Comparative Statics on the Left-Side Relatively Weak First-Degree Stochastic Dominance Order and Its Applications

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This paper proposes a new concept of the left-side relatively weak first-degree stochastic dominance (L-RWFSD) order that extends the monotone likelihood ratio (MLR) order and the left-side monotone likelihood ratio (L-MLR) order. We show that this shift is a larger subset of FSD shifts than the MLR shift and the L-MLR shift that derive the same comparative statics results.

Keywords: Left-side relatively weak first-degree stochastic dominance, Monotone likelihood ratio order, Left-side monotone likelihood ratio order

JEL Classification: D81

I. Introduction

The effects of uncertainty on an individual's choice are theoretical interesting and have significant policy implications. In fact, the attention paid to this aspect of economic decision-making has a long tradition in the history of economics. Since its introduction by von Neumann and Morgenstern (1944), expected utility theory has been the dominant framework for the economic analysis of

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uncertainty and there has been much progress in the theoretical and applied analysis of choice under uncertainty.

An important comparative static question in the study of decisions under uncertainty is how to predict the direction of change for a choice variable selected by the decision maker when a given random parameter changes. This general comparative static analysis is usually carried out by restricting the following components; (i) the changes in probability distribution function (PDF) or cumulative distribution function (CDF) of the random parameter, and/or (ii) the set of decision makers, and/or (iii) the structure of the given economic decision model.

In the context of a one-risky and one-safe asset portfolio problem, Fishburn and Porter (1976) showed that a first-degree stochastic dominance (FSD) improvement in the return of the risky asset does not necessarily induce a risk-averse investor to increase his demand for the risky asset. This implies that, only for the subsets of the set of general FSD shifts, interesting comparative static statements regarding the choice made by an arbitrary risk-averse decision maker can be made. That is, we restrict the set of FSD shifts to obtain determinate comparative statics predictions. Landsberger and Meilijson (1990) and Kim (1998) characterized such restrictions. Two recent papers by Landsberger and Meilijson (1990) and Kim (1998) introduced two special types of FSD shifts, a monotone likelihood ratio (MLR) shift and a left-side monotone likelihood ratio (L-MLR) shift, respectively. The MLR order is defined by imposing a monotonicity restriction on the ratio of a pair of PDFs. The L-MLR order that extends the MLR order relaxes the monotonicity requirement for points to the right of the crossing point.

The purpose of this paper is to present the larger subset of FSD shifts than the MLR shift and the L-MLR shift which result in the same comparative statics results. We refer to the relaxed version of the MLR and L-MLR restrictions as a left-side relatively weak FSD improvement.

This paper is organized as follows. In section II we set out the general economic model in which a decision-maker maximizes his expected utility of the outcome variable that depends on a choice variable and a random variable and give three definitions of ordering CDF's (MLR, L-MLR, and L-RWFSD). We also illustrate a graphical and numerical example to describe the definition of

L-RWFSD. Section III contains the comparative statics result for L-RWFSD orders. Finally section IV provides a concluding remark and applications of this model.

II. The Decision Model and Definitions

We use the general decision model introduced by Kraus (1979) and Katz (1981) in their work. The decision maker is assumed to choose the optimal value for a choice variable α taking the random variable x as given. He chooses α so as to maximize expected utility, where utility u depends on a scalar valued function of the choice variable and the random variable, $z(x,\alpha)$. Formally, the economic agent's decision problem is to select α to maximize $E[u(z(x,\alpha))]$. That is, $\max_{\alpha} E[u(z(x,\alpha))]$. In this decision framework, utility depends only on the outcome variable z, that is, the objective function is single dimensional. Thus, problems involving multidimensionality are avoided.

We also assume that utility function $u(z)$ is twice differentiable with respect to its argument with $u'(z)\geq 0$ and $u''(z)\leq 0$. To simplify the discussion, we follow the literature and focus on the case where $z_x(x,\alpha)\geq 0$. This assumption, combined with $u'(z)\geq 0$, indicates that higher values of the random variable are preferred to lower values. The case where $z_x(x,\alpha)\leq 0$ can be handled with appropriate modifications. Note that primes on $u(\cdot)$ are used to denote derivatives while subscripts with other functions denote partial derivatives.

We assume that the supports of the random variable x under $F(x)$ are $[x_2,x_4]$ and under $G(x)$ are $[x_1,x_3]$ where $x_1\leq x_2$ and $x_3\leq x_4$. Landsberger and Meilijson (1990) introduced the concept of a monotone likelihood ratio order that is defined by imposing a monotonicity restriction on the ratio of a pair of PDFs.

Definition 1

$F(x)$ represents a monotone likelihood ratio FSD shift from $G(x)$ (denoted by F MLR G) if there exists a non-decreasing function $h:[x_2,x_3]\rightarrow[0,\infty)$ such that $f(x)=h(x)g(x)$ for all $x\in[x_2,x_3]$.

Kim (1998) introduced the concept of the L-MLR order that, for the left-side of a given point, imposes monotonicity restrictions on the likelihood ratio. The set of L-MLR shifts includes the set of

MLR shifts defined in Landsberger and Meilijson (1990).

Definition 2
$F(x)$ represents a left-side monotone likelihood ratio FSD shift from $G(x)$ (denoted by F L-MLR G) if there exists a point $m \in [x_2, x_3]$ and a non-decreasing function $h: [x_2, m] \to [0, 1]$ such that $f(x) = h(x)g(x)$ for all $x \in [x_2, m)$ and $g(x) \leq f(x)$ for all $x \in [m, x_3]$.

The L-MLR conditions require that the PDFs f and g cross only once at the point m and that $g(x) \geq f(x)$ for all points to the left-side of m. Since the L-MLR order requires the condition of monotone likelihood ratio only for the left-side of the point m, it is more general than the MLR order. Thus, the MLR order implies the L-MLR order. A L-MLR shift specifies a probability transformation such that a decreasing proportion of probability mass of the left-side of the point m is transferred to the right-side of the point m.

We extend the set of admissible FSD shifts by relaxing a restriction on the sign of the derivative of the likelihood ratio used to define the L-MLR order. We call it a 'left-side relatively weak FSD shift' (L-RWFSD). The L-RWFSD order is formally defined as:

Definition 3
$F(x)$ represents a left-side relatively weak FSD shift from $G(x)$ (denoted by F L-RWFSD G) if

(a) There exists a point $m \in [x_2, x_3]$ such that $f(x) \leq g(x)$ for all $x \in [x_2, m)$ and $f(x) \geq g(x)$ for all $x \in [m, x_3]$

(b) When $x^* \in [x_2, m)$, one needs the following condition:

$$\frac{f(x)}{g(x)} \leq \frac{f(x^*)}{g(x^*)}, \qquad x_2 \leq x \leq x^*$$

$$\frac{f(x)}{g(x)} \geq \frac{f(x^*)}{g(x^*)}, \qquad x^* \leq x \leq m$$

where x^* denotes the value of x satisfying $[z(x, \alpha_2) - z(x, \alpha_1)] = 0$.

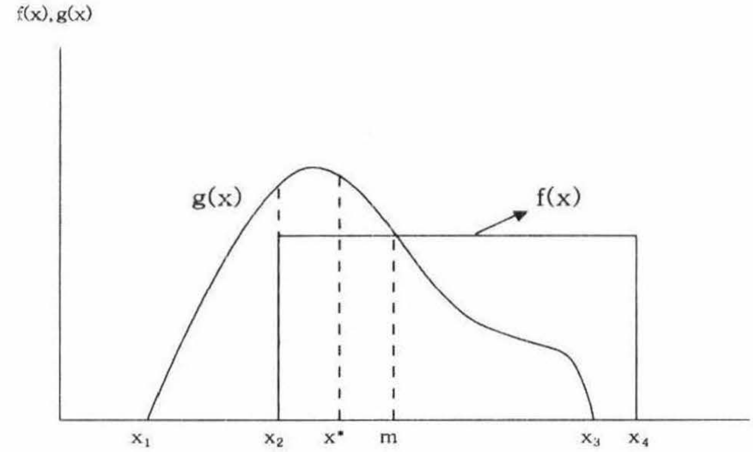

FIGURE 1
F L-RWFSD *G*.

Condition (a) imposes the restriction that the two PDFs cross only once at a point m. Conditions (a) and (b) imply that, for the left-side of the point m, the L-RWFSD order is less restrictive than that proposed by Kim, who imposes a monotonicity restriction on the likelihood ratio used to define the L-MLR order. Note that the set of L-RWFSD shifts includes the set of L-MLR shifts. These three shifts have the following relationships: F MLR $G \Rightarrow F$ L-MLR $G \Rightarrow F$ L-RWFSD G.

Figure 1 illustrates an example of a L-RWFSD shift and a case where a monotonicity restriction on the interval $x \in [x_1, m)$ to obtain a L-MLR shift is not met (see below numerical example for a practical case).

Numerical example: Consider the following two random variables with probability density functions $f(x)$ and $g(x)$, respectively; $f(x) = 1/2$ for $-1 \leq x \leq 1$ and $g(x) = 2x+3$ for $-(3/2) \leq x \leq -(5/4)$, $x+(7/4)$ for $-(5/4) \leq x \leq -(7/8)$, $-(3/5)x+(7/20)$ for $-(7/8) \leq x \leq -(1/4)$, $-(1/2)x+(3/8)$ for $-(1/4) \leq x \leq (3/4)$, where $x_1 = -(3/2)$, $x_2 = -1$, $x^* = -(2/3)$, $m = -(1/4)$, $x_3 = 3/4$, $x_4 = 1$, $\int_{-1}^{1} f(x)dx = \int_{-1}^{1} (1/2)dx = 1$ and $\int_{-(3/2)}^{3/4} g(x)dx = \int_{-(3/2)}^{-(5/4)} (2x+3)dx + \int_{-(5/4)}^{-(7/8)} ((x+(7/4))dx + \int_{-(7/8)}^{-(1/4)} (-(3/5)x+(7/20))dx + \int_{-(1/4)}^{3/4} (-(1/2)x+(3/8))dx = 1$. Note that $f(x)$ and $g(x)$ cross at the point $m(x = -$

(1/4)). Hence, $f(x)$ and $g(x)$ satisfy the following conditions in Definition 3:

(a) There exists a point $m(=-(1/4))\in[-1,(3/4)]$ such that $f(x)\leq g(x)$ for all $x\in[-1,-(1/4))$ and $f(x)\geq g(x)$ for all $x\in[-(1/4),(3/4)]$

(b) When $x^*(=-(2/3))\in[-1,-(1/4))$, one needs the following condition:

$$\frac{f(x)}{g(x)}\geq\frac{f(-(2/3))}{g(-(2/3))}=\frac{2}{3}, \qquad -1\leq x\leq-\frac{2}{3}$$

$$\frac{f(x)}{g(x)}\geq\frac{f(-(2/3))}{g(-(2/3))}=\frac{2}{3}, \qquad -\frac{2}{3}\leq x\leq-\frac{1}{4}$$

III. The Comparative Static Analysis

In this section, we provide a general comparative static statement concerning the L-RWFSD order. Using the general one-argument decision model, we follow the technique used in Landsberger and Meilijson (1990) and Kim (1998).

Theorem

Suppose that α_F and α_G maximize $E[u(z(x,\alpha))]$ under $F(x)$ and $G(x)$, respectively. For all risk-averse decision makers, $\alpha_F\geq\alpha_G$ if

(a) F L-RWFSD G

(b) $z_x\geq 0$ and $z_{\alpha x}\geq 0$.

Proof: Given the CDFs F and G, each expected utility can be expressed as a function of the decision variable α

$$EU_F(\alpha)=\int_{x_2}^{x_4}u[z(x,\alpha)]f(x)dx$$

and

$$EU_G(\alpha)=\int_{x_1}^{x_3}u[z(x,\alpha)]g(x)dx,$$

respectively. To prove $\alpha_F \geq \alpha_G$ it is sufficient to show that, for a pair of values α_1 and α_2,

$$\text{if } \alpha_2 \geq \alpha_1 \text{ and } EU_G(\alpha_2) \geq EU_G(\alpha_1), \text{ then } EU_F(\alpha_2) \geq EU_F(\alpha_1). \tag{1}$$

This is because (1) implies that $\Delta_F = EU_F(\alpha_G) - EU_F(\alpha)$ for every $\alpha \leq \alpha_G$ which in turn implies that the maximum of $EU_F(\alpha)$ exists at a value of α larger than α_G and thus $\alpha_F \geq \alpha_G$. Assuming that $\Delta_G = EU_G(\alpha_2) - EU_G(\alpha_1) \geq 0$ where $\alpha_2 > \alpha_1$, we show that the following is non-negative

$$\Delta_F = \int_{x_2}^{x_4} A(x)f(x)dx \tag{2}$$

where $A(x) = u[z(x, \alpha_2)] - u[z(x, \alpha_1)]$. Because $z_{\alpha x} \geq 0$ by assumption, the difference $z(x, \alpha_2) - z(x, \alpha_1)$ is non-decreasing in x. This implies that $z_x(x, \alpha_2) - z_x(x, \alpha_1)$ is non-negative. If the assumption $\Delta_G = \int_{x_1}^{x_3} A(x)g(x)dx \geq 0$ is satisfied for the case where $z(x, \alpha_2) - z(x, \alpha_1) \geq 0$ for all $x \in [x_2, x_3]$, the assumption $u' \geq 0$ implies that $A \geq 0$ for all $x \in [x_2, x_3]$ and thus $\Delta_F \geq 0$. If $z(x, \alpha_2) - z(x, \alpha_1) \leq 0$ for all $x \in [x_2, x_3]$, the assumption $u' \geq 0$ implies that $A \leq 0$ for all $x \in [x_1, x_3]$ and thus $\Delta_G \leq 0$ which contradicts the assumption $\Delta_G \geq 0$. Therefore we exclude the case where $z(x, \alpha_2) - z(x, \alpha_1) \leq 0$ for all $x \in [x_2, x_3]$.

Now consider the case that, with α_1, α_2 and the payoff function z given, there exists a point $x^*(\alpha_1, \alpha_2, z) \in [x_2, x_3]$ such that the difference $z(x, \alpha_2) - z(x, \alpha_1)$ is non-positive for all $x \leq x^*$ and non-negative for all $x \geq x^*$. This implies that $A \leq 0$ for all $x \leq x^*$ and $A \geq 0$ for all $x \geq x^*$. Now we consider the following two cases:

Case (i): $x^* \leq m$.
Adding and subtracting $\int_{x_1}^{x_4} A(x)g(x)dx$ in the RHS of (2) yields

$$\begin{aligned} \Delta_F &= \int_{x_1}^{x_4} A(x)[f(x) - g(x) + g(x)]dx \\ &= \Delta_G + \int_{x_1}^{x_4} A(x)[f(x) - g(x)]dx. \end{aligned} \tag{3}$$

Since $f(x) = 0$ for all $x \in [x_1, x_2)$, (3) can be rewritten as

$$\begin{aligned} \Delta_F = \Delta_G &- \int_{x_1}^{x_2} A(x)g(x)dx + \int_{x_2}^{m} A(x)[(f(x)/g(x)) - 1]g(x)dx \\ &+ \int_{m}^{x_3} A(x)[f(x) - g(x)]dx + \int_{x_3}^{x_4} A(x)f(x)dx. \end{aligned}$$

Note that $A(x)$ changes its sign from negative to positive at $x=x^*$. By the condition (b) in Definition 1, where $f(x)/g(x)\leq 1$ for $x\in[x_2,m]$,

$$\Delta_F\geq \Delta_G-\int_{x_1}^{x_2}A(x)g(x)dx+[(f(x^*)/g(x^*))-1]\int_{x_2}^{m}A(x)g(x)dx$$
$$+\int_{m}^{x_3}A(x)[f(x)-g(x)]dx+\int_{x_3}^{x_4}A(x)f(x)dx. \quad (4)$$

Since $A(x)<0$ for all $x\in[x_1,x_2]$, and $A(x)>0$ and $[f(x)-g(x)]\geq 0$ for all $x\in[m,x_4]$, (4) becomes

$$\Delta_F\geq \Delta_G+[(f(x^*)/g(x^*))-1]\int_{x_2}^{m}A(x)f(x)dx.$$

Since $\Delta_G\geq 0$ by assumption, $-1\leq[(f(x^*)/g(x^*))-1]\leq 0$ and $\Delta_G\geq \int_{x_1}^{m}A(x)g(x)dx$, we have $\Delta_F\geq 0$.

Case (ii): $m\leq x^*$.
Let's rewrite (2) as

$$\Delta_F=\int_{x_1}^{x^*}A(x)f(x)dx+\int_{x^*}^{x_3}A(x)f(x)dx+\int_{x_3}^{x_4}A(x)f(x)dx. \quad (5)$$

Integrating the first term in the RHS of (5) by parts yields

$$\Delta_F=A(x^*)F(x^*)-\int_{x_1}^{x^*}A'(x)F(x)dx+\int_{x^*}^{x_3}A(x)F(x)dx+\int_{x_3}^{x_4}A(x)f(x)dx \quad (6)$$

where $A'(x)=u'[z(x,\alpha_2)]z_x(x,\alpha_2)-u'[z(x,\alpha_1)]z_x(x,\alpha_1)$. Adding and subtracting $A(x^*)G(x^*)+\int_{x_1}^{x^*}A'(x)G(x)dx+\int_{x^*}^{x_3}A(x)g(x)dx$ in the RHS of (6) gives

$$\Delta_F=A(x^*)[F(x^*)-G(x^*)]+A(x^*)G(x^*)-\int_{x_1}^{x^*}A'(x)[F(x)-G(x)]dx$$
$$-\int_{x_1}^{x^*}A'(x)G(x)dx+\int_{x^*}^{x_3}A(x)[f(x)-g(x)]dx+\int_{x^*}^{x_3}A(x)g(x)dx+\int_{x_3}^{x_4}A(x)f(x)dx.$$

By rearranging Δ_F,

$$\Delta_F=\Delta_G+A(x^*)[F(x^*)-G(x^*)]-\int_{x_1}^{x^*}A'(x)[F(x)-G(x)]dx$$
$$+\int_{x^*}^{x_3}A(x)[f(x)-g(x)]dx+\int_{x_3}^{x_4}A(x)f(x)dx. \quad (7)$$

Note that $z(x,\alpha_2)\leq z(x,\alpha_1)$ when $x\leq x^*$, and $z_x(x,\alpha_2)\geq z_x(x,\alpha_1)$ when $x\geq x_1$ by the assumption $z_{\alpha x}\geq 0$. The assumptions $u'\geq 0$, $u''\leq 0$ and

$z_x \geq 0$ imply that $A'(x) \geq 0$ for all $x \in [x_1, x^*]$. Since $A \leq 0$ for all $x \leq x^*$ and $A \geq 0$ for all $x \geq x^*$, and the L-RWFSD condition implies that $F(x) \leq G(x)$ for all $x \in [x_1, x_4]$ and $f(x) \geq g(x)$ for all $x \in [m, x_3]$, the assumption $\Delta_G \geq 0$ implies that $\Delta_F \geq 0$.

Q.E.D.

Note that the L-MLR order implies the L-RWFSD order. Therefore, we obtain the following result derived by Kim (1998).

Corollary

Suppose that α_F and α_G maximize $E[u(z(x, \alpha))]$ under $F(x)$ and $G(x)$, respectively. For all risk-averse decision makers, $\alpha_F \geq \alpha_G$ if

(a) F L-MLR G
(b) $z_x \geq 0$ nd $z_{\alpha x} \geq 0$.

Proof: From the property that the L-MLR shift implies the L-RWFSD shift and Theorem, the proof is completed.

Theorem is a direct generalization of Corollary. The comparative statics statement made for an L-RWFSD change can also be applied for any L-MLR change without any additional cost of assumptions.

IV. Concluding Remarks and Applications

This paper introduces a new concept of the left-side relatively weak first-degree stochastic dominance order that extends the MLR order and the L-MLR order. Compared with the result in L-MLR shifts, the comparative statics result in Theorem includes a larger set of FSD changes. As a result, the L-RWFSD order represents a net improvement over the L-MLR one without any cost of additional assumptions.

We use the general decision model in this analysis and it includes a variety of economic decision problems. When we assume that the outcome variable is linear in the random variable, the simple form of $z(x, \alpha)$ may be expressed as $z(x, \alpha) \equiv \alpha(x-c)+z_0$ where z_0 and c are exogenous constants. As analyzed by Sandmo (1971), Rothschild and Stiglitz (1971), Fishburn and Porter (1976), Dionne, Eeckhoudt, and Gollier (1993a, 1993b) and Eeckhoudt and

Gollier (1995), the applications of this simple form of a decision model are numerous: the standard portfolio problem, the problem of the competitive firm with constant marginal cost under output price uncertainty, the coinsurance problem and others.

In the standard portfolio model, the payoff function can be written as $z(x, \alpha)=z_0+bW_0(x-c)$, where b is the fraction of the initial wealth W_0 allocated to the risky asset, x the random rate of return of the risky asset and $z_0 \equiv W_0(1+c)$ with c being the sure interest rate. This payoff function is equivalent to the simple form of $z(x, \alpha)$ when $\alpha \equiv bW_0$. For the competitive firm, the linear function is $z(x, \alpha)= \alpha(x-c)+z_0$, where x is the uncertain output price, c marginal cost, $-z_0$ the fixed cost and α the output level. In the standard coinsurance problem, the payoff function is given by the final wealth $z(x, \alpha)=W_0-\lambda\mu-(1-b)(x-\lambda\mu)$, where x is the amount of random loss, μ the expected loss, b coinsurance rate, b $\lambda\mu$ the insurance premium, and W_0 the initial wealth. This payoff function is equivalent to the simple form of $z(x, \alpha)$ when $z_0 \equiv W_0-\lambda\mu$, $\alpha \equiv -(1-b)$ and $c \equiv \lambda\mu$. If we limit the discussion to private insurance contracts, the coinsurance rate b belongs to the interval [0,1]. Then, by definition, α is non-positive and belongs to the interval [−1,0]. Other examples of this simple form with appropriate modifications are included in Feder (1977) who examines the problem of hiring workers and in Paroush and Kahana (1980) who investigate the cooperative firm model.

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