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Accelerated Life Testing Data Analysis using Linear Mixed-effects Model

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지도교수 이 영 조

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위 원 장 이 재 용 (인)

부위원장 이 영 조 (인)

위 원 장 원 철 (인)

Accelerated Life Time Testing Data Analysis using Linear Mixed-effects Model

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Eun Hee Hong

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College of National Sciences
Seoul National University
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Abstract

Eun Hee Hong
The Department of Statistics
The Graduate School
Seoul National University

In repeated measurement experiment, normal linear mixed-effects model is widely used to analysis data. However, in case of detecting heteroscedasticity and the non-normality of the population distribution at the same time, normal linear mixed-effects model can give improper result of analysis. To achieve more robust estimation, we used heavy-tailed linear mixed-effects model which gives more exact and reliable analysis conclusion than standard normal linear mixed-effects model. Furthermore, we will provide the results of reliability analysis as the further work.

Keywords : Repeated measurement experiment, Linear mixed-effects model, Heavy-tailed linear mixed-effects model, Reliability analysis.

Student Number : 2012-20239

Contents

Chapter 1

Introduction

Many statisticians and data analysts use linear mixed-effects model (Laird and Ware (1982)) to analyze repeated measures data. Random effects is powerful tool to consider the variety between groups or subjects. Also, it measures the difference between the average value of the response variable at each group or subject and the average value in the upper class to which groups belong.

In linear mixed-effects model, random effects and within-subject errors are usually normally distributed. However, in case that the normal assumptions are questionable, that is, the data set exhibits nonstandard characteristics like skewness and heavy tails or outliers, the distribution of the random effects may introduce biases to parameter estimates and lead to incorrect inference or increase or decrease the power of statistical tests. To deal with these difficulties, Lange and Shinsheimer (1993) showed the use of a symmetric normal/independent distributions for random effects gives a robust inference of linear mixed-effects model. Branco and Dey (2001) assumed that the random effects follow a multivariate skew-normal/independent (SNI) distributions. The SNI distribution includes skew-t, skew-slash and skew-contaminated normal distributions as special cases.

Wakefield et al. (1994) introduced the t distribution for the random effects in Bayesian setting, However, this methodology is questionable that the sensitivity in parameter estimation of random effects models could be avoided by using heavy-tailed distribution. Lee and Nelder (2004a) introduced a systematic way of modeling heavy-tailed distributions for response variables, leading that the resulting model gives the robust estimation and inference.

The aim of this paper is to show the data analysis in this robust estimation literature. we used a real data set which was derived from an acceleration experiment for fitting various linear mixed-effects models. The rest of this paper is organized in following orders. In Chapter 2, we describe the data to analyze. In Chapter 3, we exhibit the model fitting procedure and the consequential estimation result. Also, we review the usefulness of random effect in linear mixed-effects model which is the best way to describe the heterogeneity between groups or subjects. In Chapter 4, we provide the results of a reliability analysis as the further work. Discussion and conclusion are given in Chapter 5. All the results and model-checking plots of the article was obtained using the R package dhglm (Noh and Lee, 2011) in the Comprehensive R Archive Network (CRAN).

Chapter 2

Data Description

In this section, we illustrate a real data actually used to analyze. An acceleration experiment is conducted for a single type of tire. For this, the tire was mounted on different five kinds of cars. After driving, engineers record the mileages for the front wheels in every acceleration experiment. They carry same experiment out repeatedly over four times. There are thirty-nine observations because the values corresponding to the right front tire of the fifth car in the last acceleration experiment is missing. Since the acceleration experiment was not conducted until the tire is no longer of any use, we predicted tire life expectancy by fitting heavy-tailed linear mixed-effects model for this data. This will be illustrated in Chapter 4.

The data set consists of total four variables: 'id' variable means the kind of car. 'location' variable means the location of front tire on each type of car. They are categorical variables for a response variable. 'mileage' and 'depth' as continuous variables having positive numerical values indicate respectively the mileages (km) of the cars and tire groove depths (mm) in each acceleration experiment. The values corresponding to 'depth' on all kinds of car is gradually reduced as the experiment progresses as shown in Figures 2.1 and 2.2.

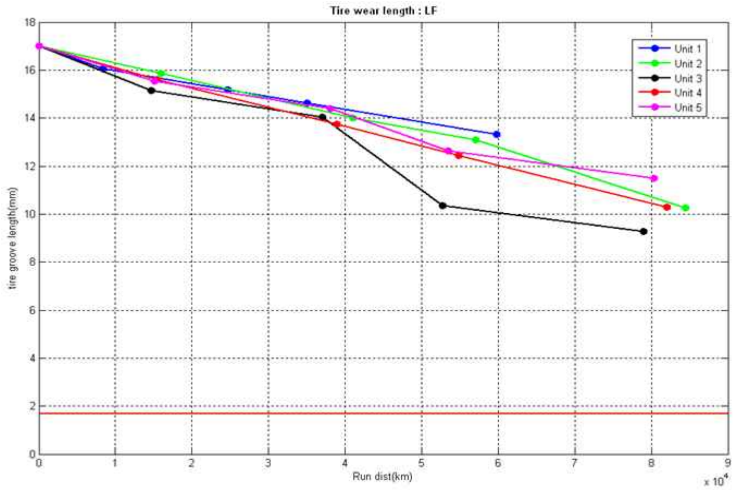

Figure 2.1: The plot about the wear level of left side tire at the front of cars. The vertical line corresponds tire groove depth (mm) on each acceleration experiment. The horizon line corresponds the mileage (km) of cars.

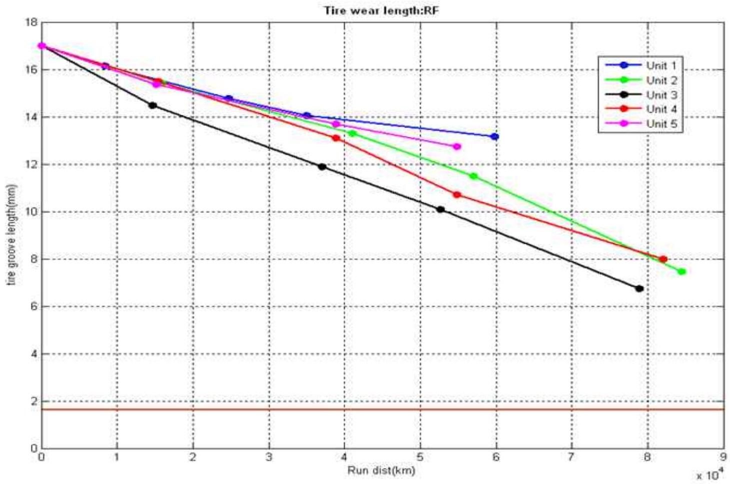

Figure 2.2: The plot about the level of right side tire wear at the front of cars. The y axis is tire groove depth (mm) and the x axis is mileages (km).

Figure 2.3: Tire figures

It is also commonly known that is the end of life of tire when tire groove depths leach to 1.7mm. The following figure displays the pattern of the data according to the kinds of car that are groups.

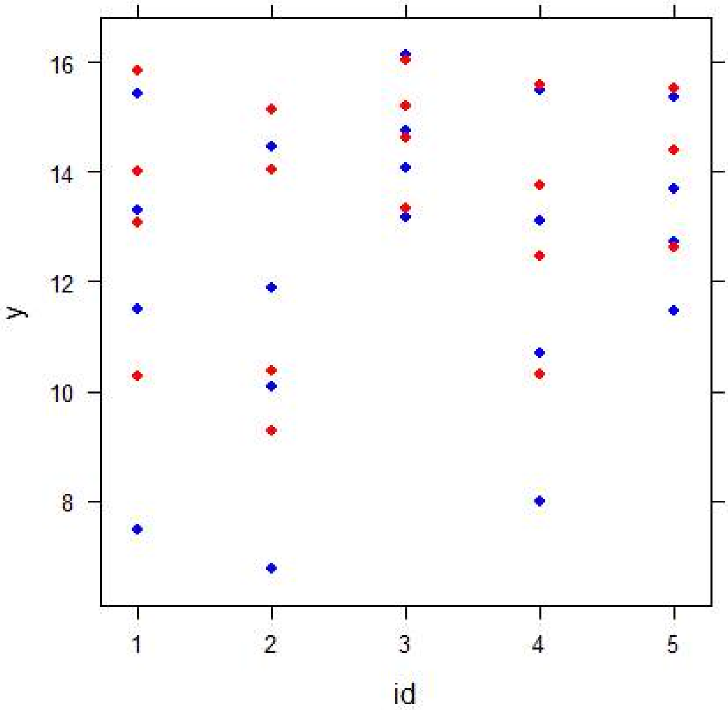

Figure 2.3: The plot describing the tire wear data: the blue point corresponds the left front tire and the red point corresponds the right front tire. The tire groove depth of fifth car on the last experiment is missing.

The vertical line on the plot in Figure 2.3 means tire groove depth on each acceleration experiment. The horizon line on the plot means the kind of car consisted of five groups. The blue point and red point means respectively the left front tire and right front tire on each type of car. As shown in the Figure 2.3, it seems very clear there is the difference of level of tire wear among the cars. That is, analysts may have to consider the kind of car as the random effects in the linear mixed-effects model. we will examine the value of consideration of random effects in chapter 3 by a REML-based likelihood ratio test for random effects using a 50:50 mixture of chi-square distributions (Verbeke and Molenberghs, 2003).

Chapter 3

Data Analysis

In this chapter, we will illustrate the model-fitting procedure. We also show how to find the linear mixed-effects model that best fits the data via the model checking plots in R package dhglm (Noh and Lee, 2011).

3.1 Model-fitting procedure

First, we implemented the simple normal linear mixed-effects model to the data which was described in Chapter 2. The model is described as follows:

$$depth_{ij} = \beta_0 + \beta_1 mileage_{1ij} + \beta_2 I(location = right)_{2ij} + v_i + e_{ij}, \qquad (3.1)$$

$$\text{where, } v_i \sim N(0, \sigma_v^2), e_{ij} \sim N(0, \sigma_e^2), i = 1, \ldots, 5, j = 1, \ldots, 4,$$

where *I()* is the indicator function which have one if the location of front tire on the car is right side. Response variable is the final groove depth of tire in each acceleration experiment. 'mileage' variable and 'location' variable are used to explanatory variables for fitting normal linear mixed-effects model (3.1). Here, v_i is the random effects for the kind of car and e_{ij} is the residual errors. Also, both random components are normally distributed.

Model-checking plot of this model is presented in Figure 3.1. As shown in the top left panel on Figure 3.1, homoscedasticity of the fitted model is violated. To achieve the equalization of variance, we tried to transform the variable in various ways. The most suitable model for homoscedasticity is the power transformation of response variable, defined as

$$depth_{ij}^2 = \beta_0 + \beta_1 mileage_{1ij} + \beta_2 I(location = right)_{2ij} + v_i + e_{ij}, \qquad (3.2)$$

$$\text{where, } v_i \sim N(0, \sigma_v^2), e_{ij} \sim N(0, \sigma_e^2), \; i = 1, \ldots, 5, j = 1, \ldots, 4.$$

The distributions of random effects and residual errors are the same as (2.1).

The Figure 3.2 exhibits the model-checking plot of the transformed model. It seems that the homogeneity of variances assumption for the errors is well satisfied. However, the Q-Q plot in Figure 3.2 indicates the nonnormality of the population distribution. The large positive and negative outliers are presented in the Q-Q plot. This means the goodness of fit test of transformed linear mixed-effects model may not be the most powerful or informative test avaliable.

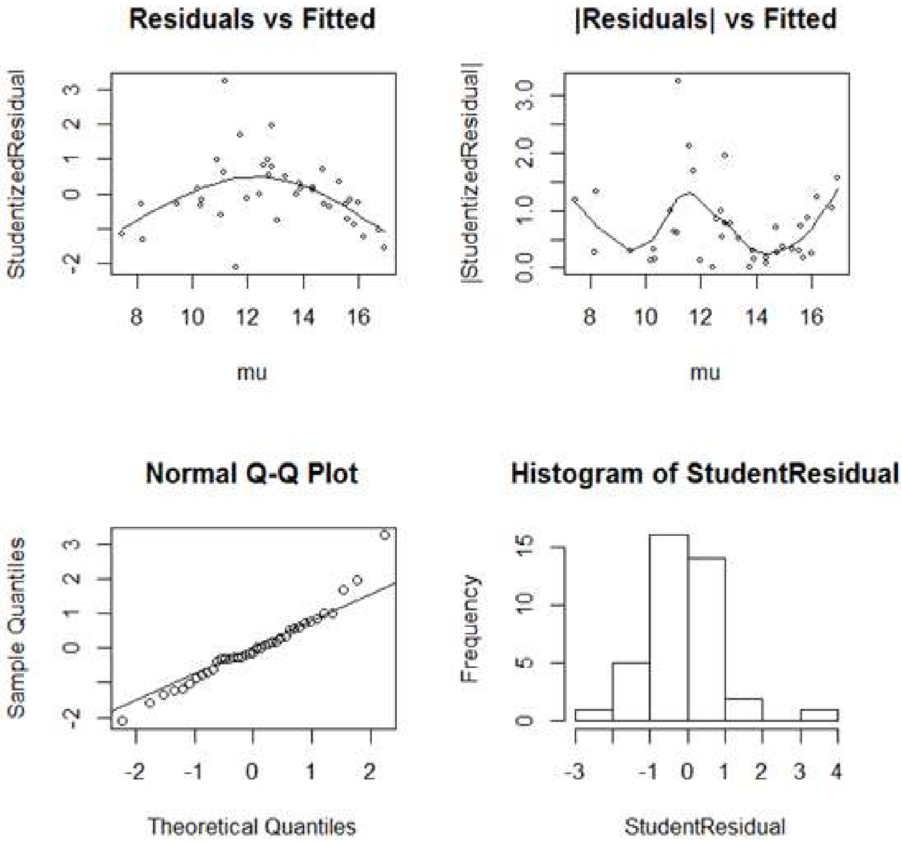

Figure 3.1: The panel on the top left side in the figure is the plot of studentized residuals versus the fitted means. The left side panel at the bottom is the normal Q-Q plot which provides a graphical way to determine the level of normality.

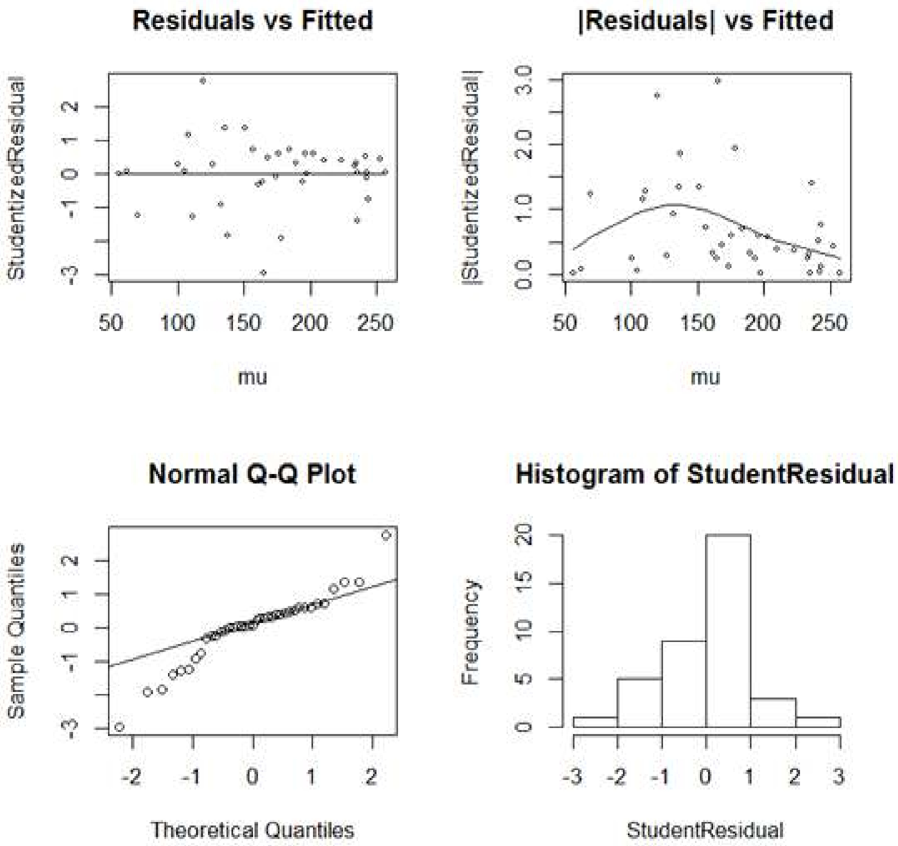

Figure 3.2: The model-checking plot of the transformed model (3.2) for homoscedasticity.

Lee and Nelder (2006) introduced various heavy-tailed distributions. Subsequently, Noh and Lee (2007) presented residual variances allowing the random effects, and showed the resulting estimators are robust against outliers. Furthermore, it is found that if the dispersion parameters corresponding to degrees of freedom in the t-distribution are estimated, the resulting estimators are consistent in the original normal distribution. Inclusion of random effects in the dispersion component enables the modelling of both skewness and kurtosis, and so provide the estimators that are less sensitive to the distributional assumption on the fitted model. This conception is well described in the article, Lee et al. (2006). Therefore, we consider the following heavy-tailed linear mixed-effects model to achieve robust estimation and inference:

$$depth_{ij}^2 = \beta_0 + \beta_1 mileage_{1ij} + \beta_2 I(location = right) + v_i + e_{ij}, \tag{3.3}$$

where $e_{ij} = \sigma_{ij} z_{ij}$, $z_{ij} \sim N(0,1)$ and $\phi_{ij} = \sigma_{ij}^2$.

$$\text{Here } \log\phi_{ij} = \gamma + b_i, \tag{3.4}$$

where $b_i \sim N(0, \tau)$.

The kurtosis of e_{ij} or y_{ij} is $3E(\phi_{ij}^2)/E(\phi_{ij})^2 \geq 3$, where the equality holds if and only if ϕ_{ij} are fixed constants. By introducing random effects b_i in the dispersion we can describe that of dispersions between groups. These can in turn describe abrupt changes among repeated measures. If also $a_i = \exp(b_i)\ k/\chi_k^2$, the error term $e_i = (e_{i1}, \ldots, e_n)^t$ follows a multivariate t-distribution (Lange, et al, 1989). When k equals one this becomes a Cauchy distribution. Thus, by allowing random effects in dispersion components, we can allow various heavy-tailed linear mixed-effects model. We assumed that e_{ij} follows the t-distribution with four degrees of freedom for fitting model (3.3) and (3.4). Model-checking plot for this heavy-tailed linear mixed-effects model (3.3) and (3.4) is presented in Figure 3.3.

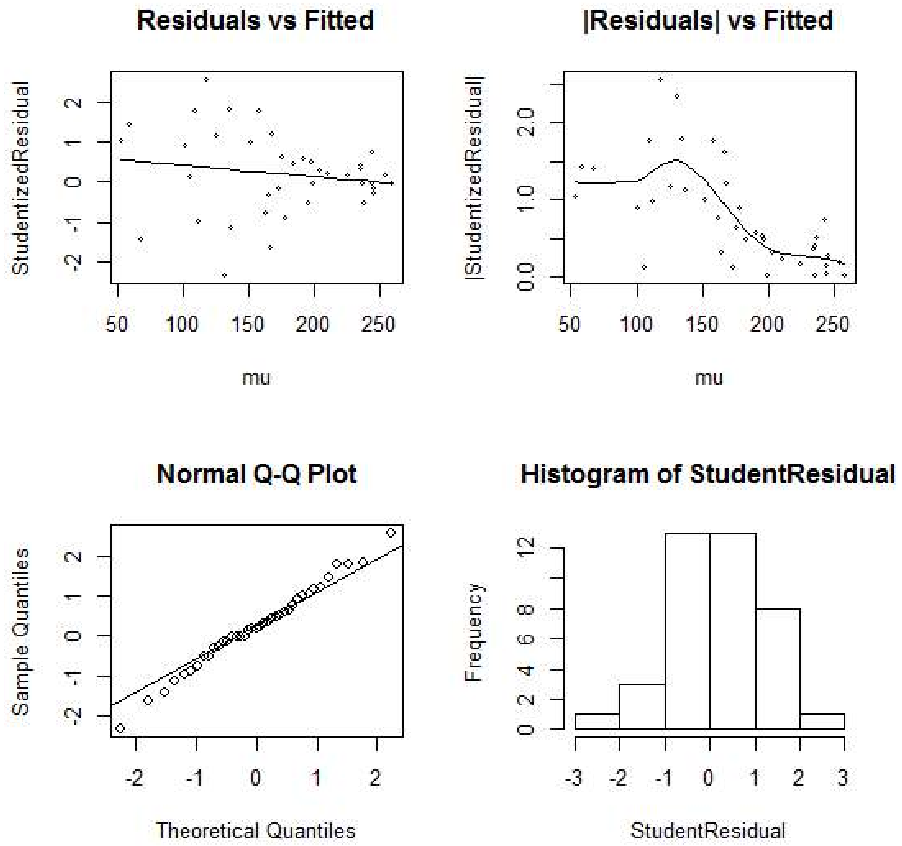

Figure 3.3: The residual plots of heavy-tailed linear mixed-effects model (3.3) and (3.4).

This model-checking plot shows the assumptions of fitted model is well satisfied, as compared the standard normal linear-mixed effect model (3.1) and transformed linear mixed-effects model (3.2). Table 3.1 shows the estimation results of models (3.3) and (3.4).

Table 3.1: The estimation results of models (3.3) and (3.4).

Variable	Estimate	Std. Error	t-value	p-value
$intercept$	279.234	4.8280	57.84	<0.0001
$mileage$	−20.632	0.7646	−26.99	<0.0001
$I(location = right)$	−5.796	4.3320	−13.38	<0.0001
$\log(\sigma_v^2)$	−2.180	0.6633		
γ	0.0221	0.2237		
$\log\tau$	−0.7975	0.5106		

From Table 3.1, we found that all of the covariates are statistically highly significant.

3.2 Testing significance of random effects

The aim of this section is to examine whether the inclusion of random effects improves the predictions from the model (3.3) and (3.4) or not. We tested a hypothesis related to the variance of the random effects. The hypotheses are described as

$$H_0 : \sigma_v^2 = 0, H_A : \sigma_v^2 > 0 \tag{3.5}$$

The null hypothesis means there are no variations between the cars. On the contrary to this, the alternative hypotheses indicates there are variations between the cars. That is, we might have to consider the effects across the kind of the car in the linear mixed-effects model.

To test hypothesis (3.5), we used a REML-based likelihood ratio test. The test statistic was calculated by subtracting the −2REML log-likelihood value for the reference model (3.3) and (3.4) including the random effects v_i from that for the nested model (3.3) and (3.4) in which the random effects are omitted from unstructured G matrix (Case3 in Verbeke and Molengerghs 2000, p.71). This difference is calculated as 22.7. Because the null hypothesis value for the variance of the random effects is on the boundary of the parameter space that is zero, the asymptotic null distribution of test statistic is a mixture of χ_0^2 and χ_1^2 distributions, each with equal weights of 0.5 (Verbeke and Molengerghs, 2000). To evaluate the significance of the test, we calculated the p-value=$0.5 \times P(\chi_0^2 \geq 22.7) + 0.5 \times P(\chi_1^2 \geq 22.7) = 0.5 \times P(\chi_1^2 \geq 22.7) < .0001$. From the result of p-value, we reject the null hypothesis and retain the random effects in model (3.3) and (3.4). Accordingly, this would mean that adding a random effect term to heavy-tailed linear mixed-effects model (3.3) and (3.4) gives a very significant improvement.

Chapter 4

Reliability Analysis

In this chapter, we will show the results of reliability analysis as the further work.

In general, it is well-known that the main goal of reliability analysis is to predict the life time of product. Therefore, we also predicted the life time of both sides of a tire at the front of the cars in the light of the random effects since the random effects in the heavy-tailed linear mixed-effects model (3.3) and (3.4) are statistically highly significant as shown in the Section 3.3.

It is normally known that the tires on a car are interchanged when the groove depth of tire reaches to 1.7 (mm). Therefore, when the value of the response variable 'depth' is 1.7 (mm) in fitted model (3.3) and (3.4), the values corresponding to 'mileage' in each car are the expected life times of a tire of cars. Tables 4.1 and 4.2 show respectively the fitted random effects and the predicted life time of both sides of a tire at the front of the cars.

Table 4.1: The fitted random effects.

id	$\hat{v}_i$
1	5.5494
2	-26.6156
3	15.6829
4	-3.1972
5	8.5805

Table 4.2: The predicted life time of both sides of a tire at the front of the cars. the unit is km.

id	left	right	left-right
1	215,035	167,873	47,162
2	199,445	155,702	43,743
3	219,946	171,707	48,239
4	210,795	164,563	46,232
5	216,504	169,020	47,484

Note that the third car shows the biggest wear difference between both sides of the front tire, while the second car show the lowest wear difference between both sides of the front tire.

Chapter 5

Conclusion and Discussion

We have found the linear mixed-effects model which best illustrates the data by using model-checking plot in the R package dhglm (Noh and Lee, 2011).

Model-checking plot gives a graphical way to easily evaluate model assumptions. Detecting heteroscedasticity and the nonormarlity of population distribution at the same time, we could find linar mixed-effects model which well satisfy standard assumptions about the random errors in turn by making the power transformation of response variable and inclusion of random effects in the dispersion. As a result, heavy-tailed linear mixed-effects model provides less sensitive estimates against outliers. Therefore, the resulting model leads to statistically robust inference and estimation.

Also, we could predict the tire life expectancy according to random effects by using the heavy-tailed linear mixed-effects model. As the result of the prediction, the third car showed the biggest wear difference between both sides of the front tire, while the second car showed the lowest wear difference between both sides of the front tire.

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국문초록

반복측정실험에서 반복 자료를 분석하는데 있어 정규 선형혼합모형(linar mixed-effects model)이 널리 사용되고 있지만 오차의 등분산성과 모수분포의 비정규성이 동시에 나타나는 경우 이 정규 선형 혼합모형이 부적절한 분석 결과를 제공하는 것으로 알려져 있다.

좀 더 강건한 추정을 위하여 본 논문에서는 표준적인 정규 선형혼합모형보다 정확하고 신뢰할만한 분석 결과를 제공하는 두터운 꼬리분포를 가지는 선형 혼합모형(heavy-tailed linear mixed-effects model)을 자료 분석을 위해 사용하였다.

또한, 선행연구로써 신뢰성분석결과를 함께 제시하였다.

주요어: 반복측정실험, 선형혼합모형, 두꺼운 꼬리를 가지는 선형혼합모형, 신뢰성분석.

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