Universal $R$-matrices and the center of the quantum generalized Kac-Moody algebras

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Abstract. We extend the result in [13] to those for the quantization of generalized Kac-Moody algebras introduced in [10]. The existence of the universal $R$-matrix is proved, and a structure theorem for the center is given.

0. Introduction

The quantum groups—more precisely, the quantization of the universal enveloping algebras of Kac-Moody algebras—were independently introduced by Drinfel'd ([6]) and Jimbo ([7]) through their investigation of $R$-matrices which are the solutions to the Yang-Baxter equation. Its importance partly comes from the fact that there exists a solution to the Yang-Baxter equation inside the quantum group, called the universal $R$-matrix, so that one can obtain various $R$-matrices as its specialization on the representations of the quantum group.

On the other hand, the notion of Kac-Moody algebras was generalized to the so-called generalized Kac-Moody algebras ([1]), and it was used crucially in Borcherds' proof of the moonshine conjecture ([2]). In [10], the first-named author extended the quantum groups to those for the generalized Kac-Moody algebras, and proved some fundamental results on their structures and their representations.

In this paper, we continue the investigation by extending the results in [13] to the quantum groups of generalized Kac-Moody algebras. In the first half of this paper, we construct an analogue of the Killing form and prove the existence of the universal $R$-matrix. The proofs are very similar to those in [13] and the analogue of the Killing form plays a crucial role. In the second half, we investigate the structure of the center of the quantum groups for generalized Kac-Moody algebras. The case of quantized universal en-
veloping algebras of ordinary Kac-Moody algebras was already treated in [4], [8], [13]. Hence we restrict ourselves to the non-ordinary case. We show that the center consists only of certain obvious elements in almost all cases. The proof is based on the reduction to the small rank cases.

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1. The Quantum Algebra $U_q(g)$

Let $F$ be a field of characteristic 0 and let $q \in F$ be transcendental over the prime subfield $\mathbb{Q}$. We assume that $F$ contains an $n$-th root of $q$ for any positive integer $n$.

Let $I$ be a countable (possibly infinite) index set and let $A = (a_{ij})_{i,j \in I}$ be a Borcherds-Cartan matrix with $a_{ij} \in \mathbb{Q}$ for all $i, j \in I$. That is, $A = (a_{ij})_{i,j \in I}$ is a rational square matrix satisfying (i) $a_{ii} = 2$ or $a_{ii} \leq 0$ for all $i \in I$, (ii) $a_{ij} \leq 0$ for $i \neq j$ and $a_{ij} \in \mathbb{Z}$ if $a_{ii} = 2$, (iii) $a_{ij} = 0$ implies $a_{ji} = 0$. Let $I^e = \{ i \in I | a_{ii} = 2 \}$, $I^{im} = \{ i \in I | a_{ii} \leq 0 \}$, and let $m = (m_i | i \in I)$ be a collection of positive integers such that $m_i = 1$ for all $i \in I^e$. We call $m$ the charge of the Borcherds-Cartan matrix $A$. We denote by $g = g(A, m)$ the generalized Kac-Moody algebra associated with the Borcherds-Cartan matrix $A$ and the charge $m$ ([1], [9], [10]).

A rational Borcherds-Cartan matrix $A = (a_{ij})_{i,j \in I}$ is called symmetrizable if there is a diagonal matrix $D = \text{diag}(s_i | i \in I)$ with $s_i \in \mathbb{Z}_{>0}$ such that $DA$ is symmetric. From now on, we assume that $A$ is a symmetrizable Borcherds-Cartan matrix.

Let $h = (\bigoplus_{i \in I} \mathbb{Q} h_i) \oplus (\bigoplus_{i \in I} \mathbb{Q} d_i)$ be the vector space with a basis $\{h_i, d_i | i \in I\}$, and let

$$P^\vee = \left( \bigoplus_{i \in I} \mathbb{Z} h_i \right) \oplus \left( \bigoplus_{i \in I} \mathbb{Z} d_i \right)$$

be the $\mathbb{Z}$-lattice of $h$. For each $j \in I$, we define the linear functionals $\alpha_j \in h^*$ by

$$\alpha_j(h_i) = a_{ij}, \quad \alpha_j(d_i) = \delta_{ij} \quad (i, j \in I).$$

Set $Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i$, $Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$, and $Q_- = -Q_+$. Let $\rho \in h^*$ be a linear functional satisfying $\rho(h_i) = \frac{1}{2} a_{ii}$ for all $i \in I$. For each $i \in I^e$, we define the simple reflection $r_i \in \text{GL}(h)$ by $r_i(h) = h - \alpha_i(h) h_i$. The subgroup $W$ of $\text{GL}(h)$
Quantum generalized Kac-Moody algebras

generated by the \( r_i \)'s is called the \emph{Weyl group} of the above Borcherds-Cartan data. It is a Coxeter group with canonical generator system \( \{ r_i | i \in I^* \} \). We denote its length function by \( l : W \to \mathbb{Z}_{\geq 0} \). The contragredient action of \( W \) on \( \mathfrak{h}^* \) is generated by \( r_i(l) = \lambda - \lambda(h_i) \alpha_i \). Since \( A \) is symmetrizable, there exists a nondegenerate symmetric bilinear form \( (\ | \ ) \) on \( \mathfrak{h} \) satisfying \( (s_i h_i | h) = \alpha_i(h) \) \( (i \in I, h \in \mathfrak{h}) \).

For each \( i \in I \), let \( \xi_i = q^{s_i} - q^{-s_i} \), \( q_i = q^{(\alpha_i, \alpha_i)/2} \), and define the \emph{q-integer} by

\[
[n]_q = \begin{cases} 
q^n - q^{-n} & \text{if } a_{ii} \neq 0, \\
q_i - q_i^{-1} & \text{if } a_{ii} = 0.
\end{cases}
\]

We also define \([n]_q! = \prod_{k=1}^{n} [k]_q \).

**Definition 1.1.** ([10]) The \emph{quantum algebra} \( \mathfrak{g}(g) \) associated with a symmetrizable Borcherds-Cartan matrix \( A = (a_{ij})_{i, j \in I} \) and a charge \( m = (m_i | i \in I) \) is an associative algebra with 1 over \( \mathbb{F} \) generated by the elements \( q^h \) \( (h \in \mathfrak{h}^\vee) \), \( e_{ik}, f_{ik} \) \( (i \in I, k = 1, 2, \ldots, m_i) \) with the defining relations

\[(R1) \quad q^0 = 1, \quad q^h q^{h'} = q^{h+h'} (h, h' \in \mathfrak{h}^\vee),
\]

\[(R2) \quad q^h e_{ik} q^{-h} = q^{(h, h)} e_{ik} (h \in \mathfrak{h}^\vee, i \in I, k = 1, 2, \ldots, m_i),
\]

\[(R3) \quad q^h f_{ik} q^{-h} = q^{-(h, h)} f_{ik} (h \in \mathfrak{h}^\vee, i \in I, k = 1, 2, \ldots, m_i),
\]

\[(R4) \quad [e_{ik}, f_{ij}] = \frac{\delta_{ij} \delta_{kl}}{\xi_l} \left( K_i - K_i^{-1} \right), \quad \text{where } K_i = q^{s_i h_i} (i, j \in I, k = 1, 2, \ldots, m_i),
\]

\[(R5) \quad \sum_{j=1}^{m_j} (-1)^j e_{ik}^{(n)} e_{ij} f_{ij} = 0 \quad \text{if } a_{ii} = 2 \text{ and } i \neq j \quad (k = 1, l = 1, 2, \ldots, m_j),
\]

\[(R6) \quad \sum_{j=1}^{m_j} (-1)^j f_{ik}^{(n)} f_{ij} = 0 \quad \text{if } a_{ii} = 2 \text{ and } i \neq j \quad (k = 1, l = 1, 2, \ldots, m_j),
\]

\[(R7) \quad [e_{ik}, e_{ij}] = 0 \quad \text{if } a_{ij} = 0.
\]

\[(R8) \quad [f_{ik}, f_{ij}] = 0 \quad \text{if } a_{ij} = 0.
\]

The algebra \( \mathfrak{g}(g) \) has a Hopf algebra structure with comultiplication \( \Delta \), counit \( \epsilon \), and antipode \( S \) defined by

\[
\Delta(q^h) = q^h \otimes q^h,
\]

\[
\Delta(e_{ik}) = e_{ik} \otimes 1 + K_i \otimes e_{ik},
\]

\[
\Delta(f_{ik}) = f_{ik} \otimes K_i^{-1} + 1 \otimes f_{ik},
\]

\[
\epsilon(q^h) = 1, \quad \epsilon(e_{ik}) = \epsilon(f_{ik}) = 0,
\]

\[
S(q^h) = q^{-h},
\]

\[
S(e_{ik}) = -K_i^{-1} e_{ik}, \quad S(f_{ik}) = -f_{ik} K_i.
\]
for \( h \in P^\vee, \ i \in \mathcal{I}, \ k = 1, \ldots, m_i \). We denote by \( U^0 \) the subalgebra of \( U = U_q(g) \) with 1 generated by \( q^h \ (h \in P^\vee) \) and \( U^+ \) (resp. \( U^- \)) the subalgebra of \( U \) generated by the elements \( e_{ik} \) (resp. \( f_{ik} \)) for \( i \in \mathcal{I}, \ k = 1, \ldots, m_i \). We also denote by \( U^{\pm 0} \) (resp. \( U^{\pm 0} \)) the subalgebra of \( U \) generated by the elements \( q^h \) and \( e_{ik} \) (resp. \( f_{ik} \)) for \( h \in P^\vee, \ i \in \mathcal{I}, \ k = 1, \ldots, m_i \). For each \( \beta \in Q^+ \), let

\[
U_{\pm \beta} = \{ x \in U^\pm | q^h x q^{-h} = q^{\pm \beta(h)} x \quad \text{for all} \quad h \in P^\vee \}.
\]

Then we have:

**Proposition 1.2.** ([10])

(a) \( U \cong U^- \otimes U^0 \otimes U^+ \).

(b) \( U^0 = \bigoplus_{h \in P^\vee} F q^h \).

(c) \( U^\pm = \bigoplus_{\beta \in Q^+} U_{\pm \beta} \).

(d) (R5) and (R7) (resp. (R6) and (R8)) are the fundamental relations for \( U^+ \) (resp. \( U^- \)).

Define a structure of directed set on \( Q^+ \) by \( \beta_1 \geq \beta_2 \) if and only if \( \beta_1 - \beta_2 \in \mathbb{Z}^\vee \), and set \( U^{+; \beta} = \bigoplus_{\gamma \in Q^+ \ s.t. \ \gamma \neq \beta} U_\gamma^+ \) for \( \beta \in Q^+ \). We define a completion \( \hat{U} \) of \( U \) by

\[
\hat{U} = \lim_{\beta} U / U^{+; \beta}.
\]

Then \( \hat{U} \) is an algebra containing \( U \). The comultiplication \( \Delta \) and the counit \( \varepsilon \) are naturally extended to those of \( \hat{U} \) ([13]).

A \( U_q(g) \)-module \( V \) is called a **highest weight module** with highest weight \( \lambda \in \mathfrak{h}^* \) if there is a nonzero vector \( v_\lambda \in V \) such that (i) \( e_{ik} v_\lambda = 0 \ (i \in \mathcal{I}, \ k = 1, \ldots, m_i) \), (ii) \( q^h v_\lambda = q^{\lambda(h)} v_\lambda \ (h \in P^\vee) \), (iii) \( V = U_q(g) v_\lambda \). Let \( \lambda \in \mathfrak{h}^* \) and consider the left ideal \( I(\lambda) \) of \( U_q(g) \) generated by \( e_{ik} \ (i \in \mathcal{I}, \ k = 1, \ldots, m_i) \) and \( q^h - q^{\lambda(h)+1} \) \( (h \in P^\vee) \). Let \( M(\lambda) = U_q(g) / I(\lambda) \) and define a \( U_q(g) \)-module structure on \( M(\lambda) \) by the left multiplication. Then \( M(\lambda) \) becomes a highest weight module with highest weight \( \lambda \) and highest weight vector \( v_\lambda = 1 + I(\lambda) \). The \( U_q(g) \)-module \( M(\lambda) \) is called the **Verma module** and it has a unique maximal submodule \( J(\lambda) \). Hence the quotient \( V(\lambda) = M(\lambda) / J(\lambda) \) is irreducible.

Let \( T \) denote the set of all imaginary roots \( \alpha_i \ (i \in I^m) \) counted with multiplicity \( m_i \).

**Proposition 1.3.** ([1], [10]) Suppose \( \lambda(h_i) \geq 0 \) for all \( i \in \mathcal{I} \) and \( \lambda(h_i) \in \mathbb{Z} \) for all \( i \in I^m \). Then we have

(a) \( \chi M(\lambda) = \prod_{\sigma \in S_\lambda} \frac{e^\lambda}{(1 - e^{-\sigma(h)\dim s_\lambda})} = e^{\lambda} \sum_{\beta \in \mathbb{Q}^+} (\dim U^-_\beta) e^{-\beta} \),

(b) \( \chi V(\lambda) = \prod_{\sigma \in S_\lambda} \frac{(-1)^{\langle w | P \rangle}}{(1 - e^{-\sigma(h)\dim s_\lambda})} \cdot \sum_{w \in \mathbb{W} \ s.t. \ P \subset T} (-1)^{\langle w | P \rangle} e^{w(\lambda + \rho - s(\mathbf{F})) + \rho} \),
where $\Delta_+$ denotes the set of all positive roots of $g$, $g_\alpha$ denotes the root space, and $F$ runs over all the finite subsets of $T$ such that $\lambda(h_i) = 0$ for $\alpha_i \in F$ and that $\alpha_i(h_j) = 0$ for $\alpha_i, \alpha_j \in F$ with $i \neq j$. We denote by $|F|$ the number of elements in $F$ and $s(F)$ the sum of elements in $F$.

**Corollary 1.4.** Let $\gamma = \sum_{i \in I} n_i \alpha_i \in Q_+$. Suppose $\lambda(h_i) > 0$ for all $i \in I$, $\lambda(h_i) \in \mathbb{Z}$ for all $h_i \in P^+$, and $\lambda(h_i) \geq n_i$ for all $i \in P^e$. Then we have a linear isomorphism $U^-_\gamma \cong V(\lambda)$ given by $u \mapsto w_\lambda$.

**Proof.** The surjectivity of the map $U^-_\gamma \rightarrow V(\lambda)$ is obvious. Hence it suffices to show $\dim U^-_\gamma = \dim V(\lambda)$. By our assumption, we have

$$\frac{\operatorname{ch} V(\lambda)}{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{|\alpha|}} = \left( \sum_{w \in W} (-1)^{|w|} e^{w(\lambda + \rho) - \rho} \right) \left( \sum_{\beta \in Q_+} (\dim U^-_{-\beta}) e^{-\beta} \right).$$

Therefore, it suffices to show that if $w(\lambda + \rho) - \rho - \beta = \lambda - \gamma$ for $w \in W, \beta \in Q_+$, then $w = 1$. Equivalently, if $w \neq 1$, then $\gamma + w(\lambda + \rho) - (\lambda + \rho) \notin Q_+$. Let us prove this by induction on the length $l(w)$ of $w$. If $w = r_i (i \in I^e)$, then

$$\gamma + r_i(\lambda + \rho) - (\lambda + \rho) = \gamma - (\lambda(h_i) + 1)\alpha_i \notin Q_+.$$

If $w = w'r_i$ and $l(w) = l(w') + 1$, then

$$\gamma + w(\lambda + \rho) - (\lambda + \rho) = \gamma + w'r_i(\lambda + \rho) - (\lambda + \rho)$$

$$= \gamma + w'(\lambda + \rho) - (\lambda + \rho) - (\lambda(h_i) + 1)w'(\alpha_i) \notin Q_+,$$

which completes the proof. □

2. The Killing Form on $U_q(g)$

The Hopf algebra structure of $U_q(g)$ defines an algebra structure on $(U^{\geq 0})^*$ with the multiplication given by $(\phi_1 \phi_2)(x) = (\phi_1 \otimes \phi_2)(A(x))$ for $\phi_1, \phi_2 \in (U^{\geq 0})^*, x \in U^{\geq 0}$. For $h \in P^+$ and $i \in I$, $k = 1, 2, \ldots, m_i$, we define the linear functionals $\phi_h, \psi_{ik} \in (U^{\geq 0})^*$ by

$$\phi_h(x q^h) = 0, \quad \psi_{ik}(x q^h) = \delta_{ik} \epsilon_{il} q^h \quad (x \in U^+, h' \in P^e),$$

(2.1)

Then it is easy to verify that there is an algebra homomorphism $\zeta: U^{\leq 0} \rightarrow (U^{\geq 0})^*$ given by $\zeta(q^h) = \phi_h, \zeta(f_{ik}) = -\frac{1}{\xi_f} \psi_{ik} (h \in P^+, i \in I, k = 1, \ldots, m_i)$. Define
a bilinear form \((\mid)
): \(U^\geq 0 \times U^\leq 0 \to F\) by

\[(2.2) \quad (x\mid y) = \langle \zeta(y), x \rangle \quad (x \in U^\geq 0, y \in U^\leq 0).\]

Then we have:

**Proposition 2.1.** The bilinear form \((\mid)\) on \(U^\geq 0 \times U^\leq 0\) defined by \((2.2)\) satisfies

\[
(x_1 x_2 \mid y_1 y_2) = (x_1 \mid y_1) (x_2 \mid y_2) \quad (x_1, x_2 \in U^\geq 0, y_1, y_2 \in U^\leq 0),
\]

\[
(q^h \mid q^{h'}) = q^{-(h \mid h')} \quad (h, h' \in P^+),
\]

\[
(e_{ik} \mid f_{jl}) = 0, \quad (e_{ik} \mid q^h) = 0,
\]

\[
(e_{ik} \mid f_{jl}) = -\frac{1}{\xi_i} \delta_{ij} \delta_{kl}
\]

for \(i, j \in I, k = 1, 2, \cdots, m_i, l = 1, 2, \cdots, m_j\).

Moreover, the bilinear form on \(U^\geq 0 \times U^\leq 0\) satisfying \((2.3)\) is uniquely determined.

The proof is similar to that of [13, Proposition 2.1.1].

The following lemmas can be proved inductively using \((2.3)\).

**Lemma 2.2.**

(a) \((S(x) \mid S(y)) = (x \mid y)\) for \(x \in U^\geq 0, y \in U^\leq 0\).

(b) \((xq^h \mid yq^{h'}) = q^{-(h \mid h')} (x \mid y)\) \((h, h' \in P^+, x \in U^+, y \in U^-)\).

(c) \((U^+_\gamma \mid U^-_{-\beta}) = 0\) if \(\gamma \neq \beta\).

For \(n \in \mathbb{Z}_{>0}\), we denote by \(\Delta_n: U_q(g) \to U_q(g)^{\otimes (n+1)}\) the algebra homomorphism defined by \(\Delta_1 = \Delta, \Delta_n = (\Delta \otimes 1) \circ \Delta_{n-1}\), and we write

\[\Delta_n(x) = \sum_{(x)_n} x_{(0)} \otimes x_{(1)} \otimes \cdots \otimes x_{(n)}.\]

**Lemma 2.3.** For \(x \in U^\geq 0, y \in U^\leq 0\), we have

\[
yx = \sum_{(x)_2, (y)_2} (x_{(0)} \mid S(y_{(0)})) (x_{(2)} \mid y_{(2)}) x_{(1)} y_{(1)},
\]

\[
xy = \sum_{(x)_2, (y)_2} (x_{(0)} \mid y_{(0)}) (x_{(2)} \mid S(y_{(2)})) y_{(1)} x_{(1)}.
\]

The following lemma is an immediate consequence of Corollary 1.4.

**Lemma 2.4.** Let \(\beta \in Q_+ \setminus \{0\}\) and \(y \in U^-_{-\beta}\). If \(e_{ik} y = y e_{ik}\) for all \(i \in I, k = 1, 2, \cdots, m_i\), then \(y = 0\).

Now we can state the main theorem of this section.
THEOREM 2.5. For \( \beta \in \mathbb{Q}^+ \), the bilinear form \(( \quad | \quad ) : U_{\beta}^0 \times U_{\beta}^0 \rightarrow \mathbb{F}\) defined by (2.2) is nondegenerate.

The proof is the same as that of [13, Proposition 2.1.4].

3. Universal \( R \)-matrix

In this section, we would like to give an explicit formula for the universal \( R \)-matrix of the quantum algebra \( U_q(\mathfrak{g}) \). We first recall the definition of quasi-triangular Hopf algebras and the pre-triangular Hopf algebras ([6], [13]). A Hopf algebra \( \mathcal{H} \) together with an element \( \mathcal{R} \in \mathcal{H} \otimes \mathcal{H} \) is called a quasi-triangular Hopf algebra if it satisfies:

(T1) \( \mathcal{R} \) is invertible,
(T2) \( \mathcal{R} \circ \Delta(a) = \Delta'(a) \circ \mathcal{R} \) for all \( a \in \mathcal{H} \),
(T3) \( (\Delta \otimes 1)(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{23} \),
(T4) \( (1 \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{12} \),

where \( \Delta' = \tau \circ \Delta \) with \( \tau(a \otimes b) = b \otimes a \) (\( a, b \in \mathcal{H} \)) and \( \mathcal{R}_{ij} \) is an element of \( \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \) such that the \((i, j)\) component is given by \( \mathcal{R} \) and the remaining component is 1. The element \( \mathcal{R} \) is called the universal \( R \)-matrix of \( \mathcal{H} \) since it satisfies the Yang-Baxter equation

\[
\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12}.
\]

A Hopf algebra together with an element \( \mathcal{C} \in \mathcal{H} \otimes \mathcal{H} \) and an algebra automorphism \( \Phi : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H} \) is called a pre-triangular Hopf algebra if it satisfies:

(P1) \( \mathcal{C} \) is invertible,
(P2) \( \mathcal{C} \circ \Delta(a) = \Phi(\Delta'(a)) \circ \mathcal{C} \) for all \( a \in \mathcal{H} \),
(P3) \( \Phi_{23} \circ \Phi_{13}(\mathcal{C}_{12}) = \mathcal{C}_{12} \),
(P4) \( \Phi_{12} \circ \Phi_{13}(\mathcal{C}_{23}) = \mathcal{C}_{23} \),
(P5) \( \Phi_{23}(\mathcal{C}_{13}) \circ \mathcal{C}_{23} = (\Delta \otimes 1)(\mathcal{C}) \),
(P6) \( \Phi_{12}(\mathcal{C}_{13}) \circ \mathcal{C}_{12} = (1 \otimes \Delta)(\mathcal{C}) \).

A pre-triangular Hopf algebra \( \mathcal{H} \) becomes a quasi-triangular Hopf algebra if there is an invertible element \( \mathcal{Z} \in \mathcal{H} \otimes \mathcal{H} \) satisfying

\[
\Phi(a \otimes b) = \mathcal{Z}(a \otimes b)\mathcal{Z}^{-1},
\]

\[
(\Delta \otimes 1)(\mathcal{Z}) = \mathcal{Z}_{23} \mathcal{Z}_{13},
\]

\[
(1 \otimes \Delta)(\mathcal{Z}) = \mathcal{Z}_{12} \mathcal{Z}_{13}.
\]

In this case, the universal \( R \)-matrix is given by \( \mathcal{R} = \mathcal{Z}^{-1} \mathcal{C} \).

We define an algebra automorphism \( \Phi : U \otimes U \rightarrow U \otimes U \) by
Seok-Jin KANG and Toshiyuki TANISAKI

\[ \Phi(q^h \otimes q'^{h'}) = q^h \otimes q'^{h'} , \]

(3.2) \[ \Phi(e_{ik} \otimes 1) = e_{ik} \otimes K_i , \quad \Phi(1 \otimes e_{ik}) = K_i \otimes e_{ik} , \]

\[ \Phi(f_{ik} \otimes 1) = f_{ik} \otimes K_{i}^{-1} , \quad \Phi(1 \otimes f_{ik}) = K_{i}^{-1} \otimes f_{ik} . \]

It can be shown that \( \Phi \) can be naturally extended to an automorphism of \( \hat{\mathcal{U}} \otimes \hat{\mathcal{U}} = (\mathcal{U} \otimes \mathcal{U})^\wedge \).

For \( \beta = \sum_{i=1}^{n} n_i \alpha_i \in Q_+ \), we denote by \( C_{\beta} \in U_\beta^+ \otimes U_\beta^+ \) the canonical element of the bilinear form \( ( \; , \; ) : U_\beta^+ \times U_\beta^- \rightarrow F \), and let \( h_\beta = \sum_{i=1}^{n} n_i s_i h_i \), \( K_\beta = q^{n_\beta} \) so that \( (h_\beta|h) = h(h) (h \in P^\vee) \). We define

\[ (3.3) \quad \mathcal{C} = \sum_{\beta \in Q_+} q^{(h_\beta|h)} (K_\beta^{-1} \otimes K_\beta) C_\beta \in \hat{\mathcal{U}} \otimes \hat{\mathcal{U}} . \]

We would like to show that \((\mathcal{U}, \mathcal{C}, \Phi)\) satisfies the conditions (P1)--(P6).

By direct calculations, we can prove the following lemmas.

**Lemma 3.1.**

(a) \( \mathcal{C} A(q^h) = \Phi(A(q^h)) \mathcal{C} \) \((h \in P^\vee)\).

(b) \( (\Phi_{23} \circ \Phi_{13}) (\mathcal{C}_{12}) = \mathcal{C}_{12} \).

(c) \( (\Phi_{12} \circ \Phi_{13}) (\mathcal{C}_{23}) = \mathcal{C}_{23} \).

**Lemma 3.2.** Let

\[ \mathcal{C}' = \sum_{\beta \in Q_+} q^{(h_\beta|h)} (1 \otimes K_\beta)(S \otimes 1) C_\beta \in \hat{\mathcal{U}} \otimes \hat{\mathcal{U}} . \]

Then \( \mathcal{C}' \mathcal{C} = \mathcal{C} \mathcal{C}' = 1 \) if and only if for any \( \beta \in Q_+ \) we have

\[ (3.4) \quad \sum_{\gamma, \delta \in Q_+} C_{\gamma} (K_{\delta} \otimes 1)(S \otimes 1)(C_{\delta}) = \delta_{\beta,0} , \]

\[ \sum_{\gamma, \delta \in Q_+} (K_{\gamma} \otimes 1)(S \otimes 1)(C_{\gamma}) C_{\delta} = \delta_{\beta,0} . \]

**Lemma 3.3.** We have

\[ \mathcal{C} A(e_{ik}) = \Phi(A'(e_{ik})) \mathcal{C} , \quad \mathcal{C} A(f_{ik}) = \Phi(A'(f_{ik})) \mathcal{C} \]

if and only if

\[ [1 \otimes e_{ik}, C_{\beta+s_i}] = C_{\beta}(e_{ik} \otimes K_i^{-1}) - (e_{ik} \otimes K_i) C_{\beta} , \]

\[ [f_{ik} \otimes 1, C_{\beta+s_i}] = C_{\beta}(K_i \otimes f_{ik}) - (K_i^{-1} \otimes f_{ik}) C_{\beta} . \]

**Lemma 3.4.** We have

\[ \Phi_{23}(\mathcal{C}_{13}) \mathcal{C}_{23} = (A \otimes 1) \mathcal{C} , \quad \Phi_{12}(\mathcal{C}_{13}) \mathcal{C}_{12} = (1 \otimes A) \mathcal{C} \]

if and only if
(3.6) 

\[ (A \otimes 1)(C_\beta) = \sum_{\gamma, \delta \in Q, \gamma + \delta = \beta} q^{-(\delta, j|\delta)}(K_\delta \otimes 1 \otimes 1)(C_\gamma)_{13}(C_\delta)_{12}. \]

Hence, in order to show that \((\tilde{U}, \mathcal{C}, \Phi)\) satisfies the conditions (P1)–(P6), it remains to show that (3.4), (3.5), and (3.6) hold. But they can be proved in an almost the same manner as in [13, Proposition 4.3.3]. Therefore, we have:

**Theorem 3.5.** Let \(\Phi: \tilde{U} \otimes \tilde{U}\) be the algebra automorphism defined by (3.2), and let \(c\) be the element of \(\tilde{U} \otimes \tilde{U}\) defined by (3.3). Then the triple \((\tilde{U}, \mathcal{C}, \Phi)\) satisfies the conditions (P1)–(P6).

**Remark.** Let \(\{h_i, d_i|i \in I\}\) and \(\{h^i, d^i|i \in I\}\) be the dual bases of \(\mathfrak{h}\) with respect to the bilinear form \((|)\) and set \(\mathcal{R} = q^{\sum h_i \otimes h^i + \sum d_i \otimes d^i}\). Then \(\mathcal{R} = \mathcal{R}^{-1}c\) gives rise to an \(R\)-matrix for any \(h\)-diagonalizable integrable representation \(V\) of the quantum algebra \(U_q(\mathfrak{g})\). Therefore, the formula (3.3) can be viewed as an explicit formula for the universal \(R\)-matrix of \(U_q(\mathfrak{g})\).

4. The center of \(U_q(\mathfrak{g})\)

In this section, we will describe the center of the quantum algebra \(U_q(\mathfrak{g})\). Let us denote by \(\mathfrak{z}(U)\) the center of \(U = U_q(\mathfrak{g})\). For each \(i \in I\) with \(a_{ii} \neq 0\), define the simple reflection \(r_i \in GL(\mathfrak{h})\) by

\[ r_i(h) = h - \frac{2}{a_{ii}} \alpha_i(h) h_i, \]

and let \(\tilde{W} = \langle r_i|i \in I, a_{ii} \neq 0 \rangle\) be the subgroup of \(GL(\mathfrak{h})\) generated by the \(r_i's\) \((i \in I, a_{ii} \neq 0)\). Let \((U^0)^W\) be the subspace of \(U^0\) consisting of the elements \(\sum_{h \in \mathfrak{c}} c_h q^h\) \((c_h \in F)\) such that \(c_h \neq 0\) implies \(w(h) \in P^+\) and \(c_{\mathfrak{w}(h)} = c_h\) for any \(w \in \tilde{W}\). We define an algebra automorphism \(\phi: U^0 \to U^0\) by \(\phi(q^h) = q^{-\rho(h)} q^h\) \((h \in P^+)\), and let \(\eta\) be the linear map given by

\[ \eta: U \overset{\sim}{\to} U^- \otimes U^0 \otimes U^+ \xrightarrow{e \otimes 1 \otimes e} U^0. \]

The linear map \(\xi: \phi \circ (\eta)_1; U^0\) is called the Harish-Chandra homomorphism.

**Proposition 4.1.**

(a) \(\xi\) is an algebra homomorphism.

(b) \(\xi\) is injective.

(c) \(\text{Im}(\xi) \subseteq (U^0)^W\).
PROOF. (a) can be proved in a standard way (for example, see [Di]), and (b) can be proved as in [13, Theorem 3.1.2].

For (c), let $M(\lambda)$ be the Verma module over $U_q(\mathfrak{g})$ with highest weight $\lambda$. Then it is easy to see that $z|_{M(\lambda)} = \chi_{\lambda+\rho}(\xi(z))I$ for all $z \in \mathfrak{g}$, where $\chi_{\lambda}: U^0 \to F (\lambda \in \mathfrak{h}^*)$ is the algebra homomorphism defined by $\chi_{\lambda}(q^h) = q^{\lambda(h)} (h \in P^\vee)$.

Moreover, if $a_{ii} \neq 0$ and $(\lambda + \rho)(h_i) \in \frac{a_{ii}}{2} \mathbb{Z}_{\geq 0}$, then $\text{Hom}_U(M(r_i(\lambda + \rho) - \rho), M(\lambda)) \neq 0$. Indeed, if $v_\lambda$ is a highest weight vector of $M(\lambda)$ with highest weight $\lambda$, then $f_{ik}^{(\frac{1}{2}a_{ii})(\lambda + \rho)(h_i)}v_\lambda$ is a highest weight vector with highest weight $r_i(\lambda + \rho) - \rho$.

Let $i \in I$ be such that $a_{ii} \neq 0$ and let $z \in \mathfrak{g}$. Then $\chi_{\lambda}(\xi(z)) = \chi_{r_i(\lambda)}(\xi(z)) = \chi_{\lambda}(r_i(\xi(z)))$ for any $\lambda \in \mathfrak{h}^*$ such that $\lambda(h_i) = \frac{a_{ii}}{2} \mathbb{Z}_{\geq 0}$. Hence $\chi_{\lambda}(\xi(z) - r_i(\xi(z))) = 0$ for any $\lambda \in \mathfrak{h}^*$ such that $\lambda(h_i) = \frac{a_{ii}}{2} \mathbb{Z}_{\leq 0}$, which implies $\xi(z) = r_i(\xi(z))$ for all $i \in I$ with $a_{ii} \neq 0$. □

For $J \subset \{(i, k)|i \in I, k = 1, 2, \ldots, m_i\}$, let $U_J = \langle e_{ik}, f_{ik}, U^0|(i, k) \in J \rangle$ be the subalgebra of $U$ generated by $U^0$ and $e_{ik}, f_{ik}$ with $(i, k) \in J$. We denote by $\mathfrak{j}_J$ the center of $U_J$ and $\xi_J: \mathfrak{j}_J \to U^0$ the Harish-Chandra homomorphism for $U_J$. We would like to show $\text{Im}(\xi) \subseteq \text{Im}(\xi_J)$. Let $U_J^+$ (resp. $U_J^-$) be the subalgebra of $U_J$ generated by $e_{ik}$ (resp. $f_{ik}$) with $(i, k) \in J$, and set

\begin{align}
R_J^+ &= \{x \in U^+|\langle x, U_J^- \rangle = 0\} = \{x \in U^+|\langle x, U_J^- U^0 \rangle = 0\}, \\
R_J^- &= \{y \in U^-|\langle U_J^+, y \rangle = 0\} = \{y \in U^-|\langle U^0 U_J^+, y \rangle = 0\}, \\
R_J &= U_J^+ U^0 U^+ + U^- U^0 U_J^+.
\end{align}

Then we have:

**LEMMA 4.2.**

(a) $U = U_J \oplus R_J$,

(b) $U_J R_J U_J \subseteq R_J$,

(c) $(\epsilon \otimes 1 \otimes \delta)(R_J) = 0$.

**PROOF.** (a) It suffices to show $U_J^+ = U_J^+ \oplus R_J^+$ for any $\gamma \in \mathbb{Q}_+$. Since

\[ R_{J, \gamma}^+ = \text{Ker}(U_J^+ \to (U^-)\ast \to (U_J^-)\ast), \]

\[ \dim R_{J, \gamma}^+ = \dim U_J^+ - \dim U_{J, \gamma}^- = \dim U_{J, \gamma}^+ - \dim U_J^+. \]

Since $\langle \cdot \rangle$ is nondegenerate on $U_J^+ \times U_J^-$, we have $R_{J, \gamma}^+ \cap U_{J, \gamma}^+ = \{0\}$.

(b) First, note that $R_J^+$ (resp. $R_J^-$) is a two-sided ideal of $U^+$ (resp. $U^-$), and that $U^0 R_J = R_J U^0$. Hence it suffices to show

\[ U_J^+ R_J^- \subseteq R_J U, \quad R_J^+ U_J^- \subseteq U R_J^+. \]
Let \( y \in R_j \). For \((i, k) \in J^\ast\), by Lemma 2.3, we have
\[
e_{ik} y = \sum_{\partial j} (e_{ik} | y(0)) (1 | S(y(2))) y(1) + \sum_{\partial j} (K_i | y(0)) (1 | S(y(2))) y(1) e_{ik}
\]
\[+ \sum_{\partial j} (K_i | y(0)) (e_{ik} | S(y(2))) y(1) K_i.
\]
Hence it suffices to show
\[
\begin{align*}
\left( x \left| \sum_{\partial j} (e_{ik} | y(0)) (1 | S(y(2))) y(1) \right. \right) = 0, \\
\left( x \left| \sum_{\partial j} (K_i | y(0)) (1 | S(y(2))) y(1) \right. \right) = 0, \\
\left( x \left| \sum_{\partial j} (K_i | y(0)) (e_{ik} | S(y(2))) y(1) \right. \right) = 0
\end{align*}
\]
for all \( x \in U^+_j \). Indeed, we have, for example,
\[
\left( x \left| \sum_{\partial j} (K_i | y(0)) (e_{ik} | S(y(2))) y(1) \right. \right) = \sum_{\partial j} (K_i | y(0)) (e_{ik} | S(y(2))) (x | y(1))
\]
\[= \sum_{\partial j} (K_i \otimes x \otimes S^{-1} (e_{ik}) | D^{(2)}(y))
\]
\[= (S^{-1} (e_{ik}) x K_i | y) = 0.
\]
The other cases can be proved in a similar way.
(c) Clear. \( \square \)

**Proposition 4.3.** \( \text{Im}(\xi) \subseteq \text{Im}(\xi_j) \).

**Proof.** Let \( z \in \mathfrak{z} \) and write \( z = z_1 + z_2 \) with \( z_1 \in U_j, z_2 \in R_j \). By Lemma 4.2 (b), \( z_1 \in \mathfrak{z}_j \), and hence by Lemma 4.2 (c), \( \xi(z) = \xi_j(z_1) \in \text{Im}(\xi_j) \). \( \square \)

We now consider the special cases when \(|I| = 1\) or \(|I| = 2\). By a direct calculation, we have:

**Proposition 4.4.** Suppose \( I = \{i\} \) and \( m_i = 1 \).

(a) If \( a_{ii} \neq 0 \), then
\[
\mathfrak{z} = \left\langle f_{i,1,1} e_{i,1} + \frac{1}{\xi_i (q_i - q_i^{-1})} (q_i K_i + q_i^{-1} K_i^{-1}), q^h | x_i(h) = 0 \right\rangle.
\]
(b) If \( a_{ii} = 0 \), then \( \mathfrak{z} \subseteq U^0 \).

**Proposition 4.5.** Assume either
(a) \( I = \{i\} \) with \( a_{ii} < 0, m_i = 2 \), or
(b) \( I = \{i, j\} \) with \( a_{ii} < 0, a_{jj} < 0, a_{ij} < 0 \) and \( m_i = m_j = 1 \).

Then \( 3 \subset U^0 \).

**Proof.** Set \( e = e_{i,1}, e' = e_{i,2}, f = f_{i,1}, f' = f_{i,2} \) in case (a), and \( e = e_{i,1}, e' = e_{i,2}, f = f_{i,1}, f' = f_{i,1} \) in case (b). Then the subalgebra \( U^+ = \langle e, e' \rangle = \bigoplus_{n=0}^{\infty} U_n^+ \) (resp. \( U^- = \langle f, f' \rangle = \bigoplus_{n=0}^{\infty} U_n^- \)) is the free associative algebra over \( F \) generated by the elements \( e, e' \) (resp. \( f, f' \)), where \( U_n^+ \) (resp. \( U_n^- \)) is the homogeneous subspace of degree \( n \) (resp. \( -n \)). Then, for \( n \geq 1 \), we have \( U_n^+ = U_{n-1}^+ e \otimes U_{n-1}^+ e' \).

Let \( z \in \mathfrak{z} \cap (\bigoplus_{k=0}^{n-1} U^- U^0 U_k^+), \) and let \( \{x_\lambda\} \) be a basis of \( U_{n-1}^+ \). Then

\[
z = \sum_{\lambda} \sum_{h \in F^i} y_{\lambda, h} q^h x_\lambda e + \sum_{\lambda} \sum_{h \in F^i} y'_{\lambda, h} q^h x_\lambda e' + y,
\]

where \( y \in \bigoplus_{k=0}^{n-1} U^- U^0 U_k^+ \), \( y_{\lambda, h}, y'_{\lambda, h} \in U^- \). Hence we have

\[
ez = \sum_{\lambda} \sum_{h \in F^i} y_{\lambda, h} q^{-e_{(h)}} q^h x_\lambda e + \sum_{\lambda} \sum_{h \in F^i} y'_{\lambda, h} q^{-e_{(h)}} q^h x_\lambda e' + z',
\]

and

\[
zh = \sum_{\lambda} \sum_{h \in F^i} y_{\lambda, h} q^h x_\lambda e^2 + \sum_{\lambda} \sum_{h \in F^i} y'_{\lambda, h} q^h x_\lambda e e' + z'',
\]

where \( z', z'' \in \bigoplus_{k=0}^{n-1} U^- U^0 U_k^+ \). Hence \( y'_{\lambda, h} = 0 \) for all \( \lambda \) and \( h \). Similarly, \( y_{\lambda, h} = 0 \) for all \( \lambda \) and \( h \). Therefore, \( z \in \mathfrak{z} \cap (\bigoplus_{k=0}^{n-1} U^- U^0 U_k^+) \), and hence, by induction, we see that \( \mathfrak{z} = \mathfrak{z} \cap U^- U^0 = \mathfrak{z} \cap U^0 \). \( \square \)

**Proposition 4.6.** Assume that \( I = \{i, j\} \) and \( a_{ii} = 2, a_{jj} < 0, a_{ij} < 0 \), and \( m_i = 1 \). Then we have \( \mathfrak{z} \subset U^0 \).

**Proof.** Let \( V' = \mathbb{Q} h_i \oplus \mathbb{Q} h_j \) and \( V = \{h \in \mathbb{h} | \alpha_i(h) = \alpha_j(h) = 0\} \). Then \( \mathfrak{h} = V \oplus V' \). Note that \( \mathfrak{w} \) preserves \( V \) and \( V' \) and that

\[
det \begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix} = a_{ii}a_{jj} - a_{ij}a_{ji} < 0.
\]

We would like to show \( \text{Im}(\xi) \subset \bigoplus_{h \in V} \mathbb{Q} h \). Since \( \text{Im}(\xi) \subset (U^0)^\mathfrak{h} \), it suffices to show \( h \in \mathfrak{h} \) and \( |\mathfrak{w}(h)| < \infty \) if and only if \( h \in V \). Hence we need only to show if \( \tilde{h} \in \mathfrak{h} \cap V \approx V' \), \( |\tilde{W}(\tilde{h})| < \infty \), then \( \tilde{h} = 0 \). Therefore, it suffices to show that the eigenvalues of \( r_i r_j |_{V'} \) are not roots of unity. Since the characteristic polynomial of \( r_i r_j |_{V'} \) is \( t^2 - \left( \frac{2a_{ij}a_{ji}}{a_{jj}} - 2 \right) t + 1, r_i r_j |_{V'} \) has an eigenvalue that is a root of unity if and only if \( \frac{2a_{ij}a_{ji}}{a_{jj}} = 0, 1, 2, 3, 4 \), which is a contradiction to our assumption. \( \square \)
**Lemma 4.7.** Assume that the Borcherds-Cartan matrix \(A = (a_{ij})_{i,j \in I}\) is indecomposable. If there is a nonempty subset \(J\) of \(\{(i, k) | i \in I, k = 1, \ldots, m_i\}\) such that \(\overline{3}_J \subset U^0\), then \(\overline{3}\) is contained in \(U^0\).

**Proof.** Let \(\overline{J} = \{i \in I | (i, k) \in J\} for some \(k\). Then we have

\[
\overline{3} \cap U^0 = \bigoplus_{h \in P^r, a_i(h) = 0 (i \in I)} F q^h, \quad \overline{3} \cap U^0 = \bigoplus_{h \in P^r, a_i(h) = 0 (i \in \overline{J})} F q^h.
\]

For \(i \in I\), set \(T_i = \bigoplus_{h \in P^r, a_i(h) = 0} F q^h\). We would like to show \(\text{Im}(\xi) \subset \overline{i} \cap \bigcap_{i \in I} T_i\). By Proposition 4.3, we have \(\text{Im}(\xi) \subset \text{Im}(\xi^2) \subset \bigcap_{i \in \overline{J}} T_i\).

If \(a_{ii} = 0\), then by Proposition 4.4 (b), \(\text{Im}(\xi) \subset \text{Im}(\xi_{(i, i)}) \subset T_i\). Hence it suffices to show that if \(a_{ij} \neq 0\), \(a_{ji} \neq 0\), then \(T_i \cap (U^0)^i \subset T_i\).

Let \(x = \sum_{h \in P^r, a_i(h) = 0} c_h q^h \in T_i \cap (U^0)^i\). Then \(x = r_f(x) = \sum_{h \in P^r, a_i(h) = 0} c_q q^{-\xi(h)}\). Hence if \(c_h \neq 0\), then \(a_i(r_f(h)) = a_i(h) = 0\), which implies \(a_i(h) = 0\). \(\square\)

By Proposition 4.4–Lemma 4.7, we have the following theorem.

**Theorem 4.8.** Suppose that the Borcherds-Cartan matrix \(A = (a_{ij})_{i,j \in I}\) is indecomposable and \(\text{Im} \neq \phi\). Then

\[
\overline{3}(U) = \bigoplus_{h \in P^r, a_i(h) = 0 (i \in I)} F q^h \subset U^0
\]

except for the case \(I\) consists of a single element \(i\) with \(a_{ii} < 0\) and \(m_i = 1\).

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