이학박사 학위논문

Higher Spin Gauge Theory and Related Issues

고차 스핀 게이지 이론과 제반 문제들

2016년 8월

서울대학교 대학원
물리천문학부
곽승호

Higher Spin Gauge Theory and Related Issues

고차 스핀 게이지 이론과 제반 문제들

지도교수 이수종

이 논문을 이학박사 학위논문으로 제출함
2016년 6월

서울대학교 대학원
물리천문학부
곽승호

곽승호의 박사 학위논문을 인준함
2016년 8월

위 원 장 ________________ (인)
부위원장 ________________ (인)
위　　원 ________________ (인)
위　　원 ________________ (인)
위　　원 ________________ (인)

Abstract

Higher Spin Gauge Theory and Related Issues

Seungho Gwak
School of Physics & Astronomy
The Graduate School
Seoul National University

Since the very earlier days of quantum field theory, the existence of the massless higher spin has been known. However, there were various obstacles to reach an interesting higher spin gauge theory. The most of all naïve try for introducing interactions between a massless higher spins and another fields had failed. Because the higher spin gauge symmetry is not conserved at all. It was hard to find the consistent interaction of higher spin gauge theory. In recent years, flourished achievements have been made. In this thesis, we try to review about obstacles and achievements, and summarised our own effort to find new interacting theories.

Keywords : higher spin, higher spin gauge, Kaluza-Klein, Kaluza-Klein with boundary, (Anti-)de Sitter space, colored gravity, colored higher spin, Chan-Paton factor
Student Number : 2008-20419

Table of Contents

Chapter 1

Motivation

How to Leave the Planet

1. Phone NASA. Their phone number is $\cdots$. 2. $\cdots$ the White House $\cdots$. 3. $\cdots$ the Kremlin$\cdot\cdots$.
4.$\cdots$ phone the Pope for guidance.$\cdots$ and I gather his switchboard is infallible.
5. If all these attempts fail, flag down a passing flying saucer and explain
that it's vitally important you get away before your phone bill arrives.
— Douglas Adams
THE HITCHHIKER'S GUIDE TO THE GALAXY

In spite of the prosperous progress in recent years, there are lots of un-explored area in higher-spin gauge theories. For instance, almost all known theory contains only gauge fields. If we compare this situation with Yang-Mills theory, it is similar to knowing only pure Yang-Mills [1]. The spectrum is totally determined by higher-spin gauge symmetry and they are in the same higher-spin multiplet. In this dissertation, the other possibilities are explored. If I use the Yang-Mills metaphor again, we want to find a Yang-Mills gauge theory with matter. Two methods are used to construct the interaction between a higher spin multiplet and matter sectors. The first one is *Kaluza-Klein method* in chapter 4 and chapter 5[2] and the second is *Higgs-like mechanism* in chapter 6 and chapter 7[3].

Kaluza-Klein method

Kaluza-Klein theory was historically introduced to understand the gravitation and electromagnetism in one unified setting [4]. It is the trial to describe the four-dimensional Einstein-Maxwell theory as Einstein theory in five-dimensional theory. This original version of Kaluza-Klein theory is known to fail to achieve its original motivation. However, their conceptual idea survives and Kaluza-Klein compactification method is widely used in theoretical physics. Our motivation of using Kaluza-Klein is, in some sense, opposite from Kaluza-Klein' original motivation: they wanted to combine a known theory to one single simpler theory but we want to know the information about unknown– an interacting higher spin with matter theory–by the known theory in higher dimension.

There are two basic difficulties in Kaluza-Klein reduction of higher spin gauge theory. The first one comes from the curvature of spacetime. In the the original version of Kaluza-Klein or their variations, a flat direction is used as a compactifying direction. However,

[1]In some sense, this statement overlooks the possibility of interacting theory which contains fermion. However, if we use Yang-Mills metaphor again, they correspond to super-Yang-Mills theory without matter fields.

[2]These chapters are based on and largely overlapped with [1].

[3]These chapters are based on and largely overlapped with [2, 3].

as we will show in chapter 2, there are lots of evidences that any interacting higher spin gauge theory is inconsistent on flat background. Therefore, we cannot use the usual flat compactification method. Instead of that we invented a new method of compactification from (A)dS$_{d+k}$ to AdS$_d$. The second difficulty comes from the spin-s indices. Higher spin-s field has at least s indices. Because of this arbitrary many indices makes the brute force reduction impossible. Therefore, we will use the gauge symmetry arguments that give us to circumvent this difficulty.

In our (A)dS compactification method, there must be boundary which is different with the original boundary of AdS. Because of existence of boundary, there are rich possibilities to consider diverse spectra depending on prescribed boundary conditions.

Colored Gravity: Higgs-like mechanism

To understand our strategy to find the higher spin with matter theory, it is good to remind the standard Higgs mechanism. In the simplest version of Higgs mechanism, the starting point is to consider the Maxwell field and complex scalar field. The scalar field minimally couple with Maxwell field. Therefore without potential, spectrum consist of one spin-1 field and two real scalar fields. After turning on the Mexican-hat-like potential, $\mathfrak{u}(1)$ symmetry spontaneously broken: the vacuum is not invariant under $\mathfrak{u}(1)$ transformation because of non-zero vacuum expectation value(VEV) of scalar field. As a result Nambu-Goldstone boson, which is massless, appears. This is the end of the story of global symmetry, but is not the end of the story in gauge symmetry case. For the gauge symmetry cases, massless spin-1 field eats the massless Nambu-Goldstone boson and become massive. This process is named after one of finder and called Higgs mechanism. If we expand the story to the non-Abelian gauge(Yang-Mills) theory with $\mathfrak{su}(N)$, the spectrum consists of massless spin-1 fields for unbroken symmetry sector massive spin-1 fields for broken symmetry sector and remaining scalar fields.

The question is that such a Higgs-like mechanism can occur in higher-spin gauge theory? The actual answer for the question is yes and we can expect that the spectrum consist of massless higher spin fields and other matter fields. As we will show in chapter 6, in colored higher spin gauge theory the coloured spin-2 can take non-zero VEV and higgs-like mechanism occur. The main difference with Higgs mechanism is that the combining field in broken sector is neither massive nor massless. They are ”partially massless”. This special type of fields are introduced in Section 2.3.

Chapter 2

Introduction to Higher Spin Gauge Theory I: Metric-like Formalism

子曰:「溫故而知新, 可以爲師矣 。」
〈爲政篇〉—《論語》

In very early times in quantum field theory, all unitary projective representation for $\mathfrak{iso}(3,1)$ is classified by Wigner [5] and by Bargmann and Wigner [6]. A massive representation with integer and half-integer spins are one possible type and a massless representation with integer and half-integer helicities are the other type. [1] The particles and forces in the standard model of particle physics, that has passed a lot of experimental tests in colliders, are described by fields with spin lower than three-half. Therefore, standard quantum field theory textbooks deal with a scalar field, a spin half field and a vector field and their representations in full detail [13]. But they do not deal with spin greater than one. The massless spin-2 field arise in different text as gravitation: because graviton is expected as a massless spin-two particle, one can easily access to contents for massless spin-2 fields through linearized gravitation sections in general relativity textbook. On the other hand, there are quite small contents about both a massive and a massless field with spins greater or equal to two. The main theme of this thesis is the massless field with spin greater than two, we first introduce the basic contents about massive and massless fields with spin greater than two.

We will review massive and massless spin-s Lagrangians in both flat spacetime and (anti-)de Sitter spacetime. First, we will focus on a flat spacetime and face with technical and conceptual problems in interacting higher spin theory. As an example, the minimal coupling with the higher spin fields and gravitation does not work, because the minimal interaction breaks the higher spin gauge symmetries. On top of that, there are no-go theorems on flat space for an interaction with massless higher spin. i.e. there are plausible signs that there is no interacting theory in flat background.

Therefore, we avert our eyes from flat spacetime to (Anti-)de Sitter space. Because the isometry in AdS is $\mathfrak{so}(3,2)$, the unitary projective representations are different with flat background. First we review the unitary representation of $\mathfrak{so}(3,2)$ which is the (A)dS counterpart of Wigner' classification. And the short and long representation arise in $\mathfrak{s}o(3,2)$ as counterparts of massive and massless representations in $\mathfrak{iso}(3,1)$. However, there is other types of field which is called "partially massless"(PM). As we advertised in Section 1, PM field will arise in chapter 6 and chapter 7. We devote Section 2.3 to explain the properties and definition of PM fields. After classified the representation, we will consider the field theoretical realization of these representations which are (A)dS version of Singh-Hagen [9] and Fronsdal [10].

[1] The little group $\mathfrak{iso}(2)$ is used for massless representation. However, the actual little group is $\mathfrak{iso}(2)$ and one can consider different representations with massless representation. These are called as "continuous spin" (see the quantum field theory books [13] section 2 for four dimension and refer to [7] for general dimensions.)

2.1 Higher Spin Fields: Flat Space

This part will deal with a free higher spin field in flat and (anti-)de Sitter background. An auxiliary variable will be introduced to express totally symmetric field in a compact way and give the explicit form of equations and calculations whenever it is possible. The contents are more or less similar to lectures by Sagontti in Solvay [14] and a review paper [19] — the more recent review and progress are given in there and references therein.

2.1.1 Massive Higher Spin Lagrangian: Flat Space

The smallest (or irreducible) field for Lagrangian is a totally symmetric and traceless field. One can try to use a mixed- symmetric field to describe a massive higher spin field [11]. However, we concentrate on the simplest case: totally symmetric and traceless fields. Fierz and Pauli showed that a totally symmetric traceless spin-s field describes the massive $\mathfrak{iso}(3,1)$ spin-s representation in [8] if the field satisfies following conditions.

$$\left(\Box - M^2\right)\, w_{\mu_1\cdots\mu_s} = 0\,, \quad \partial^{\mu_1}\, w_{\mu_1\cdots\mu_s} = 0\,. \tag{2.1}$$

The conditions are known as Fierz-Pauli conditions. The first condition is similar to Klein-Gordon equation with mass M. And the second condition implies that field must be divergence-less. If a field is not divergence-less, the divergence part of the field could contain a representation with one lower spin. The main purpose of this subsection is to find the Lagrangian whose Euler-Lagrange equation imposes Fierz-Pauli conditions.

Proca action: spin-1 example The first example is massive spin-1 action which is also known as Proca action. One might learn or can find Proca Lagrangian — which represent the massive particle — and Maxwell Lagrangian — which represents the massless particle — in standard textbooks. However, we pretend to know nothing about massive and massless spin-one Lagrangians for a while, because the procedure and the logic to find these Lagrangians will be generalized to higher spin fields. First write down the general Lorentz invariant Lagrangian for a vector field.

$$\mathcal{L}_{\text{spin-1}} = -\frac{1}{2}\,\partial_\mu\, V_\nu\, \partial^\mu\, V^\nu + \frac{c}{2}\,\partial_\mu\, V_\nu\, \partial^\nu\, V^\mu - \frac{1}{2}\, M^2\, V^\mu\, V_\mu\,. \tag{2.2}$$

We fixed the coefficient of first term by field normalization and introduce the massive scalar M in the last term. The Euler-Lagrange equation is

$$\Box\, V_\mu - c\,\partial_\mu\,\partial_\nu\, V^\nu - M^2\, V_\mu = 0\,. \tag{2.3}$$

The divergence part of the equation is

$$(1-c)\left(\Box - \frac{M^2}{1-c}\right)\partial^\mu\, V_\mu = 0\,. \tag{2.4}$$

The (2.4) implies that divergence of vector field, $\partial_\mu V^\mu$, can carry scalar degree of freedom(DoF) whenever unfixed parameter c is not equal to one. Because we want to find the free-field Lagrangian which carries only massive spin-1 DoF, the constant c must be equal to 1. For this case, the equation (2.4) impose the divergence-less condition to the vector field. Therefore, the Euler-Lagrange equation can be summarised as,

$$\left(\Box - M^2\right) V_\mu = 0\,, \quad \partial_\mu V_\mu = 0\,, \tag{2.5}$$

which coincide with Fierz-Pauli conditions.

Fierz-Pauli action: spin-2 example Let us move to a spin-2 massive representation. First, we can try to construct the Lagrangian by traceless and symmetric field with two space-time indices, $h_{\mu\nu}$. However, it will be shown that it does not produce Firez-Pauli conditions. The general form of Lorentz invariant Lagrangian of totally symmetric and traceless tensor field with two indices is

$$\mathcal{L}_{\text{massive spin-2}} = -\frac{1}{2}\,\partial_\mu\, h_{\nu\rho}\,\partial^\mu\, h^{\nu\rho} + \frac{c_1}{2}\,\partial^\nu\, h_{\nu\mu}\,\partial^\rho\, h_\rho{}^\mu - \frac{M^2}{2}\, h_{\mu\nu}\, h^{\mu\nu}\,. \tag{2.6}$$

Here, we fixed the coefficient of first term by using field normalization convention. The corresponding Euler-Lagrange equation of motion and divergence of equation of motion are

$$\Box\, h_{\mu\nu} - c_1\left(\partial_{(\mu}\,\partial_\rho\, h^\rho{}_{\nu)} - \frac{1}{d}\,\eta_{\mu\nu}\partial^\rho\,\partial^\sigma\, h_{\rho\sigma}\right) - M^2 h_{\mu\nu} = 0\,, \tag{2.7}$$

$$(2-c_1)\,\Box\,\partial^\nu\, h_{\mu\nu} + c_1\left(\frac{2}{d} - 1\right)\partial_\mu\,\partial^\rho\,\partial^\sigma\, h_{\rho\sigma} - 2\,M^2\,\partial^\nu\, h_{\mu\nu} = 0\,. \tag{2.8}$$

For any choice of c_1, we can not derive Fierz-Pauli conditions. The appropriate and the simplest way to derive Fierz-Pauli conditions was found by Singh and Hagen [9] by introducing the auxiliary scalar field: Fierz-Pauli conditions can be derived by equations of motion for the tensor field and and auxiliary scalar. Furthermore, the scalar field have zero value on-shell. Let us follows Singh and Hagen' method and introduce the auxiliary scalar field and additional Lagrangian,

$$\mathcal{L}_{\text{cross and scalar}} = \phi\,\partial^\mu\,\partial^\nu\, h_{\mu\nu} - \frac{c_2}{2}\,\partial_\mu\,\phi\,\partial^\mu\,\phi - \frac{b}{2}\,M^2\,\phi^2\,, \tag{2.9}$$

again we fix the coefficient of first term as one by the scalar field nomalization. Euler-Lagrage equation of a two-tensor field is[2]

$$\Box\, h_{\mu\nu} - c_1\left(\partial_{(\mu}\,\partial\cdot h_{\nu)} - \frac{1}{d}\,\eta_{\mu\nu}\,\partial\cdot\partial\cdot\, h\right) - M^2\, h_{\mu\nu} + \partial_\mu\,\partial_\nu\,\phi - \frac{1}{d}\,\eta_{\mu\nu}\,\Box\,\phi = 0\,. \tag{2.10}$$

We want to make the scalar field zero on-shell and spin-2 divergence-less. First let fix the scalar field as zero on-shell. The Euler-Lagrange equation for auxiliary scalar and the double divergence of the

[2] Because the spin-two tensor is traceless, the equation of motion must be traceless. Eta proportional terms appears to make the equation of motion traceless.

two-tensor equation are

$$\partial\cdot\partial\cdot h+\left(c_2\,\Box-bM^2\right)\phi=0\,,\quad\left(A\,\Box-M^2\right)\,\partial\cdot\partial\cdot h+B\,\Box^2\,\phi=0\,,\tag{2.11}$$

where $A=\left(1-\frac{d-1}{d}\,c_1\right)$ and $B=\left(1-\frac{1}{d}\right)$. Two equations (2.11) can be considered as homogeneous and simultaneous equations with determinant $\Delta=(B-c_2\,A)\,\Box^2+(b\,A+c_2)\,M^2\,\Box-b\,M^4$. If Δ does not contain any differential operators, we can algebraically solved this simultaneous equations: $\phi=0$ and $\partial\cdot\partial\cdot h=0$. Because there are two kinds of differential operators in the determinant, we obtain two constraint for unfixed constants: $B-c_2\,A=0$ and $b\,A+c_2=0$. Using on-shell conditions, $\phi=0$ and $\partial\cdot\partial\cdot h=0$, the divergence of equation (2.10) can be written as

$$\left[\left(1-\frac{c_1}{2}\right)\,\Box-b\,M^2\right]\,\partial\cdot h_\mu=0\,.\tag{2.12}$$

This constraints the constant c_1. Therefore, all unfixed constants in Lagrangian (2.8, 2.9) are fixed.

$$c_1=2\,,\quad c_2=-\frac{d-1}{d-2}\,,\quad b=-\frac{d\,(d-1)}{(d-2)^2}\,.\tag{2.13}$$

The term of kinetic term and mass term of the auxiliary scalar field is opposite with ordinary one as one can see in (2.9). However these facts do not make any problem like non-unitary or unbounded energy below. It is zero on-shell and has no physical degree of freedom. The equation of motions can be summarised as

$$\phi=0\,,\quad\partial^\nu\,h_{\mu\nu}=0\,,\quad\left(\Box-M^2\right)\,h_{\mu\nu}=0\,.\tag{2.14}$$

Therefore, Singh-Hagen Lagrangian (2.8, 2.9) with constants (2.13) derives Fierz-Pauli conditions for massive spin-2 field.

Singh-Hagen action: spin-3 example Now we can move to the higher-spin cases. We will show explicit calculation for spin-3 example. As spin-2 case, the traceless symmetric tensor with three indices are not enough to write a appropriate Lagrangian. Singh and Hagen introduce one auxiliary spin-1 and one auxiliary scalar. The general Lorentz invariant Lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{massive spin 3}}\quad=\quad&-\frac{1}{2}\,\partial_\mu\,w\cdot\partial^\mu\,w+\frac{c_1}{2}\,\left(\partial\cdot w\right)\cdot\left(\partial\cdot w\right)-\frac{1}{2}\,M^2\,w^2\\&-\frac{c_2}{2}\,\partial_\mu\,A\cdot\partial^\mu\,A+\frac{c_3}{2}\,\left(\partial\cdot\,A\right)^2+A_\mu\,\partial\cdot\partial\cdot w^\mu-\frac{b_1}{2}\,M^2\,A^2\\&-\frac{c_4}{2}\,\left(\partial\,\phi\right)^2-\frac{b_2}{2}\,M^2\,\phi^2+M\,\phi\,\partial\cdot A\,.\end{aligned}\tag{2.15}$$

Before explaining the procedure, it is a good place to comment about coefficient convention in Lagrangian (2.15). There are freedoms to chose the normalization of fields. We introduced two auxiliary fields and used two nomalization-freedoms to make coefficients of cross terms as one. And we multiplied the appropriate order of M to each term in order to make fields have canonical mass dimensions. Therefore, all undetermined constants are dimensionless quantities. The derivation of Fierz-Pauli conditions and coefficient-fixing procedure are parallel to previous massive spin-2 case.

The equation of motions are

$$\Box\, w_{\mu\nu\rho} - c_1\, \partial_{\{\mu}\, \partial \cdot w_{\nu\rho\}} - M^2\, w_{\mu\nu\rho} + \partial_{\{\mu}\, \partial_\nu\, A_{\rho\}} = 0\,, \tag{2.16}$$

$$c_2\, \Box\, A_\mu - c_3\, \partial_\mu\, \partial\cdot\, A + \partial\cdot\partial\, w_\mu - \partial_\mu\, \phi - b_1\, M^2\, A_\mu = 0\,, \tag{2.17}$$

$$c_4\, \Box\, \phi - b_2\, M^2\, \phi + \partial\cdot A = 0\,. \tag{2.18}$$

First, consider scalar quantities. i.e. $\partial\cdot\partial\cdot\partial\cdot w$, $\partial\cdot A$ and ϕ. There are three equations which consist of these scalar quantities: the scalar field equation, divergence of spin-1 equation and triple divergence of spin-3 equation. All equations are homogeneous equation. And we could conclude that these equation could be algebraically solved when there are no differential operators in determinant. The corresponding determinant is a determinant of three by three matrix, it has order three for d'Alembertian:

$$A_3\, \Box^3 + A_2\, \Box^2 + A_1\, \Box + B\,. \tag{2.19}$$

Therefore, to assure that all scalar quantities are zero, we must impose three conditions $A_i = 0$. Secondly, we must fix the vector quantities: $\partial\cdot\partial\cdot w_\mu$ and A_μ. Using the parallel argument, two constraint arise. Lastly, the divergence-less condition for spin-3 gives one condition. Total six constraints fix six undetermined constants in Lagrangian (2.15).

$$c_1 = 3\,, \quad c_2 = -\frac{d+1}{3\,d}\,, \quad c_3 = \frac{d-2}{6d} \quad c_4 = -\frac{12\,d^2\,(d-1)}{(d-2)\,(d+2)^2}\,, \tag{2.20}$$

$$b_1 = -\frac{(d+1)(d+2)}{3\,d^2}\,, \quad b_2 = \frac{18\,d^4}{(d^2-4)^2}\,. \tag{2.21}$$

After choosing these coefficient one can show that the Euler-Lagrange equation impose

$$\left(\Box - M^2\right) w_{\mu\nu\rho} = 0\,, \quad \partial_\mu\, w^{\mu\nu\rho} = 0\,, \quad A_\mu = 0\,, \quad \phi = 0\,. \tag{2.22}$$

The Fierz-Pauli conditions and zero auxiliary on-shell.

One can straightforwardly generalized above arguments to spin-s. This generalized procedures for spin-s have done in [9] and the first part of [10]. Singh and Hagen introduced the auxiliary totally symmetric and traceless spin $s-2$, $s-3$, $\cdots$, 0 fields in Lagrangian. The general form of Lagrangian is schematically written as

$$\mathcal{L} = -\sum_{k=0}^{s} \Big[\frac{\alpha_k}{2} \left(\partial\, w^k\right)\cdot\left(\partial\, w^k\right) + \frac{\beta_k}{2} \left(\partial\cdot w^k\right)\cdot\left(\partial\cdot w^k\right) - \frac{1}{2}\sigma_k\, M^2\, w^k\cdot w^k \tag{2.23}$$

$$-\gamma_k\, w^{k-2}\cdot\left(\partial\,\partial\cdot w^k\right) + \tau_k\, M\, w^{k-1}\cdot\left(\partial\cdot w^k\right) \Big]\,. \tag{2.24}$$

Here, we introduce the M terms to make fields have canonical dimension and unfixed constants are dimensionless and we assume that $w^{s-1} = 0$.

At first sight, An introducing the auxiliary fields seems un-natural. However we will see that auxiliary fields in a Lagrangian naturally arise in Kaluza-Klein of massless field as we explain in page 11.

2.1.2 Massless Higher Spin Lagrangian: Flat Space

Let us consider massless representations in flat background. The natural and historical approach for the massless representation might be the massless limit of Singh and Hagen. However, we will choose the other direction. Before going to the other direction it might be worth to summarize the historical result in [10].

Toward massless field: by massless limit Consider spin-3 Singh-Hagen action in (2.15). In a massless limit scalar sector is decoupled with spin-three and spin-one sectors. The cross therm of scalar and spin-1 is proportional to M and there is not cross term between scalar and spin-3.. The totally symmetric tensor with three indices and vector field can be combine and become one tensor with three indices without traceless condition,

$$\phi_{\mu\nu\rho} = w_{\mu\nu\rho} + c\,\eta_{(\mu\nu}\,A_{\rho)}\,. \tag{2.25}$$

What about the spin greater than three? If the spin-s and spin-$(s-2)$ system is decoupled with other system, spin-s and spin-$(s-2)$ can be combine and becomes one double-traceless tensor. The difference with spin-3 case is that there could be non-trivial cross term between two system.

$$-\gamma_{s-2}\,w^{s-4}\cdot\left(\partial\,\partial\cdot w^{(s-2)}\right) \tag{2.26}$$

Actually, one can show that $\gamma_{s-2} = 0$ as Fronsdal have shown in [10]: traceless spin-s field and traceless spin-$(s-2)$ field decoupled with the other fields in the massless limit. A traceless spin-s filed and a traceless spin-$(s-2)$ field can combine and be consider as a double traceless spin-s field. Therefore, *spin-s massless representation can be realized by double traceless spin-s field.*

$$\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}w_{\mu_1\cdots\mu_s} = 0 \tag{2.27}$$

Furthermore, as one can anticipate, *the Lagrangian is invariant under following gauge transformation.*

$$w_{\mu_1\cdots\mu_s} = \partial_{(\mu_1}\,\Lambda_{\mu_2\cdots\mu_s)}\,,\quad \text{with}\quad \eta^{\mu\nu}\,\Lambda_{\mu\nu\rho_1,\cdots,r_{s-3}} = 0\,, \tag{2.28}$$

here, the parenthesis of indices means the total symmetrization of indices. The double traceless condition of field implies that *the gauge parameter must be traceless*[3].

Toward massless field: by gauge symmetry in spin-3 example As advertised before, we will choose the other way to reach the massless field. As we have learned in spin-1 and spin-2 examples, the massless Lagrangians have gauge symmetry. Therefore, we try to the gauge symmetry as a starting point of the argument. First, we try to find the equation of motion and Lagrangian which is invariant under the higher spin gauge transformation. The gauge invariances of equation of motion will impose a traceless condition on gauge parameter . And the gauge invariance of Lagrangian will

[3]For spin-3 case, there are no trace condition for field, and it seems like that the traceless condition for gauge parameter is not necessary. However we will see in next paragraph, the gauge parameter must be traceless even for that case.

impose double traceless conditions on gauge field. As a simple starting example let us consider a tensor field with symmetric three-indices and assume that the equation of motion is invariant under the higher spin gauge transformation.

$$\mathcal{F}_{\mu\nu\rho} \equiv \Box\, w_{\mu\nu\rho} + a\, \partial_{(\mu}\, \partial \cdot w_{\nu\rho)} + b\, \partial_{(\mu}\, \partial_\nu\, w^\sigma{}_{\sigma\rho)} = 0\,, \quad w_{\mu\nu\rho} = \partial_{(\mu}\, \Lambda_{\nu\rho)}\,, \tag{2.29}$$

Gauge invariant $\mathcal{F}_{\mu\nu\rho}$ is called Fronsdal operator for spin-three. The gauge variation of Fronsdal operator is

$$\delta\, \mathcal{F}_{\mu\nu\rho} = \Big(1 + \frac{a}{3}\Big)\, \Box\, \partial_{(\mu}\, \Lambda_{\nu\rho)} + \frac{1}{3}\, (2\, a + b)\, \partial_{(\mu}\, \partial_\nu\, \partial \cdot \Lambda_{\rho)} + \frac{2\, b}{3}\, \partial_{(\mu}\, \partial_\nu\, \partial_{\rho)}\, \Lambda^\sigma{}_\sigma\,. \tag{2.30}$$

We can check that Fronsdal operator cannot be invariant under arbitrary gauge parameters. Only after imposing the traceless constraint to gauge parameters, the equation of motion for spin-three could be made as gauge invariant quantity by choosing $a = -3$ and $b = -6$. One can show that this equation coincide with spin-3 Fronsdal equation which is obtained by massless limit of Singh-Hagen.

Toward massless field: equation of motion by gauge symmetry We can apply the same logic to higher-spin cases. For an arbitrary spin, it is convenient to introduce auxiliary U variables because it keeps all totally symmetric field in compact way.

$$\Phi(X, U) = \sum_{s=0}^{\infty} \frac{1}{s!}\, \Phi_{\mu_1 \cdots \mu_s} U^{\mu_1} \cdots U^{\mu_s} \tag{2.31}$$

The auxiliary variable U is number and commute with each others. Therefore the fields are automatically symmetric tensor. [4] The taking a divergence and the taking a trace part could be changed to algebraic procedure by definition following operators.

$$\begin{gathered} \mathcal{M}_2 = U^2\,, \quad \mathcal{M}_1 = U \cdot \partial_X\,, \quad \mathcal{M}_0 = {\partial_X}^2\,, \quad \mathcal{M}_{-1} = \partial_X \cdot \partial_U\,, \quad \mathcal{M}_{-2} = {\partial_U}^2\,, \\ \mathcal{N}_X = X \cdot \partial_X\,, \quad \mathcal{N}_U = U \cdot \partial_U\,, \quad \mathcal{D}_0 = \mathcal{N}_X - \mathcal{N}_U\,, \quad \mathcal{T}_{-1} = X \cdot \partial_U\,. \end{gathered} \tag{2.32}$$

Here, the numbers in subscripts are the orders of auxiliary U variables. A multiflying the first operator $\mathcal{M}_2$ to Φ is equal a multiflying the flat metric tensor and symmetrization over all indices for all component fields. The second operator $\mathcal{M}_1$ is related with an acting derivative and an index symmetrization. The third one $\mathcal{M}_0$ is equal to d'Alembertian operator, and the fourth $\mathcal{M}_{-1}$ and fifth $\mathcal{M}_{-2}$ correspond to a taking divergence and a taking trace. The operators in second line is used in next section in "ambient-space formalism". Let us summarized the commutation relations between the operators.

$$[\mathcal{M}_{-2}, \mathcal{M}_2] = 2\, d + 4\, U \cdot \partial_U\,, \quad [\mathcal{M}_{-2}, \mathcal{M}_1] = 2\, \mathcal{M}_{-1}\,, \tag{2.33}$$

$$[\mathcal{M}_{-1}, \mathcal{M}_2] = 2\, \mathcal{M}_1\,, \quad [\mathcal{M}_{-1}, \mathcal{M}_1] = \mathcal{M}_0\,, \tag{2.34}$$

$$[\mathcal{T}_{-1}, \mathcal{M}_1] = \mathcal{D}_0\,. \tag{2.35}$$

[4] To introduce the non-symmetric fields, we must introduce many auxiliary variables like $U_1, U_2, \cdots, U_k$.

Equipped with these operators one can write the higher-spin Frondal operator, which is the higher-spin generalization of (2.29), as

$$\mathcal{F}(a,b)\,\Phi \quad \equiv \left(\partial_X{}^2 + a\,U\cdot\partial_X\,\partial_X\cdot\partial_U + b\,(U\cdot\partial_X)^2\,\partial_U{}^2\right)\Phi \tag{2.36}$$

$$\left(\mathcal{M}_0 + a\ \mathcal{M}_1\ \mathcal{M}_{-1} + b\,(\mathcal{M}_1)^2\ \mathcal{M}_{-2}\right)\Phi\ = 0\,. \tag{2.37}$$

The higher spin gauge transformation can be expressed as

$$\delta\,\Phi = U\cdot\partial_X\,\Lambda = \mathcal{M}_1\ \Phi\,. \tag{2.38}$$

The gauge invariance of the operator $\mathcal{F}(a,b)$ is

$$\delta_\Lambda\ \mathcal{F}(a,b)\,\Phi = \left((1+a)\ \mathcal{M}_1\ \mathcal{M}_0 + (a+2\,b)\ \mathcal{M}_1{}^2\ \mathcal{M}_{-1} + b\ \mathcal{M}_1{}^3\ \mathcal{M}_{-2}\right)\Lambda\,. \tag{2.39}$$

We have used the commutation relations (2.35) and changed the orders of operators in sequence of subscriptional order. We must impose $a = -1$, $b = 1/2$ and $\mathcal{M}_{-2}\ \Lambda$=0 to make operator gauge-invariant. The last condition means that the gauge parameters must be traceless. The operator $\mathcal{F} \equiv \mathcal{F}(-1,1/2)$ is called the Fronsdal operator.

Toward massless field: Lagrangian by gauge symmetry Now let us consider the gauge invariance of action. Fronsdal operator is an equation of motion for field with higher spin gauge symmetry, but it could not be an Euler-Lagrange equation. For example, zero-Ricci-tensor-condition is equal to equation of motion for graviton without matter and cosmological constant. However the Euler-Lagrange equation for graviton is zero-Einstein-tensor-condition. The zero-Einstein-tensor-condition implies the zero Ricci conditions and vice versa. The difference with Ricci and Einstein tensor is the conservation property. The conservation property comes from general covariance. Similarly, the Euler-Lagrange equation for massless higher spin could be $\mathcal{G} = \mathcal{F} + c\,U^2\,\partial_U^2\ \mathcal{F}$. The conservation property of $\mathcal{G}$ comes from the higher spin gauge invariance of an action. Then the quadratic action could be written as

$$\int d^dX\ \mathcal{L} = \int \exp\left(\partial_{U_1}\cdot\partial_{U_2}\right)\left(\Phi(X,U_1)\,\mathcal{G}\,\Phi(X,U_2)\right)|_{U_i=0}\,. \tag{2.40}$$

We introduce the two auxiliary variables U_1 and U_2 for each field. $\exp\left(\partial_{U_1}\cdot\partial_{U_2}\right)$ means contracting all indices. After contracting all possible indices we make U_i as zero. This is related with the Lorentz invariance. For the gauge variation $\delta\,\Phi = U\cdot\partial_X\,\Lambda$, the variation of action is

$$\int d^dX\,\delta_\Lambda\,\mathcal{L} = -\int \exp\left(\partial_{U_1}\cdot\partial_{U_2}\right)\left(\Lambda(X,U_1)\,\partial_X\cdot\partial_{U_2}\,\mathcal{G}\,\Phi(X,U_2)\right)|_{U_i=0} \tag{2.41}$$

We have taken one derivative the $\partial_{U_1}\cdot\partial_{U_2}$. Note that $U_2{}^2$ is effectively zero because the gauge parameter is traceless. The coefficient of Λ in the integrand can be summarized by using commutation relations (2.35).

$$\mathcal{M}_{-1}\, \mathcal{G}\, \Phi(X,U) \simeq (1+2\,c\,\Delta)\, \mathcal{B}\, \Phi + \tfrac{1}{2}\, \Delta\, (\mathcal{M}_1)^2\, (\mathcal{M}_{-2})^2\, \Phi\,, \tag{2.42}$$
$$\mathcal{B} = \left(\mathcal{M}_1\, (\mathcal{M}_{-1})^2\, \mathcal{M}_1\, \mathcal{M}_0\, \mathcal{M}_{-2} + \tfrac{1}{2}\, (\mathcal{M}_1)^2\, \mathcal{M}_{-1}\, \mathcal{M}_{-2}\right)\,, \quad \Delta = 2\,d + 4\,U\cdot\partial_U\,.$$

We ignore the U^2 terms (trace terms) and $\simeq$ means up to trace terms. Even after imposing $c = -1/(2\,\Delta)$, there are $(\mathcal{M}_{-2})^2\,\Phi$ proportional term. The term is exactly double trace part of fields. Therefore the massless field must be double traceless to make the action gauge invariant.

Back to Sing-Hagen massive field In historically the massless Fronsdal field appears as massless limit of Singh-Hagen massive field. However the stories go around and around, the reverse process is also possible. We could obtain the Singh-Hagen action from massless field in one-higher dimension. If we know that the massless field can be represented by a double-traceless field, the auxiliary fields of Singh and Hagen naturally appear in Kaluza-Klein of massless field [12]. The spin-s field w_s^{d+1}in $(d+1)$-dimension without traceless condition could be divided by trace-full fields in d-dimension. i.e. w_s^d, $w_{s-1}^d\,, \cdots, w_1^d$. The double traceless spin-s fields can be divided by w_s^d, w_{s-1}^d, w_{s-2}^d and w_{s-3}^d because other fields w_k^d with $k<(s-3)$ could be represented by these four fields by using double traceless conditions. Similarly, the spin-s gauge parameter which has $(s-1)$-Lorentz indices can be considered as trace-full gauge parameters in d dimension. i.e. Λ_{s-1}^d, Λ_{s-2}^d, $\cdots, \Lambda_0^d$. In Fronsdal case, the Λ_{s-1}^d and Λ_{s-2}^d remain because of traceless conditions. These trace-full gauge parameters are Stueckelberg gauge symmetry (see the Section 2.3) and can be used to algebraically fixed spin-$(s-1)$ field and spin-$(s-2)$ field: w_{s-1}^d and w_{s-3}^d. Therefore, trace-full w_s^d and w_{s-2}^d remain. Because the trace-full spin-k field consists of a traceless spin-k, a traceless spin-$(k-2)$, $\cdots$ and spin-$(k-2\,[k/2])$, we can see that the set of remaining fields coincides with set of fields in Singh-Hagen action. A similar but little more complicated procedure is given in Section 5.2 on (A)dS.

No-go theorems The interaction between the higher spin gauge field and other field is not a trivial procedure as in spin-1 and spin-2 cases. In lower spin cases, it could be done by changing the derivatives as covariant derivatives. For example if we know the free spin-1 field, i.e Maxwell theory, we could find interacting theory of spin-1 and spin-2 by changing the derivatives in Maxwell action as gravitational covariant derivative. The action is Einstein-Maxwell action. These procedure is called as “minimal coupling”. However the similar procedure breaks the gauge symmetry in a higher spin case. Let us consider the “minimal coupling” between a massless spin-3 and graviton. In (2.30) from the equation (2.29), we use the commutative property of normal derivatives. After “minimal coupling” the non-commutative properties of gravitational covariant derivatives breaks the gauge symmetry. On top of that, there are no-go theorem by Coleman-Mandula [15], Weinberg [16] and Weinberg-Witten [17] to prevent the interactions. Even for massive higher spin, there are some issues like Velo-Zwanziger problem [18]. The more recent review and progresses are given in [19] and reference therein.

2.2 Higher Spin Fields: (Anti-)de Sitter Space

The most achievements in recent days for the interacting higher spin gauge theory are done in curve space-time. As the same spirit in previous section, we start from the projective unitary representations of $\mathfrak{so}(2,3)$, and consider the field realizations of these representations.

2.2.1 Long and Short Higher Spin Representation: AdS Space

We concentrate on four-dimensional spacetime for simplicity and summarise the results for arbitrary dimension in Appendix B. This subsection is more or less similar to the lectures [79] and [79]. The general even dimensional analysis is given in lecture [26].

AdS$_4$ and isometries The sphere can be considered as a $\boldsymbol{SO}(3)$ invariant subspace in $\mathbf{R}^3$. It is mathematically defined as the set of points that are at the same distance R from a given point.

$$X_1^2 + X_2^2 + X_3^2 = R^2 \tag{2.43}$$

(Anti-)de Sitter space is a Lorentzian version of sphere. [5] Let us consider the spacetime $\mathbf{R}^{2|3}$ with two time direction and three space direction. It is defined as a Rimaniann manifold with a metric,

$$d\,s^2 = -d\,X_-^2 - d\,X_0^2 + d\,X_1^2 + d\,X_2^2 + d\,X_3^2\,. \tag{2.44}$$

Anti-de Sitter space is an $\boldsymbol{SO}(2,3)$ invariant subspace in $\mathbf{R}^{2|3}$, or can be mathematically defined as the set of points that are at the same distance R from a given point.

$$-X_-^2 - X_0^2 + X_1^2 + X_2^2 + X_3^2 = -\ell^2\,. \tag{2.45}$$

ℓ is called AdS radius. This space could be parametrized and from this parametrization one could get the embedded metric.

$$\begin{aligned} X_- = \ell\,\cosh\rho\,\sin\tau\,,\quad X_0 = \ell\,\cosh\rho\,\cos\tau\,, \\ X_1 = \ell\,\sinh\rho\,\sin\theta\,\cos\phi\,,\quad X_2 = \ell\,\sinh\rho\,\sin\theta\,\sin\phi\,,\quad X_3 = \ell\,\sinh\rho\,\cos\theta\,. \end{aligned} \tag{2.46}$$

As a sphere has a constant curvature, the curvature of AdS space is constant. To show that, one can deduce the AdS metric by the embedded metric from the metric of $\mathbf{R}^{2|3}$.

$$d\,s^2 = -\ell^2\,\cosh^2\rho\,d\,\tau^2 + \ell^2\,\sinh^2\rho\left(d\,\rho^2 + d\,\theta^2 + \sin^2\theta\,d\,\phi^2\right) \tag{2.47}$$

This parametrization is called global coordinate. In this embedding method, the τ coordinate is periodic. We can consider universal covering space of global AdS coordinate, and expand the range of τ from $-\infty$ to ∞. The isometry of AdS is — as almost by definition — $\boldsymbol{SO}(2,3)$. The maximal compact subgroup of these isometry group is $\boldsymbol{SO}(2) \oplus \boldsymbol{SO}(3)$. One can check that the generator

[5] We will concentrate on AdS$_4$ in this section. The (A)dS$_d$ can be considered in similar manner.

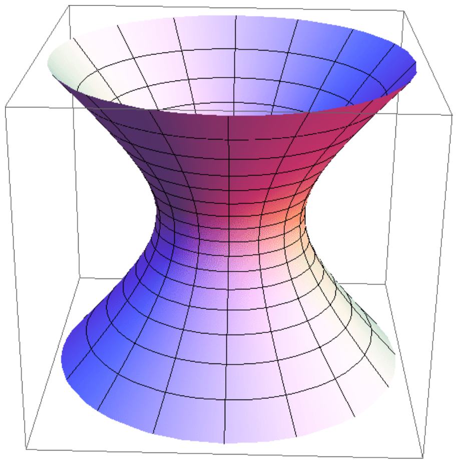

Figure 1: Global AdS_2

for $\boldsymbol{SO}(2)$ corresponds to translation for τ, and the generators for $\boldsymbol{SO}(3)$ correspond to the rotation symmetry for θ and ϕ direction.

$\mathfrak{so}(2,3)$ representation: strategy The isometry group of AdS_4 space is $\mathfrak{so}(2,3)$. Let us consider $\mathfrak{so}(2,3)$ unitary projective representation as [25] and [24]. The generalization to an arbitrary dimensional space is similar to AdS_4. One can refer [26] for a technical details and references. What we must do is basically not so different with what we have done for $\mathfrak{so}(3)$ representation in quantum mechanics course [22]. To find the general unitary representation of $\mathfrak{so}(3)$ we consider the quadratic Casimir $\mathbf{J}^2$, and choose one generator J_z to represent quantum number. The creation(raising) and the annihilation(lowering) operators $J_\pm = J_x \pm J_y$ are constructed by remaining generators. Elements in a modular space of representation are obtained by acting the creation operator to the state which has the lowest eigenvalue of J_z. And finally we get the the inter of half integer value angular momentum with finite dimensional modular space. The main difference with $\mathfrak{so}(2,3)$ and $\mathfrak{so}(3)$ is the dimension of representation(modular) space. Because $\mathfrak{so}(2,3)$ is a non-compact Lie algebra, there are no finite dimensional representation. We consider an infinite dimensional representation.

Let us define generators. We chose the anti-hermitian convention for $\mathfrak{so}(2,3)$.

$$\mathcal{M}^\dagger_{MN} = -\mathcal{M}_{MN} \ . \tag{2.48}$$

The generators of maximal compact subgroup are

$$H = -i\,\mathcal{M}_{-0}\,, \quad J_a = \epsilon_{abc}\,\mathcal{M}^{ab} \ . \tag{2.49}$$

As we showed in isometry part, these generators represent the translation to τ direction and rotation symmetry for a given τ slices. Therefore we wrote them as Hamiltonian and rotation generators. The states can be classified by eigenvalues of these operators: $|E,\, s\rangle$. Other generators can be combined and written as,

$$\mathcal{M}_a^{\pm} = -i\ \mathcal{M}_{0a} \pm \mathcal{M}_{-a}\ . \tag{2.50}$$

The commutation relations tells us that these operators act like energy creation and annihilation operators.

$$\left[H,\, \mathcal{M}_a^{\pm}\right] = \pm\, \mathcal{M}_a^{\pm}\,, \quad \left[\mathcal{M}_a^{\pm},\, \mathcal{M}_b^{\pm}\right] = 0\,, \quad \left[\mathcal{M}_a^{+},\, \mathcal{M}_b^{-}\right] = -2\ (H\,\delta_{ab} + \mathcal{M}_{ab})\ . \tag{2.51}$$

If we act $\mathcal{M}_a^{+}(\mathcal{M}_a^{-})$ on the eigenstate of energy operator, the energy of the state increase(degrease) by one unit.

$$|\xi\rangle \equiv \mathcal{M}_a^{\pm}\ |E,\, s\rangle, \quad H\,|\xi\rangle = \left[H,\, \mathcal{M}_a^{\pm}\right]\ |E,\, s\rangle + E\ \mathcal{M}_a^{\pm}\ |E,\, s\rangle = (E \pm 1)\,|\xi\rangle\,. \tag{2.52}$$

We assume that energy is bounded below. The lowest energy state, which satisfies the condition $\mathcal{M}_a^{-}\ |E_0, s\rangle = 0$, is called as lowest weight state(LWS). We can generate other states by acting the creation operators These states are called descendants. The generator $\mathcal{M}_a^{+}$ has one space index and its commutation relations with J_a imply that $\mathcal{M}_a^{+}$ are $SO(3)$ vector. By the spin summation rule, we can figure out the descendants energy and spin. The set of descendant and LWS makes Verma-module for $\mathfrak{so}(3,2)$-representation. We denote this Verma-module as $\mathcal{V}(E_0, s)$. $\mathcal{V}(E_0, s)$ could be irreducible representation or non-unitary representation. The problem arises from normalization of the descendants. During construction, we could calculate the norm of $|\xi\rangle \equiv \mathcal{M}_a^{\pm}\ |E,\, s\rangle$. First, If its norm is positive, we can alway normalize it. Secondly, sub-module arises when norm is zero. After quotient out the sub-module, we can obtain unitary irreducible representation which is called short representation. This quotient space is related with the existence of gauge symmetry. Lastly, if its norm is negative related Verma-module is non-unitary. Therefore, short representation saturate the unitary bound. There are various ways to calculate the norm of descendants. We will use the quadratic Casimir operator as a tool.

$$\mathcal{C}_2 \quad = -\tfrac{1}{2}\ \mathcal{M}^{MN}\ \mathcal{M}_{MN} = H^2 - \tfrac{1}{2}\{\mathcal{M}_a^{+},\, \mathcal{M}_a^{-}\} - \tfrac{1}{2}\,(\mathcal{M}_{ab})^2 \tag{2.53}$$

$$= H(H-3) - \tfrac{1}{2}\,(\mathcal{M}_{ab})^2 - \mathcal{M}_a^{+}\ \mathcal{M}_a^{-}\ . \tag{2.54}$$

We can calculate the quadratic Casmir by acting the $\mathcal{C}_2$ to LWS:

$$\mathcal{C}_2 = E_0(E_0 - 3) + s(s+1)\,. \tag{2.55}$$

By definition quadratic Casimir is constant in whole irreducible representation.

$\mathfrak{so}(2,3)$ representation: scalar and spin-half Let us first star with simplest case, when the LWS has zero spin. After spin summation rule, we can derive what are the possible states in this representation as in left graph of Fig.2. To consider whether we can normalize descendants or not(these diagrams are reducible or not), let us calculated the norm of thedescendant state, $|E_0 + 1, 1\rangle =$

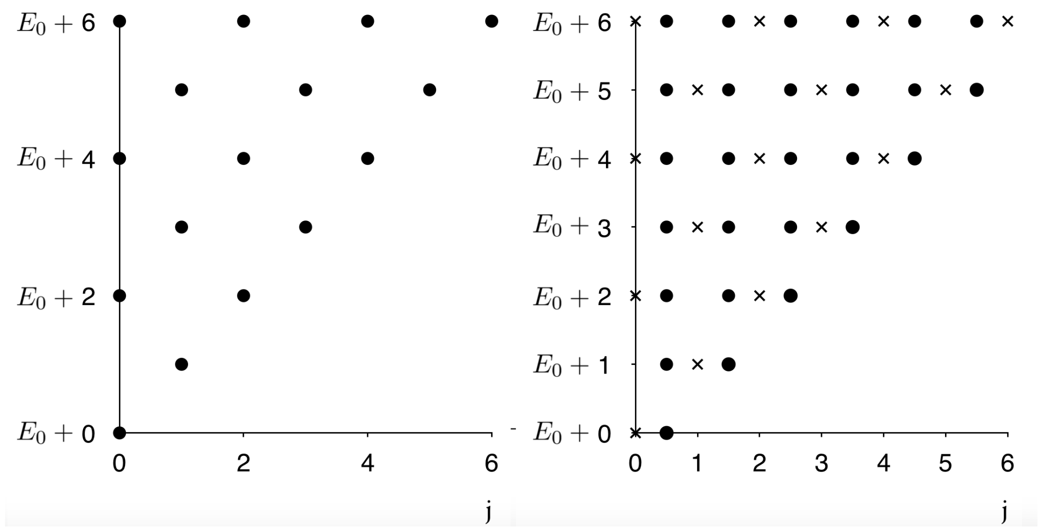

Figure 2: The weight diagram for $\mathcal{D}(E_0,\,0)$ and $\mathcal{D}(E_0,\,1/2)$. For each point there are $2j+1$ degenerated states. And the black disks correspond to states for $\mathcal{D}(E_0,\,1/2)$ in left graph. The X marks correspond to states for $\mathcal{D}(E_0,\,0)$

$N_0\, M_a^+\, |E_0, 0\rangle$ where N_0 is normalization factor.

$$\left(M_a^+\, |E_0, 0\rangle\right)^\dagger\, \mathcal{C}_2\; M_a^+\, |E_0, 0\rangle = \langle E_0, 0|\, M_a^-\; \mathcal{C}_2\; M_a^+\, |E_0, 0\rangle \tag{2.56}$$
$$= (E_0+1)\;(E_0-2) + 2 - \langle E_0+1, 1|M_a^+\, M_a^-\, |E_0+1, 1\rangle \tag{2.57}$$
$$= (E_0+1)\;(E_0-2) + 2 - |M_a^-\, |E_0+1, 1\rangle|^2 \tag{2.58}$$

Casimir is not change in whole representation, above value must be equal to $E_0(E_0-3)$. Therefore, $|M_a^-\, |E_0+1, 1\rangle|^2 = 2\, E_0$ and the value must be positive for the unitarity of representation, we conclude that $E_0 > 0$ for unitary representation. By the same procedure for other descendants, we can get the bounded values for E_0 as,

$$(E_0+2k)\,(E_0+2k-3) - E_0\,(E_0-3) \geq 0\,, \qquad \text{from the } j=0 \text{ line} \tag{2.59}$$
$$(E_0+2k+1)\,(E_0+2k-2) + 2 - E_0\,(E_0-3) \geq 0\,, \qquad \text{from the } j=1 \text{ line}\,. \tag{2.60}$$

Therefore we can get the unitary representation with the whole point in weight diagram if $E_0 > 1/2$. For the specific case $E_0 = 1/2$, we will consider in next section.

As a second case, let us consider the weight diagram with LWS $|E_0, 1/2\rangle$ as in left graph of Fig.2. We can figure out the positions of descendants in weight diagram by the spin summation rule of descents of $s=0$ case and spin half. As the same procedure with $s=0$ case, we get the conditions for the unitary representation.

$$(E_0+k)\,(E_0+k-3) - E_0\,(E_0-3) \geq 0 \quad \text{from the } j=1/2 \text{ line}\,. \tag{2.61}$$

Therefore, we can get the unitary representation with the whole point in weight diagram if $E_0 > 1$. For the specific case $E_0 = 1$, we will consider in next section.

$\mathfrak{so}(2,3)$ representation: spin-$s \geq 1$ There are new issues arise in weight diagram, for spin s cases. Some points in weight diagram is degenerated. For example let us consider $D(E_0,\,1)$, there are

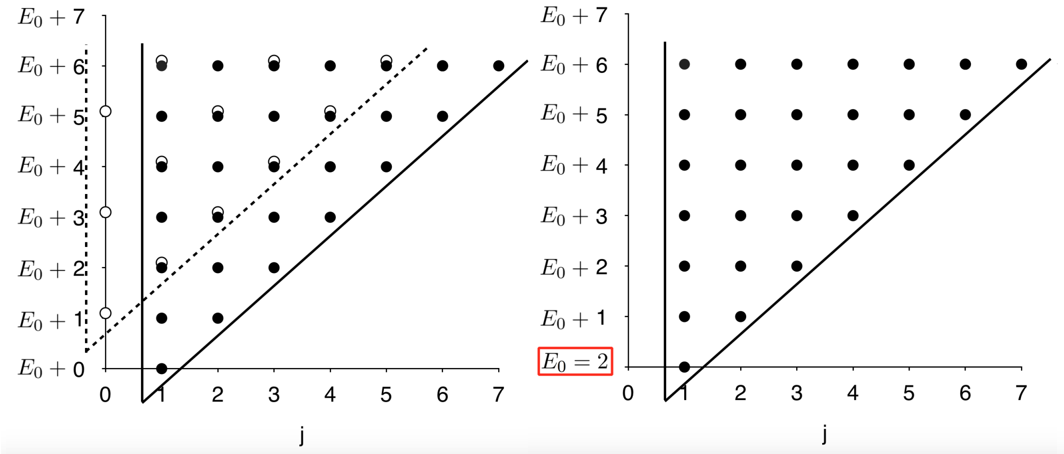

Figure 3: The weight diagram for $\mathcal{D}(E_0,\,1)$ and $\mathcal{D}(2,\,1)$. For each point there are $2j+1$ degenerated states for black circle and withe circle. When E_0 saturates the massless point $E_0 = 2$ in left graph, the scalar weight diagram which consists of white circles is decoupled. Then we get the short representation that correspond the massless representation in AdS as in right graph. The decoupled part corresponds to $\mathcal{D}(3,0)$

two states in $E_0 + 2$ and spin-one points. This structure can be obtained by the spin summation rule with descendant of $\mathcal{D}(E_0,\,0)$ and spin-one. Agin we can get the conditions for unitary representation,

$$\begin{aligned}
2\,(E_0 - s - 1) &\geq 0 && \text{from the } E = E_0 + 1 \text{ and } j = s-1 \text{ state}\,, \\
4\,(E_0 - s) &\geq 0 && \text{from the } E = E_0 + 2 \text{ and } j = s-2 \text{ state}\,, \\
&\;\vdots && \;\vdots \\
2\,k\,(E_0 + k\,(k-2) - s) &\geq 0 && \text{from the } E = E_0 + k \text{ and } j = s-k \text{ state}\,, \\
&\;\vdots && \;\vdots \\
2\,s\,(E_0 + s\,(s-2) - s) &\geq 0 && \text{from the } E = E_0 + s \text{ and } j = 0 \text{ state}\,.
\end{aligned} \tag{2.62}$$

Therefore, we can get the unitary representation with the whole point in weight diagram if $E_0 > s+1$. When the energy of LWS saturate the unitary bound. i.e. $E_0 = (s+1)$ case, the invariant subspace appears. By the construction we can conclude that the state $|s+2, s-1\rangle$ in the weight diagram has zero-norm. (For example, $|s+2, s-1\rangle = M^+_{1-2\,i}\,|s+1, s\rangle$.) The state that can be obtained by acting M^+_as are also zero-norm. We can quotient out this invariant sub-space and get the short representation. As we will show in below, this correspond the massless field in AdS_4. The sub-space that must be quotiented out corresponds to the gauge mode. The sub-space is the representation with

LWS $|s+1, s-1\rangle$.

$$\lim_{E_0 \to (s+1)^+} \mathcal{D}_{\text{long}}(E_0,\, s) = \mathcal{D}_{\text{short}}(s+1,\, s) \oplus \mathcal{D}_{\text{long}}(s+2, s-1)\,. \tag{2.63}$$

One can figure out the general structure and patterns for short representation limits in Figure.3 and Figure.4.

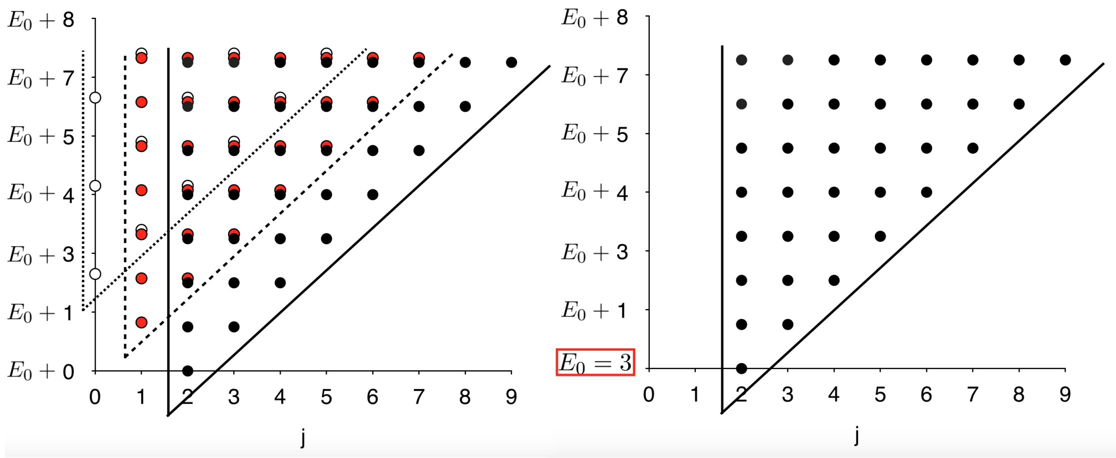

Figure 4: The weight diagram for $\mathcal{D}(E_0,\,1)$ and $\mathcal{D}(2,\,1)$. For each point there are $2j+1$ degenerated states for black, red and withe circle. When E_0 saturates the massless point $E_0=3$ in left graph, the spin-one weight diagram which LWS in $E=4$ and $j=1$ is decoupled. Then we get the short representation that correspond the massless representation in AdS as in right graph.

2.2.2 Massive and Massless Higher Spin Field on (A)dS

Natural extension from massive and massless fields in flat to the counterparts of (Anti-)de Sitter space will be considered. As in flat case, spin-s field with symmetric traceless indices will be considered with Firerz-Pauli condition on AdS$_4$:

$$\left(\Box_{AdS_4} - \kappa^2\right)\varphi_{\mu_1\cdots\mu_s}(x) = 0\,, \quad \nabla^{\nu}\,\varphi_{\nu\mu_1\cdots\mu_{s-1}} = 0\,. \tag{2.64}$$

The main purpose of this subsection is to find the relation between Fierz-Pauli mass κ in (2.64) and the conformal weight E_0 in (2.55). Field is called massless field when there are gauge symmetries. In (A)dS, the massless field has non-zero Fierz-Pauli-mass κ.

The ambient formalism (radial reduction method) is used to obtain (A)dS fields from flat fields. We have referred many contents in original paper [23], and reviews section 3 in [20] and section 2 in [21].

Field theoretical realization: scalar equation First, we apply ambient formalism to the simplest case: scalar equation in AdS$_4$. It would be convenient to write the coordinate of $\mathbf{R}^{2|3}$ as

$$X_- = R\,\cosh\rho\,\sin\tau\,,\quad X_0 = R\,\cosh\rho\,\cos\tau\,, \tag{2.65}$$
$$X_1 = R\,\sinh\rho\,\sin\theta\,\cos\phi\,,\quad X_2 = R\,\sinh\rho\,\sin\theta\,\sin\phi\,,\quad X_3 = R\,\sinh\rho\,\cos\theta\,.$$

As we summarised in page 12, AdS$_4$ space with radius ℓ can be identified with the subspace of $\mathbf{R}^{2|3}$ with constraint $R=\ell$. For any field $\varphi(x)$ on AdS$_4$, we can construct the field $\Phi(X)$ on $\mathbf{R}^{2|3}$ as follows:[6]

$$\Phi = (R/\ell)^{-\Delta}\,\varphi(x)\,,\quad X_A \simeq (R,\,\tau,\,\rho,\,\theta,\,\phi)\quad\text{and}\quad x_\mu = (\tau,\,\rho,\,\theta,\,\phi)\,. \tag{2.66}$$

Here, we abuse the coordinates $(\tau,\,\rho,\,\theta,\,\phi)$. The Klein-Gordon equation in AdS$_4$ can be derived by the massless equation of motion $\Phi(X)$ as

$$\partial_X\cdot\partial_X\,\Phi(x) = 0 \to [\Box_{AdS_4} - \Delta\,(\Delta-3)]\,\varphi(x) = 0\,. \tag{2.67}$$

In $\mathbf{R}^{2|3}$ point of view the isometry generators can be written as simple way: $\mathcal{M}_{AB} = 2\,i\,X_{[A}\,\partial^X_{B]}$. Therefore, quadratic Casimir operators can be expressed as,

$$\mathcal{C}_2 \equiv -\frac{1}{2}\,\mathcal{M}^{MN}\,\mathcal{M}_{MN} = X^2\,\partial_X^2 - X\cdot\partial_X\,(X\cdot\partial_X+3) = -\Delta\,(\Delta-3) \tag{2.68}$$

By comparing with (2.55), we conclude that $E_0 = \Delta$. Therefore, we can find the relation between Firez-Pauli mass and the conformal weight E_0:

$$\left(\Box_{AdS_4} - \kappa^2\right)\varphi(x) = 0\,,\quad \kappa^2 = E_0\,(E_0-3)\,. \tag{2.69}$$

Field theoretical realization: spin-s equation To extend the analysis to general spin, it is convenient to introduce auxiliary variable U as in subsection 2.1.2. Then the $\mathfrak{so}(2,3)$ generators can be written as

$$M_{AB} = 2\,i\,\left(X_{[A}\,\partial^X_{B]} + U_{[A}\,\partial^U_{B]}\right)\,. \tag{2.70}$$

The quadratic Casimir is

$$\begin{aligned}\mathcal{C}_2 &= -\frac{1}{2}\,M^{AB}\,M_{AB} &\quad (2.71)\\ &= X^2\,\mathcal{M}_0 - \mathcal{N}_X\,(\mathcal{N}_X+3) + 2\,X\cdot U\,\mathcal{M}_{-1} - 2\,U\cdot\partial_X\,X\cdot\partial_U \\ &\quad + U^2\,\mathcal{M}_{-2} - \mathcal{N}_U\,(\mathcal{N}_U+1)\,. &\quad (2.72)\end{aligned}$$

Here, the operators $\mathcal{N}_{X,U}$ and $\mathcal{M}_i$ are given as,

$$\mathcal{N}_X = X\cdot\partial_X\,,\quad \mathcal{N}_U = U\cdot\partial_U\,,\quad \mathcal{M}_0 = {\partial_X}^2\,,\quad \mathcal{M}_{-1} = \partial_X\cdot\partial_U\,,\quad \mathcal{M}_{-2} = {\partial_U}^2\,.$$

[6] There are many other way to construct $\Phi(X)$. For example see [85].

The Fierz-Pauli condition and traceless condition for massive field $\Phi(X,U)$ can be expressed by using operators in (2.32):

$$\left(\mathcal{M}_0 - M^2\right)\Phi = 0\,, \quad \mathcal{M}_{-1}\,\Phi = 0\,, \quad \mathcal{M}_{-2}\,\Phi = 0 \tag{2.73}$$

We want to consider similar extension from φ to Φ as (2.66). One can show that the following two conditions, which are called homogeneity and tangential conditions, are enough to define the extention:[7]

$$(X\cdot\partial_X + \Delta)\,\Phi(X,\,U) = 0\,, \quad X\cdot\partial_U\,\Phi(X,\,U) = 0\,. \tag{2.74}$$

The first condition is equivalent to scalar case and the second condition is needed to consider only spin-s field in AdS$_4$. By conditions (2.73) and (2.74), the quadratic Casimir (2.72) is given as

$$M^2 - \Delta\,(\Delta-3) - s(s+1)\,. \tag{2.75}$$

Comparing with (2.55) one can obtain $\Delta\,(\Delta-3) - M^2 = E_0\,(E_0-3)$. Then totally symmetric fields in flat $\mathbf{R}^{2|d}$ can be deduced from totally symmetric fields in (A)dS.

$$\Phi(R,x;v,u) = \left(\frac{R}{\ell}\right)^{-\Delta}\varphi(x,u)\,. \tag{2.76}$$

Here v and u are introduced to solve the tagential condition. The equation of motion for Φ is expressed in the equation in (A)dS,

$$\left(\mathcal{M}_0 - M^2\right)\,\Phi = \left(\frac{\ell}{R}\right)^{\Delta+2}\left(\Box_{AdS_4} - \kappa^2\right)\,\varphi = 0\,. \tag{2.77}$$

At last, we can derive the relation between Fierz-Pauli mass and conformal weight.

$$\kappa^2 = \frac{1}{\ell^2}\left[E_0\,(E_0-3) - s\right]\,. \tag{2.78}$$

For arbitrary dimensional results are given in Appenx B

2.3 Partially Masselss Fields on (Anti-)de Sitter Space

In spacetime with a non-zero constant curvature, there is a new kind of field which is called partially massless [164]. The physical degrees freedom(DoF) of partially massless field are bigger than DoF of the massless field with the same spin but smaller than DoF of the massive one. To make the statement more clearly let us consider spin-two particle in four dimension. A massive spin-two field has five DoF but a massless has only two DoF. Therefore, roughly speaking, a massless spin-two field needs a massless spin-one(two-DoF) and a scalar(one-DoF) to become a massive spin-two. One can ask whether we can make a new field by combining a massless spin-two field and a massless spin-one field—without a scalar field. There is no unitary representation with four-DoF in Minkowski space.

[7] For detail derivation see section 3 in [20] and section 2 in [21].

However we could find it in (A)dS representation. In this section we mainly focus on partially massless spin-two field, and review the two ways to partially massless spin-two field. i.e. gauge symmetry and Stueckelberg trick.

Before starting review the partially massless spin-two, it is worth to mention about its general properties for arbitrary spin-s. Again roughly speaking, a massless spin-s, spin-$(s-1)$, $\cdots$, spin-$(s-k)$ could be combined and makes one partially massless fields in (A)dS. There are no agreement for the naming of these kinds of field in the community yet. We will call this partially massless field as " partially massless" field with depth k in this thesis. Therefore, there are $s-2$ different kinds of partially massless fields for spin-s. In other way we could say that a massless field is–in some sense it is totally self-contradict–a partially massless field with depth with zero.

However these representations and fields are not unitary in Anti-de Sitter space. These properties can also be seen in the AdS representation in 2.2.1. For example let us consider the spin-two case in Figure.5. As we have shown in Figure.4 and the paragraph above it, the invariant sub-space appear.

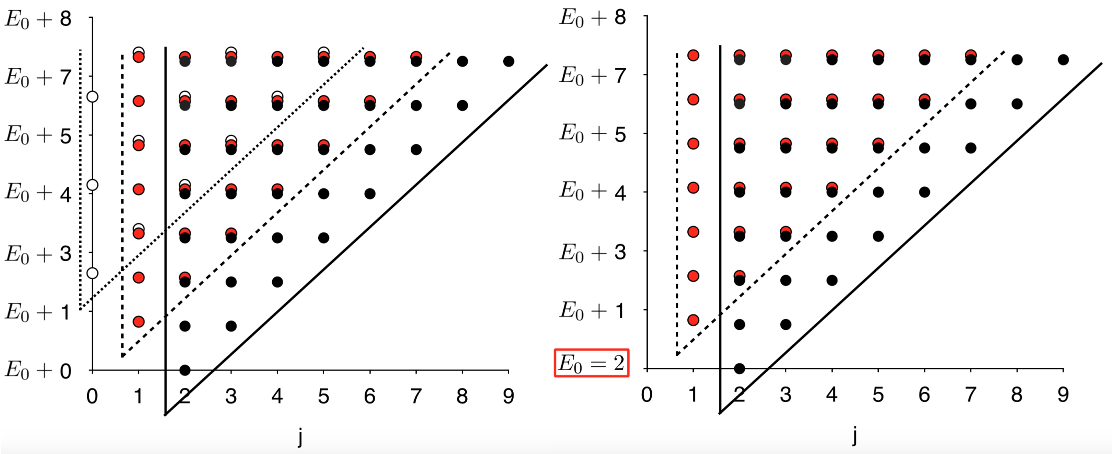

Figure 5: The weight diagram for $\mathcal{D}(E_0,\,2)$ and $\mathcal{D}(2,\,2)$

The spin-one weight diagram is decoupled with the short representation when E_0 reach the unitary bound 3. Even though it is not non-unitary representation, we could consider $\mathcal{D}(E_0,\,2)$ with $E_0<3$. The subspace appear when the norm of some state in the weight diagram goes to zero. We can figure out when these phenomenon happens from (2.62). For spin-two case, $E_0=2$ is another point when subspace appears. As one can see in Figure.5, the states that corresponds to $\mathcal{D}(4,0)$ are decoupled. Partially massless fields appear both in Chapter 4 and Chapter 6 and Chapter 7.

2.3.1 Gauge Symmetry

To introduce the partially massless spin-2 field, we go through an inverse process in Section 2.1.2. We first deforme a massless field equation and find the equation which satisfying Fierz-Palui condition (2.1) for massive field. And finally we test whether spin-2 field could have a scalar gauge transformation parameter.

There are various ways to derive the massless spin-2 equation on constant curvature background. Because massless spin-2 particle is a graviton, the equation could be derived from linearized Einstein equation. Instead of that, we will follow the gauge symmetry argument. As a first trial, let us consider the general Lorentz invariant Lagrangian,

$$\Box\, h_{\mu\nu} + c_1\, \nabla_{(\mu}\, \nabla^{\rho}\, h_{\nu)\rho} + c_2\, \nabla_\mu\, \nabla_\nu\, h + c_3\, h_{\mu\nu} + c_4\, g_{\mu\nu} h = 0\,. \tag{2.79}$$

c_i are undetermined constants, and the parenthesis means symmetrization of indices. i.e. $A_{(\mu}\, B_{\nu)} = (A_\mu\, B_\nu + A_\nu\, B_\mu)/2$. As a gauge principle, a spin-2 massless equation must be invariant under the linearized general covariance.

$$\delta\, h_{\mu\nu} = 2\, \nabla_{(\mu}\, \xi_{\nu)} \tag{2.80}$$

Because fields are in curved spacetime, the covariant derivatives does not commute.

$$[\nabla_\mu,\, \nabla_\nu]\, A_\rho = \mathfrak{R}_\rho{}^\sigma{}_{\mu\nu}\, A_\sigma = -\frac{\sigma}{\ell^2}\left(g_{\rho\mu}\, \delta^\sigma_\nu - g_{\rho\nu}\, \delta^\sigma_\mu\right) A_\sigma\,, \tag{2.81}$$

here ℓ is the (A)dS radius and σ is plus(minus) one for (A)dS. By the gauge invariant condition we can fix the all coefficient. i.e. $c_1 = -2$, $c_2 = 1$, $c_3 = \frac{2\,\sigma}{\ell^2}$ and $c_4 = -\frac{2\,\sigma}{\ell^2}$. This equation corresponds to the linearized zero Ricci condition. i.e. $\mathfrak{R}_{\mu\nu} = 0$. To get linearized Einstein equation, we must add the trace part of the equation.

$$\begin{aligned} G^{\text{linear}}_{\mu\nu} \;\; = \;\; & \Box\, h_{\mu\nu} - 2\, \nabla^\rho\, \nabla_{(\mu}\, h_{\nu)\rho} + \nabla_\mu\, \nabla_\nu\, h && (2.82)\\ & -g_{\mu\nu}\, \Box\, h + g_{\mu\nu}\, \nabla^\rho\, \nabla^\sigma\, h_{\rho\sigma} && (2.83)\\ & +2\frac{(d-1)\,\sigma}{\ell^2}\left(h_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, h\right) = 0 && (2.84)\end{aligned}$$

With tedious calculation, we can show that divergence of above quantity is zero. i.e. Bianchi identity $\nabla_\mu G^{\text{linear}}_{\mu\nu} = 0$. Therefore it is conserved tensor. Let us consider deformed equation with mass parameter m.

$$G^{\text{linear}}_{\mu\nu} - m^2\, \left(h_{\mu\nu} - a\, g_{\mu\nu}\, h\right) = 0\,. \tag{2.85}$$

The divergence, the double divergence and the trace part of the equation of motion are

$$\begin{gathered} \nabla^\mu\, h_{\mu\nu} - a\, \nabla_\nu\, h = 0\,, \quad \Box\, h - a\, \nabla^\mu\, \nabla^\nu\, h_{\mu\nu} = 0\,,\\ (d-2)(\Box\, h - \nabla^\mu\, \nabla^\nu\, h_{\mu\nu}) - \left((a\, d-1)\, m^2 - \frac{(d-1)\,(d-2)\,\sigma}{\ell^2}\right) h = 0\,. \end{gathered} \tag{2.86}$$

To obtain the Fierz-Pauli condition (2.1), a must be one. Then we can conclude that spin 2 filed is transverse and traceless for $m^2 \neq \frac{(d-2)\,\sigma}{\ell^2}$.

$$\nabla^\mu\, h_{\mu\nu} = 0\,, \quad h^\rho{}_\rho = 0. \tag{2.87}$$

Finally whole sets of equations coincide with Fierz-Pauli conditions.

$$\Box\, h_{\mu\nu} - \left(m^2 + \frac{2\,\sigma}{\ell^2}\right) h_{\mu\nu} = 0\,, \quad \nabla^\mu\, h_{\mu\nu} = 0\,, \quad h^\rho{}_\rho = 0\,. \tag{2.88}$$

For specific value of mass square $M^2 = m^2 + \frac{2\,\sigma}{\ell^2} = \frac{d\,\sigma}{\ell^2}$, all condition in (2.86) collapses to one condition,

$$\nabla^\mu\, h_{\mu\nu} - \nabla_\nu\, h = 0\,. \tag{2.89}$$

Therefore, we cannot derive Fierz-Pauli at on-shell. Instead of that the interesting phenomenon happens. The equation of motion is invariant under the scalar gauge transformation.

$$\delta h_{\mu\nu} = \nabla_\mu\, \nabla_\nu\, \lambda + \frac{\sigma}{\ell^2}\, g_{\mu\nu}\, \lambda\,. \tag{2.90}$$

This field is similar with the massless Fonsdal field in that sense that the field itself is not traceless and have gauge symmetry. However it is similar with the Singh and Hagen field or Fierz-Pauli field in that sense that the field must satisfy Fierz-Pauli-like condition (2.89). Its propagating properties and representation is given in [164].

We can approach the partially massless field in other direction. As we can derive the massless spin-2 action by the gauge symmetry argument, the partially massless equation of motion could be derived by partially massless gauge transformation. The general form of scalar gauge transformation for a spin-2 field is

$$\delta h_{\mu\nu} = \nabla_\mu\, \nabla_\nu\, \lambda + b\, g_{\mu\nu}\, \lambda\,. \tag{2.91}$$

b is a constant that will be fixed soon. Let us assume that the spin-2 field equation of motion is covariant under above gauge transformation. By the gauge transformation, we cannot impose Fierz-Pauli condition for this field. The gauge transformation of trace apart and the divergence part for spin-2 field are

$$\begin{aligned} \delta h^\rho{}_\rho &= \Box\, \lambda + b\, d\, \lambda && (2.92)\\ \delta\, \nabla^\mu\, h_{\mu\nu} &= \Box\, \nabla_\nu\, \lambda + b\, \nabla_\nu\, \lambda = \nabla_\nu\, \Box\, \lambda + \frac{\sigma\,(d-1)}{\ell^2}\, \nabla_\nu\, \lambda + b\, \nabla_\nu\, \lambda && (2.93) \end{aligned}$$

The only possible quantity that could be gauge invariant is (2.89) only when the coefficient b has a specific value. i.e. $b = \frac{\sigma}{\ell^2}$. Now we can ask what is the mass that admit this gauge variation. i.e. $\delta(\Box h_{\mu\nu} - M^2 h_{\mu\nu}) = 0$, with $M^2 = m^2 + \frac{2\,\sigma}{\ell^2}$. After using the identity of covariant derivatives commutation and $\Box\lambda + a\, d\lambda = 0$,

$$\delta(\Box\, h_{\mu\nu} - M^2\, h_{\mu\nu}) = -(a\, d - M^2 + \frac{2\,\sigma\, d}{\ell^2})\, (\nabla_\mu\, \nabla_\nu\, \lambda + b\, g_{\mu\nu}\, \lambda) \tag{2.94}$$

we get the mass of the partially masselss field. $m^2 = \frac{\sigma\,(d-2)}{\ell^2}$

2.3.2 Stueckelberg Trick

The existence of a scalar gauge variation is the main difference with the partially massless field and a massive field. It seems like that the gauge invariance popped up in some specific value of mass. In historically we first know Maxwell field and spin-1 gauge symmetry, then start to research about Proca action. Therefore the existence of gauge symmetry in a massless limit is not a surprise news in this case. However in logically we can ask why the gauge symmetry is appeared in that limit. We could answer this question by a simple trick which is called Stueckelberg trick [36]. And the existence of a scalar gauge transformation can be explained in the same way. Let us attack more simple and familiar problem. i.e. Proca and Maxwell action. Massive spin-1 field is described by Proca action.

$$\mathcal{L}_0(A|M) = \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\,m^2\,A_\mu\,A^\mu \tag{2.95}$$

Stueckelberg introduced a fake gauge symmetry by adding auxiliary scalar field through following redefinition of spin-1 field.

$$A_\mu = a_\mu - \frac{1}{m}\,\partial_\mu\,\phi\,, \tag{2.96}$$

The factor in front of the second term is introduced to make the scalar kinetic term canonical. The Porca action is changed to non-diagonalized action.

$$\frac{1}{4}\,f_{\mu\nu}f^{\mu\nu} + \frac{1}{2}\,\partial_\mu\,\phi\,\partial^\mu\,\phi + m\,\Big(\frac{m}{2}\,a_\mu\,a^\mu - \,a_\mu\,\partial^\mu\,\phi\Big) \tag{2.97}$$

Because the physically important quantity is the capital A, the Lagrangian is invariant when the capital A field is invariant. Therefore almost by definition the Lagrangian is invariant under following gauge transformation.

$$\delta\,a_\mu = \partial_\mu\,\Lambda \qquad \delta\,\phi = m\,\Lambda\,. \tag{2.98}$$

This gauge symmetry is called Stueckelberg gauge transformation and ϕ is called Stueckelberg scalar. The Stueckelberg scalar has no physical degree of freedom. As one can see in (4.5) the scalar can be algebraically fixed as zero. Therefore Stueckelberg scalar is redundant. What exactly Stueckelberg did is just adding one scalar degree and removing it by scalar gauge. However in the massless limit, Stueckelberg scalar cannot be fixed by gauge symmetry. Therefore the Stueckelberg scalar has a physical degree of freedom and massive spin-1 field goes to Maxwell field with a scalar gauge symmetry. i.e. massless field.

As the same logic let us introduce the Stueckelberg gauge to a massive spin-2 field. (We just follows the logic showing the existence of partially massless field in [168] and introducing Stueckelberg spin-2 in [54].) Fierz-Pauli massive spin 2 action is

$$\begin{aligned}\mathcal{L}_0 = \Big(\frac{1}{2}\,\nabla_\alpha\,h_{\mu\nu}\,\nabla^\alpha\,h^{\mu\nu} - \nabla_\alpha\,h_{\mu\nu}\,\nabla^\nu\,h^{\mu\alpha} + \nabla_\mu\,h\,\nabla_\nu\,h^{\mu\nu} - \frac{1}{2}\,\nabla_\mu\,h\,\nabla^\mu\,h \\ + \frac{\sigma\,(d-1)}{\ell^2}\big(h_{\mu\nu}\,h^{\mu\nu} - \frac{1}{2}\,h^2\big) - \frac{1}{2}\,m^2\big(h_{\mu\nu}\,h^{\mu\nu} - h^2\big)\Big)\,.\end{aligned} \tag{2.99}$$

As explained in the previous subsection, the massless spin-2 action is derived both by the linearized Einstein action and by the gauge principle. And the the relative factors between two terms in the mass-term is determined by the Fierz-Pauli conditions. After the introduce of spin-2 field as combination of spin-1 and scalar field, Stueckelberg gauge symmetries appear.

$$h_{\mu\nu} \to h_{\mu\nu} + 2\,\nabla_{(\mu}\,A_{\nu)} + \nabla_\mu\,\nabla_\nu\,\phi \tag{2.100}$$

$$\delta\,h_{\mu\nu} = 2\,\nabla_{(\mu}\,\xi_{\nu)}\,, \quad \delta\,A_m = -\xi_\mu + \partial_\mu\,\lambda\,, \quad \delta\,\phi = -\lambda\,, \tag{2.101}$$

Fierz-Pauli massive spin 2 action changed to non-diagonal form.

$$\Big(-\frac{1}{2}\,m^2\,F_{\mu\nu}\,F^{\mu\nu} - \frac{2\,\sigma\,(d-1)}{\ell^2}\,m^2\,A_\mu\,A^\mu - 2\,m^2\,(h_{\mu\nu}\,\nabla^\mu\,A^\nu - h\,\nabla_\mu\,A^\mu)$$
$$-\frac{4\,\sigma\,(d-1)}{\ell^2}\,m^2\,A^\mu\,\nabla_\mu\,\phi - \frac{2\,\sigma\,(d-1)}{\ell^2}\,m^2\,(\partial\phi)^2 - 2\,m^2\,(h_{\mu\nu}\,\nabla^\mu\,\nabla^\nu\,\phi - h\,\Box\,\phi)\Big)$$

The cross term between spin-2 and scalar in last can be eliminated by conformal transformation,

$$h_{\mu\nu} \to h_{\mu\nu} - \frac{2}{d-2}\,m^2\,\phi\,g_{\mu\nu}\,. \tag{2.102}$$

To make kinetic terms of spin-1 and scalar canonical, we must change the normalization of the spin-1 and scalar field as,

$$A_\mu \to \frac{1}{m}\,A_\mu\,, \quad \phi \to \frac{1}{m\,N_0}\,\phi\,, \quad {N_0}^2 = 2\,\left(\frac{d-1}{d-2}\,m^2 - \frac{\sigma\,(d-1)}{\ell^2}\right)\,. \tag{2.103}$$

Then the Stueckelberg gauge transformations are

$$\delta\,h_{\mu\nu} = 2\,\nabla_{(\mu}\,\xi_{\nu)} + \frac{2\,m}{d-2}\,g_{\mu\nu}\,\lambda\,, \quad \delta\,A_\mu = -m\,\xi_\mu + \partial_\mu\,\lambda\,, \quad \delta\,\phi = -N_0\,\lambda\,. \tag{2.104}$$

Stueckelberg spin-1 and scalar is auxiliary. They can be algebraically fixed as zero by gauge transformation, and therefore they have no physical degree of freedom and are redundant. However in special value of mass this argument is not possible. First, Stueckelberg spin-1 field cannot be fixed as zero in the massless limit as one can see in (4.57). The spin-2 Stueckelberg gauge transformation goes to the linear version of general covariance, the spin-1 and scalar change to spin1 Stueckelberg gauge transformation. (This are difference between flat massive spin-2 in (A)dS and flat in massless limt. The massive spin-2 breaks to massless spin-2, massless spin-1 and scalar in flat, but it breaks to massless spin-2 and massive spin-1 in (A)dS with a specific mass. This phenomenon also can be seen in the massless limit of representations on AdS in section 2.2.1.) In a constantly curved space, there are other specific mass. The similar phenomenon can happen at the special value of mass such that $N_0 = 0$. In this limit, Stueckelberg scalar cannot be fixed by gauge symmetry and becomes a physical scalar field. The spin-2 and spin-1 become the Stueckelberg form of "partially massless" spin-2 field. One can check that the mass is equivalent to the mass that we found in previous section,

$$m^2 = \frac{\sigma\,(d-2)}{\ell^2}\,. \tag{2.105}$$

After we fixed the spin-1 field as zero by Stueckelberg gauge,

$$A_\mu = 0 \to \delta\, A_\mu = 0 \to \xi_\mu = \frac{1}{m}\, \partial_\mu\, \lambda \tag{2.106}$$

the remnant gauge transformation of the spin-2 coincide with (2.90).

$$\delta\, h_{\mu\nu} = \frac{2}{m} \left(\nabla_\mu\, \nabla_\nu\, \lambda + \frac{\sigma}{\ell^2}\, g_{\mu\nu}\, \lambda \right) . \tag{2.107}$$

Therefore we could understand that in a partially massless limit, a massive spin-2 field breaks to a partially massless spin-2 and a scalar field.

The Stueckelberg trick can be generalized to higher spins. And the partially massless field with arbitrary depth could be deduced in similar manners. This has done in (A)dS space [168]. The Stueckelberg gauge symmetry naturally arises in the Kaluza-Klein reduction. This has done for flat case in [12]. And we have deduced the higher spin-s Stueckelberg structure on (A)dS by the new Kaluza-Klein method in Section 4.

2.3.3 First Derivative Description in Three Dimension

In three dimensions, any spin-two spectrum can be described in terms of a first-derivative Lagrangian. Again beginning with the massive Lagrangian, we can reformulate the Lagrangian into

$$\mathcal{L}_{\rm FD}(\chi, \tau) = \frac{1}{2}\, \epsilon^{\mu\nu\rho} \left(\tau_\mu{}^\lambda\, \nabla_\nu\, \chi_{\rho\lambda} + \chi_\mu{}^\lambda\, \nabla_\nu\, \tau_{\rho\lambda} \right) + \mu\, \left(\tau_{[\mu}{}^\mu\, \tau_{\nu]}{}^\nu + \frac{\sigma}{\ell^2}\, \chi_{[\mu}{}^\mu\, \chi_{\nu]}{}^\nu \right) , \tag{2.108}$$

by introducing an auxiliary field $\tau_{\mu\nu}$. Here, the tensors $\chi_{\mu\nu}$ and $\tau_{\mu\nu}$ do not have any symmetric properties. By integrating out $\tau_{\mu\nu}$ — that is by plugging in the solution $\tau_{\mu\nu}(\chi)$ of its own equation — one can show that the antisymmetric part of $\tau_{\mu\nu}$ drops and the Lagrangian (2.108) reproduces the Fierz-Pauli Lagrangian up to a factor :

$$\mathcal{L}_{\rm FD}(\chi, \tau(\chi)) = \frac{1}{\mu}\; \mathcal{L}_{\rm FP}(\chi) \, . \tag{2.109}$$

It is more convenient to recast the Lagrangian (2.108) in terms of $\varphi_{\mu\nu}$ and $\tilde\varphi_{\mu\nu}$:

$$\chi_{\mu\nu} = \sqrt{\sigma}\, \ell \left(\varphi_{\mu\nu} - \tilde\varphi_{\mu\nu} \right) , \qquad \tau_{\mu\nu} = \varphi_{\mu\nu} + \tilde\varphi_{\mu\nu} \, . \tag{2.110}$$

so that the massive spin-two Lagrangian splits into the parity breaking spin $+2$ and spin -2 parts:

$$\mathcal{L}_{\rm FD}(\chi, \tau) = \mathcal{L}_{+2}(\varphi) + \mathcal{L}_{-2}(\tilde\varphi) \, . \tag{2.111}$$

Here, the self-dual massive spin ± 2 Lagrangian [143] is given by

$$\mathcal{L}_{\pm 2}(\varphi) = \pm \sqrt{\sigma}\, \ell\, \epsilon^{\mu\nu\rho}\, \varphi_\mu{}^\lambda\, \nabla_\nu\, \varphi_{\rho\lambda} + 2\, \mu\, \varphi_{[\mu}{}^\mu\, \varphi_{\nu]}{}^\nu \, . \tag{2.112}$$

Let us remark an unusual feature of this parity breaking massive spin-two Lagrangian in three dimensions: the sign of the *mass-like* term actually determines whether the Lagrangian is ghost or not (the positive sign for the unitary case and the negative sign for the ghost), whereas the sign of the *kinetic-like* term determines the sign of the spin. For this reason, one can render a unitary spin-two field to a ghost one by only modifying its mass-like term in the first order description of three dimensional theories. Throughout this paper, we encounter three different cases: firstly, the $\mu = 1$ case corresponds to unitary massless spin-two field, whereas the $\mu = -1$ case gives ghost massless spin-two. The case of $\mu = 0$ describes partially-massless spin-two spectrum, which does not admit any two-derivative description as is clear from (2.108) and (2.109).

Chapter 3

Introduction to Higher Spin Gauge Theory II: Frame-like Formalism

Deep in his childish little heart Rumo sensed that,
if he used this silver thread of scent as a guide$\cdots$,
happiness would await him there.
— Walter Moers
RUMO: AND HIS MIRACULOUS ADVENTURES

3.1 Unfolded Equations and Vasiliev System

Even though there are many obstacles and no-go theorems for interacting higher spin gauge theories in flat background, it is generally believed that consistent interacting higher spin gauge theories can be constructed in (A)dS. As an example, Vasiliev system is believed as one of the theories.

3.1.1 Vasiliev system in four dimension

The non-linear equation in Vasiliev system makes use of fields contracted not only with higher spin oscillators $Y = (y, \bar{y})$ but also with auxiliary oscillators $Z = (z, \bar{z})$. More precisely, it consists of two fields: first the one form field $\mathcal{W}$

$$\mathcal{W}(x|Y,Z) = W_\mu(x|Y,Z)\, dx^\mu + V_\alpha(x|Y,Z)\, dz^\alpha + \bar{V}_{\dot{\alpha}}(x|Y,Z)\, d\bar{z}^{\dot{\alpha}}\,, \tag{3.1}$$

and the zero form field $B(x;Y,Z)$. As we will show below, Z independent part of one-form and zero-form are the generating functions for gauge fields and the generalized Weyl tensor. The non-linear equations are

$$\begin{aligned} &\hat{d}\,\mathcal{W} + \mathcal{W} \star \mathcal{W} = \frac{1}{4}\, B \star \left(e^{i\,\theta}\, \varkappa\, dz_\alpha \wedge dz^\alpha + e^{-i\,\theta}\, \bar{\varkappa}\, d\bar{z}_{\dot{\alpha}} \wedge d\bar{z}^{\dot{\alpha}}\right), \\ &\hat{d}\, B + \mathcal{W} \star B - B \star \pi(\mathcal{W}) = 0\,, \end{aligned} \tag{3.2}$$

where exterior derivative contains spacetime derivatives and derivatives with respect to auxiliary oscillators Z,

$$\hat{d} = dx^\mu\, \frac{\partial}{\partial x^\mu} + dz^\alpha\, \frac{\partial}{\partial z^\alpha} + d\bar{z}^{\dot{\alpha}}\, \frac{\partial}{\partial \bar{z}^{\dot{\alpha}}}\,, f \star g = \frac{1}{(2\pi)^4} \int d^4U\, d^4V \tag{3.3}$$

and the star products included the auxiliary Z oscillators are

$$f(Z+U; Y+U)\, g(Z-V; Y+V)\, \exp(i\, u_\alpha\, v^\alpha + i\, \bar{u}_\alpha\, \bar{v}^\alpha)\,. \tag{3.4}$$

The non-linear equation is covariant under the following gauge transformations.

$$\delta\,\mathcal{W} = \hat{d}\,\epsilon + \mathcal{W}\star\epsilon - \epsilon\star\mathcal{W}\,, \quad \delta B = B\star\pi(\epsilon) - \epsilon\star B\,. \tag{3.5}$$

For another form in literatures can be obtained by defining,

$$V_\alpha = \frac{1}{2\,i}\,z_\alpha + S_\alpha\,, \quad \bar{V}_{\dot\alpha} = \frac{1}{2\,i}\,\bar{z}_{\dot\alpha} + \bar{S}_{\dot\alpha}\,, \quad S = S_\alpha\,dz^\alpha + \bar{S}_{\dot\alpha}\,dz^{\dot\alpha} \tag{3.6}$$

$$d\,W + W\star W = 0\,, \quad d\,B + W\star B - B\star\pi(W) = 0\,, \tag{3.7}$$

$$d\,S + W\star S - S\star W = 0\,, \tag{3.8}$$

$$S\star B + B\star\pi(S) = 0\,, \tag{3.9}$$

$$S\star S = d\,z_\alpha\wedge d\,z^\alpha\left(i + B\star\varkappa\,e^{i\,\theta}\right) + d\,\bar{z}_{\dot\alpha}\wedge d\,\bar{z}^{\dot\alpha}\left(i + B\star\bar{\varkappa}\,e^{-i\,\theta}\right)\,, \tag{3.10}$$

The external derivative contains only spacetime derivatives. The equations (3.7) and (3.8) correspond to zero curvature condition for one-form and source free equations for scalar and spinor zero-form. The Vasiliev system (3.2) have the simple solution with the second order HS oscillators.

$$\mathcal{W}_0 = \frac{1}{4i}(\omega_{\alpha\beta}\,y^\alpha\,y^\beta + \bar{\omega}_{\dot\alpha\dot\beta}\,\bar{y}^{\dot\alpha}\,\bar{y}^{\dot\beta}) + \frac{1}{2\,i\,\ell}\,h_{\alpha\dot\beta}\,y^\alpha\,\bar{y}^{\dot\beta}\,, \quad B = 0\,. \tag{3.11}$$

Each component is a spacetime one-from and satisfies following vacuum equations.

$$d\,h_{\alpha\dot\beta} + \omega_{\alpha\gamma}\wedge\,h^\gamma{}_{\dot\beta} + \bar{\omega}_{\dot\beta\dot\gamma}\wedge\,h_\alpha{}^{\dot\gamma} = 0\,, \tag{3.12}$$

$$d\,\omega_{\alpha\beta} + \omega_{\alpha\gamma}\wedge\omega^\gamma{}_\beta + \frac{1}{\ell^2}\,h_{\alpha\dot\gamma}\wedge\,h_\beta{}^{\dot\gamma} = 0\,. \tag{3.13}$$

And there is a similar equation for $\bar{\omega}$ quantity. The one-forms h, ω and $\bar{\omega}$ could be interpreted as vielbein and spin connection. Then the first and the second equation correspond to the torsionless condition and (A)dS condition. Therefore this solution can be interpreted as (A)dS vacuum.

Let us consider the linear fluctuation. The unfolded equations for free higher spins are derived from non-linear Vasiliev system (3.2) as in [32] –see more recent reviews [33]. The similar procedure will be repeated for different vacua in this paper. The linear fluctuations around (A)dS vacuum are

$$W = \frac{1}{4\,i}\left(\omega_{\alpha\beta}\,y^\alpha\,y^\beta + \omega_{\dot\alpha\dot\beta}\,\bar{y}^{\dot\alpha}\,\bar{y}^{\dot\beta} + \frac{2}{\ell}\,h_{\alpha\dot\alpha}\,y^\alpha\,\bar{y}^{\dot\alpha}\right) + \lambda\,W_1 + \mathcal{O}(\lambda^2)\,, \tag{3.14}$$

$$S_\alpha = \frac{i}{2}\,z_\alpha + \lambda\,s_\alpha + \mathcal{O}(\lambda^2)\,, \qquad \bar{S}_{\dot\alpha} = \frac{i}{2}\,\bar{z}_{\dot\alpha} + \lambda\,\bar{s}_{\dot\alpha} + \mathcal{O}(\lambda^2)\,, \tag{3.15}$$

$$B = \lambda\,B_1 + \mathcal{O}(\lambda^2)\,. \tag{3.16}$$

It is convenient to solve algebraic equations without spacetime derivatives (3.9) and (3.10) first. They contain differentiations with respect to auxiliary oscillator z or $\bar{z}$. From the equation (3.9), we can conclude that first order of zero-form is independent of z and $\bar{z}$.

$$\frac{\partial B_1}{\partial z^\alpha} = \frac{\partial B_1}{\partial \bar{z}^{\dot\alpha}} = 0\,, \quad \therefore B_1(x|Y,Z) = C(x|Y) \tag{3.17}$$

The integrate constant $C(x|Y)$ is a generating function of the generalized Weyl tensors. The equation (3.10) gives the differential equation of s and $\bar{s}$. Spinor fields s is fixed as function of Weyl tensors by solving this equation. [1]

$$\frac{\partial s^\alpha}{\partial z^\alpha} = \frac{1}{2}\, C \star \varkappa\,, \quad \therefore s_\alpha = \frac{z_\alpha}{2}\int_0^1 dt\, t\, C(-t\,z,\bar{y})\, e^{i\,t\,z_\beta\, y^\beta + i\,\theta_0}\,, \tag{3.19}$$

Even after adding $\frac{\partial\,\epsilon}{\partial\, z^\alpha}$, s_α is also solution of the given differential equation. Therefore, we can include the general solution of the corresponding homogeneous equation to s_α. This ambiguity represents the existence of gauge symmetry which depends on auxiliary oscillator Z. This ambiguity was fixed as zero by the Z dependent gauge transformation in (3.19). The complex conjugate part for $\bar{s}_{\dot\alpha}$ also can be solved in a similar way. Because $s(\bar{s})$ is independent of $\bar{z}(z)$, $d\,z \wedge d\,\bar{z}$ part of (3.10) is trivial. (3.8) are differential equations of one-form field with respect to auxiliary oscillators z and $\bar{z}$. The one-form field can be written as a function of Weyl tensors through these differential equations.

$$\frac{\partial W_1}{\partial z^\alpha} = -[W_0, s_\alpha] - d\,s_\alpha\,, \quad \frac{\partial W_1}{\partial \bar{z}^{\dot\alpha}} = -[W_0, \bar{s}_{\dot\alpha}] - d\,\bar{s}_{\dot\alpha}\,, \tag{3.20}$$

Because we are dealing with differentiation with respect Z, the integrate constants are arbitrary functions of higher spin oscillator Y. The one-form field is the sum of special solution[2]and integrate constant. i.e. $W_1(x|Y,Z) = A(x|Y) + R(x|Y,Z)$. The special soultion is

$$R = \frac{1}{2}\int_0^1 d\tau \int_0^1 dt\,\tau\left(\tau\, t\,\omega_{\alpha\beta}\, z^\alpha\, z^\beta + \frac{i}{\ell}\, h_{\alpha\dot\beta}\, z^\alpha\, \bar{\partial}^{\dot\beta}\right) C(-t\,\tau\, z,\, \bar{y})\, e^{i\,t\,\tau\, z_\alpha\, y^\alpha + i\,\theta_0} + c.c.\,. \tag{3.22}$$

The integrate constant $A(x|Y)$ corresponds to the generating function for higher spin fields in frame-like formalism. Finally we can get the unfolded equation for free higher spin fields from the Z independent part of the zero curvature equation and the source free equation in (3.7).

$$\mathcal{D}_\Omega\ A(x|Y) = \frac{1}{2\,\ell^2}\, h^{\alpha\dot\alpha} \wedge h_\alpha{}^{\dot\beta}\, \frac{\partial^2\, C(x|0,\bar{y})}{\partial\bar{y}^{\dot\alpha}\,\partial\bar{y}^{\dot\beta}}\, e^{-i\,\theta_0} + c.c. \tag{3.23}$$

$$\tilde{\mathcal{D}}_\Omega\, C(x|Y) = 0\,. \tag{3.24}$$

---

[1]This type of differential equation has a following special solution.

$$\frac{\partial\, f^\alpha(z)}{\partial\, z^\alpha} = s(z) \quad \Rightarrow \quad f_\alpha(z) = \int_0^1 dt\, z_\alpha\, s(t\,z) + \frac{\partial\,\epsilon}{\partial\, z_\alpha}\,. \tag{3.18}$$

[2]This type of differential equation has a following special solution.

$$\frac{\partial\, f(z)}{\partial\, z^\alpha} = s_\alpha(z) \quad \Rightarrow \quad f(z) = \int_0^1 dt\, t\, z^\alpha\, s_\alpha(t\,z) + const.\,. \tag{3.21}$$

Here, $\tilde{\mathcal{D}}(\mathcal{D})$ is the covariant derivative of (un)twisted sector.

$$\tilde{\mathcal{D}}_\Omega = D^L - \frac{i}{\ell}\, h^{\alpha\dot\alpha}\left(y_\alpha \bar y_{\dot\alpha} - \partial_\alpha \bar\partial_{\dot\alpha}\right) = D^L + \frac{1}{\ell}\left(\sigma_{++} + \sigma_{--}\right)\,, \tag{3.25}$$

$$\mathcal{D}_\Omega = D^L + \frac{1}{\ell}\, h^{\alpha\dot\alpha}\left(y_\alpha \bar\partial_{\dot\alpha} + \bar y_{\dot\alpha}\partial_\alpha\right) = D^L + \frac{1}{\ell}\left(\sigma_{+-} + \sigma_{-+}\right)\,, \tag{3.26}$$

where D^L is the covariant derivative of vacum AdS metric. i.e. $D^L = d + \omega_{\alpha\beta}\, y^\alpha\, \partial^\beta + \omega_{\dot\alpha\dot\beta}\bar y_{\dot\alpha}\bar\partial_{\dot\beta}$ and we defined two-form-valued operators: $\sigma_{++} = e^{\alpha\dot\beta}\, y_\alpha\, \bar y_{\dot\beta}$, $\sigma_{--} = e^{\alpha\dot\beta}\, \partial_\alpha\, \bar\partial_{\dot\beta}$ and $\sigma_{+-} = e^{\alpha\dot\beta}\, y_\alpha\, \bar\partial_{\dot\beta}$. The untwisted covariant derivative does not mix the fields component whose have different Y total

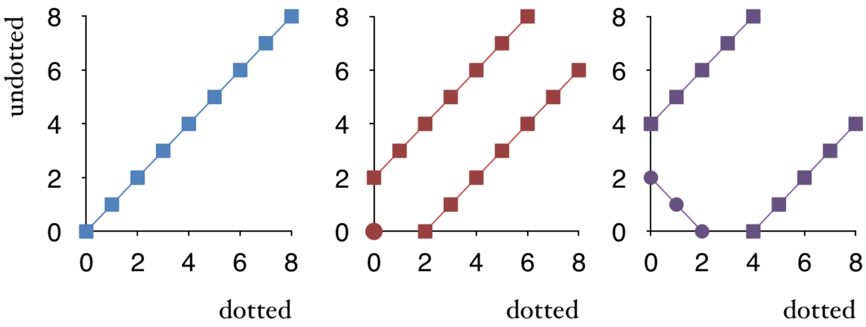

Figure 6: Figures for fields mixing through equations (3.23, 3.24). Each axis represents the number of dotted and un-dotted indices that fields carry. A square(disk) is a maker for zero-form field $C^{[n,\bar n]}$(one-form field $A^{[n,\bar n]}$).

numbers. On the other side, the fields in twisted representation only communicate with each other only when the differences between number of y and number of $\bar y$ coincide The connected pattern between zero-form and one-form in Fig.6. Infinitely many zero-forms are connected by unfolded equations, and only finitely many one-forms are connected. This connection pattern comes from the form of covariant derivatives. The equation (3.23) makes the relation between one-form and zero-form. This equation implies that some parts of $C_{\alpha(n),\dot\alpha(m)}$s are the generalized Weyl tensor for spin $|n-m|/2$. And the physical degree of freedom is in that generalized Weyl tensor.

Before conclude this subsection, let us comment about shorthand notations. p-form can be written as,

$$\eta(Y) = \sum_{n,\,\bar n=0}^{\infty} \eta^{\alpha_1\cdots\alpha_n\,\dot\alpha_1\cdots\dot\alpha_{\bar n}}\, y_{\alpha_1}\cdots y_{\alpha_n}\,\bar y_{\dot\alpha_1}\cdots\bar y_{\dot\alpha_{\bar n}} \tag{3.27}$$

$$= \sum_{n,\,\bar n} \eta^{\alpha(n)\,\dot\alpha(\bar n)}\,(y_\alpha)^n\,(\bar y_{\dot\alpha})^{\bar n} = \sum_{n,\,\bar n} \eta^{[n,\,\bar n]}\,, \tag{3.28}$$

Einstein conventions are used for each α_i and $\dot\alpha_i$ and $\alpha(n)$ and $\dot\alpha(\bar n)$ are used to express symmetric indices — $(\alpha_1\,\alpha_2\cdots\alpha_n)$ and $(\dot\alpha_1\,\dot\alpha_2\cdots\dot\alpha_{\bar n})$ — in second line. $\eta^{[n,\,\bar n]}$ will be frequently used to indicate the specific terms of η which have the order n for y and the order $\bar n$ for $\bar y$.

3.2 Analysis of Unfolded Equations

To systematically analyse the unfolded equations in (3.23, 3.24), we must inquire into not only equations of motions but also the gauge symmetries and Bianchi identities. Let us first introduce the quantities that are related with them. The equation in (3.23) and (3.24) can be written as,

$$\begin{aligned} F &\equiv \mathcal{D}_\Omega\, A = D^L\, A + \frac{1}{\ell}\,(\sigma_{+-}\, A + \sigma_{-+}\, A) \simeq e^{i\,\theta}\, \bar{E}_{\dot\alpha\dot\beta}\, \bar\partial^{\dot\alpha}\, \bar\partial^{\dot\beta}\, C(0,\bar y) - c.c\,, \quad (3.29)\\ f &\equiv \tilde{\mathcal{D}}_\Omega\, C = D^L\, C + \frac{1}{\ell}\,(\sigma_{++}\, C + \sigma_{--}\, C) \simeq 0\,, \quad (3.30)\end{aligned}$$

where $E^{\alpha\beta}$ and $\bar{E}^{\dot\alpha\dot\beta}$ are given as,

$$E^{\alpha\beta} = \frac{1}{2}\, e^{\alpha}{}_{\dot\gamma} \wedge e^{\beta\dot\gamma}\,, \quad \bar{E}^{\dot\alpha\dot\beta} = \frac{1}{2}\, e_{\gamma}{}^{\dot\alpha} \wedge e^{\gamma\dot\beta}\,, \quad (3.31)$$

and we use the symbol $\simeq$ to emphasise that the equality is an on-shell equation. Because of the nilpotency of $\mathcal{D}_\Omega$ and $\tilde{\mathcal{D}}_\Omega$, the quantities F in (3.29) and f in (3.30) satisfy the following Bianchi identities.

$$\begin{aligned} B &\equiv \mathcal{D}_\Omega\, F = D^L\, F + \frac{1}{\ell}\,(\sigma_{+-}\, F + \sigma_{-+}\, F) = 0\,, \quad (3.32)\\ b &\equiv \tilde{\mathcal{D}}_\Omega\, f = D^L\, f + \frac{1}{\ell}\,(\sigma_{++}\, f + \sigma_{--}\, f) = 0\,. \quad (3.33)\end{aligned}$$

Above equality is expressed by $=$ and not by $\simeq$ which means that it is an off-shell equality.

Because the Bianchi identities relate components of equations in (3.29) and (3.30), we do not have to impose the on-shell conditions for all components F and f: there exist a subset of equations whose element equal to zero means the on-shell condition (3.29) and (3.30). And some equations in (3.29) and (3.30) just imply that some components of fields $A^{[n,\,\bar n]}$ or $C^{[n,\,\bar n]}$ are actually auxiliary fields because they can be expressed by other fields: there exist a minimal set of fields which have the physical degree of freedom. And for one-form field, there are gauge symmetries:

$$\delta\, A \equiv D_\Omega\, \epsilon = D^L\, \epsilon + \frac{1}{\ell}\,(\sigma_{+-}\, \epsilon + \sigma_{-+}\, \epsilon)\ , \quad (3.34)$$

the equation (3.29) is invariant because of nilpotency of $\mathcal{D}_\Omega$. Through the terms which contain $\sigma_{\pm\mp}$ in gauge transformation (3.34), gauge parameters $\epsilon^{[n\mp 1,\,\bar n\pm 1]}$ can be used to algebraically fix some components of one-form fields $A^{[n,\,\bar n]}$. These components of fields can be considered as Stückelberg fields.

The purposes of the section are to clarify which components of field are auxiliary and Stückelberg fields and what is the minimal set of equations that imply the full component of equation (3.29) (3.30). To reach that purpose we start with the irreducible decompositions of differential forms.

3.2.1 Decomposition of Form into Irreducible Components

The p-form with $p \neq 0$ can be considered as a reducible tensor, because they have the space-time indices which can be changed to spinor indices by the background (A)dS vielbein. For example one-form fields can be written as,

$$A^{\beta(n)\dot{\beta}(m)} = A_\mu{}^{\beta(n)\dot{\beta}(\bar{n})}\, dx^\mu = e_{\alpha\dot{\alpha}}\, A^{\alpha\dot{\alpha}|\beta(n)\dot{\beta}(\bar{n})}\,. \tag{3.35}$$

Th spinor-indices α and $\dot{\alpha}$ which originate from a space-time index have no symmetric or anti-symmetric relation with indices $\beta(n)$ and $\dot{\beta}(\bar{n})$. Therefore, $A^{\alpha\dot{\alpha}|\beta(n)\dot{\beta}(\bar{n})}$ — which can be considered as the one-form — can divide into parts with symmetric indices and anti-symmetric indices which means that the one-form can be considered as a reducible tensor. We will review the decomposition of p-form and use results — (3.51), (3.52) and (3.53) — in the later parts.

Decomposition of Forms

The one-form in the last expression (3.35) can be decomposed into the symmetric and anti-symmetric parts under the exchanging of α and β, and each part also divides into two parts by the symmetric properties of dotted indices:

$$\begin{aligned} A^{\alpha\dot{\alpha}|\beta(n)\dot{\beta}(\bar{n})} = \quad & \mathcal{A}_{++}{}^{\left(\alpha\beta(n)\right)\,\left(\dot{\alpha}\dot{\beta}(\bar{n})\right)} + \epsilon^{\alpha\beta}\,\bar{\epsilon}^{\dot{\alpha}\dot{\beta}}\;\mathcal{A}_{--}{}^{\beta(n-1)\,\dot{\beta}(\bar{n}-1)} \\ & +\epsilon^{\alpha\beta}\;\mathcal{A}_{-+}{}^{\beta(n-1)\,\left(\dot{\alpha}\dot{\beta}(\bar{n})\right)} + \bar{\epsilon}^{\dot{\alpha}\dot{\beta}}\;\mathcal{A}_{+-}{}^{\left(\alpha\beta(n)\right)\,\dot{\alpha}(\bar{n}-1)}\,. \end{aligned} \tag{3.36}$$

where indices in the parenthesis are symmetric. A three-form is a Hodge-dual to the one-form in four dimension and has an equal number of irreducible components with the one-form. This can be explicitly shown by writing down the three-form $B(Y)$ as the following form,

$$B(Y) = B_{\mu\nu\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\rho = B_{\alpha\dot{\beta}}\,\tau^{\alpha\dot{\beta}}\,, \tag{3.37}$$

where we define the three-form-valued tensors,

$$\tau^{\alpha\dot{\beta}} \equiv \frac{1}{4}\, e^\alpha{}_{\dot{\gamma}} \wedge e_\gamma{}^{\dot{\gamma}} \wedge e^{\gamma\dot{\beta}} = \frac{1}{2}\, E^\alpha{}_\gamma \wedge e^{\gamma\dot{\beta}} = \frac{1}{2}\, \bar{E}^{\dot{\beta}\dot{\gamma}} \wedge e^\alpha{}_{\dot{\gamma}}\,. \tag{3.38}$$

The three-form also decompose into irreducible parts in the same way of one-form:

$$\begin{aligned} B^{\alpha\dot{\alpha}|\beta(n)\dot{\beta}(\bar{n})} = \quad & \mathcal{B}_{++}{}^{\left(\alpha\beta(n)\right)\,\left(\dot{\alpha}\dot{\beta}(\bar{n})\right)} + \epsilon^{\alpha\beta}\,\bar{\epsilon}^{\dot{\alpha}\dot{\beta}}\;\mathcal{B}_{--}{}^{\beta(n-1)\,\dot{\beta}(\bar{n}-1)} \\ & +\epsilon^{\alpha\beta}\;\mathcal{B}_{-+}{}^{\beta(n-1)\,\left(\dot{\alpha}\dot{\beta}(\bar{n})\right)} + \bar{\epsilon}^{\dot{\alpha}\dot{\beta}}\;\mathcal{B}_{+-}{}^{\left(\alpha\beta(n)\right)\,\dot{\alpha}(\bar{n}-1)}\,. \end{aligned} \tag{3.39}$$

The two-form can be expressed by the chiral and the anti-chiral fields with two symmetric indices,

$$F(Y) = E^{\alpha\beta}\, F_{\alpha\beta}(Y) + \bar{E}^{\dot{\alpha}\dot{\beta}}\, \tilde{F}_{\dot{\alpha}\dot{\beta}}(Y)\,, \tag{3.40}$$

where the indices α and β(or $\dot{\alpha}$ and $\dot{\beta}$) are symmetric and $E^{\alpha\beta}$ and $\bar{E}^{\dot{\alpha}\dot{\beta}}$ are given in (3.31). The chiral field $F_{\alpha\beta}(Y)$ and its complex-conjugate-counter part decompose into irreducible parts by the symmetric properties of indices as similar to previous cases.

$$F^{\alpha\beta|\gamma(n)\dot\gamma(m)} = \mathcal{F}_{+2}{}^{\left(\alpha\beta\gamma(n)\right)\dot\gamma(m)} + \epsilon^{\gamma(\alpha}\ \mathcal{F}_0{}^{\beta)\gamma(n-1)\dot\gamma(m)} + \epsilon^{\gamma\alpha}\,\epsilon^{\gamma\beta}\ \mathcal{F}_{-2}{}^{\gamma(n-2)\dot\gamma(m)}\,. \tag{3.41}$$

All chiral spinor indices in the parenthesis are symmetric.

Because the irreducible components $\mathcal{A}_{\pm\pm}$, $\mathcal{B}_{\pm\pm}$ and $\mathcal{F}_{\pm 2,\ \text{and}\ 0}$ have only symmetric chiral and anti-chiral indices, we can define the following Y-polynomials:

$$\mathcal{A}_{\pm\pm}(Y) \equiv \mathcal{A}_{\pm\pm}{}^{\alpha(n)\,\dot\alpha(\bar n)}\,(y_\alpha)^n\,(\bar y_\alpha)^{\bar n}\,, \tag{3.42}$$

$$\mathcal{B}_{\pm\pm}(Y) \equiv \mathcal{B}_{\pm\pm}{}^{\alpha(n)\,\dot\alpha(\bar n)}\,(y_\alpha)^n\,(\bar y_\alpha)^{\bar n}\,, \tag{3.43}$$

$$\mathcal{F}_{\pm 2,\ \text{and}\ 0}(Y) \equiv \mathcal{F}_{\pm 2,\ \text{and}\ 0}{}^{\alpha(n)\,\dot\alpha(\bar n)}\,(y_\alpha)^n\,(\bar y_\alpha)^{\bar n}\,. \tag{3.44}$$

The decompositions of forms are nothing but imposing the relations between forms and the polynomials of irreducible components[3]:

$$A_{\alpha\dot\alpha}(Y) = \frac{\partial^2\ \mathcal{A}_{++}}{\partial\,y^\alpha\,\partial\,\bar y^{\dot\alpha}}(Y) + \bar y_{\dot\alpha}\,\frac{\partial\ \mathcal{A}_{+-}}{\partial\,y^\alpha}(Y) + y_\alpha\,\frac{\partial\ \mathcal{A}_{-+}}{\partial\,\bar y^{\dot\alpha}}(Y) + y_\alpha\,\bar y_{\dot\alpha}\ \mathcal{A}_{--}(Y)\,, \tag{3.45}$$

$$F_{\alpha\beta}(Y) = \frac{\partial^2\ \mathcal{F}_{+2}}{\partial\,y^\alpha\,\partial\,y^\beta}(Y) + \frac{1}{2}\left(y_\alpha\,\frac{\partial\ \mathcal{F}_0}{\partial\,y^\beta} + y_\beta\,\frac{\partial\ \mathcal{F}_0}{\partial\,y^\alpha}\right)(Y) + y_\alpha\,y_\beta\ \mathcal{F}_{-2}(Y)\,. \tag{3.46}$$

The tree-form decomposition is written in the similar way with the one-form.

Let us define the following form-valued operators,

$$\sigma_{++} = e^{\alpha\dot\beta}\,y_\alpha\,\bar y_{\dot\beta}\,,\qquad \sigma_{--} = e^{\alpha\dot\beta}\,\partial_\alpha\,\bar\partial_{\dot\beta}\,,\qquad \sigma_{+-} = e^{\alpha\dot\beta}\,y_\alpha\,\bar\partial_{\dot\beta}\,, \tag{3.47}$$

$$\Sigma_{+2} = E^{\alpha\beta}\,y_\alpha\,y_\beta\,,\qquad \Sigma_0 = E^{\alpha\beta}\,y_\alpha\,\partial_\beta\,,\qquad \Sigma_{-2} = E^{\alpha\beta}\,\partial_\alpha\,\partial_\beta\,, \tag{3.48}$$

$$\tau_{++} = \tau^{\alpha\dot\beta}\,y_\alpha\,\bar y_{\dot\beta}\,,\qquad \tau_{--} = \tau^{\alpha\dot\beta}\,\partial_\alpha\,\bar\partial_{\dot\beta}\,,\qquad \tau_{+-} = \tau^{\alpha\dot\beta}\,y_\alpha\,\bar\partial_{\dot\beta}\,, \tag{3.49}$$

and we define $\bar\Sigma_{\pm 2/0}$ as the complex conjugate of $\Sigma_{\pm 2/0}$. By the conjugation relation of oscillators and the definition of $E_{\alpha\beta}$ and $\tau_{\alpha\dot\beta}$, we can show that

$$(\sigma_{ij})^\dagger = \sigma_{ji}\,,\quad (\Sigma_I)^\dagger = \bar\Sigma_I\,,\quad (\tau_{ij})^\dagger = -\tau_{ji}\,. \tag{3.50}$$

It is good place to comment about the reality conditions of the form-valued quantities we consider. Because the reality condition of one-form fields, one-form field, two-form equation and three-form Bianchi identity are pure imaginary: $A^\dagger = -A$, $F^\dagger = -F$ and $B^\dagger = -B$. By using the form-valued operators and the reality conditions, the decomposition equality (3.45) can be written as the

[3]The definitions of irreducible components in (3.36, 3.39, 3.41) and (3.45, 3.46) are different by factors that depend on the orders of y and $\bar y$. We ignored these discrepancies for simple expressions in (3.36, 3.39, 3.41). The definitions of the irreducible components in (3.45, 3.46) are used throughout the paper.

form-valued equality:

$$A = \quad \sigma_{--}\;\mathcal{A}_{++} + \sigma_{++}\;\mathcal{A}_{--} + \sigma_{+-}\;\mathcal{A}_{-+} + \sigma_{-+}\;\mathcal{A}_{+-}\,, \tag{3.51}$$

$$F = \quad \Sigma_{+2}\;\mathcal{F}_{-2} + \Sigma_0\;\mathcal{F}_0 + \Sigma_{-2}\;\mathcal{F}_{+2} - \bar{\Sigma}_{+2}\,\bar{\mathcal{F}}_{-2} - \bar{\Sigma}_0\,\bar{\mathcal{F}}_0 - \bar{\Sigma}_{-2}\,\bar{\mathcal{F}}_{+2}\,, \tag{3.52}$$

$$B = \quad \tau_{--}\;\mathcal{B}_{++} + \tau_{++}\;\mathcal{B}_{--} + \tau_{+-}\;\mathcal{B}_{-+} + \tau_{-+}\;\mathcal{B}_{+-}\,, \tag{3.53}$$

where $\bar{\mathcal{F}}_I = \mathcal{F}_I{}^\dagger$ and $\mathcal{A}_{ij}{}^\dagger = -\,\mathcal{A}_{ji}$ and $\mathcal{B}_{ij}{}^\dagger = \mathcal{B}_{ji}$.

Relations of Form Operators

The relations between form-operators in (3.47), (3.48) and (3.49) are useful to decompose the equations and Bianchi identity. The two-form identities are,

$$\begin{aligned}
\sigma_{++}\,\sigma_{--} &= -\Sigma_0\,\bar{N} - \bar{\Sigma}_0\,N\,, & \sigma_{--}\,\sigma_{++} &= \Sigma_0\,\big(2+\bar{N}\big) + \bar{\Sigma}_0\,(2+N)\,,\\
\sigma_{++}\,\sigma_{+-} &= -\Sigma_{+2}\,\bar{N}\,, & \sigma_{--}\,\sigma_{+-} &= \bar{\Sigma}_{-2}\,(2+N)\,,\\
\sigma_{+-}\,\sigma_{++} &= \Sigma_{+2}\,(2+\bar{N})\,, & \sigma_{+-}\,\sigma_{--} &= -\bar{\Sigma}_{-2}\,N\,,\\
\sigma_{-+}\,\sigma_{+-} &= -\Sigma_0\,\bar{N} + \bar{\Sigma}_0\,(2+N)\,, & \sigma_{+-}\,\sigma_{-+} &= \Sigma_0\,(2+\bar{N}) - \bar{\Sigma}_0\,N\,,
\end{aligned} \tag{3.54}$$

where $N = y^\alpha\,\partial_\alpha$ and $\bar{N} = \bar{y}^{\dot\alpha}\,\partial_{\dot\alpha}$ are the number operators for each spinors and $\sigma_{ij}^2 = 0$. And the three-form identities are,

$$\begin{aligned}
\Sigma_{+2}\,\sigma_{-\pm} &= -\tau_{+\pm}\,N\,, & \sigma_{-\pm}\,\Sigma_{+2} &= -\tau_{+\pm}\,(N+3)\,,\\
\Sigma_0\,\sigma_{+\pm} &= \frac{1}{2}\,\tau_{+\pm}\,(N+3)\,, & \sigma_{+\pm}\,\Sigma_0 &= \frac{1}{2}\,\tau_{+\pm}\,N\,,\\
\Sigma_0\,\sigma_{-\pm} &= -\frac{1}{2}\,\tau_{-\pm}\,(N-1)\,, & \sigma_{-\pm}\,\Sigma_0 &= -\frac{1}{2}\,\tau_{-\pm}\,(N+2)\,,\\
\Sigma_{-2}\,\sigma_{+\pm} &= \tau_{-\pm}\,(N+2)\,, & \sigma_{+\pm}\,\Sigma_{-2} &= \tau_{-\pm}\,(N-1)\,,
\end{aligned} \tag{3.55}$$

and $\Sigma_{+2}\,\sigma_{+\pm} = \sigma_{+\pm}\,\Sigma_{+2} = \Sigma_{-2}\,\sigma_{-\pm} = \sigma_{-\pm}\,\Sigma_{-2} = 0$. The three-form relations which contain $\bar{\Sigma}_{\pm 2/0}$ are given as the complex-conjugate of (3.55).

3.2.2 Analysis of Equation: One-form Sector

We will apply the decomposition results to Bianchi identity and the equations to analysis the one-form equation (3.23). By using Bianchi identities the minimal set of equations which imply equation (3.23) is obtained — (3.64) – (3.69) — and we conclude that they describe massless higher-spin.

Minimal Set of Equations

Let us decompose the Bianchi identity into irreducible components. Lorentz derivative part in Bianchi Identity can be changed to as follows,

$$D^L\,F = D^L\,\big(\Sigma_I\;\mathcal{F}_{-I} - \bar{\Sigma}_I\,\bar{\mathcal{F}}_{-I}\big) = \Sigma_I\,D^L\;\mathcal{F}_{-I} - \bar{\Sigma}_I\,D^L\,\bar{\mathcal{F}}_{-I}\,. \tag{3.56}$$

We used the Einstein-convention for $I = \pm 2,\, 0$. The Lorentz covariant derivative in the last expression also decompose into irreducible parts as one form (3.51):

$$D^L = \sigma_{++}\, D^L_{--} + \sigma_{+-}\, D^L_{-+} + \sigma_{-+}\, D^L_{+-} + \sigma_{--}\, D^L_{++} \,. \tag{3.57}$$

By applying formulae (3.55) which relate $\sigma_{ij}\, \Sigma_I$ with τ_{ij}, we can obtain the irreducible components of Bianchi identity in (3.53). The components which contain the equations of fields $A^{[n\pm1,\, \bar{n}\pm1]}$ are[4],

$$\begin{aligned}
(\mathcal{B}_{++})^{[n+1,\, \bar{n}+1]} &= (n+3)\, D^L_{-+}\, (\mathcal{F}_{+2})^{[n+2,\, \bar{n}]} - \frac{1}{2}\, n\, D^L_{++}\, (\mathcal{F}_0)^{[n,\, \bar{n}]} \\
&\quad + \frac{n}{\ell}\, (\mathcal{F}_{+2})^{[n+1,\, \bar{n}+1]} + (\text{terms from c.c}) = 0\,, \qquad (3.59) \\
(\mathcal{B}_{--})^{[n-1,\, \bar{n}-1]} &= -(n-1)\, D^L_{+-}\, (\mathcal{F}_{-2})^{[n-2,\, \bar{n}]} + \frac{n+2}{2}\, D^L_{--}\, (\mathcal{F}_0)^{[n,\, \bar{n}]} \\
&\quad - \frac{n+2}{\ell}\, (\mathcal{F}_{-2})^{[n-1,\, \bar{n}-1]} + (\text{terms from c.c}) = 0\,, \qquad (3.60) \\
(\mathcal{B}_{-+})^{[n-1,\, \bar{n}+1]} &= -(n-1)\, D^L_{++}\, (\mathcal{F}_{-2})^{[n-2,\, \bar{n}]} + \frac{n+2}{2}\, D^L_{-+}\, (\mathcal{F}_0)^{[n,\, \bar{n}]} \\
&\quad + (\bar{n}+3)\, D^L_{--}\, \big(\bar{\mathcal{F}}_{+2}\big)^{[n,\, \bar{n}+2]} - \frac{\bar{n}}{2}\, D^L_{-+}\, \big(\bar{\mathcal{F}}_0\big)^{[n,\, \bar{n}]} \\
&\quad + \frac{\bar{n}+3}{2\,\ell}\, \big(\bar{\mathcal{F}}_0\big)^{[n-1,\, \bar{n}+1]} - \frac{n-1}{2\,\ell}\, (\mathcal{F}_0)^{[n-1,\, \bar{n}+1]} = 0\,, \qquad (3.61)
\end{aligned}$$

where (the terms from c.c) can be obtained by changing as,

$$n \leftrightarrow \bar{n}\,, \quad D^L_{ij} \leftrightarrow D^L_{ji}\,, \quad (\mathcal{F}_{\pm2})^{[n+2,\, \bar{n}]} \leftrightarrow \big(\bar{\mathcal{F}}_{\pm2}\big)^{[n,\, \bar{n}+2]}\,, \quad (\mathcal{F}_0)^{[n,\, \bar{n}]} \leftrightarrow \big(\bar{\mathcal{F}}_0\big)^{[n,\, \bar{n}]}\,. \tag{3.62}$$

Note that these parts are not complex conjugate of the former parts. And other component $\mathcal{B}_{+-}$ can be obtained from (3.61) by changing as (3.62).

Let us inspect components of Bianchi identities to find the minimal set of equations. It is convenient to divide the infinite set of equations into following sets of equations,

$$\mathcal{S}^{[n,\, \bar{n}]} \equiv \Big\{ (\mathcal{F}_{+2})^{[n+2,\, \bar{n}]}\,, (\mathcal{F}_0)^{[n,\, \bar{n}]}\,, (\mathcal{F}_{-2})^{[n-2,\, \bar{n}]}\,, \big(\bar{\mathcal{F}}_{+2}\big)^{[n,\, \bar{n}+2]}\,, \big(\bar{\mathcal{F}}_0\big)^{[n,\, \bar{n}]}\,, \big(\bar{\mathcal{F}}_{-2}\big)^{[n,\, \bar{n}-2]} \Big\}\,,$$

whose elements are described by fields $A^{[n\pm1,\, \bar{n}\mp1]}$. If all elements in $\mathcal{S}^{[n-1,\, \bar{n}+1]}$ and $\mathcal{S}^{[n,\, \bar{n}]}$ are zero on-shell, Bianchi identities (3.59), (3.60) and the complex conjugate part of (3.61) imply the following on-shell conditions,

$$\big(\bar{\mathcal{F}}_{+2}\big)^{[n+3,\, \bar{n}-1]} \simeq 0\,, \quad (\mathcal{F}_0)^{[n+1,\, \bar{n}-1]} \simeq 0\,, \quad (\mathcal{F}_{-2})^{[n-1,\, \bar{n}-1]} \simeq 0\,. \tag{3.63}$$

---

[4]It is worth to clarify the superscript-notations of the irreducible components. We consider irreducible components as the polynomial of the oscillators and the superscript express the orders of these polynomials. For example, one-form with n order of y oscillator and $\bar{n}$ order of $\bar{y}$ decomposed as,

$$A^{[n,\, \bar{n}]} = \sigma_{--}\, \mathcal{A}^{[n+1,\, \bar{n}+1]}_{++} + \sigma_{++}\, \mathcal{A}^{[n-1,\, \bar{n}-1]}_{--} + \sigma_{+-}\, \mathcal{A}^{[n-1,\, \bar{n}+1]}_{-+} + \sigma_{-+}\, \mathcal{A}^{[n+1,\, \bar{n}-1]}_{+-}\,. \tag{3.58}$$

And the similar notations are used for other irreducible components of forms.

The minimal set of equations can be found by repeatedly applying the above arguments.

We first focus on the case $\bar{n}+n = 2\,(s-1) > 0$. To derive usual Fronsdal equation for higher-spin, it is better to assume that quantities in $S^{[s-1,s-1]}$ is zero. After that, by the inductive procedure as described in table 1, we can conclude that the whole set of equations reduces to the following on-shell

$k=$	0	1	2	$\cdots$	s-3	s-2	s-1	s	s+1	$\cdots$	2 s-4	2 s-3	2 s-2
$(\mathcal{F}_{+2})$	${}^{1}_{++}$	${}^{2}_{++}$	${}^{3}_{++}$	$\cdots$	${}^{s-2}_{++}$	$\circledcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\cdots$	$\bigcirc$	$\bigcirc$	$\bigcirc$
$(\bar{\mathcal{F}}_{+2})$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\cdots$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\circledcirc$	${}^{s}_{++}$	$\cdots$	${}^{2s-5}_{++}$	${}^{2s-4}_{++}$	${}^{2s-3}_{++}$
$(\mathcal{F}_{0})$		$\bigcirc$	$\bigcirc$	$\cdots$	$\bigcirc$	$\bigcirc$	$\bigcirc$	${}^{s-1}_{+-}$	${}^{s}_{+-}$	$\cdots$	${}^{2s-5}_{+-}$	${}^{2s-4}_{+-}$	${}^{2s-3}_{+-}$
$(\bar{\mathcal{F}}_{0})$	${}^{1}_{-+}$	${}^{2}_{-+}$	${}^{3}_{-+}$	$\cdots$	${}^{s-2}_{-+}$	${}^{s-1}_{-+}$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\cdots$	$\bigcirc$	$\bigcirc$	
$(\mathcal{F}_{-2})$			$\bigcirc$	$\cdots$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\odot$	${}^{s}_{--}$	$\cdots$	${}^{2s-5}_{--}$	${}^{2s-4}_{--}$	${}^{2s-3}_{--}$
$(\mathcal{F}_{-2})$	${}^{1}_{--}$	${}^{2}_{--}$	${}^{3}_{--}$	$\cdots$	${}^{s-2}_{--}$	$\odot$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\cdots$	$\bigcirc$		

Table 1: **Minimal set of equations**
Set of equations which imply the on-shell condition for (3.23). The quantities in the column with k are $(\mathcal{F}_{+2})^{[k+2,\,2\,(s-1)-k]}$, $\big(\bar{\mathcal{F}}_{-2}\big)^{k,\,2\,(s-1)-k+2]}$, $(\mathcal{F})^{[k,\,2\,(s-1)-k]}$, $\big(\bar{\mathcal{F}}\big)^{[k,\,2\,(s-1)-k]}$, $(\mathcal{F})^{[k-2,\,2\,(s-1)-k]}$ and $\big(\bar{\mathcal{F}}\big)^{[k,\,2\,(s-1)-k-2]}$. The quantities which are marked by ${}^{l}_{\pm\pm}$ is automatically zero by Bianchi identities $(\mathcal{B}_{\pm\pm})^{[l\pm1,\,2\,s-2-l\pm1]}$.

conditions on top of the conditions that quantities in $S^{[s-1,s-1]}$ are zero,

$$(\mathcal{F}_{+2})^{[s,\,s]} \simeq 0 \quad \text{or} \quad \big(\bar{\mathcal{F}}_{+2}\big)^{[s,\,s]} \simeq 0\,, \tag{3.64}$$

$$(\mathcal{F}_{-2})^{[s-2,\,s-2]} \simeq 0 \quad \text{or} \quad \big(\bar{\mathcal{F}}_{-2}\big)^{[s-2,\,s-2]} \simeq 0\,, \tag{3.65}$$

$$(\mathcal{F}_{+2})^{[(s-1)+k+2,\,(s-1)-k]} \simeq 0\,, \qquad \big(\bar{\mathcal{F}}_{+2}\big)^{[(s-1)-k,\,(s-1)+k+2]} \simeq 0\,, \tag{3.66}$$

$$(\mathcal{F}_{0})^{[(s-1)-k,\,(s-1)+k]} \simeq 0\,, \qquad \big(\bar{\mathcal{F}}_{0}\big)^{[(s-1)+k,\,(s-1)-k]} \simeq 0\,, \tag{3.67}$$

$$(\mathcal{F}_{-2})^{[(s-1)-l-2,\,(s-1)+l]} \simeq 0\,, \qquad \big(\bar{\mathcal{F}}_{-2}\big)^{[(s-1)+l,\,(s-1)-l-2]} \simeq 0\,, \tag{3.68}$$

for all $0 \le k \le (s-2)$ and $0 \le l \le (s-3)$ which is an integer and the equations between one-form and zero-form:

$$\begin{aligned} (\mathcal{F}_{+2})^{[2\,s,\,0]} &\simeq -e^{-i\,\theta}\, C^{[2\,s,\,0]}\,, \\ \big(\bar{\mathcal{F}}_{+2}\big)^{[0,\,2\,s]} &\simeq e^{i\,\theta}\, C^{[0,\,2\,s]}\,. \end{aligned} \tag{3.69}$$

These conditions are checked by $\bigcirc$, one of $\circledcirc$ and one of $\odot$ in table 1[5] . Except (3.64) and (3.65), all other equation is the definition of auxiliary field, as we shall show.

[5] This set is not the only one possible set. We can find other sets of equations but they gives the same physical spectrum.

Gauge Fixing and Stüceklberg Fields

The higher spin gauge transformation for the rainbow vacua are given as,

$$\delta\,\mathcal{A}_{++}^{[n+1,\,\bar{n}+1]} = D_{++}^{L}\,\epsilon^{[n,\,\bar{n}]}\,,\quad \delta\,\mathcal{A}_{+-}^{[n+1,\,\bar{n}-1]} = D_{+-}^{L}\,\epsilon^{[n,\,\bar{n}]} + \frac{1}{\ell}\,\epsilon^{[n+1\,,\bar{n}-1]}\,, \tag{3.70}$$

$$\delta\,\mathcal{A}_{--}^{[n-1,\,\bar{n}-1]} = D_{--}^{L}\,\epsilon^{[n,\,\bar{n}]}\,,\quad \delta\,\mathcal{A}_{-+}^{[n-1,\,\bar{n}+1]} = D_{-+}^{L}\,\epsilon^{[n,\,\bar{n}]} + \frac{1}{\ell}\,\epsilon^{[n-1\,,\bar{n}+1]}\,. \tag{3.71}$$

If some fields can be algebraically fixed by gauge transformation, these fields have no physical degree of freedom and can be considered as Stückelberg fields. Let us inspect the components of equations for $n+\bar{n} = 2\,(s-1)$. Because we want to compare unfolded equations and Fronsdal formulation in (A)dS, it is convenient not to fix the symmetric-double-traceless part. Again it is matter of convention, we will fix the fields as follows — or consider following field as Stüeckelberg fiels — for $0 \leq l \leq (s-1)-1$:

$$(\mathcal{A}_{+-})^{[(s-1)+l+1,\,(s-1)-l-1]} = 0 \quad \text{by gauge parameter} \quad \epsilon^{[(s-1)+l+1,\,(s-1)-l-1]}\,, \tag{3.72}$$

$$(\mathcal{A}_{-+})^{[(s-1)-l-1,\,(s-1)+l+1]} = 0 \quad \text{by gauge parameter} \quad \epsilon^{[(s-1)-l-1,\,(s-1)+l+1]}\,. \tag{3.73}$$

The Stüeckelberg fields are marked as s^m in table 2.

$k=$	0	1	2	$\cdots$	s-3	s-2	s-1	s	s+1	$\cdots$	2 s-4	2 s-3	2 s-2
$\mathcal{A}_{++}$	$^{1}_{+\bar{2}}$	$^{2}_{+\bar{2}}$	$^{3}_{+\bar{2}}$	$\cdots$	$^{s-2}_{+\bar{2}}$	$^{s-1}_{+\bar{2}}$	$\bigcirc$	$^{s-1}_{+2}$	$^{s}_{+2}$	$\cdots$	$^{2s-5}_{+2}$	$^{2s-4}_{+2}$	$^{2s-3}_{+2}$
$\mathcal{A}_{+-}$	$^{1}_{0}$	$^{2}_{0}$	$^{3}_{0}$	$\cdots$	$^{s-2}_{0}$	$^{s-1}_{0}$	s^{s}	s^{s+1}	s^{s+2}	$\cdots$	s^{2s-3}	s^{2s-2}	
$\mathcal{A}_{-+}$		s^{0}	s^{1}	$\cdots$	s^{s-4}	s^{s-3}	s^{s-2}	$^{s-1}_{\bar{0}}$	$^{s}_{\bar{0}}$	$\cdots$	$^{2s-5}_{\bar{0}}$	$^{2s-4}_{\bar{0}}$	$^{2s-3}_{\bar{0}}$
$\mathcal{A}_{--}$		$^{2}_{-2}$	$^{3}_{-2}$	$\cdots$	$^{s-2}_{-2}$	$^{s-1}_{-2}$	$\bigcirc$	$^{s-1}_{-\bar{2}}$	$^{s}_{-\bar{2}}$	$\cdots$	$^{2s-5}_{-\bar{2}}$	$^{2s-4}_{-\bar{2}}$	

Table 2: **Physical fields, Stüeckelberg fields and auxiliary fields**
The fields in positions which are marked by circle are physical fields. The fields in positions which are marked as s^l can be fixed by gauge transformation $\epsilon^{[l\,,2(s-1)-l]}$, therefore they are Stüeckelberg fields. And the fields in positions which are marked as ${}^{m}_{i}$ and ${}^{m}_{\bar{i}}$ can be expressed by other physical fields by equations $\left(\mathcal{F}^{[m\,,2(s-1)-m]}\right)_i$ and $\left(\bar{\mathcal{F}}^{[m\,,2(s-1)-m]}\right)_i$.

After gauge fixing there is remaining gauge parameter is $\epsilon^{[s-1,\,s-1]}$ which corresponds double-traceless symmetric $(s-1)$ Frondsdal spin-s gauge parameter. And the gauge variation of field $\mathcal{A}_{++}^{[s,\,s]}$ and $\mathcal{A}_{--}^{[s-2,\,s-2]}$ correspond to gauge variation of traceless part and traceless part of Fronsdal formulation. *Therefore we will consider $\mathcal{A}_{++}^{[s,\,s]}$ and $\mathcal{A}_{--}^{[s-2,\,s-2]}$ as the spin-s Fronsdal field.*

Auxiliary fields and physical equation of motion

We can obtain the irreducible components of F in (3.52) by applying formulae (3.54). Each component which contains the field $A^{[n\pm1,\,\bar n\pm1]}$ is,

$$
\begin{aligned}
(\mathcal{F}_{+2})^{[n+2,\,\bar n]} &= \bar n\, D^L_{++}\,\mathcal{A}^{[n+1,\,\bar n-1]}_{+-} - (\bar n+2) D^L_{+-}\,\mathcal{A}^{[n+1,\,\bar n+1]}_{++} - \frac{\bar n}{\ell}\,\mathcal{A}^{[n+2,\,\bar n]}_{++}\,, && (3.74)\\
(\mathcal{F}_{0})^{[n,\,\bar n]} &= \bar n\left(D^L_{++}\,\mathcal{A}^{[n-1,\,\bar n-1]}_{--} + D^L_{-+}\,\mathcal{A}^{[n+1,\,\bar n-1]}_{+-} - \frac{1}{\ell}\,\mathcal{A}^{[n,\,\bar n]}_{-+}\right)\\
&\quad -(\bar n+2)\left(D^L_{--}\,\mathcal{A}^{[n+1,\,\bar n+1]}_{++} + D^L_{+-}\,\mathcal{A}^{[n-1,\,\bar n+1]}_{-+} - \frac{1}{\ell}\,\mathcal{A}^{[n,\,\bar n]}_{+-}\right), && (3.75)\\
(\mathcal{F}_{-2})^{[n-2,\,\bar n]} &= \bar n\, D^L_{-+}\,\mathcal{A}^{[n-1,\,\bar n-1]}_{--} - (\bar n+2)\, D^L_{--}\,\mathcal{A}^{[n-1,\,\bar n+1]}_{-+} + \frac{\bar n+2}{\ell}\,\mathcal{A}^{[n-2,\,\bar n]}_{--}\,, && (3.76)
\end{aligned}
$$

and $\bar{\mathcal{F}}_I$ are given as quantities with the following changes:

$$
n \leftrightarrow \bar n\,,\quad D^L_{ij} \leftrightarrow D^L_{ji}\,,\quad (\mathcal{A}_{\pm\mp})^{[n\pm1,\,\bar n\mp1]} \leftrightarrow (\mathcal{A}_{\mp\pm})^{[n\mp1,\,\bar n\pm1]}\,. \tag{3.77}
$$

These changed quantities are not complex conjugate of (3.81, 3.82, 3.76).

Let us inspect the components of equations for $n+\bar n = 2\,(s-1)$. As we advertised, most of equations in table 2 implies that specific field is auxiliary; e.g. the equation $(\mathcal{F}_{+2})^{[s+1\,,s-1]} \simeq 0$ implies that $\mathcal{A}^{[s+1,\,s-1]}$ can be represented as physical fields as follows:

$$
\mathcal{A}^{[s+1,\,s-1]}_{++} \simeq \ell\left(D^L_{++}\,\mathcal{A}^{[s,\,s-2]}_{+-} - \frac{s+1}{s-1}\,D^L_{+-}\,\mathcal{A}^{[s,\,s]}_{++}\right) = -\frac{s+1}{s-1}\,\ell\, D^L_{+-}\,\mathcal{A}^{[s,\,s]}_{++}\,, \tag{3.78}
$$

which implies that $\mathcal{A}^{[s+1,\,s-1]}_{++}$ is auxiliary. In a similar way equation (3.66) – (3.68) are consumed to define the auxiliary field — see table 2. And equations (3.69) imply that the zero-form fields $C^{[2\,(s-1)+2,\,0]}$ and $C^{[0,\,2\,(s-1)+2]}$ are not independent field. At last but not least, we can check that the equations (3.64) and (3.65) give the second order differential equation for physical field. Even if these equation could be analysed further, but we circumvent and use the gauge symmetry.

As we told before the gauge variation of physical fields are equal to Fronsdal formulation. These gauge symmetries can fix the quadratic order action, therefore the equations (3.64) and (3.65) must coincide with Fronsdal equation.

3.2.3 Analysis of Equation: Zero-form Sector

In a similar way with previous subsection, we could analyse the zero-form equation (3.24). The Bianchi identity (3.33) of the equation (3.30) is a two-form and can be decomposed as,

$$
b = \Sigma_{+2}\,\mathfrak{b}_{-2} + \Sigma_0\,\mathfrak{b}_0 + \Sigma_{-2}\,\mathfrak{b}_{+2} - \bar\Sigma_{+2}\,\bar{\mathfrak{b}}_{-2} - \bar\Sigma_0\,\bar{\mathfrak{b}}_0 - \bar\Sigma_{-2}\,\bar{\mathfrak{b}}_{+2}\,, \tag{3.79}
$$

and the equation (3.30) for zero-form is an one-form and can be decomposed as,

$$
f = \sigma_{--}\,\mathfrak{f}_{++} + \sigma_{++}\,\mathfrak{f}_{--} + \sigma_{+-}\,\mathfrak{f}_{-+} + \sigma_{-+}\,\mathfrak{f}_{+-}\,. \tag{3.80}
$$

Then the irreducible components of the Bianchi identity is given as follows:

$$(\mathfrak{b}_{+2})^{[n+2,\,\bar{n}]} = \bar{n}\,D^L_{++}\mathfrak{f}^{[n+1,\,\bar{n}-1]}_{+-} - (\bar{n}+2)D^L_{+-}\,\mathfrak{f}^{[n+1,\,\bar{n}+1]}_{++} + \frac{\bar{n}+2}{\ell}\,\mathfrak{f}^{[n+2,\,\bar{n}]}_{+-}\,, \tag{3.81}$$

$$\begin{aligned}(\mathfrak{b}_{0})^{[n,\,\bar{n}]} &= \bar{n}\left(D^L_{++}\mathfrak{f}^{[n-1,\,\bar{n}-1]}_{--} + D^L_{-+}\,\mathfrak{f}^{[n+1,\,\bar{n}-1]}_{+-} - \frac{1}{\ell}\,\mathfrak{f}^{[n,\,\bar{n}]}_{++}\right)\\ &\quad - (\bar{n}+2)\left(D^L_{--}\mathfrak{f}^{[n+1,\,\bar{n}+1]}_{++} + D^L_{+-}\mathfrak{f}^{[n-1,\,\bar{n}+1]}_{-+} - \frac{1}{\ell}\,\mathfrak{f}^{[n,\,\bar{n}]}_{--}\right)\,,\end{aligned} \tag{3.82}$$

$$(\mathfrak{b}_{-2})^{[n-2,\,\bar{n}]} = \bar{n}\,D^L_{-+}\mathfrak{f}^{[n-1,\,\bar{n}-1]}_{--} - (\bar{n}+2)\,D^L_{--}\mathfrak{f}^{[n-1,\,\bar{n}+1]}_{-+} - \frac{\bar{n}}{\ell}\,\mathfrak{f}^{[n-2,\,\bar{n}]}_{-+}\,, \tag{3.83}$$

and $\bar{\mathfrak{b}}_I$ are given as quantities with the following changes:

$$n \leftrightarrow \bar{n}\,,\quad D^L_{ij} \leftrightarrow D^L_{ji}\,,\quad (\mathfrak{f}_{\pm\mp})^{[n\pm1,\,\bar{n}\mp1]} \leftrightarrow (\mathfrak{f}_{\mp\pm})^{[n\mp1,\,\bar{n}\pm1]}\,. \tag{3.84}$$

Let us inspect the components of Bianchi identity for $|n-\bar{n}| = 2\,s \neq 0$. We can check that after imposing $\mathfrak{f}^{[n\geq1,\,\bar{n}\geq1]}_{++}$, $\mathfrak{f}^{[1,\,2\,s-1]}_{+-}$ and $\mathfrak{f}^{[1,\,2\,s-1]}_{+-}$, all equation is automatically zero by Bianchi identity.

The irreducible components of the equation is give as follows:

$$\mathfrak{f}^{[n+1,\,\bar{n}+1]}_{++} = D^L_{++}\,C^{[n,\,\bar{n}]} - \frac{1}{\ell}\,C^{[n+1\,,\bar{n}+1]}\,,\quad \mathfrak{f}^{[n+1,\,\bar{n}-1]}_{+-} = D^L_{+-}\,C^{[n,\,\bar{n}]}\,, \tag{3.85}$$

$$\mathfrak{f}^{[n-1,\,\bar{n}-1]}_{--} = D^L_{--}\,C^{[n,\,\bar{n}]} - \frac{1}{\ell}\,C^{[n-1,\,\bar{n}-1]}\,,\quad \mathfrak{f}^{[n-1,\,\bar{n}+1]}_{-+} = D^L_{-+}\,C^{[n,\,\bar{n}]}\,. \tag{3.86}$$

We can see that $C^{[n+1,\,\bar{n}]}$ is auxiliary field because it can be expressed as function of other field by the equation $\mathfrak{f}^{[n+1,\,\bar{n}+1]}_{++}$. The only equations we must consider are $\mathfrak{f}^{[1,\,2\,s-1]}_{+-}$ and $\mathfrak{f}^{[1,\,2\,s-1]}_{+-}$:

$$D^L_{+-}\,C^{[0,\,2\,s]} \simeq 0\,,\quad D^L_{-+}\,C^{[2\,s,\,0]} \simeq 0\,. \tag{3.87}$$

Chapter 4

(Anti-)de Sitter Space Waveguide

I saw a Line that was no Line; Space that was not Space: · · ·
I shrieked loud in agony, "Either this is madness or it is Hell."
"It is neither", calmly replied the voice of the Sphere,
"it is Knowledge; it is Three Dimensions."
— A Square
FLATLAND: A ROMANCE OF MANY DIMENSIONS

The purpose of this work is to lay down a concrete theoretical framework for addressing this question and, using it, to analyze the pattern of the massive higher-spin fields as well as Higgs mechanism that underlies the mass spectrum. The idea is to utilize the Kaluza-Klein approach [4] for compactifying higher-dimensional AdS to lower-dimensional one and systematically study the Kaluza-Klein mass spectrum.

The Kaluza-Klein is a great theoretical tool to study about various aspects field theory. For example, the lower dimensional massive theory can be derived from the higher dimensional massless theory in the free-level: for scalar, spin-1 and spin-2 field theories on AdS see [38, 39] and for higher-spin field theory on flat background see [12]. Furthermore, the celebrated Yang-Mills structure could be founded by the dimensional reduction[1]. In this paper, we propose a novel method of compactification: AdS waveguide method.

Because there have been trials to use dimensional reduction in the higher-spin theory context, novel features of AdS waveguide method can be explained by denoting differences with previous works. The works [58] propose so-called 'radial reduction' that reduces higher-spin gauge theories on $(d+1)$-dimensional Minkowski spacetime to the theory on d-dimensional anti-de Sitter spacetime. *In contrast, our approach starts from higher-spin gauge theory on AdS$_{d+2}$ space and compactifies it to AdS$_{d+1}$ space*, both of which are expected to be consistent. The work [39] proposes so-called 'dimensional digression' that expands the higher-spin representations of $\mathfrak{so}(d+1,2)$ in terms of representations of $\mathfrak{so}(d,2)$. While it makes advantages of the discrete spectrum of unitary representations, the approach is rather limited since this approach is not equipped with a tunable parameter (like α which will be introduced in section 4.2.) that specifies compactification size or with a set of boundary conditions that yields different spectrum. *Our approach has both of them.* Differences and similarities are displayed in the paragraph in page 59 and section 5.2.3.

Even only concentrating on the massless spin-s waveguide, the existence of boundaries and the rich possibilities of boundary conditions permit a variety of spectra as describing in the end of section 5.2.2 and Fig. 12. These spectra reveal two interesting features:

- The first one is the existence of the partially massless fields [43] which is non-unitary in anti-de Sitter space. An intuitive picture why these non-unitary fields ariased is presented in section 5.1 through simple string system.

[1]For the historical story of Kaluza-Klein theory and Yang-Mills structure, see N. Straumann, "On Pauli's invention of nonAbelian Kaluza-Klein theory in 1953," gr-qc/0012054.

- The second one is the behaviour of masses in the reduction limit(i.e. $\alpha \to 0$). In usual Kaluza-Klein compactification, because there is no prefer mass scale in the shrinking limit, only massless field survives when a size of compactification goes to zero. Other Kaluza-Klein modes will decouple with massless modes because their masses will blow up. In contrast, owing to AdS scale, massless, partially massless and massive fields — whose mass squares are proportional to the square of inverse AdS radius — can survive when the size of the tunable parameter goes to zero. These surviving modes will be called as "*ground modes*" and the others whose masses diverse at reduction limit will be called "*Kaluza-Klein modes*".

The gauge symmetries are maximally used to figure out the physics in lower dimension It turns out that a massless spin-s gauge symmetry on AdS$_{d+2}$ can be understood as Stüeckelberg gauge symmetries [68] for spin-s on AdS$_{d+1}$ [63], which totally determine the quadratic actions. And it will be shown that considering gauge symmetries is enough to understand the lower dimensional spectrum. This is not an unanticipated result, because the quadratic action of massless spin-s is completely determined by the spin-s gauge symmetry and a dimensional reduction is nothing but a separation of variables.

The rest of the paper is organised as follows. In section 4.1, the spin-one waveguide in flat space is considered. Stüeckelberg structures arise naturally after reduction. And it is shown that only restricted sets of boundary conditions are possible for the consistency of theory and emphasise the role of boundary conditions to determine the spectrum. In section 4.2, the reduction method of AdS space is introduced. The AdS waveguides, boundary conditions and spectrums are considered for spin-1, spin-2 and spin-3 in following sections — section 4.3, 4.4 and 4.5. Reduced equations of motion and reduced gauge symmetries for each case are derived, and we shall confirm that the reduced gauge symmetries are strong enough to fix the reduced equations of motion. A variety of spectrums can be obtained depending on boundary conditions. Massless and massive fields and non-unitary partially massless(PM) field can be obtained by reduction. And we show that imposing the higher-derivative boundary condition(HD BC) is inevitable for $s \geq 3$. The physical interpretation of HD BC shall be given in next chapter.

Appendix A contains conventions for anti-de Sitter space and $\mathfrak{so}(d,2)$-modules and AdS waveguide method starting from AdS$_{d+k}$ to AdS$_d$ is demonstrated in Appendix C.

4.1 Flat Space Waveguide and Boundary Conditions

The salient feature of our approach is to compactify AdS$_{d+2}$ to AdS$_{d+1}$ times an interval whose angular size is tunable, thus forming "AdS waveguide". At each boundary of the wedge, we must impose boundary conditions which determine the spectrum in AdS$_{d+1}$.

As a primer, we study the electromagnetic — massless spin-1 — waveguide along the flat spacetime with the boundaries, paying particular attention to the relations between boundary conditions and spectrum. The flat spacetime is $\mathbb{R}^{1,\,d-1} \times I_L$, where interval $I_L \equiv \{0 \leq z \leq L\}$. The $(d+1)$-dimensional coordinates can be decomposed into parallel and perpendicular directions: $x^M = (x^\mu, z)$. And the spin-1 field in $(d+1)$-dimension can be decomposed into the spin-1 field and the scalar field in d-dimension: $A_M = (A_\mu, \phi)$. Then the equations of motion can be written as,

$$\partial^M F_{M\nu} = \partial^\mu F_{\mu\nu} + {\partial_z}^2 A_\nu - \partial_z \partial_\nu \phi = 0\,, \tag{4.1}$$

$$\partial^M F_{Mz} = \partial^\mu \partial_\mu \phi - \partial_z \partial^\mu A_\mu = 0\,, \tag{4.2}$$

and the gauge transformations are

$$\delta A_\mu = \partial_\mu \Lambda\,, \qquad \delta \phi = \partial_z \Lambda\,. \tag{4.3}$$

Both the equations of motion and the gauge transformation take the structure of Stüeckelberg system [68]. Recall that the Stüecklberg Lagrangian of massive spin-1 vector field is given by

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu \phi\, \partial^\mu \phi + m\, A_\mu \left(\frac{m}{2} A^\mu - \partial^\mu \phi \right)\,, \tag{4.4}$$

which is invariant under Stüeckelberg gauge transformation:

$$\delta A_\mu = \partial_\mu \lambda\,, \qquad \delta \phi = m\, \lambda\,. \tag{4.5}$$

The scalar field ϕ, which is called as Stueckelberg scalar, is redundant because it can be algebraically fixed as zero through the gauge transformation (4.5) whenever $m^2 \neq 0$. In the massless limit, Stüeckelberg system breaks into a massless spin-1 field and a massless scalar.

The fields are excited along the z-direction, and we mode-expand them whose coefficients are d-dimensional spin-1 fields and scalar fields. Mode functions can be any complete set of basis functions. It is natural to choose for the mode functions the eigenfunctions of $\Box := (\partial_z)^2$ with prescribed boundary conditions. And the mode functions of gauge parameters are related with mode function of fields. The structure of gauge variation (4.3) implies that mode functions must be related to each other:

$$\partial_z \text{(mode function of spin-1 field)} \propto \text{(mode function of spin-0 field)}\,. \tag{4.6}$$

What happen if one imposes the same boundary conditions for both A_μ and ϕ, either no-derivative or one-derivative? Let us consider the no-derivative boundary conditions. From the boundary conditions, $A_M(z)|_{z=0,\,L} = 0$ and from equation (4.2),

$$\left(\partial^\mu \partial_\mu \phi(z) - \partial^\mu \partial_z A_\mu(z) \right)\Big|_{z=0,\,L} = -\partial^\mu \partial_z A_\mu(z)|_{z=0,\,L} = 0\,, \tag{4.7}$$

therefore $\partial_z A_\mu(z)|_{z=0,L} = 0$. Because A_μ satisfies second order partial-differential equation, these two boundary conditions — $A_M(z)|_{z=0,\,L} = 0$ and $\partial_z A_\mu(z)|_{z=0,L} = 0$ — implies $A_\mu(z) = 0$. Also from equation (4.1), ϕ satisfies a first order partial-differential equation, therefore $\phi(z) = 0$. This means there is no field satisfying such boundary conditions.

We can anticipate this result from the fact that these boundary conditions do not preserve the relation (4.6). The relation (4.6) restricts the form of boundary conditions. For example, if we impose Robin boundary condition for scalar, (4.6) impose a higher-derivative boundary condition (HD BC) for spin-1:

$$\mathcal{M}\,\phi|_{z=0,\,L} = (a\,\partial_z + b)\,\phi|_{z=0,\,L} = 0 \quad \to \quad \mathcal{M}\,\partial_z\,A_\mu|_{z=0,\,L} = 0\,. \tag{4.8}$$

First we only consider simple cases which do not contain HD BC.

Vector boundary condition One possibility is to impose one-derivative (Neumann) boundary condition on the spin-1 field $A_\mu(x,z)$ field and zero-derivative (Dirichlet) boundary condition on spin-zero field $\phi(x,z)$ at $z = 0,\ L$. The corresponding mode expansion for A_μ and ϕ reads

$$A_\mu(z) = \sum_{n=0}^{\infty} A_\mu^{(n)} \cos\left(\frac{n\,\pi}{L}\,z\right) \qquad \text{and} \qquad \phi(z) = \sum_{n=1}^{\infty} \phi^{(n)} \sin\left(\frac{n\,\pi}{L}\,z\right), \tag{4.9}$$

so we mode-expand the field equations (4.1) and (4.2) in a suggestive form

$$\sum_{n=0}^{\infty} \cos\left(\frac{n\,\pi}{L}\,z\right)\left[\partial^\mu\,F^{(n)}{}_{\mu\nu} - \left(\frac{n\,\pi}{L}\right)\left(\frac{n\pi}{L}\,A_\nu^{(n)} + \partial_\nu\,\phi^{(n)}\right)\right] = 0\,, \tag{4.10}$$

$$\sum_{n=1}^{\infty} \sin\left(\frac{n\,\pi}{L}\,z\right)\partial^\mu\left(\frac{n\,\pi}{L}\,A_\mu^{(n)} + \partial_\mu\,\phi^{(n)}\right) = 0\,. \tag{4.11}$$

The trigonometric mode functions for $n = 0, 1, \ldots$ form a complete set of the orthogonal basis for square-integrable functions over I_L, so in the above equations, individual coefficients should vanish. The zero-mode $m = 0$ is special, since only the first equation is nonempty and gives the equation of motion for massless spin-1 field. For all Kaluza-Klein modes $n \neq 0$, it gives Stüeckelberg equation of motion for massive spin-1 field[2] with mass $n\,\pi/L$. The second equation follows from the Bianchi identity of the first equation, so this confirms consistency of the prescribed boundary conditions. In the limit $L \to 0$, all Stüeckelberg field becomes infinitely massive. Therefore, there only remains the massless spin-1 field $A_\mu^{(0)}$ with associated gauge invariance. There is also no scalar field $\phi^{(0)}$, an important fact that follows from the prescribed boundary conditions. Intuitively, $A_\mu^{(0)}$ remains massless and gauge invariant, so Stüeckelberg field $\phi^{(0)}$ is not necessary. Moreover, the spectrum is consistent with the fact that this boundary condition ensures no energy flow across the boundary.

The key idea is that the same result is obtainable from the Kaluza-Klein compactification of gauge transformations (4.3). The gauge transformation that preserves the vector boundary conditions can be expanded by the Fourier modes:

$$\Lambda = \sum_{n=0}^{\infty} \Lambda^{(n)} \cos\left(\frac{n\,\pi}{L}\,z\right)\,. \tag{4.12}$$

The gauge transformations of d-dimensional fields read

$$\delta\,A_\mu^{(n)} = \partial_\mu\,\Lambda^{(n)} \quad (\text{for } n \geq 0) \qquad \text{and} \qquad \delta\,\phi^{(n)} = -\frac{n\,\pi}{L}\,\Lambda^{(n)} \quad (\text{for } n > 0)\,. \tag{4.13}$$

[2] For flat higher-spin Kaluza-Klein compactification, these kinds of structure were founded at [12].

We note that the $n=0$ mode is present only for the gauge transformation of spin-1 field. This is the gauge transformation of a massless gauge vector field. We also note that gauge transformations of all higher $n=1,2,\cdots$ modes take precisely the form of Stüeckelberg gauge transformations. The Stüeckelberg gauge invariance fixes the quadratic action as the Stüeckelberg action with a tower of Proca fields with masses $m_n = n\,\pi/L$, $(n=1,2,\cdots)$.

We can turn around the logic reversed. Suppose we want to retain massless spin-1 field $A_\mu^{(0)}$ in d dimensions, along with associated gauge invariance. This requirement then singles out one-derivative (Neumann) boundary condition for A_μ. This and the Bianchi identity for A_μ, in turn, single out no-derivative (Dirichlet) boundary condition for ϕ.

Scalar boundary condition Alternatively, one might impose no-derivative (Dirichlet) boundary condition to A_μ and one-derivative (Neumann) boundary condition to ϕ. In this case, the equations of motion, when mode-expanded, takes exactly the same form as above except that the mode functions are interchanged:

$$\sum_{n=1}^{\infty} \sin\left(\frac{n\,\pi}{L}\,z\right)\left[\partial^\mu\,F^{(n)}{}_{\mu\nu} - \left(\frac{n\,\pi}{L}\right)\left(\frac{n\pi}{L}\,A_\nu^{(n)} - \partial_\nu\,\phi^{(n)}\right)\right] = 0\,, \tag{4.14}$$

$$\sum_{n=0}^{\infty} \cos\left(\frac{n\,\pi}{L}\,z\right)\partial^\mu\left(\frac{n\,\pi}{L}\;A_\mu^{(n)} - \partial_\mu\,\phi^{(n)}\right) = 0\,. \tag{4.15}$$

Consequently, the zero-mode $n=0$ consists of massless scalar field $\phi^{(0)}$ only ($A_\mu^{(0)}$ is absent at the outset). All Kaluza-Klein modes $n \neq 0$ are again Stüeckelberg massive spin-1 fields with mass $n\pi/L$. In the limit $L \to 0$, all Stüeckelberg field becomes infinitely massive. There only remains the massless spin-0 $\phi^{(0)}$. Once again, this is consistent with the fact that this boundary condition ensures no energy flow across the boundary.

Once again, the key idea is that the above results are obtainable from the Kaluza-Klein compactification of the gauge transformations. For a gauge transformation that preserves the scalar boundary condition, the gauge function can be expanded by the Fourier modes

$$\Lambda(x,z) = \sum_{n=1}^{\infty} \Lambda^{(n)}(x)\,\sin\left(\frac{n\,\pi}{L}\,z\right)\,. \tag{4.16}$$

With these modes, the gauge transformations of fields are

$$\delta\,A_\mu^{(n)} = \partial_\mu\,\Lambda^{(n)} \quad (\text{for } n>0)\,, \qquad \delta\,\phi^{(n)} = \frac{n\,\pi}{L}\,\Lambda^{(n)} \quad (\text{for } n>0)\,. \tag{4.17}$$

There is no $n=0$ zero-mode gauge transformation, and so no massless gauge vector field. The scalar zero-mode $\phi^{(0)}$ is invariant under gauge transformation. The gauge transformation are the Stüeckelberg gauge symmetries with masses $m_n = n\,\pi/L$.

Once again, we can turn around the logic reversed. Suppose we want to retain massless spin-0 field $\phi^{(0)}$ in d dimensions, along with associated global invariance (The global invariance is a remnant of $(d+1)$-dimensional gauge invariance). This then singles out one derivative (Neumann) boundary condition for ϕ. This and the Bianchi identity for A_μ, in turn, puts no-derivative (Dirichlet) boundary condition for A_μ.

From above examples, we can take following lessons.

- Stüeckelberg structure naturally arises by a dimensional reduction.
- Lower dimensional spectrums depend on boundary conditions.
- Boundary conditions of lower dimensional fields which come from the same higher dimensional field (for example, A_μ and ϕ from A_M), are related to each other (for example as (4.6)).

Before concluding this introductory section we comment the brane picture and S-duality for $d = 3$ case.

D3-branes ending on five-branes and S-duality

The two possible boundary conditions discussed above are universal for all dimensions d. When $d = 3$ and adjoined with sixteen supersymmetry, the two boundary conditions are related each other by electromagnetic duality. This feature can be neatly seen in the context of brane configurations in Type IIB string theory.

Consider a D3 brane ending on parallel five-branes (D5 or NS5 brane). From the viewpoint of world-volume dynamics, the stack of five-branes provides boundary conditions. The original low-energy degree of freedom of D3 brane is four-dimensional $\mathcal{N} = 4$ vector multiplet. In the presence of five-branes, half of sixteen supersymmetries is broken. At the boundary, the four-dimensional $\mathcal{N} = 4$ vector multiplet is split into three-dimensional $\mathcal{N} = 4$ vector multiplet and $\mathcal{N} = 4$ hypermultiplet. If the five-brane were D5 branes, the zero-mode is the three-dimensional $\mathcal{N} = 4$ hypermultiplet. If the five-branes were NS5 branes, the zero-mode is the three-dimensional $\mathcal{N} = 4$ vector multiplet. In terms of D3 brane world-volume theory, D5 brane sets "D5-type" boundary condition: Dirichlet boundary condition on three dimensional vector multiplet and Neumann boundary condition on three-dimensional hypermultiplet, while NS5 brane sets "NS5-type" boundary condition: Neumann boundary condition on three dimensional vector multiplet and Dirichlet boundary condition on three-dimensional hypermultiplet.

The Type IIB string theory has $SL(2,\mathbb{Z})$ duality symmetry, under which the two brane configurations are rotated each other. In terms of D3-brane world-volume dynamics, the three-dimensional $\mathcal{N} = 4$ vector multiplet and hypermultiplet are interchanged each other. This is yet another way to demonstrate the well-known mirror symmetry in three-dimensional gauge theory, which exchanges two hyperKähler manifolds provided by the vector multiplet moduli space $\mathcal{M}_{\rm V}$ and the hypermultiplet moduli space $\mathcal{M}_{\rm H}$.

4.2 Waveguide in Anti-de Sitter Space

We start from AdS$_{d+2}$ and construct a tunable AdS$_{d+1}$ waveguide – a waveguide which retains $\mathfrak{so}(d,2)$ isometry within $\mathfrak{so}(d+1,2)$ and which has a tunable thickness.

Consider the AdS_{d+2} in the Poincare patch with coordinates $(t,\, \vec{x}_{d-1},\, y,\, z) \in \mathbb{R}^{1,\,d} \times \mathbb{R}^{+}$:

$$\delta s_{d+2}{}^2 = \frac{\ell^2}{z^2}\left(-\delta t^2 + \delta\vec{x}_{d-1}^2 + \delta z^2\right) + \frac{\ell^2}{z^2}\delta y^2 = \delta s_{d+1}{}^2(g_{d+1}) + g_{yy}\delta y^2\,. \tag{4.18}$$

We freely singled out one of spatial coordinates as y. The Poincare metric is independent of y, so one might consider the slice between two timelike hypersurfaces $y = 0$ and $y = L$ and putting appropriate boundary conditions. This actually does not work. It is instructive to understand why, as we now explain. This compactification has two problems.

- The field equations satisfied by Kaluza-Klein states do not have the same structure as the field equation of massive (higher spin) fields in AdS$_d$ [38]. This implies that contrary to apparent structure, the compactification does not respect $\mathfrak{so}(d,2)$ isometry of AdS$_{d+1}$.
- When dimensionally reduced along y-direction, the $(d+2)$-dimensional tensor does *not* get reduced to a $(d+1)$-dimensional tensor. Consider, for example, a small fluctuatoin of the metric. The tensor $\nabla_\mu\, h_{\nu y}$ is dimensionally reduced to $\nabla_\mu\, A_\nu + \delta_{\mu z}\, \frac{1}{z}\, A_\nu$, where $A_\mu \equiv h_{\mu y}$. Because of the second term, the result is in no way a $(d+1)$-dimensional tensor.

One should choose internal space from isometry directions of the manifold, otherwise, there would be geometrical singularities. For all isometry, the non-covariant z dependent term always appears and it is hard to be cured. Without isometry, the values of metric (and its derivative) at two hypersurfaces are different and they cannot be identified in general. We have to introduce boundaries on hypersurfaces rather than identifying them.

Instead, we will consider the following foliation of AdS$_{d+2}$ into a semi-direct product of AdS$_{d+1}$ hypersurface and angular coordinate θ over a finite domain:

$$\boxed{\delta s_{d+2}{}^2 = \frac{1}{\cos^2\theta}\left(\delta s_{d+1}{}^2 + \ell^2\,\delta\theta^2\right)\,.} \tag{4.19}$$

This compactification bypasses the two problems that afflicted the circular compactification above. Firstly, $(d+2)$ dimensional tensor is reduced to $(d+1)$-dimensional tensor. For instance, $\nabla_\mu\, h_{\nu\theta}$ becomes $\nabla_\mu\, A_\nu - \tan\theta\, h_{\mu\nu} + \tan\theta\, \frac{1}{\ell^2}\, g_{\mu\nu}\, \phi$. In Appendix C, under plausible assumptions, we prove that semi-direct product waveguide above is the unique compactification preserving $\mathfrak{so}(d,2)$ isometry. Secondly, as we shall show completely in later part of this paper, the equations of motion of Kaluza-Klein states are exactly the same as the equations of motion of massive higher-spin fields.

We can explicitly construct the semi-direct product metric from appropriate foliations of AdS$_{d+2}$. We start from Poincaré patch of AdS$_{d+2}$ and change to polar coordinates, $z = \rho\cos\theta$, $y = \rho\sin\theta$, where z is bulk direction of AdS$_{d+2}$ and y is another arbitrary spatial direction[3]. With this parametriza-

[3]Poincare coordinate or spacial direction “y” does not play an important role in this structure. This structure can be easily generalized to any other AdS spaces. See Appendix C

tion, AdS$_{d+2}$ metric can be represented as fibration over the interval, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$:

$$\delta s_{d+2}{}^2 = \frac{\ell^2}{z^2}\left(-\delta t^2 + \delta\vec{x}^{\,2} + \delta y^2 + \delta z^2\right) = \frac{\ell^2}{\rho^2 \cos^2\theta}\left(-\delta t^2 + \delta\vec{x}^{\,2} + \delta\rho^2 + \rho^2\,\delta\theta^2\right)$$
$$= \frac{1}{\cos^2\theta}\big(\delta s_{d+1}{}^2 + \ell^2\,\delta\theta^2\big)\,. \tag{4.20}$$

The boundary of AdS_{d+1} is at $\theta = \pm\frac{\pi}{2}$. We now construct the AdS waveguide by taking the wedge $-\alpha \le \theta \le \alpha$. See Fig.7.

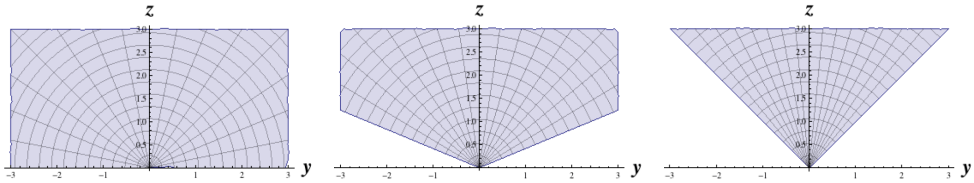

Figure 7: Anti-de Sitter waveguide: The first figure is a slice of AdS$_{d+2}$ in Poincaré coordinates (y, z). In polar coordinates (ρ, θ), the AdS boundary is located at $\theta = \pm\pi/2 (z = 0)$. The waveguide is constructed by taking the angular domain $-\alpha \le \theta \le +\alpha$ for $\alpha < \pi/2$. The waveguide with $\alpha = \pi/4$ is given at far right figure.

One might try to conceive an alternative compactification scheme based on anti-de Sitter tube by putting periodic boundary condition that identifies the two boundaries at $\theta = \pm\alpha$. This is not possible. The vector ∂_θ is not a Killing vector, so although the metric at hyper-surfaces $\theta = \pm\alpha$ are equal, their first derivatives differ each other.

Before concluding this section, we introduce the notations that will be widely used in later. We introduce the mode functions as follows:

$$\boxed{\Theta_n^{s|S}(\theta) \;=\; n\text{-th mode function for } (d+1)\text{-dimensional spin-}s\text{ component that arise from } (d+2)\text{-dimensional spin-}S\text{ field upon waveguide compactification}\,.} \tag{4.21}$$

And we also introduce the first-order differential operators $Ł_m$:

$$\boxed{Ł_n = \partial_\theta + n\,\tan\theta\,.} \tag{4.22}$$

Because theta derivatives appear through these forms in AdS waveguide, $Ł_m$ shall be used to express the Kaluza-Klein equations and boundary conditions. The tangent function comes from the form of the Christoffle symbols in Table 6.

In this paper, we shall mainly concentrate on AdS waveguide. However, we can generalized the results to dS waveguide which is introduced in Appendix C. The only technical difference is the changing $(\tan\theta)$ to $(-\tanh\theta)$ as one can see in Table 6. The results in dS are obtainable by changing tangent functions in (4.22).

4.3 Waveguide Spectrum of Spin-1 Field

In this section, we focus on the lowest spin field, spin-1, and systematically work out Kaluza-Klein compactification. In section 4.1, we learned that boundary conditions of different polarization fields are related. In our AdS compactification, where non-trivial theta dependence appears, proper boundary conditions are not clear at the beginning. We describe how to find out proper boundary conditions and this will be applied to higher spins in later sections.

Throughout examples — spin-1 in this section and spin-2 and spin-3 in later sections — two methods shall be compared: the one method using equations of motions and the other method using gauge transformations. Because the equation of motion contains many terms, the first method is hard to apply to general spin fields. However, the second method is relatively easy to deal and can be applied to general spin fields.

Furthermore, the reduced equation of motion can be derived by the second method. After compactification, the gauge transformation becomes Stüeckelberg gauge transformation. Stüeckelberg symmetries are quite restrictive and completely fix the equations of motion even for higher spin cases [63]. Therefore, it is enough to consider gauge transformations to obtain information about the reduced physics in the lower dimension.

Before displaying detail calculation, it is worth to summarize notations. The capital letters $M, N, \cdots$ will be used to represent the indices of AdS$_{d+2}$: they run from 0 to $d+1$. And the greek letters $\mu, \nu, \cdots$ are the indices of AdS$_{d+1}$: they run from 0 to d. Index for the internal direction is θ, i.e. $M = \{\mu, \theta\}$. The barred quantities are tensors of AdS$_{d+2}$ and quantities without "bar" represent tensors of AdS$_{d+1}$. Finally, we set AdS radius — ℓ — as one for the simple expression.

4.3.1 Mode function structure of spin-1 waveguide

Let us consider the method using the equation of motion. The spin-1 field equation in AdS$_{d+2}$ decompose into two equations:

$$\sec^2\theta\, \bar{g}^{MN}\, \overline{\nabla}_M\, \bar{F}_{\mu N} = \nabla^\nu\, F_{\mu\nu} - Ł_{d-2}\,(Ł_0\, A_\mu - \partial_\mu\, \phi) = 0, \tag{4.23}$$

$$\sec^2\theta\, \bar{g}^{MN}\, \overline{\nabla}_M\, \bar{F}_{\theta N} = \nabla^\mu\,(Ł_0\, A_\mu - \partial_\mu\, \phi) = 0\,, \tag{4.24}$$

where $\phi = \bar{A}_\theta$. The fields A_μ, ϕ can be expanded in terms of a complete set of mode functions on the interval $\theta \in [-\alpha, \alpha]$:

$$A_\mu = \sum_{n=0}^{\infty} A_\mu^{(n)}\, \Theta_n^{1|1}(\theta) \qquad \text{and} \qquad \phi = \sum_{n=0}^{\infty} \phi^{(n)}\, \Theta_n^{0|1}(\theta). \tag{4.25}$$

Mode functions are determined once proper boundary conditions are prescribed. As stated above, our key strategy is not to specify some boundary conditions at the outset but to require gauge invariance of various higher-spin fields and then classify possible boundary conditions that are compatible with the gauge invariances. If there is no such boundary condition, the higher-spin fields we assumed at the outset are actually absent — but as we shall show it is not.

What we learned from the example in page 43 is that boundary condition or mode function for

A_μ and ϕ must be related such that each term of equation (4.23) and (4.24) obey the same boundary condition. Otherwise, equations of motion give additional boundary conditions to each field and there is no lower dimensional degree of freedom. Therefore, each term of equation (4.23) and equation (4.24) can be expanded by the same mode function: from the equations of motion (4.23) and (4.24), we expect that

$$\boxed{Ł_{d-3}\, Ł_0\, \Theta_n^{1|1} = c_n^{11}\, \Theta_n^{1|1} \quad \text{and} \quad \begin{pmatrix} 0 & Ł_{d-3} \\ Ł_0 & 0 \end{pmatrix} \begin{pmatrix} \Theta_n^{1|1} \\ \Theta_n^{0|1} \end{pmatrix} = \begin{pmatrix} c_n^{01}\Theta_n^{1|1} \\ c_n^{10}\Theta_n^{0|1} \end{pmatrix}, \qquad (4.26)}$$

where c_n's are complex-number coefficients. The first equation indicates that the coefficient $-c_n^{11}$ is the mass-squared of spin-1 field in $(d+1)$ dimensions. These equations display the Sturm-Liouville(SL) problem at hand involves factorized differential operators, leading to

$$Ł_{d-2}\,(Ł_0\,\Theta_n^{1|1}) = c_n^{10}\, c_n^{01}\, \Theta_n^{1|1} \quad \text{such that} \quad c_n^{11} = c_n^{10} c_n^{01} \tag{4.27}$$

and also

$$Ł_0\,(Ł_{d-2}\,\Theta_n^{0|1}) = c_n^{01}\, Ł_0\, \Theta_n^{1|1} = c_n^{01}\, c_n^{10}\, \Theta_n^{0|1} \quad \text{and hence} \quad c_n^{00} = c_n^{11}. \tag{4.28}$$

By these relations, the field equations become

$$\sum_n \left[\nabla^\mu F_{\mu\nu}^{(n)} + c_n^{01}(c_n^{10}\, A_\nu^{(n)} -\, \partial_\nu\, \phi^{(n)})\right] \Theta_n^{1|1} = 0\,, \quad \sum_n \nabla^\mu \left[c_n^{10}\, A_\mu^{(n)} - \partial_\mu \phi^{(n)}\right] \Theta_n^{0|1} = 0\,. \tag{4.29}$$

From these equations, we draw two important results that the n-th Kaluza-Klein mode of spin-1 and spin-zero fields are related to each other, when their eigenvalues are equal,

$$c_n^{11} = c_n^{10} c_n^{01} = c_n^{00} = -M_n^2\,, \tag{4.30}$$

and that these spin-1 and spin-zero states combine together to trigger the Stüeckelberg mechanism. Note that central to this conclusion is the factorization property of the Sturm-Liouville second-order differential operator determining the Kaluza-Klein spectrum:

$$\boxed{Ł_{d-2} \quad \begin{matrix} \Theta_n^{1|1} \\ \upharpoonleft\downharpoonright \\ \Theta_n^{0|1} \end{matrix} \quad Ł_0 \qquad \text{with} \qquad M_n^2 = c_n^{11} = c_n^{00} = c_n^{10} c_n^{01}\,. \qquad (4.31)}$$

This shows that $Ł_{d-2}$, $Ł_0$ are raising and lowering operators between spin-1 and spin-zero states. The factorization property also makes it clear possible spectrum of massless field from the lowest

Kaluza-Klein mode $n = 0$. From the first-order relation, we see that there are two possibilities:

$c_0^{10} = 0$	no Stüeckelberg spin-zero field	Neumann
$c_0^{01} = 0$	no massless spin-1 field	Dirichlet

Attentive reader might have noticed the same underlying mathematical structure as the supersymmetry:

$$\begin{aligned} \dim \mathrm{Ker}\, Ł_{d-2} \neq 0 \qquad &\rightarrow \quad \text{Dirichlet}\,, \\ \dim \mathrm{Ker}\, Ł_0 \neq 0 \qquad &\rightarrow \quad \text{Neumann}\,. \end{aligned} \tag{4.32}$$

Note that modes in the kernel are massless fields. The modes which are not in kernel always combine together and undergo Stüeckelberg mechanism.

Let us consider method using the gauge transformations. In waveguide context, only those gauge transformations which do not change the boundary condition would make sense: gauge fields and gauge parameters should share the same boundary condition and hence the same mode functions as

$$\sum_n \delta A_\mu^{(n)}\, \Theta_n^{1|1}(\theta) = \sum_n \partial_\mu\, \Lambda^{(n)}\, \Theta_n^{1|1}(\theta)\,, \tag{4.33}$$

$$\sum_n \delta\phi^{(n)}\, \Theta_n^{0|1}(\theta) = \sum_n Ł_0\, \Lambda^{(n)}\, \Theta_n^{1|1}(\theta) = \sum_n c_n^{10}\, \Lambda^{(n)}\, \Theta_n^{0|1}(\theta). \tag{4.34}$$

It is worth to mention that the mode function structure (4.31) — which was obtained by the method using the equation of motion — can be derived by the second variation in (4.34) and the first equation in (4.26).

Collecting field equations and gauge transformations of n-th Kaluza-Klein modes, we get

$$\begin{aligned} &\nabla^\mu F_{\mu\nu}^{(n)} + c_n^{01}[c_n^{10}\, A_\nu^{(n)} - \partial_\nu\, \phi^{(n)}] = 0\,, \\ &\nabla^\mu[\, c_n^{10}\, A_\mu^{(n)} - \partial_\mu \phi^{(n)}] = 0\,, \\ &\delta A_\mu^{(n)} = \partial_\mu\, \Lambda^{(n)}\,, \quad \delta\phi^{(n)} = c_n^{10}\, \Lambda^{(n)} \end{aligned} \tag{4.35}$$

We recognize these as the Stüeckelberg equations of motion and Stüeckelberg gauge transformations that describe massive spin-1 field. Comparing with the standard form of Stüeckelberg system, we also identify the coefficients c_n's with the mass, $c_n^{10} = -c_n^{01} = M_n$.

4.3.2 Waveguide boundary conditions for spin-1 field

Let us consider boundary conditions and modes. We first concentrate on the following boundary conditions which do not contain higher-derivative boundary conditions(HD BC) [4]:

$$\begin{cases} \Theta^{(1|1)}\,|_{\theta=\pm\alpha} = 0\,, \quad Ł_{d-2}\, \Theta^{(0|1)}\,|_{\theta=\pm\alpha} = 0 & \text{Dirichlet} \\ Ł_0\, \Theta^{(1|1)}\,|_{\theta=\pm\alpha} = 0\,, \quad \Theta^{(0|1)}\,|_{\theta=\pm\alpha} = 0 & \text{Neumann} \end{cases} \tag{4.36}$$

[4] We could also consider the set of boundary conditions which contains HD BC which is the main subject of section 5.1. We postpone considering these issues after section 5.1: See page 76 and page 83.

The masses of Kaluza-Klein modes are determined by Sturm-Liouville differential equation (4.28) and above boundary conditions. The explicit value of masses are not concerned here, but it is worth to mention that the corresponding fields consist Stüeckelberg system. The ground mode of Neumann boundary condition is $\Theta_0^{1|1}(\theta) = const$ with $M_0 = 0$ and there is no scalar ground mode. For Dirichlet boundary condition, $\Theta_0^{0|1}(\theta) \propto (\sec\theta)^{2-d}$ is the ground mode with $M_0 = 0$ and there is no spin-1 ground mode. These boundary conditions are the AdS counterparts of "vector" and "scalar"

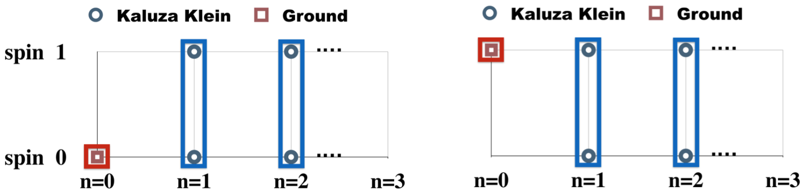

Figure 8: Spectra for boundary conditions. Each of them corresponds to Dirichlet and Neumann from left to right. x-axis is the numbering of modes and y axis represents spin. One Point corresponds to on mode function. Points in the same rectangle have the same eigenvalue and consist a Stüeckelberg system.

boundary conditions for flat spacetime in section 4.1. The spectrum of each boundary condition is summarized in Fig. 8.

Let us consider ground modes in the representation point of view. The sets of normalizable solutions of field equation consist a $\mathfrak{so}(d,2)$-module[5]. The relation between mass square and conformal weight — the Casimir of sub-algebra of $\mathfrak{so}(2)$ —,

$$m^2_{\text{spin}-0}\,\ell^2 = \Delta\,(\Delta - d)\ , \quad m^2_{\text{spin}-1}\,\ell^2 = \Delta\,(\Delta - d) + (d-1)\ , \tag{4.37}$$

tell us that the ground modes of Dirichlet and Neumann boundary conditions correspond to $\mathcal{D}\,(d,\,0)$ and $\mathcal{D}\,(d-1,\,1)$. They are irreducible parts of reducible Verma $\mathfrak{so}(d,2)$-module: $\mathcal{V}\,(d-1,\,1) = \mathcal{D}\,(d-1,\,1) \oplus \mathcal{D}\,(d,\,0)$. This pattern — ground mode comes from the irreducible parts of reducible Verma module — continues to the higher-spin cases.

As a summary, we can take following lessons.

- The mode functions of different spins in AdS$_{d+1}$ are related to each other as (4.31), which permits Stüeckelberg spin-1 structure.
- It is known that quadratic part of Stüeckelberg equation and action are uniquely determined by Stüeckelberg gauge transformation. Therefore, we could derive the lower dimensional equations of motion by only considering the gauge transformation in the lower dimension.

[5] For the conventions of the $\mathfrak{so}(d,2)$-module and representation, see Appendix B in page 139.

4.4 Waveguide Spectrum of Spin-2 Field

In this section, we extend the analysis of waveguide spectrum to a spin-2 field. The strategy is basically the same as a spin-1 case. We shall present the analysis as closely parallel as possible and highlight salient differences that are new beginning spin-2 and continue to higher spins.

4.4.1 Mode function structure of spin-2 waveguide

Let us first consider the method using the equation of motion. The equation of motion for massive spin-2 field with mass M on AdS$_{d+2}$ is given by $\mathcal{K}_{MN}(\bar h) - (d+1)\,(2\,\bar h_{MN} - \bar g_{MN}\,\bar h) - M^2\,(\bar h_{MN} - \bar g_{MN}\,\bar h) = 0$, where $\bar g_{MN}$ is the metric of AdS$_{d+2}$ and $\mathcal{K}_{MN}(\bar h)$ is the spin-2 Lichnerowicz operator:

$$\begin{aligned}\mathcal{K}_{MN}(\bar h) \;\; &= \;\; \Box\,\bar h_{MN} - \overline{\nabla}^L\,\overline{\nabla}_N\,\bar h_{ML} - \overline{\nabla}^L\,\overline{\nabla}_M\,\bar h_{NL} + \bar g_{MN}\,\overline{\nabla}_K\,\overline{\nabla}_L\,\bar h^{KL} \\ &\quad + \overline{\nabla}_M\,\overline{\nabla}_N\,\bar g^{KL}\,\bar h_{KL} - \bar g_{MN}\,\overline{\Box}\,\bar g^{KL}\,\bar h_{KL}\,. \end{aligned} \tag{4.38}$$

After compactification, the $(d+2)$-dimensional spin-2 field is decomposed to $(d+1)$-dimensional spin-2, spin-1 and scalar field:

$$h_{\mu\nu} = \bar h_{\mu\nu} + \frac{1}{d-1}\,g_{\mu\nu}\,\bar h_{\theta\theta}\,, \quad \bar h_{\mu\theta} = A_\mu\,, \quad \bar h_{\theta\theta} = \phi\,. \tag{4.39}$$

Note that the spin-2 field $h_{\mu\nu}$ is given as the linear combination of $\bar h_{\mu\nu}$ and $\bar h_{\theta\theta}$[6].

Equation of motion In terms of $(d+1)$-dimensional fields — $(h_{\mu\nu},\, A_\mu,\, \phi)$, the massless spin-2 equations in AdS$_{d+2}$ of motion read,

$$\begin{aligned}&\mathcal{K}_{\mu\nu}(h) - d\,(2\,h_{\mu\nu} - g_{\mu\nu}\,h) + \text{Ł}_{d-2}\,\text{Ł}_{-2}\,(h_{\mu\nu} - g_{\mu\nu}\,h) \\ &\quad - \text{Ł}_{d-2}\,(\nabla_\mu\,A_\nu + \nabla_\nu\,A_\mu - 2\,g_{\mu\nu}\,\nabla^\rho\,A_\rho) + \frac{d}{d-1}\,g_{\mu\nu}\,\text{Ł}_{d-2}\,\text{Ł}_{d-3}\,\phi = 0\,, \end{aligned} \tag{4.40}$$

$$\nabla^\mu\,F_{\mu\nu} - 2\,d\,A_\nu - \text{Ł}_{-2}\,(\nabla^\mu\,h_{\mu\nu} - \nabla_\nu\,h) - \frac{d}{d-1}\,\text{Ł}_{d-3}\,\nabla_\nu\,\phi = 0\,, \tag{4.41}$$

$$\Box\,\phi - \left(\frac{d+1}{d-1}\,\text{Ł}_{-1}\,\text{Ł}_{d-3} + d + 1\right)\phi - 2\,\text{Ł}_{-1}\,\nabla^\mu\,A_\mu + \text{Ł}_{-1}\,\text{Ł}_{-2}\,h = 0\,. \tag{4.42}$$

The mode expansions of each d-dimensional spin-2 field, spin-1 field and scalar field read

$$h_{\mu\nu} = \sum_n h^{(n)}{}_{\mu\nu}\,\Theta^{2|2}_n(\theta)\,, \quad A_\mu = \sum_n A^{(n)}{}_\mu\,\Theta^{1|2}_n(\theta)\,, \quad \phi = \sum_n \phi^{(n)}\,\Theta^{0|2}_n(\theta)\,. \tag{4.43}$$

From the equation (4.40, 4.41, 4.42), we can expect the relations between mode-functions which can be summarized by following two sets of factorized SL problems:

[6] There are cross terms with $\bar h$ and $\nabla^2\phi$ in the action. After this linear combination, we can remove the cross terms. This specific combination is the linearization of metric in original Kaluza-Klein reduction — $\bar g_{\mu\nu} = e^{\phi/(d-1)}\,g_{\mu\nu}$.

$$\begin{pmatrix} 0 & Ł_{d-2} \\ Ł_{-2} & 0 \end{pmatrix} \begin{pmatrix} \Theta_n^{2|2} \\ \Theta_n^{1|2} \end{pmatrix} = \begin{pmatrix} c_n^{12}\Theta_n^{2|2} \\ c_n^{21}\Theta_n^{1|2} \end{pmatrix} \tag{4.44}$$

$$\begin{pmatrix} 0 & Ł_{d-3} \\ Ł_{-1} & 0 \end{pmatrix} \begin{pmatrix} \Theta_n^{1|2} \\ \Theta_n^{0|2} \end{pmatrix} = \begin{pmatrix} c_n^{01}\Theta_n^{1|2} \\ c_n^{10}\Theta_n^{0|2} \end{pmatrix} \tag{4.45}$$

where c_n's are coefficients. The matrix equation (4.44) displays the following Sturm-Liouville problems:

$$Ł_{d-2}\,Ł_{-2}\,\Theta_n^{2|2} = c_n^{21}\,c_n^{12}\,\Theta_n^{2|2} = -M_n^2\,\Theta_n^{1|2}\,, \quad Ł_{-2}\,Ł_{d-2}\,\Theta_n^{1|2} = c_n^{12}\,c_n^{21}\,\Theta_n^{1|2} = -M_n^2\,\Theta_n^{1|2}\,. \tag{4.46}$$

The matrix equation (4.45) also displays two Sturm-Liouville problems:

$$Ł_{d-3}\,Ł_{-1}\,\Theta_n^{1|2} = c_n^{10}\,c_n^{01}\,\Theta_n^{1|2}\,, \qquad Ł_{-1}\,Ł_{d-3}\,\Theta_n^{0|2} = c_n^{01}\,c_n^{10}\,\Theta_n^{0|2}\,. \tag{4.47}$$

At first sight, $\Theta_n^{1|2}$ is the eigenfunction of two different SL problems. However we can show that two SL problems coincide by using the identity $Ł_m\,Ł_n - Ł_{n-1}\,Ł_{m+1} = (n-m-1)$ and obtain the relation for eigenvalues: $c_n^{10}c_n^{01} = c_n^{21}c_n^{12} - d + 1$. All relations can be summarized as,

$$\begin{array}{cccl} & \Theta_n^{2|2} & & \\ Ł_{d-2} & \upharpoonleft\!\downharpoonright & Ł_{-2} & : -M_{n,2|2}^2 = -M_n^2 = c_n^{21}c_n^{12} \\ & \Theta_n^{1|2} & & \\ Ł_{d-3} & \upharpoonleft\!\downharpoonright & Ł_{-1} & : -M_{n,1|2}^2 = -(M_n^2 + d - 1) = c_n^{10}c_n^{01} \\ & \Theta_n^{0|2} & & \end{array} \tag{4.48}$$

These relations define raising and lowering operators between $(d+1)$-dimensional fields of adjacent spins, a structure required for the Stüeckelberg mechanism[7]. If $M_{n,\,2|2}\,M_{n,\,1|2} \neq 0$, the corresponding modes in different spins combine and become Stüeckelberg spin-2 system. The other cases $M_{n,\,2|2}\,M_{n,\,1|2} = 0$ will be analyzed in section 4.4.2 with examples.

Gauge transformation Let us consider the method using gauge symmetries and derive the structure (4.48). The gauge transformation by gauge parameters $\bar{\xi}_M = \{\xi_\mu,\, \xi_\theta\}$, are

$$\begin{aligned} \delta h_{\mu\nu} &= \nabla_{(\mu}\,\xi_{\nu)} + \frac{1}{d-1}\,g_{\mu\nu}\,Ł_{d-2}\,\xi_\theta\,, \\ \delta A_\mu &= \frac{1}{2}\,\partial_\mu\,\xi_\theta + \frac{1}{2}\,Ł_{-2}\,\xi_\mu\,, \quad \delta\phi \;= Ł_{-1}\,\xi_\theta\,. \end{aligned} \tag{4.49}$$

[7] $M_n = M_{n,2|2}$ is the mass of the spin-2 field in the equation (4.40). On the contrary, $M_{n,1|2}$ is not related with mass-like term of spin-1 field in (4.41).

The mode functions of gauge parameter are proportional to mode functions of fields:

$$\xi_\mu = \sum_n \xi_\mu^{(n)}\, \Theta_n^{2|2}(\theta)\,, \qquad \xi_\theta = \sum_n \xi_\theta^{(n)}\, \Theta_n^{1|2}(\theta)\,. \tag{4.50}$$

By comparing the terms in (4.49), the raising and lowering operators in the structure of (4.48) can be obtained, which is derived by the form of equation of motion (4.40, 4.41, 4.42).

Stüeckelberg system After mode expansion, the equations of motions read,

$$\begin{aligned} &\mathcal{K}_{\mu\nu}(h^{(n)}) - d\left[2\, h^{(n)}{}_{\mu\nu} - g_{\mu\nu}\, h^{(n)}\right] + c_n^{21}\, c_n^{12}\left[h^{(n)}{}_{\mu\nu} - g_{\mu\nu} h^{(n)}\right] \\ &\quad - c_n^{12}\left[\nabla_\mu\, A^{(n)}{}_\nu + \nabla_\nu\, A^{(n)}{}_\mu - 2\, g_{\mu\nu}\, \nabla^\rho\, A^{(n)}{}_\rho\right] + c_n^{01}\, c_n^{12}\, \frac{d}{d-1}\, g_{\mu\nu}\, \phi^{(n)} = 0\,, \end{aligned} \tag{4.51}$$

$$\nabla^\mu\, F^{(n)}{}_{\mu\nu} - 2\, d\, A^{(n)}{}_\nu - c_n^{21}\, \nabla^\mu\left[h^{(n)}{}_{\mu\nu} - g_{\mu\nu}\, h^{(n)}\right] - c_n^{01}\, \frac{d}{d-1}\, \nabla_\nu\, \phi^{(n)} = 0\,, \tag{4.52}$$

$$\Box\, \phi^{(n)} - \left[\frac{d+1}{d-1}\, c_n^{01}\, c_n^{10} + d + 1\right]\phi^{(n)} - 2\, c_n^{10}\, \nabla^\mu\, A^{(n)}{}_\mu + c_n^{21}\, c_n^{10}\, h^{(n)} = 0 \tag{4.53}$$

and the gauge transformations read,

$$\delta h^{(n)}_{\mu\nu} = \nabla_{(\mu}\, \xi^{(n)}_{\nu)} + \frac{c_n^{12}}{d-1}\, g_{\mu\nu}\, \xi^{(n)}\,, \quad \delta A^{(n)}_\mu = \frac{1}{2}\, \partial_\mu\, \xi^{(n)} + \frac{c_n^{21}}{2}\, \xi^{(n)}_\mu\,, \quad \delta\phi^{(n)} = c_n^{10}\, \xi^{(n)}\,. \tag{4.54}$$

After defining appropriate c_n

$$c_n^{12} = -\sqrt{2}\, M_n\,, \quad c_n^{01} = -\sqrt{\frac{d}{2\,(d-1)}\, ({M_n}^2 + d - 1)}\,, \tag{4.55}$$

these system (4.51, 4.52, 4.53, 4.54) coincide the spin-2 Stüeckelberg system on AdS_d in the literature. Stüeckelberg gauge symmetries can fix all terms in quadratic parts in the action. Therefore, if we know about (4.54), the equations (4.51, 4.52, 4.53) can be deduced without any tedious calculations. And note that the modes which are neither in kernel of raising operators nor in kernel of lowering operators always combine together and undergo Stüeckelberg mechanism.

Before considering boundary conditions, it is worth to summarize Stüeckelberg spin-2 system and the breaking pattern of it. In general, Stüeckelberg spin-2 action describes the same physical degree of freedom with massive spin-2 field because spin-1 and scalar field can be algebraically fixed by the gauge transform (4.54). However these gauge fixing procedure cannot be done for the special values of masses:

$$M_n = 0 \quad \text{and} \quad M_n^2 = -\frac{(d-1)}{\ell^2}\,. \tag{4.56}$$

And Stüeckelberg system breaks into subsystems which can be deduced by the gauge transformation. For the case $M_n = 0$, the gauge transformations are

$$\delta\, h_{\mu\nu} = \nabla_{(\mu}\, \xi_{\nu)}\,, \quad \delta\, A_\mu = \frac{1}{2}\, \partial_\mu\, \lambda\,, \quad \delta\, \phi = -\sqrt{\frac{d}{2\,\ell^2}}\, \lambda\,, \tag{4.57}$$

which told us the subsystem. The spin-2 field is massless because there is the spin-2 gauge symmetry. And the existence of spin-1 Stüeckelberg gauge symmetry with $m^2 = 2\,d/\ell^2$ implies that the spin-1 and the scalar consist the corresponding Stüeckelberg spin-1 system. For the case $M_n^2 = -(d-1)/\ell^2$, the gauge transformations are

$$\delta h_{\mu\nu} = \nabla_{(\mu}\,\xi_{\nu)} + \frac{1}{\ell}\,g_{\mu\nu}\,\xi\,, \quad \delta A_\mu = \frac{1}{2}\,\partial_\mu\,\xi + \frac{1}{2\,\ell}\,\xi_\mu\,, \quad \delta\phi = 0\,. \tag{4.58}$$

The gauge symmetry of the spin-2 and spin-1 are Stüeckelberg partially massless(PM) gauge symmetry [63]. After fixing spin-1 field as zero, the remanent gauge symmetry coincides with the PM spin-2 gauge symmetry [43]: $\delta h_{\mu\nu} = \nabla_\mu\,\nabla_\nu\,\lambda - \frac{1}{\ell^2}\,g_{\mu\nu}\,\lambda$. Therefore, in this specific value, the Stüeckelberg system breaks into a spin-2 partially-massless(PM) Stüeckelberg system and a scalar[8] with $m^2 = (d+1)/\ell^2$.

This breaking pattern is parallel to the reducible pattern of Vermar $\mathfrak{so}(d,2)$-module of spin-2: $\mathcal{V}(\Delta, 2)$. For specific conformal weights — $\Delta = d$ and $\Delta = d-1$ —, Verma modules become reducible and break as,

$$\mathcal{V}\,(d,\,2) = \mathcal{D}\,(d,\,2) \oplus \mathcal{D}\,(d+1,\,1)\ , \quad \mathcal{V}\,(d-1,\,2) = \mathcal{D}\,(d-1,\,2) \oplus \mathcal{D}\,(d+1,\,0)\ . \tag{4.59}$$

The summand of the first and the second equality in (4.59) correspond to massless and partially massless representation. By using the relation between the mass squares of fields and the conformal weights[9],

$$m^2_{\text{spin}-1}\,\ell^2 = \Delta\,(\Delta - d) + (d-1)\ , \quad m^2_{\text{spin}-0,\,2}\,\ell^2 = \Delta\,(\Delta - d)\ , \tag{4.60}$$

we can show that the augend of first equality corresponds spin-1 field with $m^2 = 2\,d/\ell^2$, and the augend of second corresponds scalar field with $m^2 = (d+1)\,/\ell^2$. These results exactly match with the spectrum breaking patterns in previous paragraph.

The four kinds of fields which appear at special M_n are worth to summarize, because these four types of fields will appear as ground modes in section 4.4.2.

type	$\mathcal{D}(\Delta, s)_{\mathfrak{so}(\text{d},2)}$	Field types	mass square
type I	$\mathcal{D}(d+1, 0)$	Scalar field	$m^2 = (d+1)\,/\ell^2$
type II	$\mathcal{D}(d+1, 1)$	Stüeckelberg spin-1 system	$m^2 = 2\,d/\ell^2$
type III	$\mathcal{D}(d, 2)$	Massless spin-2	$m^2 = 0$
type IV	$\mathcal{D}(d-1, 2)$	PM spin-2 in Stüeckelberg form	$m^2 = -\,(d-1)\,/\ell^2$

Table 3: The list of fields which appear when spin-2 Stüeckelberg systems break two subsystems.

[8] The mass of scalar field is given in (4.53).

[9] We define the mass square equal to mass square in flat limit therefore, it is different with Fierz-Pauli mass square. See Appendix B in page 140.

4.4.2 Waveguide boundary conditions for spin-2 field

Boundary conditions Let us first classify the boundary conditions. As we learned in spin-1 examples, boundary conditions of different spin fields are related. This property continues in spin-2 AdS waveguide. For example, if we impose Dirichlet boundary condition for the spin-1 field in AdS_{d+1}, $\Theta^{1|2}|_{\theta=\pm\alpha}=0$, the structure of (4.48) impose the boundary conditions for other fields immediately:

$$\begin{aligned} &Ł_{-2}\,\Theta^{2|2}_n \sim \Theta^{1|2}_n\,, \qquad && Ł_{-2}\,\Theta^{2|2}|_{\theta=\pm\alpha}=0\,,\\ &Ł_{d-3}\,\Theta^{0|2}_n \sim \Theta^{1|2}_n\,, \qquad && Ł_{d-3}\,\Theta^{0|2}|_{\theta=\pm\alpha}=0\,. \end{aligned} \tag{4.61}$$

In a similar way, if we impose a boundary condition to one field, the structure (4.48) restricts all boundary conditions for other fields. We concentrate on the following boundary conditions[10]:

$$\begin{array}{llllll} \textbf{B.C. 1:} & \{ & \Theta^{2|2}|=0\,, & Ł_{d-2}\,\Theta^{1|2}|=0\,, & Ł_{d-2}\,Ł_{d-3}\,\Theta^{0|2}|=0 & \}\\ \textbf{B.C. 2:} & \{ & Ł_{-2}\,\Theta^{2|2}|=0\,, & \Theta^{1|2}|=0\,, & Ł_{d-3}\,\Theta^{0|2}|=0 & \}\\ \textbf{B.C. 3:} & \{ & Ł_{-1}\,Ł_{-2}\,\Theta^{2|2}|=0\,, & Ł_{-1}\,\Theta^{1|2}|=0\,, & \Theta^{0|2}|=0 & \} \end{array} \tag{4.62}$$

where $\Theta|$ implies the boundary values: $\Theta|_{\theta=\pm\alpha}$. The boundary conditions on each set are automatically imposed by the similar arguments with (4.61) after imposing $\Theta^{2|2}|_{\theta=\pm\alpha}=0$, $\Theta^{1|2}|_{\theta=\pm\alpha}=0$ and $\Theta^{0|2}|_{\theta=\pm\alpha}=0$ respectively. Let us inquire the spectra and the mode functions for these sets of boundary conditions.

Mode functions For simple expression, we focus on $d=2$ case, where the solutions of the Strum-Liouville differential equation are simple:

$$\Theta^{2|2} = \begin{cases} \sec\theta\,(\tan\theta\,\cos(z_n\theta)-z_n\,\sin(z_n\theta))\,, & \text{with odd condition}\\ \sec\theta\,(\tan\theta\,\sin(z_n\theta)+z_n\,\cos(z_n\theta))\,, & \text{with even condition} \end{cases} \tag{4.63}$$

$$\Theta^{1|2} = \begin{cases} \sec\theta\,\cos(z_n\theta)\,, & \text{with odd condition}\\ \sec\theta\,\sin(z_n\theta)\,, & \text{with even condition} \end{cases} \tag{4.64}$$

$$\Theta^{0|2} = \begin{cases} \sec\theta\,\sin(z_n\theta)\,, & \text{with odd condition}\\ \sec\theta\,\cos(z_n\theta)\,, & \text{with even condition} \end{cases} \tag{4.65}$$

with $z_n^2 = M_n^2+1$, and the odd(even) condition implies that mode function for spin-2 are odd function. Let us consider **B.C. 1** in detail. The boundary conditions for spin-2 and spin-1 boundary condition coincide:

$$\begin{cases} \sec\theta\,(\tan\theta\,\cos(z_n\theta)-z_n\,\sin(z_n\theta))\,|_{\theta=\pm\alpha}\,, & \text{with odd condition}\\ \sec\theta\,(\tan\theta\,\sin(z_n\theta)+z_n\,\cos(z_n\theta))\,|_{\theta=\pm\alpha}\,, & \text{with even condition} \end{cases}\,, \tag{4.66}$$

[10] Note that the first and the third conditions are higher-derivative boundary conditions(HD BC). HD BC is not self-adjoint in the functional space L^2, but is self-adjoint in expanded functional space. See section 5.1 for their properties.

and the boundary condition for scalar read

$$\begin{cases} z_n \sec\theta\,(\tan\theta\,\cos(z_n\theta) - z_n\,\sin(z_n\theta))\,|_{\theta=\pm\alpha}\,, & \text{with odd condition} \\ z_n \sec\theta\,(\tan\theta\,\sin(z_n\theta) + z_n\,\cos(z_n\theta))\,|_{\theta=\pm\alpha}\,, & \text{with even condition} \end{cases}, \tag{4.67}$$

which is equal to (4.67) except the overall term z_n[11]. In general, solutions of boundary condition, z_n, depend on α therefore corresponding modes are Kaluza-Klein modes. For this kind of solutions, there are always corresponding modes for each spins and corresponding fields combine and become spin-2 Stüeckelberg system with $M_n^2 = \sqrt{z_n^2 - 1}$.

Ground modes There are two special solution of boundary conditions: $z_n = 1$ and $z_n = 0$. These two special values are independent of α therefore, correspond to "ground mode" and have interesting features. First, the corresponding masses are equal to the mass which breaks Stüeckelberg system into subsystems in (4.56). And second, some spin modes do not exist for these values: for $z_n = 1$, there is no corresponding spin-2 mode — because $\Theta^{2|2} = 0$ in this case — and for $z_n = 0$ only corresponding scalar mode exists — because $z_n = 0$ is not a solution of (4.67). Combining these two facts we can identify corresponding fields: for $z_0 = 0$, scalar field is **type I** and for $z_1 = 1$, spin-1 and scalar combine and become **type II**. By a similar analysis, the following ground mode spectrum can be obtained:

$$\begin{array}{llll} \textbf{B.C. 1:} & \textbf{type II} & \text{and} & \textbf{type I} \\ \textbf{B.C. 2:} & \textbf{type III} & \text{and} & \textbf{type I} \\ \textbf{B.C. 3:} & \textbf{type IV} & \text{and} & \textbf{type III} \end{array} \tag{4.68}$$

The spectrum of each set of boundary conditions is summarized in Fig. 9. Let us start to consider the physical aspects of these ground spectrums.

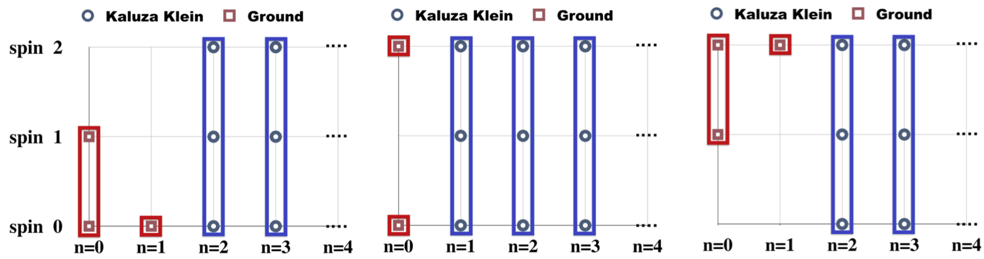

Figure 9: Spectra for boundary conditions. Each of them corresponds to **B.C.1**, **B.C.2** and **B.C.3** from left to right. x-axis is the numbering of modes and y axis represent spin. Points represent one mode function. Points in the same rectangle have the same eigenvalues and consist Stüeckelberg system. The results for an arbitrary spin is given in Fig. 12.

The ground modes of B.C. 3 We must comment on two points for the third spectrum. First, it is non-unitary because the partially massless field is not unitary in AdS [43]. We postpone the intuitive

[11] These agreements do not occur by accident but are the consequences of (4.48) and (4.61).

explanation how non-unitary spectrum can be obtained by compactification in section 5.1.2. And second, the normalized mode functions are given as

$$\begin{cases} \Theta_0^{2|2} = N_3 \, \sec\theta \, \tan\theta \,, \quad \Theta_0^{1|2} = N_4 \, \tan\theta & \textbf{type IV}\,, \\ \Theta_1^{2|2} = N_5 \, \sec^2\theta & \textbf{type III}\,, \end{cases} \tag{4.69}$$

with the normalization constants[12]:

$${N_3}^2 = -(2\alpha)^{-1}\,, \quad {N_5}^2 = -\frac{\tan\alpha}{2}\,, \quad N_4 = (\alpha - \cos\alpha\sin\alpha)^{-1/2}\,. \tag{4.70}$$

The normalization constant (4.70) indicates the ground modes in the third boundary condition have the opposite kinetic term sign, that will be confirmed in a direct way in subsection 5.1.2, and physical explanations of them are also given in the same subsection.

Decompactification limit limit As the boundary position α goes to $\pi/2$, the boundary surfaces of waveguide approach AdS_{d+2} boundary. However, the whole spectrum usually does not match with the spectrum of spin-2 on AdS_{d+2}. Most of the boundary conditions which we choose are singular therefore, some mode functions are ill-defined at this limit. For example, the spectrum for **B.C. 2** contains the massless spin-2 ground mode and scalar ground mode whose normalized mode functions are

$$\begin{cases} \Theta_0^{2|2} = N_1 \, \sec^2\theta \,, & \textbf{type III}\,, \\ \Theta_0^{0|2} = N_2 \, \sec\theta \,, & \textbf{type I}\,, \end{cases} \qquad N_1 = \sqrt{\frac{1}{2\tan\alpha}}\,, \quad N_2 = \frac{1}{\sqrt{2\alpha}}\,. \tag{4.71}$$

The ground-mode functions in (4.71) are not normalizable AdS_{d+2}: Normalized mode functions go to zero in this limit because of N_i. This explains why there is no massless spin-2 mode in "dimensional degression" method [39]. In section 5.2.3, it is shown that the "dimensional degression" spectrum [39] is equal to the spectrum of **B.C.1** in decompactification limit for general dimension and arbitrary spin.

Before concluding subsection, it is worth to summarize the lessons we can take.

- The mode function of different spins in AdS_{d+1} are related to each other as (4.48) which is completely fixed by considering the reduction of gauge transformations.
- Stüeckelberg spin-2 system breaks into subsystems for the specific masses as explained in the paragraph above Table 3. And the ground modes of (4.62) consist of the fields in Table 3 in these special subsystems

4.5 Waveguide Spectrum of Spin-3 Field

In this section, spin-3 waveguide shall be considered. The main strategy is quite similar to those are used in lower spins. The only difference is the traceless condition for spin-3 gauge parameter:

[12] We used the refined norm which is defined in section 5.1 instead of L^2 norm.

$\bar{g}^{\mu\nu}\,\bar{\zeta}_{\mu\nu}=0$ which compel us to consider linear combinations of gauge parameters.

Equation of motion The massless spin-3 equation of motion in AdS_{d+1} is $\mathcal{K}_0(w)-\frac{4(d+1)}{\ell^2}\,w_{\mu_1\mu_2\mu_3}+\frac{2(d+1)}{\ell^2}\,g_{(\mu_1\mu_2}\,w^{\lambda}{}_{\mu_3)\lambda}=0$ where $\mathcal{K}_0(w)$ is the spin-3 Lichnerowicz operator:

$$\begin{aligned}\mathcal{K}_0(w) &= \nabla^{\nu}\,\nabla_{\nu}\,w_{\mu_1\mu_2\mu_3}-\nabla^{\nu}\,\nabla_{(\mu_1}\,w_{\mu_2\mu_3)\nu}+\nabla_{(\mu_1}\,\nabla_{\mu_2}\,w_{\mu_3)\lambda}{}^{\lambda} \\ &\quad +\nabla^{\nu}\,\nabla^{\lambda}\,g_{(\mu_1\mu_2}\,w_{\mu_3)\nu\lambda}-\nabla^{\nu}\,\nabla_{\nu}\,g_{(\mu_1\mu_2}\,w^{\lambda}{}_{\mu_3)\lambda}-\frac{1}{2}\,g_{(\mu_1\mu_2}\,\nabla_{\mu_3)}\,\nabla^{\nu}\,w^{\lambda}{}_{\lambda\nu}\,.\end{aligned} \tag{4.72}$$

The $(d+2)$-dimensional spin-3 field, $\bar{w}_{M_1M_2M_3}$, decomposes to $(d+1)$-dimensional fields with different spins:

$$w_{\mu\nu\rho}=\bar{w}_{\mu\nu\rho}+\frac{1}{d+1}\,g_{(\mu\nu}\,\bar{w}_{\rho)\theta\theta},\quad h_{\mu\nu}=\bar{w}_{\mu\nu\theta}+\frac{1}{d-1}\,g_{\mu\nu}\,\bar{w}_{\theta\theta\theta}\,,\quad A_{\mu}=\bar{w}_{\mu\theta\theta}\,,\quad \phi=\bar{w}_{\theta\theta\theta}\,. \tag{4.73}$$

The linear combinations, which can be considered as counterparts of (4.39) in spin-2, are taken to remove cross terms, "$w\,\nabla^2\,A$" and "$h\,\nabla^2\,\phi$", in the action. In terms of these fields, equations of motions for $(w,\,h\,,A\,,\phi)$ read:

$$\begin{aligned}&\mathcal{K}_0(w)-4\,(d+1)\,w_{\mu_1\mu_2\mu_3}+2\,(d+1)\,g_{(\mu_1\mu_2}w^{\lambda}{}_{\mu_3)\lambda} \\ &\quad +\text{Ł}_{d-2}\,\text{Ł}_{-4}\left(w_{\mu_1\mu_2\mu_3}-g_{(\mu_1\mu_2}\,w^{\lambda}{}_{\mu_3)\lambda}\right) \\ &\quad -3\,\nabla_{(\mu_1}\,h_{\mu_2\mu_3)}+3\,\text{Ł}_{d-2}\left(\nabla_{(\mu_1}\,h-4\,\nabla^{\lambda}\,h_{\lambda(\mu_1)}\right)\,g_{\mu_2\mu_3)} \\ &\quad +\frac{3\,(d+2)}{d+1}\,\text{Ł}_{d-2}\,\text{Ł}_{d-3}\,g_{(\mu_1\mu_2}\,A_{\mu_3)}=0\,,\end{aligned} \tag{4.74}$$

$$\begin{aligned}&\mathcal{K}_0(h)-4\,(d+1)\,h_{\mu_1\mu_2}+(3\,d+4)\,g_{\mu_1\mu_2}\,h-\frac{3}{2}\,\text{Ł}_{-4}\,\text{Ł}_{d-2}\,g_{\mu_1\mu_2}\,h \\ &\quad -\frac{1}{2}\,\text{Ł}_{-4}\left(g_{\mu_1\mu_2}\,\nabla^{\lambda}\,w_{\lambda\nu}{}^{\nu}+2\,\nabla^{\lambda}\,w_{\lambda\mu_1\mu_2}-\nabla_{(\mu_1}\,w_{\mu_2)\lambda}{}^{\lambda}\right) \\ &\quad -\frac{d+2}{d+1}\,\text{Ł}_{d-3}\,(\nabla_{\mu}\,A_{\nu}+\nabla_{\nu}\,A_{\mu}-2\,g_{\mu\nu}\,\nabla^{\rho}\,A_{\rho})+\frac{d+2}{d-1}\,\text{Ł}_{d-3}\,\text{Ł}_{d-4}\,\phi=0\,,\end{aligned} \tag{4.75}$$

$$\begin{aligned}&\nabla^{\rho}\,F_{\rho\mu}-\frac{d+3}{d+1}\,\text{Ł}_{-3}\,\text{Ł}_{d-3}\,A_{\mu}-3\,(d+1)\,A_{\mu}+\text{Ł}_{-3}\,\text{Ł}_{-4}\,w^{\rho}{}_{\rho\mu} \\ &\quad -2\,\text{Ł}_{-3}\,(\nabla^{\rho}\,h_{\rho\mu}-\nabla_{\mu}\,h)-\frac{d+1}{d-1}\,\text{Ł}_{d-4}\,\nabla_{\mu}\,\phi=0\,,\end{aligned} \tag{4.76}$$

$$\begin{aligned}&\Box\,\phi-\frac{2\,(d+2)}{d-1}\,\text{Ł}_{-2}\,\text{Ł}_{d-4}\,\phi-2\,(d+2)\,\phi \\ &\quad +3\,\text{Ł}_{-2}\,\text{Ł}_{-3}\,h-3\,\text{Ł}_{-2}\,\nabla^{\lambda}\,A_{\lambda}=0\,.\end{aligned} \tag{4.77}$$

The theta derivatives appear only through the form of Ł_n, which implies the relations between modes functions:

$$
\begin{array}{ccl}
 & \Theta_n^{3|3} & \\
Ł_{d-2} & \uparrow\!\downarrow & Ł_{-4} \; : -M^2_{n,3|3} = -M^2_n \\
 & \Theta_n^{2|3} & \\
Ł_{d-3} & \uparrow\!\downarrow & Ł_{-3} \; : -M^2_{n,2|3} = -\big(M^2_n + d + 1\big) \\
 & \Theta_n^{1|3} & \\
Ł_{d-4} & \uparrow\!\downarrow & Ł_{-2} \; : -M^2_{n,1|3} = -\big(M^2_n + 2\,d\big) \\
 & \Theta_n^{0|3} &
\end{array}
\tag{4.78}
$$

Gauge transformation In this paragraph, we argue the followings: the structure (4.78) can be derived by the reduced gauge transformation, which is strong enough to fix the reduced equations of motion (4.74, 4.75, 4.76, 4.77). To achieve this goal, the reduced gauge parameters must be identified with the higher-spin Stüeckelberg gauge parameters which are traceless:

$$
\xi_{\mu\nu} \equiv \bar{\zeta}_{\mu\nu} + \frac{1}{d+1}\, g_{\mu\nu}\, \bar{\zeta}_{\theta\theta}\,, \quad \xi_\mu \equiv \frac{3}{2}\,\bar{\zeta}_{\mu\theta}\,, \quad \xi \equiv 3\,\bar{\zeta}_{\theta\theta}\,. \tag{4.79}
$$

The traceless Fronsdal gauge parameter conditions, $\bar{g}^{\mu\nu}\,\bar{\zeta}_{\mu\nu} = 0$, implies that $\xi_{\mu\nu}$ is traceless. Then the gauge transformations of the fields in (4.73) read

$$
\begin{aligned}
\delta\, w_{\mu\nu\rho} &= \partial_{(\mu}\,\xi_{\nu\rho)} + 3\, g_{(\mu\nu}\, Ł_{d-2}\,\xi_{\rho)}\,, \qquad & \delta\, h_{\mu\nu} &= \partial_{(\mu}\,\xi_{\nu)} + \frac{1}{3}\, Ł_{-4}\,\xi_{\mu\nu} + c\, g_{\mu\nu}\, Ł_{d-3}\,\xi\,, \\
\delta\, A_\mu &= \partial_\mu\,\xi + Ł_{-3}\,\xi_\mu\,, & \delta\,\phi &= 3\, Ł_{-2}\,\xi\,,
\end{aligned} \tag{4.80}
$$

where $c = \frac{2\,(d+2)}{(d^2-1)}$. By comparing the terms, we can deduce the relation between the mode functions:

$$
\Theta^{3|3} \sim Ł_{d-2}\,\Theta^{2|3}\,, \qquad \Theta^{2|3} \sim Ł_{-4}\,\Theta^{3|3} \sim Ł_{d-3}\,\Theta^{1|3}\,, \tag{4.81}
$$

$$
\Theta^{1|3} \sim Ł_{-3}\,\Theta^{2|3}\,, \qquad \Theta^{0|3} \sim Ł_{-2}\,\Theta^{1|3}\,, \tag{4.82}
$$

which can be summarised as (4.78). Furthermore, after mode-expansion, these gauge transformations can be identified with spin-3 Stüeckelberg gauge symmetry [63]. This procedure provides all information for physical spectrum; the eigenvalue of $Ł_{d-2}\, L_{-4}$ can be identified with the minus mass square of spin-3.

Boundary condition and ground modes An interesting feature of spin-3 is that the higher-derivative boundary condition is inevitable. After imposing a boundary condition for one field, the structure (4.78) automatically impose the boundary conditions for other-spin fields as we showed in section 4.4.2. For example, if we impose Dirichlet boundary condition for one field — $\Theta^{k|3}|_{\theta=\pm\alpha} =$

0, we will obtain following sets of boundary conditions:

$$
\begin{array}{lllll}
\{ & \Theta^{3|3}| = 0 & , \text{Ł}_{d-2}\,\Theta^{2|3}| = 0 & , \text{Ł}_{d-2}\,\text{Ł}_{d-3}\,\Theta^{1|3}| = 0 & , \text{Ł}_{d-2}\,\text{Ł}_{d-3}\,\text{Ł}_{d-4}\,\Theta^{0|3}| = 0\} \\
\{ & \text{Ł}_{-4}\,\Theta^{3|3}| = 0 & , \Theta^{2|3}| = 0 & , \text{Ł}_{d-3}\,\Theta^{1|3}| = 0 & , \text{Ł}_{d-3}\,\text{Ł}_{d-4}\,\Theta^{1|3}| = 0 \quad \} \\
\{ & \text{Ł}_{-3}\,\text{Ł}_{-4}\,\Theta^{3|3}| = 0 & , \text{Ł}_{-3}\,\Theta^{2|3}| = 0 & , \Theta^{1|3}| = 0 & , \text{Ł}_{d-4}\,\Theta^{0|3}| = 0 \quad \} \\
\{ & \text{Ł}_{-2}\,\text{Ł}_{-3}\,\text{Ł}_{-4}\,\Theta^{3|3}| = 0 & , \text{Ł}_{-3}\,\text{Ł}_{-4}\,\Theta^{2|3}| = 0 & , \text{Ł}_{-2}\,\Theta^{1|3}| = 0 & , \Theta^{0|3}| = 0 \quad \}
\end{array}
$$

Here, $\Theta|$ implies the boundary value: $\Theta|_{\theta=\pm\alpha}$. Above cases always contain at least one higher-derivative boundary condition(HD BC). The physical interpretation of HD BC is the subject of section 5.1.

The features of mode-functions are similar to spin-2. All Kaluza-Klein modes compose spin-3 Stüeckelberg system and the ground modes consist of the irreducible parts of reducible Verma $\mathfrak{so}(d,2)$-module:

$$
\mathcal{V}\,(d+k-1,\,3) = \mathcal{D}\,(d+k-1,\,3) \oplus \mathcal{D}\,(d+2,\,k) \qquad \text{for } k = 0,\,1,\,2\,. \tag{4.83}
$$

By the relation between mass square and conformal dimension in (4.60), we can recognize the possible type of fields that appear as ground mode:

type	$\mathcal{D}(\Delta, s)_{\mathfrak{so}(d,2)}$	Field types	mass square
type I	$\mathcal{D}(d+2,0)$	Scalar field	$m^2 = 2\,(d+2)\,/\ell^2$
type II	$\mathcal{D}(d+2,1)$	Stüeckelberg spin-1 system	$m^2 = 3\,(d+1)\,/\ell^2$
type III	$\mathcal{D}(d+2,2)$	Stüeckelberg spin-2 system	$m^2 = 2\,(d+2)\,/\ell^2$
type IV	$\mathcal{D}(d+1,3)$	Massless spin-3	$m^2 = 0$
type V	$\mathcal{D}(d,3)$	PM spin-3 with depth-1	$m^2 = -\,(d+1)\,/\ell^2$
type VI	$\mathcal{D}(d-1,3)$	PM spin-3 with depth-2	$m^2 = -2\,d/\ell^2$

Table 4: The list of fields which appear when spin-2 Stüeckelberg systems break two subsystems. "PM" implies partially massless and they are expressed in Stüeckelberg form.

And the spectra can be summarized in Fig. 10. which will be proved in section 5.2.

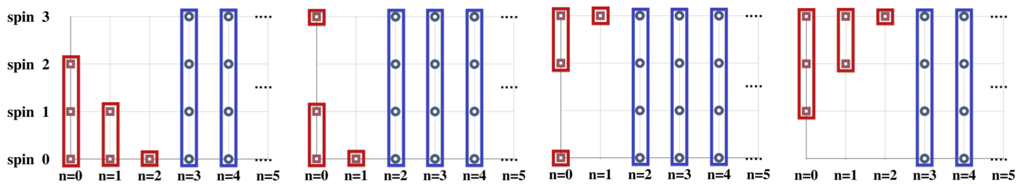

Figure 10: Spectra for spin-3 cases. x-axis is the numbering of modes and y axis represent spins. Point represents one mode function. Points in the same rectangle consist Stüeckelberg system.

Chapter 5

(Anti-)de Sitter Space Higher-Spin Waveguide

"I couldn't do it. I couldn't reduce it to the freshman level.
That means we don't really understand it."
— Richard Feynman

In previous chapter, we introduced the (A)dS waveguide and analyse the boundary conditions and spectrums of spin-1, 2 and 3 cases. And we showed that imposing the higher-derivative boundary condition is inevitable for $s \geq 3$. Therefore, before considering generalization for arbitrary spin-s, we give a simple example is considered in section 5.1 to provide the physical meaning of the higher derivative boundary conditions. This example provides the intuitive picture why non-unitary representation appears by reduction. All procedure extend to spin-s in section 5.2. As a clue for consistent reduction of interacting higher spin theory, reduction of frame-like higher spin theory and reduced higher spin algebra is considered in 5.3. Section 5.4 provides the possible extensions and discussion.

5.1 Higher-Derivative Boundary Condition

We encountered equations of motions and eigenvalue problems with higher-derivative boundary condition(HD BC). Especially for fields with spin greater than three, HD BC is unavoidable consequence. With HD BC, an Strum-Liouville differential operator is not self-adjoint on L^2 functional space and eigenfunctions are not orthogonal nor complete. Suppose we have an arbitrary function $f(\theta)$ and eigenfunctions $\Theta_n(\theta)$ from eigenvalue problem with HD BC. In previous sections, we just assumed $f(\theta)$ can be expanded by $\Theta_n(\theta)$ — $f(\theta) = \sum f_n \Theta_n(\theta)$ — and $f(\theta) = 0$ gives $f_n = 0$ for every n. Without orthogonality and completeness of $\Theta_n(\theta)$, each step is unclear and even we cannot get f_n from $f(\theta)$. How can we get $f_n = 0$ which correspond to lower dimensional equation of motions?

There is another issue about HD BC. When there have been boundaries, the variation of action usually contains surface terms, which should vanish by boundary condition for well defined variational principle. If we use an action from dimensional reduction without additional boundary action, the surface term will not vanish by HD BC. How can we decide proper boundary action?

There is the mathematical literature which studies about eigenvalue problems with HD BC (or eigen-parameter dependent BC) [65, 66] and provides answers to above two questions. The functional space is expanded from L^2 to $L^2 \oplus R^N$ and the refined norm for expanded functional space is defined. Then eigenvalue problem becomes self-adjoint for HD BC. Therefore, our fields and eigenfunction should be in $L^2 \oplus R^N$ and additional N real numbers will be interpreted as boundary degrees of freedom. Using this refined norm, we can do mode expansion of equation of motion, and also, can determine boundary action which gives well defined variational principle.

In the first subsection, we introduce an example of a vibrating string with second derivative boundary condition. It is simple and physically clear so one can see the relation between HD BC and boundary degrees of freedom. In the second subsection, we discuss spin-2 example with second derivative boundary condition which is analogous to string example.

5.1.1 Simple Example: Vibrating String System

First, we study a simple physical system which consist of a string, point particles and springs.[1] From this system, a higher-derivative boundary condition appears naturally and we can learn physical meaning of it.

Let us consider a vibrating string in three-dimensional spacetime — $(x,\, y,\, t)$ — whose end points are attached to point particles living in two-dimensional boundaries — $(0,\, y,\, t)$ and $(l,\, y,\, t)$. And length-less springs are connected with point particles and the fixed positions, $(x, y) = (0,\, 0)$ and $(l,\, 0)$, as described in Fig. 11 and their spring constants are $k_1,\, k_2$. The action of total system is

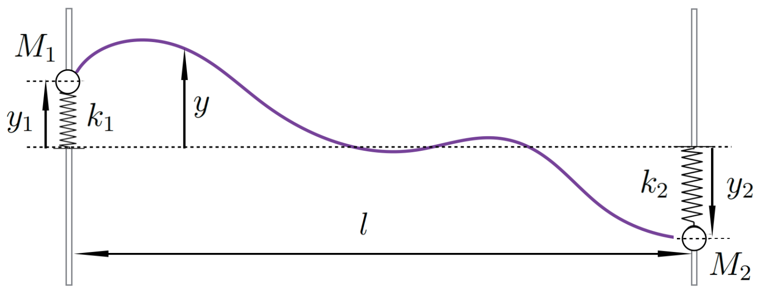

Figure 11: String, point particle and spring system.

$I = I_{\text{string}} + I_{\text{particle},\,1} + I_{\text{particle},\,2}$.

$$I_{\text{string}} = \frac{\tau}{2} \int \mathrm{d}t \int_0^l \mathrm{d}x \,\left(\dot{y}^2 - y'^2\right)\,, \quad I_{\text{particle, i}} = \frac{1}{2} \int \mathrm{d}t\, \left(M_i\, \dot{y}_i^2 - k_i\, {y_i}^2\right)\,, \tag{5.1}$$

where $M_{i=1,\,2}$ and $y_{1,\,2}(t)$ are the masses and the positions of point particles and τ is the string tension. "Prime" denotes the derivative with respect to x. End points of string are attached to point particles therefore we have constraints

$$y(0, t) = y_1(t), \quad y(l, t) = y_2(t). \tag{5.2}$$

Consider variation of action to get the equation of motions:

$$\delta I = -\tau \int \mathrm{d}t\mathrm{d}x\, \delta y(\ddot{y} - y'') + \tau \int \mathrm{d}t\, [\delta y\, y']_0^l - \sum_{i=1,2} \int \mathrm{d}t\, \delta y_i (M_i \ddot{y}_i + k_i y_i) \tag{5.3}$$

By above constraints $\delta y(0, t) = \delta y_1(t)$ and $\delta y(l, t) = \delta y_2(t)$, we obtain equation of motion for string

[1] This example was introduced at [67].

$\ddot{y} = y''$ and equations of motion for point particles,

$$M_1\,\ddot{y_1} + k_1\,y_1 - \tau\,y'|_{x=0} = 0\,, \quad M_2\,\ddot{y_2} + k_2\,y_2 + \tau\,y'|_{x=l} = 0\,. \tag{5.4}$$

Apply the constraints for every y_i and use string equation of motion at end points, then total degrees of freedom reduce to only the string and equations of motion for point particles become following higher-derivative boudnary conditions for the string.

$$\begin{cases} M_1\,y'' - \tau\,y' + k_1\,y = 0\,, & \text{for } x = 0 \\ M_2\,y'' + \tau\,y' + k_2\,y = 0\,, & \text{for } x = l \end{cases} \tag{5.5}$$

From this example, we could learn that higher-derivative boundary condition is related with additional boundary degrees of freedom. Evidently $M_{1,2} = 0$, which means no boundary degrees of freedom, is correspond to ordinary Robin boundary condition.

We can do above process reversely. Actually this reverse process is what we should do for higher spin equations. Suppose we have equation of motion for string $\ddot{y} = y''$ and HD BC (5.5). Variation of action which gives this system is,

$$\begin{aligned} \delta I \sim -\tau \int \mathrm{d}t \int_0^l \mathrm{d}x\; \delta y(\ddot{y} - y'') + \mathcal{N}_1 \int \mathrm{d}t\; \delta y(M_1 y'' - \tau y' + k_1 y)|_{x=0} \\ + \mathcal{N}_2 \int \mathrm{d}t\; \delta y(M_2 y'' + \tau y' + k_2 y)|_{x=l} \end{aligned} \tag{5.6}$$

where $\mathcal{N}$'s are normalization factors. To find the whole action from its variation, we should deal with $\delta y\, y''$ and $\delta y\, y'$ of surface terms because they cannot be written as a total variation of a term. Actually, $\delta y\, y'$ is the necessary surface term to get bulk action.

$$-\int \mathrm{d}t \int_0^l \mathrm{d}x\, \delta y\, y'' + \int \mathrm{d}t [\delta y\, y']_0^l = \delta \left(\frac{1}{2} \int \mathrm{d}t \int_0^l \mathrm{d}x\, {y'}^2 \right) \tag{5.7}$$

Therefore, $\mathcal{N}_1 = \mathcal{N}_2 = -\tau$. For $\delta y\, y''$, as we already learned from previous process, we should use equation of motion at end points.[2] In terms of action, using equation of motion is equivalent to adding one more possible term $\int \mathrm{d}t\, \delta y(\ddot{y} - y'')|_{x=0,l}$ to variation (5.6).

$$\int \mathrm{d}t\, \delta y\, y'' + \int \mathrm{d}t\, \delta y(\ddot{y} - y'') = \int \mathrm{d}t\, \delta y\, \ddot{y} = -\delta \left(\frac{1}{2} \int \mathrm{d}t\, \dot{y}^2 \right) \tag{5.8}$$

Then we get the action:

$$I = \frac{\tau}{2} \int \mathrm{d}t \int_0^l \mathrm{d}x\, (\dot{y}^2 - {y'}^2) + \frac{1}{2} \int \mathrm{d}t\, (M_1 \dot{y}^2 - k_1 y^2)|_{x=0} + \frac{1}{2} \int \mathrm{d}t\, (M_2 \dot{y}^2 - k_2 y^2)|_{x=l} \tag{5.9}$$

By introducing boundary degree $y_{1,2}$ together with the constraints (5.2), we get nice interpretation of boundary action as point particles and restore original action (5.1).

[2] The other possibility is $\delta y\, y'' + y\, \delta y'' = \delta(y\, y'')$ but we don't have and we don't want $y\, \delta y''$.

We want to do the similar analysis on a higher spin field. We have the equation of motion and boundary condition, therefore, we have to do inverse process. However, an inverse process which was introduced was tricky and hard to generalize to a complicated system. We introduce a more systematic inverse process. Let us first consider Robin boundary condition. To solve a string system, we expand string wave function by eigenfunctions of ∂_x^2. Then the action becomes summation of action for coefficient of each mode: $y(x,t) = \sum T_n(t) X_n(x)$ when $X_n'' = -\lambda_n X_n$ and

$$I_{\text{string}} = \sum_n \frac{\tau}{2} \int \mathrm{d}t \left(\dot{T}_n^2 - \lambda_n T_n^2 \right) . \tag{5.10}$$

Such mode expansion was possible because the action can be written in terms of L^2 norm,

$$I_{\text{string}} = \frac{\tau}{2} \int \mathrm{d}t \left(\langle \dot{y} | \dot{y} \rangle + \langle y | \partial_x^2 y \rangle \right), \quad \langle f(x) | g(x) \rangle \equiv \int_0^l \mathrm{d}x\, f(x)\, g(x) \tag{5.11}$$

and the operator ∂_x^2 is self-adjoint on L^2 space. Motivated by this observation, even for HD BC, let us require the total action $I = I_{\text{string}} + I_{\text{boundary}}$ to be a following form,

$$I = \frac{\tau}{2} \int \mathrm{d}t \left(\langle \dot{y} | \dot{y} \rangle_R + \langle y | \partial_x^2 y \rangle_R \right) \tag{5.12}$$

where $\langle \, | \, \rangle_R$ is the refined norm which makes ∂_x^2 self-adjoint for HD BC. Such the refined norm can be found in mathematical literatures, [65, 66]. It can be defined naturally on expanded function space $L^2 \oplus R^2$. A general element of $L^2 \oplus R^2$ and its norm with respect to HD BC (5.5) is

$$\vec{f} = \begin{pmatrix} f(x) \\ f_1 \\ f_2 \end{pmatrix}, \qquad \vec{f} \cdot \vec{g} = \int_0^l \mathrm{d}x\, f(x)\, g(x) + \frac{M_1}{\tau} f_1 g_1 + \frac{M_2}{\tau} f_2 g_2 . \tag{5.13}$$

Roughly speaking, two new real numbers $f_{1,2}$ correspond to boundary value of $f(x)$ which are not determined by the eigenvalue equation and HD BC. The boundary conditions on element of $L^2 \oplus R^2$ is (5.5) for $f(x)$ together with $f_1 = f(0)$ and $f_2 = f(l)$. With these boundary conditions we can define the refined norm $\langle \, | \, \rangle_R$.

$$\langle f(x) | g(x) \rangle_R \equiv \vec{f} \cdot \vec{g}\,|_{b.c.} = \int_0^l \mathrm{d}x\, f(x)\, g(x) + \frac{M_1}{\tau} f(0) g(0) + \frac{M_2}{\tau} f(l) g(l). \tag{5.14}$$

Then one can show that $\langle f | \partial_x^2 g \rangle_R - \langle \partial_x^2 f | g \rangle_R$ is a surface term

$$\begin{aligned} &\frac{1}{\tau} f (M_1 g'' - \tau g' + k_1 g)|_{x=0} - \frac{1}{\tau} (M_1 f'' - \tau f' + k_1 f) g|_{x=0} \\ &+ \frac{1}{\tau} f (M_2 g'' + \tau g' + k_2 g)|_{x=l} - \frac{1}{\tau} (M_2 f'' + \tau f' + k_2 f) g|_{x=l} \end{aligned} \tag{5.15}$$

which vanish by HD BC (5.5) and ∂_x^2 is self-adjoint. Then the action (5.12) becomes the original action (5.1). Note that above boundary conditions $f_1 = f(0)$ and $f_2 = f(l)$ are the same with (5.2), therefore $f_{1,2}$ are identified as boundary degrees of freedom.

For the second inverse process, we just assumed mode expansion of total action rather than a well-defined variational problem. And surprisingly, we could restore the original action. Actually imposing mode expansion is stronger than imposing a well-defined variational problem because variation with respect to the coefficient of each mode gives the well-defined variational problem. The refined norm gives answers to the questions arisen at the beginning of this section. We can do mode expansion of equation of motion and also we can find proper boundary action. Therefore, we will concentrate on finding proper refined norm for higher spin fields.

Before moving on to higher spin fields, let us emphasize a remark on the refined norm. An interesting point of the refined norm (5.14) is that the norm can be negative when point particle mass is negative $\frac{M}{\tau} < 0$. For the case with $M_1 = M_2 = M < 0 < \tau$ and $k_1 = k_2 = 0$, there is always at least one mode,

$$X_0(x) = N_0 \sinh\left(m_0\left(x - \frac{l}{2}\right)\right), \quad \text{with} \quad \frac{1}{m_0} = -\frac{\tau}{M}\tanh\frac{m_0\, l}{2}, \tag{5.16}$$

whose norm is negative:

$$\langle X_0|X_0\rangle_R = {N_0}^2\left(-\frac{l}{2} + \frac{M}{\tau}\sinh^2\frac{m_0\, l}{2}\right) < 0\,. \tag{5.17}$$

In this string system, the cause of the unstable mode with negative norm is obvious: a negative mass of point particle. In the same way, we can suppose the cause of the opposite sign of kinetic term and the appearance of PM field in page 58: an opposite sign of boundary action. We show that this supposition is right in next section.

5.1.2 Spin-2 Waveguide with Higher-Derivative Boundary Conditions

We will consider spin-2 example with sets of boundary conditions which contain higher derivatives — **B.C. 1** and **B.C. 3** in section 4.4.2 — and provide the reason why non-unitarity rises in AdS$_d$. From the example of the string system, we learned that the refined norm for HD BC gives correct boundary action for well-defined variational principle and makes it possible to expand equations of motions. Therefore, let us introduce the refined norm for spin-2 example.

The refined norm and negative norm states

The eigenvalue problems that we will consider have the following form:

$$Ł_b\, Ł_a\, \Theta_n = -\lambda\, \Theta_n\,, \quad Ł_c\, Ł_a\, \Theta_n\big|_{\theta=\pm\alpha} = 0\,. \tag{5.18}$$

Note that the eigenvalue equation and the boundary condition share the same operator $Ł_a$. From dimensional reduction of spin-2 action, we get an L^2 norm with weight factor [3]:

$$\langle \Theta_m | \Theta_n \rangle = \int_{-\alpha}^{\alpha} \mathrm{d}\theta \, (\sec\theta)^{a+b} \, \Theta_m(\theta) \, \Theta_n(\theta) \,. \tag{5.19}$$

We can show that the differential operator $Ł_b \, Ł_a$ is not self-adjoint on L^2 functional space by using the following relation twice,

$$\int_{-\alpha}^{\alpha} \mathrm{d}\theta \, (\sec\theta)^c \, \Theta_m (Ł_a \Theta_n) = - \int_{-\alpha}^{\alpha} \mathrm{d}\theta \, (\sec\theta)^c \, (Ł_{c-a} \Theta_m) \, \Theta_n + (\sec\alpha)^c \big[\Theta_m \, \Theta_n \big]_{-\alpha}^{+\alpha} \,, \tag{5.20}$$

$$\langle \Theta_m | Ł_b \, Ł_a \, \Theta_n \rangle - \langle Ł_b \, Ł_a \, \Theta_m | \Theta_n \rangle = (\sec\alpha)^{a+b} \big[\Theta_m (Ł_a \Theta_n) - (Ł_a \Theta_m) \Theta_n \big]_{-\alpha}^{+\alpha} \neq 0 \,. \tag{5.21}$$

Refined norm which makes $Ł_b Ł_a$ self-adjoint is,

$$\langle \Theta_m | \Theta_n \rangle_R \equiv \langle \Theta_m | \Theta_n \rangle + \mathcal{N}_+ \Theta_m \Theta_n |_{+\alpha} + \mathcal{N}_- \Theta_m \Theta_n |_{-\alpha} \,. \tag{5.22}$$

where $\mathcal{N}_+ = \mathcal{N}_- = \frac{1}{c-b} \frac{(\sec\,\alpha)^{a+b}}{\tan\,\alpha}$. We can show that $Ł_b Ł_a$ is self-adjoint under the refined norm by the following calculation:

$$\begin{aligned}
&\langle \Theta_m | Ł_b \, Ł_a \Theta_n \rangle_R - \langle Ł_b \, Ł_a \Theta_m | \Theta_n \rangle_R \\
&= (\sec\alpha)^{a+b} \Big(\Theta_m (Ł_a \Theta_n) - (Ł_a \Theta_m) \Theta_n \Big) \big|_{+\alpha} - (\sec\alpha)^{a+b} \Big(\Theta_m (Ł_a \Theta_n) - (Ł_a \Theta_m) \Theta_n \Big) \big|_{-\alpha} \qquad (5.23) \\
&\quad + \mathcal{N}_+ \Big(\Theta_m Ł_b Ł_a \Theta_n - (Ł_b Ł_a \Theta_m) \, \Theta_n \Big) \big|_{+\alpha} + \mathcal{N}_- \Big(\Theta_m Ł_b Ł_a \Theta_n - (Ł_b Ł_a \Theta_m) \, \Theta_n \Big) \big|_{-\alpha} \\
&= (\sec\alpha)^{a+b} \big[\Theta_m (Ł_c Ł_a \Theta_n) - (Ł_c Ł_a \Theta_m) \Theta_n \big]_{-\alpha}^{+\alpha} \qquad (5.24)
\end{aligned}$$

Note that the last expression vanishes by the boundary conditions in (5.18). Let us apply the refined norm to the ground modes for **B.C. 1** and **B.C. 3** in section 4.4.2.

Consider the normalization of mode functions: $\langle \Theta_n | \Theta_n \rangle_R \overset{?}{=} 1$, which is always possible for positive-norm states. For **B.C. 1**, the constant $\mathcal{N}_\pm$ in (5.22) is positive and the refined norm is positive-definite. In contrast, the constant $\mathcal{N}_\pm$ for **B.C. 3** is negative and the refined norm is not positive-definite in this case. To see the explicit example, let us concentrate on ground modes in **B.C. 3**:

$$\begin{cases} \Theta_1^{2|2} = N_1 \, \sec\theta \, \tan\theta \,, \quad \Theta_1^{1|2} = N_2 \, \sec\theta & \textbf{type IV} \text{ in Table 3} \,, \\ \Theta_0^{2|2} = N_3 \, \sec^2\theta & \textbf{type II} \text{ in Table 3} \,, \end{cases} \tag{5.25}$$

which correspond to PM and massless fields. Because spin-1 mode function is in L^2 space and norm is positive-definite, appropriate N_2 always can be found. In contrast, the normalization conditions for

[3]The spin-2 action in AdS_{d+2} has two source of $\sec\theta$: $\sqrt{-\bar{g}}$ and $\bar{g}^{-1}$. Each of them gives the factor $(\sec\theta)^{d+2}$ and $(\sec\theta)^{-6}$ which coincide with the weight factor because $a + b = d - 4$ in spin-2 cases

spin-2 modes, $\langle\Theta_n|\Theta_n\rangle_R = 1$, imply that

$$N_1{}^{-2} = \int_{-\alpha}^{\alpha} d\theta\, \sec^{d-2}\theta\, \tan^2\theta - \frac{2}{d-1}\, \sec^{d-2}\alpha\, \tan\alpha\,, \tag{5.26}$$

$$N_3{}^{-2} = \int_{-\alpha}^{\alpha} d\theta\, \sec^{d}\theta - \frac{2\,\sec^d\alpha}{(d-1)\,\tan\alpha}\,. \tag{5.27}$$

It can be shown that $N_1{}^2$, which corresponds to PM mode, is always negative[4]. Therefore the PM mode in **B.C. 3** have the same properties with unstable mode in string system (5.16).

Boundary action

The action of free massless spin-2 field $\bar{h}_{MN}$ on AdS$_{d+2}$ background is

$$\begin{aligned} &\int \mathrm{d}^{d+2}x\, \mathcal{L}_2\left(\bar{h}_{MN};\, d+2\right) \\ =& -\int \mathrm{d}^{d+2}x\, \Big(\frac{1}{2}\,\bar{\nabla}^L\,\bar{h}^{MN}\,\bar{\nabla}_L\,\bar{h}_{MN} - \bar{\nabla}^M\,\bar{h}^{NL}\,\bar{\nabla}_N\,\bar{h}_{ML} + \bar{\nabla}^M\,\bar{h}_{MN}\,\bar{\nabla}^N\,\bar{h} - \frac{1}{2}\,\bar{\nabla}^L\,\bar{h}\,\bar{\nabla}_L\,\bar{h} \\ &+ (d+1)\left(\bar{h}^{MN}\,\bar{h}_{MN} - \frac{1}{2}\,\bar{h}^2\right)\Big) \end{aligned} \tag{5.28}$$

After reduction, each term of (5.28) is divided into several quadratic terms of $h_{\mu\nu}$, A_μ and ϕ which can be represented as L^2 norm (5.19) with $a+b = d-4$. For example,

$$\int \mathrm{d}^{d+1}x\,\sqrt{-g}\,\int_{-\alpha}^{\alpha} \mathrm{d}\theta\,(\sec\theta)^{d-4}\,\nabla^\rho\,h^{\mu\nu}\,\nabla_\rho\,h_{\mu\nu} = \int \mathrm{d}^{d+1}x\,\sqrt{-g}\,\langle\,\nabla^\rho\,h^{\mu\nu}|\nabla_\rho\,h_{\mu\nu}\,\rangle\,.$$

Actually, the action (5.28) is just bulk action which is the counterpart of string action in (5.1). Introducing boundaries (or boundary conditions), boundary actions, which are the counterparts of point particle action in (5.1), should be added to (5.28) to define the variational principle appropriately. As we learned from the string example, one way of doing it is using the refined norm. Contrast to the string example, we have fields with various boundary conditions together with L_m operators. Let us classify terms of action by boundary conditions of fields. Even before specific boundary conditions are assigned, we know there are three kinds of boundary conditions.

- spin 2 like, expanded by $\Theta_n^{2|2}$: $h_{\mu\nu}$, $\text{Ł}_{d-2}A_\mu$, $\text{Ł}_{d-2}\text{Ł}_{d-3}\phi$
- spin 1 like, expanded by $\Theta_n^{1|2}$: $\text{Ł}_{-2}h_{\mu\nu}$, A_μ, $\text{Ł}_{d-3}\phi$
- spin 0 like, expanded by $\Theta_n^{0|2}$: $\text{Ł}_{-1}\text{Ł}_{-2}h_{\mu\nu}$, $\text{Ł}_{-1}A_\mu$, ϕ

The crucial observation is that every term in the reduced action is composed of two fields with the same boundary condition. One can check that from the explicit form of the reduced Lagrangian in

[4] The inverse square of other constant, $N_2{}^{-2}$, is negative for $\alpha \sim 0$ and positive for $\alpha \sim \pi/2$. When one of Kaluza-Klein mass hits zero mass, N_1 becomes infinity. In this specific value of α, there is not massless spin-2 field, **type III**, in the spectrum and **type II** appear instead.

AdS$_{d+1}$:

$$
\begin{aligned}
&\int \mathrm{d}\theta\,(\sec\theta)^{d-4}\,\mathcal{L}_2\,(h_{\mu\nu};\,d+1) - \frac{1}{2}\,\langle F^{\mu\nu}|F_{\mu\nu}\rangle + d\,\langle A^{\mu}|A_{\mu}\rangle - \frac{B}{2}\,(\langle\nabla^{\mu}\,\phi|\nabla_{\mu}\,\phi\rangle + (d+1)\,\langle\phi|\phi\rangle) \\
&+ \frac{1}{2}\langle \text{Ł}_{-2}\,h^{\mu\nu}|\text{Ł}_{-2}\,h_{\mu\nu}\rangle - \frac{1}{2}\langle \text{Ł}_{-2}\,h|\text{Ł}_{-2}\,h\rangle + \frac{d(d+1)}{(d-1)^2}\langle \text{Ł}_{d-4}\phi|\text{Ł}_{d-4}\phi\rangle \qquad (5.29) \\
&+ \langle \text{Ł}_{-2}h^{\mu\nu}|\nabla_{\mu}A_{\nu} + \nabla_{\nu}A_{\mu} - 2g_{\mu\nu}\nabla^{\rho}A_{\rho}\rangle - B\,\langle \text{Ł}_{-2}h|\text{Ł}_{d-4}\phi\rangle + B\,\langle \text{Ł}_{-1}A^{\mu}|\nabla_{\mu}\phi\rangle\,,
\end{aligned}
$$

where $B = d/(d-1)$. We can classify every term into following three types: $\langle\Theta^{2|2}|\Theta^{2|2}\rangle$, $\langle\Theta^{1|2}|\Theta^{1|2}\rangle$ and $\langle\Theta^{0|2}|\Theta^{0|2}\rangle$[5]; $\langle \text{Ł}_{-1}A^{\mu}|\nabla_{\mu}\phi\rangle$ is classified as $\langle\Theta^{0|2}|\Theta^{0|2}\rangle$. Now we are going to assign specific boundary conditions and get the boundary actions.

For **B.C. 1**, $\text{Ł}_{-1}\text{Ł}_{d-3}\Theta^{0|2} = -\lambda\Theta^{0|2}$ and $\text{Ł}_{d-2}\text{Ł}_{d-3}\Theta^{0|2}\big|_{\theta=\pm\alpha} = 0$, we should use the refined norm for $\langle\Theta^{0|2}|\Theta^{0|2}\rangle$ type terms: the kinetic and mass terms of ϕ and the cross term $\langle \text{Ł}_{-1}A^{\mu}|\nabla_{\mu}\phi\rangle$. Note that the boundary conditions for other type terms are the Dirichlet and Robin boundary condition and L^2 norm is proper norm for mode expansion. Apply (5.22) with $a = d-3$, $b = -1$, $c = d-2$, then we get the boundary actions $\langle\,|\,\rangle_R - \langle\,|\,\rangle$ which are the counterparts of the point particle actions in (5.1):

$$
\begin{aligned}
-\int \mathrm{d}^{d+1}x\sqrt{-g}\;\frac{d\,(\sec\alpha)^{d-4}}{(d-1)^2\tan\alpha}\Big[\Big(\frac{1}{2}\nabla^{\mu}\phi\nabla_{\mu}\phi + \frac{d+1}{2}\phi^2\Big)\big|_{\theta=+\alpha} - \Big(\text{Ł}_{-1}A^{\mu}\nabla_{\mu}\phi\Big)\big|_{\theta=+\alpha} \\
+\Big(\frac{1}{2}\nabla^{\mu}\phi\nabla_{\mu}\phi + \frac{d+1}{2}\phi^2\Big)\big|_{\theta=-\alpha} - \Big(\text{Ł}_{-1}A^{\mu}\nabla_{\mu}\phi\Big)\big|_{\theta=-\alpha}\Big]\,. \quad (5.30)
\end{aligned}
$$

Motivated by previous string example, let us introduce boundary fields $\phi^{\pm}$ which is related with bulk field ϕ by

$$
\phi^{\pm} = \left(\frac{d}{(d-1)^2}\frac{(\sec\alpha)^{d-4}}{\tan\alpha}\right)^{1/2}\phi\big|_{\theta=\pm\alpha} \qquad (5.31)
$$

Then the first line of boundary action become the kinetic term and mass term of $\phi^{\pm}$. $\phi^{\pm}$ become the dynamical degree of freedom like point particle in string example. Note that the sign of kinetic term is equal to standard one in our convention, which is the consequence of $\mathcal{N}_{\pm} = \frac{1}{d-1}\frac{(\sec\alpha)^{d-4}}{\tan\alpha} > 0$. This is compatible with the fact that after dimensional reduction, every spectrum is the unitary representation. For the second line of boundary action, there are two ways of interpretation. As $\text{Ł}_{-1}A_{\mu}\big|_{\theta=\pm\alpha}$ do not have the kinetic term in boundary action, it is not a dynamical degree of freedom. $\text{Ł}_{-1}A_{\mu}\big|_{\theta=\pm\alpha}$ is just determined by the dynamics of bulk field A_{μ}. Therefore, one interpretation of the cross term is an interaction (or mixing) between bulk field A_{μ} and boundary field $\phi^{\pm}$. Or we can delete this term by writing bulk cross term as $\langle A^{\mu}|\text{Ł}_{d-3}\nabla_{\mu}\phi\rangle_{L^2}$. This is related with the ambiguity which is explained in footnote[5]. It will be discussed at the end of this section.

For **B.C. 3**, $\text{Ł}_{d-2}\text{Ł}_{-2}\Theta^{0|2} = -\lambda\Theta^{2|2}$ and $\text{Ł}_{-1}\text{Ł}_{-2}\Theta^{2|2}\big|_{\theta=\pm\alpha} = 0$, we should use the refined norm for $\langle\Theta^{2|2}|\Theta^{2|2}\rangle$ type terms: the kinetic and mass-like terms of $h_{\mu\nu}$. Apply (5.22) with $a = -2$, $b = d-2$, $c = -1$, then we get the boundary actions which are the counterparts of the point particle

[5] This classification is ambiguous. $\langle \text{Ł}_{d-3}\phi|\text{Ł}_{d-3}\phi\rangle$ is $\langle\Theta^{1|2}|\Theta^{1|2}\rangle$ but its another form $\langle\phi|\text{Ł}_{-1}\text{Ł}_{d-3}\phi\rangle$, which can be obtained by integration by parts, is $\langle\Theta^{0|2}|\Theta^{0|2}\rangle$. We will show that the total action is the same.

actions in (5.1):

$$-\int \mathrm{d}^{d+1}x\sqrt{-g}\ \left[\mathcal{L}_2\left(h^{+}_{\mu\nu};\, d+1\right)+\mathcal{L}_2\left(h^{-}_{\mu\nu};\, d+1\right)\right]\ , \tag{5.32}$$

where $h^{\pm}_{\mu\nu}$ is the boundary degrees of freedom, $h^{\pm}_{\mu\nu}=\left(\frac{1}{d-1}\frac{(\sec\alpha)^{d-4}}{\tan\alpha}\right)^{1/2} h_{\mu\nu}\big|_{\theta=\pm\alpha}$. Note that the minus sign in front of standard massless spin-2 action in (5.32). We cannot absorb -1 into h^2 while keeping the reality of h. This is the consequence of $\mathcal{N}_{\pm}=-\frac{1}{d-1}\frac{(\sec\alpha)^{d-4}}{\tan\alpha}<0$. This opposite sign to standard in kinetic term is the origin of the existence of non-unitary spectrum (partially massless) after dimensional reduction. Furthermore, also, the kinetic term for the partially massless field has the wrong sign (analogous to negative norm state of string example).

Here is the summary of this section.

- For the higher-derivative boundary condition(HD BC), we must extend the functional space from L^2 to $L^2\oplus R^N$ to make operator self-adjoint. This extension can be physically understood as adding boundary degrees of freedom.
- From the refined norm, we obtained the “boundary action” for a given HD BC. With boundary action, we could see the pathology of boundary conditions more directly. For **B.C. 3** in section 4.4.2, we got a boundary action with kinetic term of wrong sign which is the origin of non-unitarity of lower dimensional spectrum.
- The boundary actions for **B.C. 1** are equal to the action of scalar field with mass of **type I** in Table 3. And the “boundary actions” for **B.C. 3** are equal to the action of massless spin-2 field with opposite sign.

Before concluding this subsection, let us discuss about ambiguity which is mentioned in in footnote[5]. Consider **B.C. 1** and a term $\langle Ł_{d-3}\phi|Ł_{d-3}\phi\rangle$. Such term was classified as $\langle\Theta^{1|2}|\Theta^{1|2}\rangle$ and do not have to be refined. Using the integration by part (5.20), the same term can be written as $-\langle\phi|Ł_{-1}Ł_{d-3}\phi\rangle$ with surface term $\left[(\sec\alpha)^{d-4}\,\phi Ł_{d-3}\phi\right]\Big|_{-\alpha}^{+\alpha}$. Now it is classified as $\langle\Theta^{0|2}|\Theta^{0|2}\rangle$ and have to be refined. This situation looks ambiguous but it is not.

$$\begin{aligned}\langle\phi|Ł_{-1}Ł_{d-3}\phi\rangle_R &= \langle\phi|Ł_{-1}Ł_{d-3}\phi\rangle+\mathcal{N}\Big(\phi\,Ł_{-1}Ł_{d-3}\phi\big|_{+\alpha}+\phi\,Ł_{-1}Ł_{d-3}\phi\big|_{-\alpha}\Big)\\ &= -\langle Ł_{d-3}\phi|Ł_{d-3}\phi\rangle+\mathcal{N}\Big(\phi\,Ł_{d-2}Ł_{d-3}\phi\big|_{+\alpha}+\phi\,Ł_{d-2}Ł_{d-3}\phi\big|_{-\alpha}\Big)\end{aligned} \tag{5.33}$$

$\langle\phi|Ł_{-1}Ł_{d-3}\phi\rangle_R$ and $-\langle Ł_{d-3}\phi|Ł_{d-3}\phi\rangle$ are the same up to boundary conditions. One can start with any bulk action, and the refined action is the same. There is no ambiguity.

5.2 Waveguide Spectrum of Spin-s Field

From spin-1, spin-2 and spin-3 examples, we learned that even after dimensional reduction, the quadratic equation of motions are fixed by gauge transformation. Therefore, in this section, we focus on reduction of gauge transformation which is much simpler than a reduction of an equation of motion.

In Fronsdal formulation, indices of spin-s gauge field is totally symmetric and double traceless and indices of gauge parameter is totally symmetric and traceless. As we showed in previous examples, naive reduction does not give Fronsdal fields of lower dimension because of these (double) traceless conditions. In this section, we find out the correct linear combination of gauge fields and gauge parameters and their gauge transformation. The result is similar to previous low spin examples: Stüeckelberg spin-s and relation between mode functions arise naturally. Comparing with work done in [63], we can determine the equation of motions and the spectrum with prescribed boundary conditions.

5.2.1 Dimensional reduction of gauge transformation

In this subsection we will show that the gauge variation of double traceless field $\phi^{(k)}$ — which is the linear combination of $d+2$ dimensional spin-s fields $\bar{\phi}_{M_1,\,M_2,\cdots,\,M_s}$ for $0 \leq k \leq s$ — is,

$$\delta\phi^{(k)}_{\mu_1\cdots\mu_k} = \frac{k}{s}\,\nabla_{(\mu_1}\,\xi^{(k)}_{\mu_2\cdots\mu_k)} + a_1\,Ł_{-(s+k-1)}\,\xi^{(k+1)}_{\mu_1\cdots\mu_k} + a_2\,Ł_{d-(s-k)-2}\,g_{(\mu_1\mu_2}\,\xi^{(k-1)}_{\mu_3\cdots\mu_k)}\,, \tag{5.34}$$

$$a_1 = \frac{s-k}{s}\,, \quad a_2 = \frac{k\,(k-1)\,(d+s+k-3)}{s\,(d+2k-5)\,(d+2k-3)}\,,$$

The gauge transformation (5.34) exactly coincides with the higher-spin Stüeckelberg gauge transformation.

At first we find the correct linear combinations that make field double traceless. A spin-s field $\bar{\phi}^{(s)}_{M_1\cdots M_s}$ is totally symmetric and double traceless and gauge parameter $\bar{\xi}^{(s)}_{M_2\cdots M_s}$ is totally symmetric and traceless in $d+1$ dimension. A naive lower dimensional fields and parameters are defined as[6], $\psi^{(s-n)}_{\mu_1\cdots\mu_{s-n}} \equiv \bar{\phi}^{(s)}_{\mu_1\cdots\mu_{s-n}\theta(n)}$, $\zeta^{(s-n)}_{\mu_1\cdots\mu_{s-n-1}} \equiv \bar{\xi}^{(s)}_{\mu_1\cdots\mu_{s-n-1}\theta(n)}$. Because of the (double) traceless conditions of $\bar{\xi}$ ($\bar{\phi}$), ζ (ψ) is not (double) traceless. To get the d dimensional double traceless fields $\phi^{(k)}$, we have to consider linear combination of $\psi^{(k)}$s. The followings is ansatz for double traceless field:

$$\phi^{(k)}_{\mu_1\cdots\mu_k} = \sum_{n=0}^{[k/2]} c_{n,k}\,\psi^{(n,k)}_{\mu_1\cdots\mu_k}\,, \quad \psi^{(n,k)} = g_{(\mu_1\mu_2}\cdots g_{\mu_{2n-1}\mu_{2n}}\,\psi^{(k-2n)}_{\mu_{2n+1}\mu_{2n+2}\cdots\mu_k)}\,. \tag{5.35}$$

The trace part of ψ fields is not included in linear combination because it would include the divergence term $\nabla^\mu\xi_{\mu\nu\cdots}$ in gauge variation which we do not want to include. The double traceless condition of $\phi^{(k)}$ gives the recursion relations. If we chose $c_{0,k}=1$, their solution is

$$c_{n,k} \;=\; \frac{1}{4^n\,n!}\frac{\Gamma(k+1)\,\Gamma\left(k+(d+1)/2-2-n\right)}{\Gamma(k-2n+1)\,\Gamma\left(k+(d+1)/2-2\right)}. \tag{5.36}$$

These equations hold for $4 \leq k \leq s$. Through similar process we can get the traceless linear combi-

[6] $\theta(n)$ is used to denote n θs.

nation of gauge parameters.

$$\xi^{(k)}_{\mu_1\cdots\mu_{k-1}} \equiv \sum_{n=0}^{[k/2]} \frac{k-2n}{k}\, c_{n,k}\, \zeta^{(n,k-1,0)}_{\mu_1\cdots\mu_{k-1}} \tag{5.37}$$

With these fields and gauge parameters, the gauge transformations reduce the gauge transformations (5.34).

5.2.2 Waveguide boundary conditions for spin-s field

Gauge transformation

At first, we derive the relations between expansion modes $\Theta^{k|s}_n$s and their differential equations. Each term in gauge variations (5.34) must be expanded by the same mode functions:

$$\boxed{\begin{pmatrix} 0 & Ł_{d-(s-k)-2} \\ Ł_{-(s+k-2)} & 0 \end{pmatrix} \begin{pmatrix} \Theta^{k|s}_n \\ \Theta^{k-1|s}_n \end{pmatrix} = \begin{pmatrix} c^{k-1|k}_n\, \Theta^{k|s}_n \\ c^{k|k-1}_n\, \Theta^{k-1|s}_n \end{pmatrix}, \qquad (5.38)}$$

These relations determine the Sturm-Liouville(SL) differential equations for $\Theta^{k|s}_n$'s:

$$Ł_{d-(s-k)-2}\, Ł_{-(s+k-2)}\, \Theta^{k|s}_n = c^{k|k-1}\, c^{k-1|k}\, \Theta^{k|s}_n\,, \tag{5.39}$$
$$Ł_{-(s+k-1)}\, Ł_{d-(s-k)-1}\, \Theta^{k|s}_n = c^{k|k+1}\, c^{k+1|k}\, \Theta^{k|s}_n\,. \tag{5.40}$$

$-M^2_{n,k|s}$ is used to represent the eigenvalue of equation (5.39). We can show that the SL differential equations (5.39) and (5.40) coincide by the identity $Ł_m\, Ł_n - Ł_{n-1}\, Ł_{m+1} = (n-m-1)$ when the eigenvalues satisfy followings:

$$M^2_{n,k|s} = M^2_{n,k+1|s} + d + 2k - 3. \tag{5.41}$$

All relations can be summarized as following:

$$\boxed{\begin{array}{rcll} & \Theta^{s|s}_n & & \\ Ł_{d-2} & \uparrow\downarrow & Ł_{-(2s-2)} & -M^2_{n,s|s} = -M^2_n \\ & \vdots & & \\ Ł_{d-(s-k-1)-2} & \uparrow\downarrow & Ł_{-(s+(k+1)-2)} & -M^2_{n,k+1|s} = -\big(M^2_n + (s-k-1)(d+s+k-3)\big) \\ & \Theta^{k|s}_n & & \qquad\qquad\qquad\qquad\qquad\qquad (5.42) \\ Ł_{d-(s-k)-2} & \uparrow\downarrow & Ł_{-(s+k-2)} & -M^2_{n,k|s} = -\big(M^2_n + (s-k)(d+s+k-4)\big) \\ & \vdots & & \\ Ł_{d-s-1} & \uparrow\downarrow & Ł_{-(s-1)} & -M^2_{n,1|s} = -\big(M^2_n + (s-1)(d+s-3)\big) \\ & \Theta^{0|s}_n & & \end{array}}$$

We can see that $M^2_{n,s|s}$ is mass square of nth mode of spin s field. By using the relations between mode expansions, the gauge transformation (5.34) can be interpreted as Stüeckelberg gauge transformation. After fixing relative constant in relation (5.42) as follows,

$$Ł_{-(s+k-2)}\,\Theta_n^{k|s} = -a_{k|s}\,M_{n,k|s}\,\Theta_n^{k-1|s}, \quad Ł_{d-(s-k)-2}\,\Theta_n^{k-1|s} = \frac{M_{n,k|s}}{a_{k|s}}\,\Theta_n^{k|s} \tag{5.43}$$

$a_{k|s}^2 = \frac{k\,(d+s+k-3)}{(s-k+1)\,(2k+d-3)}$, we can show that (5.34) coincide with the Stüeckelberg spin s gauge variations in AdS$_{d+1}$ that are given in [63],

$$\begin{aligned}
&\delta\phi^{(k)}_{\mu_1\cdots\mu_k} = \frac{k}{s}\,\nabla_{(\mu_1}\,\xi^{(k)}_{\mu_2\cdots\mu_k)} + \alpha_k\,\xi^{(k+1)}_{\mu_1\cdots\mu_k} + \beta_k\,g_{(\mu_1\mu_2}\xi^{(k-1)}_{\mu_3\cdots\mu_k)}\,,\\
&\alpha_k^2 = \frac{(k+1)\,(s-k)\,(d+s+k-2)}{s^2\,(d+2k-1)}\,\big(M^2+(s-k-1)\,(d+s+k-3)\big)\,,\\
&\beta_k = -\frac{k-1}{d+2k-5}\,\alpha_{k-1}.
\end{aligned} \tag{5.44}$$

Gauge transformation of each mode is identified as Stüeckelberg gauge transformation with mass M_n which is determined by boundary conditions. These gauge transformations completely fixed the equation of motion for each mode.

For the special cases with $\alpha_k = 0$ which coincide with conditions $M_{n,k+1|s} = 0$, Stüeckelberg system decompose into two subsystems[7]:

- The upper system which is consisted of $\big(\phi^{(s)},\,\phi^{(s-1)},\,\cdots\,\phi^{(k+1)}\big)$ forms Stüeckelberg form of partially massless field with depth-$t = (s-k-1)$ whose mass square is $M^2 = -t\,(d+2s-t-4)\,/\ell^2$.
- The lower part which is consisted of $\big(\phi^{(k)},\,\phi^{(k-1)},\,\cdots\,\phi^{(0)}\big)$ forms Stüeckelberg spin k field with mass square $M^2 = (s-k+1)(d+s+k-3)/\ell^2$.

As in previous examples, these field will appear as ground modes. These breaking pattern can be understood by Verma $\mathfrak{so}(d,2)$-modules. Except specific values, Verma module $\mathcal{V}\,(\Delta,\,s)$ is irreducible, but for specific values $\Delta = d+k-1$, it breaks [85, **?**]:

$$\mathcal{V}\,(d+k-1,\,s) = \mathcal{D}\,(d+k-1,\,s) \oplus \mathcal{D}\,(d+s-1,\,k)\;. \tag{5.45}$$

The summand of (5.45) represents the partially massless fields. And by the relation between a mass square of spin-s field and a conformal dimension Δ ,

$$m^2\,\ell^2 = \Delta\,(\Delta-d) - (s-2)\,(d+s-2)\;, \tag{5.46}$$

we can show the augend correspond the massive spin-$(k+1)$ field with the prescribed mass.

[7]We will call massless field as partially massless field with depth-0. And we have used the “mass” square of field as it is zero when higher spin gauge symmetry exist. It is different with the mass square which appears in Fierz-Pauli equation in AdS: $\left(\nabla^2+\kappa^2\right)\phi_{\mu_1\,\mu_2\,\cdot\mu_s} = 0$.

Boundary conditions and ground modes

Let us consider spectra and boundary conditions. We only consider boundary conditions which are derived from $\Theta^{k|s}|_{\theta=\pm\alpha}$ with $0 \leq k \leq s$. The relations in (5.42) fix all boundary condition for other fields:

$$Ł_{-(s+k-1)} \cdots Ł_{s+l-2}\, \Theta^{l\geq k|s}|_{\theta=\pm\alpha} = 0\,, \quad Ł_{d-(s-k)-2} \cdots Ł_{d-(s-l-1)-2}\, \Theta^{l\geq k|s}|_{\theta=\pm\alpha} = 0\,. \tag{5.47}$$

The remaining of this section is devoted to show the spectrum structure in Fig. 12. First let us define some notations as $\Theta^l = \Theta^{k+l|s}$, $M^2_{n,l} = M^2_{n,\,k+l|s}$, $U_l = Ł_{d-s-2+l+k}$ and $D_l = Ł_{-s+2-l-k}$. Then sub-diagram of (5.42) is written as following form.

$$\begin{array}{ccc} & \Theta^l_n & \\ U_l & \upharpoonleft\!\downharpoonright & D_l \\ & \Theta^{l-1}_n & \end{array} \quad : -M^2_{n,l} = -\left(M^2_n + (s-k-l)\,(d+s+k+l-4)\right) \tag{5.48}$$

Because of this relation, there is one-to-one map between Θ^l_n and Θ^{l-1}_n for $M^2_{n,l} \neq 0$. And for $M^2_{n,l} = 0$, there exist one additional mode Θ^l_0 (Θ^{l-1}_0) when l is positive (negative). This additional mode satisfies $D_l\,\Theta^l_0 = 0$ for positive l, $U_l\,\Theta^{l-1}$ for negative l. After inductively applying this result from $l = 0$, we can show that structure of mode function is,

$$\{\Theta^{k+l|s}\} = \{K^l_i,\, G^l_{a=1,\,2,\,\cdots\,l}\}\,, \quad \{\Theta^{k-l|s}\} = \{K^{-l}_i,\, G^{-l}_{a=1,\,2,\,\cdots\,l}\}\,. \tag{5.49}$$

K^l_is are Kaluza-Klein modes, and G^l_as are ground modes which satisfy,

$$\begin{cases} D_a\, D_{a+1} \cdots D_l\, G^l_a = 0 & \text{with } \; D_k\, D_{k+1} \cdots D_l\, G^l_a \neq 0 \quad \text{for all } \; a < k \\ U_{-a+1}\, U_{-a} \cdots U_{-l+1}\, G^{-l}_a = 0 & \text{with } \; U_k\, U_{k+1} \cdots U_{-l+1}\, G^{-l}_a \neq 0 \quad \text{for all } \; a < -k+1 \end{cases} \tag{5.50}$$

The ground modes G^l_i with the same subscript i have the same eigenvalue. Their eigenvalue can be obtained by first order differential equation $D_l\, G^l_l = 0$ and $U_{-l+1} G^{-l}_l$ for positive l. Finally fields correspond to $G^k_a = (s-k,\, \cdots a)$ consist Stüeckelberg form of partially massless spin s field with depth-$(s-k-a+1)$, as explained in below (5.44). And fields correspond $G^k_a = (-a,\, \cdots - k)$ consist massive spin $k-a$ Stüeckelberg field with mass $M^2 = (s-k+a+1)(d+s+k-a-3)/\ell^2$. The spectrum structure is summarized in Fig. 12.

Higher-derivative boundary conditions and boundary action

As it was shown in section 5.1, higher derivative boundary conditions imply the existence of boundary degrees of freedom and "boundary action". To analysis this in detail, we must know the mathematical structure — for example the extended functional space and the norm — for arbitrary higher-derivative boundary conditions which is not completely known yet — even though the factorization structure of differential equation (5.39) makes problems simpler than general cases. However it is worth to mention about the expected features for general spin-s.

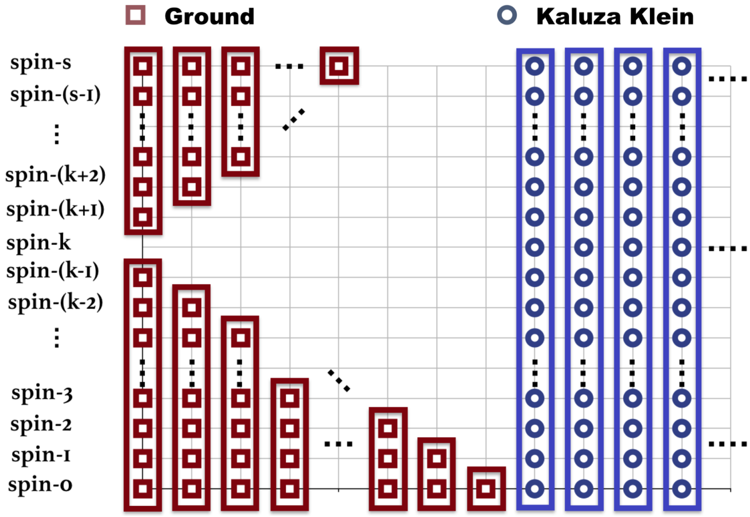

Figure 12: Spectrum for set of boundary conditions which are derived by Dirichlet boundary condition at $\Theta^{k|s}$. Point represents mode function. Points in the same rectangle consist Stüeckelberg system with the highest spin. The the upper triangle parts consist of Stüeckelberg system of partially massless field, and the lower triangle parts consist of Stüeckelberg spin $l = 0,\, 1, \cdots k-1$ as described in (5.51).

- For the boundary conditions derived by $\Theta^{k|s}|_{\theta=\pm\alpha}$ with $k \leq s-2$, it is expected that the "boundary actions" are equal to the minus times the standard actions, This is the origin of non-unitarity and partially massless fields. In contrast, for the cases with $k \geq s-1$, it is expected that the "boundary actions" are equal to the standard actions and the theory and spectrum are unitary in AdS$_d$.
- It is expected that the forms of "boundary action" are proportional to the action of the fields with the mass which is given in page 74 box. For example, it is expected that the "boundary action" is equal to **type II** in Table 4 for the first case in Figure. 10

It is worth to summarize the result.

- Spectrum for the case with Dirichlet boundary condition at $\Theta^{k|s}$ consist three part. The first one is a set of massive spin-s Kaluza-Klein tower whose mass squares are given as eigenvalue of Sturm-Liouville problem. The second is a set of partially massless spin s field with depth-$(0,\, 1,\, 2, \cdots,\, s-k-1)$. The last one is consisted of massive Stüeckelberg spin $l = 0,\, 1, \cdots,\, k-1$ with mass square, $M^2 = (s-l+1)(d+s+l-3)/\ell^2$.
- By the relation (5.46), we can obtain the $\mathfrak{so}(d,2)$ representations of ground modes:

$$\begin{cases} \mathcal{D}\,(d+s-t-2,\, s) & \text{for} \quad t = 0,\, 1, \cdots,\, s-k-1 \\ \mathcal{D}\,(d+s-1,\, l) & \text{for} \quad l = 0,\, 1, \cdots,\, k-1 \end{cases} \tag{5.51}$$

Extension of boundary condition The more general form of boundary conditions can be considered. For example, the patterns of the raising and lowering operators between adjoint spins in (5.42) suggest interesting sets of boundary conditions. To consider these kinds of boundary conditions, let us define $\Theta^{k|s}|_{\theta=\pm\alpha} = 0$ for $k > s$ or $k < 0$, although there is no corresponding physical fields;

$$\text{Ł}_{d-2} \quad \begin{array}{c} \Theta_n^{s+1|s} \\ \upharpoonleft\downharpoonright \\ \Theta_n^{s|s} \end{array} \quad \text{Ł}_{-2s+1} \quad -M^2_{n,s+1|s} = -\left(M_n^2 - 2\,s - d + 3\right) \ . \tag{5.52}$$

After imposing Dirichlet boundary condition on $\Theta^{k|s}$, the expanded structure of (5.42) automatically imposes the boundary conditions for all fields. By the same analysis of previous case, we could derive the form of the ground modes and related $\mathfrak{so}(d,2)$-modules: ground modes for $k > s$ consist the ground modes for $k = s$ and followings,

$$M^2 = (l-s)\,(d+s+l-4)\,/\ell^2\,, \quad \mathcal{D}(d+l-2,s) \quad \text{for} \;\; l = s+1,\, s+2\,,\cdots, k \tag{5.53}$$

and ground modes for $k < 0$ consist the ground modes for $k = 0$ and followings,

$$M^2 = (l-s)\,(d+s+l-4)\,/\ell^2\,, \quad \mathcal{D}(d+l-2,s) \quad \text{for} \;\; l = 0,\, -1\,,\cdots, -|k|+1\,. \tag{5.54}$$

The $k < 0$ cases contain the non-unitary fields. For more forms of extended boundary conditions, see page 83 in conclusion section 5.4.

5.2.3 Decompactification limit: $\alpha \to \pi/2$

In this subsection, the physics with $\alpha \to \pi/2$, which will be called as decompactification limit , is investigated. Because the $\alpha = \pi/2$ corresponds the AdS$_{d+2}$ boundary, the AdS waveguide physics is naïvely expected to approach the AdS$_{d+2}$ physics. However, it is shown that the spin-s field spectrum in AdS$_{d+2}$ can be recovered only for the special set of boundary conditions.

First, let us consider the AdS$_{d+2}$ massless spin-s spectrum in one lower dimensional point of view. The normalizable mode solutions of the massless spin-s equation consist the $\mathfrak{so}(d+1,2)$-module: $\mathcal{D}(d+s-1,s)_{\mathfrak{so}(\mathrm{d+1,2})}$. We can decompose this module into $\mathfrak{so}(d,2)$-modules by group theoretical consideration [39]:

$$\mathcal{D}(d+s-1,s)_{\mathfrak{so}(\mathrm{d+1,2})} = \bigoplus_{n=0}^{\infty} \mathcal{D}(d+n+s-1,s)_{\mathfrak{so}(\mathrm{d,2})} \oplus \bigoplus_{l=0}^{s-1} \mathcal{D}(d+s-1,l)_{\mathfrak{so}(\mathrm{d,2})}\,, \tag{5.55}$$

which can be also checked by the field theoretical consideration with the foliation Fig.7. There are two kinds of $\mathfrak{so}(d,2)$-modules in left hand side of (5.55): the modules in the first kind have the same spin, spin-s, and different conformal weights and the modules in the second kind have the same conformal weight and different spins starting from 0 to $s-1$. The set of modules in the second kinds in (5.55) coincide the set of ground modes for $k = s$ in (5.51). We shall show that, in decompactification limit, the Kaluza-Klein modes for $k = s$ case approach to the fields with corresponding to modules in the first kind.

In the $k = s$ case, the masses of spin-s field are determined by Sturm-Liouville equation (5.39) and the boundary condition for mode functions $\Theta^{(s|s)} = f$:

$$Ł_a\, Ł_b\, f = -m^2 f\,, \quad f|_{\theta=\pm\alpha} = 0\,, \quad \text{with } a = d-2 \text{ and } b = -2\,(s-1)\;, \tag{5.56}$$

whose differential equation can be solved as

$$\Theta^{(s|s)} \equiv f(\theta) = (\cos\theta)^{\mu}\ \left(c_1\, P^{\mu}_{\nu}(\sin\theta) + c_1\, Q^{\mu}_{\nu}(\sin\theta)\right)\;, \tag{5.57}$$

where P^{μ}_{ν} and Q^{μ}_{ν} are the associated Legendre functions with arguments, $\mu = \frac{1}{2}\,(d+2\,s-3)$ and $\nu\,(\nu+1) = m^2 - \frac{1}{4}\left(1-(d+2s-4)^2\right)$. In decompactification limit, the boundary conditions $0 = f|_{\theta=\pm\alpha}$ can be expressed[8],

$$\begin{aligned} 0 &= -\frac{\pi}{2}\,\sin A\left((P^{\mu}_{\nu})^2 - \frac{4}{\pi^2}\,(Q^{\mu}_{\nu})^2\right) - 2\cos A\, P^{\mu}_{\nu}\, Q^{\mu}_{\nu} \qquad (5.58)\\ &\simeq \begin{cases} -\frac{1}{2\,\pi}\,\sin A\,(\cos(\mu\,\pi)\,\Gamma(\mu))^2\left(\frac{2}{\epsilon}\right)^{\mu} & \text{for even } d \\ -\frac{\pi}{2}\,\sin A\left(\frac{1}{\Gamma(1-\mu)}\right)^2\left(\frac{2}{\epsilon}\right)^{\mu} & \text{for odd } d \end{cases} \qquad (5.59)\end{aligned}$$

where $A = \pi\,(\mu+\nu)$ and $1 \gg \epsilon = 1-\sin\alpha > 0$. Therefore, $\mu+\nu \simeq N$ with integer N in decompactification limit. By using masses and conformal weights relation (5.46), we can conclude that the corresponding module for Kaluza-Klein modes are $\bigoplus_{n=0}^{\infty}\,\mathcal{D}(d+n+s-1,s)$:

> Spectrum for the cases of $(k = s)$ goes to the spectrum of “dimensional degression [39]” in decompactification limit(i.e. $\alpha \to \pi/2$).

For $k \neq s$, some ground modes in (5.51) contains the modules which are not in massless spin-s module in AdS_{d+2}: $\mathcal{D}(d+s-1,s)_{\mathfrak{so}(d+1,2)}$ in (5.55). The mode functions corresponding these kinds will blow up — or equivalently the nomalization factor goes to zero – in decompactification limit; as in spin-2 case (4.71), which explains why there are no corresponding mode in “dimensional degression [39]”.

5.3 Toward Interaction

In previous sections, dimensional reduction of metric-like formalism is considered. In this section, dimensional reduction of frame-like equations is considered. It can be considered as a first step for the reduction of Vasiliev system [45]. As a positive sign that dimensional reduction procedure works well in interacting level, we derive Lie algebra of $\mathrm{hs}_3(1/2)\oplus\mathrm{hs}_3(1/2)$ as remaining gauge symmetry for dimensional reduction from AdS_4 to AdS_3.

[8]For mathematical formulae, see *NIST Digital Library of Mathematical Functions*.

5.3.1 Gauge Symmetries in (A)dS Waveguide: Hihger Spin Generalization

As in metric-like formalism, we only consider the reduction of gauge variation. The massless frame-like Lagrangian on AdS$_{d+1}$ is invariant under following gauge transformation:

$$\delta\,\bar{\Phi}_M{}^{A(s-1)} = \quad \nabla_M\,\bar{\xi}^{A(s-1)} + \bar{E}_{M\,B}\,\bar{\eta}^{B,A(s-1)}\,, \tag{5.60}$$

$$\delta\,\bar{\Omega}_M{}^{B,\,A(s-1)} = \quad \nabla_M\,\bar{\eta}^{B,A(s-1)} + \frac{(d+s-3)\,(s-1)}{d-1}\Big[E_M{}^B\,\bar{\xi}^{A(s-1)} - E_M{}^A\,\bar{\xi}^{A(s-2)B}\Big] -(\text{traces})\,, \tag{5.61}$$

Linear combinations

Higher spin vielbein The first task is to identify which field on AdS$_{d+1}$ corresponds to Stueckelberg frame-like field on AdS$_d$. The frame-like massless spin-s fields on AdS$_{d+1}$, $\bar{\Phi}_M{}^{A(s-1)}$, have symmetric and traceless internal indices:

$$\bar{g}_{B(2)}\bar{\Phi}_M{}^{B(2)\,A(s-3)} = 0\,. \tag{5.62}$$

The frame-like Stueckelberg structure on AdS$_d$ consists of the fields $\Phi_\mu{}^{a(k-1)}$ (for $s \geq k \geq 1$) — with traceless and symmetric internal indices — and scalar field Φ. Because of the traceless conditions, the naïve reduction of fields $\Psi^{a(k-1)} \equiv \bar{\Phi}^{a(k-1)D(s-k)}$ cannot be candidates for Stueckelberg frame-like fields. To obtain the fields with traceless and symmetric indices, we must consider following linear combinations of Ψs:

$$\Phi_\mu{}^{a(k-1)} \equiv \sum_{n=0}^{[k/2]} a_{n,k}\,({}_{n,\,k-1}\Psi)_\mu{}^{a(k-1)}\,, \quad ({}_{n,\,k}\Psi)_\mu{}^{a(k)} = \eta^{a(2)}\cdots\eta^{a(2)}\Psi_\mu{}^{a(k-2\,n)}\,. \tag{5.63}$$

with $a_{n,\,k} = \left(-\frac{1}{4}\right)^n \frac{[1-k]_{2n}}{n!\,[3-k-d/2]_n}$ which is chosen to satisfy the traceless conditions. By the totally same logic, the linear combinations of naïve gauge parameters, $\zeta^{a(k-1)} \equiv \bar{\xi}^{a(k-1)\,D(s-k)}$, are the candidates of Stueckelberg gauge parameters:

$$\xi^{a(k-1)} \equiv \sum_{n=0}^{[k/2]} a_{n,k}\,({}_{n,\,k-1}\zeta)^{a(k-1)}\,, \quad ({}_{n,\,k}\zeta)^{a(k)} = \eta^{a(2)}\cdots\eta^{a(2)}\zeta^{a(k-2\,n)}\,. \tag{5.64}$$

Higher spin connection In the frame-like formalism higher spin connection on AdS$_{d+1}$, $\bar{\Omega}_M{}^{B\,A(s-1)}$, satisfies $(1, s-1)$ Young-symmetry:

$$\bar{g}_{B(2)}\bar{\Omega}_M{}^{B,\,BA(s-2)} = 0\,, \quad \bar{g}_{B(2)}\bar{\Omega}_M{}^{A,\,B(2)A(s-3)} = 0\,, \quad \bar{\Omega}_M{}^{A,\,A(s-1)} = 0\,. \tag{5.65}$$

The frame-like Stueckelberg structure on AdS$_d$ consists of the fields $\Omega_\mu{}^{b,\,a(k-1)}$ (for $s \geq k \geq 1$) — with $(1, k-1)$ Young symmetric properties —, spin-1 field strength F^{ab} and the derivative of scalar Π^a. Because of the $(1, k-1)$ Young symmetric properties, the naïve reduction of fields $\Upsilon^{b,\,a(k-1)} \equiv$

$\bar{\Omega}^{b,\,a(k-1)D(s-k)}$ cannot be candidates for Stueckelberg frame-like fields, ${\Omega_\mu}^{b,\,a(k-1)}$. To obtain the fields with traceless and symmetric indices, we must consider following linear combinations of Υs:

$$
\begin{aligned}
{\Omega_\mu}^{b,\,a(k-1)} \equiv \sum_{n=0}^{[k/2]} \quad & \Big[b_{n,k} \left((g^{a(2)})^n \, {\Upsilon_\mu}^{a,\,ba(k-2\,n-2)} - ({}_{n,k}\Upsilon)_\mu{}^{b,\,a(k-1)} \right) && (5.66)\\
& + c_{n,k} \left(g^{ab} \, ({}_{n-1,k}\Upsilon)_\mu{}^{a,\,a(k-3)} - ({}_{n,k}\Upsilon)_\mu{}^{b,\,a(k-1)} \right) \Big] \,, && (5.67)
\end{aligned}
$$

where we used the short notation,

$$
({}_{n,k}\Upsilon)_\mu{}^{b,\,a(k)} = (g^{a(2)})^n \, {\Upsilon_\mu}^{b,\,a(k-2\,n)} \,, \tag{5.68}
$$

and the sequences $b_{n,\,k}$ and $c_{n,\,k}$ is chosen to satisfy $(1, k-1)$ Young symmetric properties:

$$
b_{n,\,k}/a_{n,\,k} = \frac{s\,(k-2\,n+d-4) - k\,(k+d-4)}{(k+d-4)\,(k-2\,n-s)\,(k-1)}\,(k-2\,n-1)\,, \tag{5.69}
$$

$$
c_{n,\,k}/a_{n-1,\,k} = \frac{s\,[k-2n]_2}{(k+d-4)\,(k-2\,n-s)\,(k-1)}\,. \tag{5.70}
$$

By the totally same logic, the linear combinations of naïve gauge parameters,
$\varsigma^{b,\,a(k-1)} \equiv \bar{\eta}^{b,\,a(k-1)\,D(s-k)}$, are the candidates of Stueckelberg gauge parameters:

$$
\begin{aligned}
\eta^{b,\,a(k-1)} \equiv \quad & \alpha \sum_{n=0}^{[k/2]} \Big[b_{n,k} \left((g^{a(2)})^n \, \varsigma^{a,\,ba(k-2\,n-2)} - ({}_{n,k}\varsigma)^{b,\,a(k-1)} \right) \\
& \qquad + c_{n,k} \left(g^{ab} \, ({}_{n-1,k}\varsigma)^{a,\,a(k-3)} - ({}_{n,k}\varsigma)^{b,\,a(k-1)} \right) \Big] \,,
\end{aligned} \tag{5.71}
$$

where $({}_{n,k}\varsigma)^{b,\,a(k)} = (g^{a(2)})^n \, \varsigma^{b,\,a(k-2\,n)}$. The overall term α will be fixed in next subsection.

Dimensional reduction of gauge transformation Let us consider the gauge transformations of fields ${\Phi_\mu}^{a(k-1)}$. From the form of gauge variations (5.60) and the linear combinations (5.63, 5.64, 5.71), we could obtain

$$
\begin{aligned}
\delta \, {\Phi_\mu}^{a(k-1)} \quad = \quad & \nabla_\mu \, \xi^{a(k-1)} \\
& + \tan\theta \, \beta \left({E_\mu}^a \, \xi^{a(k-2)} - \frac{k-2}{2\,k+d-6} \, E_{\mu\,b} \xi^{ba(k-3)} \, g^{a(2)} \right) \\
& - \tan\theta \, (s-k) \, E_{\mu\,b} \, \xi^{b,\,a(k-1)} \\
& + \sec\theta \, E_{\mu\,b} \sum_n a_{n,\,k} \, ({}_{n,\,k-1}\varsigma)^{ba(k-1)} \,,
\end{aligned} \tag{5.72}
$$

$$
\begin{aligned}
\delta \, \Phi_\theta^{a(k-1)} \quad = \quad & \partial_\theta \, \xi^{a(k-1)} \\
& + \sec\theta \sum_n \frac{k-2n-1}{s-k+2n+1} a_{n,\,k} ({}_{n,\,k-2}\varsigma)^{a,a(k-2)} \,,
\end{aligned} \tag{5.73}
$$

where $\beta = \frac{(k-1)\,(s+k+d-4)}{2\,k+d-4}$. After choosing $\alpha = \frac{k}{k-1}\,\sec\theta$ in (5.71), one can show that

$$\delta\left[\Phi_\mu{}^{a(k-1)} + \kappa\left(E_\mu{}^a\,\Phi_\theta{}^{a(k-2)} - \frac{k-2}{2k+d-6}\,g^{aa}\,E_{\mu b}\,\Phi_\theta{}^{ba(k-3)}\right) + \lambda\,E_{\mu b}\,\Phi_\theta{}^{ba(k-1)}\right] \tag{5.74}$$

$$\begin{aligned} = \quad & \nabla_\mu\,\xi^{a(k-1)} + E_{\mu\,b}\,\eta^{b,\,a(k-1)} \\ & +\kappa\,Ł_{k+d-4}\left(E_\mu{}^a\,\xi^{a(k-2)} - \frac{k-2}{2\,k+d-6}\,E_{\mu\,b}\xi^{ba(k-3)}\,g^{a(2)}\right) + \lambda\,Ł_{-k}\,E_{\mu\,b}\,\xi^{ba(k-1)} \end{aligned} \tag{5.75}$$

where $\kappa = \frac{(k-1)\,(s+k+d-4)}{(k+d-4)\,(2\,k+d-4)} = \frac{\beta}{k+d-4}$ and $\lambda = \frac{s-k}{k}$. After considering the relations between metric-like and frame-like field, $\Phi_{\mu_1\cdots\mu_s} = \Phi_{\mu_1}^{a_2\cdots a_s}\,E_{\mu_2\,a_2}\cdots E_{\mu_s\,a_s}$, one can check that the form of $Ł_m$ predict the form of (5.42). These gauge variation forms (5.75) exactly match with the result in [64].

5.3.2 Reduced higher spin algebra and Chern-Simon equation: d=2 case

After dimensional reduction, there are remaining gauge symmetry that can be interpreted as lower dimensional higher-spin symmetry. We can ask what is the remaining gauge parameters form a close Lie algebra. In this subsection it is shown that they form a close Lie algebra which is isomorphic to $\mathrm{hs}_3(1/2)\oplus\mathrm{hs}_3(1/2)$. and the massless sector has well-known Blencowe-Chern-Simons forms [44]. In four dimensional higher spin gauge theory, zeroth order gauge variations are given as following:

$$\begin{aligned} \delta\bar\omega_\mu(y,\bar y|x,\theta) \quad &= \quad \partial_\mu\bar\xi - \Omega_{\alpha\beta}(y^\alpha\partial^\beta + \bar y^\alpha\bar\partial^\beta)\bar\xi \\ & \qquad -i\tan\theta E_{\alpha\beta}(y^\alpha\partial^\beta - \bar y^\alpha\bar\partial^\beta)\bar\xi - \frac{1}{\cos\theta}E_{\alpha\beta}(\bar y^\alpha\partial^\beta + y^\alpha\bar\partial^\beta)\bar\xi \end{aligned} \tag{5.76}$$

$$\delta\bar\omega_\theta(y,\bar y|x,\theta) \quad = \quad \partial_\theta\bar\xi - \frac{i}{2\cos\theta}(\bar y^\alpha\partial_\alpha - y^\alpha\bar\partial_\alpha)\bar\xi\,, \tag{5.77}$$

where $\bar\omega_M$ is the generating function of higher-spin fields and generalized spin-connection and $\bar\xi$ is the generating function of higher-spin gauge parameters. Because we assign the boundary condition, gauge parameters remain gauge parameters in three dimension only when they preserve boundary conditions. In other words, they must satisfy $\delta\bar\omega_\theta = 0$ and $\partial_\alpha\bar\partial^\alpha\delta\bar\omega_\mu = 0$, which are the spinor version:

$$\partial_\alpha\bar\partial^\alpha\bar\xi = 0, \quad \partial_\theta\bar\xi - \frac{i}{2\cos\theta}(\bar y^\alpha\partial_\alpha - y^\alpha\bar\partial_\alpha)\bar\xi = 0, \tag{5.78}$$

$$E_{\alpha\beta}\big(2i\sin\theta\partial^\alpha\bar\partial^\beta + \partial^\alpha\partial^\beta - \bar\partial^\alpha\bar\partial^\beta\big)\bar\xi = 0 \tag{5.79}$$

After try and error, we can find general solutions for above conditions.

$$\xi(x|y) = \xi_+(x|y_+) + \xi_-(x|y_-), \quad y_\pm = \frac{1}{\sqrt{2\cos\theta}}(e^{\mp i\theta/2}y \pm e^{\pm i\theta/2}\bar y) \tag{5.80}$$

$\xi_\pm(x|y_\pm)$ are arbitrary even polynomials of $y_\pm$ and depend on three dimensional space-time coordinates. It can be shown that $y_\pm$ satisfying following commutation relation,

$$\left[y^\alpha_\pm, y^\beta_\pm\right] = \left[y^\alpha, y^\beta\right] = \left[\bar{y}^\alpha, \bar{y}^\beta\right] = 2i\epsilon^{\alpha\beta}, \quad \left[y^\alpha_\pm, y^\beta_\mp\right] = 0\,. \tag{5.81}$$

This reduced Lie algebra is isomorphic to Lie algebra of $\mathrm{hs}_3(1/2) \oplus \mathrm{hs}_3(1/2)$. Because higher spin Lie algebra is related with interaction, the algebra closeness is a non-trivial clue for the existence and the consistence of three dimensional higher spin gauge theory by AdS dimensional reduction.

The algebra is strong enough to fix the interaction at leat between for massless fields. To explain more precisely, let us consider four dimensional Vasiliev system. The generating function for higher spin gauge potentials is space-time 1form, $\bar{\mathcal{W}}$. The 0-form $\mathcal{B}$ is a generating function for scalar field and for generalized Weyl-tensor. And the space-time 0-form s serve as a tool to turn on interacting. Vacuum solution $\bar{\mathcal{W}}_0$ in four dimensional Vasiliev system can be as vacuum solution in AdS_3 plus one-form in θ direction.[9]

$$\begin{aligned}
\bar{\mathcal{W}}_{0,4D} &= \frac{1}{4i}\Omega_{\alpha\beta}(y^\alpha y^\beta + \bar{y}^\alpha\bar{y}^\beta) + \frac{1}{4i}E_{\alpha\beta}\big(i\tan\theta(\bar{y}^\alpha\bar{y}^\beta - y^\alpha y^\beta) + 2\sec\theta y^\alpha\bar{y}^\beta\big)\\
&\quad + \frac{1}{4}\sec\theta\epsilon_{\alpha\beta}y^\alpha\bar{y}^\beta d\theta\\
&= \frac{1}{4i}\Omega_{\alpha\beta}(y^\alpha_+ y^\beta_+ + y^\alpha_- y^\beta_-) + \frac{1}{4i}E_{\alpha\beta}(y^\alpha_+ y^\beta_+ - y^\alpha_- y^\beta_-) - \frac{1}{4}\sec\theta y^+_\alpha y^\alpha_- d\theta \qquad (5.82)\\
&= \mathcal{A}^{\alpha\beta}_{0+}y^+_\alpha y^+_\beta + \mathcal{A}^{\alpha\beta}_{0-}y^-_\alpha y^-_\beta - \frac{1}{4}\sec\theta y^+_\alpha y^\alpha_- d\theta = \mathcal{W}_{0,3D} - \frac{1}{4}\sec\theta y^+_\alpha y^\alpha_- d\theta \qquad (5.83)
\end{aligned}$$

By remaining gauge transformation argument, we can figure out massless higher spin field in 3D can be written as,

$$\mathcal{W}_1 = \mathcal{A}^+_1(y_+|x) + \mathcal{A}^-_1(y_-|x). \tag{5.84}$$

The above form can be also deduced from the result in a previous solution of torsionless conditions. We can identify Blencowe higher spins as sums of vacuum value and linear order correction. i.e. $\mathcal{A}_\pm = \mathcal{A}^0_\pm + \mathcal{A}^1_\pm$. In contrast with higher dimension, three-dimensional massless modes are rather trivial. Because higher spin has no physical degree of freedom in 3 dimensions, 0-form $\mathcal{B}$ and corrections of s_α and $\bar{s}_\alpha$ are zero. As a result at least tree point massless higher spin self-interacting level, reduced equations are equal to Blencowe-Chern-Simon equation.

5.4 Comment

This paper attempts to obtain higher-spin with matter theory by dimensional reduction. After dimensional reduction, two types of fields appear. The first-type fields, which are called as Kaluza-Klein modes, have masses that diverge as two boundaries close to each other. This type has Stüeckelberg gauge symmetries in AdS_{d+1} which are originated from the gauge symmetry in AdS_{d+2}. The second-type fields, which are called as ground modes in this paper, have masses which are independent of the distance between two boundaries. Masses of ground mode could be deduced by the breaking pattern

[9] Notation for AdS space and vacuum solution in spinor form are given in Appendix A.

of Verma $\mathfrak{so}(d,2)$-modules. They can be massless, partially massless and Stüeckelberg massive with specific masses, which are described in (5.51).

Extension

In this paper we mainly concentrate dimensional reduction of massless spin-s from AdS$_{d+2}$ to AdS$_{d+1}$ with the specific boundary conditions like (4.62) and (5.47). We can extend the scope.

- First for the boundary conditions. We could consider sets of boundary conditions which is derived from $\mathcal{M}\,\Theta^{k|s}|_{\theta=\pm\alpha}$ with arbitrary differential operator $\mathcal{M}$. Then, we would get following sets of boundary conditions instead of (5.47).

$$\begin{aligned} \mathcal{M}\,\text{Ł}_{-(s+k-1)}\cdots\text{Ł}_{s+l-2}\,\Theta^{l\geq k|s}|_{\theta=\pm\alpha} &= 0\,, \\ \mathcal{M}\,\text{Ł}_{d-(s-k)-2}\cdots\text{Ł}_{d-(s-l-1)-2}\,\Theta^{l\geq k|s}|_{\theta=\pm\alpha} &= 0\,. \end{aligned} \tag{5.85}$$

  Even in these cases, the set of ground modes includes the ground modes we discussed in section 5.2 by construction.

- Only symmetric boundary conditions with range $-\alpha<\theta<\alpha$ is considered in this paper, we can also generalize the range $\alpha_1<\theta<\alpha_2$ and impose an asymmetric boundary conditions for each boundary.

- Second compactifying more directions. We provide the dimensional reduction method from AdS$_{d+k}$ to AdS$_d$ in Appendix C. It is AdS generalization of [**?**]. As the Kaluza-Klein reduction from $R^{1,d+k}$ to $R^{1,d}\otimes S^k$ taught us about Yang-Mills like structure, we could expect that reduction methods with $k\neq 1$ might provide a novel structure of colored higher-spin theory [123, 3].

- Third extension to fermion. We only consider the bosonic cases. After considering the fermion cases, we can extend this procedure to the supersymmetric cases and ask what are the boundary conditions which preserve some supersymmetries.

- We can consider also partially massless fields as a starting point. The gauge symmetry has the central role in dimensional reduction. Partially massless fields or its Stüeckelberg form has enough symmetries to fix the quadratic action. It would be interesting to search the boundary conditions which gives the massless fields, and interpret them in physical way.

- The analysis of boundary action for arbitrary spin-s. The mathematical structure for second derivative boundary conditions provide us the tool to analyse the boundary action for spin-2 case in section 5.1. The mathematical structure for arbitrary higher-derivative boundary condition is more complicated — even the factorization structure (5.39) make things easier — than second derivative cases. Making these mathematical structure clear and analysing boundary action might be interesting.

- The AdS waveguide method are partially applied to frame-like formalism in section 5.3.1: only for massless sectors. One can expand this to the general boundary conditions and general sectors. The role of Stüeckelberg spin-s symmetries in metric-like formalism [63] might be replaced by its extension [64] of the frame-like formalism.

Discussion

After dimensional reduction of a massless spin-s field in AdS$_{d+2}$ with the specific boundary conditions , we can get a massless spin-s field and massive spin-$(s-2)$, -$(s-3)$, ... -0 fields as ground modes. There are Kaluza-Klein modes, but they will be decoupled with ground modes as a distance between two plane goes to zero. In known consistent higher spin theory — Vasiliev theory [46] —, there is only one trajectory, i.e. massless spin s. We could anticipate the interacting theory with there are infinite trajectories after reduction of the interacting higher-spin theory: It is more stringy.

To check whether our reduction procedure is applicable to an interacting case, it might be interesting to calculate on-shell three point function between massive modes and massless modes. This is little more cumbersome and interesting than Chern-Simons like interaction as in section 5.3.2. But we give some clues that our procedure is applicable in some interaction higher-spin theory on four dimension: Tour dimensional higher spin algebra is reduced to three dimensional one. The reduced algebra is isomorphic to hs$_3(1/2) \oplus$ hs$_3(1/2)$. The followings are possible extensions.

- It is known that Blencowe-Chern-Simons higher-spin theory has W_∞ as an asymptotical symmetry [81, 83]. It is interesting to check that the reduced interacting higher-spin matter theory has the same symmetry. And we can consider a similar problem to supersymmetric one as in [82]. And It would be interesting if one can find the method whose reduced algebra is hs$_3(\lambda) \oplus$ hs$_3(\lambda)$ for general lambda.
- It is possible to consider about the reduced higher-spin algebra for AdS$_{d+k}$ to AdS$_d$ reduction. As far as we know, it is hard to find the hs$_d$ subalgebra in a hs$_{d+1}$ for general d-dimension. It might give some clue to find the sub-algebra[10]. And there are one parameter family of higher spin algebra in five dimensions — see [87] and reference therein —, it could be also another possible way to get hs$_3(\lambda) \oplus$ hs$_3(\lambda)$ as a reduced algebra.
- In metric-like point of view, the boundary conditions for the theory in section 5.3 correspond to $\Theta^{s-1|s}|_{\theta=\pm\alpha} = 0$ and derived boundary conditions. It would be interesting to consider more general cases and check what is the reduced gauge algebra.

It is very interesting subject to consider AdS waveguide in AdS/CFT view point.

- Let us first concentrate on the case with the case with boundary conditions which are derived by $\Theta^{s-1|s}|_{\theta=\pm\alpha} = 0$. Then by the AdS$_{d+1}$/CFT$_d$ dictionary [59] and the results (5.51), the spectrum of dual CFT$_d$ is,

$$\bigoplus_{s=0}^{\infty} \mathcal{D}(d+s-2,s) \oplus \bigoplus_{s=1}^{\infty} \bigoplus_{l=0}^{s-2} \mathcal{D}(d+s-1,l)\,, \tag{5.86}$$

  where there is an ambiguity for scalar spectrum in summand because there is no gauge symmetry on scalar field.

[10] We thank E. Joung and K. Mkrtchya for discussing these points.

- In geometrical point of view the $O(N)$ vector model [144, 145] with boundary is plausible dual CFT. Finding the precise boundary condition for $O(N)$ vector model would be interesting subject.
- The other interesting feature of these procedure is emergence of partially massless fields. For partially massless field there are conjectured dual CFT in [173, 85, 84]. It will be also interesting to research about the reduction of a partially massless field on AdS_{d+2}. Because there is HS/CFT conjecture for the bulk partially massless field, it may be helpful to find dual CFT_d for our reduced model.
- In the context of AdS/CFT, a free energy of both side should match and especially 1-loop free energy of AdS side correspond to 1/N correction of free energy of CFT side. The Casimir/free energy calculation [60] will provide the clue of the dual CFT[11].

It has been known that Kaluza-Klein compactification of $\mathbb{R}^{d+1} \to \mathbb{R}^d \times \mathbb{S}^1$ of Einstein gravity gives rise to coupled algebra of Kac-Moody extension of the Poincare algebra and Virasoro extension of internal symmetry algebra [62]. So it would be interesting to find a similar structure for higher-spin generalization.

[11] For these calculation on HS/CFT duality see [citation for Casimir energy calcuations].

Chapter 6

Colored Gravity

"It's time to try. Defying gravity.
I think I'll try. Defying gravity.
And you can't pull me down!"
— Gregory Maguire
'Wicked: The Life and Times of the Wicked Witch of the West'

6.1 Introduction

The Einstein's theory of gravity is known to be rigid. Variety of modifications has been challenged with diverse motivations, yet no concrete result of success has been reported so far (for related readings, see e.g. [116] and references therein). Recently, two situations defying the rigidity of Einstein gravity were actively explored. One is the massive modification of gravity [117], along with numerous variants in three dimensions [118]. Another is higher-derivative modifications of the gravity [119].

In this work, we investigate the modification of Einstein gravity to a multi-graviton theory: *the color decoration*. In spite of previous negative results [120, 121], certain models of colored gravity can be consistently constructed by introducing other field contents than massless spin-two fields only. Moreover, the color-decoration we study is not limited to the Einstein gravity and can be applied to various extensions of it. In particular, all higher-spin theories formulated in [122] can be straightforwardly color-decorated, whose first steps were conceived in [123]. In the next chapter 7, we study a three-dimensional color-decorated higher-spin gravity.

The color decoration of gravity evokes various conceptual issues. Clearly, the colored gravity is analogous to Yang-Mills theory were if the Einstein gravity compared to Maxwell theory. Besides the presence of multiple gauge bosons in the system, the Yang-Mills theory as color-decorated Maxwell theory has far-reaching consequences that are not shared by the Maxwell theory.[1] Likewise, we anticipate that color-decorated gravity brings out surprising new features one could not simply guess on a first look. In this chapter, we define and study a version of the color-decorated Einstein gravity in three dimensions, and uncover remarkable new features not shared by the Einstein gravity itself. Most interestingly, we will find that this color-decorated gravity admits a number of (A)dS backgrounds with different cosmological constants as classical vacua.

In anaylzing our model of three-dimensional color-decorated gravity, we shall make use of both the Chern-Simons formulation [124] and the metric formulation. Various features of the theory are more transparent in one formulation over the other. For instance, the existence of multiple (A)dS vacua

[1] The story goes that, during C.N. Yang's seminar at the Institute for Advanced Study at Princeton in 1953, Wolfgang Pauli commented that he first discovered non-Abelian gauge theory in this manner, but then immediately dismissed it because vector bosons are massless and hence "unphysical". We acknowledge Stanley Deser for straightening us up for details of this history.

with different cosmological constants can be understood more intuitively in the metric formulation, whereas consistency of the theory is more manifest in the Chern-Simons formulation. The latter makes use of the gauge algebra,

$$\mathfrak{g} \,=\, (\mathfrak{gl}_2 \oplus \mathfrak{gl}_2) \otimes \mathfrak{u}(N)\,, \tag{6.1}$$

where the $\mathfrak{u}(N)$ and $\mathfrak{gl}_2 \oplus \mathfrak{gl}_2$ correspond respectively to the color gauge algebra and the extended isometry algebra governing the gravitational dynamics. We stress that, compared to the usual gravity with $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ gauge algebra, the color-decorated gravity has two additional identity generators from each of $\mathfrak{gl}_2$. They are indispensable for the consistency of color decoration and correspond to two additional Chern-Simons gauge fields on top of the graviton. Hence, when colored-decorated, we get a massless spin-two field and two non-Abelian spin-one fields, both taking adjoint values of $\mathfrak{u}(N)$. Let us also remark that compared to the spin-one situation where the Abelian Maxwell theory turns into the non-Abelian Yang-Mills theory once color-decorated, Einstein gravity is already non-Abelian, while color decoration enlarges the gauge algebra of the theory.

Re-expressing the theory in metric formulation makes it clear that, among N^2 massless spin-two fields, only the singlet one plays the role of genuine graviton, viz. the first fundamental form, whereas the rest rather behave as *colored spinning matter* fields with minimal covariant coupling to the gravity as well as to the $\mathfrak{u}(N)$ gauge fields. We derive the explicit form of Lagrangian for these *colored spinning matter* fields and find that they have a strong self-coupling compared to the gravitational one by the factor of $\sqrt{N}$. Analyzing the potential of the Lagrangian, we identify all the extrema: there are $[\frac{N+1}{2}]$ number of them and they have different cosmological constants,

$$\left(\frac{N}{N-2k}\right)^2 \Lambda\,, \tag{6.2}$$

where $k = 0, \ldots, [\frac{N-1}{2}]$ is the label of the extrema and Λ is the cosmological constant of the vacuum with maximum radius (corresponding to $k = 0$). Note that not only (A)dS but also any exact gravitational backgrounds such as BTZ black holes [125] lie multiple times with different cosmological constants (6.2) in the vacua of the colored gravity. All extrema except the $k = 0$ vacuum spontaneously break the color symmetry $U(N)$ down to $U(N-k) \times U(k)$. When this symmetry breaking takes place, the corresponding $2\,k\,(N-k)$ spin-two Goldstone modes are combined with the gauge fields to become the partially-massless spin-two fields [126]: the latter spectrum does not have any propagating degrees of freedom (DoF) similarly to the massless ones. Instead in AdS case, they describe 'four' boundary DoF which originate from the boundary modes of the colored massless spin-two and spin-one fields.

The organization of the chapter is as follows. In Section 6.2, we recapitulate the no-go theorem of interacting theory of multiple massless spin-two fields. In Section 6.3, we define the color-decorated (A)dS$_3$ gravity in Chern-Simons formulation, and discuss how this theory evades the no-go theorem. In Section 7.4, we recast the Chern-Simons action into metric formulation by solving torsion condition and obtain the Lagrangian for the colored massless spin-two fields. In Section 7.3, we solve the equations of motion and find a class of classical vacua with varying degrees of color symmetry breaking. We show that these (A)dS vacua have different cosmological constants. We explicitly investigate the simplest example of $k = 1$ vacuum in $N = 3$ case. In Section 6.6, we expand the theory around a color non-singlet vacuum and analyze the field spectrum contents. We demonstrate that the fields corresponding to the broken part of the color symmetry describe the spectrum of partially-massless spin-two field. Section 7.7 contains discussions of our results and outlooks.

6.2 No-Go Theorem on Multiple Spin-Two Theory

Einstein gravity describes the dynamics of massless spin-two field on a chosen vacuum. Conversely, it can also be verified that the Einstein gravity is the only interacting theory of a massless spin-two field (see e.g. [127]). In this context, one may ask whether there exists a non-trivial theory of multiple massless spin two fields. This possibility has been examined in [120, 121], leading to a no-go theorem. We shall begin our discussion by reviewing this result.[2]

The no-go theorem asserts that there is no interacting theory of multiple massless spin-two fields, without inclusion of other fields. The first point to note in this consideration is that any gauge-invariant two-derivative cubic interactions among the spin-two fields is in fact equivalent to that of Einstein-Hilbert (EH) action, modulo *color-decorated* cubic coupling constants g_{IJK} :

$$g_{IJK}\left(h^I_{\mu\rho}\,\partial^\rho\, h^J_{\nu\lambda}\partial^\lambda h^{K\,\mu\nu}+\cdots\right). \tag{6.3}$$

Here, $h^I_{\mu\nu}$ are the massless spin-two fields with *color* index I , and the tensor structure inside of the bracket is that of the EH cubic vertex. For the consistency with the color indices, it is required that the coupling constants are fully symmetric: $g_{IJK} = g_{(IJK)}$. Moreover, the gauge invariance requires that these constants define a Lie algebra spanned by the colored isometry generators. For instance, in the Minkowski spacetime, the colored generators P^I_μ and $M^I_{\mu\nu}$ obey

$$[\,M^I_{\mu\nu}\,,\,P^J_\rho\,] = 2\,g^{IJ}{}_K\,\eta_{\rho[\nu}\,P^K_{\mu]}\,,\qquad [\,M^I_{\mu\nu}\,,\,M^J_{\rho\lambda}\,] = 4\,g^{IJ}{}_K\,\eta_{[\nu[\rho}\,M^K_{\lambda]\mu]}\,. \tag{6.4}$$

Relating these colored generators to the usual isometry ones as $P^I_\mu = P_\mu\otimes\boldsymbol{T}^I$ and $M^I_{\mu\nu} = M_{\mu\nu}\otimes\boldsymbol{T}^I$, one can straightforwardly conclude that the color algebra $\mathfrak{g}_c$ generated by $\boldsymbol{T}^I$ must be *commutative* and *associative* [120]. Moreover, one can even show that $\mathfrak{g}_c$ necessarily reduces to a direct sum of one-dimensional ideals [121]: $\boldsymbol{T}^I\,\boldsymbol{T}^J = 0$ for $I \neq J$. Therefore, in this set-up, the only possibility is the simple sum of several copies of Einstein gravity which do not interact with each other.

This no-go theorem can be evaded with a slight generalization of the setup. Firstly, if the isometry algebra can be consistently extended from a Lie algebra to an associative one, then the commutativity condition on the color algebra $\mathfrak{g}_c$ can be relaxed. The associative extension of isometry algebra typically requires to include other spectra, such as spin-one and possibly higher spins [123]. Moreover, it is not necessary to require that the structure constants g_{IJK} of $\mathfrak{g}_c$ be totally symmetric, but sufficient to assume that the totally symmetric part is non-vanishing, $g_{(IJK)} \neq 0$, so that massless spin-two fields have non-trivial interactions among themselves.

Hence, an interacting theory of multiple massless spin-two fields might be viable once other fields

[2]See also related discussion in [128].

are added and coupled to them. As the next consistency check, one can examine the fate of the general covariance in such a theory: if there exists a genuine metric field among these massless spin-two fields, the others should be subject to interact covariantly with gravity. Moreover, one can also examine whether the multiple massless spin two fields can be color-decorated bona fide by carrying non-Abelian charges. In principle, a theory can be made to covariantly interact with gravity or non-Abelian gauge field by simply replacing all its derivatives by the covariant ones with respect to both the diffeomorphism transformation and the non-Abelian gauge transformation. However, as in the diffeomorphism-covariant interactions of higher-spin fields, such replacements spoil the gauge invariance of the original system [129]. The problematic term in the gauge variation is proportional to the curvatures, namely, Riemann tensor $R_{\mu\nu\rho\lambda}$ or non-Abelian gauge field strength $F_{\mu\nu}$. In three-dimensions, fortuitously, this is not a problem as these curvatures are just proportional to the field equations of Eintein gravity or Chern-Simons gauge theory, respectively. In higher dimensions, these terms can be compensated by introducing a non-trivial cosmological constant, but at the price of adding higher-derivative interactions [130, 131].

So, we conclude that, to have a consistent interacting theory of color-decorated massless spin-two fields, we need an (A)dS isometry gauge algebra which can be extended to an associative one. An immediate candidate is higher-spin algebra in any dimensions, since Vasiliev's higher-spin theory can be consistently color-decorated, as mentioned before. Other option is to take the isometry algebras of $(A)dS_3$ and $(A)dS_5$ which are isomorphic to $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and $\mathfrak{sl}_4$ and can be extended to associative ones, $\mathfrak{gl}_2 \oplus \mathfrak{gl}_2$ and $\mathfrak{gl}_4$ by simply adding unit elements corresponding to spin-one fields.

6.3 Color-Decorated (A)dS$_3$ Gravity: Chern-Simons Formulation

Let us now move to the explicit construction of a theory of colored gravity. In this chapter, we focus on the case of three-dimensional gravity.

6.3.1 Color-Decorated Chern-Simons Gravity

In the uncolored case, it is known that the three-dimensional gravity can be formulated as a Chern-Simons theory with the action

$$S[\mathcal{A}] = \frac{\kappa}{4\pi} \int \mathrm{Tr}\left(\mathcal{A} \wedge d\,\mathcal{A} + \frac{2}{3}\, \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right), \tag{6.5}$$

for the gauge algebra $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. The constant κ is the level of Chern-Simons action. We are interested in color-decorating this theory. Physically, this can be done by attaching Chan-Paton factors to the gravitons. Mathematically, this amounts to requiring the fields to take values in the tensor-product space $\mathfrak{g}_i \otimes \mathfrak{g}_c$, where the $\mathfrak{g}_i$ is the isometry part of the algebra including $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and the $\mathfrak{g}_c$ is a finite-dimensional Lie algebra of a matrix group $\mathfrak{G}_c$. For generic Lie algebras $\mathfrak{g}_i$ and $\mathfrak{g}_c$, their tensor product do not form a Lie algebra, as is clear from the commutation relations:

$$[\, M_X \otimes \boldsymbol{T}_I, M_Y \otimes \boldsymbol{T}_J \,] = \frac{1}{2}\,[\, M_X, M_Y \,] \otimes \{\, \boldsymbol{T}_I, \boldsymbol{T}_J \,\} + \frac{1}{2}\,\{\, M_X, M_Y \,\} \otimes [\, \boldsymbol{T}_I, \boldsymbol{T}_J \,]\,. \tag{6.6}$$

The anticommutators $\{\boldsymbol{T}_I, \boldsymbol{T}_J\}$ and $\{M_X, M_Y\}$ do not make sense within the Lie algebras. Instead, if we start from associative algebras $\mathfrak{g}_i$ and $\mathfrak{g}_c$, their direct product $\mathfrak{g}_i \otimes \mathfrak{g}_c$ will form an associative algebra, from which we can also obtain the Lie algebra structure. Hence, in this chapter, we will consider associative algebras for $\mathfrak{g}_i$ and $\mathfrak{g}_c$. For the color algebra $\mathfrak{g}_c$, we take the matrix algebra $\mathfrak{u}(N)$. For the isometry algebra $\mathfrak{g}_i$, we take $\mathfrak{g}_i = \mathfrak{gl}_2 \oplus \mathfrak{gl}_2$ (instead of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$). The trace Tr of (7.6) should be defined also in the tensor product space and is given by the product of two traces as

$$\mathrm{Tr}(\mathfrak{g}_i \otimes \mathfrak{g}_c) := \mathrm{Tr}(\mathfrak{g}_i)\ \mathrm{Tr}(\mathfrak{g}_c)\,. \tag{6.7}$$

We also need for the fields to obey Hermicity conditions compatible with the real form of the complex algebra.[3]

Therefore, our model of colored gravity is the Chern-Simons theory (7.6) where the one-form gauge field $\mathcal{A}$ takes value in

$$\mathfrak{g} = (\mathfrak{gl}_2 \oplus \mathfrak{gl}_2) \otimes \mathfrak{u}(N)\ \ominus\ \mathrm{id} \otimes \boldsymbol{I}\,. \tag{6.8}$$

Notice that we have subtracted the $\mathrm{id} \otimes \boldsymbol{I}$, where id and $\boldsymbol{I}$ are the centers of $\mathfrak{gl}_2 \oplus \mathfrak{gl}_2$ and $\mathfrak{u}(N)$, respectively: it corresponds to an Abelian vector field (described by Chern-Simons action) which does not interact with other fields.[4] As a complex Lie algebra, $\mathfrak{g}$ in (7.7) is in fact isomorphic to $\mathfrak{sl}_{2N} \oplus \mathfrak{sl}_{2N}$. This can be understood from the fact that the tensor product of 2×2 and $N \times N$ matrices gives $2N \times 2N$ matrix. It would be worth to remark as well that the algebra $\mathfrak{g}$ necessarily contains elements in $\mathrm{id} \otimes \mathfrak{su}(N)$ which correspond to the gauge symmetries of $\mathfrak{su}(N)$ Chern-Simons theory. In this sense, this $\mathfrak{su}(N)$ will be referred to as the color algebra.

It turns out useful[5] to decompose the algebra $\mathfrak{g}$ (7.7) into two orthogonal parts as

$$\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{c}\,, \qquad \text{such that} \qquad \mathrm{Tr}(\mathfrak{b}\,\mathfrak{c}) = 0\,, \tag{6.9}$$

where $\mathfrak{b}$ is the subalgebra:

$$[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{b}\,, \tag{6.10}$$

corresponding to the *gravity plus gauge* sector (mediating gravity and gauge forces), whereas $\mathfrak{c}$ corresponds to the *matter* sector — including all colored spin-two fields — subject to the covariant

[3] Note that if the isometry algebra $\mathfrak{g}_i$ is not associative — as is the case with Poincaré algebra discussed in [120, 121] — then the requirement of the closure of the algebra is that the color algebra $\mathfrak{g}_c$ be associative (for the first term in (7.1) to be in the product algebra) and commutative (for the second term in (7.1) to vanish).

[4] In the Introduction, we sketched our model without taking into account this subtraction for the sake of simplicity.

[5] Later, we will take advantage of this decomposition in solving the torsionless condition to convert Chern-Simons formulation into metric formulation.

transformation,

$$[\mathfrak{b}, \mathfrak{c}] \subset \mathfrak{c}\,. \tag{6.11}$$

Corresponding to the decomposition (7.29), the one-form gauge field $\mathcal{A}$ can be written as the sum of two parts

$$\mathcal{A} = \mathcal{B} + \mathcal{C}\,, \tag{6.12}$$

where $\mathcal{B}$ and $\mathcal{C}$ takes value in $\mathfrak{b}$ and $\mathfrak{c}$, respectively. In terms of $\mathcal{B}$ and $\mathcal{C}$, the Chern-Simons action (7.6) is reduced to

$$S[\mathcal{B},\mathcal{C}] = \frac{\kappa}{4\pi} \int \mathrm{Tr}\left(\mathcal{B} \wedge d\,\mathcal{B} + \frac{2}{3}\,\mathcal{B} \wedge \mathcal{B} \wedge \mathcal{B} + \mathcal{C} \wedge D_{\mathcal{B}}\,\mathcal{C} + \frac{2}{3}\,\mathcal{C} \wedge \mathcal{C} \wedge \mathcal{C}\right), \tag{6.13}$$

where $D_{\mathcal{B}}$ is the the $\mathcal{B}$-covariant derivative:

$$D_{\mathcal{B}}\,\mathcal{C} = d\,\mathcal{C} + \mathcal{B} \wedge \mathcal{C} + \mathcal{C} \wedge \mathcal{B}\,. \tag{6.14}$$

This splitting will prove to be a useful guideline in keeping manifest covariance with respect to the diffeomorphism and the non-Abelian gauge transformation.

6.3.2 Basis of Algebra

For further detailed analysis, we set our conventions and notations of the associative algebra involved. The $\mathfrak{sl}_2$ has three generators J_0, J_1, J_2. Combining them with the center generator J, one obtains $\mathfrak{gl}_2 = \mathrm{Span}\{J, J_0, J_1, J_2\}$ with the product,

$$J_a\, J_b = \eta_{ab}\, J + \epsilon_{abc}\, J^c \qquad\qquad [\,a, b, c = 0, 1, 2\,]\,. \tag{6.15}$$

The η_{ab} is the flat metric with mostly positive signs and ϵ_{abc} is the Levi-civita tensor of $\mathfrak{sl}_2$ with sign convention $\epsilon_{012} = +1$. The generators of the other $\mathfrak{gl}_2$ will be denoted by $\tilde{J}_a$ and $\tilde{J}$. In the case of AdS_3 background, the real form of the isometry algebra corresponds to $\mathfrak{so}(2,2) \simeq \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$, which satisfy

$$(J_a, \tilde{J}_a)^\dagger = -(J_a, \tilde{J}_a)\,, \qquad (J, \tilde{J}\,)^\dagger = (J, \tilde{J}\,)\,. \tag{6.16}$$

In the case of dS_3 background, the real form of the isometry algebra corresponds to $\mathfrak{so}(1,3) \simeq \mathfrak{sl}(2,\mathbb{C})$, which satisfy

$$(J_a, \tilde{J}_a)^\dagger = -(\tilde{J}_a, J_a)\,, \qquad (J, \tilde{J}\,)^\dagger = (\tilde{J}, J\,)\,. \tag{6.17}$$

Defining the Lorentz generator M_{ab} and the translation generator P_a as

$$M_{ab} = \frac{1}{2}\,\epsilon_{ab}{}^{c}\left(J_c + \tilde{J}_c\right), \qquad P_a = \frac{1}{2\sqrt{\sigma}}\left(J_a - \tilde{J}_a\right), \tag{6.18}$$

where $\sigma = +1$ for AdS_3 and $\sigma = -1$ for dS_3, we recover the standard commutation relations

$$[\, M_{ab}, M_{cd}\,] = 2\left(\eta_{d[a}\, M_{b]c} + \eta_{c[b}\, M_{a]d}\right), \quad [\, M_{ab}, P_c\,] = 2\,\eta_{c[b}\, P_{a]}\,, \quad [\, P_a, P_b\,] = \sigma\, M_{ab}\,, \tag{6.19}$$

of $\mathfrak{so}(2,2)$ and $\mathfrak{so}(1,3)$ for $\sigma = +1$ and -1, respectively. The reality structure of $\mathfrak{gl}_2$ determines that of the full algebra $\mathfrak{g}$ in (7.7). As we remarked before, the latter is isomorphic to $\mathfrak{sl}_{2N} \oplus \mathfrak{sl}_{2N}$, hence the conditions (6.16) and (6.17) define which real form of $\mathfrak{sl}_{2N}$ we are dealing with.

The color algebra $\mathfrak{su}(N)$ can be supplemented with the center $\boldsymbol{I}$ to form the associative algebra $\mathfrak{u}(N)$, with the product

$$\boldsymbol{T}_I\, \boldsymbol{T}_J = \frac{1}{N}\, \delta_{IJ} \boldsymbol{I} + \left(g_{IJ}{}^K + i\, f_{IJ}{}^K\right) \boldsymbol{T}_K \qquad [I, J, K = 1, \ldots, N^2 - 1]\,. \tag{6.20}$$

The totally symmetric and anti-symmetric structure constants g_{IJK} and f_{IJK} are both real-valued.

We normalize the center generators of both algebras such that their traces are given by[6]

$$\mathrm{Tr}(J) = 2\sqrt{\sigma}, \qquad \mathrm{Tr}(\tilde{J}) = -2\sqrt{\sigma}, \qquad \mathrm{Tr}(\boldsymbol{I}) = N\,. \tag{6.21}$$

The traces of all other elements vanish. This also defines the trace convention in the Chern-Simons action (7.6). With the associative product defined in (6.15) , these traces yield all the invariant multilinear forms. For instance, we get the bilinear forms,

$$\mathrm{Tr}(J_a\, J_b) = 2\,\sqrt{\sigma}\,\eta_{ab}\,, \qquad \mathrm{Tr}(\tilde{J}_a\, \tilde{J}_b) = -2\,\sqrt{\sigma}\,\eta_{ab}\,, \qquad \mathrm{Tr}(\,\boldsymbol{T}_I\, \boldsymbol{T}_J) = \delta_{IJ}\,, \tag{6.22}$$

which extract the quadratic part of the action.

In the Chern-Simons formulation, the equation of motion is the zero curvature condition: $\mathcal{F} = 0$. In searching for classical solutions, we choose to decompose the subspaces $\mathfrak{b}$ and $\mathfrak{c}$ in (7.29) as

$$\mathfrak{b} = \mathfrak{b}_{\mathrm{GR}} \oplus \mathfrak{b}_{\mathrm{Gauge}}\,, \qquad \mathfrak{c} = \mathfrak{iso} \otimes \mathfrak{su}(N)\,. \tag{6.23}$$

Here, the gravity plus gauge sector corresponds to

$$\mathfrak{b}_{\mathrm{GR}} = \mathfrak{iso} \otimes \boldsymbol{I}\,, \qquad \mathfrak{b}_{\mathrm{Gauge}} = \mathrm{id} \otimes \mathfrak{su}(N)\,, \tag{6.24}$$

in which $\mathfrak{iso}$ stands for the isometry algebra of the (A)dS_3 space:

$$\mathfrak{iso} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2\,. \tag{6.25}$$

There is a trivial vacuum solution where the connection $\mathcal{A}$ is nonzero only for the color-singlet component:

$$\mathcal{B} = \left(\frac{1}{2}\,\omega^{ab}\, M_{ab} + \frac{1}{\ell}\, e^a\, P_a\right) \boldsymbol{I}\,, \qquad \mathcal{C} = 0\,. \tag{6.26}$$

The zero-curvature condition imposes to ω^{ab} and e^a the usual zero (A)dS curvature and zero torsion

[6]We use the same notation Tr for the traces of both the isometry algebra and the color algebra.

conditions:

$$d\,\omega^{ab}+\omega^{a}{}_{c}\wedge\omega^{cb}+\frac{\sigma}{\ell^2}\,e^a\wedge e^b=0\,, \tag{6.27}$$

$$d\,e^a+\omega^{ab}\wedge e_b=0\,, \tag{6.28}$$

which define the (A)dS$_3$ space with the radius ℓ, or equivalently with the cosmological constant $\Lambda=-(\sigma/\ell^2)$.

For a general solution, we again decompose $\mathcal{A}=\mathcal{B}+\mathcal{C}$ according to (6.23). The gravity plus gauge sector takes the form

$$\mathcal{B}=\left[\frac{1}{2}\left(\omega^{ab}+\frac{1}{\ell}\,\Omega^{ab}\right)M_{ab}+\frac{1}{\ell}\,e^a\,P_a\right]\boldsymbol{I}+\boldsymbol{A}+\tilde{\boldsymbol{A}}\,, \tag{6.29}$$

where $\boldsymbol{A}=A^I\,J\,\boldsymbol{T}_I$ and $\tilde{\boldsymbol{A}}=\tilde{A}^I\,\tilde{J}\,\boldsymbol{T}_I$ are two copies of $\mathfrak{su}(N)$ gauge field with

$$(\boldsymbol{A},\tilde{\boldsymbol{A}})^\dagger=-\left\{\begin{array}{ll}(\boldsymbol{A},\tilde{\boldsymbol{A}}) & [\sigma=+1]\\ (\tilde{\boldsymbol{A}},\boldsymbol{A}) & [\sigma=-1]\end{array}\right.\,. \tag{6.30}$$

In (6.29), the splitting $\omega^{ab}+\frac{1}{\ell}\Omega^{ab}$ in the gravity part is arbitrary and is purely for later convenience. The matter sector is composed of

$$\mathcal{C}=\frac{1}{\ell}\left(\boldsymbol{\varphi}^a\,J_a+\tilde{\boldsymbol{\varphi}}^a\,\tilde{J}_a\right). \tag{6.31}$$

Here, the colored massless spin-two fields $\boldsymbol{\varphi}^a=\varphi^{a,I}\,\boldsymbol{T}_I$ and $\tilde{\boldsymbol{\varphi}}^a=\tilde{\varphi}^{a,I}\,\boldsymbol{T}_I$ take value in $\mathfrak{su}(N)$ carrying the adjoint representation. They satisfy

$$(\boldsymbol{\varphi}^a,\tilde{\boldsymbol{\varphi}}^a)^\dagger=+\left\{\begin{array}{ll}(\boldsymbol{\varphi}^a,\tilde{\boldsymbol{\varphi}}^a) & [\sigma=+1]\\ (\tilde{\boldsymbol{\varphi}}^a,\boldsymbol{\varphi}^a) & [\sigma=-1]\end{array}\right.\,. \tag{6.32}$$

Note that the above has a sign difference from (6.30).

We may find solutions by demanding that (6.29) and (6.31) solve for the zero curvature condition. While this procedure straightfowardly yields nontrivial solutions, for better physical interpretations, we shall first recast the Chern-Simons formulation to the metric formulation and then obtain these nontrivial solutions by solving the latter's field equations.

6.4 Color-Decorated (A)dS$_3$ Gravity: Metric Formulation

So far, we described the theory in terms of the gauge field $\mathcal{A}$, so the fact that we are dealing with color-decorated gravity is not tangible. For the sake of concreteness and the advantage of intuitiveness, we shall recast the theory in metric formulation.

We first need to solve the torsionless conditions. This is technically a cumbersome step. Here, we take a short way out from this problem. The idea is that, instead of solving the torsionless conditions for all the colored fields, we shall do it only for the singlet graviton, which we identified above with the metric. This will still allow us to write the action in a metric form but, apart from the gravity, all other colored fields will be still described by a first-order Lagrangian.

In three dimensions, any spectrum with spin greater than zero can be written as a first-order Lagrangian which describes only one helicity mode. If one solves the torsionless conditions for the remaining non-gravity fields, the two fields describing helicity positive and negative modes will combine to generate a single field with a standard second-order Lagrangian. However, this last step appears not necessary and even impossible for certain spectra.

In the following, we will derive the full metric action for the first-order Lagrangian description. For the second-order Lagrangian description, we shall only identify the potential term, leaving aside the explicit form of kinetic terms.

6.4.1 Colored Gravity around Singlet Vacuum

Starting from the Chern-Simons formulation, described in terms of e^a, $\omega^{ab} + \Omega^{ab}/\ell$, $(\boldsymbol{A}, \tilde{\boldsymbol{A}})$ and $(\boldsymbol{\varphi}, \tilde{\boldsymbol{\varphi}})$, we construct a metric formulation by solving the torsionless condition of the gravity sector. This condition is given by

$$de^a + \left(\omega^{ab} + \frac{1}{\ell}\,\Omega^{ab}\right) \wedge e_b + \frac{\sqrt{\sigma}}{N\,\ell}\,\epsilon^{abc}\,\mathrm{Tr}\left(\boldsymbol{\varphi}_b \wedge \boldsymbol{\varphi}_c - \tilde{\boldsymbol{\varphi}}_b \wedge \tilde{\boldsymbol{\varphi}}_c\right) = 0\,, \tag{6.33}$$

where we require ω^{ab} to satisfy the standard torsionless condition (6.28) . This forces $\Omega^{ab} = \epsilon^{abc}\,\Omega_c$ to satisfy

$$\Omega^{[a} \wedge e^{b]} - \frac{\sqrt{\sigma}}{N}\,\mathrm{Tr}\left(\boldsymbol{\varphi}^a \wedge \boldsymbol{\varphi}^b - \tilde{\boldsymbol{\varphi}}^a \wedge \tilde{\boldsymbol{\varphi}}^b\right) = 0\,. \tag{6.34}$$

With the above condition together with the standard torsionless condition (6.28) , the action (7.6) can be recast to the sum of three parts:

$$S = S_{\mathrm{Gravity}} + S_{\mathrm{CS}} + S_{\mathrm{Matter}}\,. \tag{6.35}$$

The first term S_{Gravity} is the action for the (A)dS$_3$ gravity, given by[7]

$$\begin{aligned} S_{\mathrm{Gravity}}[g] &= \frac{\kappa\,N}{4\pi\,\ell}\int \epsilon_{abc}\,e^a \wedge \left(d\omega^{bc} + \omega^{bd} \wedge \o_d{}^c + \frac{\sigma}{3\,\ell^2}\,e^b \wedge e^c\right) \\ &= \frac{1}{16\pi\,G}\int d^3x\sqrt{|g|}\left(R + \frac{2\,\sigma}{\ell^2}\right), \end{aligned} \tag{6.36}$$

where the Chern-Simons level is related to the Newton's constant G, the (A)dS$_3$ radius ℓ and the rank of the color algebra N by

$$\kappa = \frac{\ell}{4\,N\,G}\,. \tag{6.37}$$

[7] In our normalization, $d^3x\sqrt{|g|} = \frac{1}{6}\,\epsilon_{abc}\,e^a \wedge e^b \wedge e^c$.

The second term $S_{\rm CS}$ is the Chern-Simons action for $\mathfrak{su}(N) \oplus \mathfrak{su}(N)$ gauge algebra:

$$S_{\rm CS}[\boldsymbol{A}\tilde{\boldsymbol{A}}] = \frac{\kappa\sqrt{\sigma}}{2\pi}\int\left[{\rm Tr}\left(\boldsymbol{A}\wedge d\boldsymbol{A}+\frac{2}{3}\boldsymbol{A}\wedge\boldsymbol{A}\wedge\boldsymbol{A}\right)-{\rm Tr}\left(\tilde{\boldsymbol{A}}\wedge d\tilde{\boldsymbol{A}}+\frac{2}{3}\tilde{\boldsymbol{A}}\wedge\tilde{\boldsymbol{A}}\wedge\tilde{\boldsymbol{A}}\right)\right]. \tag{6.38}$$

In the uncolored Chern-Simons gravity, it is unclear whether the Chern-Simons level κ has to be quantized since the gauge group is not compact. However, in the case of colored Chern-Simons gravity, the level κ should take an integer value for the consistency of $S_{\rm CS}$ (6.38) under a large $SU(N)\times SU(N)$ gauge transformation.

The last term $S_{\rm Matter}$ is the action for the colored massless spin-two fields $\boldsymbol{\varphi}^a$ and $\tilde{\boldsymbol{\varphi}}^a$. To derive it, we use the decompositions (6.29) and (6.31), and simplify by using (6.34). We get

$$S_{\rm Matter}[\boldsymbol{\varphi},\tilde{\boldsymbol{\varphi}}] = \frac{1}{16\,\pi\,G}\int\left[\frac{1}{N}\,L[\boldsymbol{\varphi},\tilde{\boldsymbol{\varphi}},\ell]-\frac{1}{\ell^2}\,\epsilon_{abc}\,e^a\wedge\Omega^b(\boldsymbol{\varphi},\tilde{\boldsymbol{\varphi}})\wedge\Omega^c(\boldsymbol{\varphi},\tilde{\boldsymbol{\varphi}})\right]\,, \tag{6.39}$$

where the three-form Lagrangian $L[\boldsymbol{\varphi},\tilde{\boldsymbol{\varphi}};\ell]$ is given by

$$\begin{aligned}
&L[\boldsymbol{\varphi},\tilde{\boldsymbol{\varphi}},\ell] = L_+[\boldsymbol{\varphi},\ell]-L_-[\tilde{\boldsymbol{\varphi}},\ell]\,, \qquad (6.40)\\
&L_\pm[\boldsymbol{\varphi},\ell] = 2\sqrt{\sigma}\;{\rm Tr}\left[\frac{1}{\ell}\,\boldsymbol{\varphi}_a\wedge D\,\boldsymbol{\varphi}^a+\frac{1}{\ell^2}\,\epsilon_{abc}\left(\frac{\pm 1}{\sqrt{\sigma}}\,e^a\wedge\boldsymbol{\varphi}^b\wedge\boldsymbol{\varphi}^c+\frac{2}{3}\,\boldsymbol{\varphi}^a\wedge\boldsymbol{\varphi}^b\wedge\boldsymbol{\varphi}^c\right)\right].
\end{aligned}$$

In this expression, the covariant derivative D is with respect to both the Lorentz transformation and the $\mathfrak{su}(N)$ gauge transformation:

$$\begin{aligned}
D\,\boldsymbol{\varphi}^a &= d\,\boldsymbol{\varphi}^a+\omega^{ab}\wedge\boldsymbol{\varphi}_b+\boldsymbol{A}\wedge\boldsymbol{\varphi}^a+\boldsymbol{\varphi}^a\wedge\boldsymbol{A}\,,\\
D\,\tilde{\boldsymbol{\varphi}}^a &= d\,\tilde{\boldsymbol{\varphi}}^a+\omega^{ab}\wedge\tilde{\boldsymbol{\varphi}}_b+\tilde{\boldsymbol{A}}\wedge\tilde{\boldsymbol{\varphi}}^a+\tilde{\boldsymbol{\varphi}}^a\wedge\tilde{\boldsymbol{A}}\,.
\end{aligned} \tag{6.41}$$

The last term in (6.39) is an implicit function of $\boldsymbol{\varphi}^a$ and $\tilde{\boldsymbol{\varphi}}^a$. It is proportional to

$$\epsilon_{abc}\,e^a\wedge\Omega^b\wedge\Omega^c = \frac{1}{3}\,\epsilon_{abc}\,e^a\wedge e^b\wedge e^c\,\Omega_{[d}{}^{,d}\,\Omega_{e]}{}^{,e}\,, \tag{6.42}$$

where $\Omega^a = \Omega_b{}^a\,e^b$. From (6.34), they are determined to be

$$\Omega_a{}^b = \frac{1}{N}\,W_a{}^b(\boldsymbol{\varphi},\tilde{\boldsymbol{\varphi}}) = \Omega_a{}^b(\boldsymbol{\varphi},\tilde{\boldsymbol{\varphi}})\,, \tag{6.43}$$

where $W_a{}^b(\boldsymbol{\varphi},\tilde{\boldsymbol{\varphi}})$ is given by

$$\begin{aligned}
&W_a{}^b(\boldsymbol{\varphi},\tilde{\boldsymbol{\varphi}}) = W_a{}^b(\boldsymbol{\varphi})-W_a{}^b(\tilde{\boldsymbol{\varphi}})\,,\\
&W_a{}^b(\boldsymbol{\varphi}) = 4\sqrt{\sigma}\;{\rm Tr}\left(\boldsymbol{\varphi}_{[a}{}^b\,\boldsymbol{\varphi}_{c]}{}^c-\frac{1}{4}\,\delta_a^b\,\boldsymbol{\varphi}_{[c}{}^c\,\boldsymbol{\varphi}_{d]}{}^d\right).
\end{aligned} \tag{6.44}$$

Here, $\boldsymbol{\varphi}_b{}^a$ are the components of $\boldsymbol{\varphi}^a$: $\boldsymbol{\varphi}^a\;=\;\boldsymbol{\varphi}_b{}^a\,e^b$. Notice that only the term (6.42) — which is quartic in $\boldsymbol{\varphi}^a$ and $\tilde{\boldsymbol{\varphi}}^a$ — gives the cross couplings between $\boldsymbol{\varphi}$'s and $\tilde{\boldsymbol{\varphi}}$'s.

6.4.2 First-order Description

Gathering all above results and replacing the dreibein e^a in terms of the metric $g_{\mu\nu}$, the colored gravity action reads

$$S = S_{\rm CS} + \frac{1}{16\pi\, G} \int d^3x \sqrt{|g|} \left[R - V(\boldsymbol{\varphi}, \tilde{\boldsymbol{\varphi}}) + \frac{2\sqrt{\sigma}}{N\,\ell}\, \epsilon^{\mu\nu\rho}\ {\rm Tr} \left(\boldsymbol{\varphi}_\mu{}^\lambda D_\nu \boldsymbol{\varphi}_{\rho\lambda} - \tilde{\boldsymbol{\varphi}}_\mu{}^\lambda D_\nu \tilde{\boldsymbol{\varphi}}_{\rho\lambda} \right) \right], \tag{6.45}$$

where the covariant derivative is given by

$$D_\mu \boldsymbol{\varphi}_{\nu\rho} = \nabla_\mu \boldsymbol{\varphi}_{\nu\rho} + [\boldsymbol{A}_\mu, \boldsymbol{\varphi}_{\nu\rho}] \tag{6.46}$$

and the scalar potential function is given by

$$\begin{aligned} & V(\boldsymbol{\varphi}, \tilde{\boldsymbol{\varphi}}) \\ = \quad & -\frac{1}{N\,\ell^2}\ {\rm Tr}\ \Big[2\,\sigma\, \boldsymbol{I} + 4 \left(\boldsymbol{\varphi}_{[\mu}{}^\mu\, \boldsymbol{\varphi}_{\nu]}{}^\nu + \tilde{\boldsymbol{\varphi}}_{[\mu}{}^\mu\, \tilde{\boldsymbol{\varphi}}_{\nu]}{}^\nu \right) + 8\sqrt{\sigma} \left(\boldsymbol{\varphi}_{[\mu}{}^\mu\, \boldsymbol{\varphi}_\nu{}^\nu\, \boldsymbol{\varphi}_{\rho]}{}^\rho - \tilde{\boldsymbol{\varphi}}_{[\mu}{}^\mu\, \tilde{\boldsymbol{\varphi}}_\nu{}^\nu\, \tilde{\boldsymbol{\varphi}}_{\rho]}{}^\rho \right) \Big] \\ & - \frac{16\,\sigma}{N^2\,\ell^2}\ {\rm Tr} \left(\boldsymbol{\varphi}_{[\mu}{}^\nu\, \boldsymbol{\varphi}_{\rho]}{}^\rho - \tilde{\boldsymbol{\varphi}}_{[\mu}{}^\nu\, \tilde{\boldsymbol{\varphi}}_{\rho]}{}^\rho \right)\ {\rm Tr} \left(\boldsymbol{\varphi}_{[\nu}{}^\mu\, \boldsymbol{\varphi}_{\lambda]}{}^\lambda - \tilde{\boldsymbol{\varphi}}_{[\nu}{}^\mu\, \tilde{\boldsymbol{\varphi}}_{\lambda]}{}^\lambda \right) \\ & + \frac{6\,\sigma}{N^2\,\ell^2} \left[{\rm Tr} \left(\boldsymbol{\varphi}_{[\mu}{}^\mu\, \boldsymbol{\varphi}_{\nu]}{}^\nu - \tilde{\boldsymbol{\varphi}}_{[\mu}{}^\mu\, \tilde{\boldsymbol{\varphi}}_{\nu]}{}^\nu \right) \right]^2 . \end{aligned} \tag{6.47}$$

The scalar potential function consists of single-trace and double-trace parts. The single-trace part originates from the Chern-Simons cubic interaction, while the double-trace part originates from solving the torsionless conditions. For a general configuration, all terms in the potential have the same order in large N as the other terms in (6.45).

Already at this stage, the content of the colored gravity is clearly demonstrated: it is a theory of colored massless left-moving and right-moving spin-two fields, as seen from the kinetic term in (6.40) or (6.45). They interact covariantly with the color singlet gravity and also with the Chern-Simons color gauge fields. Moreover, they interact with each other through the potential $V(\boldsymbol{\varphi}, \tilde{\boldsymbol{\varphi}})$. The self-interaction is governed by the constant $1/N$. The single-trace cubic interaction is stronger than the gravitational cubic interaction by the factor of $\sqrt{N}$. Therefore, at large N and for fixed Newton's constant, the colored massless spin-two fields will be strongly coupled to each other.

6.4.3 Second-order Description

In principle, we could also solve the torsionless condition for the colored spin-two fields and obtain a second-order Lagrangian (although this spoils the minimal interactions to the $\mathfrak{su}(N)$ gauge fields $\boldsymbol{A}$ and $\tilde{\boldsymbol{A}}$). It amounts to taking linear combinations

$$\boldsymbol{\chi}_{\mu\nu} = \sqrt{\sigma} \left(\boldsymbol{\varphi}_{\mu\nu} - \tilde{\boldsymbol{\varphi}}_{\mu\nu} \right), \qquad \boldsymbol{\tau}_{\mu\nu} = \boldsymbol{\varphi}_{\mu\nu} + \tilde{\boldsymbol{\varphi}}_{\mu\nu}\,, \tag{6.48}$$

and integrating out the *torsion* part $\boldsymbol{\tau}_{\mu\nu}$, while keeping $\boldsymbol{\chi}_{\mu\nu}$. The resulting action is given by

$$S = S_{\rm CS} + \frac{1}{16\pi\, G} \int d^3x \sqrt{|g|}\ [\, R - V(\boldsymbol{\chi}) + \mathcal{L}_{\rm CM}(\boldsymbol{\chi}, \nabla \boldsymbol{\chi}, \boldsymbol{A}, \tilde{\boldsymbol{A}})\,]\,. \tag{6.49}$$

The Lagrangian $\mathcal{L}_{\rm CM}$ reads

$$\mathcal{L}_{\rm CM}(\boldsymbol{\chi}, \nabla\boldsymbol{\chi}, \boldsymbol{A}, \tilde{\boldsymbol{A}}) = \frac{1}{N}\,\mathrm{Tr}\left(2\,\boldsymbol{\chi}_{\mu\nu}\,\nabla^2\,\boldsymbol{\chi}^{\mu\nu} + \cdots\right), \tag{6.50}$$

where the ellipses include other tensor contractions together with higher-order terms of the form, $\boldsymbol{\chi}^n\,(\nabla\,\boldsymbol{\chi})^2$ with $n \geq 1$ as well as couplings to the gauge fields $\boldsymbol{A}$ and $\tilde{\boldsymbol{A}}$. We do not attempt to obtain the complete structure of these derivative terms.

The potential function $V(\boldsymbol{\chi})$ corresponds to the extremum of

$$\begin{aligned} V(\boldsymbol{\chi},\boldsymbol{\tau}) \;\;=\;\; & -\frac{2\,\sigma}{N\,\ell^2}\,\mathrm{Tr}\left(\boldsymbol{I} + \boldsymbol{\chi}_{[\mu}{}^{\mu}\,\boldsymbol{\chi}_{\nu]}{}^{\nu} + \sigma\,\boldsymbol{\tau}_{[\mu}{}^{\mu}\,\boldsymbol{\tau}_{\nu]}{}^{\nu} + \boldsymbol{\chi}_{[\mu}{}^{\mu}\,\boldsymbol{\chi}_{\nu}{}^{\nu}\,\boldsymbol{\chi}_{\rho]}{}^{\rho} + 3\,\sigma\,\boldsymbol{\chi}_{[\mu}{}^{\mu}\,\boldsymbol{\tau}_{\nu}{}^{\nu}\,\boldsymbol{\tau}_{\rho]}{}^{\rho}\right) \\ & -\frac{4}{N^2\,\ell^2}\,\mathrm{Tr}\left(\boldsymbol{\chi}_{[\mu}{}^{\nu}\,\boldsymbol{\tau}_{\rho]}{}^{\rho} + \boldsymbol{\tau}_{[\mu}{}^{\nu}\,\boldsymbol{\chi}_{\rho]}{}^{\rho}\right)\,\mathrm{Tr}\left(\boldsymbol{\chi}_{[\nu}{}^{\mu}\,\boldsymbol{\tau}_{\lambda]}{}^{\lambda} + \boldsymbol{\tau}_{[\nu}{}^{\mu}\,\boldsymbol{\chi}_{\lambda]}{}^{\lambda}\right) \\ & +\frac{6}{N^2\,\ell^2}\,\left[\mathrm{Tr}\left(\boldsymbol{\chi}_{[\mu}{}^{\mu}\,\boldsymbol{\tau}_{\nu]}{}^{\nu}\right)\right]^2, \end{aligned} \tag{6.51}$$

along the $\boldsymbol{\tau}_{\mu\nu}$ direction. As the extremum equation for $\boldsymbol{\tau}_{\mu\nu}$ is linear in $\boldsymbol{\tau}_{\mu\nu}$,

$$\mathcal{M}(\boldsymbol{\chi})\cdot\boldsymbol{\tau} = 0\,, \tag{6.52}$$

it must be that the unique solution is $\boldsymbol{\tau}_{\mu\nu} = 0$ for a generic configuration of $\boldsymbol{\chi}_{\mu\nu}$.[8] Proceeding with this situation, we end up with the cubic potential for $\boldsymbol{\chi}_{\mu\nu}$:

$$V(\boldsymbol{\chi}) = -\frac{2\,\sigma}{N\,\ell^2}\,\mathrm{Tr}\left(\boldsymbol{I} + \boldsymbol{\chi}_{[\mu}{}^{\mu}\,\boldsymbol{\chi}_{\nu]}{}^{\nu} + \boldsymbol{\chi}_{[\mu}{}^{\mu}\,\boldsymbol{\chi}_{\nu}{}^{\nu}\,\boldsymbol{\chi}_{\rho]}{}^{\rho}\right). \tag{6.53}$$

This potential has a noticeably simple form, but also has rich implications as we shall discuss in the next sections.

6.5 Classical Vacua of Colored Gravity

6.5.1 Identification of Vacuum Solutions

Having identified the action in metric formulation, we now search for classical vacua that solve the field equations of motion:

$$-\frac{\delta\,\mathcal{L}_{\rm CM}}{\delta g_{\mu\nu}} = G_{\mu\nu} - \frac{1}{2}V(\boldsymbol{\chi})\,g_{\mu\nu}\,, \qquad \frac{\delta\,\mathcal{L}_{\rm CM}}{\delta\boldsymbol{\chi}_{\mu\nu}} = \frac{\partial V(\boldsymbol{\chi})}{\partial\boldsymbol{\chi}_{\mu\nu}}\,, \tag{6.54}$$

$$-\frac{N}{2\sqrt{\sigma}\,\ell}\,\frac{\delta\,\mathcal{L}_{\rm CM}}{\delta\boldsymbol{A}_\mu} = \epsilon^{\mu\nu\rho}\,\boldsymbol{F}_{\nu\rho}\,, \qquad -\frac{N}{2\sqrt{\sigma}\,\ell}\,\frac{\delta\,\mathcal{L}_{\rm CM}}{\delta\tilde{\boldsymbol{A}}_\mu} = \epsilon^{\mu\nu\rho}\,\tilde{\boldsymbol{F}}_{\nu\rho}\,. \tag{6.55}$$

[8] There can also exist nontrivial $\boldsymbol{\tau}_{\mu\nu}$ solutions at special values of $\boldsymbol{\chi}_{\mu\nu}$, corresponding to kernel of $\mathcal{M}$ in (6.52). They break the parity symmetry spontaneously, and hence of special interest. We relegate complete classification of these null solutions in a separate paper.

In order to find their solutions, we assume that the colored massless spin-two fields are covariantly constant with the trivial $\mathfrak{su}(N)$ gauge connection,

$$\boldsymbol{A} = 0\,, \qquad \tilde{\boldsymbol{A}} = 0\,, \qquad \nabla_\rho\, \boldsymbol{\chi}_{\mu\nu} = 0\,. \tag{6.56}$$

This can be satisfied by

$$\boldsymbol{\chi}_{\mu\nu} = g_{\mu\nu}\, \boldsymbol{X} \qquad \text{for} \qquad \boldsymbol{X} = \text{constant} \,\in\, \mathfrak{su}(N)\,. \tag{6.57}$$

Physically, we interpret this as the colored spin-two matter acting as Higgs field. In Poincaré invairant field theory, the vacuum is Poincaré invariant, so only a scalar field φ (which is proportional to an identity operator $\varphi\,\propto\,\mathbb{I}$) can take a vacuum expectation value, $\langle\varphi\rangle\,=\,v$. On the other hand, fields with nonzero spin cannot develop a nonzero expectation value since it is incompatible with the Lorentz invariance. In generally covariant field theory, where the background metric $g_{\mu\nu}$ plays the role of first fundamental form, the spin-2 field $\boldsymbol{\chi}_{\mu\nu}$ can similarly develop a nonzero vacuum expectation value $\langle\chi_{\mu\nu}\rangle = v\, g_{\mu\nu}$ proportional to the metric $g_{\mu\nu}$, while all other fields of higher spin cannot. We thus refer this phenomenon to as ‘gravitational Higgs mechanism’.

With (6.56) and (6.57), the equations in the second line (6.55) trivialize and the rest reduce to

$$G_{\mu\nu} - \frac{1}{2} V(\boldsymbol{X})\, g_{\mu\nu} = 0 \qquad \text{and} \qquad \frac{\partial V(\boldsymbol{X})}{\partial \boldsymbol{X}} = 0\,, \tag{6.58}$$

where $V(\boldsymbol{X}) = V(\boldsymbol{\chi}_{\mu\nu} = g_{\mu\nu}\, \boldsymbol{X})$ is given by

$$V(\boldsymbol{X}) = -\frac{2\,\sigma}{N\,\ell^2}\, \mathrm{Tr}\left(\boldsymbol{I} + 3\,\boldsymbol{X}^2 + \boldsymbol{X}^3\right). \tag{6.59}$$

From (6.58), the extremum of the potential defines the corresponding cosmological constant:

$$\Lambda = \frac{1}{2}\, V(\boldsymbol{X})\,. \tag{6.60}$$

Although cubic, being a matrix-valued function, the potential $V(\boldsymbol{X})$ may admit a large number of nontrivial extrema that depends on the color algebra $\mathfrak{su}(N)$. If exists, each of such extrema will define a distinct vacuum with a different cosmological constant (6.60). As an illustration of this potential, consider the function $f(\boldsymbol{X}) = \frac{1}{N}\,\mathrm{Tr}\left(\boldsymbol{I} + 3\,\boldsymbol{X}^2 + \boldsymbol{X}^3\right)$ for the $\boldsymbol{X}$ belonging to $\mathfrak{su}(3)$. The 3×3 matrix $\boldsymbol{X}$ can be diagonalized by a $SU(3)$ rotation to

$$\boldsymbol{X} = a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} + b \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{6.61}$$

We plot the function $f(a,b)$ in Fig.13. It clearly exhibits four extremum points: $(0,0)$, $(2,0)$, $(0,2)$ and $(-2,-2)$. The first point at the origin gives $f=1$, whereas the other three points all give $f=9$. In fact, these three points are connected by $SU(3)$ rotation.

We now explicitly identify the extrema of potential function (6.59) for arbitrary value of N. The

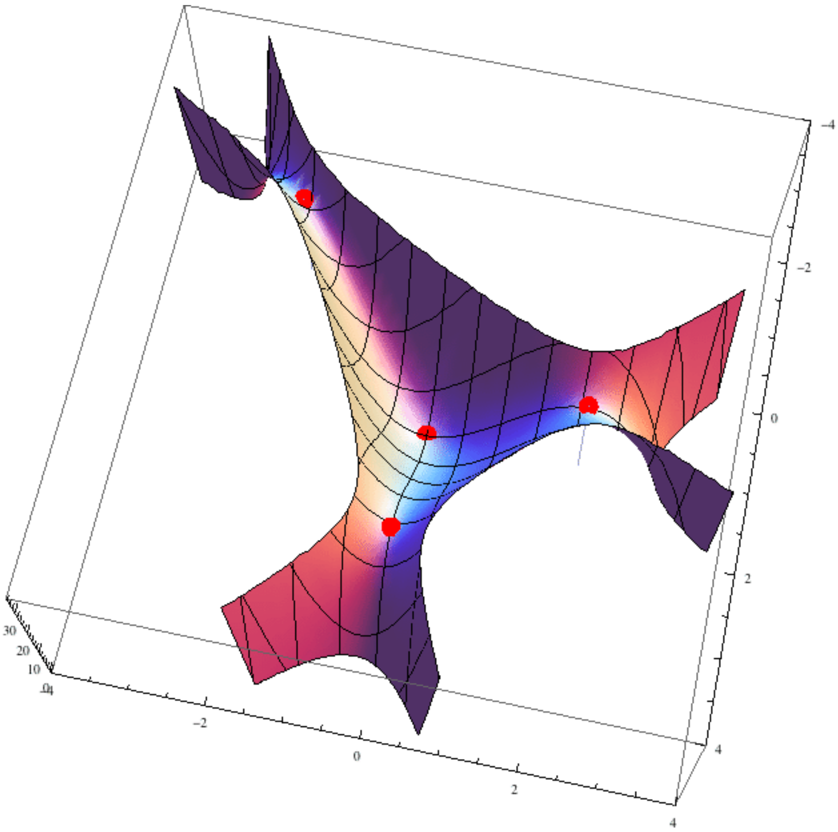

Figure 13: *The shape of the potential function for* $\mathfrak{su}(3)$.

extremum points are defined by the equation:

$$\delta V(\boldsymbol{X}) = -\frac{6\,\sigma}{N\,\ell^2}\ \mathrm{Tr}\left[(2\,\boldsymbol{X} + \boldsymbol{X}^2)\,\delta\boldsymbol{X}\right] = 0\,. \tag{6.62}$$

Since $\boldsymbol{X}$ is traceless, it follows that $\delta\boldsymbol{X}$ is also traceless. Thus, the equation reads

$$2\,\boldsymbol{X} + \boldsymbol{X}^2 = \frac{1}{N}\ \mathrm{Tr}\left(2\,\boldsymbol{X} + \boldsymbol{X}^2\right)\boldsymbol{I}\,. \tag{6.63}$$

Since $\mathrm{Tr}\,(\boldsymbol{I} + \boldsymbol{X})^2 \neq 0$ — otherwise it would follow from (6.63) that the matrix $\boldsymbol{I} + \boldsymbol{X}$ is nilpotent while having a non-trivial trace — one can redefine the matrix $\boldsymbol{X}$ in terms of $\boldsymbol{Z}$:

$$\boldsymbol{Z} = \sqrt{\frac{N}{\mathrm{Tr}(\boldsymbol{I} + \boldsymbol{X})^2}}\ (\boldsymbol{I} + \boldsymbol{X})\,, \tag{6.64}$$

or equivalently,

$$\boldsymbol{X} = \frac{N}{\mathrm{Tr}(\boldsymbol{Z})}\,\boldsymbol{Z} - \boldsymbol{I}\,. \tag{6.65}$$

This simplifies the equation (6.63) as

$$\boldsymbol{Z}^2 = \boldsymbol{I}\,. \tag{6.66}$$

Complete solutions of this equation, up to $SU(N)$ rotations, are given by

$$\boldsymbol{Z}_k = \begin{bmatrix} \boldsymbol{I}_{(N-k)\times(N-k)} & 0 \\ 0 & -\boldsymbol{I}_{k\times k} \end{bmatrix}, \qquad k = 0, 1, \ldots, \left[\tfrac{N-1}{2}\right]. \tag{6.67}$$

where the upper bound of k is fixed by $[\frac{N-1}{2}]$ due to the property that $\boldsymbol{X}_{N-k}$ is a $SU(N)$ rotation of $\boldsymbol{X}_k$. Notice also that, when N is even, $k = \frac{N}{2}$ is excluded since it leads to $\mathrm{Tr}(\boldsymbol{Z}) = 0$ for which $\boldsymbol{X}$ is ill-defined. Plugging the solutions (6.67) to the potential, we can identify the values of the potential at the extrema as

$$V(\boldsymbol{X}_k) = -\frac{2\,\sigma}{\ell^2}\left(\frac{N}{\mathrm{Tr}\,(\boldsymbol{Z}_k)}\right)^2 = -\frac{2\,\sigma}{\ell^2}\left(\frac{N}{N-2k}\right)^2. \tag{6.68}$$

These values play the role of the cosmological constant at the k-th extremum, according to (6.60).

Let us discuss more on the potential (6.59). Firstly, the cubic form shows that the potential is not bounded from below or above. Secondly, the overall factor σ shows that the overall sign of the potential depends whether we consider AdS$_3$ or dS$_3$ background. Thirdly, we can understand better the stability of the extrema we found by considering the second variation of the potential,

$$\delta^2 V(\boldsymbol{X}_k) = -\frac{12\,\sigma}{(N-2k)\,\ell^2}\;\mathrm{Tr}\left(\boldsymbol{Z}_k\,\delta\boldsymbol{X}^2\right). \tag{6.69}$$

The Hessian is not positive(or negative)-definite for an arbitrary $\delta\boldsymbol{X}$ except the singlet vacuum $k = 0$. So, all $k \neq 0$ vacua are saddle points and the $k = 0$ vacuum is the minimum/maximum in dS$_3$/AdS$_3$ space.

6.5.2 $N = 3$ Example and Linearized Spectrum

In the standard Higgs mechansim, the gauge fields combine with the Goldstone bosons to become massive vector fields. In the following, we will analyze the analogous mechanism in our model of colored gravity. For the concreteness, let us consider the $k = 1$ vacuum solution (6.67) in $N = 3$ case. This solution has a non-zero background for the colored matter fields which breaks the $SU(3)$ symmetry down to $SU(2)\times U(1)$. We linearize the colored matter fields $(\boldsymbol{\varphi}, \tilde{\boldsymbol{\varphi}})$ around this vacuum as

$$\boldsymbol{\varphi}_{\mu\nu} = \frac{\boldsymbol{X}_1}{2\sqrt{\sigma}}\,g_{\mu\nu} + \boldsymbol{\varphi}^{\text{fluc}}_{\mu\nu}\,, \qquad \tilde{\boldsymbol{\varphi}}_{\mu\nu} = -\frac{\boldsymbol{X}_1}{2\sqrt{\sigma}}\,g_{\mu\nu} + \tilde{\boldsymbol{\varphi}}^{\text{fluc}}_{\mu\nu}\,, \tag{6.70}$$

where the background value of $(\boldsymbol{\varphi}, \tilde{\boldsymbol{\varphi}})$ is proportional to the matrix $\boldsymbol{X}_1$ (6.65), whose explicit form reads

$$\boldsymbol{X}_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{bmatrix}. \tag{6.71}$$

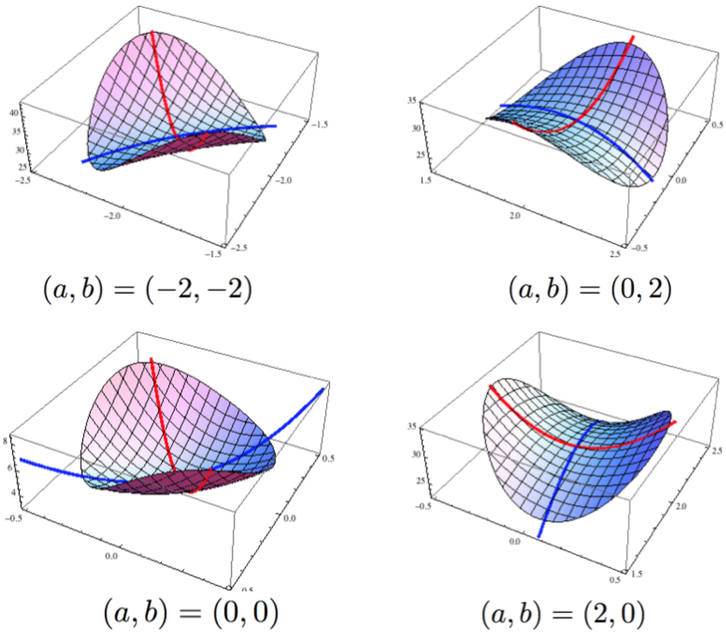

Figure 14: *Potentials around each rainbow vacua with* $N = 3$. *The* $(a, b) = (0, 0)$ *vacuum is the minimum/maximum in dS*$_3$*/AdS*$_3$ *space. The others, connected by* $SU(3)$ *transformation, are all saddle points.*

The fluctuation parts of the colored matter fields and the spin-one Cherns-Simons gauge fields can be decomposed as

$$\begin{aligned}
\boldsymbol{\varphi}^{\text{fluc}}_{\mu\nu} &= \frac{3}{\sqrt{2}}\,\varrho^{a}_{\mu\nu}\,\boldsymbol{T}^{a}_{\mathfrak{su}(2)} + \sqrt{\frac{3}{2}}\left(\tilde{\psi}_{\mu\nu} - 2\,\psi_{\mu\nu}\right)\boldsymbol{T}_{\mathfrak{u}(1)} + \frac{3}{\sqrt{2}}\,\phi^{i}_{\mu\nu}\,\boldsymbol{T}^{i}_{\text{BS}}\,, \\
\tilde{\boldsymbol{\varphi}}^{\text{fluc}}_{\mu\nu} &= \frac{3}{\sqrt{2}}\,\tilde{\varrho}^{a}_{\mu\nu}\,\boldsymbol{T}^{a}_{\mathfrak{su}(2)} + \sqrt{\frac{3}{2}}\left(\psi_{\mu\nu} - 2\,\tilde{\psi}_{\mu\nu}\right)\boldsymbol{T}_{\mathfrak{u}(1)} + \frac{3}{\sqrt{2}}\,\tilde{\phi}^{i}_{\mu\nu}\,\boldsymbol{T}^{i}_{\text{BS}}\,,
\end{aligned} \tag{6.72}$$

$$\begin{aligned}
\boldsymbol{A}_{\mu} &= A^{a}_{\mu}\,\boldsymbol{T}^{a}_{\mathfrak{su}(2)} + A_{\mu}\,\boldsymbol{T}_{\mathfrak{u}(1)} + \frac{1}{\sqrt{8}}\,A^{i}_{\mu}\,\boldsymbol{Z}_{1}\,\boldsymbol{T}^{i}_{\text{BS}}\,, \\
\tilde{\boldsymbol{A}}_{\mu} &= \tilde{A}^{a}_{\mu}\,\boldsymbol{T}^{a}_{\mathfrak{su}(2)} + \tilde{A}_{\mu}\,\boldsymbol{T}_{\mathfrak{u}(1)} + \frac{1}{\sqrt{8}}\,\tilde{A}^{i}_{\mu}\,\boldsymbol{Z}_{1}\,\boldsymbol{T}^{i}_{\text{BS}}\,,
\end{aligned} \tag{6.73}$$

in terms of the $SU(3)$ generators:

$$\boldsymbol{T}^a_{\mathfrak{su}(2)}=\tfrac{1}{\sqrt{2}}\begin{bmatrix}\sigma^a & 0\\ 0 & 0\end{bmatrix},\quad \boldsymbol{T}_{\mathfrak{u}(1)}=\tfrac{1}{\sqrt{6}}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-2\end{bmatrix},\tag{6.74}$$

$$\boldsymbol{T}^1_{\rm BS}=\tfrac{1}{\sqrt{2}}\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix},\quad \boldsymbol{T}^2_{\rm BS}=\tfrac{1}{\sqrt{2}}\begin{bmatrix}0&0&-i\\0&0&0\\i&0&0\end{bmatrix},\tag{6.75}$$

$$\boldsymbol{T}^3_{\rm BS}=\tfrac{1}{\sqrt{2}}\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix},\quad \boldsymbol{T}^4_{\rm BS}=\tfrac{1}{\sqrt{2}}\begin{bmatrix}0&0&0\\0&0&-i\\0&i&0\end{bmatrix},$$

where $\sigma^1,\sigma^2,\sigma^3$ are the Pauli matrices. Various factors in (6.72) and (6.73) have been introduced for latter convenience. By plugging (6.72) into the original action (6.45) and expanding the action up to quadratic order in the fluctuations, we obtain the perturbative Lagrangian around the $k=1$ vacuum. We first expand the potential as

$$V(\boldsymbol{\varphi},\tilde{\boldsymbol{\varphi}})=-\frac{2}{\ell_1^2}\left(\sigma+\varrho^{a}_{[\mu}{}^{\mu}\,\varrho^{a}_{\nu]}{}^{\nu}+\tilde{\varrho}^{a}_{[\mu}{}^{\mu}\,\tilde{\varrho}^{a}_{\nu]}{}^{\nu}-\psi_{[\mu}{}^{\mu}\,\psi_{\nu]}{}^{\nu}-\tilde{\psi}_{[\mu}{}^{\mu}\,\tilde{\psi}_{\nu]}{}^{\nu}\right)+\mathcal{O}(\Phi^3)\,,\tag{6.76}$$

and the kinetic part as

$$\begin{aligned}&\frac{2\sqrt{\sigma}}{9\,\ell_1}\,\epsilon^{\mu\nu\rho}\;\mathrm{Tr}\left(\boldsymbol{\varphi}_{\mu}{}^{\lambda}D_\nu\boldsymbol{\varphi}_{\rho\lambda}-\tilde{\boldsymbol{\varphi}}_{\mu}{}^{\lambda}D_\nu\tilde{\boldsymbol{\varphi}}_{\rho\lambda}\right)\\&=\frac{\sqrt{\sigma}}{\ell_1}\,\epsilon^{\mu\nu\rho}\left(\varphi'^{a\,\lambda}_{\mu}\nabla_\nu\varphi'^{a}_{\rho\lambda}+\psi_\mu{}^{\lambda}\nabla_\nu\psi_{\rho\lambda}+\phi^{i\;\lambda}_{\mu}\nabla_\nu\phi^i_{\rho\lambda}+\frac{1}{\sqrt{\sigma}}\,\phi^i_{\mu\nu}\,A^i_\rho\right)+c.c+\mathcal{O}(\Phi^3)\,.\end{aligned}\tag{6.77}$$

Here the $\ell_1=\ell/3$ is the radius of the $k=1$ (A)dS solution, and $\mathcal{O}(\Phi^3)$ means the cubic-order terms in the fluctuation fields. Combining (6.76) and (6.77), the colored gravity action (6.45) becomes

$$S=S_{\rm CS}+\frac{1}{16\pi\,G}\int d^3\sqrt{|g|}\left(R+\frac{2\,\sigma}{\ell_1^2}+\mathcal{L}_{\rm RS}+\mathcal{L}_{\rm BS}+\mathcal{O}(\Phi^3)\right),\tag{6.78}$$

where the Lagragian for the residual symmetry part is given by

$$\begin{aligned}\ell_1^2\;\mathcal{L}_{\rm RS}\quad&=\quad\sqrt{\sigma}\,\ell_1\,\epsilon^{\mu\nu\rho}\varrho^{a\,\lambda}_{\mu}\nabla_\nu\varrho^a_{\rho\lambda}+2\,\varrho^{a}_{[\mu}{}^{\mu}\,\varrho^{a}_{\nu]}{}^{\nu}+c.c.\\&\quad+\sqrt{\sigma}\,\ell_1\,\epsilon^{\mu\nu\rho}\,\psi_\mu{}^{\lambda}\nabla_\nu\psi_{\rho\lambda}-2\,\psi_{[\mu}{}^{\mu}\,\psi_{\nu]}{}^{\nu}+c.c.\end{aligned}\tag{6.79}$$

and that for the broken symmetry part by

$$\ell_1^2\;\mathcal{L}_{\rm BS}=\sqrt{\sigma}\,\ell_1\,\epsilon^{\mu\nu\rho}\left(\phi^{i\;\lambda}_{\mu}\nabla_\nu\phi^i_{\rho\lambda}+\frac{1}{\sqrt{\sigma}}\,\phi^i_{\mu\nu}\,A^i_\rho\right)+c.c.\tag{6.80}$$

Several remarks are in order:

- In the Lagrangian $\mathcal{L}_{\rm RS}$ (6.79), the fields $\varrho^a_{\mu\nu}$ — associated with the $\mathfrak{su}(2)$ generators — describe the standard massless spin-two fields. On the contrary, the field $\psi_{\mu\nu}$ — associated with the $\mathfrak{u}(1)$ generator — describes a *ghost* massless spin-two due to the sign flip of the no-derivative term (why this sign determines whether the spectrum is ghost or not is explained in section 2.3).

- In the Lagrangian $\mathcal{L}_{\rm BS}$ (6.80) — associated with the broken part of the symmetry — has an unusual cross term with the spin-one Cherns-Simons gauge field A^i_μ. In fact, A^i_μ behaves as a Stueckelberg field hence can be removed by a spin-two gauge transformation. Let us remark that this gauge choice is analogous to the unitary gauge in the standard Higgs mechanism. As a result, the Chern-Simons action $S_{\rm CS}$ reduces from $SU(3)$ to $SU(2)\times U(1)$, and the field $\phi^i_{\mu\nu}$ inherits the gauge symmetries of A^i_μ as a second-derivative form:

$$\delta\,\phi^i_{\mu\nu} = \left(\nabla_\mu\,\nabla_\nu - \frac{\sigma}{\ell_1^2}\,g_{\mu\nu}\right)\xi^i\,. \tag{6.81}$$

  This spectrum clearly combines the massless spin-two mode with the spin-one mode in an irreducible manner. It actually corresponds to so-called partially-massless spin-two field [126]. Since our system is after all a Chern-Simons theory, there is no propagating DoF such as a scalar field. Hence, it is clear that we cannot have a massive spin-two as a result of symmetry breaking because it would require not only spin-one but also a scalar mode. We postpone more detailed analysis to the next section.

6.6 Colored Gravity around Rainbow Vacua

We learned that there are $[\frac{N+1}{2}]$ many distinct vacua having different cosmological constants. In this section, we study the colored gravity around each of these vacua and analyze the spectrum. In principle, we can proceed in the same way as we did for the $N=3$ example in Section 6.5.2, but there is a more systematic way relying on the Chern-Simons formulation.

6.6.1 Decomposition of Algebra Revisited

For an efficient treatment of the colored gravity at each distinct vacuum in the Chern-Simons formulation, it is important to identify the proper decomposition of the algebra (7.29). For that, we revisit the isometry and the color algebra decompositions. The isometry algebra can be divided into the rotation part $\mathcal{M}$ and the translation part $\mathcal{P}$ as

$$\mathfrak{iso} = \mathcal{M}\oplus\mathcal{P}\,, \tag{6.82}$$

the same as the trivial vacuum. For the color algebra, each vacuum spontaneously breaks the Chan-Paton $\mathfrak{su}(N)$ gauge symmetry down to $\mathfrak{su}(N-k)\oplus\mathfrak{su}(k)\oplus\mathfrak{u}(1)$, and hence the original algebra admits the decomposition:

$$\mathfrak{su}(N)\simeq\mathfrak{su}(N-k)\oplus\mathfrak{su}(k)\oplus\mathfrak{u}(1)\oplus\mathfrak{bs}\,. \tag{6.83}$$

Here, $\mathfrak{bs}$ is the vector space corresponding to the *broken symmetry*, spanned by $2k(N-k)$ generators. It is important to note that each part commutes or anti-commutes with the background matrix $\boldsymbol{Z}_k$ (6.67) as

$$\big[\,\boldsymbol{Z}_k\,,\,\mathfrak{su}(N-k)\oplus\mathfrak{su}(k)\oplus\mathfrak{u}(1)\,\big]=0\,,\qquad \big\{\,\boldsymbol{Z}_k\,,\,\mathfrak{bs}\,\big\}=0\,. \tag{6.84}$$

We now decompose the entire algebra (7.7) according to (7.29) in terms of the gravity plus gauge sector $\mathfrak{b}$ and the matter sector $\mathfrak{c}$. The former has again two parts similarly to the singlet vacuum case as $\mathfrak{b}=\mathfrak{b}_{\rm GR}\oplus\mathfrak{b}_{\rm Gauge}$, but the algebras to which the gravity and the gauge sectors correspond differ from (6.24). They are

$$\mathfrak{b}_{\rm GR}=\big(\mathcal{M}\otimes\boldsymbol{I}\big)\oplus\big(\mathcal{P}\otimes\boldsymbol{Z}_k\big)\,,\qquad \mathfrak{b}_{\rm Gauge}={\rm id}\otimes\Big(\mathfrak{su}(N-k)\oplus\mathfrak{su}(k)\oplus\mathfrak{u}(1)\Big)\,. \tag{6.85}$$

The gauge sector is concerned only with the unbroken part of the color algebra. The algebra of the gravity sector is deformed by $\boldsymbol{Z}_k$, but still satisfies the same commutation relations with the generators:

$$\boldsymbol{M}_{ab}=M_{ab}\,\boldsymbol{I}\,,\qquad \boldsymbol{P}_a=P_a\,\boldsymbol{Z}_k\,. \tag{6.86}$$

The one-form gauge fields associated with these sectors are given correspondingly by

$$\begin{aligned}\mathcal{B}_{\rm GR}&=\frac{1}{2}\left(\omega^{ab}+\Omega^{ab}\right)\boldsymbol{M}_{ab}+\frac{1}{\ell_k}\,e^a\,\boldsymbol{P}_a\,,\\ \mathcal{B}_{\rm Gauge}&=\boldsymbol{A}_++\boldsymbol{A}_-+\tilde{\boldsymbol{A}}_++\tilde{\boldsymbol{A}}_-+\big(A+\tilde{A}\big)\,\boldsymbol{Y}_k\,,\end{aligned} \tag{6.87}$$

where the k-vacuum radius ℓ_k is related to the singlet one as

$$\ell_k:=\left(\frac{N-2k}{N}\right)\ell\,, \tag{6.88}$$

and $\boldsymbol{Y}_k$ is the traceless matrix:

$$\boldsymbol{Y}_k=\frac{k\,\boldsymbol{I}_+-(N-k)\,\boldsymbol{I}_-}{N}\,. \tag{6.89}$$

Here again, the spin connection ω^{ab} is the standard one satisfying (6.28), whereas Ω^{ab} will be determined in terms of other fields from the torsionless conditions. The gauge fields $\boldsymbol{A}_\pm$ and $\tilde{\boldsymbol{A}}_\pm$ take values in $\mathfrak{su}(N-k)$ for the subscript $+$ and $\mathfrak{su}(k)$ for the subscript $-$, whereas A and $\tilde{A}$ are Abelian gauge fields taking values in $\mathfrak{u}(1)$.

In the case of non-singlet vacua, the matter sector space has two parts:

$$\mathfrak{c}=\mathfrak{c}_{\rm CM}\oplus\mathfrak{c}_{\rm BS}\,. \tag{6.90}$$

For the introduction of each elements, let us first define the generators of $\mathfrak{gl}_2\oplus\mathfrak{gl}_2$ deformed by $\boldsymbol{Z}_k$ as

$$\begin{aligned}\boldsymbol{J}_a&=J_a\,\boldsymbol{I}_++\tilde{J}_a\,\boldsymbol{I}_-\,,\qquad &\boldsymbol{J}&=J\,\boldsymbol{I}_++\tilde{J}\,\boldsymbol{I}_-\,,\\ \tilde{\boldsymbol{J}}_a&=J_a\,\boldsymbol{I}_-+\tilde{J}_a\,\boldsymbol{I}_+\,,\qquad &\tilde{\boldsymbol{J}}&=J\,\boldsymbol{I}_-+\tilde{J}\,\boldsymbol{I}_+\,,\end{aligned} \tag{6.91}$$

where $\boldsymbol{I}_\pm$ are the identities associated with $\mathfrak{u}(N-k)$ and $\mathfrak{u}(k)$, respectively:

$$\boldsymbol{I}_\pm = \frac{1}{2}\left(\boldsymbol{I} \pm \boldsymbol{Z}_k\right). \tag{6.92}$$

These deformed $\mathfrak{gl}_2 \oplus \mathfrak{gl}_2$ generators satisfy also the same relation as (6.15), and they are related to $\boldsymbol{M}_{ab}$ and $\boldsymbol{P}_a$ (7.37) analogously to (6.18) by

$$\boldsymbol{M}_{ab} = \frac{1}{2}\,\epsilon_{ab}{}^{c}\left(\boldsymbol{J}_c + \tilde{\boldsymbol{J}}_c\right) \qquad \text{and} \qquad \boldsymbol{P}_a = \frac{1}{2\sqrt{\sigma}}\left(\boldsymbol{J}_a - \tilde{\boldsymbol{J}}_a\right). \tag{6.93}$$

Therefore, if we define the matter fields using $\boldsymbol{J}_a$ and $\tilde{\boldsymbol{J}}_a$, then they will have the standard interactions with the gravity.

We now introduce each elements of (6.90). The first one $\mathfrak{c}_{\rm CM}$ is the residual color symmetry:

$$\mathfrak{c}_{\rm CM} = \mathfrak{iso} \otimes \Big(\mathfrak{su}(N-k) \oplus \mathfrak{su}(k) \oplus \mathfrak{u}(1)\Big), \tag{6.94}$$

describing colored spin-two fields associated with the one form

$$\mathcal{C}_{\rm CM} = \frac{1}{\ell_k}\left[\left(\boldsymbol{\varphi}^a_+ + \boldsymbol{\varphi}^a_-\right)\boldsymbol{J}_a + \left(\tilde{\boldsymbol{\varphi}}^a_+ + \tilde{\boldsymbol{\varphi}}^a_-\right)\tilde{\boldsymbol{J}}_a + \Big(\psi^a\,\boldsymbol{J}_a + \tilde{\psi}^a\,\tilde{\boldsymbol{J}}_a\Big)\,\boldsymbol{Y}_k\,\boldsymbol{Z}_k\right]. \tag{6.95}$$

The fields $\boldsymbol{\varphi}^a_+$ and $\tilde{\boldsymbol{\varphi}}^a_+$ take values in $\mathfrak{su}(N-k)$, whereas $\boldsymbol{\varphi}^a_-$ and $\tilde{\boldsymbol{\varphi}}^a_-$ in $\mathfrak{su}(k)$, both transforming in the adjoint representations. The fields ψ^a and $\tilde{\psi}^a$ are charged under $\mathfrak{u}(1)$. The matrix factor $\boldsymbol{Y}_k\,\boldsymbol{Z}_k$ is inserted to ensure $\mathrm{Tr}(\mathfrak{b}_{\rm GR}\,\mathfrak{c}_{\rm CM}) = 0$, equivalently,

$$\mathrm{Tr}\left(\boldsymbol{J}\,\boldsymbol{Y}_k\,\boldsymbol{Z}_k\right) = 0 = \mathrm{Tr}\left(\tilde{\boldsymbol{J}}\,\boldsymbol{Y}_k\,\boldsymbol{Z}_k\right). \tag{6.96}$$

The second element $\mathfrak{c}_{\rm BS}$ is what corresponds to the broken part of the color symmetries:

$$\mathfrak{c}_{\rm BS} = (\mathrm{id} \oplus \mathfrak{iso}) \otimes \mathfrak{bs}. \tag{6.97}$$

Unlike the fields in $\mathcal{C}_{\rm CM}$, this part does not describe massless spin-two fields. Rather, it describes so-called partially-massless spin-two fields [126], as we shall demonstrate in the following. The corresponding one form is given by

$$\mathcal{C}_{\rm BS} = \frac{1}{\ell_k}\left(\boldsymbol{\phi}\,\boldsymbol{J} + \boldsymbol{\phi}^a\,\boldsymbol{J}_a + \tilde{\boldsymbol{\phi}}\,\tilde{\boldsymbol{J}} + \tilde{\boldsymbol{\phi}}^a\,\tilde{\boldsymbol{J}}_a\right), \tag{6.98}$$

where the fields $\boldsymbol{\phi}^a$, $\boldsymbol{\phi}$, $\tilde{\boldsymbol{\phi}}^a$ and $\tilde{\boldsymbol{\phi}}$ take values in $\mathfrak{bs}$, carrying the bi-fundamental representations of $\mathfrak{su}(N-k)$ and $\mathfrak{su}(k)$, as well as the representation of $\mathfrak{u}(1)$. Because these fields anti-commute with $\boldsymbol{Z}_k$, they also intertwine the left-moving and the right-moving $\mathfrak{gl}_2$'s. For instance,

$$\boldsymbol{\phi}^a\,\boldsymbol{J}_b = \tilde{\boldsymbol{J}}_b\,\boldsymbol{\phi}^a. \tag{6.99}$$

As a consequence, they transform differently under Hermitian conjugate:

$$(\boldsymbol{\phi}, \boldsymbol{\phi}^a, \tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\phi}}^a)^\dagger = \begin{cases} (-\tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\phi}}^a, -\boldsymbol{\phi}, \boldsymbol{\phi}^a) & [\sigma = +1] \\ (-\boldsymbol{\phi}, \boldsymbol{\phi}^a, -\tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\phi}}^a) & [\sigma = -1] \end{cases}, \tag{6.100}$$

compared to the massless ones (6.32).

6.6.2 Colored Gravity around Non-Singlet Vacua

With the precise form of the fields (7.38), (6.95), (6.98), we now rewrite the Chern-Simons action into a metric form. It is given by the sum of three terms as in (6.35). Firstly, we have the standard gravity action

$$S_{\text{Gravity}} = \frac{1}{16\pi\, G} \int d^3x \sqrt{|g|} \left(R + \frac{2\,\sigma}{\ell_k^2} \right), \tag{6.101}$$

with a k-dependent cosmological constant, set by (6.88). The Chern-Simons action S_{CS} for the gauge fields $\boldsymbol{A}_+$ for $\mathfrak{su}(N-k)$, $\boldsymbol{A}_-$ for $\mathfrak{su}(k)$ and A for $\mathfrak{u}(1)$ are given analogously to (6.38). Finally, the action for the matter sector takes the following form:

$$\begin{aligned} S_{\text{Matter}} \;\;=\;\; & \frac{1}{16\,\pi\, G} \int \frac{1}{N-2k} \Big(L[\boldsymbol{\varphi}_+, \tilde{\boldsymbol{\varphi}}_+, \ell_k] - L[\boldsymbol{\varphi}_-, \tilde{\boldsymbol{\varphi}}_-, \ell_k] + L_{\text{BS}}[\boldsymbol{\phi}, \tilde{\boldsymbol{\phi}}, \ell_k] + L_{\text{cross}} \Big) \\ & \qquad - \frac{k(N-k)}{N^2}\, L[\psi, \tilde{\psi}, \ell_k] - \frac{1}{\ell_k^2}\, \epsilon_{abc}\, e^a \wedge \Omega^b \wedge \Omega^c \,, \end{aligned} \tag{6.102}$$

where L is the massless Lagrangian given in (6.40) whereas L_{BS} is given by

$$L_{\text{BS}}[\boldsymbol{\phi}, \tilde{\boldsymbol{\phi}}, \ell] = \frac{4\sqrt{\sigma}}{\ell}\, \text{Tr} \left[\left\{ \tilde{\boldsymbol{\phi}} \wedge \left(D\,\boldsymbol{\phi} - \frac{1}{\sqrt{\sigma}\,\ell}\, e^a \wedge \boldsymbol{\phi}_a \right) - \tilde{\boldsymbol{\phi}}_a \wedge \left(D\,\boldsymbol{\phi}^a - \frac{1}{\sqrt{\sigma}\,\ell}\, e^a \wedge \boldsymbol{\phi} \right) \right\} \boldsymbol{Z}_k \right]. \tag{6.103}$$

The covariant derivatives $D\boldsymbol{\phi}^a$ and $D\boldsymbol{\phi}$ are given by

$$\begin{aligned} D\boldsymbol{\phi}^a \;&=\; D_{\o}\boldsymbol{\phi}^a + \left(\tilde{\boldsymbol{A}}_+ + \tilde{\boldsymbol{A}}_- + \tilde{A}\,\boldsymbol{Y}_k \right) \wedge \boldsymbol{\phi}^a - \boldsymbol{\phi}^a \wedge (\boldsymbol{A}_+ + \boldsymbol{A}_- + A\,\boldsymbol{Y}_k)\,, \\ D\boldsymbol{\phi} \;&=\; d\boldsymbol{\phi} + \left(\tilde{\boldsymbol{A}}_+ + \tilde{\boldsymbol{A}}_- + \tilde{A}\,\boldsymbol{Y}_k \right) \wedge \boldsymbol{\phi} \;-\; \boldsymbol{\phi} \wedge (\boldsymbol{A}_+ + \boldsymbol{A}_- + A\,\boldsymbol{Y}_k)\,, \end{aligned} \tag{6.104}$$

and similarly for the tilde counter parts. The other terms in (7.53) give additional interactions: the last term gives quartic interaction through $\Omega_a{}^{,b}$:

$$\Omega_a{}^{,b} = \frac{1}{N-2k} \left[W^a_b(\boldsymbol{\varphi}_+, \tilde{\boldsymbol{\varphi}}_+) + W^a_b(\boldsymbol{\varphi}_-, \tilde{\boldsymbol{\varphi}}_-) + W_{\text{BS}}{}^a_b(\boldsymbol{\phi}, \tilde{\boldsymbol{\phi}}) \right] - \frac{1}{k(N-k)}\, W^a_b(\psi, \tilde{\psi})\,, \tag{6.105}$$

where W^a_b is given by (6.44) and $W_{\text{BS}}{}^a_b$ by

$$W_{\text{BS}}{}^a_b(\boldsymbol{\phi}, \tilde{\boldsymbol{\phi}}) = 8\sqrt{\sigma}\, \text{Tr} \left[\left(\tilde{\boldsymbol{\phi}}_{[a}{}^{[b}\, \boldsymbol{\phi}_{c]}{}^{c]} - \frac{1}{4}\, \delta^b_a\, \tilde{\boldsymbol{\phi}}_{[c}{}^{c}\, \boldsymbol{\phi}_{d]}{}^{d} \right) \boldsymbol{Z}_k \right]. \tag{6.106}$$

The term $L_{\rm cross}$, given by

$$L_{\rm cross} \;=\; \frac{4\sqrt{\sigma}}{\ell_k^2}\,{\rm Tr}\Big[\left(\boldsymbol{\varphi}_+^a-\boldsymbol{\varphi}_-^a+\psi^a\,\boldsymbol{Y}_k\right)\wedge\Big(\tilde{\boldsymbol{\phi}}\wedge\boldsymbol{\phi}_a+\tilde{\boldsymbol{\phi}}_a\wedge\boldsymbol{\phi}\Big)+ \\ +\,\epsilon_{abc}\left(\boldsymbol{\varphi}_+^a\wedge\boldsymbol{\varphi}_+^b+\boldsymbol{\varphi}_-^a\wedge\boldsymbol{\varphi}_-^b\right)\wedge\psi^c\,\boldsymbol{Y}_k\,\boldsymbol{Z}_k\Big]-\Big([\ ]\leftrightarrow\widetilde{[\ ]}\Big)\,,\tag{6.107}$$

is the cross terms originating from the Chern-Simons cubic interactions.

In principle, we can further simplify the action as we did in the singlet vacuum case. However, already at this level, we can extract a lot of physics.

- We have a scalar potential as a function of four fields $\boldsymbol{\varphi}_\pm$, ψ, $\boldsymbol{\phi}$ (and their tilde counter parts) and the point where all fields vanish correspond to the extremum point whose potential value gives the cosmological constant $-\sigma/\ell_k^2$. This potential should be a shift of the potential $V(\boldsymbol{\varphi},\tilde{\boldsymbol{\varphi}})$ (6.47) defined around the singlet vacuum, hence it will admit all other vacua as extrema.

- The interaction strength for each field can be easily read off from the action. The gravity and gauge interaction have the same strength controlled by G and κ as in the singlet vacuum case. The interaction of colored spin two fields $\boldsymbol{\varphi}_\pm$ is weakened — the coefficient changed from N to $N-2k$. The same for the broken-symmetry field $\boldsymbol{\phi}$. Finally, ψ has interaction strength controlled by $N^2/[k(N-k)]$. Therefore, when the color symmetry is maximally broken, that is $N-2k\sim 1$, the interaction between all these fields becomes as weak as the gravitational one.

- Let us conclude this section with the summary of the field content around the k-vacuum. At first, we have the graviton and $\mathfrak{su}(N-k)\oplus\mathfrak{su}(k)\oplus\mathfrak{u}(1)$ Chern-Simons gauge fields. Next, about the colored matter fields, there are $(N-k)^2-1$ fields for $(\boldsymbol{\varphi}_+,\tilde{\boldsymbol{\varphi}}_+)$, k^2-1 for $(\boldsymbol{\varphi}_-,\tilde{\boldsymbol{\varphi}}_-)$ and 1 for $(\psi,\tilde{\psi})$. They are all massless spin-two fields, but $(\boldsymbol{\varphi}_-,\tilde{\boldsymbol{\varphi}}_-)$ and $(\psi,\tilde{\psi})$ — hence k^2 fields — are in fact ghost. For the broken symmetry part, we have $2k(N-k)$ fields for $(\boldsymbol{\phi},\tilde{\boldsymbol{\phi}})$. The latter describes so-called partially-massless fields and its proper analysis is the subject of the next section.

6.6.3 Partially Massless Spectrum Associated with Broken Color Symmetry

Around a non-singlet vacuum, the fields $\boldsymbol{\varphi}_\pm$ and ψ both describe massless spin-two fields having the same quadratic Lagrangian given by (6.40). On the other hand, the fields $\boldsymbol{\phi}$ corresponding to the broken part of the color symmetries have different quadratic Lagrangian (7.54), hence describe different spectrum. We have already mentioned that they correspond to partially-massless fields [126]. In this section, we analyze the quadratic Lagrangian (7.54) to prove this statement. Here, we concentrate on AdS_3. To get the dS_3 result, it is sufficient to replace ℓ by $i\,\ell$.

Though the Lagrangian (7.54) has a rather non-standard form involving cross term between $\boldsymbol{\phi}$ and $\tilde{\boldsymbol{\phi}}$ together with an insertion of $\boldsymbol{Z}_k$, it can always be diagonalized with the help of the Hermiticity property (6.100). Therefore, for the spectrum analysis, it will suffice to consider $S_{\rm BS}[\phi,\phi^a]$ taking the following expression:

$$S_{\rm BS}[\phi,\phi^a]=\int \phi\wedge\left(d\phi-\frac{1}{\ell}\,e_a\wedge\phi^a\right)-\phi_a\wedge\left(D\phi^a-\frac{1}{\ell}\,e^a\wedge\phi\right), \tag{6.108}$$

with the AdS dreibein and spin connection (e^a,ω^{ab}) . We first note that this action admits the gauge symmetries with parameters $(\varepsilon,\varepsilon^a)$,

$$\delta\,\phi=d\,\varepsilon-\frac{1}{\ell}\,e^a\,\varepsilon_a\,,\qquad \delta\,\phi^a=D\,\varepsilon^a-\frac{1}{\ell}\,e^a\,\varepsilon\,, \tag{6.109}$$

which come from the Chern-Simons gauge symmetries.

For a closer look of this action involving three fields $h_{\mu\nu}\;=\;e^a{}_{(\mu}\,\phi_{\nu)a}$, $f_{\mu\nu}\;=\;e^a{}_{[\mu}\,\phi_{\nu]a}$ and $\phi_\mu=e^a{}_\mu\,\phi_a$, we consider two different but equivalent paths:

- We first derive the equation of motion for one-form fields ϕ^a and ϕ . They are given by

$$D\phi^a-\frac{1}{\ell}e^a\wedge\phi=0\,,\qquad d\phi-\frac{1}{\ell}e^a\wedge\phi_a=0\,. \tag{6.110}$$

  The second equation implies that the antisymmetric field $f_{\mu\nu}$ is the field strength of ϕ_μ : $f_{\mu\nu}=\ell\,\partial_{[\mu}\phi_{\nu]}$. Then, by gauge fixing ϕ_μ to zero with the gauge parameter ε^a, the field $f_{\mu\nu}$ decouples from the first equation. We thus end up with only one field $h_{\mu\nu}$ satisfying the equation of motion,

$$\nabla_{[\mu}h_{\nu]\rho}=0\,, \tag{6.111}$$

  and the gauge symmetry,

$$\delta\,h_{\mu\nu}=\left(\nabla_\mu\,\nabla_\nu-\frac{1}{\ell^2}\,g_{\mu\nu}\right)\varepsilon\;. \tag{6.112}$$

  This coincides with the gauge symmetry of partially-massless spin-two field [126].

- Instead of first deriving the equation and then gauge fixing to $\phi_\mu=0$, one can reverse the procedure. We first gauge fix and eliminate ϕ_μ field in the action and obtain

$$S_{\rm PM}[\phi_{\mu\nu}]=\int d^3x\sqrt{|g|}\,\epsilon^{\mu\nu\rho}\,\phi^\lambda{}_\mu\,\nabla_\nu\,\phi_{\rho\lambda}\,, \tag{6.113}$$

  modulo a boundary term. We note that the field $\phi_{\mu\nu}$ contains both of the symmetric part $h_{\mu\nu}$ and the antisymmetric part $f_{\mu\nu}$. Only $h_{\mu\nu}$ admits the gauge symmetries (6.112). The equation of motion is now given by

$$C_{\mu\nu,\rho}:=\nabla_{[\mu}h_{\nu]\rho}+\nabla_{[\mu}f_{\nu]\rho}=0\,. \tag{6.114}$$

  The totally anti-symmetric part $C_{[\mu\nu,\rho]}=\partial_{[\mu}f_{\nu\rho]}=0$ can be readily solved as $f_{\mu\nu}=\partial_{[\mu}\,a_{\nu]}$. With the field redefinition $h_{\mu\nu}\to h_{\mu\nu}-\nabla_{(\mu}a_{\nu)}$, the trace of the above equation, $C^\rho{}_{\mu,\rho}=0$, gives

$$a_{\mu}=\frac{\ell^{2}}{2}\,(\nabla^{\rho}h_{\mu\rho}-\nabla_{\mu}h^{\rho}{}_{\rho})\,, \tag{6.115}$$

Taking now a divergence of $C_{\mu\nu,\rho}$, we arrive at the second-order equation,

$$\nabla^{\rho}C_{\rho(\mu,\nu)}=G^{\rm lin}_{\mu\nu}+\frac{1}{\ell^{2}}\,(h_{\mu\nu}-g_{\mu\nu}\,h^{\rho}{}_{\rho})=0\,, \tag{6.116}$$

with the linearized Einstein tensor $G^{\rm lin}_{\mu\nu}$. One can also check that the mass in the above equation corresponds to that of a partially-massless field. Furthermore, using Bianchi identity, we deduce that the left-hand side of (6.115) vanishes, so does $f_{\mu\nu}$. Therefore, we end up with the same equation (6.111).[9]

6.7 Discussions

In this chapter, we proposed a Chan-Paton color-decorated gravity in three dimensions and studied its properties. We have shown that the theory describes a gravitational system of colored massless spin-two matter fields coupled to $\mathfrak{su}(N)$ gauge fields. These matter fields have a non-trivial potential whose extrema have $[\frac{N+1}{2}]$ different values of cosmological constant. All the extremum points but the origin spontaneously break the $\mathfrak{su}(N)$ color symmetry down to $\mathfrak{su}(N-k)\oplus\mathfrak{su}(k)\oplus\mathfrak{u}(1)$. We found that the spin-two Goldstone modes corresponding to the broken part of the symmetries are combined with the gauge fields and become partially-massless spin-two fields. In the vacua with large $k\sim N/2$, the interactions of the matter fields are as weak as the gravitational one. In the small k vacua, their interaction becomes strong by the factor of $\sqrt{N}$.

Considering the dS_3 branch, the potential takes a spiral stairwell shape (Fig.15) with $[\frac{N+1}{2}]$ many steps, having split cosmological constants that range from $\Lambda=1/\ell^{2}$ at the lowest step all the way up to $\sim N^{2}\,\Lambda$ at the highest step. The spacing gets dense in lower steps, while sparse in higher steps. If such features continue to hold in higher dimensions, the colored gravity with large N might be very relevant for the early universe cosmology in that the universe begins in an inflationary epoch with a large cosmological constant at a very high stairstep. The colored matter are weakly coupled there, and hence they are not confined. As the state of the universe decays towards lower stairsteps, the effective cosmological constant decreases sequentially and eventually exits the inflation. The colored matter fields start to interact stronger and eventually form heavy color-neutral composites. It is in this synopsis that the spin-two colored matter fields might play a novel role in the current paradigm of the inflationary cosmology.

We also speculate on a novel approach to the three-dimensional quantum colored gravity. At large N, the contribution of the $\mathcal{O}(N/2)$ multiple vacua in the path integral might be captured by the

[9]Strictly speaking, the equation (6.116) alone is weaker than the first-order one (6.111). The former describes one propagating degrees of freedom, while the latter does not have any bulk mode and corresponds to the spectrum described by (6.108). Note that the latter partially-massless spectrum is what the three-dimensional conformal gravity contains analogously to the four-dimensional case [132, 133]. To recapitulate, in three dimensions (not in higher dimensions), there are two kinds of partially-massless fields for the maximal depth, which includes the spin-two partially-massless spectrum. We shall discuss more about this subtlety in next chapter 7.

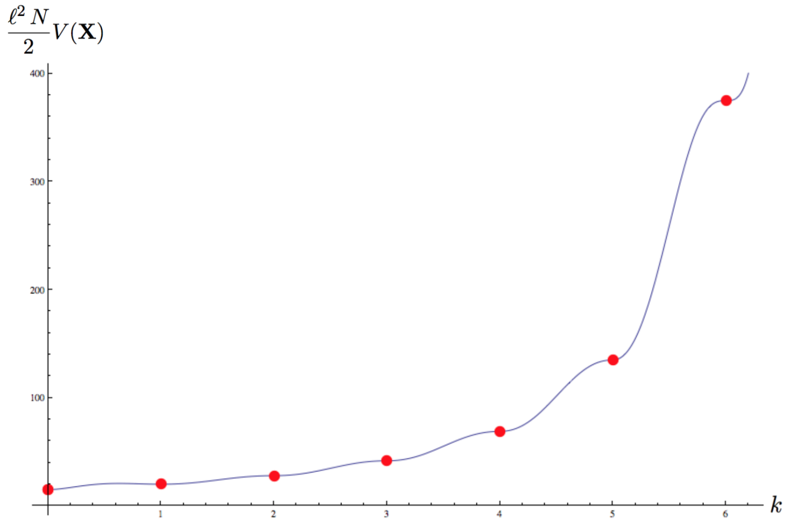

Figure 15: Potential of the colored gravity in dS ($N = 15$): k is the parameter of a curve in $\mathfrak{su}(15)$ that passes through the extremum points.

$\mathfrak{su}(N)$ random matrix model given by

$$\mathcal{Z}_{\rm MM} = \int d\boldsymbol{X}\, \exp\left[i\, c\, V(\boldsymbol{X})\right]. \tag{6.117}$$

It would be also interesting to explore ab initio definition of the three-dimensional quantum gravity starting from tensor-field valued matrix models.

This work brings in many open problems worth of further investigation. First of all, extensions to (higher-spin) supergravity as well as the analysis of the asymptotic symmetries [134, 135] are imminent. Further extensions to color-decoration of the known higher-spin gravity in three-dimensional Lifshitz spacetime [136] and flat spacetime [137] are also straightforward. Extension to higher-dimensional spacetime is also highly interesting. A version of such situation was already studied in the context of AdS/CFT correspondence [138]. Vasiliev equations for color-decorated higher-spin theories needs to be better understood, along with higher-dimensional counterpart of the stairstep potential we found in three dimensions. As the color dynamics is described by Chern-Simons gauge theory, one might anticipate to formulate colored gravity in any dimensions in terms of a version of Chern-Simons formulation, perhaps, along the lines of [139] and [140]. Quantum aspects of color-decorated gravity is an avenue to be explored. In particular, consequences and implications of strong color interactions among colored spin-two fields. Turning to the inflationary cosmology, it would be interesting to understand how the color-decoration modifies the infrared dynamics of interacting massless spin-two fields at super-horizon scales. This brings one to investigate stochastic dynamics of these fields, as would be described by color-decorated version of the Langevin dynamics [141, 142].

Chapter 7

Colored Higher Spin

Whenever a theory appears to you as the only possible one,
take this as a sign that you have neither understood the theory
nor the problem which the theory was intended to solve.
Carl Popper, 'Objective Knowledge: Evolutionary Approach' (1972)

7.1 Introduction

In recent years, higher-spin gravity has been studied intensively. In particular, much progress has been made related to the higher-spin AdS/CFT duality [144, 145]. In three dimensional theories, the breakthroughs took place in several places consecutively: first, the asymptotic symmetry of higher-spin gravity has been identified as nonlinear W-algebras [146, 147], then it led to the conjecture of the W_N minimal models as dual CFTs [148]. Blackhole-like exact solutions were constructed [149]. Most of these results are about the $hs(\lambda) \oplus hs(\lambda)$ Chern-Simons higher-spin gravity [150, 151] or the Prokushkin-Vasiliev theory [152] which contains the former as the gauge sector. Many variant models of three-dimensional higher-spin gravity have been considered later on and many other interesting features were discovered (see [153, 154, 155, 156, 157, 158, 159] for a non-complete list of references).

In previous chapter 6, we proposed an extension of the three-dimensional gravity to multi-graviton system — color-decorated three-dimensional gravity. The purpose of this chapter is to extend this analysis to higher spins. More precisely, we consider the Chern-Simons formulation of higher-spin (A)dS gravity, leaving aside the matter coupling issue of the Prokushkin-Vasiliev theory. In fact, the possibility of color decoration appears as a rather natural generalization in the Vasiliev's approach to higher-spin gravity, where the higher-spin algebra plays the key role for consistency of the theory and the color decoration is one of the simplest extensions of higher-spin algebra. This was first pointed out in [161, 162], and further discussed in [163]. Actually, all the Vasiliev's nonlinear higher-spin equations can be consistently color decorated with the same mechanism. Judging from this and also from the dual CFT considerations, we anticipate that the issue of color decoration might be more consequential in the context of higher-spin gravity.

We analyze the colored higher-spin gravity in three dimensions and show, in particular, that the salient aspect of the colored gravity persists in the higher-spin extension: when the higher-spin gravity is color decorated, there appears a staircase potential with a number of extrema. Each of these extrema provides an (A)dS background solution with a different cosmological constant. For $SU(N)$ color symmetry, there exist $[\frac{N+1}{2}]$ different vacua — henceforth, referred to as *rainbow vacua* — which spontaneously break the color symmetry down to $SU(N-k)\times SU(k)\times U(1)$ $(k = 0, 1, \dots, [\frac{N-1}{2}])$. When this symmetry breaking happens, the Goldstone modes — the spin two fields corresponding to the broken part of the symmetries — combine with all other spins and become a longer spectrum. For the models of higher-spin gravity involving massless spins up to M, that is the $\mathfrak{gl}_M \oplus \mathfrak{gl}_M$ Chern-

Simons model, the symmetry broken part of the spectrum forms maximal-depth partially massless spin M representation [164]. This representation has as many degrees of freedom (DoF) as the collection of massless spins from 1 to M . Therefore, for a generic model of colored higher-spin gravity — involving arbitrary higher spins — we might end up with a rather exotic spectrum: maximal-depth partially-massless fields of infinite spin, which is reminiscent of a Regge trajectory. In three dimensions, the (partially-)massless fields have only boundary DoF, but in the limit of infinite spin, the dimension of their phase-space becomes infinite. They correspond in dual conformal field theory living at the boundary to infinite-tower of (partially-)conserved global currents. We would like to emphasize that the color-decorated higher-spin gravity, or any variant/generalization of the latter, is a plausible bulk theory when considering a bi-vector type free CFT and looking for its bulk dual in the context of AdS/CFT. In this set-up, the rainbow vacua and the spontaneous symmetry breaking are generic and unneglectable phenomena.

The organization of the chapter is as follows. In Section 7.2, we review how the consistent color decoration works in the models of higher-spin gravity. In Section 7.3, the rainbow vacua with different cosmological constants are identified. In Section 7.4, we expand the theory around one of the rainbow vacua by solving the torsionless condition. In Section 7.5, we analyze the spectrum resulting from the symmetry breaking and show that it corresponds to the maximal-depth partially-massless field. In Section 7.6, we provide an account for the partially massless representations in three dimensions. Finally, Section 7.7 contains further discussion.

7.2 Color Decoration of Higher-Spin (A)dS$_3$ Gravity

7.2.1 Color Decoration

We first recapitulate how one can consistently color-decorate a given higher-spin theory, extending our previous chapter 6.

Suppose we are given an uncolored (higher-spin) gravity theory, defined either by an action or by a set of field equations. Assume that elementary fields take values in an (higher-spin) isometry Lie algebra $\mathfrak{g}_i$. The idea is that in order to color-decorate this theory by attaching Chan-Paton factors to the fields, we require these fields to take values in the tensor product algebra $\mathfrak{g}_i \otimes \mathfrak{g}_c$. The $\mathfrak{g}_c$ is the color symmetry algebra. However, though we may start with Lie algebras $\mathfrak{g}_i$ and $\mathfrak{g}_c$, the tensor product algebra does not automatically provide a Lie algebra $\mathfrak{g}_i \otimes \mathfrak{g}_c$ because the anticommutators are not defined. This point should be clear from

$$[\,M_X \otimes \boldsymbol{T}_I, M_Y \otimes \boldsymbol{T}_J\,] = \frac{1}{2}\,[\,M_X, M_Y\,] \otimes \{\,\boldsymbol{T}_I, \boldsymbol{T}_J\,\} + \frac{1}{2}\,\{\,M_X, M_Y\,\} \otimes [\,\boldsymbol{T}_I, \boldsymbol{T}_J\,]\,. \tag{7.1}$$

We conclude that, if an associative product can be defined in $\mathfrak{g}_i$ and $\mathfrak{g}_c$, the color-decoration through the Chan-Paton factors can be achieved.

One can always take $\mathfrak{g}_c = \mathfrak{u}(N)$ as color symmetry, and hence the associativity of $\mathfrak{g}_c$ is satisfied. However, the relevant isometry algebras $\mathfrak{so}(d,2)$ or $\mathfrak{so}(d+1,1)$ for (A)dS$_{d+1}$ space do not have an associative structure for general d. The way out of this problem is to consider a larger algebra $\mathfrak{g}_i$ which contains the isometry algebra. In this way, the $\mathfrak{g}_i$ would contain more generators, and so the corresponding uncolored theory would involve more fields than the pure (A)dS$_d$ Einstein gravity. For instance, in the previous chapter 6, we considered the three-dimensional (anti)-de Sitter gravity, where the original isometry algebra $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ was first extended to $\mathfrak{g}_i = \mathfrak{gl}_2 \oplus \mathfrak{gl}_2$. Apart from (A)dS$_3$ isometries, this algebra contains generators corresponding to two spin-one fields, whose dynamics is described by Chern-Simons action.

The higher-spin theories are particularly suited for the color-decoration, as discussed earlier in [161, 162, 163]. The higher-spin algebra in which the higher-spin fields take values is typically an associative algebra unless one deliberately truncates the theory to the so-called minimal spectrum, containing only spins of even integers. In fact, the color-decoration necessarily requires fields of odd integer spins in the spectrum (spin-one for the pure Chern-Simons (A)dS$_3$ gravity as studied in the previous chapter 6). As such, it is not possible to truncate the spectrum of the colored higher-spin theory to even spins only.

It was also noticed in [162] that including fermion generators necessarily requires non-trivial color algebra, therefore realistic models of higher-spin theory in four dimensions should be given by color-decorated theories, possibly with additional color symmetry breaking pattern that leaves only one massless graviton in the spectrum. This is not, however, our concern in this work.

7.2.2 Color-Decorated (A)dS$_3$ Higher-Spin Theory

In this work, we shall consider the simplest class of higher-spin theory and study their color-decoration. The theory we shall study is the colored version of the Chern-Simons formulation of the higher-spin (A)dS$_3$ theory whose gauge algebra is given by the infinite-dimensional algebra labelled by a continuous parameter λ:

$$\mathfrak{g}_i = hs(\lambda) \oplus hs(\lambda)\,. \tag{7.2}$$

To render the conceptual problem simpler, we shall often restrict ourselves to the truncated algebras,

$$\mathfrak{g}_i = \mathfrak{gl}_M \oplus \mathfrak{gl}_M \qquad (M = 2, 3, 4, \ldots) \tag{7.3}$$

or even to the simplest higher-spin algebra, the $M = 3$ case. The gauge algebra of spin-two, leading to $(A)dS_3$ Einstein gravity, corresponds to $M = 2$.

Let us further discuss aspects of the colored higher-spin (A)dS$_3$ gravity in the Chern-Simons formulation. The theory is based on the gauge field taking value in $\mathfrak{g}_i \otimes \mathfrak{g}_c$:

$$\mathcal{A} = A^{X,I}\, M_X \otimes \boldsymbol{T}_I\,, \tag{7.4}$$

where M_X are the generators of higher-spin algebra $\mathfrak{g}_i$ and the index X is the shorthand notation for the set of indices $A_1B_1,\ldots,A_rB_r$ of higher-spin generators. The color algebra $\mathfrak{g}_c$ is spanned by the generators $\boldsymbol{T}_I$. The gauge field strength is given by

$$\mathcal{F} = d\,\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = F^{X,I}\, M_X \otimes \boldsymbol{T}_I\,. \tag{7.5}$$

Up to this point, it is clear that all elements of the theory can be straightforwardly color-decorated by adjoining the Chan-Paton indices. Hence, if a theory can be defined solely in terms of $\mathcal{A}$ and $\mathcal{F}$ as in the Chern-Simons formulation — or together with more elements which can be equally well color-decorated — then the theory can be consistently generalized to a color-decorated version.

The action of the (A)dS$_3$ higher-spin theory is given in the Chern-Simons formulation by

$$S = \frac{\kappa}{4\pi} \int \mathrm{Tr}_{\mathfrak{g}} \left(\mathcal{A} \wedge d\,\mathcal{A} + \frac{2}{3}\, \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right), \tag{7.6}$$

where the constant κ is the Chern-Simons level. More concretely, we take the full algebra on which the theory will be based on as

$$\mathfrak{g} = (hs(\lambda) \oplus hs(\lambda)) \otimes \mathfrak{u}(N) \ \ominus\ \mathrm{id} \otimes \boldsymbol{I}\,. \tag{7.7}$$

Note that we subtracted the $\mathrm{id} \otimes \boldsymbol{I}$ — where id and $\boldsymbol{I}$ are the centers of $hs(\lambda) \oplus hs(\lambda)$ and $\mathfrak{u}(N)$, respectively. This generator corresponds to an Abelian Chern-Simons field which does not interact with other fields in the theory. Since gauge field $\mathcal{A}$ takes value in the subspace of the tensor product space, the trace Tr of (7.6) should be defined in the tensor product space and it is given by the product of two traces as $\mathrm{Tr}(\mathfrak{g}_i \otimes \mathfrak{g}_c) = \mathrm{Tr}(\mathfrak{g}_i)\ \mathrm{Tr}(\mathfrak{g}_c)$.

Higher-Spin Algebra

In the uncolored case, the Chern-Simons theory with the algebra $hs(\lambda) \oplus hs(\lambda)$ can be interpreted as a theory of massless fields with spins $s = (1), 2, 3, 4, \ldots$, where spin 1 may or may not be present, depending on whether $hs(\lambda)$ includes the identity or not. When the parameter λ takes an integer value, say M, then the $hs(M)$ develops an ideal. The quotient of $hs(M)$ by the ideal is the finite-dimensional algebra $\mathfrak{gl}_M$, whose generators can be organized as

$$\mathfrak{gl}_M = \mathrm{Span}\{\, J\,,\ J_a\,,\ J_{a_1 a_2}\,,\ \ldots\,,\ J_{a_1 \cdots a_{M-1}}\,\}\,, \tag{7.8}$$

whereas the other $\mathfrak{gl}_M$ is spanned by $\tilde{J}_{a_1 \cdots a_n}$ $(n = 0, \ldots, M-1)$. J and $\tilde{J}$ are the identities of two copies of $\mathfrak{gl}_M$. The basis is chosen in a way that makes manifest that the Chern-Simons action with $\mathfrak{gl}_M \oplus \mathfrak{gl}_M$ algebra describes a system of massless spins $1, 2, 3, \ldots, M$.

In order to simplify the use of multi indices, let us employ the following notation,

$$\begin{aligned} &A_{a(n)} \equiv A_{a \cdots a} \quad \leftrightarrow \quad A_{a_1 \cdots a_n}\,, \\ &A_{a(n)}\, B_{a(m)} \quad \leftrightarrow \quad A_{\{a_1 \cdots a_n}\, B_{a_{n+1} \cdots a_{n+m}\}}\,, \end{aligned} \tag{7.9}$$

where the index operation $\{-\}$ means the *traceless* symmetrization:

$$\begin{aligned} &A_{\{a_1 \cdots a_n}\, B_{a_{n+1} \cdots a_{n+m}\}} = A_{(a_1 \cdots a_n}\, B_{a_{n+1} \cdots a_{n+m})} - (\mathrm{trace})\,, \\ &\eta^{a_1 a_2}\, A_{\{a_1 \cdots a_n}\, B_{a_{n+1} \cdots a_{n+m}\}} = 0\,. \end{aligned} \tag{7.10}$$

The algebraic structure of $hs(\lambda)$ is given by the product,

$$J_{a(m)}\, J_{b(n)} = \eta_{a(m),b(n)}\, J + \frac{1}{l!}\, c_{a(m),b(n),c(l)}\, J^{c(l)}\,, \tag{7.11}$$

where $c_{a(m),b(n),c(l)}$ are the structure constants and $\eta_{a(m),b(n)}$ is defined by

$$\eta_{a(m)}{}^{,b(n)} = \delta_{mn}\left(\delta_a^b\right)^n. \tag{7.12}$$

For our analysis, it is not necessary to identify all explicit forms of $c_{a(m),b(n),c(l)}$. It is sufficient to know the following product

$$J_{a(n)}\, J_b = c_n\, \eta_{ba}\, J_{a(n-1)} + \epsilon_{ab}{}^c\, J_{ca(n-1)} + c_{n+1}\, J_{ba(n)}\,, \tag{7.13}$$

where c_n can be found e.g in [165] (see [166] and references therein for recent works):

$$c_n = \sqrt{\frac{n(\lambda^2-n^2)}{2n+1}}\,. \tag{7.14}$$

Under the Hermitian conjugation, we get

$$(J_{a(n)}, \tilde{J}_{a(n)})^\dagger = (-1)^n \begin{cases} (J_{a(n)}, \tilde{J}_{a(n)}) & [\sigma=+1] \\ (\tilde{J}_{a(n)}, J_{a(n)}) & [\sigma=-1] \end{cases}\,. \tag{7.15}$$

We also define the trace of the identity element as

$$\mathrm{Tr}(J) = 2\,\sqrt{\sigma} = -\,\mathrm{Tr}(\tilde{J})\,. \tag{7.16}$$

Note that the overall factor 2 of the above trace is a matter of convention, which is related to the quantization condition of the Chern Simons level κ . It is chosen such that the consistent κ are integers. We shall come back to this point in Section 7.4.2.

7.3 Color Symmetry Breaking and Rainbow Vacua in General Dimensions

In the previous chapter 6, we showed that the dynamics of color-decorated (A)dS_3 gravity is rich enough to trigger a spontaneous color symmetry breaking as the colored spin-two matter fields take nonzero vacuum expectation values proportional to the color-singlet component $g_{\mu\nu}$. Note that, in chapter 6, we identified this color-single component as the first fundamental form, viz. the metric. By analyzing the potential of these colored spin-two fields, we identified multiple vacua – named as *rainbow vacua* — having different values of cosmological constants.

The existence of panoramic rainbow vacua is not a feature unique to the (A)dS_3 gravity. This feature actually holds true for a generic class of color-decorated (higher-spin) gravity theories in any dimensions. In the following, we shall make this point clear, while keeping generality of our discussion.

We look for the (A)dS$_D$ solutions to the equations of the colored higher-spin gravity, such as Chern-Simons and Vasiliev theory. As an ansatz, we consider the configuration for which only spin-two components of $\mathcal{A}$ are non-trivial. All other components — the one-forms of the other spins and also the zero forms of the Vasiliev's equation — are set to zero. This ansatz is actually the only consistent one with the isometry of $(A)dS_D$. First of all, any odd spin fields cannot take a non-trival vacuum expectation value because there is no odd rank tensor compatible with the $(A)dS_D$ isometry. About even spin fields, one may consider

$$\langle \varphi_{\mu_1\cdots\mu_{2n}} \rangle = v\, g_{\mu_1\mu_2}\, \cdots\, g_{\mu_{2n-1}\mu_{2n}}\,, \tag{7.17}$$

where $\varphi_{\mu_1\cdots\mu_{2n}}$ are higher-spin fields in the Fronsdal description. However, since fields of spins greater than four are subject to double-traceless constraint, only scalar and spin-two fields can take a vacuum expectation value compatible with the $(A)dS_D$ isometry. Therefore, we take an ansatz for the one-form gauge field as

$$\mathcal{A} = (\omega^{ab}\, \boldsymbol{I} + \boldsymbol{\Omega}^{ab})\, M_{ab} + \frac{1}{\ell}\, (e^a\, \boldsymbol{I} + \boldsymbol{E}^a)\, P_a\,, \tag{7.18}$$

where $\boldsymbol{\Omega}^{ab}$ and $\boldsymbol{E}^a$ take values in the Chan-Paton algebra, $\mathfrak{su}(N)$.

The idea for finding classical solutions is to require that the configuration does not lead to back-reaction onto the other components of the one-form field, viz. either spin-one or higher-spin (and the zero-form field). It is straightforward to see that these requirements are met if we impose the conditions

$$\boldsymbol{\Omega}^{ab} \wedge \boldsymbol{\Omega}^{cd} \left\{M_{ab}, M_{cd}\right\} = 0\,, \quad \left\{\boldsymbol{\Omega}^{ab} \overset{\wedge}{,} \boldsymbol{E}^c\right\}\left\{M_{ab}, P_c\right\} = 0\,, \quad \boldsymbol{E}^a \wedge \boldsymbol{E}^b \left\{P_a, P_b\right\} = 0\,. \tag{7.19}$$

Were if these conditions not met, anti-commutators $\{M_{AB}, M_{CD}\}$ would contribute[1] and give rise to the generators of fields with spin different from two.

We take the ansatz, corresponding to the tensor product structure, for (7.19) as

$$\boldsymbol{\Omega}^{ab} = 0 \qquad \text{and} \qquad \boldsymbol{E}^a = e^a\, \boldsymbol{X}\,, \tag{7.20}$$

Here, $\boldsymbol{X}$ is a particular element of the $\mathfrak{su}(N)$ to be determined. Note that we could add a factor in the latter equation but it can be simply absorbed into the matrix $\boldsymbol{X}$. So, our final ansatz for the gauge field takes the form

$$\mathcal{A} = \omega^{ab}\, M_{ab}\, \boldsymbol{I} + \frac{1}{\ell}\, e^a\, P_a\, (\boldsymbol{I} + \boldsymbol{X})\,. \tag{7.21}$$

With the above ansatz, the only non-trivial equation is the zero curvature equation, $\mathcal{F} = 0$, which reads

$$\left(d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}\right) \boldsymbol{I} + \frac{\sigma}{\ell^2}\, e^a \wedge e^b\, (\boldsymbol{I} + \boldsymbol{X})^2 = 0\,, \qquad (de^a + \omega^a{}_c \wedge e^c)\, (\boldsymbol{I} + \boldsymbol{X}) = 0\,, \tag{7.22}$$

[1]We denote by M_{AB} generators of $(A)dS_D$ algebra while M_{ab} and P_a are Lorenz and translation generators, respectively.

where σ is a $\pm$ sign, positive for AdS$_D$ and negative for dS$_D$.

The first equation in (7.22) clearly shows that the Chan-Paton gauge symmetry acts as a new source to the spacetime curvature. To see this more explicitly, let us decompose the above $N \times N$ matrix equations into the singlet $\mathfrak{u}(1)$ part and the $\mathfrak{su}(N)$ part, which in string theory would originate from the *closed string* part and the *open string* part, respectively. The *closed string* part of the equation of motion reads

$$d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} - \frac{2}{(D-1)(D-2)}\,\Lambda\, e^a \wedge e^b = 0\,, \qquad de^a + \omega^a{}_c \wedge e^c = 0\,, \tag{7.23}$$

where the cosmological constant Λ (measured in units of D-dimensional Newton's constant) is given by

$$\Lambda = -\frac{(D-1)(D-2)\,\sigma}{2\,N\,\ell^2}\,\mathrm{Tr}(\boldsymbol{I}+\boldsymbol{X})^2\,, \tag{7.24}$$

On the other hand, the *open string* part of the equation of motion is given by

$$e^a \wedge e^b \left(2\,\boldsymbol{X} + \boldsymbol{X}^2 - \frac{\mathrm{Tr}(2\,\boldsymbol{X}+\boldsymbol{X}^2)}{N}\,\boldsymbol{I}\right) = 0\,. \tag{7.25}$$

We note that, for non-degenerate e^a's, (7.25) is identical to the condition for a critical point of the scalar potential, projected to $\mathfrak{su}(N)$ part:

$$V(\boldsymbol{X}) = -\frac{(D-1)(D-2)\,\sigma}{N\,\ell^2}\,\mathrm{Tr}\left(\boldsymbol{I} + 3\,\boldsymbol{X}^2 + \boldsymbol{X}^3\right). \tag{7.26}$$

The cosmological constant in (7.24) is given by $\Lambda = V(\boldsymbol{X})/2$ evaluated at extremum points. Putting $D = 3$, these are precisely what we found in the analysis of the color-decorated (A)dS$_3$ gravity in chapter 6. This suggests that the stairwell potential of the color-decorated (A)dS$_3$ gravity would persist to exist in the generic colored (higher-spin) gravity theory in any dimensions.

The complete set of solutions to (7.25) can be found precisely the same way as in the color-decorated (A)dS$_3$ gravity analyzed in chapter 6. We simply state the result: the solution $\boldsymbol{X}$ for (7.25) is given by

$$\boldsymbol{X}_k = \frac{N}{\mathrm{Tr}(\boldsymbol{Z}_k)}\,\boldsymbol{Z}_k - \boldsymbol{I}\,, \tag{7.27}$$

with

$$\boldsymbol{Z}_k = \begin{bmatrix} \boldsymbol{I}_{(N-k)\times(N-k)} & 0 \\ 0 & -\boldsymbol{I}_{k\times k} \end{bmatrix} \tag{7.28}$$

modulo a $SU(N)$ rotation. Extrema of the potential are labelled by $k = 0, 1, \ldots, [\frac{N-1}{2}]$. Moreover, $\boldsymbol{X}_k$ at the k-th vacuum — which shifts the gauge field $\mathcal{A}$ from the $k = 0$ (A)dS vacuum to (C.2) — spontaneously breaks the $\mathfrak{su}(N)$ color symmetry down to $\mathfrak{su}(N-k) \oplus \mathfrak{su}(k) \oplus \mathfrak{u}(1)$. We thus conclude from (C.2) that the vacuum expectation values of colored spin-two fields act as the order parameter of the color symmetry breaking.

7.4 Metric Formulation

Coming back to our model of colored higher-spin theory in three dimensions (7.6 , 7.7), let us notice that the appearance of the non-trivial potential with multiple extrema and the field contents at such vacua are better treatable in the metric form. The exact expression of the staircase potential can be computed along the same lines as in chapter 6, so we shall not aim to repeat the derivation. Rather, we shall focus on the identification of perturbative spectrum around each extremum.

7.4.1 Decomposition of Associative Algebra

Once the frame-like formulation of higher-spin gravity is given, rewriting it in the metric form is in principle possible. However, it is technically cumbersome to get exact expressions in the metric-like variables. Here, we shall reformulate the Chern-Simons action to metric form solving the torsionless condition for the genuine gravition, while leaving the other field contents in the first-order formulation. For this task, it is convenient to decompose the algebra $\mathfrak{g}$ (7.7) into two pieces $\mathfrak{b}$ and $\mathfrak{c}$:

$$\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{c}\,, \qquad \mathrm{Tr}(\mathfrak{b}\,\mathfrak{c}) = 0\,, \tag{7.29}$$

in a proper way. The rule of the decomposition is that $\mathfrak{b}$ forms a subalgebra under which $\mathfrak{c}$ carries an adjoint representation, that is,

$$[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{b}\,, \qquad [\mathfrak{b}, \mathfrak{c}] \subset \mathfrak{c}\,. \tag{7.30}$$

Correspondingly to this decomposition of the algebra, we also split the one-form gauge field into two parts,

$$\mathcal{A} = \mathcal{B} + \mathcal{C}\,, \tag{7.31}$$

where $\mathcal{B}$ and $\mathcal{C}$ take values in $\mathfrak{b}$ and $\mathfrak{c}$ respectively. In terms of $\mathcal{B}$ and $\mathcal{C}$, the Chern-Simons action (7.6) reduces to

$$S = \frac{\kappa}{4\pi} \int \mathrm{Tr}\left(\mathcal{B}\wedge d\,\mathcal{B} + \frac{2}{3}\,\mathcal{B}\wedge\mathcal{B}\wedge\mathcal{B} + \mathcal{C}\wedge D_{\mathcal{B}}\,\mathcal{C} + \frac{2}{3}\,\mathcal{C}\wedge\mathcal{C}\wedge\mathcal{C}\right), \tag{7.32}$$

with $D_{\mathcal{B}}\,\mathcal{C} = d\,\mathcal{C} + \mathcal{B}\wedge\mathcal{C} + \mathcal{C}\wedge\mathcal{B}$. Properly selecting $\mathfrak{b}$ and $\mathfrak{c}$ from the full algebra $\mathfrak{g}$ (7.7) , we can conveniently handle the manifest covariance with respect to diffeomorphism and non-Abelian gauge transformation.

The choice of the decomposition (7.29) reflects the symmetry of the background around which we are expanding the theory. Instead of analysing the spectrum separately for the singlet vacuum and the colored vacua, we directly consider the latter case since it also covers the former for a special value $k = 0$. In order to begin with the proper decomposition (7.29), we first split the $(A)dS_D$ isometry algebra (the $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ subalgebra of $hs(\lambda) \oplus hs(\lambda)$) into the Lorentz part $\mathcal{M}$ and the translation part $\mathcal{P}$ as

$$\mathfrak{iso} \simeq \mathcal{M} \oplus \mathcal{P}\ . \tag{7.33}$$

For the color algebra, we take advantage of the fact that, at k-th extremum, the $\mathfrak{su}(N)$ symmetry is broken down to $\mathfrak{su}(k) \oplus \mathfrak{su}(N-k) \oplus \mathfrak{u}(1)$. In accordance with this symmetry breaking, we decompose the space of the $\mathfrak{su}(N)$ as

$$\mathfrak{su}(N) \simeq \mathfrak{su}(N-k) \oplus \mathfrak{su}(k) \oplus \mathfrak{u}(1) \oplus \mathfrak{bs}\,, \tag{7.34}$$

where $\mathfrak{bs}$ is the $2k(N-k)$ dimensional vector space corresponding to the *broken gauge symmetry* generators. Then, the background matrix $\boldsymbol{Z}_k$ (7.28) enjoys either commutation or anti-commutation properties with each of these generators:

$$\big[\,\boldsymbol{Z}_k\,,\,\mathfrak{su}(N-k) \oplus \mathfrak{su}(k) \oplus \mathfrak{u}(1)\,\big] = 0\,, \qquad \big\{\,\boldsymbol{Z}_k\,,\,\mathfrak{bs}\,\big\} = 0\,. \tag{7.35}$$

Taking advantage of these two decompositions (7.33) and (7.34) of $\mathfrak{iso}$ and $\mathfrak{g}_c$, we now decompose the full algebra $\mathfrak{g}$ (7.7) according to (7.29) : the gravity plus gauge sector $\mathfrak{b}$ and the matter sector $\mathfrak{c}$.

The gravity plus gauge sector has two parts $\mathfrak{b} = \mathfrak{b}_{\rm GR} \oplus \mathfrak{b}_{\rm Gauge}$:

$$\mathfrak{b}_{\rm GR} = \big(\,\mathcal{M} \otimes \boldsymbol{I}\big) \oplus \big(\,\mathcal{P} \otimes \boldsymbol{Z}_k\big)\,, \qquad \mathfrak{b}_{\rm Gauge} = {\rm id} \otimes \Big(\mathfrak{su}(N-k) \oplus \mathfrak{su}(k) \oplus \mathfrak{u}(1)\Big)\,. \tag{7.36}$$

In the gravity sector, the isometry algebra is deformed by $\boldsymbol{Z}_k$ as

$$\boldsymbol{M}_{ab} = M_{ab}\,\boldsymbol{I}\,, \qquad \boldsymbol{P}_a = P_a\,\boldsymbol{Z}_k\,, \tag{7.37}$$

but still satisfies the same commutation relations as the undeformed one. We now specify the fields corresponding to the $\mathfrak{b}$ sector as

$$\mathcal{B}_{\rm GR} = \frac{1}{2}\left(\omega^{ab} + \Omega^{ab}\right)\boldsymbol{M}_{ab} + \frac{1}{\ell_k}\,e^a\,\boldsymbol{P}_a\,, \qquad \mathcal{B}_{\rm Gauge} = \boldsymbol{A}_+ + \boldsymbol{A}_- + \tilde{\boldsymbol{A}}_+ + \tilde{\boldsymbol{A}}_- + (A+\tilde{A})\,\boldsymbol{Y}_k\,, \tag{7.38}$$

where the traceless matrix

$$\boldsymbol{Y}_k = \frac{k\,\boldsymbol{I}_+ - (N-k)\,\boldsymbol{I}_-}{N}\,, \tag{7.39}$$

corresponds to $\mathfrak{u}(1)$ symmetry. The tensor Ω^{ab} has been introduced so that the spin connection ω^{ab} is determined only by e^a when solving the torsionless condition. Consequently, the Ω^{ab} contains the contributions from other matter and higher-spin fields. The gauge fields $\boldsymbol{A}_\pm$ and $\tilde{\boldsymbol{A}}_\pm$ take value in $\mathfrak{su}(N-k)$ for the subscript $+$ and $\mathfrak{su}(k)$ for the subscript $-$. Finally, the deformed radius ℓ_k is related to the undeformed one as

$$\ell_k = \frac{N-2k}{N}\,\ell\,. \tag{7.40}$$

The matter sector $\mathfrak{c}$ has four parts:

$$\mathfrak{c} = \mathfrak{c}_{\rm CM} \oplus \mathfrak{c}_{\rm BS} \oplus \mathfrak{c}_{\rm HS}\,. \tag{7.41}$$

For the introduction of each element, we need to define first the deformed higher-spin algebra generators analogously to the spin-two sector (7.37) as

$$\boldsymbol{J}_{a(n)} = J_{a(n)}\,\boldsymbol{I}_+ + \tilde{J}_{a(n)}\,\boldsymbol{I}_-\,, \qquad \tilde{\boldsymbol{J}}_{a(n)} = J_{a(n)}\,\boldsymbol{I}_- + \tilde{J}_{a(n)}\,\boldsymbol{I}_+\,, \tag{7.42}$$

where $\boldsymbol{I}_\pm$ are the identities associated with $\mathfrak{u}(N-k)$ and $\mathfrak{u}(k)$, respectively:

$$\boldsymbol{I}_\pm = \frac{1}{2}\left(\boldsymbol{I} \pm \boldsymbol{Z}_k\right). \tag{7.43}$$

Analogous to the deformation of the spin-two part (7.37), the deformed higher-spin generators (7.42) still form $hs(\lambda) \oplus hs(\lambda)$ algebra. In terms of these generators, we define the one form fields corresponding to the colored matter $\mathcal{C}_{\rm CM}$ and the color-neutral matter $\mathcal{C}_{\rm NM}$ as

$$\mathcal{C}_{\rm CM} = \frac{1}{\ell_k}\sum_{n\geq 1}\frac{1}{n!}\Big[\left(\boldsymbol{\varphi}_+^{a(n)} + \boldsymbol{\varphi}_-^{a(n)}\right)\boldsymbol{J}_{a(n)} + \left(\tilde{\boldsymbol{\varphi}}_+^{a(n)} + \tilde{\boldsymbol{\varphi}}_-^{a(n)}\right)\tilde{\boldsymbol{J}}_{a(n)}\Big] \tag{7.44}$$

$$\mathcal{C}_{\rm NM} = \frac{1}{\ell_k}\sum_{n\geq 1}\frac{1}{n!}\left(\psi^{a(n)}\,\boldsymbol{J}_{a(n)} + \tilde{\psi}^{a(n)}\,\tilde{\boldsymbol{J}}_{a(n)}\right)\boldsymbol{Y}_k\,\boldsymbol{Z}_k\,, \tag{7.45}$$

where the fields $\boldsymbol{\varphi}_{+/-}^{a(n)}$ and $\tilde{\boldsymbol{\varphi}}_{+/-}^{a(n)}$ take value in $\mathfrak{su}(N-k)/\mathfrak{su}(k)$ and $\psi^{a(n)}$ and $\tilde{\psi}^{a(n)}$ in $\mathfrak{u}(1)$. The last matrix factor $\boldsymbol{Y}_k$ has been introduced so that ${\rm Tr}(\mathfrak{b}_{\rm GR}\,\mathfrak{c}_{\rm CM}) = 0$. Equivalently,

$$\mathrm{Tr}\left(\boldsymbol{J}\,\boldsymbol{Y}_k\boldsymbol{Z}_k\right) = 0 = \mathrm{Tr}\left(\tilde{\boldsymbol{J}}\,\boldsymbol{Y}_k\boldsymbol{Z}_k\right). \tag{7.46}$$

The one-form corresponding to the $\mathfrak{c}_{\rm BS}$ sector is given by

$$\mathcal{C}_{\rm BS} = \frac{1}{\ell_k}\sum_{n\geq 0}\frac{1}{n!}\left(\boldsymbol{\phi}^{a(n)}\,\boldsymbol{J}_{a(n)} + \tilde{\boldsymbol{\phi}}^{a(n)}\,\tilde{\boldsymbol{J}}_{a(n)}\right), \tag{7.47}$$

where $\boldsymbol{\phi}^{a(n)}$ and $\tilde{\boldsymbol{\phi}}^{a(n)}$ take values in $\mathfrak{bs}$ (7.34). Notice that the summation starts from $n=0$ so it involves not only higher-spin generators but also the identity piece corresponding to spin one. Lastly, we have the singlet higher-spin sector:

$$\mathcal{C}_{\rm HS} = \frac{1}{\ell_k}\sum_{n\geq 2}\frac{1}{n!}\left(\varphi^{a(n)}\,\boldsymbol{J}_{a(n)} + \tilde{\varphi}^{a(n)}\,\tilde{\boldsymbol{J}}_{a(n)}\right), \tag{7.48}$$

which, in principle, could be treated together with the gravity plus gauge sector. But, since we do not know any natural form of higher-spin covariant interactions in metric-like form, they are treated here as extra matter fields.

7.4.2 Action in Metric Formulation

Putting all the above results into the Chern-Simons action, we get

$$S = S_{\rm CS} + S_{\rm HSG} + S_{\rm Matter}\,, \tag{7.49}$$

where the first term $S_{\rm CS}$ is the two copies of the Chern-Simons action with the levels κ and $-\kappa$, respectively, and with $\mathfrak{su}(N-k)\oplus\mathfrak{su}(k)\oplus\mathfrak{u}(1)$ gauge algebra. In the uncolored Chern-Simons (higher-spin) gravity, it is not clear whether the level has to be quantized because the gauge group is non-compact. In the color-decorated cases, the level κ ought to take a discrete value for the consistency of large color gauge transformations.

The second term S_{HSG} is the action in metric form — only for the gravity part — for higher-spin gravity,

$$S_{\text{HSG}} = \frac{1}{16\pi\, G} \int \left[d^3x \sqrt{|g|} \left(R + \frac{2\,\sigma}{\ell_k^2} \right) + \frac{2\,\sqrt{\sigma}}{N-2k} L(\varphi, \tilde{\varphi}, \ell_k) \right], \tag{7.50}$$

where L is for single higher-spin fields, given by

$$\begin{aligned} L[\varphi, \tilde{\varphi}, \ell] &= L_+[\varphi, \ell] - L_-[\tilde{\varphi}, \ell]\,, \\ L_\pm[\varphi, \ell] &= \sum_n \frac{1}{n!} \left[\frac{1}{\ell}\, \varphi^{a(n)} \wedge \left(D\, \varphi_{a(n)} \pm \frac{1}{\sqrt{\sigma}\,\ell}\, c_{a(n)bc(n)}\, e^b \wedge \varphi^{c(n)} \right) \right. \\ &\qquad \left. + \frac{2}{3\,\ell^2}\, c_{a(m)b(n)c(l)}\, \varphi^{a(m)} \wedge \varphi^{b(n)} \wedge \varphi^{c(l)} \right]. \end{aligned} \tag{7.51}$$

The derivation D is covariant both with respect to Lorentz transformations and non-Abelian gauge transformations (in case it acts on color charged fields). At quadratic order, components with different n are independent and describe massless spin $(n+1)$. The gravitational constant G is fixed in terms of the Chern-Simons level by

$$\kappa = \frac{\ell}{4\,N\,G}\,. \tag{7.52}$$

Finally, the matter action takes the form,

$$\begin{aligned} S_{\text{Matter}} \;\; = \;\; & \frac{1}{16\,\pi\, G} \int \frac{1}{N-2k}\, \text{Tr} \left(L[\boldsymbol{\varphi}_+, \tilde{\boldsymbol{\varphi}}_+, \ell_k] - L[\boldsymbol{\varphi}_-, \tilde{\boldsymbol{\varphi}}_-, \ell_k] \right) \\ & - \frac{k(N-k)}{N^2}\, L[\psi, \tilde{\psi}, \ell_k] + \frac{1}{N-2k}\, L_{\text{BS}}[\boldsymbol{\phi}, \tilde{\boldsymbol{\phi}}, \ell_k] + L_{\text{int}}\,, \end{aligned} \tag{7.53}$$

where the new three form L_{BS} has fully correlated components as opposed to L (7.51) :

$$\begin{aligned} & L_{\text{BS}}[\boldsymbol{\phi}, \tilde{\boldsymbol{\phi}}, \ell] \\ = \; & \frac{4\,\sqrt{\sigma}}{\ell} \sum_n \frac{1}{n!}\, \text{Tr} \left[\tilde{\boldsymbol{\phi}}_{a(n)} \wedge \left(D\, \boldsymbol{\phi}^{a(n)} + \frac{c_n}{\sqrt{\sigma}\,\ell}\, e_a \wedge \boldsymbol{\phi}^{a(n-1)} + \frac{c_{n+1}}{\sqrt{\sigma}\,\ell}\, e_a \wedge \boldsymbol{\phi}^{a(n+1)} \right) \boldsymbol{Z}_k \right]. \end{aligned} \tag{7.54}$$

The term L_{int} in the second line of (7.53) concerns exclusively the interaction terms and it contains the cross couplings from the Chern-Simons cubic interaction and the quadratic terms in Ω^{ab} (which itself is quadratic in fields, hence these terms represent quartic couplings). Note that, in k-th vacua, the colored matter interacts with other fields with strength proportional to powers of $\sqrt{N-2k}$, while the neutral matter $\psi, \tilde{\psi}$ interact with strength proportional to powers of $1/\sqrt{k(N-k)}$.

In describing the action of colored higher-spin gravity above, we omitted the explicit expression for the structure constants $c_{a(m),b(n),c(l)}$, L_{int} and Ω^{ab} . Identifying their form is straightforward in principle but not necessary for our purpose: we are more interested in the pattern of mass spectra around rainbow vacua and the qualitative structure of interactions. In the following, we elaborate more on these aspects.

- The spectrum consists of spin-one Chern-Simons gauge fields, spin-two gravity, and higher-spin fields, whose dynamics are governed by S_{CS} and S_{HSG} . In particular, the latter S_{HSG} coincides with the action of the uncolored Chern-Simons higher-spin theory in three-dimensions. The colored higher-spin fields $\boldsymbol{\varphi}_{\pm}^{a(n)}$ and singlet higher-spin fields $\psi^{a(n)}$ (and their tilde counter parts) share the same structure of the quadratic Lagrangian L (7.51). As such, they describe massless spin-$(n+1)$ fields, described in first-order formalism.

- From the signs of the one-derivative term and masslike term in the first-order Lagrangian (7.51)[2], it follows that the colored matter $\boldsymbol{\varphi}_{+}^{a(n)}$, $\tilde{\boldsymbol{\varphi}}_{+}^{a(n)}$ are unitary, but $\boldsymbol{\varphi}_{-}^{a(n)}$, $\tilde{\boldsymbol{\varphi}}_{-}^{a(n)}$ and singlet matter $\psi^{a(n)}$, $\tilde{\psi}^{a(n)}$ in (7.53) behave as non-unitary ghosts. This reflects that all of the $0 < k < N/2$ rainbow vacua are actually saddle points and so they have unstable, runaway directions in the field configuration space.

- The remaining bi-fundamental matter fields $\boldsymbol{\phi}^{a(n)}$ and $\tilde{\boldsymbol{\phi}}^{a(n)}$ corresponds to the broken part of the color symmetries. They are all massive. More precisely, as we shall demonstrate in detail in the next section, these fields are the partially-massless fields [164] of maximal-depth.

- The color symmetry breaking mechanism through colored higher-spin gravity should be contrasted against the more familiar standard Higgs mechanism through colored scalar field dynamics. The role of the Higgs field and its symmetry-breaking potential is now played by the $\mathfrak{su}(N)$-valued spin-two matter field and its symmetry-breaking potential $V(\boldsymbol{X})$ (7.26). Much as colored spin-0 field condensate respects the Poincaré invariant vacuum in the former, colored spin-2 field condensate respects the generally covariant vacuum in the latter. When this field takes a nonzero vacuum expectation value corresponding to one of the saddle points (labeled by k as in (7.28)), its components split into two parts, each of which retains the residual $\mathfrak{su}(N-k) \oplus \mathfrak{su}(k) \oplus \mathfrak{u}(1)$ symmetry. The symmetry preserving part remains massless, while the symmetry breaking part — analogous to the Goldstone bosons — combine with the same part of all other higher-spins. Hence, the massless spin-2 Goldstone field combines with masless fields of spin $1, 3, 4, \ldots, M$ in case of the $\mathfrak{gl}_M \oplus \mathfrak{gl}_M$ higher-spin gravity. The resulting spectrum is the spin-M partially-massless fields of maximal-depth. We expect the same pattern continues to hold for $hs(\lambda) \oplus hs(\lambda)$ higher-spin gravity: the massless colored spin-2 Goldstone field combines to massless colored fields of all other spins and form a partially massless Regge trajectory.

- The interaction among the above fields is set not only by the gravitational constant but also by k, N. The structure of interaction in the color non-singlet vacua is analogous to the colored (A)dS$_3$ gravity we studied in chapter 6. All the fields are coupled to gravity in the diffeomorphism invariant manner. All the colored ihgher-spin fields — adjoints $\boldsymbol{\varphi}_{\pm}$ and bi-fundamentals $\boldsymbol{\phi}$ — have covariant gauge couplings to the Chern-Simons gauge fields. There are also nonlinear self-couplings among the matter fields with coupling constants controlled by N and k (7.53). These nonlinear interactions become strong for small k (small symmetry breaking) and as weak as gravity for large $k \sim [N/2]$ (large symmetry breaking).

[2]In $(2+1)$ dimensions, the sign of one-derivative term in first-order Lagrangian signifies 'chirality' , while the sign of zero-derivative 'mass' term determines (non)unitarity. Details of this peculiar feature is explained in chapter 6, Section 5.2 and Appendix A.

7.5 Mass Spectrum of the Broken Color Symmetries

We already noted that the color symmetry breaking triggers the mass generation as well. Moreover, it suggested a mechanism for emergent Regge trajectory out of massless higher-spin fields. This is an important aspect by itself, so we analyze below the spectrum of these "massive" components.

7.5.1 General Structure

We now analyze the action for the bi-fundamental higher-spin fields $\boldsymbol{\phi}^{a(n)}$ and $\tilde{\boldsymbol{\phi}}^{a(n)}$ corresponding to the broken part of color symmetries, paying special attention to their mass spectra. For definiteness, we concentrate on the AdS space. To get analogous result for dS space, we simply replace the AdS radius to $\sqrt{-1}$ times the dS radius.

It turns out that all these fields with different n are correlated. Furthermore, even the left-movers and the right-movers have cross-couplings in the quadratic action. However, we can always diagonalize the acton. Taking, for clarity of the analysis, the $\mathfrak{gl}_M$ higher-spin algebra, we can reduce the action to a collection of $S_{\rm BS}$ given by

$$
\begin{aligned}
&S_{\rm BS}[\phi, \phi^a, \ldots, \phi^{a(M-1)}] = \\
&\quad = \sum_{n=0}^{M-1} (-1)^n \int \phi_{a(n)} \wedge \left[D\,\phi^{a(n)} + \frac{1}{\ell} \left(c_n\, e^a \wedge \phi^{a(n-1)} + c_{n+1}\, e_a \wedge \phi^{a(n+1)} \right) \right] .
\end{aligned} \tag{7.55}
$$

Notice that the one-form fields contributing to the action are truncated to the first M fields. The above action also admits gauge symmetries with parameters $(\varepsilon, \varepsilon^a, \ldots, \varepsilon^{a(M-1)})$ as

$$
\delta\,\phi^{a(n)} = D\,\varepsilon^{a(n)} + \frac{1}{\ell}\left(c_n\, e^a\, \varepsilon^{a(n-1)} + c_{n+1}\, e_a\, \varepsilon^{a(n+1)} \right). \tag{7.56}
$$

For the analysis of equations of motion, we consider the decomposition of $\phi^{a(n)}$ into

$$
\begin{aligned}
\bar{h}_{\mu(n+1)} &= \left(e_\mu{}^a\right)^n \phi_{\mu\, a(n)} , \\
h'_{\mu(n-1)} &= \left(e_\mu{}^a\right)^{n-1} e^{\mu b}\, \phi_{\mu\, a(n-1) b} , \\
f_{\mu(n),\nu} &= \left(e_\mu{}^a\right)^n \phi_{\nu\, a(n)} + e_\nu{}^a \left(e_\mu{}^a\right)^{n-1} \phi_{\mu\, a(n)} ,
\end{aligned} \tag{7.57}
$$

where $\bar{h}_{\mu(n+1)}$ and $h'_{\mu(n-1)}$ are totally symmetric traceless fields and $f_{\mu(n),\nu}$'s are the traceless fields of the Young-symmetry type $\{n,1\}$. Note that we are using the same repeated index convention as (7.9) and (7.10).

The procedure of the analysis can be summarized in the following steps:

- We first gauge-fix $\bar{h}_{\mu(n)}$ from $n=1$ to $M-1$ using the gauge transformations (7.56) with the parameters $\varepsilon_{\mu(n)}$ from $n=1$ to $M-1$.
- Using the equations of motions, all the hook fields $f_{\mu(n),\nu}$ can be algebraically determined in terms of the rest. At this stage, the residual field contents are

$$
\bar{h}_{\mu(M)}\,, \qquad h'_{\mu(n)} \quad [n = 0, \ldots, (M-2)]\,, \tag{7.58}
$$

and these fields combine to form two traceful fields of spin M and $M-3$, respectively. This is the field content of massive higher-spin fields along the lines taken by Singh and Hagen [9], except that, in our case, we also have a gauge symmetry with the scalar parameter ε. This already suggests that the spectrum described by this system corresponds to the maximal-depth partially-massless spin-M field.

- Other equations can be used to algebraically determine $h'_{\mu(n)}$ from $n=1$ to $(M-2)$. Hence, after this step, we end up only with $\bar{h}_{\mu(M)}$ and $h' \equiv h'_{\mu(0)}$, modulo the gauge equivalence given by the scalar parameter ε. In the $M=2$ case, $\bar{h}_{\mu\nu}$ and h' can combine to a single traceful field $h_{\mu\nu}$.

The final equation is of first-order type and involves the fields $\bar{h}_{\mu(M)}$ and h'. These fields have gauge symmetries involving M derivatives for $\bar{h}_{\mu(M)}$ and of second-order for h'. To analyze further, instead of proceeding with the generic value of M, we shall consider the $M=3$ example in detail. The analysis for generic values of M is a straightforward generalization.

7.5.2 Example: $\mathfrak{gl}_3 \oplus \mathfrak{gl}_3$

For more concrete understanding, let us explicitly analyze the $M=3$ case. From (7.57), we get seven fields

$$\bar{h}_{\mu\nu\rho}\,, \quad \bar{h}_{\mu\nu}\,, \quad \bar{h}_{\mu}\,, \quad f_{\mu\nu,\rho}\,, \quad f_{\mu\nu}\,, \quad h'_{\mu}\,, \quad h'\,. \tag{7.59}$$

They admit the equations of motion,

$$\nabla_{[\mu}\,\bar{h}_{\nu]}{}^{\rho\sigma} + \frac{4}{3}\,\nabla_{[\mu}\,f^{\rho\sigma}{}_{,\nu]} - \frac{3}{5}\,\delta^{\{\rho}_{[\mu}\,\nabla_{\nu]}\,h'^{\sigma\}} + \frac{2\sqrt{2}}{\ell}\,\delta^{\{\rho}_{[\mu}\,\bar{h}_{\nu]}{}^{\sigma\}} = 0\,, \tag{7.60}$$

$$\nabla_{[\mu}\bar{h}_{\nu]\rho} + \nabla_{[\mu}f_{\nu]\rho} + g_{\rho[\mu}\left(\frac{1}{\ell}\,\bar{h}_{\nu]} - \frac{1}{3}\,\partial_{\nu]}h'\right) = \frac{2\sqrt{2}}{3\,\ell}\left(f_{\rho[\nu,\mu]} - \frac{3}{4}\,g_{\rho[\nu}\,h'_{\mu]}\right), \tag{7.61}$$

$$\partial_{[\mu}\bar{h}_{\nu]} = \frac{8}{3\,\ell}\,f_{\mu\nu}\,, \tag{7.62}$$

where (7.61) and (7.62) simply imply that $f_{\mu\nu,r}$, $f_{\mu\nu}$ and h'_{μ} are determined by the rest. The remaining fields have the gauge symmetries,

$$\begin{aligned} \delta\bar{h}_{\mu\nu\rho} &= \nabla_{\{\mu}\,\varepsilon_{\nu\rho\}}\,, & \delta\bar{h}_{\mu\nu} &= \nabla_{\{\mu}\,\varepsilon_{\nu\}} + \frac{1}{\sqrt{2}\,\ell}\,\varepsilon_{\mu\nu}\,, \\ \delta\bar{h}_{\mu} &= \partial_{\mu}\,\varepsilon + \frac{8}{3\,\ell}\,\varepsilon_{\mu}\,, & \delta h' &= \nabla^{\rho}\varepsilon_{\rho} + \frac{3}{\ell}\,\varepsilon\,. \end{aligned} \tag{7.63}$$

One can first gauge fix $\bar{h}_{\mu\nu}$ and $\bar{h}_{\mu}$ using the gauge transformations with the parameters $\varepsilon_{\mu\nu}$ and ε_{ν}. This gauge fixing will relate the latter gauge parameters to the scalar one ε as

$$\varepsilon_{\mu\nu} = \frac{3\,\ell^2}{4\sqrt{2}}\,\nabla_{\{\mu}\partial_{\nu\}}\,\varepsilon\,, \qquad \varepsilon_{\mu} = -\frac{3\,\ell}{8}\,\partial_{\mu}\,\varepsilon\,. \tag{7.64}$$

Finally, the remaining equations of motions and gauge transformations are given by

$$\nabla_{[\mu}\,\bar{h}_{\nu]\rho\sigma} + \frac{\sqrt{2}\,\ell}{5}\,\nabla_{[\mu}\,g_{\nu]\{\rho}\,\nabla_{\sigma\}}\,h' = 0\,, \tag{7.65}$$

and

$$\delta\bar{h}_{\mu\nu\rho} = \ell^2\,\nabla_{\{\mu}\nabla_{\nu}\nabla_{\rho\}}\,\varepsilon\,, \qquad \delta h' = -\frac{\ell}{\sqrt{2}}\left(\nabla^2 - \frac{8}{\ell^2}\right)\varepsilon\,. \tag{7.66}$$

These gauge transformations precisely coincide with those of the maximal-depth partially-massless field, which has been studied e.g. in [164] and [168]. Hence, this $M = 3$ example demonstrates that the spectrum corresponding to the broken part of color symmetry is indeed the maximal-depth partially-massless fields of the highest spin in the theory.

7.6 Partially-Massless Fields in Three Dimensions

A novel aspect of the color-decorated (higher-spin) (A)dS$_3$ gravity is that the fields in the broken symmetry sector, which acquired masses via Higgs mechanism through the color symmetry breaking, are all partially-massless. In this section, we discuss salient features of these states in (A)dS$_3$ space.

Partially-massless fields carry irreducible representations of the isometry algebra of non-vanishing constant curvature background [169, 170, 171][3]. In dS space, these states are unitary. In AdS space, even though their energy is bounded from below, these states are nonunitary because of negative norm states involved. For a given spin s, there are s different partially-massless fields labelled by depth $t = 0, 1, \ldots, s-1$, where $t = 0$ case corresponds to the massless field. In the flat limit, depth t partially-massless field is reduced to a set of massless fields with helicities $s, s-1, \ldots, s-t$. This pattern manifests the number of degrees of freedom (DoF) they have [164]: the number interpolates between those of massless and massive fields. In the case of minimal-depth with $t = 0$, we already mentioned that the partially-massless field is a massless spin-s field. In the case of maximal-depth with $t = s-1$, the partially-massless field contains just one less DoF — corresponding to a scalar field — compared to a massive spin-s field.

AdS_3 case

Let us first consider AdS case $\mathfrak{so}(2,d)$ and its lowest-weight representation $\mathcal{V}_{\mathfrak{so}(2,d)}(\Delta, s)$ labeled by the lowest energy Δ and spin s. The unitarity bound $\Delta = s + d - 2$ corresponds to the massless field (for $s \geq 1$) and the depth t partially-massless fields corresponds to $\Delta = s + d - 2 - t$. In these cases, we have to factor out invariant subspaces corresponding to gauge modes in order to describe irreducible representations .

In three dimensions, the lowest-weight representation $\mathcal{V}_{\mathfrak{so}(2,2)}(\Delta, s)$ is decomposed into those of

[3]For framelike formulation of partially-massless higher-spin fields, see [172]. For early discussion on (A)dS/CFT correspondence of partially-massless fields, see [173].

two $\mathfrak{so}(2,1)$ in $\mathfrak{so}(2,2) \simeq \mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1)$ as

$$\mathcal{V}_{\mathfrak{so}(2,2)}(h_1+h_2, h_1-h_2) = \big[\, \mathcal{V}_{\mathfrak{so}(2,1)}(h_1) \otimes \mathcal{V}_{\mathfrak{so}(2,1)}(h_2)\,\big] \oplus \big[\, \mathcal{V}_{\mathfrak{so}(2,1)}(h_2) \otimes \mathcal{V}_{\mathfrak{so}(2,1)}(h_1)\,\big]\,, \tag{7.67}$$

where we identify the lowest-weights and spin of $\mathfrak{so}(2,2)$ with those of $\mathfrak{so}(2,1)$'s as $\Delta = h_1+h_2$ and $s = |h_1 - h_2|$. Here, we focus on the parity-invariant representations, and so include both of the $\pm s$ helicities assuming $s \neq 0$. If $s = 0$, then we simply get $\mathcal{V}_{\mathfrak{so}(2,2)}(2h,0) = \mathcal{V}_{\mathfrak{so}(2,1)}(h) \otimes \mathcal{V}_{\mathfrak{so}(2,1)}(h)$. In terms of $V_h := \mathcal{V}_{\mathfrak{so}(2,1)}(h)$, it is simpler to understand the appearance of invariant subspace: when h takes a non-positive half-integer value, the representation splits into

$$V_{-h} = R_h \oplus V_{h+1} \qquad [2h \in \mathbb{N}]\,, \tag{7.68}$$

where V_{h+1} is the infinite-dimensional invariant subspace and R_h is the $(2h+1)$-dimensional representation.

Now considering the lowest-weight of partially-massless fields, $\Delta = s - t$, the lowest-weight representation $\mathcal{V}_{\mathfrak{so}(2,2)}(\Delta, s)$ reduces to

$$\begin{aligned} \mathcal{V}_{\mathfrak{so}(2,2)}(s-t,s) &= \Big(V_{s-\frac{t}{2}} \otimes V_{-\frac{t}{2}}\Big) \oplus \Big(V_{-\frac{t}{2}} \otimes V_{s-\frac{t}{2}}\Big) \\ &= \Big[V_{s-\frac{t}{2}} \otimes \Big(R_{\frac{t}{2}} \oplus V_{\frac{t}{2}+1}\Big)\Big] \oplus \Big[\Big(R_{\frac{t}{2}} \oplus V_{\frac{t}{2}+1}\Big) \otimes V_{s-\frac{t}{2}}\Big]\,, \end{aligned} \tag{7.69}$$

which involve an invariant subspace,

$$\mathcal{V}_{\mathfrak{so}(2,2)}(s+1, s-t-1) = \Big(V_{s-\frac{t}{2}} \otimes V_{\frac{t}{2}+1}\Big) \oplus \Big(V_{\frac{t}{2}+1} \otimes V_{s-\frac{t}{2}}\Big)\,, \tag{7.70}$$

corresponding to the gauge modes. After factoring out this, the remaining representation corresponds to the partially-massless ones:

$$\mathcal{D}_{\mathfrak{so}(2,2)}(s-t,s) = \Big(V_{s-\frac{t}{2}} \otimes R_{\frac{t}{2}}\Big) \oplus \Big(R_{\frac{t}{2}} \otimes V_{s-\frac{t}{2}}\Big)\,. \tag{7.71}$$

Distinct from the massless case where R_0 is the trivial representation, due to $R_{t/2}$, the partially-massless field cannot be decomposed neatly into left-moving and right-moving (or holomorphic and anti-holomorphic) parts . Moreover, $R_{t/2}$ is the finite-dimensional representation, so it is not unitary apart from the trivial one with $t = 0$. Hence, all partially-massless fields with $t \neq 0$ are non-unitary in AdS space.

Note that the partially-massless field $\mathcal{D}_{\mathfrak{so}(2,2)}(s-t,s)$ does not have any bulk DoF (as one of two $\mathfrak{so}(2,1)$'s has a finite-dimensional representation $R_{t/2}$) but it has $2(t+1)$ boundary DoFs. On the other hand, the gauge mode $\mathcal{V}_{\mathfrak{so}(2,2)}(s+1, s-t-1)$ has two bulk DoFs as it has infinite-dimensional representation for both of $\mathfrak{so}(2,1)$. The maximal-depth case with $t = s-1$ is special here. Even though the partially-massless field $\mathcal{D}_{\mathfrak{so}(2,2)}(1,s)$ follows the same pattern as generic t , its gauge mode is given by *two copies* of a parity-invariant scalar mode, $(V_{(s+1)/2} \otimes V_{(s+1)/2})^{\otimes 2} = [\mathcal{V}_{\mathfrak{so}(2,2)}(s+1,0)]^{\otimes 2}$. This particularity of the maximal-depth can be understood as well from simple field-theoretical consideratons.

For better understanding, consider the simplest example of spin-one particle. The maximal-depth partially-massless field coincides with the massless field for spin-one particle. The massive spin-one field is usually described by Proca action,

$$S_{\rm Proca}[A] = \int d^3x\sqrt{g}\left(\frac{1}{4e^2}\,F_{\mu\nu}\,F^{\mu\nu} + \frac{m^2}{2}\,A_\mu\,A^\mu\right). \tag{7.72}$$

In three dimensions, it can also be described as two copies of a *self-dual massive* action [174],

$$S_{\rm SDM}[A^\pm] = \int d^3x\left(\frac{1}{2}\,\epsilon^{\mu\nu\rho}\,A^\pm_\mu\,\partial_\nu A^\pm_\rho \pm \sqrt{g}\,\frac{me}{2}\,A^\pm_\mu\,A^{\pm\mu}\right), \tag{7.73}$$

where $A^\pm$ separately describe the $\pm$ helicity modes. In the massless limit, both actions acquire gauge symmetries and lose DoFs.

- The Proca action (7.72) acquires one gauge symmetry removing only one mode $\mathcal{V}_{\mathfrak{so}(2,2)}(2,0)$ from $\mathcal{V}_{\mathfrak{so}(2,2)}(1,1)$ (7.69), leaving $\mathcal{D}_{\mathfrak{so}(2,2)}(1,1)\oplus\mathcal{V}_{\mathfrak{so}(2,2)}(2,0)$. Hence, together with two boundary modes, it also describes a bulk scalar mode.
- The self-dual action (7.73) acquires two gauge symmetries: one for A^+ and the other for A^-. Each removes $V_{\frac{s+1}{2}}\otimes V_{\frac{s+1}{2}}$, so we end up with two copies of Abelian Chern-Simons, describing $\mathcal{D}_{\mathfrak{so}(2,2)}(1,1)$ with only boundary degrees of freedom.

In the higher-spin cases, one can still construct a one-derivative self-dual action for a massive field that is parity odd. For $t\neq s-1$ partially-massless field, there is also an equivalent two-derivative parity preserving description. For $t=s-1$ the situation is similar to Maxwell field: the one-derivative description is not equivalent to the two-derivative one. In the partially-massless limit (including $t=s-1$) of self-dual action, their gauge symmetry eliminates both $V_{s-\frac{t}{2}}\otimes V_{\frac{t}{2}+1}$ and $V_{\frac{t}{2}+1}\otimes V_{s-\frac{t}{2}}$, so completely removes the parity-invariant gauge mode $\mathcal{V}_{\mathfrak{so}(2,2)}(s+1,s-t-1)$ (7.70) , leaving only the boundary DoF $\mathcal{D}_{\mathfrak{so}(2,2)}(s+1,s-t-1)$ (7.71). On the other hand, beginning with a two-derivative massive action (see e.g. [168]), the partially-massless limit attains one *parity-invariant* gauge symmetry. When the depth is not maximal, $t\neq s-1$, it again removes the parity-invariant combination of gauge modes $\mathcal{V}_{\mathfrak{so}(2,2)}(s+1,s-t-1)$. On the contrary, in the maximal-depth case with $t=s-1$, the gauge symmetry removes only one mode among two $\mathcal{V}_{\mathfrak{so}(2,2)}(s+1,0)$'s . Hence, the left-over DoFs $\mathcal{D}_{\mathfrak{so}(2,2)}(1,s)\oplus\mathcal{V}_{\mathfrak{so}(2,2)}(s+1,0)$ contain a bulk scalar.

The colored higher-spin gravity of $\mathfrak{gl}_M\oplus\mathfrak{gl}_M$ makes use of the self-dual description for (partially-)massless fields. Hence, the maximal-depth partially-massless field of spin-M analyzed in Section 7.5 does not carry a bulk DoF but $2\,M$ boundary DoF (that is, M left-moving and M right-moving DoF). This number matches with the total boundary DoF of massless spin $1,2,\ldots,M$.

dS_3 case

In the dS$_3$ space, the isometry algebra is given by $\mathfrak{so}(1,3)\simeq\mathfrak{so}(3)\oplus\overline{\mathfrak{so}(3)}$, and we begin with the representations of $\mathfrak{so}(3)$ and $\overline{\mathfrak{so}(3)}$. Differently from the AdS_3 case, we do not assume that these representations are of lowest-weight type because in dS we do not have an invariant notion of energy to which we can impose a bound condition. Still, the representations can be labelled by $\mathbb{C}$ numbers

h and h^* (for now h^* is different from the complex conjugate $\bar{h}$) which parameterize the Casimir operators of $\mathfrak{so}(3)$ and $\overline{\mathfrak{so}(3)}$ as

$$C = h(h+1)\,, \qquad C^* = h^*(h^*+1)\,. \tag{7.74}$$

From the compactness of $\widetilde{SO(3)} \simeq SU(2)$ in $\widetilde{SO(1,3)} \simeq SL(2,\mathbb{C})$, we get the quantization condition:

$$h - h^* = s \in \tfrac{1}{2}\,\mathbb{Z}\,, \tag{7.75}$$

which is related to the spin of a particle in dS. For convenience, let us define the other combination of h and h^* as

$$h + h^* + 1 = \mu\,, \tag{7.76}$$

and μ is an arbitrary complex number for the moment. Since $\overline{\mathfrak{so}(3)}$ is the complex conjugate of $\mathfrak{so}(3)$, for unitarity, their representations should also be related by complex conjugate:

$$\bar{C} = C^* \quad \Leftrightarrow \quad (\bar{h} - h^*)(\bar{h} + h^* + 1) = 0\,. \tag{7.77}$$

There are two options to satisfy this unitarity condition:

$$\bar{h} + h^* + 1 = 0 \qquad \Rightarrow \qquad \text{any } s\,, \quad \mu \in i\,\mathbb{R}\,, \tag{7.78}$$

$$\bar{h} - h^* = 0 \qquad \Rightarrow \qquad s = 0\,, \quad \mu \in \mathbb{R}\,. \tag{7.79}$$

The first case (7.78) corresponds to the usual massive spin-s representation with mass-squared given by μ^2. The second case (7.79) corresponds to the special mass region only allowed for the scalar.[4] Since the representation space does not develop any invariant subspace in both of the cases (7.78, 7.79), we do not find any unitary short representation in dS background. In a sense, this is consistent with the fact that dS space does not have any Lorentzian boundary where the short representation can live.

7.7 Discussions

In this chapter, we have analyzed the theory of colored higher-spin gravity in three dimensions. We showed that this theory can be viewed as a theory of higher-spin gravity and Chern-Simons gauge fields coupled to matter fields consisting of massless higher spins. The matter fields introduce multiple saddle point vacua with different cosmological constants to the theory, exactly like in the case of (A)dS$_3$ colored gravity in chapter 6. On each of these vacua, the gauge symmetry breaking takes place which affects the spectrum of the theory.

The mechanism of gauge symmetry breaking and the resulting spectrum are interesting. First, the Goldstone modes, which are spin-two fields corresponding to the broken part of the color symmetry, are not simply eaten by one of the other fields but by "all" other fields. In a sense, it is more correct to describe this as if Goldstone modes devours all other spectrum so that they combine altogether to

[4] In the complete analysis [175], the unitary condition further restricts the allowed value of μ to $|\mu| \leq 1$.

become a single irreducible Regge trajectory of a maximal-depth partially-massless field.

The nature of partially-massless field is also intriguing. For the algebra $\mathfrak{gl}_M \oplus \mathfrak{gl}_M$, it is the spin M maximal-depth partially-massless field, which contains all the modes of massless spins from 1 to M . In other words, it behaves almost like a massive spin M field but lacks only one DoF, the scalar mode. However, in three dimensions, all (partially) massless fields with spin greater or equal to one (considering Chern-Simons as spin one) do not have propagating DoF. They still have boundary modes. The scalar mode is special as it is the only propagating DoF in the three dimensional bulk. Interestingly, when considering a generic $hs(\lambda)$ rather than $\mathfrak{gl}_M$, we do not have any bound on the highest spin, suggesting that the maximal-depth partially massless fields, appearing in the symmetry-broken phase of colored higher-spin gravity, might have an infinite tower of spin. The entire multiplet is a kind of Regge trajectory, whose slope is set by the nonzero expectation value of the colored spin-2 field and the intercept is set by the depleted spin component state. We believe these states are better described when formulated in the Prokushkin-Vasiliev theory (they correspond to the so-called twisted sector).

Chapter 8

Conclusion

“ You’re really not going to like it,” observed Deep Thought. “Tell us!” “ All right,”
said Deep Thought. “The Answer to the Great Question...” “ Yes...!”
“Of Life, the Universe and Everything...” said Deep Thought.
“Yes...!” “ Is...” said Deep Thought. “Yes...!!!...?”
“ Forty-two,” said Deep Thought, with infinite majesty and calm.
— Douglas Adams
THE HITCHHIKER’S GUIDE TO THE GALAXY

The higher spin gauge symmetry is quite rigid. Therefore, it is incredibly hard not only to deformed the interaction but also to change the spectrum. In this dissertation, we made a small step to find the tilted interaction and spectrum.

In first chapter, we summarised the basic knowledge of free massive and massless higher-spin fields. One of the interacting high spin gauge theory, Vasiliev system in four-dimension, is reviewed in next chapter with a full calcualtion. In the chapter chapter 4 and chapter 5, the Kaluza-Klein method is used to find new interacting theories. In spite of the fully consistency of the reduced theory is abstruse in this moment, we provide a nice clue for the consistence.

The different way to construct the interacting higher spin gauge theory is provided in chapter 6 and chapter 7: higher spin gauge theory with Chan-Paton factor, colored higher spin theory, and Higgs-like mechanism. Even though the colored higher spin theory is known before, finding new vacuum solution, which is named as rainbow vacua, is our own contribution. Rainbow vacua is interpreted as circumstance that colored massless spin-2 fields take non-zero VEV. As a consequence of non-zero VEV, part of colored higher spin is broken. In a similar way with Higgs-mechanism, massless fields combine and become more massive field: partially massless field. For the fields in unbroken higher-spin sector, fields remain as massless. Therefore, around rainbow vacua, the colored gravity can be understood as higher spin massless theory with the partially massless matters.

We cannot insist that the new interacting higher spin gauge theory is achieved. In this dissertation, the very small step to find more broad model in higher spin theories is taken.

Appendices

Chapter A

AdS space and related conventions

Conventions are demonstrated in d dimension. Space-time coordinate indices μ, ν and tangent indices a, b run from 0 to d. We will use most positive sign for metric and $\eta_{ab} = \text{diag}(-1, 1, \cdots, 1)$. We will call $e_\mu{}^a$ and (sometimes its inverse $e_a{}^\mu$ or $e^a = e_\mu{}^a\, dx^\mu$) as vielbein if it satisfies $e_a{}^\mu\, \eta_{ab}\, e_b{}^\nu = g^{\mu\nu}$. Covariant derivatives are defined as follow. $\omega^{ab} = \omega_\mu{}^{ab}\, dx^\mu$ is spin connection.

$$\nabla_\mu A^\rho = \partial_\mu A^\nu + \Gamma^\nu_{\rho\mu} A^\rho, \qquad \nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\rho_{\mu\nu} A_\rho \tag{A.1}$$

$$\nabla_\mu A^a = \partial_\mu A^a + \omega_\mu{}^a{}_b A^b, \qquad \nabla_\mu A_a = \partial_\mu A_a - \omega_\mu{}^b{}_a A_b. \tag{A.2}$$

(A)dS$_{d+1}$ metric and Riemann tensor are given as following.

$$ds^2 \;=\; g_{\mu\nu} dx^\mu dx^\nu = -\frac{\ell^2}{z^2}(dt^2 - d\vec{x}^2_{d-1} - dz^2), \tag{A.3}$$

$$\mathfrak{R}_{\mu\nu\rho\sigma} \;=\; \frac{\sigma}{\ell^2}\left(-g_{\mu\rho}\, g_{\nu\sigma} + g_{\mu\sigma}\, g_{\nu\rho}\right) = \frac{2\,\Lambda}{d\,(d-1)}\left(g_{\mu\rho}\, g_{\nu\sigma} - g_{\mu\sigma}\, g_{\nu\rho}\right), \tag{A.4}$$

where ℓ is $(A)dS$ radius and σ is $(+1)$ for AdS and (-1) for dS. $\Lambda = -\frac{\sigma}{2\,\ell^2}\, d\,(d-1)$ is the cosmological constant. In section 5.3, spinorial form is used. Star product and epsilon tensor conventions are given as,

$$\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}, \quad \epsilon_{\dot\alpha\dot\beta} = -\epsilon_{\dot\beta\dot\alpha}, \quad \epsilon_{12} = -1, \tag{A.5}$$

$$\epsilon_{\gamma\beta}\epsilon^{\gamma\alpha} = \delta^\alpha_\beta, \quad A^\alpha = \epsilon^{\alpha\beta} A_\beta, \quad A_\alpha = A^\beta \epsilon_{\beta\alpha}, \tag{A.6}$$

$$y^\alpha * y^\beta = i\epsilon^{\alpha\beta}, \quad \bar{y}^\alpha * \bar{y}^\beta = i\epsilon^{\alpha\beta}, \quad y^\alpha * \bar{y}^\beta = 0. \tag{A.7}$$

AdS vacuum value of one-form in four dimensional Vasiliev system is,

$$\bar{\mathcal{W}}_0 = \frac{1}{4i}(\bar\Omega_{\alpha\beta} y^\alpha y^\beta + \bar{\tilde{\Omega}}_{\dot\alpha\dot\beta}\bar{y}^{\dot\alpha}\bar{y}^{\dot\beta}) + \frac{1}{2i\ell}\bar{E}_{\alpha\dot\beta} y^\alpha \bar{y}^{\dot\beta}. \tag{A.8}$$

They satisfy following vacuum equations.

$$dE_{\alpha\dot\beta} + \Omega_{\alpha\gamma}\wedge\epsilon^{\gamma\lambda}E_{\lambda\dot\beta} + \tilde\Omega_{\dot\beta\dot\gamma}\wedge\epsilon^{\dot\gamma\dot\lambda}E_{\alpha\dot\lambda} = 0, \tag{A.9}$$

$$d\Omega_{\alpha\beta} + \Omega_{\alpha\gamma}\wedge\epsilon^{\gamma\lambda}\Omega_{\lambda\beta} + \frac{1}{\ell^2}E_{\alpha\dot\gamma}\wedge\epsilon^{\dot\gamma\dot\lambda}E_{\beta\dot\lambda} = 0. \tag{A.10}$$

Vielbein and spin connection of AdS$_4$ and AdS_3 are used various times in main text. Four-dimensional vielbien and spin connection is,

$$
\begin{aligned}
\bar{E}_{\alpha\dot{\alpha}} &= \frac{1}{2\rho\cos\theta}\begin{pmatrix} dt+d\rho & dx-i\rho d\theta \\ dx+i\rho d\theta & dt-d\rho \end{pmatrix}, && \text{(A.11)}\\
\bar{\Omega}_{\alpha\beta} &= -\frac{1}{2\rho}\begin{pmatrix} dx & dt \\ dt & dx \end{pmatrix} + \frac{i\tan\theta}{2\rho}\begin{pmatrix} dt+d\rho & dx \\ dx & dt-d\rho \end{pmatrix}, && \text{(A.12)}\\
\bar{\tilde{\Omega}}_{\dot{\alpha}\dot{\beta}} &= -\frac{1}{2\rho}\begin{pmatrix} dx & dt \\ dt & dx \end{pmatrix} - \frac{i\tan\theta}{2\rho}\begin{pmatrix} dt+d\rho & dx \\ dx & dt-d\rho \end{pmatrix}. && \text{(A.13)}
\end{aligned}
$$

In three-dimensional vacuum values are,

$$
\begin{aligned}
E^{\alpha\beta} &= \bar{E}^a\sigma_a^{\alpha\beta} = \frac{1}{2\rho}\begin{pmatrix} dt+d\rho & dx \\ dx & dt-d\rho \end{pmatrix}, && \text{(A.14)}\\
\Omega^{\alpha\beta} &= \Omega^{AB}(\sigma_A\sigma_B)^{\alpha\beta} = -\frac{1}{2\rho}\begin{pmatrix} dx & dt \\ dt & dx \end{pmatrix}. && \text{(A.15)}
\end{aligned}
$$

We chose σ_2 as a compactifying direction. They related with each other. i.e. $\bar{E}_{\alpha\dot{\beta}} = \frac{1}{\cos\theta}E_{\alpha\beta} + \frac{i}{2\cos\theta}d\theta\epsilon_{\alpha\beta}$, $\bar{\Omega}_{\alpha\beta} = \Omega_{\alpha\beta} + i\tan\theta E_{\alpha\beta}$ and $\bar{\tilde{\Omega}}_{\alpha\beta} = \Omega_{\alpha\beta} - i\tan\theta E_{\alpha\beta}$. The relation $\bar{\Omega}_\mu^{a,D} = \tan\theta {E_\mu}^a$ is used in explicit calculation.

Chapter B

Verma module and partailly massless(PM) field

To define the Vemar $\mathfrak{so}(d,2)$-module, let us consider a finite dimensional module of sub-algebra $\mathfrak{so}(2)\oplus\mathfrak{so}(d)$: $\mathcal{Y}(\Delta,\,Y)$. We used the Δ to represent the "conformal weight" and Y represents a Young diagram for $\mathfrak{so}(d-1)$. We can consider the general form of $\mathcal{V}(\Delta,\,Y)$, but concentrate on the case Young diagram is made of a single row of length s. The Vemar $\mathfrak{so}(d,2)$-module $\mathcal{V}(\Delta,\,s)$ is the space which is generated by acting raising operators to $\mathcal{Y}(\Delta,\,Y)$. Following the prevailing notation in physics literature, $\mathcal{D}(\Delta,s)$ is used to represent the irreducible quotient of $\mathcal{V}(\Delta,\,s)$. Except specific values, Verma module $\mathcal{V}(\Delta,\,s)$ is irreducible and therefore coincide with $\mathcal{D}(\Delta,s)$. However, for a specific values, it has a submodule. For example, $\Delta=d+k-1$ with an integer $0\le k\le s-1$, there is a submodule $\mathcal{D}\left(d+s-1,\,k\right)$. Therefore, $\mathcal{D}\left(d+k-1,\,s\right)$ is not equal to Verma module but is to the quotient of Verma module:

$$\mathcal{V}\left(d+k-1,\,s\right)\simeq\mathcal{D}\left(d+k-1,\,s\right)\oplus\mathcal{D}\left(d+s-1,\,k\right)\,,\quad\mathcal{D}\left(d+k-1,\,s\right)\simeq\frac{\mathcal{V}\left(d+k-1,\,s\right)}{\mathcal{D}\left(d+s-1,\,k\right)}\,. \tag{B.1}$$

The massless spin-s field in AdS is the field theoretical realization of $\mathcal{D}(d+(s-1)-1,s)$, which is unitary. For $0\le k<s-1$ cases, the representations are not unitary and their field theoretical realizations are called the partially massless(PM) fields[1] with depth-$(s-k-1)$. For more general cases, see [85, **?**]. The action for PM field has the PM gauge symmetry which contains covariant derivatives up to order $t+1$:

$$\delta\,\phi_{\mu_1\mu_2\cdots\mu_s}=\nabla_{(\mu_1}\cdots\nabla_{\mu_{t+1}}\,\xi_{\mu_{t+2}\cdots\mu_s)}+\cdots \tag{B.2}$$

which can be derived by Stueckelberg form of PM field. See paragraph below (4.58) for PM spin-2 case. The followings are properties of PM field:

Field type	Δ_+	m^2	Gauge variation: $\delta\,\phi_{\mu_1\mu_2\cdots\mu_s}$
depth-t PM field	$d+s-t-2$	$-\frac{\sigma}{\ell^2}\,t\,(d+2s-t-4)$	$\nabla_{(\mu_1}\cdots\nabla_{\mu_{t+1}}\,\xi_{\mu_{t+2}\cdots\mu_s)}+\cdots$

Table 5: Partially massless(PM) field

It is better to comment the "mass" convention in Table 5. In main text, the "mass" square of field is equal the mass square in flat limit. Therefore, it is zero when the higher spin gauge symmetry exist.

[1] We can call the massless field is the PM with depth-0 in our convention.

In this notation, the relation between mass square and conformal weight are given as,

$$m^2\,\ell^2 = \Delta\,(\Delta - d) - (s-2)\,(d+s-2)\ . \tag{B.3}$$

And it is different with the mass square which appears in Fierz-Pauli equation in AdS [176]:

$$\left(\nabla^2 + \kappa^2\right)\phi_{\mu_1\,\mu_2\,\cdot\mu_s} = 0\,, \quad \text{with}\kappa^2\,\ell^2 = \Delta\,(\Delta - d) - s\,. \tag{B.4}$$

Interestingly $\mathfrak{so}(d+1,2)$-module for massless spin-s can be decomposed into $\mathfrak{so}(d,2)$-modules by the following branching rules [39]:

$$\mathcal{D}(d+s-1,s)_{\mathfrak{so}(\mathrm{d}+1,2)} = \bigoplus_{n=0}^{\infty} \mathcal{D}(d+n+s-1,s)_{\mathfrak{so}(\mathrm{d},2)} \oplus \bigoplus_{l=0}^{s-1} \mathcal{D}(d+s-1,l)_{\mathfrak{so}(\mathrm{d},2)} \tag{B.5}$$

In main text we sometimes omit subscripts ${}_{\mathfrak{so}(\mathrm{d},2)}$ for compact expression.

Chapter C

Reduction Method from (A)dS$_{d+k}$ to (A)dS$_d$

In section 4.2, we have introduced the AdS waveguide by using Poincare coordinate of AdS$_{d+1}$. But it can be generalized to other coordinate system and it can be shown that the physical features do not change. Further more the more general waveguide method shall be introduced in this section.

The main condition for the waveguide is the covariance condition: covariant derivatives of tensors in $d+k$ dimension are also tensors in a lower dimensional point of view. A covariant derivative of tensor in higher dimension can be written as,

$$\bar{\nabla}_\mu \bar{B}_\nu{}^a \quad = \quad \nabla_\mu B_\nu a - \bar{\Gamma}^{\theta_i}{}_{\mu\nu} B_{\theta_i} a + \bar{\Omega}^a{}_{\mu\, m} A_\nu{}^m , \tag{C.1}$$

where $X^M = (x^\mu, \theta^i)$ represent higher dimensional coordinates, μ, ν are for lower dimensional indices and θ_i internal space indices in lower dimensional space-time point of view. And $A = (a, m)$ are defined in similar ways: a, b are indices for lower dimensional local space-time indices, m and n is for internal. "Bar" are used to represent that quantities are tensors or covariant derivatives in $d+k$ dimension. At last, $\bar{\Omega}$ is spin connection of $d+k$ dimension and $\bar{\Gamma}$ is Christoffel symbol. By the covariance condition for $\bar{\nabla}_\mu B_\nu{}^a$, $\Gamma^{\theta_i}{}_{\mu\nu}$ and $\bar{\Omega}_\mu{}^{a\,m}$ must be tensor in the lower dimension. Let us consider an ansatz about spin connection $\bar{\Omega}^{AB}$ and vielbein $\bar{E}^A$:

$$\bar{E}^a = f_0\, E^a , \quad \bar{E}^m = M^m{}_n\, d\theta^n\, , \quad \bar{\Omega}^{ab} = \Omega^{ab}\, , \quad \bar{\Omega}^{am} = N^m\, E^a\, , \quad \bar{\Omega}^{mn} = L^{mn}\, , \tag{C.2}$$

here f_0, N^m and $M^m{}_n$ are functions of θ^i and independent of x^μ and L^{mn} is an one-form tensor which is independent of x^μ. Because we want reduction from (A)dS$_{d+k}$ to (A)dS$_d$, barred and unbarred vielbein and spin connection are those corresponding (A)dS space:

$$dE^a + \Omega^{ab} \wedge E_b = 0, \quad d\,\Omega^{ab} + \Omega^{ac} \wedge \Omega_c{}^b + \sigma\, E^a \wedge E^b = 0\, , \tag{C.3}$$

where σ is $+1$ for AdS and -1 for dS. The first equation is the torsionless condition and second is (A)dS conditions. The same conditions for higher dimensional quantities constraint unknowns in the ansatz (C.2):

$$\delta\, f_0 = N^m\, \bar{E}_m\, , \quad \delta\, N^m = N_n\, \bar{\Omega}^{nm} + \sigma\, f_0\, \bar{E}^m\, , \quad \sigma\, f_0^2 - N^m\, N_m = \sigma'\, , \tag{C.4}$$

$$\delta\, \bar{E}^m + \bar{\Omega}^{mn} \wedge \bar{E}_n = 0\, , \quad \delta\, \bar{\Omega}^{mn} + \bar{\Omega}^n{}_l \wedge \bar{\Omega}^{ln} + \sigma\, \bar{E}^m \wedge \bar{E}^n = 0\, , \tag{C.5}$$

where δ represents the external derivatives in internal space θ^i. i.e. $\delta f = \sum_{i=1}^k \frac{\partial f}{\partial \theta^i} d\theta^i$. And the σ' is introduced to represent the curvature sign of (A)dS$_d$. The equations in (C.5) imply that $\bar{E}^m$ and $\bar{\Omega}^{mn}$ are vielbein and spin connection for space with negative constant curvature whose curvature of internal space is equal to that of AdS. Let us consider explicit forms for $k = 1$ and $k = 2$.

($k = 1$ case) In this case $M^m{}_n \to M$ and $N^m \to N$ and (C.5) is automatically true. The constraints (C.4), Cristoffle symbols and (A)dS$_{d+1}$ metric are,

$$\sigma\, {f_0}^2 - N^2 = \sigma' \,, \quad \frac{d\,f_0}{d\,\theta} = N\,M \,, \quad \frac{d\,N}{d\,\theta} = \sigma f_0\, M \,. \tag{C.6}$$

$$\bar{\Gamma}^\theta{}_{\nu\lambda} = -\frac{f_0\,N}{M} g_{\nu\lambda}, \quad \bar{\Gamma}^\lambda{}_{\nu\theta} = \frac{M\,N}{f_0}\delta^\lambda{}_\nu \,, \quad \bar{\Gamma}^\theta{}_{\theta\theta} = \frac{1}{M}\frac{d\,M}{d\,\theta} \,, \tag{C.7}$$

$$d\,s^2_{d+1} = f_0^2\,d\,s^2_d + M^2\,d\theta^2 \,. \tag{C.8}$$

It can be shown that the waveguide from AdS$_{d+1}$ to AdS$_d$, from dS$_{d+1}$ to dS$_d$ and from AdS$_{d+1}$ to dS$_d$ can be constructed. In contrast, we cannot obtain the waveguide from dS$_{d+1}$ to AdS$_d$ in this method because of the last equation in (C.4). Let us first consider a waveguide from AdS$_{d+1}$ to AdS$_d$: $\sigma = 1$ and $\sigma' = 1$. It can be checked that $f_0 = M = \sec\theta$ and $N = \tan\theta$ are the possible solutions of constraints (C.6)[1]. The higher dimensional metric can be written as $ds^2_{\text{AdS}_{d+1}} = \frac{1}{\cos^2\theta}(ds^2_{\text{AdS}_d} + \ell^2 d\theta^2)$ where $ds^2_{\text{AdS}_d}$ is an arbitrary locally AdS metric and do not have to Poincare in AdS$_d$ as in main text. For other cases, see Table 6.

Waveguide(σ/σ')	f_0	M	N	$\bar{\Gamma}_1$	$\bar{\Gamma}_2$	$\bar{\Gamma}^\theta{}_{\theta\theta}$
AdS$_{d+1}$ to AdS$_d$ (1/1)	$\sec\theta$	$\sec\theta$	$\tan\theta$	$-\tan\theta$	$\tan\theta$	$\tan\theta$
dS$_{d+1}$ to dS$_d$ (-1/-1)	$\operatorname{sech}\theta$	$-\operatorname{sech}\theta$	$\tanh\theta$	$\tanh\theta$	$-\tanh\theta$	$-\tanh\theta$
AdS$_{d+1}$ to dS$_d$ (1/-1)	$\tan\theta$	$\sec\theta$	$\sec\theta$	$-\tan\theta$	$\sec^2\theta\,\cot\theta$	$\tan\theta$

Table 6: Various waveguide from (A)dS$_{d+1}$ to (A)dS$_d$ and related factors. $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ are the parts of $\bar{\Gamma}^\theta{}_{\mu\nu}$ and $\bar{\Gamma}^\mu{}_{\nu\theta}$: $\bar{\Gamma}^\theta{}_{\mu\nu} = \bar{\Gamma}_1\, g_{\mu\nu}$, $\bar{\Gamma}^\mu{}_{\nu\theta} = \bar{\Gamma}_2\,\delta^\mu{}_\nu$.

($k = 2$ case) Ansatz for vielbein and spin connection in d+1 dimension are,

$$\bar{E}^a = f_0 E^a, \quad \bar{E}^i = M^i{}_j d\theta^i, \quad \bar{\Omega}^{ai} = N^i E^a, \tag{C.9}$$

where i and j represent 1 or 2. Constraints (C.4) and (C.5) can be solved as

$$\bar{E}^i = \begin{pmatrix} \sinh\rho\, du \\ d\rho \end{pmatrix}, \quad \bar{\Omega}^{ij} = \cosh\rho \begin{pmatrix} 0 & du \\ -du & 0 \end{pmatrix}, \quad f_0 = \cosh\rho\,, \quad N_1 = 0\,, \quad N_2 = \sinh\rho\,.$$

And metric in $d + 2$ dimension can be written as

$$ds^2_{\text{AdS}_{d+2}} = \cosh^2\rho\,(ds^2_{\text{AdS}_d}) + d\,\rho^2 + \sinh^2\rho\, du^2. \tag{C.10}$$

We can easily generalize these to an arbitrary k.

[1]There are other cases, for examples $f_0 = \cosh\theta$, $M = 1$ and $N = \sinh\theta$ or $f_0 = 1/\tanh\theta$, $M = -1/\sinh\theta$ and $N = 1/\sinh\theta$. But they are just coordinate change of first one.

Bibliography

[1] S. Gwak, J. Kim and S. J. Rey, *Massless and Massive Higher Spins from Anti-de Sitter Space Waveguide*, [arXiv:1605.06526].

[2] S. Gwak, E. Joung, K. Mkrtchyan and S. J. Rey, *Rainbow vacua of colored higher-spin (A)dS$_3$ gravity*, JHEP **1605** (2016) 150 doi:10.1007/JHEP05(2016)150 [arXiv:1511.05975 [hep-th]]. [arXiv:1511.05975].

[3] S. Gwak, E. Joung, K. Mkrtchyan and S. J. Rey, *Rainbow Valley of Colored (Anti) de Sitter Gravity in Three Dimensions*, JHEP **1604** (2016) 055 doi:10.1007/JHEP04(2016)055 [arXiv:1511.05220].

[4] T. Kaluza, *Zum Unitätsproblem der Physik*, *Sitz. Preuss. Akad. Wiss. Phys.* Berlin. (Math. Phys.): (1921) 966–972, O. Klein, *Quantentheorie und fünfdimensionale Relativitätstheorie*, *Zeits. Phys.* **37** (1926) 895.

[5] E. Wigner, *Annals Math.* **40**, 149 (1939) and *Nucl. Phys. Proc. Suppl.* **6**, 9 (1989).

[6] V. Bargmann and E. Wigner, *Proc. Natl. Acad. Sci.* USA **34** (1948) 211.

[7] L. Brink, A. Khan, P. Ramond, X. Xiong, *Continuous Spin Representations of the Poincaré and Super-Poincaré Groups*, *J.Math.Phys.***43** (2002) 6279. [arXiv:0205145].

[8] M. Fierz and W. Pauli, *On Relativistic Wave Equations for Particles of Arbitrary Spin in an Electromagnetic Field*, *Proc. Roy. Soc. Lond.***A 173**, 211 (1939).

[9] L. P. S. Singh and C. R. Hagen, *Lagrangian formulation for arbitrary spin. 1. The boson case*, *Phys. Rev.* **D9** (1974) 898–909.

[10] C. Fronsdal, *Massless fields with integer spin*, *Phys. Rev.* **D 18**, 3624 (1978).

[11] D. Freedman, P. van Nieuwenhuizen and S. Ferrara, *Phys. Rev.***D 13** (1976) 3214; S. Deser and B. Zumino, *Phys. Lett.***B 62** (1976) 335; P. de Medeiros and C. Hull, *Commun. Math. Phys.***235** (2003) 255 [arXiv:hepth/0208155]; X. Bekaert and N. Boulanger, *Commun. Math. Phys.***245**, 27 (2004) [arXiv:hep-th/0208058]; X. Bekaert and N. Boulanger, *Phys. Lett.***B 561**, 183 (2003) [arXiv:hep-th/0301243].

[12] C. Aragone, S. Deser and Z. Yang, *Massive Higher Spin from Dimensional Reduction of Gauge Fields*, *Annals of physics* **179**, 76-96 (1987), and for lower spin examples: S. D. Rindani and M. Sivakumar, *Gauge - Invariant Description Of Massive Higher - Spin Particles By Dimensional Reduction Phys. Rev.* **D 32** 3238 (1985).

[13] S. Weinberg, *The Quantum theory of fields. Vol. 1: Foundations*, (1995) *Cambridge University Press*, **ISBN**: 9780521670531

[14] N. Bouatta, G. Compere, A. Sagnotti, *An Introduction to Free Higher-Spin Fields*, *“Higher spin gauge theories: Proceedings, 1st Solvay Workshop” : Brussels, Belgium, 12-14 May, (2004)* **C04-05-12.2**, p.79–99 [arXiv:0409068].

[15] S. Coleman and J. Mandula, *All possible symmetries of the s matrix*, *Phys. Rev.* **159** (1967) 1251–1256.

[16] S. Weinberg, *Photons and Gravitons in s Matrix Theory: Derivation of Charge Conservation and Equality of Gravitational and Inertial Mass Phys. Rev.* **135** (1964) B1049–B1056.

[17] S. Weinberg and E. Witten, *Limits on Massless Particles*, *Phys. Lett.***B96** (1980) 59.

[18] G. Velo and D. Zwanziger, *Noncausality and other defects of interaction lagrangians for particles with spin one and higher*, *Phys. Rev.* **188** (1969) 2218–2222.

[19] X. Bekaert, N. Boulanger and P. Sundell, *How higher-spin gravity surpasses the spin two barrier: no-go theorems versus yes-go examples*, *Rev. Mod. Phys.* **84**, 987– [arXiv:1007.0435].

[20] E. Joung and M. Taronna *Cubic interactions of massless higher spins in (A)dS: metric-like approach*, *Nucl.Phys* **B 861** (2012) 145—174.

[21] E. Joung, L. Lopez and M. Taronna, *On the cubic interactions of massive and partially-massless higher spins in (A)dS*, *JHEP***07** (2012) 04, section 2. [arXiv:1203.6578].

[22] J. Sakurai and J. Napolitano, *Modern Quantum Mechanics*, *Boston, USA: Addison-Wesley* (2011) 550 p **ISBN**-9780805382914

[23] C. Fronsdal *Singletons and massless, integral-spin fields on de Sitter space*, *Phys. Rev.* **D 20**, 848 (1979).

[24] E. Angelopoulos, M. Flato, C. Fronsdal, and C. Sternheimer, *Massless particles, conformal group and De Sitter universe*, *Phys. Rev.* **D23** (1981), 1278–1289.

[25] B. de Wit and I. Herger, *Anti-de Sitter Supersymmetry*, *Towards Quantum Gravity Proceedings of the XXXV International Winter School on Theoretical Physics*, in Polanica, Poland, 2-11 February (1999) by J. Kowalski-Glikman (Ed.)

[26] R. R. Metsaev, *Arbitrary spin massless bosonic fields in d-dimensional anti-de Sitter space*, *Lect. Notes Phys.***524** (1999) 331-340, [hep-th/9810231].

[27] P. Dirac, *A remarkable representation of the 3 + 2 de Sitter group*, *J. Math. Phys.* **4** (1963), 901–909

[28] M. Flato, C. Fronsdal, D. Sternheimer, *Singleton physics*, [arXiv:9901043].

[29] M. Vasiliev, *Higher Spin Superalgebras in any Dimension and their Representations JHEP* 0412:046,2004 [arXiv:0404124].

[30] X. Bekaert and M. Grigoriev, *Higher order singletons, partially massless fields and their boundary values in the ambient approach,* Nucl. Phys. B **876**, 667 (2013) [hep-th/1305.0162].

[31] X. Bekaert, *Singletons and their maximal symmetry algebras*, *Modern Mathematical Physics. Proceedings, 6th Summer School : Belgrade, Serbia*, September 14-23, 2010, B. Dragovich(ed.) and Z. Rakic (ed.), [`arXiv:1111.4554`].

[32] M. A. Vasiliev, *More on equations of motion for interacting massless fields of all spins in* $3+1$ *dimensions*, *P*hy. Lett. **285** (1992) 225–234,

[33] V. E. Didenko and E. D. Skvortsov, *Elements of Vasiliev theory*, [`arXiv:1401.2975`].

[34] S. Deser and A. Waldron, *Gauge Invariances and Phases of Massive Higher Spins in (A)dS*, *Phys. Rev. Lett.* **87**, 031601 (2001), [`hep-th/0102166`]; *Partial Masslessness of Higher Spins in (A)dS*, *Nucl. Phys.***B 607** (2001) 577 [`hep-th/0103198`]; *Stability of Massive Cosmological Gravitons*, *Phys. Lett.***B 508**, 347 (2001) [`hep-th/0103255`]; *Null propagation of partially massless higher spins in (A)dS and cosmological constant speculations*, *Phys. Lett.* **B513** (2001) 137–141, [`hep-th/0105181`]; *Arbitrary Spin Representations in de Sitter from dS/CFT with Applications to dS Supergravity*, *Nucl. Phys.***B 662** (2003) 379 [`hep-th/0301068`];

[35] Yu. M. Zinoviev, *On massive high spin particles in AdS*, [`hep-th/0108192`].

[36] E. C. G. Stueckelberg, *D̈ie Wechselwirkungskräfte in der Elektrodynamik und in der Feldtheorie der Kräfte*, *Helv. Phys. Acta.***11** (1938) 225;

[37] K. Hinterbichler *Theoretical Aspects of Massive Gravity*, *Rev. Mod. Phys.* **84**, 671-710 (2012) [`hep-th/1105.3735v2`].

[38] R. R. Metsaev, "Massive fields in AdS(3) and compactification in AdS spacetime", hep-th/0103088.

[39] M. A. Vasiliev, "On Dimensional Degression in AdS$_d$", arXiv: hep-th/0810.2065

[40] P. A. Dirac, "A Remarkable representation of the 3 + 2 de Sitter group," J.Math.Phys. 4 (1963) 901.

[41] P. A. M. Dirac, "Wave equations in conformal space, Annals Math. 37 (1936) 429.

[42] J. M. Maldacena, "The Large N Limit of Superconformal Field Theories and Supergravity", Adv. Theor. Math. Phys. 2 (1998) 231, hep-th/9711200.

[43] S. Deser, A. Waldron, "Gauge Invariances and Phases of Massive Higher Spins in (A)dS", PhysRevLett.87.031601, hep-th/0102166, "Null Propagation of Partially Massless Higher Spins in (A)dS and Cosmological Constant Speculations", hep-th/0105181,

[44] M.P. Blencowe, "A consistent interacting massless higher-spin field theory in $D = 2 + 1$", Class. Quantum Grav. 6 (1989) 443.

[45] M. A. Vasiliev, "More on equations of motion for interacting massless fields of all spins in (3+1)-dimensions", Phys. Lett. B285 (1992) 225.

[46] M. A. Vasiliev, “Higher-spin gauge theories in four, three and two dimensions,” Int. J. Mod. Phys. D5 (1996) 763, hep-th/9611024, and “Higher spin gauge theories: Star-product and AdS space”, hep-th/9910096.

[47] S. F. Prokushkin and M.A. Vasiliev, “Higher spin gauge interactions for massive matter fields in 3d AdS space-time,” Nucl. Phys. B 545 (1999) 385 [arXiv:hep-th/9806236 [hep-th]].

[48] S. F. Prokushkin and M.A. Vasiliev, “3-d higher spin gauge theories with matter,” arXiv:hep-th/9812242 [hep-th].

[49] S. Giombi and X. Yin, “Higher spin gauge theory and holography: the three-point functions,” JHEP 1009 (2010) 115 [arXiv:0912.3462 [hep-th]]

[50] S. Giombi and X. Yin, “Higher spins in AdS and twistorial holography,” JHEP 1104 (2011) 086 [arXiv:1004.3736 [hep-th]].

[51] S. Giombi and X. Yin, “The higher spin/vector model duality,” arXiv:1208.4036 [hep-th].

[52] M. R. Gaberdiel and R. Gopakumar, “An AdS3 Dual for Minimal Model CFTs,” Phys.Rev. D83 (2011) 066007, 1011.2986.

[53] M. Gaberdiel, R. Gopakumar, “Minimal Model Holography”, arXiv:1207.6697 [hep-th]

[54] K. Hinterbichler “Theoretical Aspects of Massive Gravity”, hep-th/1105.3735v2

[55] E. Joung, M. Taronna,“Cubic interactions of massless higher spins in (A)dS: metric-like approach”, arXiv:1110.5918 [hep-th]

[56] E. Joung, L. Lopez, M. Taronna, “Solving the Noether procedure for cubic interactions of higher spins in (A)dS”, arXiv:1207.5520 [hep-th]

[57] E. Joung, L. Lopez, M. Taronna, “On the cubic interactions of massive and partially-massless higher spins in (A)dS”, hep-th/1203.6578.

[58] R. Manvelyan, R. Mkrtchyan and W. Rühl, “Radial Reduction and Cubic Interaction for Higher Spins in (A)dS space”, arXiv:1210.7227 [hep-th]

[59] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, Int. J. Theor. Phys. **38**, 1113 (1999) [Adv. Theor. Math. Phys. **2**, 231 (1998)] [hep-th/9711200].

[60] J. Kim, S. Gwak, to appear.

[61] Simone Giombi, Igor R. Klebanov, *One Loop Tests of Higher Spin AdS/CFT*, JHEP **1312** (2013) 68 [hep-th/1308.2337]; Simone Giombi, Igor R. Klebanov, Arkady A. Tseytlin, *Partition Functions and Casimir Energies in Higher Spin AdS_{d+1}/CFT_d*, Phys. Rev. D **90** (2014) 2, 024048 [hep-th/1402.5396]; M. Beccaria, Arkady A. Tseytlin, *On Higher Spin Partition Functions*, J. Phys. A **48** (2015) 27, 275401 [hep-th/1503.08143].

[62] L. Dolan and M. J. Duff, “Kac-Moody Symmetries of Kaluza-Klein Theories”, Phys.Rev.Lett. 52 (1984) 14

[63] Yu. M. Zinoviev “On Massive High Spin Particles in (A)dS”, hep-th/0108192.

[64] Yu. M. Zinoviev “Frame-like gauge invariant formulation for massive high spin particles”, hep-th/0808.1778.

[65] C. T. Fulton, *Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics 77.3-4 (1977): 293-308.

[66] P. A. Binding, P. J. Browne, K. Seddighi, *Sturm-Liouville problems with eigenparameter dependent boundary conditions*, Proceedings of the Edinburgh Mathematicla Society (1993) 37, 57-72

[67] J. W. Strutt, B. Rayleigh, he Theory of Sound (vol.1), London, Macmillan and co. (1877): p. 200-201

[68] E. C. G. Stueckelberg, D̈ie Wechselwirkungskräfte in der Elektrodynamik und in der Feldtheorie der Kräfte”, Helv. Phys. Acta. 11 (1938) 225; V. I. Ogievetskii and I. V. Polubarinov, “A Gauge Invariant Formulation of Neutral Vector

[69] Klaus Kirsten, “SPECTRAL FUNCTIONS in MATHEMATICS and PHYSICS”, 2002 by Chapman and Hall/CRC.

[70] M. Borda a, U. Mohideen, V.M. Mostepanenko,“New Developments in the Casimir Effect”, quant-ph/0106045.

[71] R. Camporesi and A. Higuchi, “Arbitrary spin effective potentials in anti-de Sitter space-time”, Phys.Rev. D47 (1993) 3339.

[72] R. Camporesi and A. Higuchi, “Spectral functions and zeta functions in hyperbolic spaces”, J.Math.Phys. 35 (1994) 4217.

[73] Simone Giombi, Alexander Maloney and Xi Yin, “One-loop Partition Functions of 3D Gravity”, hep-th/0804.1773.

[74] Justin R. David, Matthias R. Gaberdiel and Rajesh Gopakumar, “The Heat Kernel on AdS_3 and its Applications”, hep-th/0911.5085

[75] Simone Giombi, Igor R. Klebanov and Arkady A. Tseytlin, “Partition Functions and Casimir Energies in Higher Spin $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$”, hep-th/1402.5396

[76] Simone Giombi, Igor R. Klebanov and Benjamin R. Safdi “Higher Spin $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ at One Loop”, hep-th/1401.0825

[77] Matteo Beccaria, Xavier Bekaert and Arkady A. Tseytlin, “Partition function of free conformal higher spin theory”, hep-th/1406.3542

[78] “Higher spins in AdS5 at one loop: vacuum energy, boundary conformal anomalies and AdS/CFT”, hep-th/1410.3273

[79] H. Nicolai, “Representation of Supersymmetry in Anti-De Sitter Space” Lecture in ICTP Trieste, April 1984.

[80] B. de Wit, I. Herger, “Anti de Sitter supersymmetry”, Lect. Notes Phys. 541 (2000) 79 [arXiv:hep-th/9908005] Section 5.

[81] M. Henneaux and S.J. Rey,“Nonlinear W(infinity) algebra as asymptotic symmetry of three-dimensional higher spin Anti-de Sitter gravity,” JHEP 1012 (2010) 007 [arXiv:1008.4579 [hep-th]]

[82] M. Henneaux, G. Lucena Gomez, J. Park and S.-J. Rey,“Super- W(∞) asymptotic symmetry of higher-spin AdS3 supergravity,” JHEP 1206 (2012) 037 [arXiv:1203.5152 [hep-th]].

[83] A. Campoleoni, S. Fredenhagen and S. Pfenninger, Theisen, “Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields”, JHEP 1011 (2010) 007 [arXiv:1008.4744 [hep-th]] and “Asymptotic W-symmetries in three-dimensio- nal higher-spin gauge theories,” JHEP 1109 (2011) 113 [arXiv:1107.0290 [hep-th]].

[84] T. Basile, X. Bekaert, N. Boulanger“Flato-Fronsdal theorem for higher-order singletons”, arXiv:1410.7668 [hep-th]

[85] X. Bekaert, M. Grigoriev, “Higher order singletons, partially massless fields and their boundary values in the ambient approach”, arXiv:1305.0162 [hep-th]

[86] Work on progress.

[87] E. Joung, K. Mkrtchyan, “Notes on higher-spin algebras: minimal representations and structure constants”, arXiv:1401.7977 [hep-th].

[88] Y. M. Cho,“Higher dimensional unifications of gravitation and gauge theories”, J. Math. Phys. 16, 2029 (1975).

[89] S. N. Gupta, “Quantization of Einstein’s gravitational field: general treatment,” Proc. Phys. Soc. A **65** (1952) 608.

[90] R. P. Feynman, F. B. Morinigo, W. G. Wagner and B. Hatfield, “Feynman lectures on gravitation,” Reading, USA: Addison-Wesley (1995) 232 p. Original by California Institute of Technology 1963.

[91] V. I. Ogievetsky and I. V. Polubarinov, “Interacting field of spin 2 and the einstein equations”, Ann. Phys., NY **35** (1965) 167.

[92] R. M. Wald, “Spin-2 Fields and General Covariance,” Phys. Rev. D **33** (1986) 3613.

[93] C. Cutler and R. M. Wald, “A New Type of Gauge Invariance for a Collection of Massless Spin-2 Fields. 1. Existence and Uniqueness,” Class. Quant. Grav. **4** (1987) 1267.

[94] R. M. Wald, “A New Type of Gauge Invariance for a Collection of Massless Spin-2 Fields. 2. Geometrical Interpretation,” Class. Quant. Grav. **4** (1987) 1279.

[95] N. Boulanger, T. Damour, L. Gualtieri and M. Henneaux, “Inconsistency of interacting, multi-graviton theories,” Nucl. Phys. B **597** (2001) 127 [hep-th/0007220].

[96] M. A. Vasiliev, “Extended Higher Spin Superalgebras and Their Realizations in Terms of Quantum Operators,” Fortsch. Phys. **36** (1988) 33.

[97] E. S. Fradkin and M. A. Vasiliev, "Cubic Interaction in Extended Theories of Massless Higher Spin Fields," Nucl. Phys. B **291** (1987) 141.

[98] S. E. Konstein and M. A. Vasiliev, "Extended Higher Spin Superalgebras and Their Massless Representations," Nucl. Phys. B **331** (1990) 475.

[99] S. F. Prokushkin and M. A. Vasiliev, "Higher spin gauge interactions for massive matter fields in 3-D AdS space-time," Nucl. Phys. B **545** (1999) 385 [hep-th/9806236].

[100] L. P. S. Singh and C. R. Hagen, "Lagrangian formulation for arbitrary spin. I. The boson case", Phys. Rev. D 9, **898** (1974)

[101] S. Deser, A. Waldron, "Gauge Invariances and Phases of Massive Higher Spins in (A)dS", Phys Rev Lett. 87. 031601, hep-th/0102166, "Null Propagation of Partially Massless Higher Spins in (A)dS and Cosmological Constant Speculations", hep-th/0105181,

[102] Y . M. Zinoviev, "On Massive High Spin Particles in (A)dS," [arXiv:0108192 [hep-th]].

[103] M. Henneaux and S. J. Rey, "Nonlinear $W_{infinity}$ as Asymptotic Symmetry of Three-Dimensional Higher Spin Anti-de Sitter Gravity," JHEP **1012** (2010) 007 [arXiv:1008.4579 [hep-th]].

[104] M. Henneaux, G. Lucena GÃ³mez, J. Park and S. J. Rey, "Super- W(infinity) Asymptotic Symmetry of Higher-Spin AdS_3 Supergravity," JHEP **1206** (2012) 037 [arXiv:1203.5152 [hep-th]].

[105] I. Bars and S. J. Rey, "Noncommutative Sp(2,R) gauge theories: A Field theory approach to two time physics," Phys. Rev. D **64** (2001) 046005 [hep-th/0104135].

[106] S. R. Das and S. J. Rey, "Dilaton Condensates and Loop Effects in Open and Closed Bosonic Strings," Phys. Lett. B **186** (1987) 328.

[107] M. Gary, D. Grumiller, S. Prohazka and S. J. Rey, "Lifshitz Holography with Isotropic Scale Invariance," JHEP **1408** (2014) 001 [arXiv:1406.1468 [hep-th]].

[108] A. LeClair and M. Neubert, "Semi-Lorentz invariance, unitarity, and critical exponents of symplectic fermion models," JHEP **0710** (2007) 027 [arXiv:0705.4657 [hep-th]].

[109] D. J. Robinson, E. Kapit and A. LeClair, "Lorentz Symmetric Quantum Field Theory for Symplectic Fermions," J. Math. Phys. **50** (2009) 112301 [arXiv:0903.2399 [hep-th]].

[110] D. Anninos, T. Hartman and A. Strominger, "Higher Spin Realization of the dS/CFT Correspondence," arXiv:1108.5735 [hep-th].

[111] P. Ouyang, "Toward Higher Spin dS3/CFT2," arXiv:1111.0276 [hep-th].

[112] D. Das, S. R. Das, A. Jevicki and Q. Ye, "Bi-local Construction of Sp(2N)/dS Higher Spin Correspondence," JHEP **1301** (2013) 107 [arXiv:1205.5776 [hep-th]].

[113] J. B. Bae and S. J. Rey, "Conformal Bootstrap Approach to O(N) Fixed Points in Five Dimensions," arXiv:1412.6549 [hep-th].

[114] C. Vafa, "Non-Unitary Holography," arXiv:1409.1603 [hep-th].

[115] A. M. Polyakov, "Decay of Vacuum Energy," Nucl. Phys. B **834** (2010) 316 [arXiv:0912.5503 [hep-th]].

[116] T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, *Modified Gravity and Cosmology*, Phys. Rept. **513** (2012) 1 [arXiv:1106.2476 [astro-ph.CO]];
A. De Felice and S. Tsujikawa, *$f(R)$ Theories*, Living Rev. Rel. **13** (2010) 3 [arXiv:1002.4928 [gr-qc]];
S. Capozziello and M. De Laurentis, *Extended Theories of Gravity*, Phys. Rept. **509** (2011) 167 [arXiv:1108.6266 [gr-qc]];
D. G. Boulware and S. Deser, *Can gravitation have a finite range?*, Phys. Rev. D **6** (1972) 3368;
P. Creminelli, A. Nicolis, M. Papucci and E. Trincherini, *Ghosts in Massive Gravity*, JHEP **0509** (2005) 003 [hep-th/0505147];
E. Joung, W. Li and M. Taronna, *No-Go Theorems for Unitary and Interacting partially-massless Spin-Two Fields*, Phys. Rev. Lett. **113** (2014) 091101 [arXiv:1406.2335 [hep-th]].

[117] C. de Rham and G. Gabadadze, *Generalization of the Fierz-Pauli Action*, Phys. Rev. D **82** (2010) 044020 [arXiv:1007.0443 [hep-th]];
C. de Rham, G. Gabadadze and A. J. Tolley, *Resummation of Massive Gravity*, Phys. Rev. Lett. **106** (2011) 231101 [arXiv:1011.1232 [hep-th]];
S. F. Hassan and R. A. Rosen, *Resolving the Ghost Problem in non-Linear Massive Gravity*, Phys. Rev. Lett. **108** (2012) 041101 [arXiv:1106.3344 [hep-th]], *Confirmation of the Secondary Constraint and Absence of Ghost in Massive Gravity and Bimetric Gravity*, JHEP **1204** (2012) 123 [arXiv:1111.2070 [hep-th]].

[118] E. A. Bergshoeff, O. Hohm and P. K. Townsend, *Massive Gravity in Three Dimensions*, Phys. Rev. Lett. **102** (2009) 201301 [arXiv:0901.1766 [hep-th]];
E. Bergshoeff, W. Merbis, A. J. Routh and P. K. Townsend, *The Third Way to 3D Gravity*, arXiv:1506.05949 [gr-qc].

[119] X. O. Camanho, J. D. Edelstein, J. Maldacena and A. Zhiboedov, *Causality Constraints on Corrections to the Graviton Three-Point Coupling*, arXiv:1407.5597 [hep-th];
H. Lu, A. Perkins, C. N. Pope and K. S. Stelle, *Black Holes in Higher-Derivative Gravity*, Phys. Rev. Lett. **114** (2015) 17, 171601 [arXiv:1502.01028 [hep-th]].

[120] R. M. Wald, *Spin-2 Fields and General Covariance*, Phys. Rev. D **33** (1986) 3613;
C. Cutler and R. M. Wald, *A New Type of Gauge Invariance for a Collection of Massless Spin-2 Fields. 1. Existence and Uniqueness*, Class. Quant. Grav. **4** (1987) 1267;
R. M. Wald, *A New Type of Gauge Invariance for a Collection of Massless Spin-2 Fields. 2. Geometrical Interpretation*, Class. Quant. Grav. **4** (1987) 1279.

[121] N. Boulanger, T. Damour, L. Gualtieri and M. Henneaux, *Inconsistency of interacting, multi-graviton theories*, Nucl. Phys. B **597** (2001) 127 [hep-th/0007220].

[122] M. A. Vasiliev, *Consistent Equation for Interacting gauge Fields of All Spins in (3+1)-Dimensions*, Phys. Lett. B **243** (1990) 378;

S. F. Prokushkin and M. A. Vasiliev, *Higher Spin Gauge Interactions for Massive Matter Fields in 3-D AdS Spacetime*, Nucl. Phys. B **545** (1999) 385 [hep-th/9806236];
M. A. Vasiliev, *Nonlinear Equations for Symmetric Massless Higher Spin Fields in (A)dS(d)*, Phys. Lett. B **567** (2003) 139 [hep-th/0304049].

[123] M. A. Vasiliev, *Extended Higher Spin Superalgebras and Their Realizations in Terms of Quantum Operators*, Fortsch. Phys. **36** (1988) 33;
S. E. Konshtein and M. A. Vasiliev, *Massless Representations and Admissibility Condition for Higher Spin Superalgebras*, Nucl. Phys. B **312** (1989) 402;
M. A. Vasiliev, *Consistent Equations for Interacting Massless Fields of All Spins in the First Order in Curvatures*, Annals Phys. **190** (1989) 59;
S. E. Konstein and M. A. Vasiliev, *Extended Higher Spin Superalgebras and Their Massless Representations*, Nucl. Phys. B **331** (1990) 475;
M. A. Vasiliev, *Higher Spin Superalgebras in Any Dimension and Their Representations*, JHEP **0412** (2004) 046 [hep-th/0404124].

[124] A. Achucarro and P. K. Townsend, "A Chern-Simons Action for Three-Dimensional anti-De Sitter Supergravity Theories," Phys. Lett. B **180** (1986) 89;
E. Witten, "(2+1)-Dimensional Gravity as an Exactly Soluble System," Nucl. Phys. B **311** (1988) 46.

[125] M. Banados, C. Teitelboim and J. Zanelli, *The Black Hole in Three-Dimensional Spacetime*, Phys. Rev. Lett. **69** (1992) 1849 [hep-th/9204099].

[126] S. Deser, A. Waldron, *Gauge Invariances and Phases of Massive Higher Spins in (A)dS*, Phys Rev Lett. 87. 031601, [hep-th/0102166], *Null Propagation of partially-massless Higher Spins in (A)dS and Cosmological Constant Speculations*, Phys. Lett. B **513**, 137 (2001), [hep-th/0105181].

[127] S. N. Gupta, *Quantization of Einstein's Gravitational Field: General Treatment*, Proc. Phys. Soc. A **65** (1952) 608;
R. H. Kraichnan, "Special-Relativistic Derivation of Generally Covariant Gravitation Theory," Phys. Rev. **98** (1955) 1118;
V. I. Ogievetsky and I. V. Polubarinov, *Interacting Field of Spin 2 and the Einstein Equations*, Ann. Phys., NY **35** (1965) 167;
S. Deser, "Selfinteraction and gauge invariance," Gen. Rel. Grav. **1** (1970) 9 [gr-qc/0411023];
R. P. Feynman, F. B. Morinigo, W. G. Wagner and B. Hatfield, *Feynman Lectures on Gravitation*, Reading, USA: Addison-Wesley (1995) 232 p. Original by California Institute of Technology 1963.

[128] N. Boulanger and L. Gualtieri, *An Exotic Theory of Massless Spin Two Fields in Three-Dimensions*, Class. Quant. Grav. **18** (2001) 1485 [hep-th/0012003];
S. C. Anco, *Parity Violating Spin-Two Gauge Theories*, Phys. Rev. D **67** (2003) 124007 [gr-qc/0305026].

[129] C. Aragone and S. Deser, *Consistency Problems of Hypergravity*, Phys. Lett. B **86** (1979) 161, *Consistency Problems of Spin-2 Gravity Coupling*, Nuovo Cim. B **57** (1980) 33.

[130] E. S. Fradkin and M. A. Vasiliev, *On the Gravitational Interaction of Massless Higher Spin Fields*, Phys. Lett. B **189** (1987) 89, *Cubic Interaction in Extended Theories of Massless Higher Spin Fields*, Nucl. Phys. B **291** (1987) 141.

[131] E. Joung and M. Taronna, *Cubic-Interaction-Induced Deformations of Higher-Spin Symmetries*, JHEP **1403** (2014) 103 [arXiv:1311.0242 [hep-th]].

[132] S. Deser, E. Joung and A. Waldron, *Gravitational- and Self- Coupling of partially massless Spin 2*, Phys. Rev. D **86** (2012) 104004 [arXiv:1301.4181 [hep-th]], *Partial Masslessness and Conformal Gravity*, J. Phys. A **46** (2013) 214019 [arXiv:1208.1307 [hep-th]].

[133] E. Joung and K. Mkrtchyan, *Partially-Massless Higher-Spin Algebras and Their Finite-Dimensional Truncations*, arXiv:1508.07332 [hep-th].

[134] M. Henneaux and S. J. Rey, *Nonlinear W_∞ as Asymptotic Symmetry of Three-Dimensional Higher Spin Anti-de Sitter Gravity*, JHEP **1012** (2010) 007 [arXiv:1008.4579 [hep-th]];
A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, *Asymptotic Symmetries of Three-Dimensional Gravity Coupled to Higher-Spin Fields*, JHEP **1011** (2010) 007 [arXiv:1008.4744 [hep-th]].

[135] M. Henneaux, G. Lucena Gamez, J. Park and S. J. Rey, *Super-$W(\infty)$ Asymptotic Symmetry of Higher-Spin AdS_3 Supergravity*, JHEP **1206** (2012) 037 [arXiv:1203.5152 [hep-th]].

[136] M. Gary, D. Grumiller, S. Prohazka and S. J. Rey, *Lifshitz Holography with Isotropic Scale Invariance*, JHEP **1408** (2014) 001 [arXiv:1406.1468 [hep-th]].

[137] H. Afshar, A. Bagchi, R. Fareghbal, D. Grumiller and J. Rosseel, *Spin-3 Gravity in Three-Dimensional Flat Space*, Phys. Rev. Lett. **111** (2013) 12, 121603 [arXiv:1307.4768 [hep-th]];
H. A. Gonzalez, J. Matulich, M. Pino and R. Troncoso, "Asymptotically flat spacetimes in three-dimensional higher spin gravity," JHEP **1309** (2013) 016 [arXiv:1307.5651 [hep-th]].

[138] O. Aharony, M. Berkooz and S. J. Rey, *Rigid holography and six-dimensional $\mathcal{N} = (2,0)$ theories on AdS$_5 \times \mathbb{S}^1$*, JHEP **1503** (2015) 121
[arXiv:1501.02904 [hep-th]].

[139] I. Bars and S. J. Rey, *Noncommutative Sp(2,R) Gauge Theories: A Field Theory Approach to Two Time Physics*, Phys. Rev. D **64** (2001) 046005 [hep-th/0104135].

[140] R. Bonezzi, O. Corradini, E. Latini and A. Waldron, *Quantum Gravity and Causal Structures: Second Quantization of Conformal Dirac Algebras*, Phys. Rev. D **91** (2015) 12, 121501 [arXiv:1505.01013 [hep-th]].

[141] A. A. Starobinsky, *Stochastic De Sitter (inflationary) Stage In The Early Universe*, Lect. Notes Phys. **246** (1986) 107.

[142] S. J. Rey, *Dynamics of Inflationary Phase Transition*, Nucl. Phys. B **284** (1987) 706.

[143] C. Aragone and A. Khoudeir, *Selfdual Massive Gravity*, Phys. Lett. B **173**, 141 (1986).

[144] I. Klebanov and A. Polyakov, *AdS dual of the critical O(N) vector model*, *Phys.Lett.* **B550** (2002) 213–219, [hep-th/0210114].

[145] E. Sezgin and P. Sundell, *Massless higher spins and holography*, *Nucl.Phys.* **B644** (2002) 303–370, [hep-th/0205131].

[146] M. Henneaux and S.-J. Rey, *Nonlinear $W_{infinity}$ as Asymptotic Symmetry of Three-Dimensional Higher Spin Anti-de Sitter Gravity*, *JHEP* **1012** (2010) 007, [arXiv:1008.4579].

[147] A. Campoleoni, S. Fredenhagen, S. Pfenninger, and S. Theisen, *Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields*, *JHEP* **1011** (2010) 007, [arXiv:1008.4744].

[148] M. R. Gaberdiel and R. Gopakumar, *An AdS_3 Dual for Minimal Model CFTs*, Phys. Rev. D **83**, 066007 (2011), [arXiv:1011.2986].

[149] M. Ammon, M. Gutperle, P. Kraus, and E. Perlmutter, *Black holes in three dimensional higher spin gravity: A review*, J. Phys. A **46**, 214001 (2013), [arXiv:1208.5182].

[150] M. P. Blencowe, *A Consistent Interacting Massless Higher Spin Field Theory in D = (2+1),* Class. Quant. Grav. **6** (1989) 443.

[151] M. A. Vasiliev, *Equations of motion for d = 3 massless fields interacting through Chern-Simons higher spin gauge fields,* Mod. Phys. Lett. A **7** (1992) 3689.

[152] S. F. Prokushkin and M. A. Vasiliev, *Higher spin gauge interactions for massive matter fields in 3-D AdS space-time*, *Nucl. Phys.* **B545** (1999) 385, [hep-th/9806236].

[153] H. Afshar, A. Bagchi, R. Fareghbal, D. Grumiller, and J. Rosseel, *Spin-3 Gravity in Three-Dimensional Flat Space*, *Phys. Rev. Lett.* **111** (2013), no. 12 121603, [arXiv:1307.4768].

[154] H. A. Gonzalez, J. Matulich, M. Pino and R. Troncoso, *Asymptotically flat spacetimes in three-dimensional higher spin gravity*, *JHEP* **1309** (2013) 016, [arXiv:1307.5651].

[155] A. Campoleoni, S. Fredenhagen, S. Pfenninger, and S. Theisen, *Towards metric-like higher-spin gauge theories in three dimensions*, *J. Phys.* **A46** (2013) 214017, [arXiv:1208.1851].

[156] N. Boulanger, D. Ponomarev, E. Sezgin, and P. Sundell, *New unfolded higher spin systems in AdS_3*, *Class. Quant. Grav.* **32** (2015), no. 15 155002, [arXiv:1412.8209].

[157] A. Campoleoni and M. Henneaux, *Asymptotic symmetries of three-dimensional higher-spin gravity: the metric approach*, *JHEP* **03** (2015) 143, [arXiv:1412.6774].

[158] J. de Boer and J. I. Jottar, *Entanglement Entropy and Higher Spin Holography in AdS_3*, *JHEP* **04** (2014) 089, [arXiv:1306.4347].

[159] I. L. Buchbinder, T. V. Snegirev, and Yu. M. Zinoviev, *On gravitational interactions for massive higher spins in AdS_3*, *J. Phys.* **A46** (2013) 214015, [arXiv:1208.0183].

[160] S. Gwak, E. Joung, K. Mkrtchyan, and S.-J. Rey, *Rainbow Valley of Colored (Anti) de Sitter Gravity in Three Dimensions*, arXiv:1511.05220.

[161] M. A. Vasiliev, *Extended Higher Spin Superalgebras and Their Realizations in Terms of Quantum Operators*, *Fortsch. Phys.* **36** (1988) 33–62.

[162] S. Konstein and M. A. Vasiliev, *Extended Higher Spin Superalgebras and Their Massless Representations*, *Nucl.Phys.* **B331** (1990) 475–499.

[163] M. A. Vasiliev, *Higher spin superalgebras in any dimension and their representations*, *JHEP* **12** (2004) 046, [hep-th/0404124].

[164] S. Deser and A. Waldron, *Null propagation of partially massless higher spins in (A)dS and cosmological constant speculations*, *Phys. Lett.* **B513** (2001) 137–141, [hep-th/0105181].

[165] M. A. Vasiliev, *Higher Spin Algebras and Quantization on the Sphere and Hyperboloid*, *Int. J. Mod. Phys.* **A6** (1991) 1115-1135.

[166] E. Joung and K. Mkrtchyan, *Notes on higher-spin algebras: minimal representations and structure constants*, *JHEP* **05** (2014) 103, [hep-th/1401.7977].

[167] S. Gwak, E. Joung and S.-J. Rey, to appear.

[168] Yu. M. Zinoviev, *On massive high spin particles in AdS*, [hep-th/0108192].

[169] S. Deser and R. I. Nepomechie, *Anomalous Propagation of Gauge Fields in Conformally Flat Spaces*, Phys. Lett. B **132** (1983) 321.

[170] S. Deser and R. I. Nepomechie, *Gauge Invariance Versus Masslessness in De Sitter Space*, Annals Phys. **154** (1984) 396.

[171] S. Deser and A. Waldron, *Partial masslessness of higher spins in (A)dS*, Nucl. Phys. B **607** (2001) 577 [hep-th/0103198].

[172] E. D. Skvortsov and M. A. Vasiliev, *Geometric formulation for partially massless fields*, Nucl. Phys. B **756** (2006) 117 [hep-th/0601095].

[173] L. Dolan, C. R. Nappi and E. Witten, *Conformal operators for partially massless states*, JHEP **0110** (2001) 016 [hep-th/0109096].

[174] P. K. Townsend, K. Pilch and P. van Nieuwenhuizen, *Selfduality in Odd Dimensions,* Phys. Lett. B **136** (1984) 38 [Phys. Lett. B **137** (1984) 443].

[175] Harish-Chandra, *Infinite irreducible representations of the Lorentz group,* Proc. R. Soc. **A189** 1018 (1947) 372-401.

[176] R. Camporesi and A. Higuchi, *Arbitrary spin effective potentials in anti-de Sitter space-time,* Phys. Rev. D **47** (1993) 3339; *Spectral functions and zeta functions in hyperbolic spaces,* J. Math. Phys. **35** (1994) 4217.

초 록

양자 장론의 발전 초기 단계에서 부터 질량이 없고 고차 스핀을 가지는 입자의 존재에 대해서 알려져 있었다. 하지만 흥미로운 이론을 찾기가 어려웠는데, 그 이유는 다른 입자와 상호 작용을 하는 이론을 구성하는데 어려움이 있었기 때문이다. 최근에 다양한 이론들이 제시되었다. 이 논문에서는 여러가지 어려움들과 최근의 발전 상황을 소개하고, 새로운 상호작용하는 이론을 찾으려는 저자의 노력에 대해 정리할 것이다.

주요어 : 고차 스핀, 고차 스핀 게이지, 칼루자 클라인, 경계가 있는 칼루자 클라인, 반 디지터 공간, 색이 있는 중력, 찬-패톤 팩터
학번 : 2008-20419