *and that* $f : f^{-1}(D_{0,i}) \to D_{0,i}$ *is isomorphic to* $g_i : g_i^{-1}(D_i) \to D_i$ *for each* $1 \leq i \leq n$*, where each* $D_{0,i} \subset B_0$ *is a disk containing* $q_1^{(0)}, \cdots q_{k_i}^{(0)}$ *and* $D_i \subset B_i$ *is a disk containing* $q_1^{(i)}, \cdots, q_{k_i}^{(i)}$*.*

In order to use the subtraction method explained above, we need to construct the building blocks X *and* Y_i*'s. First, we describe various gluing pieces* Y_i*.*

Proposition 4.2.1. *[14] Let* $f \geq 3$ *and let* a *be a simple closed curve on* Σ_f*. In the mapping class group* $\mathrm{Mod}(\Sigma_f)$*,*
(a) t_a^2 *can be written as a product of two commutators,*

(b) if a *is nonseparating, then* t_a^4 *can be written as a product of three commutators.*

Remark 4.2.2. This proposition gives us two genus $f \geq 3$ Lefschetz fibrations $Y_1 \to \Sigma_2$ and $Y_2 \to \Sigma_3$ whose monodromy factorizations are given by $[f_1, g_1][f_2, g_2]t_a^2 = 1$ and $[f_3, g_3][f_4, g_4][f_5, g_5]t_a^4 = 1$ for some mapping classes $f_i, g_i \in \mathrm{Mod}(\Sigma_f)$ for $1 \leq i \leq 5$. Generally, for every n, we can obtain a Lefschetz fibration which has n singular fibers and the monodromy t_a^n using a daisy relation.

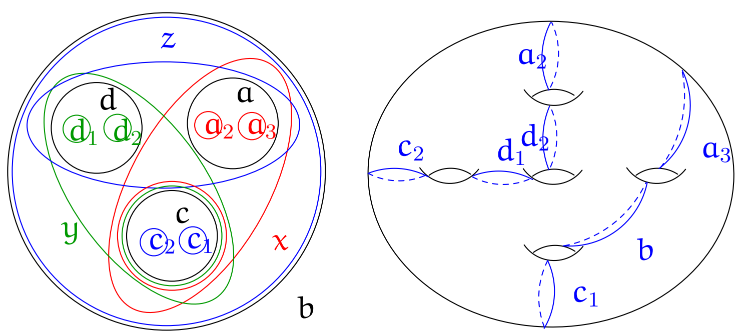


Figure 4.1: Supports of four lantern relations and an embedding of Σ_0^7 into a genus 5 surface

The following two propositions allow us to glue building blocks along more complicated monodromies in the sense that they are products of Dehn twists along distinct simple closed curves.

Proposition 4.2.3. *Let* $f \geq 5$ *and let* b,c *be disjoint simple closed curves on* Σ_f *such that* $\Sigma_f - b - c$ *is connected. In* $\mathrm{Mod}(\Sigma_f)$*,* $t_b^2 t_c^2$ *can be written as a product of three commutators.*

Proof. We may assume b and c are embedded, as shown in Figure 1, because any pair of simple closed curves whose complement in Σ_f is connected is topologically equivalent. On the planar surface Σ_0^7 in Figure 1, the following four lantern relations hold. $L_1 := t_a^{-1}t_b^{-1}t_c^{-1}t_d^{-1}t_y t_x t_z$, $L_2 := t_d t_{D_2} t_{D_1} t_{d_1}^{-1} t_{d_2}^{-1} t_c^{-1} t_y^{-1}$, $L_3 := t_x^{-1} t_{a_2}^{-1} t_{a_3}^{-1} t_c^{-1} t_a t_{A_3} t_{A_2}$, $L_4 := t_z^{-1} t_{c_1}^{-1} t_{c_2}^{-1} t_b^{-1} t_c t_{C_2} t_{C_1}$. Here, D_1 is an interior curve surrounding two boundary curves except d_1, and all other curves denoted by capital letters are defined similarly. After embedding Σ_0^7 into Σ_f with $f \geq 5$, as shown in Figure 1, we have $1 = L_1 \cdot L_2^{t_y t_x t_z} \cdot L_3^{t_z} \cdot L_4 = t_b^{-1} t_c^{-1} t_{D_2} t_{d_2}^{-1} t_{D_1} t_{d_1}^{-1} t_c^{-1} t_{A_3} t_{a_3}^{-1} t_{A_2} t_{a_2}^{-1}\ t_b^{-1} t_{C_2} t_{c_2}^{-1} t_{C_1} t_{c_1}^{-1}$ in $\mathrm{Mod}(\Sigma_f)$. Since both $\Sigma_f - D_2 - d_2$ and $\Sigma_f - D_1 - d_1$ are connected, $\{d_2, D_2\}$ and $\{D_1, d_1\}$ are topologically equivalent and then $t_{D_2} t_{d_2}^{-1} t_{D_1} t_{d_1}^{-1} = [t_{D_2} t_{d_2}^{-1}, \phi_1]$ for some $\phi_1 \in \mathrm{Mod}(\Sigma_f)$. Similarly, $t_{A_3} t_{a_3}^{-1} t_{A_2} t_{a_2}^{-1} = [t_{A_3} t_{a_3}^{-1}, \phi_2]$ and $t_{C_2} t_{c_2}^{-1} t_{C_1} t_{c_1}^{-1} = [t_{C_2} t_{c_2}^{-1}, \phi_3]$ for some $\phi_2, \phi_3 \in \mathrm{Mod}(\Sigma_f)$.
Therefore, $t_b^2 t_c^2 = [t_{D_2} t_{d_2}^{-1}, \phi_1]^{(t_b t_c)^{-1}} [t_{A_3} t_{a_3}^{-1}, \phi_2]^{t_b^{-1}} [t_{C_2} t_{c_2}^{-1}, \phi_3]$. □

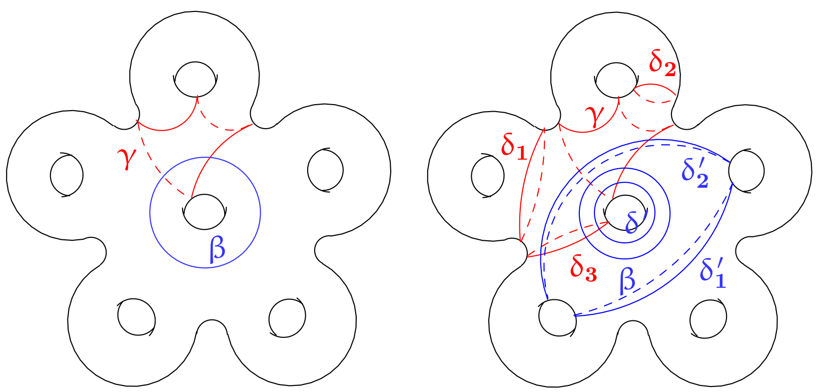


Figure 4.2: Supports of two lantern relations embedded in a genus 6 surface

Proposition 4.2.4. *Let* $f \geq 6$ *and let* β, γ *be simple closed curves on* Σ_f *embedded, as shown in Figure* 2. *In* $\mathrm{Mod}(\Sigma_f)$, $t_\beta t_\gamma$ *can be written as a product of three commutators.*

Proof. Choose two lantern relations with their supports on Σ_f, as shown in Figure 2: $L_1 := t_\gamma^{-1} t_{\delta_1}^{-1} t_{\delta_2}^{-1} t_{\delta_3}^{-1} t_y t_x t_z$ and $L_2 := t_{x'} t_{z'} t_{y'} t_{\delta'}^{-1} t_{\delta_1'}^{-1} t_{\delta_2'}^{-1} t_\beta^{-1}$. For interior curves, see Figure 3. It follows that $1 = L_1 \cdot L_2 = t_\gamma^{-1} t_{\delta_2}^{-1} t_y t_{\delta_3}^{-1} t_x t_z t_{\delta_1}^{-1} \cdot t_{x'} t_{\delta_2'}^{-1} t_{z'} t_{\delta'}^{-1} t_{y'} t_{\delta_1'}^{-1} t_\beta^{-1}$. In Figure 2 and Figure 3, we can see that δ_1 and x' are separating curves on Σ_f and that both $\Sigma_f - z - \delta_1$ and $\Sigma_f - \delta_2' - x'$ are homeomorphic to $\Sigma_1^1 \cup \Sigma_{f-2}^3$. Hence, we have $t_z t_{\delta_1}^{-1} t_{x'} t_{\delta_2'}^{-1} = [t_z t_{\delta_1}^{-1}, \phi_2]$ for some ϕ_2. Similarly, we have $t_{\delta_2}^{-1} t_y t_{\delta_3}^{-1} t_x = [t_{\delta_2}^{-1} t_y, \phi_1]$ and $t_{z'} t_{\delta'}^{-1} t_{y'} t_{\delta_1'}^{-1} = [t_{z'} t_{\delta'}^{-1}, \phi_3]$ for some ϕ_1 and ϕ_3.
Therefore, $t_\beta t_\gamma = [t_{\delta_2}^{-1} t_y, \phi_1]^{t_\beta^{-1}} [t_z t_{\delta_1}^{-1}, \phi_2]^{t_\beta^{-1}} [t_{z'} t_{\delta'}^{-1}, \phi_3]^{t_\beta^{-1}}$. □

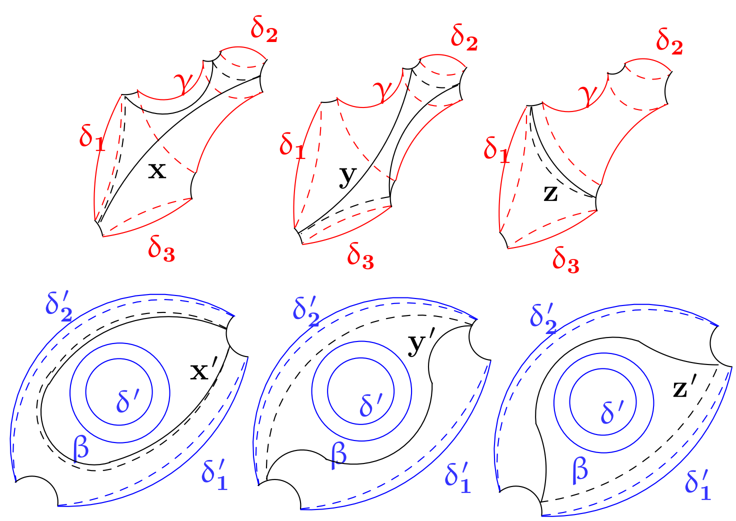


Figure 4.3: Interior curves for two lantern relations

In Proposition 11 *of [14], they constructed a genus* $f \geq 3$ *Lefschetz fibration over a torus with* 10 *singular fibers using a two-holed torus relation which is* *also called a 3-chain relation. In the following three Propositions, we generalize this construction of a Lefschetz fibration.*

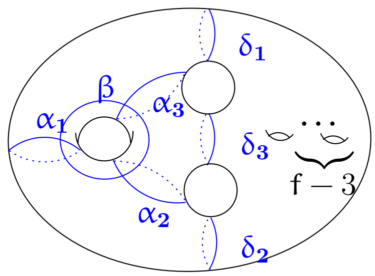


Figure 4.4: Support of a star relation

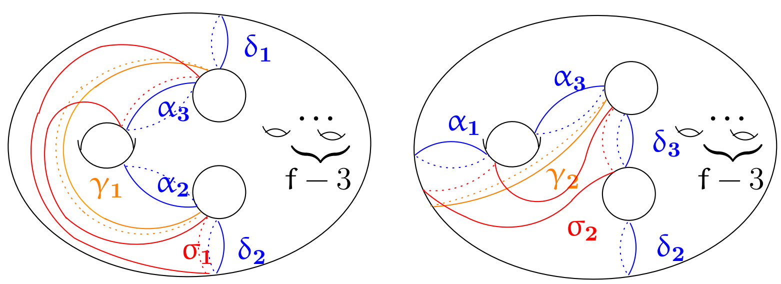


Figure 4.5: Supports of two lantern relations

Proposition 4.2.5. *Let $f \geq 3$ and let $\{\alpha_1, \alpha_2\}$ be any pair of nonseparating simple closed curves on Σ_f such that $\Sigma_f - \alpha_1 - \alpha_2$ is connected. Then there exists a genus f Lefschetz fibration X over Σ_3 which has six singular fibers, four of which have monodromy t_{α_1} and two of which have monodromy t_{α_2}.*

Proof. We use the star relation $E := t_{\delta_3}^{-1} t_{\delta_2}^{-1} t_{\delta_1}^{-1} (t_{\alpha_1} t_{\alpha_2} t_{\alpha_3} t_\beta)^3$ supported on $\Sigma_1^3 \hookrightarrow \Sigma_f$ (Figure 4). Also, consider the following lantern relations whose supports are given in Figure 5 : $L_1 := t_{\alpha_3}^{-1}t_{\alpha_2}^{-1}t_{\delta_1}^{-1}t_{\delta_2}^{-1}t_{\sigma_1}t_{\alpha_1}t_{\gamma_1}$, $L_2 := t_{\alpha_3}^{-1}t_{\alpha_1}^{-1}t_{\delta_2}^{-1}t_{\delta_3}^{-1}$ $t_{\sigma_2}t_{\alpha_2}t_{\gamma_2}$. Let $W_0 := t_\beta(t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta)^2$, $W_1 := t_\beta t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta$, and $W_2 := t_\beta$. Then, by using commutativity relations and braid relations,

$$
\begin{aligned}
1 &= E\cdot(W_0^{-1}L_1W_0)\cdot(W_1^{-1}L_1W_1)\cdot(W_2^{-1}L_2W_2)\\
&= t_{\delta_3}^{-1}t_{\delta_2}^{-1}t_{\delta_1}^{-1}t_{\alpha_1}t_{\delta_1}^{-1}t_{\delta_2}^{-1}t_{\sigma_1}t_{\alpha_1}t_{\gamma_1}t_\beta t_{\alpha_1}t_{\delta_1}^{-1}t_{\delta_2}^{-1}t_{\sigma_1}t_{\alpha_1}t_{\gamma_1}t_\beta t_{\alpha_2}t_{\delta_2}^{-1}t_{\delta_3}^{-1}t_{\sigma_2}t_{\alpha_2}t_{\gamma_2}t_\beta\\
&= t_{\alpha_1}t_{\delta_1}^{-1}t_{\sigma_1}t_{\delta_2}^{-1}t_{\alpha_1}t_{\gamma_1}t_{\delta_1}^{-1}t_\beta t_{\alpha_1}t_{\delta_1}^{-1}t_{\sigma_1}t_{\delta_2}^{-1}t_{\alpha_1}t_{\gamma_1}t_{\delta_2}^{-1}t_\beta t_{\alpha_2}t_{\delta_2}^{-1}t_{\sigma_2}t_{\delta_3}^{-1}t_{\alpha_2}t_{\gamma_2}t_{\delta_3}^{-1}t_\beta\\
&= t_{\alpha_1}^2\{t_{\delta_1}^{-1}t_{t_{\alpha_1}^{-1}(\sigma_1)}t_{\delta_2}^{-1}t_{\gamma_1}t_{\delta_1}^{-1}t_\beta\}t_{\alpha_1}^2\{t_{\delta_1}^{-1}t_{t_{\alpha_1}^{-1}(\sigma_1)}t_{\delta_2}^{-1}t_{\gamma_1}t_{\delta_2}^{-1}t_\beta\}\\
&\quad t_{\alpha_2}^2\{t_{\delta_2}^{-1}t_{t_{\alpha_2}^{-1}(\sigma_2)}t_{\delta_3}^{-1}t_{\gamma_2}t_{\delta_3}^{-1}t_\beta\}\\
&= t_{\alpha_1}^2[t_{\delta_1}^{-1}t_{t_{\alpha_1}^{-1}(\sigma_1)}t_{\delta_2}^{-1},\phi_1]t_{\alpha_1}^2[t_{\delta_1}^{-1}t_{t_{\alpha_1}^{-1}(\sigma_1)}t_{\delta_2}^{-1},\phi_2]t_{\alpha_2}^2[t_{\delta_2}^{-1}t_{t_{\alpha_2}^{-1}(\sigma_2)}t_{\delta_3}^{-1},\phi_3]
\end{aligned}
$$

For the last equality, we need to verify that there exists a self-homeomorphism ϕ_1 of Σ_f sending δ_1, $t_{\alpha_1}^{-1}(\sigma_1)$, and δ_2 to β, δ_1, and γ_1, respectively. First, it is easy to check that $\sigma_1 = t_\beta^{-1}t_{\alpha_2}^{-1}t_{\alpha_1}t_{\alpha_3}^{-1}(\beta)$. Hence, the self-homeomorphism $t_{\alpha_3}t_{\alpha_1}^{-1}t_{\alpha_2}t_\beta t_{\alpha_1}$ sends δ_1, $t_{\alpha_1}^{-1}(\sigma_1)$, and δ_2 to δ_1, β, and δ_2, respectively. Also, there exists a homeomorphism sending δ_1, β, and δ_2 to β, δ_1, and γ_1, respectively, because both $\Sigma_f - \delta_1 - \beta - \delta_2$ and $\Sigma_f - \beta - \delta_1 - \gamma_1$ are homeomorphic to Σ_{f-3}^6. The composition of these two homeomorphisms is the required ϕ_1. The existence of ϕ_2 and ϕ_3 can be proven in a similar way because $\sigma_2 = t_\beta^{-1}t_{\alpha_1}^{-1}t_{\alpha_3}^{-1}t_{\alpha_2}(\beta)$. Finally, we get the required Lefschetz fibration over Σ_3 with fiber Σ_f whose monodromy factorization is given by

$$
[t_{\delta_1}^{-1}t_{t_{\alpha_1}^{-1}(\sigma_1)}t_{\delta_2}^{-1},\phi_1]^{t_{\alpha_1}^{-2}}[t_{\delta_1}^{-1}t_{t_{\alpha_1}^{-1}(\sigma_1)}t_{\delta_2}^{-1},\phi_2]^{t_{\alpha_1}^{-4}}[t_{\delta_2}^{-1}t_{t_{\alpha_2}^{-1}(\sigma_2)}t_{\delta_3}^{-1},\phi_3]^{t_{\alpha_2}^{-2}t_{\alpha_1}^{-4}}t_{\alpha_1}^4t_{\alpha_2}^2 = 1
$$

.

□

Proposition 4.2.6. *Let $f \geq 4$ and let $\{\alpha_2, \alpha_3\}$ be any pair of nonseparating simple closed curves on Σ_f such that $\Sigma_f - \alpha_2 - \alpha_3$ is connected. Then there is a genus f Lefschetz fibration Z over Σ_4 which has four singular fibers, two of which have monodromy t_{α_2} and another two of which have monodromy t_{α_3}.*

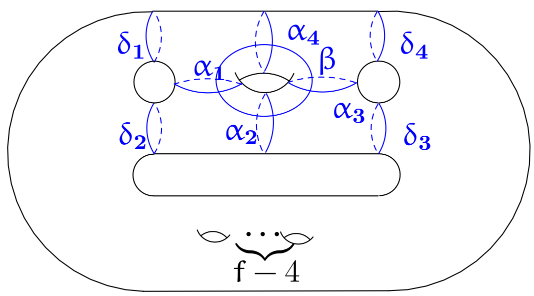


Figure 4.6: Support of a four-holed torus relation embedded in a genus 4 surface

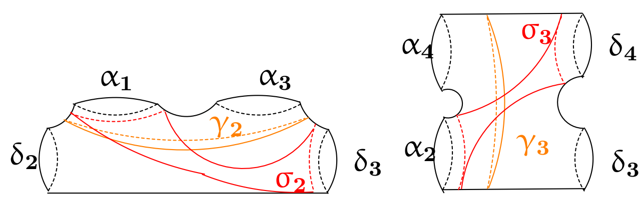


Figure 4.7: Supports of two lantern relations

Proof. We use the 4-holed torus relation [34] and lantern relations. Let $E_2 := t_{\delta_4}^{-1}t_{\delta_3}^{-1}t_{\delta_2}^{-1}t_{\delta_1}^{-1}t_{\alpha_1}t_{\alpha_3}t_{\beta}t_{\alpha_2}t_{\alpha_4}t_{\beta}t_{\alpha_1}t_{\alpha_3}t_{\beta}t_{\alpha_2}t_{\alpha_4}t_{\beta}$. We embed the support of this relation into Σ_f, as shown in Figure 6. Let $L_5 := t_{\alpha_3}^{-1}t_{\alpha_1}^{-1}t_{\delta_2}^{-1}t_{\delta_3}^{-1}t_{\sigma_2}t_{\alpha_2}t_{\gamma_2}$ and $L_6 := t_{\alpha_4}^{-1}t_{\alpha_2}^{-1}t_{\delta_3}^{-1}t_{\delta_4}^{-1}t_{\sigma_3}t_{\alpha_3}t_{\gamma_3}$. For the supports of lanterns, see Figure 7. Let $w_1 := t_{\beta}t_{\alpha_2}t_{\alpha_4}t_{\beta}t_{\alpha_1}t_{\alpha_3}t_{\beta}t_{\alpha_2}t_{\alpha_4}t_{\beta}$, $w_2 := t_{\beta}t_{\alpha_1}t_{\alpha_3}t_{\beta}t_{\alpha_2}t_{\alpha_4}t_{\beta}$, and $w_3 := t_{\beta}t_{\alpha_2}t_{\alpha_4}t_{\beta}$.

Then, from commutativity relations and braid relations,

$$\begin{aligned}
1 &= \mathsf{E}_2\cdot \mathsf{L}_5^{w_1}\cdot \mathsf{L}_6^{w_2}\cdot \mathsf{L}_5^{w_3}\cdot \mathsf{L}_6^{t_\beta}\\
&= t_{\delta_4}^{-1}t_{\delta_3}^{-1}t_{\delta_2}^{-1}t_{\delta_1}^{-1}(t_{\delta_2}^{-1}t_{\delta_3}^{-1}t_{\sigma_2}t_{\alpha_2}t_{\gamma_2}t_\beta)(t_{\delta_3}^{-1}t_{\delta_4}^{-1}t_{\sigma_3}t_{\alpha_3}t_{\gamma_3}t_\beta)\\
&\quad (t_{\delta_2}^{-1}t_{\delta_3}^{-1}t_{\sigma_2}t_{\alpha_2}t_{\gamma_2}t_\beta)(t_{\delta_3}^{-1}t_{\delta_4}^{-1}t_{\sigma_3}t_{\alpha_3}t_{\gamma_3}t_\beta)\\
&= (t_{\delta_2}^{-1}t_{\sigma_2}t_{\delta_3}^{-1}t_{\alpha_2}t_{\gamma_2}t_{\delta_1}^{-1}t_\beta)(t_{\delta_3}^{-1}t_{\sigma_3}t_{\delta_4}^{-1}t_{\alpha_3}t_{\gamma_3}t_{\delta_4}^{-1}t_\beta)\\
&\quad (t_{\delta_2}^{-1}t_{\sigma_2}t_{\delta_3}^{-1}t_{\alpha_2}t_{\gamma_2}t_{\delta_2}^{-1}t_\beta)(t_{\delta_3}^{-1}t_{\sigma_3}t_{\delta_4}^{-1}t_{\alpha_3}t_{\gamma_3}t_{\delta_3}^{-1}t_\beta)\\
&= t_{\alpha_2}(t_{\delta_2}^{-1}t_{t_{\alpha_2}^{-1}(\sigma_2)}t_{\delta_3}^{-1}t_{\gamma_2}t_{\delta_1}^{-1}t_\beta)t_{\alpha_3}(t_{\delta_3}^{-1}t_{t_{\alpha_3}^{-1}(\sigma_3)}t_{\delta_4}^{-1}t_{\gamma_3}t_{\delta_4}^{-1}t_\beta)\\
&\quad t_{\alpha_2}(t_{\delta_2}^{-1}t_{t_{\alpha_2}^{-1}(\sigma_2)}t_{\delta_3}^{-1}t_{\gamma_2}t_{\delta_2}^{-1}t_\beta)t_{\alpha_3}(t_{\delta_3}^{-1}t_{t_{\alpha_3}^{-1}(\sigma_3)}t_{\delta_4}^{-1}t_{\gamma_3}t_{\delta_3}^{-1}t_\beta)\\
&= t_{\alpha_2}[t_{\delta_2}^{-1}t_{t_{\alpha_2}^{-1}(\sigma_2)}t_{\delta_3}^{-1},\phi_1]t_{\alpha_3}[t_{\delta_3}^{-1}t_{t_{\alpha_3}^{-1}(\sigma_3)}t_{\delta_4}^{-1},\phi_2]\\
&\quad t_{\alpha_2}[t_{\delta_2}^{-1}t_{t_{\alpha_2}^{-1}(\sigma_2)}t_{\delta_3}^{-1},\phi_3]t_{\alpha_3}[t_{\delta_3}^{-1}t_{t_{\alpha_3}^{-1}(\sigma_3)}t_{\delta_4}^{-1},\phi_4]\\
&= [t_{\delta_2}^{-1}t_{t_{\alpha_2}^{-1}(\sigma_2)}t_{\delta_3}^{-1},\phi_1]^{t_{\alpha_2}^{-1}}[t_{\delta_3}^{-1}t_{t_{\alpha_3}^{-1}(\sigma_3)}t_{\delta_4}^{-1},\phi_2]^{t_{\alpha_3}^{-1}t_{\alpha_2}^{-1}}\\
&\quad [t_{\delta_2}^{-1}t_{t_{\alpha_2}^{-1}(\sigma_2)}t_{\delta_3}^{-1},\phi_3]^{t_{\alpha_3}^{-1}t_{\alpha_2}^{-2}}[t_{\delta_3}^{-1}t_{t_{\alpha_3}^{-1}(\sigma_3)}t_{\delta_4}^{-1},\phi_4]^{t_{\alpha_3}^{-2}t_{\alpha_2}^{-2}}t_{\alpha_2}^2t_{\alpha_3}^2
\end{aligned}$$

For the fifth equality, we need to find certain ϕ_1,ϕ_2,ϕ_3 and ϕ_4. For ϕ_1, it is sufficient to verify that $\{\delta_2, t_{\alpha_2}^{-1}(\sigma_2),\delta_3\}$ is topologically equivalent to $\{\beta,\delta_1,\gamma_2\}$. This is because $\{\delta_2, t_{\alpha_2}^{-1}(\sigma_2),\delta_3\}$ is topologically equivalent to $\{\delta_2,\beta,\delta_3\}$, and then $\{\delta_2,\beta,\delta_3\}$ to $\{\beta,\delta_1,\gamma_2\}$. The arguments for ϕ_2,ϕ_3, and ϕ_4 are similar. For these, we can check that $\{\delta_3, t_{\alpha_3}^{-1}(\sigma_3),\delta_4\}$ is topologically equivalent to $\{\beta,\delta_4,\gamma_3\}$, $\{\delta_2, t_{\alpha_2}^{-1}(\sigma_2),\delta_3\}$ is topologically equivalent to $\{\beta,\delta_2,\gamma_2\}$, and $\{\delta_3, t_{\alpha_3}^{-1}(\sigma_3),\delta_4\}$ is topologically equivalent to $\{\beta,\delta_3,\gamma_3\}$.

□

Proposition 4.2.7. *Let $f \geq 6$ and let β, γ be simple closed curves on Σ_f embedded, as shown in Figure 2. Then there is a genus f Lefschetz fibration W over Σ_3 which has two singular fibers, one of which has monodromy t_β and another has monodromy t_γ.*

Proof. There is a 9-holed torus relation $\mathsf{E}_7 := t_{\delta_1}^{-1}t_{\delta_2}^{-1}\dots t_{\delta_8}^{-1}t_{\gamma_9}^{-1}t_{\beta_8}t_{\sigma_3}t_{\sigma_6}t_{\alpha_{10}}t_{\beta_5}$ $t_{\sigma_4}t_{\sigma_7}t_{\alpha_6}t_{\beta_2}t_{\sigma_5}t_{\sigma_8}t_{\alpha_3}$ (see its support in orange in Figure 8 and see Figure 9 for its interior curves), where we use the idenfication $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9) \to (\alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_{10}, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ to go from Figure 9 in [34] to Figure 9 in this article. Here, each $\beta_i = t_{\alpha_i}(\beta)$ as in [34]. If we combine this relation E_7 and one more lantern relation $L_8 := t_{\delta_9}^{-1} t_{\delta_{10}}^{-1} t_{\gamma_9} t_{\sigma_9} t_{\alpha_9} t_{\alpha_8}^{-1} t_{\alpha_{10}}^{-1}$ (see its support in blue in Figure 8), then we get the following 10-holed torus relation $E_8 := t_{\delta_1}^{-1} t_{\delta_2}^{-1} \dots t_{\delta_{10}}^{-1} t_{\alpha_8}^{-1} t_{\alpha_{10}}^{-1} t_{\beta_8} t_{\sigma_3} t_{\sigma_6} t_{\alpha_{10}} t_{\beta_5} t_{\sigma_4} t_{\sigma_7} t_{\alpha_6} t_{\beta_2} t_{\sigma_5} t_{\sigma_8} t_{\alpha_3} t_{\sigma_9} t_{\alpha_9}$. Let $\beta_5' = (t_{\sigma_4} t_{\sigma_7} t_{\alpha_6} t_{\sigma_5} t_{\sigma_8} t_{\alpha_3} t_{\sigma_9} t_{\alpha_9})^{-1}(\beta_5)$ and $\beta_2' = (t_{\sigma_5} t_{\sigma_8} t_{\alpha_3} t_{\sigma_9} t_{\alpha_9})^{-1}(\beta_2)$. Then, by using commutativity relations and braid relations,

$$
\begin{aligned}
1 &= t_{\delta_1}^{-1} t_{\delta_2}^{-1} \dots t_{\delta_{10}}^{-1} t_{\beta_8} t_{\sigma_3} t_{\sigma_6} t_{\alpha_{10}} t_{\sigma_4} t_{\sigma_7} t_{\alpha_6} t_{\sigma_5} t_{\sigma_8} t_{\alpha_3} t_{\sigma_9} t_{\alpha_9} t_{\beta_5'} t_{\alpha_8}^{-1} t_{t_{\alpha_8}(\beta_2')} t_{\alpha_{10}}^{-1} \\
&= t_{\delta_1}^{-1} t_{\delta_2}^{-1} \dots t_{\delta_{10}}^{-1} t_{\beta_8} t_{\sigma_3} t_{\sigma_4} t_{\sigma_5} t_{\sigma_6} t_{\sigma_7} t_{\sigma_8} t_{\alpha_{10}} t_{\alpha_6} t_{\alpha_3} t_{\alpha_9} t_{t_{\alpha_9}^{-1}(\sigma_9)} t_{\beta_5'} t_{\alpha_8}^{-1} t_{t_{\alpha_8}(\beta_2')} t_{\alpha_{10}}^{-1} \\
&= t_{\delta_1}^{-1} t_{\delta_2}^{-1} \dots t_{\delta_{10}}^{-1} (t_{\sigma_3} t_{\sigma_4} t_{\sigma_5} t_{\sigma_6} t_{\sigma_7} t_{\sigma_8} t_{\alpha_{10}} t_{\alpha_6} t_{\alpha_3} t_{\alpha_9})^{t_{\beta_8}^{-1}} t_{\beta_8} t_{t_{\alpha_9}^{-1}(\sigma_9)} t_{\beta_5'} t_{\alpha_8}^{-1} t_{t_{\alpha_8}(\beta_2')} t_{\alpha_{10}}^{-1} \\
&= \{t_{\delta_1}^{-1} \cdot t_{t_{\beta_8}(\sigma_3)} \cdot t_{\delta_3}^{-1} \cdot t_{t_{\beta_8}(\sigma_4)} \cdot t_{\delta_{10}}^{-1} \cdot t_{t_{\beta_8}(\sigma_5)} \cdot t_{\delta_2}^{-1} \cdot t_{t_{\beta_8}(\sigma_6)} \cdot t_{\delta_7}^{-1} \cdot t_{t_{\beta_8}(\sigma_7)} \cdot t_{\delta_9}^{-1} \cdot t_{t_{\beta_8}(\sigma_8)}\} \\
&\quad \{t_{\delta_5}^{-1} \cdot t_{t_{\beta_8}(\alpha_3)} \cdot t_{\delta_8}^{-1} \cdot t_{t_{\beta_8}(\alpha_{10})} \cdot t_{\delta_6}^{-1} \cdot t_{t_{\beta_8}(\alpha_6)} \cdot t_{\delta_4}^{-1} \cdot t_{t_{\beta_8}(\alpha_9)}\}\{t_{\beta_8} \cdot t_{t_{\alpha_9}^{-1}(\sigma_9)}\}\{t_{\beta_5'} \cdot t_{\alpha_8}^{-1} \cdot t_{t_{\alpha_8}(\beta_2')} \cdot t_{\alpha_{10}}^{-1}\} \\
&= [t_{\delta_1}^{-1} \cdot t_{t_{\beta_8}(\sigma_3)} \cdot t_{\delta_3}^{-1} \cdot t_{t_{\beta_8}(\sigma_4)} \cdot t_{\delta_{10}}^{-1} \cdot t_{t_{\beta_8}(\sigma_5)}, \phi_1][t_{\delta_5}^{-1} \cdot t_{t_{\beta_8}(\alpha_3)} \cdot t_{\delta_8}^{-1} \cdot t_{t_{\beta_8}(\alpha_{10})}, \phi_2] \\
&\quad \cdot t_{\beta_8} \cdot t_{t_{\alpha_9}^{-1}(\sigma_9)} \cdot [t_{\beta_5'} t_{\alpha_8}^{-1}, \phi_3]
\end{aligned}
$$

For the last equality, we need to verify that $\{\delta_1, t_{\beta_8}(\sigma_3), \delta_3, t_{\beta_8}(\sigma_4), \delta_{10}, t_{\beta_8}(\sigma_5)\}$ is topologically equivalent to $\{t_{\beta_8}(\sigma_8), \delta_9, t_{\beta_8}(\sigma_7), \delta_7, t_{\beta_8}(\sigma_6), \delta_2\}$. This follows from the fact that both $\Sigma_f - \delta_1 - \delta_3 - \delta_{10} - \sigma_3 - \sigma_4 - \sigma_5$ and $\Sigma_f - \delta_2 - \delta_7 - \delta_9 - \sigma_6 - \sigma_7 - \sigma_8$ are connected. For ϕ_2 and ϕ_3, it is easy to check that $\Sigma_f - \delta_5 - \alpha_3 - \delta_8 - \alpha_{10} \approx \Sigma_{f-4}^8 \approx \Sigma_f - \alpha_9 - \delta_4 - \alpha_6 - \delta_6$ and that $\{\beta_5', \alpha_8\}$ is topologically equivalent to $\{\beta, \alpha_8\}$ and $\{\alpha_{10}, t_{\alpha_8}(\beta_2')\}$ is topologically equivalent to $\{\alpha_{10}, \beta\}$. Finally, observe that $\{\beta_8, t_{\alpha_9}^{-1}(\sigma_9)\}$ is topologically equivalent to $\{\beta, t_{\alpha_9}^{-1}(\sigma_9)\}$ and $t_{\alpha_9}^{-1}(\sigma_9) = \gamma$. $\square$

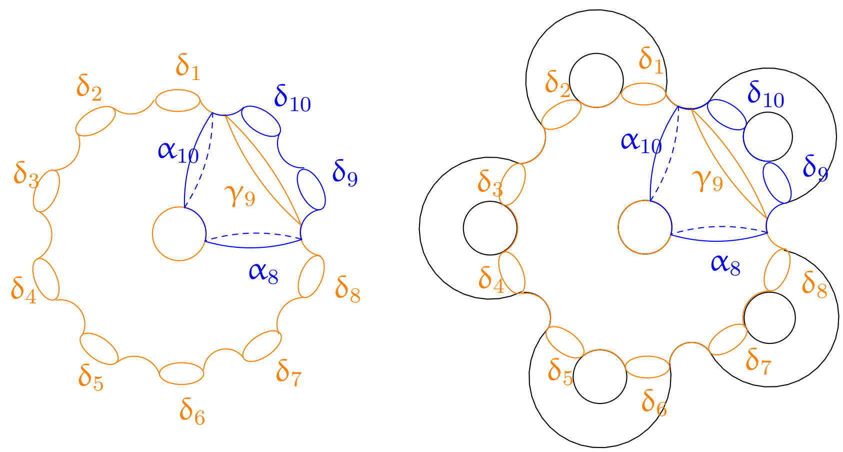


Figure 4.8: Supports for a 9-holed torus relation and a lantern relation and their embeddings into a genus 6 surface

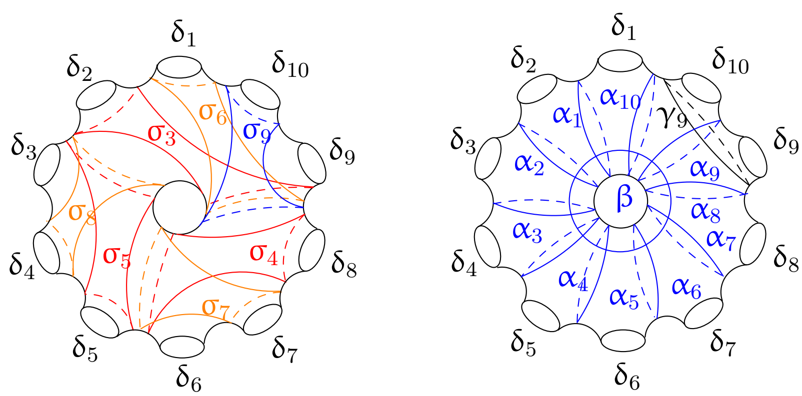


Figure 4.9: Interior curves for a 10-holed torus relation

4.3 Signature computation

In order to compute the signature of the total space of surface bundles, we first review the definition of Meyer's signature cocycle.

Definition 4.3.1. For any given $A, B \in Sp(2g, \mathbb{R})$, consider the subspace

$$V_{A,B} := \{(x, y) \in \mathbb{R}^{2g} \times \mathbb{R}^{2g} | (A^{-1} - I_{2g})x + (B - I_{2g})y = 0\}$$

of the real vector space $\mathbb{R}^{2g} \times \mathbb{R}^{2g}$ and the bilinear form $\langle , \rangle_{A,B} : (\mathbb{R}^{2g} \times \mathbb{R}^{2g}) \times (\mathbb{R}^{2g} \times \mathbb{R}^{2g}) \to \mathbb{R}$ defined by $\langle (x_1, y_1), (x_2, y_2) \rangle_{A,B} := (x_1 + y_1) \cdot J(I_{2g} - B)y_2$, where $\cdot$ is the inner product of $\mathbb{R}^{2g}$ and J is the matrix representing the multiplication by $-\sqrt{-1}$ on $\mathbb{R}^{2g} = \mathbb{C}^g$. Since the restriction of $\langle , \rangle_{A,B}$ on $V_{A,B}$ is symmetric, we can define $\tau_g(A, B) := \mathrm{sign}(\langle , \rangle_{A,B}, V_{A,B})$.

We denote by $\psi : \mathrm{Mod}(\Sigma_g) \to Sp(2g; \mathbb{R})$ *the symplectic representation of the mapping class group.*

Theorem 4.3.2. *[42] Let* $E_{A,B} \to P$ *be an oriented* Σ_g *bundle over a pair of pants* P *whose monodromy representation* χ *composed with the symplectic representation* ψ *is given by* $\psi \circ \chi : \pi_1(P, *) \to Sp(2g; \mathbb{R})$ *sending one generator to* A *and the other to* B*. Then* $\sigma(E_{A,B}) = -\tau_g(A, B)$*.*

We can easily check that τ_g *is a 2-cocycle on the symplectic group* $Sp(2g, \mathbb{R})$ *using Novikov's additivity. We call this* τ_g *Meyer's signature cocycle. The pants decomposition of any base surface gives the following signature formula.*

Theorem 4.3.3. *[43] Let* $f : E \to \Sigma_h^r$ *be an oriented surface bundle with fiber* Σ_g *and monodromy representation* $\chi : \pi_1(\Sigma_h^r) \to \mathrm{Mod}(\Sigma_g)$*. Fix a standard presentation of* $\pi_1(\Sigma_h^r)$ *as follows:*

$$\pi_1(\Sigma_h^r) = \langle a_1, b_1, \cdots, a_h, b_h, c_1, \cdots, c_r | \prod_{i=1}^{h} [a_i, b_i] \prod_{j=1}^{r} c_j = 1 \rangle$$

and let τ_g *be Meyer's signature cocycle. Then the signature of* E *is given by the formula*

$$\sigma(E) = \sum_{i=1}^{h} \tau_g(\kappa_i, \beta_i) - \sum_{i=2}^{h} \tau_g(\kappa_1 \cdots \kappa_{i-1}, \kappa_i) - \sum_{j=1}^{r-1} \tau_g(\kappa_1 \cdots \kappa_h \gamma_1 \cdots \gamma_{j-1}, \gamma_j)$$

where $\alpha_i = \psi \circ \chi(a_i), \beta_i = \psi \circ \chi(b_i), \gamma_i = \psi \circ \chi(c_i)$ *and* $\kappa_i = [\alpha_i, \beta_i]$.

By applying this formula, we can compute the signatures of surface bundles obtained by taking out some neighborhoods of singular fibers from the Lefschetz fibrations constructed in Section 3. *We used Mathematica for computing each term in the above formula.*

Meyer also provided another interpretation of the above signature formula. For this, we start with the following diagram.

$$\begin{array}{ccccccccc} 1 & \to & \widetilde{R} & \to & \widetilde{F} & \xrightarrow{\widetilde{\pi}} & \pi_1(\Sigma_h) & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow \chi & & \\ 1 & \to & R & \to & F & \xrightarrow{\pi} & \mathrm{Mod}(\Sigma_g) & \to & 1 \end{array}$$

Here, $\pi_1(\Sigma_h) = \langle a_1, \cdots, a_h, b_1, \cdots, b_h | \prod_{i=1}^{h} [a_i, b_i] = 1 \rangle$, $\widetilde{F} = \langle \widetilde{a_1}, \cdots, \widetilde{a_h}, \widetilde{b_1}, \cdots, \widetilde{b_h} \rangle$, $\widetilde{R}$ *is the normal closure of* $\widetilde{r} = \prod_{i=1}^{h} [\widetilde{a}_i, \widetilde{b}_i]$, *and* $\widetilde{\pi} : \widetilde{a}_i \mapsto a_i, \widetilde{b}_i \mapsto b_i$. *The second row corresponds to the finite presentation of* $\mathrm{Mod}(\Sigma_g)$ *due to Wajnryb.* $F = F(S)$, *where* $S = \{y_1, y_2, u_1, \cdots, u_g, z_1, \cdots, z_{g-1}\}$ *and* R *is the normal closure of* $A_{i,j}^k$*'s,* B_i^k*'s,* C^1, D^1, E^1 *(cf.[16] §3). If we have a monodromy representation* $\chi : \pi_1(\Sigma_h) \to \mathrm{Mod}(\Sigma_g)$, *then there exists a homomorphism* $\widetilde{\chi} : \widetilde{F} \to F$ *such that* $\chi \circ \widetilde{\pi} = \pi \circ \widetilde{\chi}$ *since* π *is surjective and* $\widetilde{F}$ *is free. Hence we have* $\widetilde{\chi}(\widetilde{r}) \in R \cap [F, F]$. *Now define the* 1*-cochain* $c : F \to \mathbb{Z}$ *cobounding the* 2*-cocycle* $-\pi^* \psi^* (\tau_g)$ *as follows.*

$$c(x) := \sum_{j=1}^{m} \tau_g(\psi(\pi(\widetilde{x_{j-1}})), \psi(\pi(x_j)))$$

$$(x = \prod_{j=1}^{m} x_j, \quad x_j \in S \cup S^{-1}, \quad \widetilde{x_j} = \prod_{i=1}^{j} x_i)$$

Since $\pi^*\psi^*(\tau_g)\mid_{R\times R}=0$*, the restriction* $c\mid_R : R \to \mathbb{Z}$ *is a homomorphism. The values of* c *for the relations of Wajnryb's presentation were calculated in [16].*

Theorem 4.3.4. *[43]*

Let $p : E \to \Sigma_h$ *be a* Σ_g*- bundle over* Σ_h *and* $\chi : \pi_1(\Sigma_h) \to \mathsf{Mod}(\Sigma_g)$ *be its monodromy homomorphism. Then the signature of the total space* E *is given as follows :*

$$\sigma(E) = -c\mid_R (\widetilde{\chi}(\widetilde{r})) \quad (= -\langle \psi^*[\tau_g], \widetilde{\chi}(\widetilde{r})[R, F] \rangle)$$

where $\langle , \rangle$ *is a pairing on the second cohomology and homology of* $\mathsf{Mod}(\Sigma_g)$*.*

Now, we are ready to prove our main theorem.

Proof of Theorem 4.0.6. (a) We apply the subtraction operation to the Lefschetz fibrations $X \to \Sigma_3$, $Y_1 \to \Sigma_2$, and $Y_2 \to \Sigma_3$ constructed in Propositions 3.5 and Proposition 3.1. Let $N_1 \subset X$ be the neighborhood of four singular fibers with coinciding vanishing cycles and $N_2 \subset X$ be the neighborhood of two singular fibers with coinciding vanishing cycles. Then the complement $X \setminus N_1 \setminus N_2$ is the Σ_f bundle over Σ_3^2, and its signature can be computed by applying Theorem 4.2 to this bundle. More precisely to its monodromy representation $\chi : \pi_1(\Sigma_3^2) \to \mathsf{Mod}(\Sigma_f)$ given by $\chi(a_1) = (t_{\delta_1}^{-1} \cdot t_{t_{\alpha_1}^{-1}(\sigma_1)} \cdot t_{\delta_2}^{-1})^{t_{\alpha_1}^{-2}}$, $\chi(b_1) = (\phi_1)^{t_{\alpha_1}^{-2}}$, $\chi(a_2) = (t_{\delta_1}^{-1} \cdot t_{t_{\alpha_1}^{-1}(\sigma_1)} \cdot t_{\delta_2}^{-1})^{t_{\alpha_1}^{-4}}$, $\chi(b_2) = (\phi_2)^{t_{\alpha_1}^{-4}}$, $\chi(a_3) = (t_{\delta_2}^{-1} \cdot t_{t_{\alpha_2}^{-1}(\sigma_2)} \cdot t_{\delta_3}^{-1})^{t_{\alpha_1}^{-4} t_{\alpha_2}^{-2}}$, $\chi(b_3) = (\phi_3)^{t_{\alpha_1}^{-4} t_{\alpha_2}^{-2}}$, $\chi(c_1) = t_{\alpha_1}^4$, and $\chi(c_2) = t_{\alpha_2}^2$. Now, by computations using Mathematica we have $\tau(\kappa_1, \beta_1) = \tau(\kappa_2, \beta_2) = \tau(\kappa_3, \beta_3) = 2$, $-\tau(\kappa_1, \kappa_2) = -\tau(\kappa_1\kappa_2, \kappa_3) = -2$, and $-\tau(\kappa_1\kappa_2\kappa_3, \gamma_1) = 0$. Hence, $\sigma(X \setminus N_1 \setminus N_2) = 3 \cdot 2 + 2 \cdot (-2) + 0 = 2$. By taking out the neighborhood M_i of all singular fibers from Y_i (for $i = 1, 2$), we get $Y_i \setminus M_i$, the Σ_f bundles (one over Σ_2^1 and another over Σ_3^1), both with signature -1. For signature computation, we can directly apply Theorem 4.2 to these two bundles as above. Alternatively, we can first compute the signature of Lefschetz fibrations : $\sigma(Y_1) = -2$ and $\sigma(Y_2) = -4$ (cf. Proposition 15 and Proposition

16 in [14]). In order to compute the signature of taken out parts, apply Theorem 4.1 several times and use the fact that $\sigma(N(\text{a nonseparating singular fiber})) = 0$ (cf.[48]). From these, we have $\sigma(Y_1 \setminus M_1) = (-2)-(-1) = -1$ and $\sigma(Y_2 \setminus M_2) = (-4)-(-3) = -1$. Therefore, $X - Y_1 - Y_2$ is the $\Sigma_{f\geq 3}$ bundle over Σ_8, and $\sigma(X - Y_1 - Y_2) = \sigma(X \setminus N_1 \setminus N_2) + \sigma(\overline{Y_1 \setminus M_1}) + \sigma(\overline{Y_2 \setminus M_2}) = 2+1+1 = 4$ by Novikov additivity. Moreover, if we pullback this bundle (or, with opposite orientation) to unramified coverings of Σ_8 of degree $|n|$, then we get $b(f \geq 3, n) \leq 7|n| + 1$.

(b) Apply the subtraction operation to the Lefschetz fibrations $Z \to \Sigma_4$ and $Y_3 \to \Sigma_3$, constructed in Proposition 3.6 and Proposition 3.3, respectively. Then, $Z - Y_3$ is the required $\Sigma_{f\geq 5}$ bundle over Σ_7. Let N be the neighborhood of all singular fibers in Z and let M be the neighborhood of all singular fibers in Y_3. By applying Theorem 4.2 to two surface bundles $Z \setminus N$ and $Y_3 \setminus M$, we get $\sigma(Z - Y_3) = \sigma(Z \setminus N) + \sigma(\overline{Y_3 \setminus M}) = 2+2 = 4$. Let me give you another proof for verifying $\sigma(Z - Y_3) = 4$ using Theorem 4.3. From Proposition 3.6 and Proposition 3.3, we have $\widetilde{\chi}(\widetilde{r}) \equiv (E_2 \cdot L_5^{w_1} \cdot L_6^{w_2} \cdot L_5^{w_3} \cdot L_6^{t_\beta})(L_1 \cdot L_2^{t_y t_x t_z} \cdot L_3^{t_z} \cdot L_4)^g$ modulo commutativity and braid relations, where g is a self-homeomorphism of $\Sigma_{f\geq 5}$ such that $g(\alpha_3) = b$ and $g(\alpha_2) = c$. Moreover, from [34], $E_2 \equiv L_{10} \cdot (L_9 \cdot ((C^1)^{-1})^{z_0})^{z_1}$ for some mapping classes z_0, z_1, modulo commutativity and braid relations. Observe that for each L_i, four boundary curves are nonseparating and $\Sigma_f \setminus \mathrm{supp}(L_i)$ is connected. Since the same holds for the relation $(D^1)^{-1}$, there exists a self-homeomorphism f_i of Σ_f sending the $\mathrm{supp}((D^1)^{-1})$ to the $\mathrm{supp}(L_i)$ for each i. Therefore, $\widetilde{\chi}(\widetilde{r}) \equiv ((D^1)^{-1})^{f_{10}}((D^1)^{-1})^{f_9 \circ z_1} \cdot ((C^1)^{-1})^{z_0 \circ z_1} \cdot ((D^1)^{-1})^{f_5 \circ w_1} \cdot ((D^1)^{-1})^{f_6 \circ w_2} \cdot ((D^1)^{-1})^{f_5 \circ w_3} \cdot ((D^1)^{-1})^{f_6 \circ t_\beta} \cdot ((D^1)^{-1})^{f_1 \circ g} \cdot ((D^1)^{-1})^{f_2 \circ (t_y t_x t_z) \circ g} \cdot ((D^1)^{-1})^{f_3 \circ t_z \circ g} \cdot ((D^1)^{-1})^{f_4 \circ g}$ modulo commutativity and braid relations and hence $\sigma(Z - Y_3) = -c(\widetilde{\chi}(\widetilde{r})) = c(C^1) + 10 \cdot c(D^1) = (-6) + 10 = 4$. For the upper bound for the genus function $b(f \geq 5, n)$, use the same argument as before.

(c) Apply the subtraction operation to the Lefschetz fibrations $W \to \Sigma_3$ and $Y_4 \to \Sigma_3$, constructed in Proposition 3.7 and Proposition 3.4, respectively.

Then $W - Y_4$ is the required $\Sigma_{f\geq 6}$ bundle over Σ_6 with signature 4. From Proposition 3.4 and Proposition 3.7, $\widetilde{\chi}(\widetilde{r}) \equiv E_8 \cdot (L_1 \cdot L_2)^h$ modulo braid and commutativity relations, where h is a self-homeomorphism of Σ_f such that $h\{\beta_8, t_{\alpha_9}^{-1}(\sigma_9)\} = \{\beta, \gamma\}$. Moreover, $E_8 \equiv (\prod_{j=1}^{8}((D^1)^{-1})^{z_j}) \cdot ((C^1)^{-1})^{z_0}$ for some $z_0, \ldots, z_8$ (cf.[34] and Proposition 3.7). Therefore, $\sigma(W - Y_4) = -c(\widetilde{\chi}(\widetilde{r})) = c(C^1) + 10 \cdot c(D^1) = (-6) + 10 = 4$. For the upper bound for the genus function $b(f \geq 6, n)$, use the same argument as before.

$\square$

Proof of Theorem 4.0.8. Every odd genus surface is a covering of genus three surface. By Morita[45], after replacing a given surface bundle by a pullback to some covering of the base, the resulting surface bundle admits a fiber-wise covering of any given degree. After applying this to the genus 3 surface bundle over $\Sigma_{b_3(1)}$ with signature 4 and the degree of the covering $\Sigma_f \to \Sigma_3$, we obtain $b_f(\frac{f-1}{2}n) \leq n(b_3(1) - 1) + 1$. Hence, $G_f := \lim_{n\to\infty} \frac{b_f(n)}{n} \leq \lim_{n\to\infty} \frac{2n(b_3(1)-1)+2}{(f-1)n} \leq \lim_{n\to\infty} \frac{14n+2}{(f-1)n} = \frac{14}{f-1}$. $\square$

Remark 4.3.5. In [26, 36, 50], it was proven that $H_2(\mathrm{Mod}(\Sigma_g); \mathbb{Z}) \cong \mathbb{Z}$ for every $g \geq 4$ and $H_2(\mathrm{Mod}(\Sigma_g); \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ for $g = 3$. Meyer[43] proved that each generator of $H_2(\mathrm{Mod}(\Sigma_g))/\mathrm{Tor}$ gives us signature 4 relying on the Theorem 4.3. In order to prove this, Meyer used Birman-Hilden's presentation of $\mathrm{Mod}(\Sigma_g)$, and Endo[16] reproved this using a simpler presentation due to Wajnryb[53]. By taking $\widetilde{\chi}(\widetilde{r})$ as different representatives for a generator of $H_2(\mathrm{Mod}(\Sigma_g))/\mathrm{Tor}$, we can construct various surface bundles with a fixed signature 4 as we have seen in the proof of Theorem 1.2. Therefore, the problem to determine $b(f, n)$ is to find the most effective representative $\tilde{\chi}(\tilde{r})$, in the sense of commutator length, for n times generator of $H_2(\mathrm{Mod}(\Sigma_f))/\mathrm{Tor}$.

Chapter 5

Double Kodaira fibartions with small signature

5.1 Kodaira fibrations

In 1967*, Kodaira [35] constructed examples of complex surfaces with positive signature which admit fibrations which are differentiable but not holomorphic fiber bundles. In his honour, such fibrations are nowadays called Kodaira fibrations.*

Definition 5.1.1. A *Kodaira fibration* is a holomorphic submersion $\phi \colon S \to B$ from a compact complex surface to a compact complex curve which is not isotrivial, that is, not all fibres are isomorphic.

It is well known that every Kodaira fibration has positive signature and hence the genus of the base is at least 2 *and the genus of the fiber is at least* 3 *[43].*

Definition 5.1.2. A double Kodaira fibration is a finite surjective map $f \colon S \to B_1 \times B_2$ from a compact complex surface to a product of two curves such that the composition with the projections onto the factors induces two different Kodaira fibrations $f_i : S \to B_i$.

Definition 5.1.3. Let $f\colon S \to B_1 \times B_2$ be a double Kodaira fibration and let $D \subset B_1 \times B_2$ be the branch divisor. We call S double étale if the induced projections $D \to B_i$ are both unramified coverings. We say S is of graph type if D is a disjoint union of graphs of maps from $B_1 \to B_2$. The graph type S is called the automorphism type if there is a map $g\colon B_1 \to B_2$ and $\mathcal{S} \subset \mathrm{Aut}(B_2)$ such that D is the pullback of a disjoint union of graphs of automorphisms in $\mathcal{S}$ by g, that is $D = \cup_{\sigma\in\mathcal{S}}\Gamma_{\sigma\circ g}$. If the ramified cover $f\colon S \to B_1 \times B_2$ is G-Galois, i.e. the quotient map for the action of a finite group G, then we call S a G-Galois double Kodaira fibration.

Kodaira, Atiyah, and Hirzebruch constructed Kodiara fibrations as the cyclic ramified coverings of the product of two Riemann surfaces with disjoint graphs as the branch divisor. To generalize their construction, we introduce the following notion.

Definition 5.1.4. A virtual Kodaira fibration $\mathcal{A}$ consists of the following data:

- A product $B \times F$ of smooth curves of genus at least 2.
- A curve $D \subset B \times F$ such that both projections restrict to unramified coverings on D.
- A surjective homomorphism $\theta : \pi_1(\hat{F}) \to G$, where G is a finite group and $\hat{F} = F \setminus F \cap D$, satisfying the following two conditions:

(1) (ramification condition) For each loop γ in $\hat{F}$ around an intersection point of F and D, the order of $\theta(\gamma)$ in G is at least 2.
(2) (liftability condition) Suppose D_0 is a component of D and γ and γ' are loops in $\hat{F}$ around two points x and x' of $F \cap D_0$, respectively. Then $\theta(\gamma)$ and $\theta(\gamma')$ are conjugate in G.

Remark 5.1.5. Here such a θ corresponds to a Galois G- covering of the fiber F ramified over $F \cap D$. The liftability condition is necessary for extending $\theta\colon \pi_1(\hat{F}) \to G$ to $\Theta\colon \pi_1(B \times F \setminus D) \to G$. This follows from the fact that meridians of a connected divisor in a complex surface are conjugate to each other in the fundamental group of the complement.

Definition 5.1.6. We call a virtual Kodaira fibration $\mathcal{A}$ realizable if $\theta\colon \pi_1(\hat{F}) \to G$ is the restriction of a homomorphism $\Theta\colon \pi_1(B \times F \setminus D) \to G$.

Proposition 5.1.7. *If* $f\colon S \to B \times F$ *is a double étale Kodaira fibration which is* G*-Galois then the unramified* $G-$*cover* $S \setminus f^{-1}D \to B \times F \setminus D$ *induces a realisable virtual Kodaira fibration* $\mathcal{A}(S) = (B \times F, D, \theta\colon \pi_1(\hat{F}) \to G)$. *Conversely, for every realisable vitual Kodaira fibration* $(B \times F, D, \theta\colon \pi_1(\hat{F}) \to G)$*, there exists a double étale Kodaira fibration* $f\colon S \to B \times F$ *which is a smooth* G*-cover.*

Proof. It follows from the Riemann existence theorem. □

Definition 5.1.8. Let $\mathcal{A} = (B \times F, D, \theta\colon \pi_1(\hat{F}) \to G)$ be a (double étale) virtual Kodaira fibration, r_i be the ramification order at $D_i \cap F$, and d_i(or e_i) be the degree of the projection $D_i \to B$(or $D_i \to F$), respectively. We define the virtual Chern classes, signature, and slope of $\mathcal{A}$ to be

$$c_2(\mathcal{A}) = |G|e(B)\left(e(F) - \sum_{i=1}^{m} \frac{r_i - 1}{r_i} d_i\right),$$

$$c_1^2(\mathcal{A}) = 2c_2(S) - |G|e(B)\sum_{i=1}^{m} \frac{r_i^2 - 1}{r_i^2} d_i,$$

$$\sigma(\mathcal{A}) = \frac{1}{3}\left(c_1^2(\mathcal{A}) - 2c_2(\mathcal{A})\right) = -\frac{|G|e(B)}{3}\sum_{i=1}^{m} \frac{r_i^2 - 1}{r_i^2} d_i = -\frac{|G|e(F)}{3}\sum_{i=1}^{m} \frac{r_i^2 - 1}{r_i^2} e_i,$$

$$\nu(\mathcal{A}) = \frac{c_1^2(\mathcal{A})}{c_2(\mathcal{A})}.$$

If in particular $\mathcal{A}$ is of graph type, then

$$\sigma(\mathcal{A}) = -\frac{|G|e(B)}{3}\sum_{i=1}^{m} \frac{r_i^2 - 1}{r_i^2}.$$

Remark 5.1.9. Note that for a realisable virtual Kodaira fibraion these invariants coincide with the invariants for a double étale Galois double Kodaira fibration.

Definition 5.1.10. Let $\mathcal{A} = (B \times F, D, \theta)$ be a virtual Kodaira fibration. Let $g\colon \tilde{B} \to B$ be a finite étale map of minimal degree such that the pull-back $g^*\mathcal{A} = (\tilde{B} \times F, (g \times id_F)^*D, \theta)$ is realisable. We call $\tilde{b} = g(\tilde{B})$ the realisation genus of $\mathcal{A}$ and $\tilde{\sigma} = \sigma(g^*\mathcal{A})$ the realisation signature of $\mathcal{A}$.

Definition 5.1.11. An $(m+1)$- tuple $(q \mid r_1, \cdots, r_m)$ is called the ramification type of a Galois G-cover $B \to B/G$ if the genus of the quotient curve B/G is equal to q and the multiplicity of each ramification point is given by r_i for each $i = 1, \cdots, m$, where m is the number of branch points. If in particular, G is a cyclic group generated by an automorphism ϕ of B, then we call the corresponding $(m+1)$-tuple the ramification type of an automorphism ϕ.

5.2 Effective tautological construction

In [12] the tautological construction was used to show that every virtual Kodaira fibration becomes realisable after a finite étale pullback. In that paper the focus was on the slope, which is invariant under pullback, so computing the degree of the pullback was not important. However, to compute the smallest possible signature of Kodaira fibraions and to obtain the minimal base genus we'll find the minimal degree pullback for the realisability of a given virtual Kodaira fibration in this section. Before we begin, let us fix some notations first. Let S *be a product of curves,* $p\colon S \to B$ *be the projection to one factor, and* F *be its general fiber. Let* $D \subset S$ *be a divisor such that* $p|_D$ *is unramified. We let* $\hat{S} = S \setminus D$ *and* $\hat{F} = F \setminus F \cap D$*. Then* $p|_{\hat{S}}\colon \hat{S} \to B$ *is a differentiable fiber bundle with the fiber* $\hat{F}$*. Let* $\theta : \pi_1(\hat{F}) \to G$ *be a surjective homomorphism to a finite group which defines* $\tilde{F} \to F$*, the ramified cover of the fiber* F *branched over* $F \cap D$ *satisfying the liftability condition mentioned in definition 5.1.4. Now* *we would like to extend this cover of the fiber* F *to the whole product* S*, and Proposition 5.1.7 tells us that the virtual Kodaira fibration corresponds to an actual Kodaira fibration which is* G*-Galois if and only if* θ *is the restriction of a homomorphism* $\Theta : \pi_1(\hat{\mathsf{S}}) \to \mathsf{G}$.

5.2.1 Construction: if D contains a graph

First, we restate the Catanese-Rollenske's result in our flavor.

Proposition 5.2.1. *Assume that* D *contains graphs.*
Then θ *is the restriction of a homomorphism* $\Theta : \pi_1(\hat{\mathsf{S}}) \to \mathsf{G}$ *if and only if there exists a graph* $\mathsf{D}_0 \subset \mathsf{D}$ *and a homomorphism* $\theta' : \pi_1(\mathsf{T}_0) \to \mathsf{G}$*, where* $\mathsf{T}_0 = \mathsf{T} \setminus \mathsf{D}_0$ *with* T *a tubular neighborhood of* D_0*, with the following properties:*

1. $\theta(\gamma_0) = \theta'(\gamma_0)$
2. *For all* $x \in \pi_1(\hat{\mathsf{F}})$ *and all* $y \in \pi_1(\mathsf{T}_0)$ *we have*

$$\theta(x) = \theta'(y^{-1})\theta(yxy^{-1})\theta'(y)$$

.

In the case G *is abelian, there exists* θ' *satisfying the above conditions if and only if* $\theta(\gamma_0^{2b-2}) = 0$ *and* θ *is invariant under the monodromy action of* $\pi_1(\mathsf{B})$ *on* $\mathsf{Hom}(\pi_1(\hat{\mathsf{F}}), \mathsf{G})$.

Proof. First observe that $\pi_1(\hat{\mathsf{S}})$ is generated by $\pi_1(\hat{\mathsf{F}})$ and $\pi_1(\mathsf{T}_0)$. From this fact and since $\pi_1(\hat{\mathsf{F}})$ is the normal subgroup, every element of a of $\pi_1(\hat{\mathsf{S}})$ can be written as $a = xy$ with $x \in \pi_1(\hat{\mathsf{F}})$ and $y \in \pi_1(\mathsf{T}_0)$. If Θ exists then we can define $\theta' \colon \pi_1(\mathsf{T}_0) \to \mathsf{G}$ by its restriction and have

$$\Theta(a) = \Theta(x)\Theta(y) = \theta(x)\theta'(y). \tag{5.2.1}$$

Conversely, if we are given $\theta' \colon \pi_1(T_0) \to G$ then we can define $\Theta \colon \pi_1(\hat{S}) \to G$ by (5.2.3). If a has another expression as $a = x'y'$ with $x' \in \pi_1(\hat{F})$ and $y' \in \pi_1(T_0)$, then the well-definedness of Θ independent of this choice is equivalent to the condintion (1) since $\pi_1(\hat{F}) \cap \pi_1(T_0) = \langle \gamma_0 \rangle$. Moreover, Θ is a group homomorphism if and only if the condition (2) holds. If G is abelian, then from the defining relation of $\pi_1(T_0) = \langle \alpha_1, \beta_1, \cdots, \alpha_b, \beta_b, \gamma_0 \mid \prod [\alpha_i, \beta_i] = \gamma_0^{2b-2} \rangle$ and from $yxy^{-1} = \rho(\chi(y))(x)$ in $\pi_1(\hat{S})$ where χ is a monodromy of $\hat{S} \to B$ we get the last statement. $\square$

5.2.2 Construction: general case

The lesson from the special case when D contains a graph component and the geometric intuition tell us that the condition that the monodromy action stabilizes the cover of the fiber does not restrict to the special case. Instead, that condition is necessary and sufficient to extend θ over the wedge of circles in the base. In this section, we give a rigorous proof for the general situation that D does not necessarily contain a graph component.

Lemma 5.2.2. *Let b be a genus of B and choose disjoint based loops $\alpha_1, \beta_1, \cdots, \alpha_b, \beta_b$ in B such that their images in homology are a symplectic basis. Let $\iota \colon \bigvee_{j=1}^{2b} S^1 \hookrightarrow B$ be the inclusion of the wedge of the chosen loops and $\hat{\iota} \colon \iota^* \hat{S} \to \hat{S}$ be the induced inclusion. Then there is a commutative diagram with exact rows and columns.*

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
 & & \pi_1(\hat{F}) \arrow[r, equal] \arrow[d] & \pi_(\hat{F}) \arrow[d] & \\
1 \arrow[r] & \mathrm{Ker}(\hat{\iota}_*) \arrow[r] \arrow[d, "\cong"] & \pi_1(\iota^*\hat{S}) \arrow[r, "\hat{\iota}_*"] \arrow[d, "p_*"] & \pi_1(\hat{S}) \arrow[r] \arrow[d, "p_*"] & 1 \\
1 \arrow[r] & \langle\langle \prod_i [\alpha_i, \beta_i] \rangle\rangle \arrow[r] & \langle \alpha_i, \beta_i \rangle \arrow[r, "\iota_*"] \arrow[d] & \pi_1(B) \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd}$$

Proof. Two columns are the homotopy exact sequences associated to the fiber bundle $\hat{S} \to B$ and its pull-back by ι, and the last row is the usual presentation of the fundamental group of the base curve B.

□

Proposition 5.2.3. *In the above situation the following holds:*

1. *The homomorphism θ is the restriction of a homomorphism $\tilde{\Theta} \colon \pi_1(\iota^*\hat{S}) \to G$ if and only if there exist loops $\tilde{\alpha}_i, \tilde{\beta}_i \in \pi_1(\iota^*\hat{S})$ with $p_*\tilde{\alpha}_i = \alpha_i$ and $p_*\tilde{\beta}_i = \beta_i$ and a homomorphism $\theta' \colon \langle \tilde{\alpha}_i, \tilde{\beta}_i \rangle \to G$ satisfying*

$$\theta(x) = \theta'(y^{-1})\theta(yxy^{-1})\theta'(y) \text{ for all } x \in \pi_1(\hat{F}) \text{ and for all } y \in \langle \tilde{\alpha}_i, \tilde{\beta}_i \rangle_{i=1}^{b} \tag{5.2.2}$$

2. *If the extension $\tilde{\Theta}$ exists then it descends to a homomorphism $\Theta \colon \pi_1(\hat{S}) \to G$ if and only if $\tilde{\Theta}$ is trivial on* $\ker \hat{\iota}_*$.

Proof. From the middle column in the diagram 5.2.2, $\pi_1(\iota^*\hat{S})$ is isomorphic to the semi-direct product of $\pi_1(\hat{F})$ and $\langle \alpha_i, \beta_i \rangle$ since $\langle \alpha_i, \beta_i \rangle$ is free. Since $\pi_1(\hat{F})$ is the normal subgroup of $\pi_1(\iota^*\hat{S})$, every element of a of $\pi_1(\iota^*\hat{S})$ can be written as $a = xy$ with $x \in \pi_1(\hat{F})$ and $y \in \langle \tilde{\alpha}_i, \tilde{\beta}_i \rangle$.

If the homomorphism $\tilde{\Theta}$ extending θ exists then we can define a homomorphism $\theta'\colon \langle\tilde{\alpha}_i, \tilde{\beta}_i\rangle \to G$ by its restriction and have

$$\tilde{\Theta}(a) = \tilde{\Theta}(x)\tilde{\Theta}(y) = \theta(x)\theta'(y). \tag{5.2.3}$$

Conversely, if we are given any homomorphism $\theta'\colon \langle\tilde{\alpha}_i, \tilde{\beta}_i\rangle \to G$ then we can define $\tilde{\Theta}\colon \pi_1(\iota^*\hat{S}) \to G$ by (5.2.3). Moreover, $\tilde{\Theta}$ defined in this way is a group homomorphism if and only if we can find θ' satisfying the condition (2). This follows from the definition of the semi-direct product. Finally, the middle row in the diagram 5.2.2 implies that the homomorphism $\tilde{\Theta}$ descends to a homomorphism $\Theta\colon \pi_1(\hat{S}) \to G$ if and only if it is trivial on the kernel. $\square$

Next, we would like to investigate the obstruction for descending $\tilde{\Theta}\colon \pi_1(\iota^*\hat{S}) \to G$ *to* $\Theta\colon \pi_1(\hat{S}) \to G$ *in more detail.*

For abelian group G*, the global extension obstruction of* θ *is definded by*

$$o(\theta) = \sum_{i=1}^{|D|} \deg(D_i \to F)\theta(\gamma_i)$$

where we fix a small loop γ_i *in* $\hat{F}$ *around one of the points in* $F \cap D_i$ *for each component* D_i *of* D*. The element* $o(\theta)$ *in* G *does not depend on the choice of* γ_i*'s nor the order of the components of* D *.*

Corollary 5.2.4. *Assume* G *is abelian. Then*

1. *Given* $\theta\colon \pi_1(\hat{F}) \to G$ *is the restriction of a homomorphism* $\tilde{\Theta}\colon \pi_1(\iota^*\hat{S}) \to G$ *if and only if* θ *is invariant under the monodromy action of* $\pi_1(B)$ *on* $\mathrm{Hom}(\pi_1(\hat{F}), G)$*.*

2. *There exists such an extension* $\tilde{\Theta}\colon \pi_1(\iota^*(\hat{S})) \to G$ *of* θ *which descends to a homomorphism* $\Theta\colon \pi_1(S) \to G$ *if and only if in addition* $o(\theta) = 0$ *in* G*.*

Proof. In the case G is abelian, equation (5.2.2) is equivalent to the condition that θ is invariant under the monodromy action.

Now in order to prove the second claim, let F_0 be the fixed fiber over b_0, B_0 be the fixed horizontal curve isormorphic to the base curve B, and T be the tubular neighborhood of D. Take a subspace Z of $\hat{S}$ given by $Z = T \cup F_0 \cup B_0 \setminus D$. Then we have in $\pi_1(Z)$ and thus in $\pi_1(\hat{S})$ we have the relation

$$\prod \delta_j \prod [\tilde{\alpha}_i, \tilde{\beta}_i] = 1$$

where for each intersection point y_j in $B_0 \cap D$, δ_j is a small loop in B_0 around y_j, and $\tilde{\alpha}_i$ is a lift of α_i to B_0. Moreover, modulo commutators in $\pi_1(Z)$

$$\prod \delta_j = \prod \gamma_i^{\deg(D_i \to F)}$$

since δ_j is homologous to γ_i.

Since $\ker \iota_*$ is normally generated by $p_*(\prod \gamma_i^{\deg(D_i \to F)} \prod [\tilde{\alpha}_i, \tilde{\beta}_i])$, $\ker \hat{\iota}_*$ is normally generate by $\prod \gamma_i^{\deg(D_i \to F)} \prod [\tilde{\alpha}_i, \tilde{\beta}_i]$.

Therefore, $\tilde{\Theta} \colon \pi_1(\iota^* \hat{S}) \to G$ descends to $\Theta \colon \pi_1(\hat{S}) \to G$ if and only if

$$\tilde{\Theta}(\prod \gamma_i^{\deg(D_i \to F)} \prod [\tilde{\alpha}_i, \tilde{\beta}_i]) = 0. \tag{5.2.4}$$

For abelian G, the left hand side of Equation 5.2.4 is exactly the global extension obstruction.

□

Corollary 5.2.5. *Let* $\mathcal{A} = (B \times F, D, \theta \colon \pi_1(\hat{F}) \to G)$ *be a virtual Kodaira fibration with* G *an abelian group. Then there exists an étale cover* $g \colon \tilde{B} \to B$ *such that* $g^*\mathcal{A}$ *is realisable. The minimal degree of such a* g *is the least common muptiple of* $[\pi_1(B) \colon \mathrm{Stab}_\theta]$ *and the order of* $o(\theta)$ *in* G, *where* Stab_θ *is the stabilizer of* θ *under the action of* $\pi_1(B)$ *on* $\mathrm{Hom}(\pi_1(\hat{F}), G)$.

Proof. By Cor 5.2.4 we need that the monodromy action fixes θ and that the global extension obstruction vanishes for $g^*\mathcal{A}$. Let H be the subgroup of

$\pi_1(B)$ corresponding to g. Then the first condition is satisfied if and only if $H \subset \mathrm{Stab}_\theta$, and the second condition is satisfied if and only if $\deg(g) \cdot o(\theta) = 0$. Since the fundamental group of a curve of positive genus has subgroups of every finite index, one can always find a subgroup of Stab_θ which has the required index.

□

5.3 Method to compute monodromy action

In this section we compute the monodromy action of $\pi_1(B)$ *on* $\mathrm{Hom}(\pi_1\hat{F}, G)$ *for a finite abelian group* G*. So let* $\mathcal{A} = (B \times F, D, \theta\colon \pi_1(\hat{F}) \to G)$ *be a virtual Kodaira fibration with* G *finite abelian. Let* $F \cap D = \{x_0, \cdots, x_k, \cdots, x_{d-1}\}$ *and for each* k*, let* γ_k *be a small loop in* $\hat{F}$ *going around one point* x_k*.*

Lemma 5.3.1. *Suppose* G *is abelian, then* $\theta \in \mathrm{Hom}(\pi_1(\hat{F}), G)$ *corresponds uniquely to an element in*

$$H_1(F, F \cap D; G)$$

Proof. Since G is abelian, θ factors uniquely through a homomorphism $H_1(\hat{F}; \mathbb{Z}) \to G$. By Lefschetz-Poincare duality and the Excision theorem,

$$H_1(\hat{F}; \mathbb{Z}) \cong H^1(\hat{F}, \partial\hat{F}) \cong H^1(F, F \cap D; \mathbb{Z})$$

where $\hat{F}$ was abused to denote the compact surface with boundary obtained from F by deleting the small open neighborhoods around the intersection points $F \cap D$. Hence by the Universal coefficient theorem,

$$\mathrm{Hom}(H_1(\hat{F}; \mathbb{Z}), G)) \cong \mathrm{Hom}(H^1(F, F \cap D; \mathbb{Z}), G) \cong H_1(F, F \cap D; G). \quad (5.3.1)$$

Therefore, the claim immediately follows. □

Let $\chi\colon \pi_1(B) \to \mathrm{Mod}(F, F \cap D)$ *be the monodromy homomorphism of the bundle* $(B \times F, D) \to B$ *with the fiber* $(F, F \cap D)$*, the surface with* d *distinguished points, where* $\mathrm{Mod}(F, F \cap D)$ *is the marked mapping class group.*

Proposition 5.3.2. *The monodromy homomorphism of the marked surface bundle* $(B \times F, D) \to B$ *with the fiber* $(F, F \cap D)$ *is given as follows:*

$$\chi : \pi_1(B) \to \mathrm{Mod}(F, \{x_0, x_1, \cdots, x_{d-1}\})$$
$$\alpha \mapsto \mathrm{Push}(q_*\{\tilde{\alpha}^{x_0}, \cdots, \tilde{\alpha}^{x_{d-1}}\})$$

where each $\tilde{\alpha}^{x_i}$ *is the unique lift of* α *under the covering* $p|_D : D \to B$ *starting at* x_i*,* $\{\tilde{\alpha}^{x_0}, \cdots, \tilde{\alpha}^{x_{d-1}}\}$ *is a* d*-stranded surface braid which lives in* $\pi_1(\mathcal{C}(D, d))$*, and* $q : B \times F \to F$ *is the projection.*

Proof. It follows from the following generalized Birman exact sequence.

$$1 \to \pi_1(\mathcal{C}(F, d)) \xrightarrow{\text{Push}} \mathrm{Mod}(F, \{x_0, x_1, \cdots, x_{d-1}\}) \xrightarrow{\mathrm{Forget}} \mathrm{Mod}(F) \to 1$$

Since the surface bundle $B \times F \to B$ is trivial, the image of χ is contained in the $\ker(\mathrm{Forget}) = \mathrm{Im}(\mathrm{Push})$. Moreover, over each base loop α, the trace of d distinguished points in the fiber F under the isotopy from the identity to the representative homeomorphism of $\chi(\alpha)$ is $q_*\{\tilde{\alpha}_{x_0}, \cdots, \tilde{\alpha}_{x_{m-1}}\}$. $\square$

Since we are interested in the subgroup of $\pi_1(B)$ *which stabilizes* θ*, we will need the following homomorphism measuring the difference between before and after taking the monodromy action on the* θ*.*

Proposition 5.3.3. *Assume that* $\theta \in H_1(F, F \cap D; G)$ *satisfies the liftability condition that is,* $\theta(\gamma_j) = \theta(\gamma_k)$ *whenever the loops* γ_j, γ_k *go around the intersection points* $x_j, x_k \in F \cap D_i$*, for the same component* D_i*, respectively. Then the map*

$$\iota \colon \pi_1(B) \to H_1(F, F \cap D; G), \quad \alpha \mapsto \chi(\alpha)_*\theta - \theta$$

is a homomorphism with its image contained in the subgroup $H_1(F; G)$*.*

Proof. Consider the following long exact sequence of the pair $(F, F \cap D)$.

$$0 \to H_1(F; G) \to H_1(F, F \cap D; G) \xrightarrow{\partial} H_0(F \cap D; G) \to H_0(F; G) \to 0$$

First, in order to prove that the image of ι is contained in $H_1(F; G)$, it's enough to prove $\partial(\chi(\alpha)_*\theta - \theta) = 0$ and this is equivalent to prove $\delta^*(\chi(\alpha)_*\theta - \theta) = 0$ by the following commutative diagram :

$$\begin{tikzcd}
H_0(F \cap D; G) \arrow[d, "\cong"] & H_1(F, F \cap D; G) \arrow[l, "\partial"'] \arrow[d, "\cong"] \\
\mathrm{Hom}(H^0(F \cap D; \mathbb{Z}), G) & \mathrm{Hom}(H^1(F, F \cap D), G) \arrow[l, "\delta^*"']
\end{tikzcd}$$

Here the verical isomorphisms are given by the Universal Coefficient theorem.

Now consider the following commutative diagram with Lefschetz-Poincare duality as the vertical isomorphisms. [22]

$$\begin{tikzcd}
\arrow[r] & H^0(\partial\hat{F}) \arrow[r, "\delta"] \arrow[d, "\cong"] & H^1(\hat{F}, \partial\hat{F}) \arrow[r] \arrow[d, "\cong"] & H^1(\hat{F}) \arrow[r] \arrow[d, "\cong"] & {} \\
\arrow[r, "\partial"] & H_1(\partial\hat{F}) \arrow[r, "i_*"] & H_1(\hat{F}) \arrow[r] & H_1(\hat{F}, \partial\hat{F}) \arrow[r] & {}
\end{tikzcd}$$

Then the restriction of $\chi(\alpha)_*\theta - \theta$ to the image of the coboundary δ corresponds to the restriction of $\chi(\alpha)_*\theta - \theta$ to the image of i_* by the vertical isomorphism. Therefore, the problem reduces to show that the restriction of $\chi(\alpha)_*\theta - \theta$ to the subgroup L of $H_1(\hat{F})$, the image of i_*, is identically zero. This is true because the monodromy action might permute the puntures x_k (correspondingly, the small loops γ_k) but preserves the component of D to which they belong (and the orientaions of the loops).

Now it remains to prove ι is a homomorphism. First recall that the monodromy action on $\mathrm{im}(\iota) \subset H_1(F; G)$ is trivial. Therefore,

$$\begin{aligned}\iota(\alpha\beta) &= \chi(\alpha\beta)_*(\theta) - \theta \\ &= \chi(\alpha)_*(\chi(\beta)_*\theta - \theta) + \chi(\alpha)_*(\theta) - \theta \\ &= \chi(\beta)_*(\theta) - \theta + \chi(\alpha)_*(\theta) - \theta \\ &= \iota(\alpha) + \iota(\beta)\end{aligned}$$

$\square$

We have already seen that the monodromy action $\chi(\alpha)_*$ *is trivial on* $H_1(F;G) \subset H_1(F, F \cap D; G)$. *This suggests that we only need to know* θ *modulo* $H_1(F;G)$.

Lemma 5.3.4. *Let* G *be an abelian group and consider* θ *as an element in* $H_1(F, F \cap D; G)$. *Then*

$$\theta \equiv \sum_{i=1}^{d-1} \delta_i \otimes \theta(\gamma_i) \qquad \textit{mod } H_1(F;G)$$

$$\textit{and} \quad \partial\theta = \sum_{i=0}^{d-1} x_i \otimes \theta(\gamma_i) \qquad \in H_0(F \cap D; G)$$

where we denote the homology class of x_i *by the same symbol.*

Proof. We denote the Hom-dual of γ_i by γ_i^* and the class of a path from x_0 to x_i by δ_i. Since $H_1(\hat{F}) \cong \langle \alpha_j, \beta_j, \gamma_i \mid \sum_{i=0}^{d-1} \gamma_i = 0 \rangle$, modulo $\mathrm{Hom}(H_1(F), G)$

$$\begin{aligned}\theta &\equiv \sum_{i=1}^{d-1} \gamma_i^* \otimes \theta(\gamma_i) \qquad \in \mathrm{Hom}(H_1(\hat{F}), G) \\ &\cong \sum_{i=1}^{d-1} \delta_i \otimes \theta(\gamma_i) \qquad \in H_1(F, F \cap D; G)\end{aligned}$$

by Lefschetz-Poincare duality. Therefore,

$$\begin{aligned}\partial\theta &= \sum_{i=1}^{d-1}(x_i - x_0)\otimes\theta(\gamma_i)\\ &= \sum_{i=0}^{d-1} x_i \otimes \theta(\gamma_i).\end{aligned}$$

The last equality comes from the well-definedness of θ, that is, $\theta(\sum_{i=0}^{d-1}\gamma_i) = 0$. $\square$

In order to describe the ι *explicitly, we first introduce the notion of weighted transfer pairing. For a loop* α *in* B *based at* b_0 *and* $x \in D \cap F$*, let* $\tilde{\alpha}^x$ *be the unique lift of* α *to* D *starting at* x*. Now, consider the bilinear pairing defined by*

$$H_1(B;\mathbb{Z}) \times H_0(F\cap D; G) \to H_1(F, F\cap D; G), \quad (\alpha, x) \mapsto \hat{\alpha}(x) := q_*\tilde{\alpha}^x,$$

where we identify $x \in F \cap D$ *with its homology class. We call this the weighted transfer pairing since (for* $\mathbb{Z}$*-coefficients)* $\hat{\alpha}(x_0 + \cdots + x_{d-1}) = q_* p^! \alpha$ *where* $p^!\colon H_1(B) \to H_1(B, b_0) \to H_1(D, D\cap F)$ *is the transfer map.*

Theorem 5.3.5. *Let* G *be an abelian group and* $D \subset B \times F$ *a divisor in a product of curves such that the projection to* B *is étale. Then the monodromy action of* $\alpha \in \pi_1(B)$ *on an element* $\theta \in H_1(F, F\cap D; G)$ *is given by the weighted transfer pairing of the homology class of* α *with the boundary of* θ *as follows:*

$$\chi(\alpha)_*\theta - \theta = \hat{\alpha}(\partial\theta)$$

Proof. From Proposition 5.3.2 and Lemma5.3.4, we only need to compute the monodromy action on each path $\delta_i \otimes \theta(\gamma_i)$. Once we prove the following formula

$$\chi(\alpha)_*(\delta_i) = [\delta_i] - [q_*\tilde{\alpha}^{x_0}] + [q_*\tilde{\alpha}^{x_i}], \tag{5.3.2}$$

we can derive that

$$\chi(\alpha)_*\theta - \theta = \sum_{i=1}^{d-1} (q_*\tilde{\alpha}^{x_i} - q_*\tilde{\alpha}^{x_0}) \otimes \theta(\gamma_i) = \sum_{i=1}^{d-1} \hat{\alpha}(\partial\delta_i \otimes \theta(\gamma_i))$$

Now, for proving the formula 5.3.2, we assume θ is a path δ_i connecting two marked points x_0 to x_i. Then $\chi(\alpha)_*\theta$ is homotopic relative to the end points to the composition of three paths $(\tilde{\alpha}^{x_0})^{-1}, \delta_i$ and $\tilde{\alpha}^{x_i}$ in $F \times I$. The projection to F gives us a homotopy equivalence and thus the claimed formula follows. □

By Cor 5.2.4, the degree of the minimal pullback of the base for the realisability of the virtual Kodaira fibration is given by the index of the subgroup Stab_θ *in* $\pi_1(B)$.

Corollary 5.3.6. *Assume* G *is finite abelian and* θ *satisfies the liftability condition, that is* θ *is constant, say* g_i*, along the same component* D_i*. Write* $F \cap D_i = \{x_{ij}\}_j$ *so that* $F \cap D = \{x_{ij}\}_{i,j}$ *and let* γ_{ij} *be a small loop in* $\hat{F}$ *around a point* x_{ij}*. Then the above* $\iota\colon \pi_1(B) \to H_1(F, F\cap D; G)$ *induces and is determined by*

$$\iota\colon H_1(B;\mathbb{Z}) \to H_1(F;G), \qquad \alpha \mapsto \hat{\alpha}(\sum_{i,j} x_{ij} \otimes \theta(\gamma_{ij})) = q_* \sum_i p_i^! \alpha \otimes g_i \quad (5.3.3)$$

Consequently, the index $[\pi_1(B)\colon \mathsf{Stab}_\theta]$ *equals to the cardinality of the quotient group* $H_1(B;\mathbb{Z})/\ker(\iota) \cong \mathrm{Im}\,\iota$.

Moreover, if $D = \cup \Gamma_{\phi_i}$ *is a disjoint union of graphs, then* $\iota(\alpha) = \sum_i \phi_{i*}\alpha \otimes g_i$.

Proof. Since ι is a homomorphism and $H_1(F;G)$ is abelian, the commutator subgroup $[\pi_1(B), \pi_1(B)]$ lies in the kernel of ι. Therefore, ι induces the homormophism from $H_1(B;\mathbb{Z})$ and by the third isomorphism theorem in group theory the claim follows. The formula 5.3.3 follows from the above Theorem 5.3.5, $H_1(\hat{F}) = \langle \alpha_j, \beta_j, \gamma_i \mid \sum_{i=0}^{d-1} \gamma_i = 0 \rangle$ and the well-definedness of θ. □

Remark 5.3.7. In Theorem 5.3.5, we have seen the monodromy action of $\alpha \in \pi_1(B)$ on an element $\theta \in H_1(F, F \cap D)$. In this remark, we'll explain the monodromy action in more detail and then by applying it to the specific element θ it becomes clear that which conditions are needed for stabilizing θ. Let $D \subset B \times F$ be a smooth divisor which is unramified over B. Then the pair $(B \times F, D) \xrightarrow{p} B$ is a locally trivial fibration with the fiber $(F, F \cap D)$, the surface F with d distinguished points $\{x_0, \cdots, x_{d-1}\}$. Its monodromy homomorphism $\chi\colon \pi_1(B) \to \mathsf{Mod}(F, F \cap D)$, which goes to the marked mapping class group, takes values in the subgroup $\mathsf{Br}_d(F)$, the d-stranded surface braid group, because $B \times F \to B$ is a trivial bundle. This follows from the generalised Birman exact sequence[4, 19]

$$1 \to \mathsf{Br}_d(F) \xrightarrow{push} \mathsf{Mod}(F, d) \xrightarrow{forget} \mathsf{Mod}(F) \to 1.$$

Therefore, the monodromy action of the marked mapping class $\chi(\alpha) \in \mathsf{Mod}(F, F \cap D)$ on the relative homology group $H_1(F, F \cap D)$ is determined by the action of the push map along the surface braid $\beta(\alpha) \in \mathsf{Br}_d(F)$, where $\beta\colon \pi_1(B) \to \pi_1(C(F, d),) = \mathsf{Br}_d(F)$ is given by $[\alpha(t)] \mapsto [q_*(p^{-1}(\alpha(t)) \cap D)]$. On the other hand, we have an exact sequence $1 \to H_1(F) \to H_1(F, F \cap D) \xrightarrow{\partial} H_0(F \cap D) \to H_0(F) \to 1$. Hence $H_1(F, F \cap D)$ is isomorphic to a direct sum of a natural subgroup $H_1(F)$ and a (noncanonical) complement H isomorphic to $\mathrm{Im}\partial$. We can take H as the subgroup generated by the paths δ_i in a fixed disk connecting two marked points x_0 and x_i. With respect to a basis subordinate to this decomposition, the induced action on the relative homology is given by

$$\chi(\alpha)_* = \left[\begin{array}{c|c} \mathrm{Id} & \psi(\alpha) \\ \hline 0 & \sigma(\alpha) \end{array}\right]$$

since the action is trivial on $H_1(F)$. In order to look at the action on H, consider the short exact sequence $1 \to \mathsf{PBr}_d(F) \to \mathsf{Br}_d(F) \to S_d \to 1$. Then every surface braid $\beta(\alpha)$ can be written as a product of a pure surface braid $\bar{\beta}(\alpha)$ and a braid $\pi_\beta(\alpha)$ supported on the disk used above for the definition of

H. $\pi_\beta(\alpha)$ contributes as the associated permutation matrix $\sigma(\alpha)$ and $\bar{\beta}(\alpha)$ constributes as a matrix $\psi(\alpha)$ where the i-th column gives the difference in homology $H_1(F)$ of the loops traced by x_i and x_0. To verify this, first write $\bar{\beta}(\alpha) = (\bar{\beta}_0(\alpha), \bar{\beta}_1(\alpha), \cdots, \bar{\beta}_{d-1}(\alpha))$ so that the i-th strand $\bar{\beta}_i(\alpha)$ is the trace of x_i. Then the push map along each strand of the pure braid acts trivially on the subgroup L of $H_1(\hat{F})$ generated by γ_i's, and for each $\beta \in H_1(F) \subset H_1(\hat{F})$,

$$\mathsf{Push}(\bar{\beta}_i(\alpha))_*(\beta) = \beta + \langle \bar{\beta}_i(\alpha), \beta \rangle \gamma_i.$$

Hence, for any $\theta \in H_1(F, F \cap D; G)$,

$$\mathsf{Push}(\bar{\beta}_i(\alpha))_*\theta = \theta + [(\bar{\beta}_i(\alpha)] \otimes \theta(\gamma_i)$$

by Poincare duality. Therefore, by [20],

$$\mathsf{Push}(\bar{\beta}(\alpha))_*\theta - \theta = \sum_{i=0}^{d-1} [\bar{\beta}_i(\alpha)] \otimes \theta(\gamma_i) = \sum_{i=1}^{d-1} = ([\bar{\beta}_i(\alpha)] - [\bar{\beta}_0(\alpha)]) \otimes \theta(\gamma_i)$$

5.4 Virtual Kodaira fibrations with small realisation signature

5.4.1 Numerical classification of virtual Kodaira fibrations of virtual signature 4

By Proposition 5.1.7, we have a one-to-one correspondence between the set of double étale double Kodaira fibrations $S \to B \times F$ *which are* G*- Galois and the set of realisable double étale virtual Kodaira fibrations* $\mathcal{A} = (B \times F, D, \theta\colon \pi_1(F) \to G)$*. Therefore we can classify double étale Galois double Kodaira fibrations with signature* 4 *by first making a list of all possible candidates of double étale virtual Kodaira fibrations with virtual signature* 4*, and then check the realisability of* *each case in the list. Recall from Definition 5.1.8 that the virtual signature of the virtual Kodaira fibration is defined to be*

$$\sigma(\mathcal{A}) = -\frac{|G|e(B)}{3}\sum_{i=1}^{m}\frac{r_i^2-1}{r_i^2}d_i = -\frac{|G|e(F)}{3}\sum_{i=1}^{m}\frac{r_i^2-1}{r_i^2}e_i,$$

so that is coincides with the siganature of the actual Kodaira fibration if $\mathcal{A}$ is realisable. Hence if we restrict our attention to signature 4, then from this signature formula, we can find the numerical restrictions on $(|G|, r = (r_i), f, e_i, b, d_i, g(D_i))$ *as the following proposition.*

Proposition 5.4.1. *The possible numerical invariants of a virtual double étale Kodaira fibration of virtual signature 4 are as follows: for each component D_i of D the ramification $r_i = 2$ and the other invariants can be (up to reordering the D_i) given as in Table5.1, where we also collect partial information on realisability.*

Proof. $4 = \frac{-|G|e(F)}{3}\sum_{i=1}^{m}\frac{r_i^2-1}{r_i^2}e_i$ implies $f \geq 2$ and $|G| \leq 8$. We split into two cases depending on the number of components of D.
(1) case 1: $m = 1$
In this case, $r_1 = 2$ and $8 = |G| \cdot (f-1) \cdot e_1$. This together with $(f-1)\cdot e_1 = (b-1)\cdot d_1$ gives a complete list of possibilities for $m = 1$ case in the table : G_1 type and $C_1, C_2, \cdots, C_7$ type.
(2) case 2: $m \geq 2$
In this case, $4 = \frac{-|G|e(F)}{3}\sum_{i=1}^{m}\frac{r_i^2-1}{r_i^2}e_i$ implies $|G| \leq 4$.
(case 2-1) If $|G| = 4$, then $\sum_{i=1}^{m}(1-\frac{1}{r_i^2})\cdot e_i \leq \frac{3}{2}$ and the equality is realised if and only if $m = 2, r_1 = r_2 = 2, e_1 = e_2 = 1, f = 2$. This gives us a single possible case: G_2 type.
(case 2-2) $|G| = 3$ is impossible.
(case 2-3) If $|G| = 2$, then $r_i = 2$ for all i, and hence $4 = (f-1)\cdot\sum_{i=1}^{m} e_i$. This together with $(f-1)\cdot e_i = (b-1)\cdot d_i$ gives us a complete list of possibil-

Table 5.1: virtual Kodaira fibraions with virtual signature 4

type	b	f	$\|G\|$	$g(D_i)$	(d_i, e_i)	realisability	$((\tilde{f}, \tilde{b}), (\tilde{f}_2, \tilde{b}_2), \tilde{\sigma})$
G_1	2	2	8	2	$(1,1)$	no	((11,9),(81,2),32)
G_2	2	2	4	$(2,2)$	$(1,1),(1,1)$	no	((7,17),(97,2),64)
							((4,17),(49,2),32) $G = \mathbb{Z}/2$)
G_3	3	3	2	$(3,3)$	$(1,1),(1,1)$	no	((6,9),(21,3),16),
							((6,33),(81,3),64)
G_4	3	2	2	$(3,3)$	$(1,2),(1,2)$	no	((4,17),(49,2),32)
C_1	2	2	2	5	$(4,4)$	?	
C_2	3	2	2	5	$(2,4)$	?	
C_3	2	3	2	5	$(4,2)$	?	
C_4	3	3	2	5	$(2,2)$	no	((6,9),(21,3),16)
							((6,17),(41,3),32),
							((6,65),(161,3),128)
C_5	2	5	2	5	$(4,1)$	no	((11,9),(21,5),32)
C_6	3	5	2	5	$(2,1)$	no	((10,65),(145,5),128)
C_7	2	2	4	3	$(2,2)$	no	((7,9),(49,2),32)
							((4,9),(25,2),16) $G = \mathbb{Z}/2$)
C_8	2	3	4	3	$(2,1)$	no	((11,9),(41,3),32)
C_9	2	2	2	$(3,3)$	$(2,2),(2,2)$	?	
C_{10}	2	2	2	$(4,2)$	$(3,3),(1,1)$	?	
C_{11}	2	3	2	$(3,3)$	$(2,1),(2,1)$	no	((7,9),(25,3),32)
C_{12}	2	2	2	$(2,2,3)$	$(1,1),(1,1),(2,2)$	?	

ities: G_3, C_9, C_{10}, C_{11}, and C_{12} type. Four disjoint graph case is impossible by Proposition 5.3 in [12]. □

5.4.2 Automorphisms without fixed points on curves of small genus

In order to classify the examples of graph type, in this section we classify fixed-point-free automorphisms on curves of genus up to nine. Let B *be a curve of genus* $b \leq 9$ *and assume that* $\phi \in \mathrm{Aut}(B)$ *acts on* B *without fixed points. Let* d *be the order of* ϕ *and* $(q|r_1, \cdots, r_m)$ *be the ramification type of the quotient map* $B \to B/\langle\phi\rangle$ *which is a* $\mathbb{Z}/d$ *cover. We first classify the possible ramification types of a free automorphism* ϕ *and then classify its topological types.*

Proposition 5.4.2. *The ramification types of a fixed-point-free automorphism* ϕ *of order* d *on a curve of genus* $b \leq 9$ *are exactly given in the Table5.4.2:*

Lemma 5.4.3. *If* $B \to B/\langle\phi\rangle$ *is étale, then we have* $b - 1 = d(q-1)$*, which gives the unramified cases listed in the Table5.4.2.*

Now we need to consider the ramified cases. The ramified cover $B \to B/\langle\phi\rangle$ *over* m *branch points* $\{P_1, \cdots, P_m\}$ *corresponds to a surjection*

$$\eta : \pi_1(B/\langle\phi\rangle\backslash\{P_1, \cdots, P_m\}) = \langle \alpha_1, \cdots, \beta_q, \gamma_1, \cdots, \gamma_m \mid \prod[\alpha_j, \beta_j]\prod \gamma_i = 1\rangle \to \mathbb{Z}/d \tag{5.4.1}$$

with $\eta(\gamma_i) \in \mathbb{Z}/d\backslash\{0\}$*. Denoting* $a_i = \eta(\gamma_i)$*, the surjection* η *gives an* $m-$*tuple* $(a_1, \cdots, a_m) \in (\mathbb{Z}/d \setminus \{0\})^m$ *such that* $\sum_{i=1}^m a_i = 0$ *(in particular* $m \geq 2$*).*

Lemma 5.4.4. *Let* $d, m \geq 2$ *and* $q \geq 0$ *be integers and* $(a_1, \cdots, a_m) \in (\mathbb{Z}/d \setminus \{0\})^m$ *such that* $\sum_{i=1}^m a_i = 0$*. Then*

1. *There exists a surjection* η *giving rise to the* m*-tuple* $(a_1, \cdots, a_m)$ *if either* $q > 0$ *or* $q = 0$ *and* $\{a_i\}_{i=1}^m$ *generate* $\mathbb{Z}/d$*.*

Table 5.2: free automorphisms of curves of small genus

genus b	order d	ramification type
9	2	$(5 \mid -)$
9	4	$(3 \mid -)$
9	8	$(2 \mid -)$
9	4	$(2 \mid 2^4)$
9	16	$(1 \mid 2^2)$
9	12	$(1 \mid 3^2)$
9	10	$(1 \mid 5^2)$
9	8	$(1 \mid 2^4)$
9	8	$(1 \mid 2, 4^2)$
9	6	$(1 \mid 3^4)$
9	4	$(1 \mid 2^8)$
9	6	$(0 \mid 2^4, 3^4)$
9	10	$(0 \mid 2^4, 5^2)$
9	12	$(0 \mid 2, 3^2, 4^2)$
8	7	$(2 \mid -)$
8	14	$(1 \mid 2^2)$
8	6	$(1 \mid 2^2, 3^2)$
8	18	$(0 \mid 2^2, 9^2)$
8	15	$(0 \mid 3^2, 5^2)$
8	12	$(0 \mid 4^2, 6^2)$
8	10	$(0 \mid 2^2, 5^3)$
8	6	$(0 \mid 2^2, 3^5)$
8	6	$(0 \mid 2^6, 3^2)$
6	5	$(2 \mid -)$
6	10	$(1 \mid 2^2)$
6	14	$(0 \mid 2^2, 7^2)$
6	12	$(4^2, 3^2)$
6	6	$(0 \mid 2^2, 3^4)$

genus b	order d	ramification type
7	2	$(4 \mid -)$
7	3	$(3 \mid -)$
7	6	$(2 \mid -)$
7	4	$(2 \mid 2^2)$
7	12	$(1 \mid 2^2)$
7	9	$(1 \mid 3^2)$
7	8	$(1 \mid 4^2)$
7	6	$(1 \mid 2^4)$
7	6	$(1 \mid 3^3)$
7	4	$(1 \mid 2^6)$
7	12	$(0 \mid 3, 4^2, 6)$
7	6	$(0 \mid 2^4, 3^3)$
5	2	$(3 \mid -)$
5	4	$(2 \mid -)$
5	8	$(1 \mid 2^2)$
5	6	$(1 \mid 3^2)$
5	4	$(1 \mid 2^4)$
5	6	$(0 \mid 2^4, 3^2)$
4	3	$(2 \mid -)$
4	6	$(1 \mid 2^2)$
4	6	$(0 \mid 2^2, 3^3)$
4	10	$(0 \mid 2^2, 5^2)$
3	2	$(2 \mid -)$
3	4	$(1 \mid 2^2)$
2	6	$(0 \mid 2^2, 3^2)$

2. *Assume (1) holds and let* $H_i = \langle a_i \rangle$. *Then a generator of* $\mathbb{Z}/d$ *corresponds to a fixed-point-free automorphism* ϕ *on the ramified cover induced by* η *if and only if* H_i *is a proper subgroup of* $\mathbb{Z}/d$ *for all* i.

Proof. Since $\mathbb{Z}/d$ is abelian, once we have $\sum_i a_i = \eta(\prod \gamma_i) = 0$ we can construct a well-defined homomorphism η by choosing the images of α_i and β_i arbitrarily. If $q > 0$, then we can always make η be a surjection by setting $\eta(\alpha_1) = 1$, a generator. If $q = 0$, then the image of η is generated by a_i's and thus we need the condition stated above. Finally, if we require a generator of $\mathbb{Z}/d$ to be free as an automorphism of the ramified cover, then it cannot be contained in the stabiliser subgroup of any ramification points. □

Proof of Proposition 5.4.2.. Now it's enough to study the possible $d, m, q, (a_1, \cdots, a_m)$ satisfying the conditions from Lemma 5.4.4. Refer to [39] for the remaining proof. □

Classification of the finite order automorphisms up to topological equivalence was studied by Nielsen [47]; Let ϕ *be an automorphism of* B *of order* d *and ramification type* $(q \mid r_1, \cdots, r_m)$. *Then the quotient map* $B \to B/\langle\phi\rangle$ *induces a monodromy homomorphism* $\rho\colon \langle \alpha_1, \beta_1, \cdots \alpha_q, \beta_q, \gamma_1, \cdots \gamma_m \mid \prod_i [\alpha_i, \beta_i] \prod_j \gamma_j = 1\rangle \to \langle\phi\rangle \cong \mathbb{Z}/d$ *as in 5.4.1.*

What Nielsen proved is that two automorphisms of the same order are topologically equivalent if and only if they have the same monodromy exponents $(\rho(\gamma_1), \cdots, \rho(\gamma_m))$ *at the branch points up to permutation. We call this unordered tuple* $(\rho(\gamma_1), \cdots, \rho(\gamma_m))$ *the Nielsen type.*

For each ramification type in Proposition 5.4.2 the possible Nielsen types can be easily analysed yielding the following result.

Proposition 5.4.5. *The ramification types of Proposition 5.4.2 are uniquely realized by a Nielsen type except for the following cases:*

genus b	*order* d	*ramification type*	*Nielsen type (modulo ~)*
9	10	$(1 \mid 5^2)$	$(2,8),\ (4,6)$
9	8	$(1 \mid 2,4^2)$	$(4,2,2) \sim (4,6,6)$
9	12	$(0 \mid 2,3^2,4^2)$	$(6,4,8,3,3) \sim (6,4,8,9,9)$
9	10	$(0 \mid 2^4,5^2)$	$(5,5,5,5,2,8),\ (5,5,5,5,4,6)$
8	18	$(0 \mid 2^2,9^2)$	$(9,9,2,16),\ (9,9,4,14),\ (9,9,8,10)$
8	15	$(0 \mid 3^2,5^2)$	$(5,10,3,12),\ (5,10,6,9)$
8	10	$(0 \mid 2^2,5^3)$	$(5,5,2,2,6) \sim (5,5,8,8,4),$
			$(5,5,2,4,4) \sim (5,5,8,6,6)$
8	6	$(0 \mid 2^2,3^5)$	$(3,3,2,2,2,2,4) \sim (3,3,2,4,4,4,4)$
7	6	$(1 \mid 3^3)$	$(2,2,2) \sim (4,4,4)$
7	12	$(0 \mid 3,4^2,6)$	$(4,3,3,2) \sim (8,9,9,10),$
			$(4,9,9,2) \sim (8,3,3,10)$
7	6	$(0 \mid 2^4,3^3)$	$(3,3,3,3,2,2,2) \sim (3,3,3,3,4,4,4)$
6	14	$(0 \mid 2^2,7^2)$	$(7,7,2,12),\ (7,7,4,10),\ (7,7,6,8)$
4	6	$(0 \mid 2^2,3^3)$	$(3,3,2,2,2) \sim (3,3,4,4,4)$
4	10	$(0 \mid 2^2,5^2)$	$(5,5,2,8),\ (5,5,4,6)$

The equivalence relation ~ on the Nielsen types is given by the multiplication by -1. *The Nielsen types related by ~ correspond to topologically equivalent configurations of the form* $\Gamma_{\mathsf{id}} \cup \Gamma_{\phi} \subset \mathsf{B} \times \mathsf{B}$.

5.5 Computing realisation signature: examples

In this section, we want to check the realisability of possible candidates of double étale virtual Kodaira fibrations of virtual signature 4.

5.5.1 Examples of graph type

The most typical examples come from the case when D *is a disjoint union of graphs of maps* $\phi_1, \cdots, \phi_m \colon B \to F$*. First we restrict our attention to the case* $B = F$*, and in the last paragraph of this section we'll consider the case of graphs of non-automorphisms. Now we would like to find automorphisms* $\phi_1, \cdots, \phi_m$ *of* B *whose graphs are disjoint. We may assume that the first component is the graph of* id_B*, by precomposing with* ϕ_1^{-1}*. Thus, if* $m = 1$ *then there exists the only configuration* $(B \times B, \Gamma_{\mathrm{id}})$*. If* $m = 2$ *then the second component* Γ_{ϕ_2} *is disjoint from the first component if and only if the automorphism* ϕ_2 *has no fixed points. In this case* $D = \Gamma_{\mathrm{id}} \cup \Gamma_{\phi}$ *for some fixed point free automorphism* ϕ*. Fixed point free automorphisms on curves of small genus are classified in the Proposition 5.4.2. Thus in case* $m = 2$*, there is a unique case if* $g(B) = 2$*, and two different cases if* $g(B) = 3$ *by Proposition 5.4.2, 5.4.5. First note that the global extension obstruction always vanishes for the graph type when* G *is abelian. So we only need to compute the index* $[\pi_1(B) \colon \mathrm{Stab}_\theta]$ *for each case to determine the realisability.*

Example 5.5.1 (G_2 type: the free automorphism of a curve of genus two). Consider the unique free automorphism ϕ on a curve $B = F$ of genus 2, which has order 6 and the ramification type $(0 \mid 2, 2, 3, 3)$ (See Table 5.4.2). We can easily check that there is a unique homomorphism with this ramification type and hence the unique topological model of ϕ realised by this surjection

$$\langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \mid \gamma_1\gamma_2\gamma_3\gamma_4 \rangle \to \mathbb{Z}/6, \quad (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \mapsto (\phi^3, \phi^3, \phi^2, \phi^4).$$

We can observe that ϕ^5 is also a fixed point free automorphism of order 6, and $\phi^5 = \tau \circ \iota$, where $\tau = \phi^2$ is an order 3 automorphism with 4 fixed points and $\iota = \phi^3$ is a an involution with 6 fixed points. If we consider the surface of genus two as two 2−spheres connected by three 1- handles, then τ is given by rotation by $\frac{2\pi}{3}$ about the axis passing through two 2-spheres and

ι is a hyperelliptic involution. Since ι and τ act on the homology $H_1(B;\mathbb{Z})$ by $\iota_* = -\mathrm{id}$ and

$$\tau_* = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

with respect to the basis $\{m_1, m_2, l_1, l_2\}$, we have

$$\phi_*^5 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We now consider $D = \Gamma_{\mathrm{id}} \cup \Gamma_{\phi^5} \subset B \times F$ where $B = F$ and any surjective homomorphism θ which defines the double branched cover of F branched over $F \cap D$. For $G = \mathbb{Z}/2 = \{0, 1\}$, there is the unique choice for $\theta(\gamma_1) = \theta(\gamma_2) = 1$. Hence by Cor5.3.6 we can compute

$$\iota \colon H_1(B;\mathbb{Z}) \to H_1(F;\mathbb{Z}/2), \quad (\mathrm{id} + \phi_*^5)(\alpha) \otimes 1$$

and $\mathrm{ker}(\iota) = \langle 2m_1, 2m_2, 2l_1, 2l_2 \rangle$ which implies

$$H_1(B;\mathbb{Z})/\mathrm{ker}(\iota) \cong (\mathbb{Z}/2)^4$$

and thus the index $[\pi_1(B) \colon \mathrm{Stab}_\theta]$ is equal to 16. Therefore, the degree of the minimal pullback is 16, the realising genus $\tilde{b} = 17$, the realization signature $\tilde{\sigma} = \frac{2}{3} \cdot 2 \cdot 16 \cdot (\frac{3}{4} + \frac{3}{4}) = 32$, and the fiber genus $\tilde{f} = 4$.

If we consider the case where $G = \mathbb{Z}/4$ or $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ as in the case G_2 in the Table 5.1, we can use the similar argument to show $\tilde{b} = 17, \tilde{\sigma} = 64$, and $\tilde{f} = 7$ as above since the ramification order at every branch point is 2. Therefore, G_2 type is not realisable.

Example 5.5.2 (G_3 type: the free involution on a curve of genus 3). Let $B = F$ be a curve of genus 3 admitting a fixed-point-free involution ϕ. By Proposition

5.4.5 there exists unique such ϕ up to topological equivalence realised by the surjection

$$\langle \alpha_1, \beta_1, \alpha_2, \beta_2 \mid [\alpha_1, \beta_1][\alpha_2, \beta_2] \rangle \to \mathbb{Z}/2 \quad : (\alpha_1, \beta_1, \alpha_2, \beta_2) \mapsto (1, 1, 1, \phi).$$

If we arrange three holes in a row and consider the 180° rotation about the axis passing through the middle hole, then the induced action on the homology $H_1(B; \mathbb{Z})$ is given by

$$\phi_* = \begin{pmatrix} 1 & 0 & & & & \\ 0 & 1 & & & & \\ & & & & 1 & 0 \\ & & & & 0 & 1 \\ & & 1 & 0 & & \\ & & 0 & 1 & & \end{pmatrix}$$

with respect to the basis $\{m_2, l_2, m_1, l_1, m_3, l_3\}$. *Hence,*

$$\iota : H_1(B; \mathbb{Z}) \to H_1(F; \mathbb{Z}/2), \quad \alpha \mapsto (\alpha + \phi_* \alpha) \otimes 1$$

has $\ker(\iota) = \langle m_2, l_2, 2m_1, 2m_3, 2l_1, 2l_2, m_1 - m_3, l_1 - l_3 \rangle$ *which implies*

$$H_1(B; \mathbb{Z}) / \ker(\iota) \cong (\mathbb{Z}/2)^2$$

and thus the index $[\pi_1(B) \colon \mathrm{Ker}(\iota)]$ *is equal to* 4*. Therefore, the degree of the minimal pullback is* 4*, the realization genus* $\tilde{b} = 9$*, the realization signature* $\tilde{\sigma} = 16$*, the fiber genus* $\tilde{f} = 6$*.*

Remark 5.5.3. This example was first considered in [1, 27] and improved in [7]. We can observe that the minimal degree covering of the base we found coincides with the covering appeared in [7] guaranteeing the divisibility of the divisor.

Example 5.5.4 (G_3 type: the free automorphism of order 4 on a curve of genus 3). Let ϕ be a fixed-point-free automorphism of order 4 on a curve $B = F$ of genus 3. By Proposition 5.4.5, there is a unique topological type, realised by the surjection

$$\langle \alpha, \beta, \gamma_1, \gamma_2 \mid [\alpha, \beta]\gamma_1\gamma_2 \rangle \to \mathbb{Z}_4, \qquad (\alpha, \beta; \gamma_1, \gamma_2) \mapsto (1, \phi; \phi^2, \phi^2).$$

From this, we can find the topological model as in Figure 5.1. Now consider $D = \Gamma_{id} + \Gamma_\phi \subset B \times F$ and let x_0 (or x_1) be the intersection point of Γ_{id} (or Γ_ϕ) and the fixed fibre F over b_0, respectively. Consider any θ which defines the double cover of F branched over $F \cap D$.

By Cor 5.3.6, the degree of the minimal pullback is given by

$$|H_1(B; \mathbb{Z}) / \ker \iota|$$

where

$$\iota \colon H_1(B; \mathbb{Z}) \to H_1(F; \mathbb{Z}/2), \quad \alpha \mapsto (\alpha + \phi_* \alpha) \otimes 1.$$

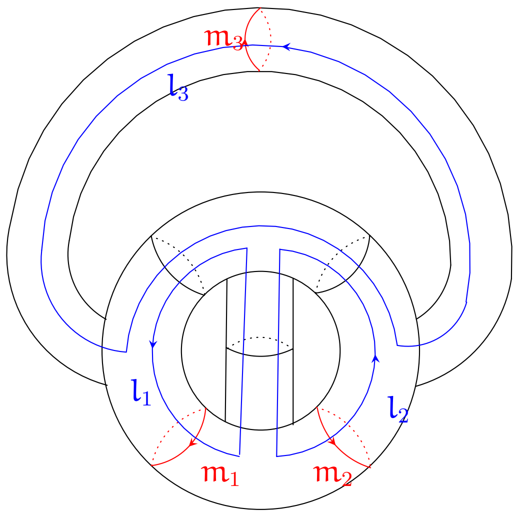
Since

$$\phi_* = \begin{pmatrix} 0 & 0 & -1 & & & \\ 1 & 1 & 1 & & & \\ 0 & -1 & 0 & & & \\ & & & 1 & 0 & -1 \\ & & & 1 & 0 & 0 \\ & & & 1 & -1 & 0 \end{pmatrix}$$

with respect to the basis $\{m_1, m_2, m_3, l_1, l_2, l_3\}$ of $H_1(F; \mathbb{Z})$ depicted in Figure 5.1, the degree of the minimal pullback is 16, and hence the realisation genus $\tilde{b} = 33$, the realisation signature $\tilde{\sigma} = 64$, and the fiber genus $\tilde{f} = 6$. Therefore, this example together with Example 5.5.2 tells us that virtual Kodaira fibration corresponding to the G_3 type is not realisable.

Example 5.5.5 (G_4 type: graphs of non-automorphisms). First we can find one configuration of G_4 type in the complex category as follows. Let ϕ be the unique free automorphism of F of genus 2 which has order 6 and consider $D_0 = \Gamma_{id} \cup \Gamma_\phi \subset F \times F$. After taking the pull-back of D_0 by any étale double

Figure 5.1: Genus three surface with $\mathbb{Z}/4$- symmetry



cover $\pi\colon B \to F$, we get $D = \Gamma_\pi \cup \Gamma_{\phi\circ\pi} \subset B \times F$. Let x_0 respectively x_1 be the intersection point of Γ_π respectively $\Gamma_{\phi\circ\pi}$ and the fixed fiber F over b_0. Consider any $\theta\colon \pi_1(\hat{F}) \to \mathbb{Z}/2 = \{0,1\}$ which corresponds to the double cover of F branched over $F \cap D$. Then the surface braid $\beta(\alpha) = \{\pi_*(\alpha), \phi_*\pi_*(\alpha)\}$ and $-\theta(\gamma_0) = \theta(\gamma_1)$ in an abelian group $\mathbb{Z}/2$. Hence,

$$\iota\colon H_1(B;\mathbb{Z}) \to H_1(F;\mathbb{Z}/2), \quad \alpha \mapsto (\mathrm{id} + \phi_*)\pi_*\alpha \otimes 1.$$

Since $\pi_*(m_1) = m_1 = \pi_*(m_3)$, $\pi_*(m_2) = m_2$, $\pi_*(l_1) = l_1 = \pi_*(l_3)$, $\pi_*(l_2) = 2l_2$ and

$$\mathrm{id} + \phi_* = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

with respect to the basis $\{m_1, m_2, l_1, l_2\}$, we have

$$\iota(m_1) = \iota(m_3) = (m_1 + m_2) \otimes 1, \quad \iota(m_2) = m_1 \otimes 1$$

$$\iota(l_1) = \iota(l_3) = l_2 \otimes 1, \quad \iota(l_2) = 0.$$

Hence, the degree of the minimal pull back for the realisability is $|H_1(B;\mathbb{Z})/\ker\iota| = |(\mathbb{Z}/2)^3| = 8$, the realisation genus $\tilde{b} = 17$, the fiber genus $\tilde{f} = 4$, and the realisation signature is $\tilde{\sigma} = 32$. Now it remains to verify the topological uniqueness of the configurarions of G_4 type to conclude that G_4 type is not realisable. If there exists such a configuration, then we should have a surface of genus 3 with two free involutions σ_1 and σ_2 whose composition is also fixed-point-free. Therefore, it reduces to the same problem with classification of configurations of $C7$ type as explained in 5.5.9.

Example 5.5.6 (G_1 type: non-abelian group of order 8)*.* There are two non-abelian groups of order 8: one is the dihedral group $D_4 = \{1, \rho_1, \rho_2, \rho_3, \mu_1, \mu_2, \delta_1, \delta_2\}$ and the other is the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. First let's see if the virtual kodaira fibration $\mathcal{A} = (B \times F, \Delta, \theta : \pi_1(F \setminus pt) \to D_4)$ corresponding to the G_1 case is realisable. Since the commutator subgroup $[D_4, D_4]$ is the subgroup generated by ρ_2, there is the unique choice for $\theta(\gamma)$ which is ρ_2. Hence the quotient group of D_4 by the subgroup N of rotations acts freely on S/N and thus any realisation $S \to B \times F$ of $\mathcal{A}$ factors through the unramified double cover $B \times \tilde{F} \to B \times F$. Therefore if $\mathcal{A}$ corresponding to G_1 is realisable then so is $\mathcal{A}' = (B \times \tilde{F}, \pi^*(\Delta), \pi_1(\tilde{F} \setminus \text{2points}) \to \mathbb{Z}/4)$ corresponding to C_8. Now we can immediately say that $\mathcal{A}'$, and thus $\mathcal{A}$, is not realisable because $|D| = 1$, $e = 1$, and abelianness of G imply that there's no ramified cover of the horizontal curve B. For the Q_8 case instead of D_4, we can use the same argument since the commutator subgroup $[Q_8, Q_8]$ is the subgroup generated by -1.

Example 5.5.7 (C_8 type: realisation signature)*.* Let B be a curve of genus 2 and $\pi\colon F \to B$ be a double étale cover. If we pull-back the diagonal $\Delta \subset B \times B$ by $id_B \times \pi$, we get a divisor $D \subset B \times F$. This configuration $(B \times F, D)$ together with any surjection $\theta\colon \pi_1(F \setminus D) \to \mathbb{Z}/4$, which defines the degree 4 cover of F branched along $F \cap D$ with ramification order 2 at every branch point, gives a virtual Kodaira fibration of type C_8. To compute the index $[\pi_1(B)\colon \mathrm{Stab}_\theta]$, first denote by $\mathsf{p}\colon \mathsf{D} \subset \mathsf{B} \times \mathsf{F} \to \mathsf{B}$ and $\mathsf{q}\colon \mathsf{D} \subset \mathsf{B} \times \mathsf{F} \to \mathsf{F}$ the projections to each factor. If we use the topological model for the free involution on a surface of genus 3 described in Example 5.5.2, then with respect to the basis $\{\mathfrak{m}_1, \mathfrak{l}_1, \mathfrak{m}_2, \mathfrak{l}_2\}$ of $\mathsf{H}_1(\mathsf{B};\mathbb{Z})$ we can compute

$$\iota\colon \mathsf{H}_1(\mathsf{B};\mathbb{Z}) \to \mathsf{H}_1(\mathsf{F};\mathbb{Z}/4), \quad \alpha \mapsto \mathsf{q}_*\mathsf{p}^!\alpha \otimes 2 (\mathrm{mod}\ 4) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix} \otimes 2 (\mathrm{mod}\ 4)$$

Therefore, the index $[\pi_1(\mathsf{B})\colon \mathsf{Stab}_\theta] = 8$, the realisation genus $\tilde{\mathsf{b}} = 9, \tilde{\mathsf{f}} = 11$, and the realisation signature $\tilde{\sigma} = 32$.

Example 5.5.8 (G_1 type: realisation signature). In Example 5.5.6, we have seen that the realisability of G_1 implies the realisability of C_8. In this example, we would like to prove the converse: the realisation $\tilde{\mathsf{S}} \to \mathsf{B} \times \mathsf{F}_1$ of $\mathcal{A}' = (\mathsf{S}_1 = \mathsf{B} \times \mathsf{F}_1, \pi^*(\Delta), \pi_1(\mathsf{F}_1 \setminus 2\text{points}) \to \mathbb{Z}/4)$ which is of C_8 type composed with the unramified double cover $\mathsf{p}\colon \mathsf{S}_1 := \mathsf{B} \times \mathsf{F}_1 \to \mathsf{B} \times \mathsf{F} =: \mathsf{S}$ gives the realisation $\tilde{\mathsf{S}} \to \mathsf{B} \times \mathsf{F}$ of $\mathcal{A} = (\mathsf{S} = \mathsf{B} \times \mathsf{F}, \Delta, \theta : \pi_1(\mathsf{F} \setminus \mathsf{pt}) \to \mathsf{D}_4)$ which is of G_1 type. Suppose we have $\Theta_1\colon \pi_1(\mathsf{S} \setminus \Delta) \to \mathbb{Z}/2$ extending $\theta_1\colon \pi_1(\mathsf{F} \setminus \Delta) \to \mathbb{Z}/2$ and $\Theta_2\colon \pi_1(\mathsf{S}_1 \setminus \pi^*(\Delta) =: \hat{\mathsf{S}}_1) \to \mathsf{N}\langle\phi\rangle$ extending $\theta_2\colon \pi_1(\mathsf{F}_1 \setminus \pi^*(\Delta) =: \hat{\mathsf{F}}_1) \to \mathbb{Z}/4$.

$$
\begin{tikzcd}
 & & 1 & 1 & \\
 & & \mathbb{Z}/2 \arrow[u] \arrow[r, equal] & \mathbb{Z}/2 \arrow[u] & \\
1 \arrow[r] & \mathsf{Ker}(\theta) \arrow[r] & \pi_1(\hat{F}) \arrow[u, "\theta_1"'] \arrow[r, "\theta"] & D_4 \arrow[u] \arrow[r] & 1 \\
1 \arrow[r] & \mathsf{Ker}(\theta_2) \arrow[u, "\cong", "p_*"'] \arrow[r] & \pi_1(\hat{F}_1) \arrow[u, "p_*"'] \arrow[r, "\theta_2"] & N\langle\phi\rangle \arrow[u] \arrow[r] & 1 \\
 & & 1 \arrow[u] & 1 \arrow[u] &
\end{tikzcd}
$$

Now for a given surjection $\theta\colon \pi_1(F \setminus \Delta =: \hat{F}) \to D_4$ satisfying the commutative diagram 5.5.8, we would like to define a homomorphism

$$\Theta\colon \pi_1(\hat{S} := S \setminus \Delta) \to D_4$$

which is an extension of $\theta\colon \pi_1(\hat{F}) \to D_4$ satisfying the following commutative diagram 5.5.8.

$$
\begin{tikzcd}
 & & 1 & 1 & \\
 & & \mathbb{Z}/2 \arrow[u] \arrow[r, equal] & \mathbb{Z}/2 \arrow[u] & \\
1 \arrow[r] & \mathsf{Ker}(\Theta) \arrow[r] & \pi_1(\hat{S}) \arrow[u, "\Theta_1"'] \arrow[r, "\Theta"] & D_4 \arrow[u] \arrow[r] & 1 \\
1 \arrow[r] & \mathsf{Ker}(\Theta_2) \arrow[u, "p_*"'] \arrow[r] & \pi_1(\hat{S}_1) \arrow[u, "p_*"'] \arrow[r, "\Theta_2"] & N\langle\phi\rangle \arrow[u] \arrow[r] & 1 \\
 & & 1 \arrow[u] & 1 \arrow[u] &
\end{tikzcd}
$$

Since $\pi_1(\hat{S})/\pi_1(\hat{S_1}) \cong \mathbb{Z}/2$, $\pi_1(\hat{S}) = K \coprod K \cdot x$, where $K = \pi_1(\hat{S_1})$. Define Θ to be a homomorphism when we restrict to each coset. Precisely, for any $k \in K$ and $k \cdot x \in K \cdot x$, we define $\Theta(k) := (\Theta_2(k), 1)$ and $\Theta(k \cdot x) := (\Theta_2(k), \Theta_1(x))$ where we identify D_4 with the semi-direct product $N\langle\phi\rangle \rtimes_\psi \mathbb{Z}/2\langle\overline{\sigma}\rangle$.

In order to check that Θ is a homomorphism on the whole $\pi_1(\hat{S})$, first observe that $\Theta(k_1 \cdot (k_2 \cdot x)) = \Theta(k_1) \cdot \Theta(k_2 \cdot x)$ by the assumption that Θ_2 is a homomorphism. Secondly, $\Theta((k_1 \cdot x) \cdot k_2) = \Theta(k_1 \cdot x) \cdot \Theta(k_2)$ because the right-hand side equals to $(\Theta_2(k_1), \Theta_1(x)) \cdot (\Theta_2(k_2), 1) = (\Theta_2(k_1) \cdot \psi(\Theta_1(x))(\Theta_2(k_2)), \Theta_1(x))$ and the left-hand side equals to $\Theta(k_1 \cdot x k_2 x^{-1} \cdot x) = (\Theta_2(k_1)\Theta_2(x k_2 x^{-1}), \Theta_1(x)) = (\Theta_2(k_1) {\cdot} \Theta(x)\Theta(k_2 x^{-1}), \Theta_1(x))$. Moreover, $\Theta|_{\pi_1(\hat{F})} = \theta \colon \pi_1(\hat{F}) \to D_4$ follows from the following two.

1. $\Theta|_{\pi_1(\hat{F}_1)} = (\Theta|_{\pi_1(\hat{S}_1)})|_{\pi_1(\hat{F}_1)} = \Theta_2|_{\pi_1(\hat{F}_1)} = \theta_2 = \theta|_{\pi_1(\hat{F}_1)}$
2. $q \circ \Theta|_{\pi_1(\hat{F})} = \Theta_1|_{\pi_1(\hat{F})} = \theta_1 = q \circ \theta$

Finally, observe that

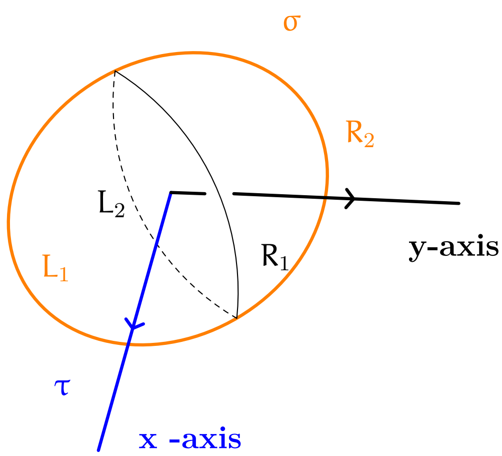
$$p_*(\mathrm{Ker}(\Theta_2 \colon \pi_1(\hat{S_1}) \to N\langle\phi\rangle)) = p_*\pi_1(\hat{S_1}) \cap \mathrm{Ker}\Theta$$

where $\mathrm{Ker}(\Theta \colon \pi_1(\hat{S}) \to D_4)$. This implies that the realisation $\tilde{S} \to S_1$ of C_8 type coincides with the realisation of $\tilde{S} \to S$ of G_1 type.

5.5.2 Examples of correspondence type

Example 5.5.9 (C_7 type: double bisection). If we have a curve D_0 of genus 3 inside a product of two curves of genus 2 whose projection to both direction has degree 2, then we have two étale double covers $\pi_\sigma \colon D_0 \to B$ and $\pi_\tau \colon D_0 \to F$ corresponding to free involutions σ and τ in $\mathrm{Aut}(D_0)$, respectively. If in addition, $\phi = \tau \circ \sigma^{-1}$ is fixed point free, then $\pi_\sigma \times \pi_\tau \colon D_0 \to B \times F$ gives an embedding. By Table 5.4.2, the order of such a ϕ is either 2 or 4. If the order of $\tau \circ \sigma$ has order 2, then the group H generated by σ and τ is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ and then H, a group of order 4, acts freely on a curve of genus 3,

Figure 5.2: D_4 symmetry on a surface of genus 3



which contradicts to the Hurwitz formula. Hence, $\tau\circ\sigma$ has order 4 and this leads to $H = \{1, \sigma, \tau, \tau\sigma, (\tau\sigma)^2, (\tau\sigma)^3, \sigma\tau\sigma, \tau\sigma\tau\}$, which is isomorphic to the dihedral group $D_4 = \langle x, y \mid x^2 = y^4 = 1,\ xyx^{-1} = y^{-1}\rangle$ via $\sigma \mapsto x, \tau\sigma \mapsto y$. By [6], D_4 acts on a curve of genus 3 uniquely in such a way that each of σ, τ, and $\tau\sigma$ acts freely. Precisely, the epimorphism $\langle \alpha, \beta, \gamma \mid [\alpha, \beta]\gamma = 1\rangle \to D_4$ given by $\alpha \mapsto x, \beta \mapsto xy, \gamma \mapsto y^2$ gives us a complex curve D_0 of genus 3 with $\mathrm{Aut}(D_0) = D_4$. Therefore, we have a configuration $(B \times F, D = \pi_\sigma \times \pi_\tau(D_0))$ corresponding to C_7 type in the complex category and it is unique up to topological equivalence.

For a group G of order 2 or 4, we consider any surjective homomorphism $\theta\colon \pi_1(\hat{F}) \to G$ satisfying the ramification condition and the liftability condition. In order to compute the realisation signature of the virtual Kodaira fibration $\mathcal{A} = (B \times F, D = (\pi_\sigma \times \pi_\tau)(D_0), \theta\colon \pi_1(\hat{F}) \to G)$, we first need to investigate the global extension obstruction. Actually, the global extension obstruction $o(\theta) = 2\theta(\gamma_0)$ vanishes because the ramification order at every branch point is 2.

Now for computing the index $[\pi_1(B)\colon \mathtt{stab}_\theta]$, we need to find the topological model for the required free involutions σ and τ on a surface of genus 3 such that their composition is also free. Such a pair of free involutions appeared in [46] to study the self-intersection number of multi-sections of any Σ_g bundle over Σ_h. Take a graph Γ as the intersection of the standardly embedded 2-sphere S^2 and $\{(x+y)(x-y)=0\}$ in $\mathbb{R}^3$, and realize a surface Σ_3 of genus 3 as the smooth boundary of a thin regular neighborhood of Γ in $\mathbb{R}^3$. In Figure 5.2, $S^2 \cap \{x+y=0\}$ is drawn in orange and $S^2 \cap \{x-y=0\}$ is drawn in black. We can think of the rotation of the surface Σ_3 by π, denoted by σ, around the great circle in orange and another π-rotation of Σ_3, denoted by τ, around the x-axis which is in coordinates $(x,y,z) \mapsto (x,-y,-z)$. Under the action of the first involution σ on Σ_3, the torus around the great circle in orange is invariant and rotated by π around the core circle, while the other two 1-handles connecting the regions close to the poles are exchanged. The second involution τ on Σ_3 is nothing but the π-rotation around the axis passing through the middle hole. Now it is obvious that σ, τ, and $\sigma\tau$ are all fixed-point-free.

Now we are ready to compute $\iota\colon H_1(B;\mathbb{Z}) \to H_1(F;G)$. Let D be a curve of genus 3 with the group action of $D_4 = \langle \sigma,\tau \mid \sigma^2=\tau^2=1, (\sigma\tau)^4=1\rangle$ such that each of σ,τ, and $\tau\sigma$ acts freely. Let $B = D/\sigma$ and $F = D/\tau$. We can take a basis of $H_1(D;\mathbb{Z})$ as meridians and longitudes of $\{L_1R_1, L_1R_2, L_1L_2\}$. Then we have the induced bases of $H_1(B;\mathbb{Z})$ and $H_1(F;\mathbb{Z})$, which are blockwisely $\{L_1R_1, L_1R_2\}$ and $\{R_1,R_2\}$, respectively. With respect to these bases, we can compute

$$\pi^!_\sigma = \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & 1 & 0 \\ & & 0 & 2 \\ 1 & 0 & & \\ 0 & 1 & & \end{pmatrix}, \quad (\pi_\tau)_* = \begin{pmatrix} 1 & & 0 & & 0 & \\ & 2 & & 1 & & 1 \\ & & 1 & & 1 & \\ & & & 1 & & -1 \end{pmatrix}$$

and hence we get $\iota\colon \mathsf{H}_1(\mathsf{B};\mathbb{Z}) \to \mathsf{H}_1(\mathsf{F};\mathbb{Z}/2)$

$$\iota = (\pi_\tau)_* \pi_\sigma^! \otimes 1 = \begin{pmatrix} 1 & & 0 & \\ & 3 & & 2 \\ 1 & & 1 & \\ & -1 & & 2 \end{pmatrix} (\mathrm{mod} 2)$$

Therefore, we get the index $[\pi_1(\mathsf{B})\colon \mathsf{Stab}_\theta] = 8$, the realisation genus $\tilde{\mathsf{b}} = 9, \tilde{\mathsf{f}} = 4$ and the realisation signature $\tilde{\sigma} = 16$. For a group G of order 4, by the same argument, $[\pi_1(\mathsf{B})\colon \mathsf{Stab}_\theta] = 8$, $\tilde{\mathsf{b}} = 9$, $\tilde{\mathsf{f}} = 7$, and $\tilde{\sigma} = 32$. Therefore, C_7 type is not realisable.

Example 5.5.10 (C_4 type: topological classification of the configurations). Assume we have a curve D_0 of genus 5 and two étale double covers $\pi_1\colon \mathsf{D}_0 \to \mathsf{B}$ and $\pi_2\colon \mathsf{D}_0 \to \mathsf{F}$ corresponding to free involutions σ and τ, respectively. They generate the dihedral group $\mathsf{D}_m \cong \langle \sigma, \tau \mid \sigma^2 = 1, \tau^2 = 1, (\sigma \circ \tau)^m = 1 \rangle$ as a subgroup of the automorphism group $\mathsf{Aut}(\mathsf{D}_0)$. Moreover, if we require that $\pi_1 \times \pi_2\colon \mathsf{D}_0 \to \mathsf{B} \times \mathsf{F}$ gives an embedding, then the composition $\sigma \circ \tau$ also must be a free automorphism. By Table 5.4.2, the order m of $\phi = \sigma \circ \tau$ is even and at most 8. In any cases, we have a free involution on the quotient curve $\mathsf{D}_0/\langle \phi \rangle$ induced by σ because both σ and τ acts freely on D_0 and thus so does every reflection.

In the case ϕ has order $m = 8$, by Table 5.4.2, the quotient of D_0 by the cyclic subgroup $\langle \phi \rangle$ generated by ϕ has genus 1 and two branch points of multiplicity 2. Observe that we have a free involution on the quotient $\mathsf{D}_0/\langle \phi \rangle$ induced by σ. Hence we have an epimorphism $\eta\colon \langle \alpha, \beta, \gamma \mid [\alpha, \beta]\gamma \rangle \to \mathsf{D}_8$ with the ramification type $(1 \mid 2)$. This implies that the monodromy around the unique branch point is the order 2 elemenet in the commutator subgroup $[\mathsf{D}_8, \mathsf{D}_8] \cong \langle \phi^2 \rangle$. Thus η induces an epimorphism $\langle \alpha, \beta \mid [\alpha, \beta] \rangle \to \mathsf{D}_8/\langle \phi^4 \rangle \cong \mathsf{D}_4$ which contradicts to the nonabelianness of D_4.

In the case $m = 6$, the quotient map of D_0 by the cyclic group of order 6 generated by $\phi = \sigma\tau$ has ramification type either $(1 \mid 3^2)$ or $(0 \mid 2^4, 3^2)$. The second case is impossible because there's no free involution on a sphere. In the first case, $D_0/\langle\phi\rangle \to B$ is double étale, and hence the quotient map $D_0 \to B$ by D_6 action has ramification type $(1 \mid 3)$. There exists a single equivalence class of epimorphism realised by

$$\langle\alpha, \beta, \gamma \mid [\alpha, \beta]\gamma\rangle \to D_6, \quad (\alpha, \beta; \gamma) \mapsto (\sigma, \tau; \phi^4).$$

In the case it has order $m = 4$, by Table 5.4.2 the quotient mapof D_0 by the cyclic group of order 4 generated by $\phi = \sigma\tau$ has ramification type either $(2 \mid -)$ or $(1 \mid 2^4)$. Since there's no free involution on a curve of genus two, the first case contradicts to the assumption that σ, τ are free. In the second case, on the quotient $D_0/\langle\phi^2\rangle$, which has genus two, we should have a free action of the quotient group $D_4/\langle\phi^2\rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. However, it contradicts to the Hurwitz formula.

Finally if $m = 2$, then D_m is abelian and $\langle\sigma, \tau\rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. In this case, the whole group acts freely, hence by the result of Edmonds [13] there are two topologically distinct cases corresponding to equivalence classes of epimorphisms

$$\langle\alpha_1, \beta_1, \alpha_2, \beta_2 \mid [\alpha_1, \beta_1][\alpha_2, \beta_2]\rangle \to \mathbb{Z}/2 \times \mathbb{Z}/2$$

represented by maps

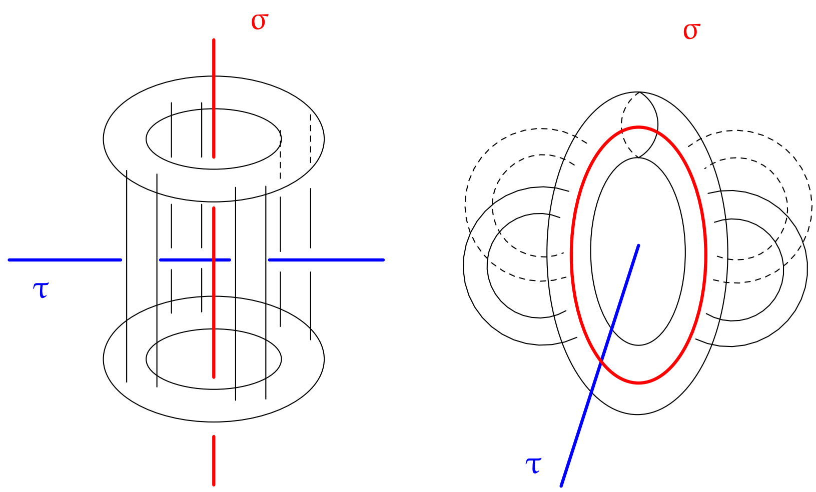
$$\alpha_1, \alpha_2 \mapsto (0, 0), \beta_1 \mapsto (1, 0), \beta_2 \mapsto (0, 1), \quad \alpha_2, \beta_2 \mapsto (0, 0), \alpha_1 \mapsto (1, 0), \beta_1 \mapsto (0, 1).$$

Thus we have a complete topological classification of type C_4.

Example 5.5.11 (C_4 type: two distinct $\mathbb{Z}/2 \times \mathbb{Z}/2$ actions on a curve of genus 5). Let D be a curve of genus 5 with a free action of $G = \mathbb{Z}/2 \times \mathbb{Z}/2 = \langle\sigma, \tau\rangle$. Let $B = D/\sigma$ and $F = D/\tau$. Then the natural projections embed $D \to B \times F$. In both cases, the global extension obstruction vanishes because the ramification order is 2. So we only need to compute the index $[\pi_1(B) \colon \mathrm{Stab}_\theta]$. First consider the action realised by the epimorphism

$$\alpha_1, \alpha_2 \mapsto (0, 0), \beta_1 \mapsto (1, 0), \beta_2 \mapsto (0, 1)$$

Figure 5.3: $\mathbb{Z}/2 \times \mathbb{Z}/2$ symmetries on a surface of genus 5



From Figure 5.3, we can take a basis of $\mathsf{H}_1(\mathsf{D};\mathbb{Z})$ as meridians and longitudes from the North torus, the South torus, and then from the middle, precisely, the West, the Front, and the East. Then we have the induced bases in $\mathsf{H}_1(\mathsf{B};\mathbb{Z})$ and $\mathsf{H}_1(\mathsf{F};\mathbb{Z})$, which are blockwisely {North, South, West} and {North, West, East}, respectively. With respect to these bases, we can compute

$$(\pi_\sigma^!)^\mathsf{T} = \begin{pmatrix} 2 & 0 & & & & & & & & \\ 0 & 1 & & & & & & & & \\ & & 2 & 0 & & & & & & \\ & & 0 & 1 & & & & & & \\ & & & & 1 & 0 & 0 & 0 & 1 & 0 \\ & & & & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}^\mathsf{T}, \quad (\pi_\tau)_* = \begin{pmatrix} 1 & 0 & 1 & 0 & & & * & * & & \\ 0 & 1 & 0 & 1 & & & * & * & & \\ & & & & 1 & 0 & * & * & & \\ & & & & 0 & 2 & * & * & & \\ & & & & & & * & * & 1 & 0 \\ & & & & & & * & * & 0 & 2 \end{pmatrix}$$

and hence we get $\iota\colon \mathsf{H}_1(\mathsf{B};\mathbb{Z}) \to \mathsf{H}_1(\mathsf{F};\mathbb{Z}/2)$

$$\iota = (\pi_\tau)_*\pi_\sigma{}^! \otimes 1 = \begin{pmatrix} 2 & & 2 & & & \\ & 1 & & 1 & & \\ & & & & 1 & \\ & & & & & 2 \\ & & & & 1 & \\ & & & & & 2 \end{pmatrix} (\mathrm{mod} 2)$$

Therefore we get the index $[\pi_1(\mathsf{B})\colon \mathtt{Stab}_\theta] = 4$, the realisation genus $\tilde{\mathsf{b}} = 9, \tilde{\mathsf{f}} = 6$ and the realisation signature $\tilde{\sigma} = 16$.

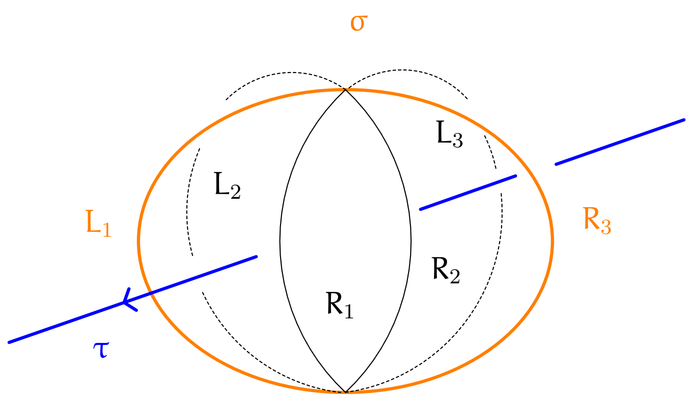
Now move to the second action realised by the epimorphism

$$\alpha_2, \beta_2 \mapsto (0,0), \alpha_1 \mapsto (1,0), \beta_1 \mapsto (0,1).$$

From Figure 5.3, we can choose a basis of $\mathsf{H}_1(\mathsf{D};\mathbb{Z})$ as meridians and longitudes from the Core torus, and then from four handles, precisely, the Front-left, the Front-right, the Back-left, and the Back-right. Then we have the induced bases in $\mathsf{H}_1(\mathsf{B};\mathbb{Z})$ and $\mathsf{H}_1(\mathsf{F};\mathbb{Z})$, which are blockwisely {Core, Front-left, Front-right} and {Core, Front-left, Back-left}, respectively. With respect to these bases, we can compute

$$(\pi^!_\sigma)^\mathsf{T} = \begin{pmatrix} 1 & 0 & & & & & & & & \\ 0 & 2 & & & & & & & & \\ & & 1 & 0 & & & 1 & 0 & & \\ & & 0 & 1 & & & 0 & 1 & & \\ & & & & 1 & 0 & & & 1 & 0 \\ & & & & 0 & 1 & & & 0 & 1 \end{pmatrix}^\mathsf{T}, \ (\pi_\tau)_* = \begin{pmatrix} 1 & 0 & & & & & & & & \\ 0 & 2 & & & & & & & & \\ & & 1 & 0 & 1 & 0 & & & & \\ & & 0 & 1 & 0 & 1 & & & & \\ & & & & & & 1 & 0 & 1 & 0 \\ & & & & & & 0 & 1 & 0 & 1 \end{pmatrix}$$

Figure 5.4: D_6 symmetry on a surface of genus 5



and hence $\iota\colon H_1(B;\mathbb{Z}) \to H_1(F;\mathbb{Z}/2)$

$$\iota = (\pi_\tau)_* \pi_\sigma^! \otimes 1 = \begin{pmatrix} 1 & 0 & & & & \\ 0 & 4 & & & & \\ & & 1 & 0 & 1 & 0 \\ & & 0 & 1 & 0 & 1 \\ & & 1 & 0 & 1 & 0 \\ & & 0 & 1 & 0 & 1 \end{pmatrix} (\mathrm{mod} 2)$$

Therefore we get the index $[\pi_1(B)\colon \mathtt{Stab}_\theta] = 8$, the realisation genus $\tilde{b} = 17, \tilde{f} = 6$ and the realisation signature $\tilde{\sigma} = 32$.

Example 5.5.12 (C_4 type: D_6 symmetry on a surface of genus 5). Let D be a genus 5 curve with an action of dihedral group $G = D_6 = \langle \sigma, \tau \rangle$ realised by the equivalence class of an epimorphism

$$\langle \alpha, \beta, \gamma \mid [\alpha, \beta]\gamma \rangle \to D_6, \quad (\alpha, \beta; \gamma) \mapsto (\tau\sigma\tau, \tau; (\sigma\tau)^2).$$

In particular, we can easily check this epimorphism is equivalent to the epimorphism $(\alpha, \beta; \gamma) \mapsto (\sigma, \tau; (\sigma\tau)^4)$ using a push map. We can find its topological model as in the Figure 5.4. Consider a graph Γ in $\mathbb{R}^3$ consisting of $S^2 \cap \{y+x=0\}$ in orange and $S^2 \cap \{x \cdot (y-x) = 0\}$ in black. Now realise a surface of genus 5 as the smooth boundary of a thin regular neighborhood of Γ in $\mathbb{R}^3$. We take a free involution σ as the rotation by π about the circle in orange, and another free involution τ as the rotation by π about the x- axis in blue. Then they generate the dihedral group D_6 because $\sigma\tau$ has order 6.

Let $B = D/\sigma$ and $F = D/\tau$. From the Figure, we can choose a basis of $H_1(D; \mathbb{Z})$ as meridians and longitudes of $\{L_1R_1, L_1R_2, L_1R_3, L_1L_3, L_1L_2\}$. Then we have the induced bases of $H_1(B; \mathbb{Z})$ and $H_1(F; \mathbb{Z})$, which are blockwisely $\{L_1R_1, L_1R_2, L_1R_3\}$ and $\{R_1, R_2, R_3\}$, respectively. With respect to these bases, we can compute

$$(\pi^!_\sigma)^T = \begin{pmatrix} 1 & 0 & & & & & & & 1 & 0 \\ 0 & 1 & & & & & & & 0 & 1 \\ & & 1 & 0 & & & 1 & 0 & & \\ & & 0 & 1 & & & 0 & 1 & & \\ & & & & 1 & 0 & & & & \\ & & & & 0 & 2 & & & & \end{pmatrix}^T, \quad (\pi_\tau)_* = \begin{pmatrix} 1 & 0 & 0 & & 0 & & 0 & & 0 & \\ 0 & 2 & & 1 & & 1 & & 1 & & 1 \\ & & 1 & 0 & & & & & 1 & 0 \\ & & 0 & 1 & & & & & 0 & 1 \\ & & & & 1 & 0 & 1 & 0 & & \\ & & & & 0 & 1 & 0 & 1 & & \end{pmatrix}$$

and hence $\iota \colon H_1(B; \mathbb{Z}) \to H_1(F; \mathbb{Z}/2)$

$$\iota = (\pi_\tau)_* \pi^!_\sigma \otimes 1 = \begin{pmatrix} 1 & 0 & & & & \\ 0 & 3 & & 2 & & 2 \\ 1 & 0 & 1 & 0 & & \\ 0 & 1 & 0 & 1 & & \\ & & 1 & 0 & 1 & 0 \\ & & 0 & 1 & 0 & 2 \end{pmatrix} (\text{mod} 2)$$

Therefore we get the index $[\pi_1(B) : \mathrm{Stab}_\theta] = 32$, the realisation genus $\tilde{b} = 65, \tilde{f} = 5$ and the realisation signature $\tilde{\sigma} = 128$.

Example 5.5.13 (C_{11} type: symmetric to G_4 type). As in the Example 5.5.1, we take B as a curve of genus two with the free automorphism ϕ and $\mathsf{D}_0 \subset \mathsf{B}\times\mathsf{B}$ be a divisor consisting of two disjoint graphs $\mathsf{D}'_1 = \Gamma_{\mathrm{id}}$ and $\mathsf{D}'_2 = \Gamma_{\phi}$. Consider a double étale cover $\pi\colon \mathsf{F} \to \mathsf{B}$ and the pull-back divisor $\mathsf{D} = (\mathrm{id}_{\mathsf{B}} \times \pi)^*(\mathsf{D}_0) \subset \mathsf{B}\times\mathsf{F}$. Denote the first component of D coming from D'_1 by D_1 and the second component coming from D'_2 by D_2. Then we get a configuration $(\mathsf{B}\times\mathsf{F}, \mathsf{D} = \mathsf{D}_1 \cup \mathsf{D}_2)$ of type C_{11}. For any surjective homomorphism $\theta\colon \pi_1(\hat{\mathsf{F}}) \to \mathbb{Z}/2$ corresponding to the double branched covering of F branched along $\mathsf{F}\cap\mathsf{D}$, we would like to determine the realisability of the virtual Kodaira fibration $\mathcal{A} = (\mathsf{B}\times\mathsf{F}, \mathsf{D}, \theta)$. In fact, we can deduce the non-realisability of this $\mathcal{A}$ from the non-realisability of the G_4 type (Example 5.5.5) because of the symmetry of those two types coming from exchanging the role of B and F. However, in this example we explicitly compute the realisation signature and realisation genera. By Corollary 5.3.6,

$$\iota\colon \mathsf{H}_1(\mathsf{B};\mathbb{Z}) \to \mathsf{H}_1(\mathsf{F};\mathbb{Z}/2), \quad \alpha \mapsto \sum_{i=1}^{2} q_* p_i^! \alpha \otimes 1.$$

where each $p_i\colon \mathsf{D}_i \subset \mathsf{B}\times\mathsf{F} \to \mathsf{B}$ is the projection to the first factor.
Moreover, we can compute the transfer maps as follows using the explicit model for the free involution on a surface of genus 3. (Example 5.5.2)

$$q_*p_1^! = \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & 2 & 0 \\ & & 0 & 1 \\ 1 & 0 & & \\ 0 & 1 & & \end{pmatrix}, \quad q_*p_2^! = q_*p_1^!\phi_* = \begin{pmatrix} & 1 & & \\ -1 & 1 & & \\ & & 2 & -2 \\ & & 1 & 0 \\ & 1 & & \\ -1 & 1 & & \end{pmatrix}$$

$$\iota = \begin{pmatrix} 1 & 1 & & \\ 1 & 0 & & \\ & & 0 & 0 \\ & & 1 & 1 \\ 1 & 1 & & \\ 1 & 0 & & \end{pmatrix} (\text{in } \mathbb{Z}/2)$$

Therefore, the minimal degree is 8, the realisation genus $\tilde{b} = 9$, $\tilde{f} = 7$, and the realisation signature $\tilde{\sigma} = 32$.

Example 5.5.14 (C_6 type). Let B be a curve of genus three and $\pi\colon F \to B$ be a double étale cover. If we pull-back the diagonal $\Delta \subset B \times B$ by $\mathrm{id}_B \times \pi$ we get a divisor $D \subset B \times F$. This configuration $(B \times F, D)$ together with any surjection $\theta\colon \pi_1(F \setminus D) \to \mathbb{Z}/2$, which defines the double cover of F branched along $F \cap D$ gives a virtual Kodaira fibration of type C_6. Denote by $p\colon D \subset B \times F \to B$ and $q\colon D \subset B \times F \to F$ the projections to each factor. By using the topological model for the free involution on a surface of genus 5, we can compute

$$\iota\colon H_1(B;\mathbb{Z}) \to H_1(F;\mathbb{Z}/2)$$

$$\iota = q_* p^! = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 2 & \\ & & & & & 1 \\ & & 1 & & & \\ & & & 1 & & \\ 1 & & & & & \\ & 1 & & & & \end{pmatrix} (\text{mod} 2)$$

Therefore, the minimal degree is 2^5, the realisation signature $\tilde{\sigma} = 128$, and the realisation genus $\tilde{b} = 65$, $\tilde{f} = 10$.

Example 5.5.15 (C_5 type). Let B be a curve of genus two and $\pi \colon \mathsf{F} \to \mathsf{B}$ be a degree 4 étale cover. If we pull-back the diagonal $\Delta \subset \mathsf{B} \times \mathsf{B}$ by $\mathrm{id}_{\mathsf{B}} \times \pi$, we get a divisor $\mathsf{D} \subset \mathsf{B} \times \mathsf{F}$. This configuration $(\mathsf{B} \times \mathsf{F}, \mathsf{D})$ together with any surjection $\theta \colon \pi_1(\mathsf{F} \setminus \mathsf{D}) \to \mathbb{Z}/2$, which defines the double cover of F branched along $\mathsf{F} \cap \mathsf{D}$ gives a virtual Kodaira fibration of type C_5. By using the topological model for the free action of the cyclic group $\mathbb{Z}/4$, we can compute

$$\iota \colon \mathsf{H}_1(\mathsf{B};\mathbb{Z}) \to \mathsf{H}_1(\mathsf{F};\mathbb{Z}/2), \quad \iota = \begin{pmatrix} 1 & & & \\ & 1 & & \\ 1 & & & \\ & 1 & & \\ 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 1 \end{pmatrix} (\mathrm{mod} 2)$$

Here the third column follows from $q_* p^! m_2 = x_1 + x_2 + x_3 + x_4 = 2x_1 + 2x_2 = 0$ in $\mathbb{Z}/2$. Therefore, the minimal degree is 8, the realisation signature $\tilde{\sigma} = 32$, and the realisation genus $\tilde{b} = 9$, $\tilde{f} = 11$.

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국문초록

이 논문에서 우리는 부호수가 0 *이 아닌 곡면 위의 곡면 번들 구조를 갖는 매끄러운 4차원 다양체 X(혹은 복소곡면 X)의 부호수와 오일러수* $(\sigma(X), e(X))$ *에 대한 위상적 제약조건을 연구한다. 첫번째 주된 결과는 파이버 종수* f *와 부호수* $4n$ *이 주어졌을 때 정의되는 최소의 기저 종수* $b(f,n)$ *의 개선된 상한에 관한 것이다. 특별히 우리는 레프세츠 파이브레이션들을 빼는 방법으로, 부호수가* 4 *이고 오일러 수가 작은 곡면 위의 곡면 번들인 새로운 매끄러운* 4 *차원 다양체들을 건설한다. 그 중에는 지금까지 알려진 부호수가* 0 *이 아닌 곡면 번들 중에 가장 작은 오일러 수를 갖는 예도 포함된다. 두번째로 우리는 두 복소 곡선들의 곱의 분기된 덮개로서, 곡면 위의 곡면 번들인 작은 부호수를 갖는 코다이라 파이브레이션들을 건설할 수 있는 가능성에 관해 탐구한다. 최소의 기저 종수와 가능한 가장 작은 부호수를 얻기 위해서 우리는 구분된 점들이 있는 (복소) 곡선들의 파이브레이션의 모노드로미의 작용을 조사한다. 논문 전반에 걸쳐 우리는 곡면 사상류 군이 건설 방법과 위상적 불변량의 조절 모두에 중요한 역할을 한다는 것을 보게 될 것이다.*

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