교육학석사 학위논문

The (i, j)-step competition graphs of oriented complete bipartite graphs

(방향 지어진 완전이분그래프의 (i, j)-step 경쟁그래프)

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Abstract

In this thesis, we study the (i, j)-step competition graph of an oriented complete bipartite graph. Kim *et al.* studied the competition graph of an oriented complete bipartite graph. We take a further step to study the $(1, 2)$-step competition graph of an oriented complete bipartite graph by extending the results given by Kim *et al*. Then we study (i, j)-step competition graph, a more general version of $(1, 2)$-step competition graph. Finally, we deal with the limit of (i, j)-step competition graph.

Key words: complete bipartite graphs; $(1, 2)$-step competition graphs; $(1, 2)$-step competition-realizable; (i, j)-step competition graphs; limits of (i, j)-step competition graphs; orientations.

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Contents

Chapter 1

Introduction

1.1 The notion of $(1, 2)$-step competition graphs

Given a digraph D, we denote by $N_D^+(u)$ (resp. $N_D^-(u)$) the set of out-neighbors (resp. in-neighbors) of a vertex u in D. The *out-degree* (resp. *in-degree*) of u in D is defined to be $|N_D^+(u)|$ (resp. $|N_D^-(u)|$).

The *distance* from u to v in a digraph D, denoted by $d_D(u, v)$, is defined as the minimum number of arcs in a directed (u, v)-walk. The digraph $D - u$ is the graph obtained from D by removing vertex u and all arcs incident with u.

A *tournament* is an oriented complete graph. An *n-tournament* is a tournament on n vertices. A digraph D is *strongly connected* if, for every pair u, v of distinct vertices in D, there exist a directed (u, v)-walk and a directed (v, u)-walk.

The *competition graph* $C(D)$ of a digraph D is the graph G defined by $V(G) = V(D)$ and $E(G) = \{uv \mid u, v \in V(D),\ u \neq v,\ N_D^+(u) \cap N_D^+(v) \neq \emptyset\}$. We say that u and v *compete* provided there exists $w \in N_D^+(u) \cap N_D^+(v)$. The competition graph of a digraph, introduced by Cohen [6] to study "food web" models in 1968, has been extensively studied in connection to some biological models and some radio communication networks. For a comprehensive introduction to competition graphs, see [7, 20]. A variety of generalizations of the notion of competition graph have also been introduced, including the common enemy graph in [21,24], the competition-common enemy graph in [15, 19, 22, 23], the niche graph in [3, 4, 11], the p-competition graph in [1, 16, 18], and the m-step competition graph in [5, 13].

Among the variants, the notion of m-step competition graph introduced by Cho *et al.* [5] is closely related to the topic $(1, 2)$-step competition graph of this thesis. The *m-step competition graph* $C^m(D)$ of a digraph D is a graph on the vertex set of D with an edge uv if there is a vertex w in D such that both an (u, w)-path and a (v, w)-path of length m exist.

Ho introduced two new variants of competition graphs in [14]: same-step competition graph and any-step competition graph both of which are motivated by the introduction of m-step competition graph. The *same-step competition graph* $C^S(D)$of a digraph D has the same vertex set as D and has an edge between vertices u and v if and only if there exists some vertex w such that there exist directed walks from u to w and from v to w of the same length. A same-step competition graph can be written in terms of m-step competition graphs: for any digraph D, $C^S(D) = \bigcup_{m=1}^{\infty} C^m(D)$. The *any-step competition graph* $C^*(D)$ of a digraph D has the same vertex set as D and has an edge between vertices u and v if and only if there exist nontrivial directed walks from u and v to some common vertex.

The notion "(i, j)-step" is originated from the paper [12] authored by Hefner (Factor) *et al.* in 1991 in which the notion of (i, j) competition graph is introduced. The introduction of (i, j) competition graph was followed by that of the $(1, 2)$-domination graph by Factor and Langley in [8]. In 2011, Factor, one of the authors, and Merz introduced the the $(1, 2)$-step competition graph in [9]. They completely characterized the $(1, 2)$-step competition graphs of tournaments and extended the results to the (i, k)-step competition graphs.

The $(1, 2)$-*step competition graph* of a digraph D, denoted by $C_{1,2}(D)$, is a graph on $V(D)$ where $uv \in E(C_{1,2}(D))$ if and only if there exists a vertex $w \neq u, v$ such that either $d_{D-v}(u, w) = 1$ and $d_{D-u}(v, w) \leq 2$ or $d_{D-u}(v, w) = 1$ and $d_{D-v}(u, w) \leq 2$. We say that u and v $(1, 2)$-*compete* provided there exists $w \neq u, v$ such that either $d_{D-v}(u, w) = 1$ and $d_{D-u}(v, w) = 2$ or $d_{D-u}(v, w) = 1$ and $d_{D-v}(u, w) = 2$. Thus, $uv \in E(C_{1,2}(D))$ provided u and v compete or $(1, 2)$-compete.

Factor and Merz *et al.* [9] showed that in a strong tournament T, $uv \notin E(C_{1,2}(T))$ if and only if $N^+(u) = \{v\}$ or $N^+(v) = \{u\}$. Furthermore, they completely characterized the $(1, 2)$-step competition graphs of tournaments. A graph G with n vertices is the $(1, 2)$-step competition graph of some tournament if and only if G is one of the following graphs:

(i) K_n where $n \neq 2, 3, 4$,

(ii) $K_{n-1} \cup K_1$, where $n > 1$,

(iii) $K_n - E(P_3)$ where $n > 2$,

(iv) $K_n - E(P_2)$ where $n \neq 1, 4$, or

(v) $K_n - E(K_3)$ where $n \geq 3$.

Then they obtained the following corollary: if T is a tournament, the maximum number of edges missing from the $(1, 2)$-step competition graph of a tournament on $n \geq 4$ vertices is $n - 1$. Especially, a graph G on $n \geq 5$ vertices is the $(1, 2)$-step competition graph of some strong tournament if and only if G is K_n, $K_n - E(P_3)$, or $K_n - E(P_2)$.

1.2 The competition graphs of oriented complete bipartite graphs

A graph is *bipartite* if its vertex set can be partitioned into two subsets V_1 and V_2 so that every edge has one end in V_1 and the other end in V_2; such a partition (V_1, V_2) is called a *bipartition* of the graph, and V_1 and V_2 are called its *parts*. If a bipartite graph is simple and every vertex in one part is joined to every vertex in the other part, then the graph is called a *complete bipartite graph*. We denote by $K_{m,n}$ a complete bipartite graph with bipartition (V_1, V_2) if $|V_1| = m$ and $|V_2| = n$.

Let G_1 and G_2 be graphs with m vertices and n vertices, respectively. The pair (G_1, G_2) is said to be *competition-realizable through* $K_{m,n}$ if the disjoint union of G_1 and G_2 is the competition graph of an orientation of $K_{m,n}$ with bipartition $(V(G_1), V(G_2))$.

Kim *et al.* [17] studied the competition graphs of oriented complete bipartite graphs. They showed that the competition graph of D has no edges between the vertices in V_1 and the vertices in V_2 for an orientation D of $K_{m,n}$ with bipartition (V_1, V_2) where $|V_1| = m$ and $|V_2| = n$.

1.3 A preview of thesis

A graph is said to be $(1,2)$*-step competition-realizable through* $K_{m,n}$ if it is the $(1,2)$-step competition graph of an orientation of $K_{m,n}$. For an illustration, see Figure 1.1.

Suppose that G is $(1,2)$-step competition-realizable through $K_{m,n}$. By definition, G is the $(1,2)$-step competition graph of an orientation D of $K_{m,n}$. Let (V_1, V_2) be the bipartition of $K_{m,n}$ where $|V_1| = m$ and $|V_2| = n$. Then $V(G) = V_1 \cup V_2$. Kim *et al.* [17] showed that the competition graph of an orientation of a complete

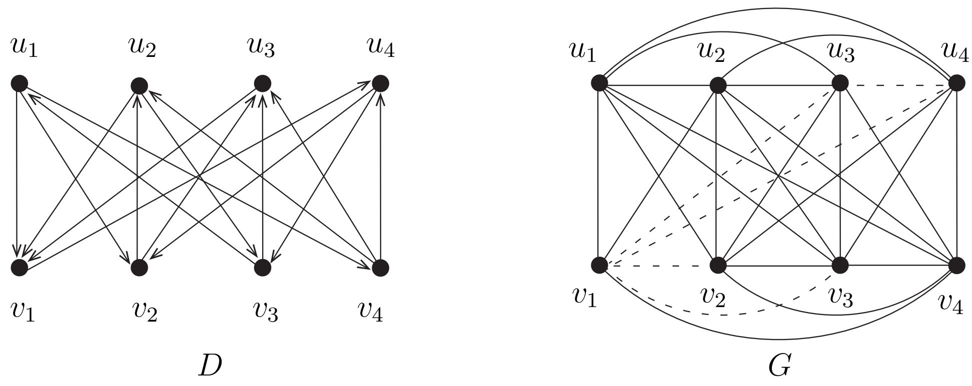

Figure 1.1: The graph G is $(1,2)$-step competition-realizable through $K_{4,4}$ since it is the $(1,2)$-step competition graph of an orientation D of $K_{4,4}$. The excluded edges of G and H are represented by the dotted lines.

bipartite graph is the union of two graphs whose vertex sets are the parts of the complete bipartite graph, respectively. Therefore a subgraph of $G[V_1] \cup G[V_2]$ is the competition graph of D. By the way, $G[V_1] \cup G[V_2]$ is the competition graph of D.

Throughout this thesis, we denote by (V_1, V_2) the bipartition of $K_{m,n}$ where $|V_1| = m$ and $|V_2| = m$ unless otherwise stated.

Proposition 1.3.1. *Let D be a digraph whose underlying graph is a complete bipartite graph $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$ for positive integers m and n, and G be the $(1,2)$-step competition graph of D. Then $G[V_1] \cup G[V_2]$ is the competition graph of D.*

Proof. If there is no edge in $G[V_1] \cup G[V_2]$, then $G[V_1] \cup G[V_2]$ is the competition graph of D by the observation that a subgraph of $G[V_1] \cup G[V_2]$ is the competition graph of D. Thus we may assume that $G[V_1] \cup G[V_2]$ has an edge and take an edge e in $G[V_1] \cup G[V_2]$. Let u and v be the ends of e. Then u and v belong to the same part, that is, either V_1 or V_2. By definition, there exists a vertex w satisfying one of the following: in D,

(i) there are arcs (u, w) and (v, w);

(ii) there exist an arc (u, w) and a directed (v, w)-walk of length 2;

(iii) there exist a directed (u, w)-walk of length 2 and an arc (v, w).

Then $w \in V_1$ or $w \in V_2$. Since D is an orientation of $K_{m,n}$, which is a bipartite graph, and u and v belong to the same part, the lengths of a directed (u, w)-walk and a directed (v, w)-walk have the same parity. Therefore only (i) is possible and the proposition follows. □

Corollary 1.3.2. *Vertices u and v belong to distinct parts in $K_{m,n}$ and are adjacent in the $(1, 2)$-step competition graph of an orientation D of $K_{m,n}$ if and only if u and v $(1, 2)$-compete in D.*

Proof. The 'only if' part immediately follows from Proposition 1.3.1. To show the 'if' part, suppose that u and v $(1, 2)$-compete in D. Then there exists a vertex w in D such that there is a (u, w)-walk of length 1 and a (v, w)-walk of length 2, or vice versa, which implies that there exist a (u, w)-walk and a (v, w)-walk of lengths with different parities. Therefore u and v must belong to distinct parts. □

1.4 (i, j)-step competition graphs

Factor and Merz extended the results to the (i, j)-step competition graphs. Let i and j be positive integers such that $i \geq 1$ and $j \geq 2$. The (i, j)-*step competition graph* of a digraph D, denoted by $C_{i,j}(D)$, is a graph on $V(D)$ where $uv \in E(C_{i,j}(D))$ if and only if there exists a vertex $w \neq u, v$ such that either $d_{D-v}(u, w) \leq i$ and $d_{D-u}(v, w) \leq j$ or $d_{D-u}(v, w) \leq i$ and $d_{D-v}(u, w) \leq j$. We say that u and v

(i,j)-*compete* provided there exists $w \neq u, v$ such that either $d_{D-v}(u,w) = i$ and $d_{D-u}(v,w) = j$ or $d_{D-u}(v,w) = i$ and $d_{D-v}(u,w) = j$. An any-step competition graph can be written in terms of (i,j)-step competition graphs: for any digraph D, $C^*(D) = \bigcup_{i,j=1}^{\infty} C_{i,j}(D)$.

They showed the following result similar to that of $(1,2)$-step competition graph: let T be a strongly connected tournament with $i \geq 1$ and $j \geq 2$. Edge $uv \notin E(C_{i,j}(T))$ if and only if $N^+(u) = \{v\}$ or $N^+(v) = \{u\}$.

Furthermore, if T is an n-tournament, $i \geq 1$ and $j \geq 2$, then $C_{i,j}(T) = C_{1,2}(T)$. Thus the (i,j)-step competition graph has no more edges than the $(1,2)$-step competition graph.

Chapter 2

$(1, 2)$-step competition graphs

2.1 A characterization of $(1, 2)$-step competition graphs

Theorem 2.1.1. *Let D be a digraph whose underlying graph is a complete bipartite graph $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$ for integers $m \geq n \geq 2$, and G be the $(1, 2)$-competition graph of D. Then, for vertices u and v belonging to distinct parts in $K_{m,n}$, u and v do not $(1, 2)$-compete in D if and only if one of the following holds:*

$$N_D^+(u) = \emptyset; N_D^+(v) = \emptyset; N_D^+(u) = \{v\}; N_D^+(v) = \{u\}.$$

Proof. By the symmetry of u and v, without loss of generality, we may assume that $u \in V_1$ and $v \in V_2$. As the 'if' part is obviously true, it suffices to show the 'only if' part. Suppose that u and v do not $(1, 2)$-compete, $N_D^+(u) \neq \emptyset$ and $N_D^+(v) \neq \emptyset$. The proof is complete if we show that $N_D^+(u) = \{v\}$ or $N_D^+(v) = \{u\}$. Since D is an orientation of $K_{m,n}$, either $v \in N_D^+(u)$ or $u \in N_D^+(v)$. We assume that $v \in N_D^+(u)$. To the contrary, suppose that there exists $v' \neq v$ such that $v' \in N_D^+(u)$. By our assumption that $N_D^+(v) \neq \emptyset$, there exists $u' \in N_D^+(v)$. Since $v \in N_D^+(u)$, $u' \neq u$. Noting that $N_D^+(v) \subset V_1$ sand $N_D^+(u) \subset V_2$, we have $u' \in V_1$ and $v' \in V_2$. Since D

is an orientation of $K_{m,n}$, (u', v') or (v', u') is an arc of D. If (u', v') is an arc of D, then $(u, v') \in A(D)$ and $v \to u' \to v'$ is a directed walk in D, that is, u and v $(1, 2)$-compete and we reach a contradiction. If (v', u') is an arc of D, $(v, u') \in A(D)$ and $u \to v' \to u'$ is a directed walk in D, that is, u and v $(1, 2)$-compete and we reach a contradiction. Therefore $N_D^+(u) = \{v\}$. By applying a similar argument, we may show that if $u \in N_D^+(v)$, then $N_D^+(v) = \{u\}$. □

Corollary 2.1.2. *Let G be the $(1, 2)$-step competition graph of an orientation D of $K_{m,n}$ for some positive integers m and n. Then each vertex has out-degree at least two in D if and only if the edges of G not belonging to $G[V_1] \cup G[V_2]$ induce $K_{m,n}$.*

Proof. To show the 'only if' part, take two vertices u and v in distinct parts of $K_{m,n}$. Then the out-degree of each of u and v is at least two, so none of the four cases in the statement of Theorem 2.1.1 holds. Thus u and v $(1, 2)$-compete and therefore they are adjacent in G. Since u and v are arbitrarily chosen, the edges not belonging to $G[V_1] \cup G[V_2]$ induce $K_{m,n}$. To show the 'if' part by contradiction, suppose that there exists a vertex v such that v has out-degree at most one. If v has no out-neighbor, then v is isolated in G and we reach a contradiction. Thus v has exactly one out-neighbor, say u, and so, by Theorem 2.1.1 u and v do not $(1, 2)$-compete in D. Hence u and v are not adjacent in G and we reach a contradiction. This completes the proof. □

The *independence number* of a graph G is the maximum number of vertices in G which are pairwise nonadjacent, and is denoted by $\alpha(G)$.

Proposition 2.1.3. *Let G be the $(1, 2)$-step competition graph of an orientation D of $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$ for some positive integers m and n. Then one of the following holds:*

(i) $\alpha(G[V_1]) = \alpha(G[V_2]) = 2$;

(ii) $\alpha(G[V_1]) = 1$ or $\alpha(G[V_2]) = 1$.

Proof. Kim *et al.* [17] showed that for a competition-realizable pair (G_1, G_2), one of the following holds:

(i) $\alpha(G_1) = \alpha(G_2) = 2$;

(ii) $\alpha(G_1) = 1$ or $\alpha(G_2) = 1$.

By Proposition 1.3.1, the proposition immediately follows. □

2.2 Structural characterization of $(1,2)$-step competition graphs

Let G be the $(1,2)$-step competition graph of an orientation D of $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$. By proposition 1.3.1,

> when $G[V_1]$ and $G[V_2]$ are identified, we only need to figure out the edges not belonging to $G[V_1] \cup G[V_2]$ to characterize G. $(*)$

When (ii) holds in Proposition 2.1.3, we may assume that $\alpha(G[V_2]) = 1$ without loss of generality. In the rest of this thesis, we will assume that $\alpha(G[V_2]) = 1$ whenever (ii) holds in Proposition 2.1.3. Since $G[V_1]$ has either an isolated vertex or no isolated vertex, if (ii) holds in Proposition 2.1.3, then $G[V_1]$ has an isolated vertex and $\alpha(G[V_2]) = 1$ or $G[V_1]$ has no isolated vertex and $\alpha(G[V_2]) = 1$. Based on this observation, we may conclude that it is sufficient to consider the following three cases to investigate the $(1,2)$-step competition graph of an orientation D of $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$ for some positive integers m and n.

- $\alpha(G[V_1]) = \alpha(G[V_2]) = 2$;
- $G[V_1]$ has an isolated vertex and $\alpha(G[V_2]) = 1$;
- $G[V_1]$ has no isolated vertex and $\alpha(G[V_2]) = 1$.

In the following, we study the structure of $(1, 2)$-step competition graphs of orientations of $K_{m,n}$ corresponding to one of the above cases.

Kim *et al.* [17] showed the following theorem.

Theorem 2.2.1 ([17]). *The pair (C_m, C_n) is competition-realizable if and only if $(m, n) = (4, 4)$.*

Theorem 2.2.2. *Let m and n be integers such that $m \geq n \geq 3$ and G be the $(1, 2)$-step competition graph of an orientation D of $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$. Then $G[V_1] = C_m$ and $G[V_2] = C_n$ if and only if $(m, n) = (4, 4)$ and the edges of G not belonging to $G[V_1] \cup G[V_2]$ induce $K_{4,4}$.*

Proof. Suppose that $G[V_1] = C_m$, $G[V_2] = C_n$. Then, by Proposition 1.3.1 and Theorem 2.2.1, $(m, n) = (4, 4)$. Let $C_4 := u_1u_2u_3u_4u_1$ and $C_4 := v_1v_2v_3v_4v_1$. Fix i for $i = 1, 2, 3, 4$. Then u_i and u_{i-1} have a common out-neighbor, say w_i, in D and u_i and u_{i+1} also have a common out-neighbor, say w'_i for each where subscripts are reduced to modulo 4. Since u_{i-1} and u_{i+1} are not adjacent in G, w_i and w'_i are distinct and so the out-degree of u_i is at least two in D. Thus we have shown that the out-degree of u_i is at least two in D for each $i = 1, 2, 3, 4$. By the same argument, we may show that v_j has out-degree at least two in D for each $j = 1, 2, 3, 4$ and the 'only if' part is true.

To show the 'if' part, suppose that $(m, n) = (4, 4)$ and the edges not belonging to $G[V_1] \cup G[V_2]$ induce $K_{4,4}$. Then, by Corollary 2.1.2, each vertex has out-degree at least two in D. Then each vertex has in-degree at most two in D. Now

$$2 \times 8 \leq \sum_{v \in V(D)} |N_D^+(v)| = \sum_{v \in V(D)} |N_D^-(v)| \leq 2 \times 8.$$

Therefore every vertex has in-degree exactly two in D. Then each vertex in V_1 has exactly two in-neighbors in V_2, so $G[V_2] = C_4$. For the same reason, $G[V_1] = C_4$. $\square$

Theorem 2.2.3 ([17]). *The pair (P_m, P_n) is competition-realizable if and only if (m, n) is one of $(1, 1), (2, 1), (3, 3)$ and $(4, 3)$.*

Theorem 2.2.4. *Let m and n be positive integers with $m \geq n$ and G be a $(1, 2)$-step competition graph of an orientation D of $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$. If $G[V_1] = P_m$ and $G[V_2] = P_n$, then one of the following holds:*

(i) $(m, n) = (1, 1)$ *and there is no edge not belonging to $G[V_1] \cup G[V_2]$;*

(ii) $(m, n) = (2, 1)$ *and there is no edge not belonging to $G[V_1] \cup G[V_2]$;*

(iii) $(m, n) = (3, 3)$ *and the edges not belonging to $G[V_1] \cup G[V_2]$ induce $K_{3,3} - E(C_4)$;*

(iv) $(m, n) = (4, 3)$ *and the edges not belonging to $G[V_1] \cup G[V_2]$ induce $K_{4,3} - E(P_2) \times 2$ where $E(P_2) \times 2$ means a set of two nonadjacent edges.*

The converse is also true only for the cases (i) and (iii).

Proof. To show the 'only if' part, suppose that $G[V_1] = P_m$ and $G[V_2] = P_n$. Then by Proposition 1.3.1 and Theorem 2.2.3, (m, n) is one of $(1, 1), (2, 1), (3, 3)$ and $(4, 3)$.

Suppose that (m, n) is $(1, 1)$ or $(2, 1)$. Since $(1, 2)$-competition requires at least four vertices by definition, there exist no vertices which $(1, 2)$-compete. Thus $G = G[V_1] \cup G[V_2]$.

Suppose that (m, n) is $(3, 3)$. Let $P_3 := u_1u_2u_3$ and $P_3 := v_1v_2v_3$. Then exactly four pairs $\{u_1, u_2\}$, $\{u_2, u_3\}$, $\{v_1, v_2\}$ and $\{v_2, v_3\}$ compete in D. Since u_1 and u_3 do not compete, u_2 has out-degree at least two in D. Since v_1 and v_3 do not compete, v_2 has out-degree at least two in D. Thus both u_2 and v_2 have in-degree at most one, and so they cannot be a common in-neighbor of other vertices in D. Therefore u_1, v_1, u_3 and v_3 are common in-neighbors of the competing pairs in D, so they have in-degree of two or three. However, each of these vertices competes with a vertex, so it has out-degree at least one. Thus those vertices have in-degree of two and out-degree of one. Hence any pair of them does not $(1, 2)$-compete in D by Theorem 2.1.1. Since u_1 and u_3 are common in-neighbors of the competing pairs $\{v_1, v_2\}$ and $\{v_2, v_3\}$, $u_1, u_3 \in N_D^+(v_2)$. Thus by Theorem 2.1.1 u_1 and v_2 $(1, 2)$-compete and so do u_3 and v_2. Similarly, we can show that v_1 and u_2, v_3 and u_2 $(1, 2)$-compete, respectively. Since both u_2 and v_2 have out-degree at least two, by Theorem 2.1.1 u_2 and v_2 $(1, 2)$-compete. Thus the edges not belonging to $G[V_1]$ and $G[V_2]$ induce the graph deleting the edges u_1v_1, v_1u_3, u_3v_3 and v_3u_1 from $K_{3,3}$, that is, $K_{3,3} - E(C_4)$.

Suppose that (m, n) is $(4, 3)$. Let $P_4 := u_1u_2u_3u_4$ and $P_3 := v_1v_2v_3$. Then exactly three pairs $\{u_1, u_2\}$, $\{u_2, u_3\}$ and $\{u_3, u_4\}$ in V_1 and exactly two pairs $\{v_1, v_2\}$ and $\{v_2, v_3\}$ in V_2 compete in D. Since u_1 and u_3 do not compete in D, u_2 has out-degree at least two in D. Suppose that u_2 has out-degree of three. Then u_2 is an in-neighbor of each vertex of V_2 in D. Since u_4 has out-degree at least one, u_2 and u_4 must compete in D, which contradicts $G[V_1] = P_4$. Thus u_2 has out-degree of two in D. Similarly, we can show that u_3 has out-degree of two in D. Therefore both u_2 and u_3 have in-degree of one in D, and so they cannot be a common in-neighbor of other vertices in D. Thus u_1 and u_4 are common in-neighbors of the competing pairs $\{v_1, v_2\}$ and $\{v_2, v_3\}$ in D. By symmetry, we may assume that u_1 is a common in-neighbor of v_1 and v_2 and u_4 is a common in-neighbor of v_2 and v_3. Since u_1 and u_4 have out-degree at least one, v_3 is the out-neighbor of u_1 and v_1 is the out-neighbor of u_4. Moreover, v_2 has out-degree at least two in D. Since u_1 and u_2 compete and v_3 is the only out-neighbor of u_1, v_3 is also an out-neighbor of u_2. Since u_3 and u_4 compete and v_1 is the only out-neighbor of u_4, v_1 is also an out-neighbor of u_3. Then v_2 is a common out-neighbor of u_2 and u_3, so v_2 has out-degree of two. Thus v_1 and v_3 have out-degree of two in D. Thus the vertices u_1 and u_4 have out-degree of one and the others have out-degree of two in D. Therefore neither u_1 and v_3 nor u_4 and v_1 do not $(1, 2)$-compete in D by Theorem 2.1.1. Thus the edges not belonging to $G[V_1]$ and $G[V_2]$ induce the graph with two edges u_1v_3 and u_4v_1 deleted from $K_{4,3}$, that is, $K_{4,3} - 2 \times E(P_2)$.

To show the converse, suppose that $(m, n) = (1, 1)$ and the edges not belonging to $G[V_1] \cup G[V_2]$ induce an empty set. Then, obviously, $G[V_1] = G[V_2] = P_1$.

Suppose that $(m, n) = (3, 3)$ and the edges not belonging to $G[V_1] \cup G[V_2]$ induce $K_{3,3} - E(C_4)$. Let $C_4 := u_1v_1u_3v_3u_1$ be the graph deleted from $K_{3,3}$ where $V_1 := \{u_1, u_2, u_3\}$ and $V_2 := \{v_1, v_2, v_3\}$. Since there is no isolated vertex in $K_{3,3} - C_4$, every vertex has out-degree at least one in D. Since u_1 and v_1 do not $(1, 2)$-compete, $N^+_D(u_1) = \{v_1\}$ or $N^+_D(v_1) = \{u_1\}$ by Theorem 2.1.1. By symmetry, we may assume that $N^+_D(u_1) = \{v_1\}$. Then $N^-_D(u_1) = \{v_2, v_3\}$. Since v_1 and u_3 do not $(1, 2)$-compete, $N^+_D(v_1) = \{u_3\}$ or $N^+_D(u_3) = \{v_1\}$ by Theorem 2.1.1. If $N^+_D(u_3) = \{v_1\}$, then $N^+_D(v_1) = \{u_2\}$, which contradicts the assumption that v_1 and u_2 are adjacent vertices in $K_{3,3} - E(C_4)$. Thus $N^+_D(v_1) = \{u_3\}$ and $N^-_D(v_1) = \{u_1, u_2\}$. Similarly, we can show that $N^+_D(u_3) = \{v_3\}$, $N^-_D(u_3) = \{v_1, v_2\}$, $N^+_D(v_3) = \{u_1\}$

and $N^-_D(v_3) = \{u_2, u_3\}$.

Therefore $u_1, u_3 \in N^+_D(v_2)$ and $v_1, v_3 \in N^+_D(u_2)$. Thus the competing pairs in D are $\{u_1, u_2\}, \{u_2, u_3\}, \{v_1, v_2\}$ and $\{v_2, v_3\}$. Hence $G[V_1] = G[V_2] = P_3$.

To show the converse of *(ii)* is not true, let $V_1 = \{u_1, u_2\}$ and $V_2 = \{v_1\}$. Now suppose that an orientation of $K_{2,1}$ has the arcs (u_1, v_1) and (v_1, u_2). Then there is no edge not belonging to $G[V_1] \cup G[V_2]$, but $G[V_1] = I_2$ and $G[V_2] = I_1$. Thus the converse of *(ii)* is not true.

Now suppose the case *(iv)*. Let $V_1 = \{u_1, u_2, u_3, u_4\}$, $V_2 = \{v_1, v_2, v_3\}$ and $\{(u_1, v_1), (u_1, v_2), (u_2, v_1), (u_2, v_3), (u_3, v_1), (u_3, v_3), (u_4, v_2), (u_4, v_3), (v_1, u_4), (v_2, u_2), (v_2, u_3), (v_3, u_1)\}$ be the set of arcs of an orientation of $K_{4,3}$. Then it can be checked that only the pairs $\{u_1, v_3\}$, $\{u_4, v_1\}$ satisfy the condition given in Theorem 2.1.1. Therefore the edges not belonging to $G[V_1] \cup G[V_2]$ exclude exactly two nonadjacent edges u_1v_3 and u_4v_1 from $K_{4,3}$. Since $N^-_D(v_1) = \{u_1, u_2, u_3\}$, $N^-_D(v_3) = \{u_2, u_3, u_4\}$ and $N^-_D(v_2) = \{u_1, u_4\}$, $G[V_1] = K_4$. Since $N^+_D(v_1) = \{u_4\}$, $N^+_D(v_2) = \{u_2, u_3\}$ and $N^+_D(v_3) = \{u_1\}$, no two of v_1, v_2 and v_3 have a common out-neighbor and so, by Proposition 1.3.1, $G[V_2] = I_3$. Therefore the converse of *(iv)* is not true. For an illustration, see Figure 2.1. □

Corollary 2.2.5. *Let G be the $(1, 2)$-step competition graph of an orientation D of $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$ where $m \geq n$. If both $G[V_1]$ and $G[V_2]$ are trees, then $(G[V_1], G[V_2])$ is one of $(P_1, P_1), (P_2, P_1), (P_3, P_3)$ and (P_4, P_3).*

Proof. Suppose that both $G[V_1]$ and $G[V_2]$ are trees. By Proposition 2.1.3, both $G[V_1]$ and $G[V_2]$ have the independence number at most two. Note that if a tree T is not a path, then T has a vertex of degree at least three whose neighbors, therefore, form an independent set of size at least three. Thus both $G[V_1]$ and $G[V_2]$ are paths and then the corollary immediately follows from Theorem 2.2.4. □

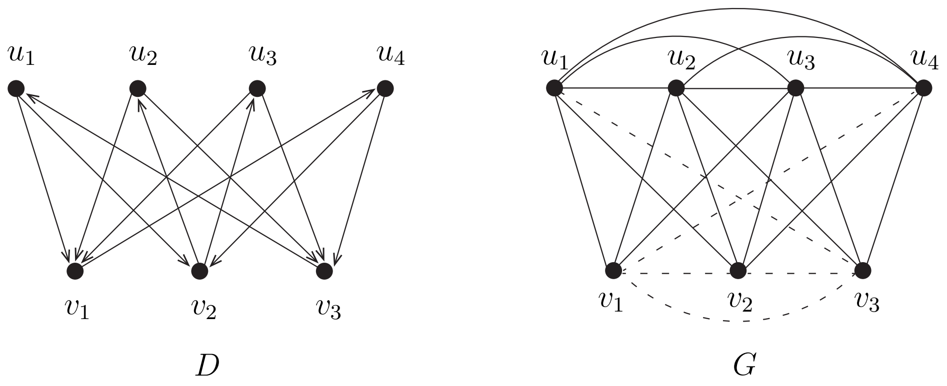

Figure 2.1: The graph G is the $(1, 2)$-step competition graph of an orientation D of $K_{4,3}$. The excluded edges of G are represented by the dotted lines and $G[V_1] = K_4$ and $G[V_2] = I_3$.

Theorem 2.2.6. *Let m and n be positive integers with $m \geq 3$ and G be the $(1, 2)$-step competition graph of an orientation D of $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$. Let j, k and l $(j+k+l = m)$ be the number of vertices of V_1 which have out-degree of zero, one and at least two in D, respectively. If $G[V_1] = I_m$ and $G[V_2] = K_n$, then the edges not belonging to $G[V_1] \cup G[V_2]$ induce $K_{m,n} - \{E(K_{1,n}) \times j + E(P_2) \times k\}$ where $E(K_{1,n}) \times j$ means a set of j edge-disjoint complete bipartite graphs $K_{1,n}$ and $E(P_2) \times k$ means a set of k edges.*

Proof. Since $G[V_1] = I_m$, any pair of elements in V_1 does not compete. Thus each vertex in V_2 has in-degree at most one and so it has out-degree at least $m - 1 \geq 2$. Hence, by Theorem 2.1.1, whethere or not a vertex in V_2 is adjacent to a vertex u in V_1 is determined by the out-neighborhood of u.

We first consider the case $N_D^+(u) = \emptyset$. Then no vertex in V_2 $(1, 2)$-competes with u in D. Thus the edges incident with u in $K_{m,n}$, that is, the edges of $K_{1,n}$, do not exist in G.

Second, we consider the case $N_D^+(u)$ consists of one element, say v. Then by Theorem 2.1.1, u and v do not $(1,2)$-compete in D while u $(1,2)$-competes with the rest of V_2 in D. Therefore the edge uv does not exist in G.

Finally, we consider the case $N_D^+(u)$ consists of at least two elements. By Corollary 2.1.2, u is adjacent to each vertex of V_2 in G.

Thus the edges not belonging to $G[V_1] \cup G[V_2]$ induce $K_{m,n} - \{E(K_{1,n}) \times j + E(P_2) \times k\}$. $\square$

An *m-partite graph* is one whose vertex set can be partitioned into m subsets, or *parts*, in such a way that no edge has both ends in the same part. An m-partite graph is *complete* if any two vertices in different parts are adjacent. We denote by K_n^m the complete m-partite graph in which each part has n vertices.

Theorem 2.2.7. *Let m and n be integers such that $2 \leq m < n$ and G be the $(1,2)$-step competition graph of an orientation D of $K_{2m,n}$ with $|V_1| = 2m$ and $|V_2| = n$. Suppose that $G[V_1] = K_2^m$ and $G[V_2] = K_n$. Then the edges not belonging to $G[V_1] \cup G[V_2]$ induce $K_{2m,n}$.*

Proof. By Corollary 2.1.2, it suffices to show that every vertex of $K_{2m,n}$ has out-degree at least two in D.

Fix two distinct elements i, j in $[m]$. Let $\{u_{1,i}, u_{2,i}\}$ be the ith part of K_2^m. Then $u_{1,i}$ compete with $u_{1,j}$ and $u_{2,j}$. However, since $u_{1,j}$ and $u_{2,j}$ do not compete, no common out-neighbor of $u_{1,i}$ and $u_{1,j}$ can be a common out-neighbor of $u_{1,i}$ and $u_{2,j}$. Therefore $u_{1,i}$ has out-degree at least two in D. By similarity, $u_{2,i}$ also has out-degree at least two in D. Thus every vertex in V_1 has out-degree at least two in D.

Since $G[V_1] = K_2^m$ and K_m is the maximum clique in K_2^m, every vertex in V_2 has in-degree at most m in D and so each has out-degree at least $m \geq 2$ in D.

Therefore every vertex of $K_{m,n}$ has out-degree at least two in D. $\square$

2.3 Extremal $(1,2)$-step competition graphs

Theorem 2.3.1 ([17]). *Let m and n be integers such that $m \geq n \geq 6$. Then, the pair (K_m, K_n) is competition-realizable.*

Theorem 2.3.2. *Let m and n be positive integers such that $m \geq n \geq 6$. Then K_{m+n} is $(1,2)$-step competition-realizable through $K_{m,n}$.*

Proof. By Proposition 1.3.1 and Theorem 2.3.1, there exists an orientation D of $K_{m,n}$ whose $(1,2)$-step competition graph G has the induced subgraphs $G[V_1] = K_m$ and $G[V_2] = K_n$. Thus we have to show that every vertex in V_1 is adjacent to every vertex in V_2 in G. By Corollary 2.1.2, it suffices to show that each vertex has out-degree at least two in D.

To the contrary, suppose that a vertex u in V_1 has out-degree at most one in D. If u has no out-neighbor in D, then u is an isolated vertex in $G[V_1]$, which contradicts $G[V_1] = K_m$. Suppose that u has the only out-neighbor v in D. Since $G[V_1] = K_m$, v is also the out-neighbor of the remaining vertices of V_1 in D. Thus v has no out-neighbor in D, and so v is an isolated vertex in $G[V_2]$, which contradicts $G[V_2] = K_n$. Thus every vertex in V_1 has out-degree at least two in D. Similarly, we can show that every vertex in V_2 has out-degree at least two in D. Therefore every vertex in V_1 is adjacent to every vertex in V_2 in G. $\square$

Corollary 2.3.3. *Let m and n be positive integers and G be the $(1,2)$-step competition graph of an orientation D of $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$. If $m \geq n \geq 6$, then*

$$|E(G)| \leq \frac{1}{2}(m+n)(m+n-1)$$

Proof. The corollary follows from Theorem 2.3.2. $\square$

Chapter 3

(i, j)-step competition graphs

The notion of (i, j)-step competition graph is a generalization of $(1, 2)$-step competition graph.

Lemma 3.0.4. *Let H be the (i, j)-step competition graph of an orientation D of $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$ for some positive integers m and n. Let u and v be the vertices of D which (k, l)-compete for some integers $1 \leq k \leq i$ and $2 \leq l \leq j$. Then k and l have the same parity if and only if uv is an edge in H whose ends belong to the same part that is either V_1 or V_2.*

Proof. If two vertices u and v in D are in the same part and can reach a vertex w via directed paths P and Q, respectively, then the length of P and Q have the same parity. If two vertices u and v in D are in the distinct parts and can reach a vertex x via directed paths R and S, respectively, then the length of R and S have the distinct parities. Therefore the lemma immediately follows. □

Theorem 3.0.5. *Let G be the $(1, 2)$-step competition graph and H be the (i, j)-step competition graph of an orientation D of $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$ for* *some positive integers m and n. Let G' (resp. H') be the graph induced by the edges not belonging to $G[V_1] \cup G[V_2]$ (resp. $H[V_1] \cup H[V_2]$). Then the following holds:*

(i) $G[V_\alpha] \subseteq H[V_\alpha]$ *for each* $\alpha = 1, 2$*;*

(ii) $G' = H'$.

Proof. By the definition of the (i, j)-step competition graph, $G[V_\alpha] \subseteq H[V_\alpha]$ for each $\alpha = 1, 2$.

We show that $G' = H'$. By definition of the (i, j)-step competition graph, $G \subseteq H$ and so $G' \subseteq H'$. Let $uv \in E(H')$. Then u and v belong to distinct parts V_1 and V_2. Thus, by Lemma 3.0.4, u and v (k, l)-compete for some integers $1 \leq k \leq i$ and $2 \leq l \leq j$ which have distinct parities. Suppose that u and v do not $(1, 2)$-compete. We may assume that $k \leq l$. Then we consider the following two cases:
(i) $k \geq 2$. Since no special conditions are imposed on u and v, we may assume that there exist a directed (u, w)-path of length k which does not contain v and a directed (v, w)-path of length l which does not contain u for some vertex $w \neq u, v$ in D. Let u_1 be the vertex immediately following u on the directed (u, w)-path. In addition, let v_1 be the vertex immediately following v on the directed (v, w)-path in D. By the choice of the directed paths, $u \neq v_1$ and $v \neq u_1$. Since u and v belong to distinct parts, u_1 and v_1 are also in distinct parts. Since D is an orientation of a complete bipartite $K_{m,n}$, either (u_1, v_1) or (v_1, u_1) is an arc in D. Suppose $(u_1, v_1) \in A(D)$. Then $u \rightarrow u_1 \rightarrow v_1$ is a (u, v_1)-directed walk of length two. Since (v, v_1) is an arc in D, u and v $(1, 2)$-compete, a contradiction. Suppose $(v_1, u_1) \in A(D)$. Then $v \rightarrow v_1 \rightarrow u_1$ is a (v, u_1)-directed walk of length two. Since (u, u_1) is an arc in D, u and v $(1, 2)$-compete, a contradiction.
(ii) $k = 1$. Then, since k and l have different parities, l is even. Furthermore, by our assumption, $l \neq 2$. We may assume that there exist an arc (u, w) and a directed

(v, w)-path of length l which does not contain u for some vertex $w \neq u, v$ in D. Let v_1 be the vertex immediately following v on the directed (v, w)-path in D. By the choice of the directed path, $u \neq v_1$. Since u and v belong to distinct parts, w and v_1 are also in distinct parts. Since D is an orientation of a complete bipartite $K_{m,n}$, either (w, v_1) or (v_1, w) is an arc in D. Suppose $(w, v_1) \in A(D)$. Then $u \to w \to v_1$ is a (u, v_1)-directed walk of length two. Since (v, v_1) is an arc in D, u and v $(1, 2)$-compete, a contradiction. Suppose (v_1, w) is an arc. Then $v \to v_1 \to w$ is a (v, w)-directed walk of length two. Since (u, w) is an arc in D, u and v $(1, 2)$-compete, a contradiction.

Therefore u and v $(1, 2)$-compete and so $uv \in E(G')$ by Corollary 1.3.2. □

Proposition 3.0.6. *Let D be an orientation of $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$ for some positive integers $m \geq n \geq 3$. Let $G = C_{i,j}(D)$ where $m + n - 1 \leq i \leq j$ and u and v be the vertices that belong to the same part V_1 or V_2. If $N_D^+(v) \neq \emptyset$ and there exists a directed (u, v)-path of length at least four in D, then $uv \in E(G)$.*

Proof. Without loss of generality, we may assume $u, v \in V_1$. Let P be a directed (u, v)-path of length at least four in D. We denote it by $P := x_0 y_0 x_1 y_1 \cdots x_{l-1} y_{l-1} x_l$ where $u = x_0$ and $v = x_l$. If there exists an arc (v, y_k) for some $k \in \{0, 1, 2, \ldots l - 2\}$, then u and v $(1, 2k+1)$-compete, so u and v are adjacent in G. It remains to consider the case where $(y_0, v), (y_1, v) \ldots (y_{l-2}, v)$ are arcs in D (see Figure 3.1). Since $N_D^+(v) \neq \emptyset$, there exists a vertex $z \in V_2$ such that $(v, z) \in A(D)$. If $(x_1, z) \in A(D)$, then u and v $(1, 3)$-compete for z. If $(z, x_1) \in A(D)$, then u and v $(2, 2)$-compete for x_1. Therefore u and v are adjacent in G. □

For a digraph D, we denotes by D^2 the digraph such that $V(D^2) = V(D)$ and $(u, v) \in A(D^2)$ if and only if there exists an directed (u, v)-path of length two in D. Note that D^2 is acyclic.

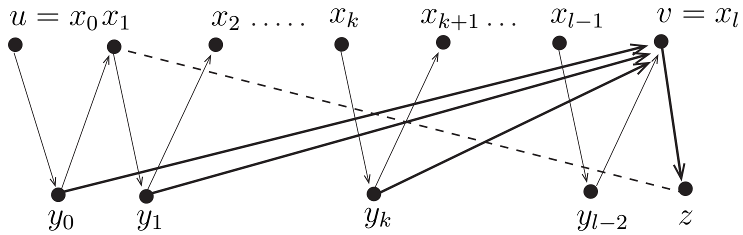

Figure 3.1: The case in which $(y_0, v), (y_1, v) \ldots (y_{l-2}, v)$ are arcs in D.

Theorem 3.0.7. *Let D be an orientation of $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$ for some integers $m \geq n \geq 3$. We construct a graph G_1 as follows:*

(i) $V(G_1) = V_1$;

(ii) $uv \in E(G_1)$ *if and only if each of u, v has an out-neighbor in D and one of the following holds:*

(1) u *and* v *have a common out-neighbor;*

(2) there is an out-neighbor of u in D^2 other than v;

(3) there is an out-neighbor of v in D^2 other than u.

Then $G_1 = G[V_1]$ where $G = C_{i,j}(D)$ for any $2 \leq i \leq j$, $3 \leq j$.

Proof. Take two distinct vertices u and v in V_1.It suffices to show that $uv \in E(G_1)$ if and only if $uv \in E(G)$.

First we suppose that $uv \in E(G_1)$. Then, by the way in which G_1 is constructed, $N^+_D(u) \neq \emptyset$, $N^+_D(v) \neq \emptyset$, and one of (1), (2), (3) holds. If (1) holds, then u and v compete in D and so they are adjacent in G. Suppose that (2) holds. Let w be a vertex in $N^+_{D^2}(u)$ distinct from v. Then there is a (u, w)-directed path $u \to x \to w$ for some $x \in V_2$. Since $N^+_D(v) \neq \emptyset$, we may take $y \in N^+_D(v)$. If $y \in N^+_D(u)$, then

y is a common out-neighbor of u and v, and so $uv \in E(G)$. Therefore we may assume $y \notin N_D^+(u)$. Then, since $x \in N_D^+(u)$, $x \neq y$. Since $w \in V_1$ and $y \in V_2$, either $(w, y) \in A(D)$ or $(y, w) \in A(D)$. In the former, u and v $(1, 3)$-compete via paths $u \to x \to w \to y$ and $v \to y$. In the latter, u and v $(2, 2)$-compete via paths $u \to x \to w$ and $v \to y \to w$. Thus u and v are adjacent in G. If (3) holds, we can apply a similar argument to deduce the same conclusion. Thus $uv \in E(G)$.

To show the converse, we suppose that $uv \in E(G)$. Then there exist a directed (u, w)-path P of length i^* and a directed (v, w)-path Q of length j^* for some $w \in V(G)$ satisfying the following properties:

(i) v is not on P;

(ii) u is not on Q;

(iii) $i^* \in \{1, 2, \ldots, i\}$ and $j^* \in \{1, 2, \ldots, j\}$, or $i^* \in \{1, 2, \ldots, j\}$ and $j^* \in \{1, 2, \ldots, i\}$.

For the property (iii), without loss of generality, we may assume $i^* \in \{1, 2, \ldots, i\}$ and $j^* \in \{1, 2, \ldots, j\}$.

If $i^* = j^* = 1$, then (1) holds. If $i^* \geq 2$ or $j^* \geq 2$, then (2) or (3) holds, respectively, by the property (i) or (ii), respectively. Thus $uv \in E(G_1)$. $\square$

The following corollary immediately follows.

Corollary 3.0.8. *Let D be an orientation D of $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$. Then $C_{i,j}(D) = C_{2,3}(D)$ for any positive integers i and j with $2 \leq i \leq j$, $3 \leq j$.*

Recall that a digraph D is *strongly connected* if, for every pair u, v of distinct vertices in D, there exist a directed (u, v)-walk and a directed (v, u)-walk.

Theorem 3.0.9. *Let D be an orientation of $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$ for some positive integers $m \geq n \geq 3$. Let $G = C_{2,3}(D)$. If D is strongly connected, then $G[V_1] = K_m$ and $G[V_2] = K_n$.*

Proof. Take two distinct vertices $u, v \in V_1$. Since D is strongly connnected, $N_D^+(u) \neq \emptyset$ and $N_D^+(v) \neq \emptyset$. To show that u and v are adjacent in G, it suffices to show that at least one of (1), (2), (3) in the statement of Theorem 3.0.7 holds. Suppose, to the contrary, that none of (1), (2), (3) holds.

Let $W = V_1 \setminus \{u, v\}$. Then $W \neq \emptyset$ as $m \geq 3$. Since neither (2) nor (3) holds, every vertex in $N_D^+(u) \cup N_D^+(u)$ is an out-neighbor of every vertex in W (see Figure 3.2). Then any vertex in $N_D^+(u) \cup N_D^+(u)$ cannot reach a vertex in W via a directed path, which contradicts that D is strongly connnected.

By applying a similar argument, we can show that any two vertices in V_2 are adjacent in G. $\square$

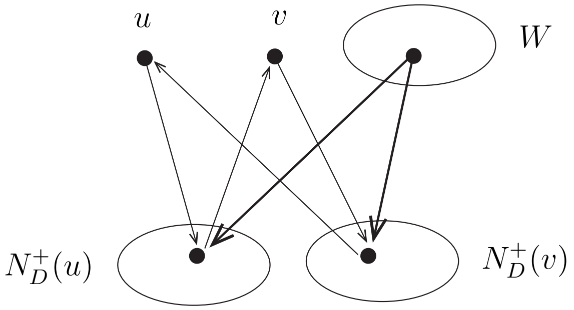

Figure 3.2: An illustration for the case where every vertex in $N_D^+(u) \cup N_D^+(u)$ is an out-neighbor of every vertex in W.

Corollary 3.0.10. *Let D be an orientation of a complete bipartite graph $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$ for some positive integers m and n. Let $G = C_{2,3}(D)$.*

If D is strongly connected and every vertex of D has out-degree at least two, then $G = K_{m+n}$.

Proof. The corollary follows from Corollary 2.1.2 and Theorem 3.0.9. $\square$

Suppose that i, i', j and j' are positive integers satisfying $1 \leq i \leq i', 2 \leq j \leq j'$. Then, for any orientation D of $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$, it certainly holds, by the definition of (i, j)-step competition graph, that $C_{i,j}(D) \subseteq C_{i',j'}(D) \subseteq K_{m+n}$. Therefore, it is clear that $C_{i,j}(D)$ converges to a graph G with $m+n$ vertices as i and j go to infinity. We call G the *limit graph*, or the *limit* of $\{C_{i,j}(D)\}_{i,j}$. Now, two natural questions arise: (i) Can we describe G?; (ii) What is a condition for G to have a special form? For example, we are interested in the condition for $G[V_1]$ or $G[V_2]$ being a complete graph. For an illustration, see Figure 3.3.

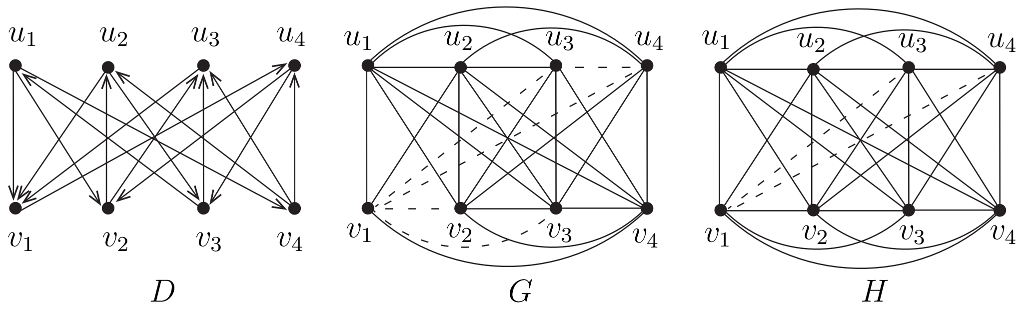

Figure 3.3: G is the $(1, 2)$-step competition graph of D and H is the limit of $\{C_{i,j}(D)\}_{i,j}$.The excluded edges of G and H are represented by the dotted lines.

Corollary 3.0.11. *Let D be an orientation of $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$. Let G be the limit of $\{C_{i,j}(D)\}_{i,j}$. Then $G = C_{2,3}(D)$.*

Proof. By Corollary 3.0.8, $C_{i,j}(D) = C_{2,3}(D)$ as i and j go to infinity. $\square$

Corollary 3.0.12. *Let D be an orientation of $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$ for some positive integers m and n. Let G be the limit of $\{C_{i,j}(D)\}_{i,j}$. If D is strongly connected, then $G[V_1] = K_m$ and $G[V_2] = K_n$.*

Proof. The corollary follows from Theorem 3.0.9. $\square$

Chapter 4

Closing remarks

We assume that the vertices of D are labeled $v_1, v_2, \ldots, v_n$ in some arbitrary but fixed manner. The *adjacency matrix* $M = [m_{ij}]$ of a digraph D is an $n \times n$ matrix such that $m_{ij} = 1$ if $v_i \rightarrow v_j$ and $m_{ij} = 0$ otherwise.

Remark 4.0.13. *Let M be the adjacency matrix of D. Then*

1. *Theorem 2.1.1 may be stated as the following: u and v do not $(1, 2)$-compete in D if and only if one of the following holds:*

   (1) The row corresponding to either u or v in M is zero row;

   (2) the row corresponding to u in M has only 1 in the column corresponding to v in M;

   (3) the row corresponding to v in M has only 1 in the column corresponding to u in M.

2. *Similary, Corollary 2.1.2 may be stated as the following: each row in M has 1's in the at least two columns if and only if the edges of G not belonging to $G[V_1] \cup G[V_2]$ induce $K_{m,n}$.*

3. *In Theorem 3.0.7, the condition (ii) may be replaced with the equivalent one: (ii) $uv \in E(G_1)$ if and only if the rows corresponding to u and v are not zero rows, and either (1) or (2) holds:*

   (1) *there is a vertex in V_2 whose corresponding column in M has 1's in the rows corresponding to u and v, or*

   (2) *there is a vertex in V_1 other than u, v whose corresponding column in M^2 has 1 in a row corresponding to u or v.*

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국문초록

이 논문에서는 방향 지어진 완전이분그래프의 (i, j)-step 경쟁그래프를 연구한다. Kim *et al.* 은 방향 지어진 완전이분그래프의 경쟁그래프를 연구하였다. Kim *et al.* 의 연구 결과를 확장하기 위하여 본 논문은 먼저 방향 지어진 완전이분그래프의 $(1, 2)$-step 경쟁그래프를 다룬다. 다음으로는 $(1, 2)$-step 경쟁그래프의 일반적인 형태인 (i, j)-step 경쟁그래프를 연구한다. 마지막으로 (i, j)-step 경쟁그래프의 극한을 다룬다.

주요 어휘: 완전이분그래프; $(1, 2)$-step 경쟁그래프; $(1, 2)$-step 경쟁-실현 가능한; (i, j)-step 경쟁그래프; (i, j)-step 경쟁그래프의 극한; 방향.
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