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이학석사 학위논문

Comparison of L^1 and L^2 Approximation in Estimating the Implied Volatility Surface

(내재변동성 표면 추정에 있어서 L^1, L^2 근사 비교)

2013년 2월

서울대학교 대학원

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이 논문을 이학석사 학위논문으로 제출함

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김원세의 이학석사 학위논문을 인준함

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Comparison of L^1 and L^2 Approximation in Estimating the Implied Volatility Surface

A Dissertation
Submitted to the Faculty of
the Graduate School
of
Seoul National University
in Partial Fulfillment of
the Requirements for the Degree of
Master of Philosophy

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Abstract

In this paper, we approximate the implied volatility surface of KOSPI200 call option in the assumption that the implied volatility surface can be approximated by bivariate polynomial function of degree 4. In the approximation , we adopt two different methods, L^1 approximation and L^2 approximation. Finally, We discuss about the accuracy of the approximation methods by comparing the numerical results of them.

Key words: the Implied Volatility Surface, L^1 Approximation, L^2 Approximation

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Chapter 1

Introduction

As written in the Wilmott [1], in the Black -Scholes world of constant volatility, the value of a European call option is simply

$$V(S,t;\sigma,r,;E,T) = SN(d_1) - Ee^{-r(t-t)}N(d_2),$$

where

$$d_1 = \frac{log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = \frac{log(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Of the variables of V, the first two are the independent variables, the second two are parameters of the asset and the financial world, and the last two are specific to the contract considered. All but σ are easy to measure(r may be a bit inaccurate but the price is typically not too sensitive to this). If we know σ, then we can calculate the option price. Conversely, if we know the option price V, then we can calculate σ. We can do this because the value of a call option is monotonic in the volatility. Provided that the market value of the call option is greater than $max(S - E^{-r(T-t)}, 0)$ and less than S there

is one, and only one, value for σ that makes the theoretical option value and the market price the same, This is called 'the implied volatility'. And then, We can show the implied volatility against both maturity and strike in a three-dimensional plot. That is, we can get a surface. This surface is called the implied volatility surface.

In this paper, we approximate the implied volatility surface of KOSPI200 call option using two different methods,L^1&L^2 approximation, and compare the numerical results of them.

In Chapter 2, We will introduce some basic setting of my study.

In Chapter 3, We will show how we have performed the two approximating methods, L^1&L^2 approximation, and show the numerical results that we have gotten by applying these two methods to KOSPI200 call option data.

In Chapter 4, We will compare the numerical results of these two different approximating methods

In Chapter 5, We will conclude that L^1 approximation may produce much better approximating results than that of L^2 approximation if the approximating model is well made that it reflects the original date nicely.

Chapter 2

Basic Setting

2.1 Moneyness

Many text book generally chose the time to maturity and strike price as the two variables of the implied volatility surface. However, according to Cont(2002) [2], the implied volatility patterns across moneyness vary less in time than when expressed as a function of the strike. So, we will approximate the implied volatility surface as a function of moneyness and time to maturity instead of a function of strike and time to maturity.

The moneyenss is defined according to Gross and Waltners (1995) [3] as below

$$M(E,T) = \frac{lnS/E + (r-\delta)T}{\sqrt{T}},$$

where δ is 'the dividend rate' and is assumed to be 0 in my study and the other indexes will be explained at the next section.

2.2 Index

By the black-scholes model, for performing this study, we need six index;$S, E, r, T, \sigma^{imp}, \sigma_{imp}$.

S is the price of a underlying asset. In my study, S is KOSPI 200 INDEX.

E is the strike price of European call option. In my study, E is the strike price of KOSPI 200 call option.

r is the risk free rate. In my study, we adopt 91-days CD rate as r because of the data used for σ^{imp} in this study. I will explain about this in detail when I explain about index σ^{imp}.

T is the time to maturity of KOSPI 200 call option

σ^{imp} is the implied volatility of European call option. In my study, σ^{imp} is the implied volatility of KOSPI 200 call option. these data is calculated from KRX(Korea exchange) and shown in every HTS(Home Trading System) . When calculating σ^{imp}, KRX use 91-days CD as a risk free rate in the Black-scholes model. This is the reason why I used 91-days CD as a risk free rate in my study.

σ_{imp} is the approximation polynomial function of σ^{imp}. About this index, I will explain in detail in the next section.

2.3 Approximating Polynomial Function of the Implied Volatility Surface

In this study, I set a bivariate polynomial function

$$\sigma_{imp}(M,T) = \sum_{i=0}^{4}\sum_{j=0}^{i} a_{I_{ij}} M^{i-j} T^{j}, \tag{1}$$

where M is moneyness in 2.1 and $I_{ij} \in \{0, 1, 2, ...14\}$ for given $0 \leq i \leq j \leq 4$, as the approximating function of the implied volatility surface.

There are three reason that I take (1) as the approximating polynomial function σ_{imp}.

First, by a several of empirical test, even degree of polynomial function have made much better performance than odd degree of polynomial function in approximating the implied volatility surface. The cause of this phenomenon may be due to 'the volatility smile'.

Second, by the limit of number of data, 4 was the maximum degree for the approximating polynomial function σ_{imp} that I could select.

Third, degree 4 produces much better performance than degree 2 in approximation.

On this basic setting, I chose two different methods for the approximation, L^1 approximation and L^2 approximation. For the elementary explanation of L^1 approximation and L^2 approximation, I will treat in detail in next chapter.

Lastly, the data used in my study is given in the next page.

the Implied Volatility(%) of KOSPI 200 Call Option(data:2012.11.20)

	Time to maturity (annualized)				
STRIKE PRICE (K)		0.065753	0.142466	0.238356	0.315068
	227.5	19.64			
	230	18.34			9.52
	232.5	17.45			11.87
	235	17.1		10.26	13.18
	237.5	17.14		11.31	13.46
	240	16.91	9.84	11.92	13.66
	242.5	16.38	10.89	12.28	13.55
	245	15.31	11.74	12.24	14.37
	247.5	15.41	12	12.92	14.51
	250	14.97	12.45	13.13	14.81
	252.5	14.74	12.49	12.91	14.8
	255	14.45	12.87	13.33	15.1
	257.5	14.47	13.01	13.71	
	260	14.53	13.04	13.52	
	262.5	14.59	13.19	13.88	
	265	14.87	13.34	14.02	
	267.5	15.09	13.77		
	270	15.4	14.14		
	272.5	15.92	14.18		
	275	16.65			
	277.5	17.11			
	280	18.32			

Figure 2.1: data출처: 키움증권 HTS

the Implied Volatility(%) of KOSPI 200 Call Option(data:2012.11.30)

	Time to maturity (annualized)				
STRIKE PRICE (K)		0.065753	0.142466	0.238356	0.315068
	222.5	59			
	225	56.42			
	227.5	52.78			
	230	40.31		10.91	13.85
	232.5	45.47		12.76	
	235	42.2		13.49	14.61
	237.5	38.45		13.81	
	240	27.12		14.07	14.85
	242.5	24.58	12.26	14.1	
	245	22.93	10.94	14.22	14.93
	247.5	21.05	11.75	14.36	
	250	18.38	11.3	13.48	13.65
	252.5	17.66	11.48	13.47	
	255	16.25	11.94	14.09	14.5
	257.5	15.57	12.28	13.84	
	260	14.9	12.23	13.58	14.3
	262.5	14.53	12.44	13.66	
	265	14.52	12.4	13.81	14.9
	267.5	14.51	12.52	14	
	270	14.86	12.62	13.68	14.7
	272.5	15.25	12.81		
	275	16.17			14.92
	277.5	16.52			
	280	18.05			15.08

Figure 2.2: data출처: 키움증권 HTS

Chapter 3

L^1 & L^2 Approximation and Numerical Results

3.1 L^1 Approximation

For methodology of L^1approximation, I adopted the method of the Majid Amirfakhrian [4]. I briefly explain the method.

For L^1 approximation, we have to find $a_0, a_1, ...a_{14}$ such that

$$Minimize \sum_{i=1}^{N} |\sigma_{imp}(M_i, T_i) - \sigma_i^{imp}|,$$

where N is the number of data.

For each $i = 1, 2, ...N$, define two variable u_i, v_i as follows;

$$e_i = \sigma_{imp}(M_i, T_i) - \sigma_i^{imp} \tag{2}$$

and

$$e_i = u_i - v_i, \tag{3}$$

where u_i and v_i are defined as

$$\begin{cases} u_i = e_i, \quad v_i = 0; & e_i \geq 0 \\ u_i = 0, \quad v_i = -e_i; & e_i < 0 \end{cases}$$

Now we define β and λ as follows

$$\begin{aligned} \beta &= min(a_0, a_1, ..., a_{14}) \\ \lambda &= \begin{cases} -\beta \ ; & \beta < 0 \\ 0 \ ; & \beta \geq 0 \end{cases} \end{aligned}$$

For $i = 0, 1, 2, ..., 14$, we define $\overline{a_i}$ as follows;

$$\overline{a_i} := a_i + \lambda, \qquad i = 0, 1, 2, ..., 14.$$

By the definition, $\sigma_{imp}(M, T)$ has the following form

$$\sigma_{imp}(M, T) = \sum_{i=0}^{4} \sum_{j=0}^{i} \overline{a}_{I_{ij}} M^{i-j} T^j - \lambda(\sum_{i=0}^{4} \sum_{j=0}^{i} M^{i-j} T^j).$$

From (2) and (3), we have

$$u_i - v_i + \sigma_{imp}(M_i, T_i) = \sigma_i^{imp}.$$

Therefore, L^1 approximation problem reduces to the following linear programming problem:

$$min \sum_{i=1}^{N} (u_i + v_i) \tag{4}$$

with the following constraint:

$$\begin{cases} u_i - v_i + \sum_{i=0}^{4} \sum_{j=0}^{i} \overline{a}_{I_{ij}} M^{i-j} T^j - \lambda(\sum_{i=0}^{4} \sum_{j=0}^{i} M^{i-j} T^j) = \sigma_i^{imp} \\ u_i \geq 0, v_i \geq 0 \\ \overline{a}_i \geq 0 \\ \lambda \geq 0 \end{cases}$$

The number of all variables in above linear programming is $2N$(the number of u_i, v_i) + 15(the number of coefficient) + 1(λ).

By performing the above process using MATLAB, We can get the following numerical results.

Optimal Coefficients(data: 2012.11.20)

a_0	a_1	a_2	a_3	a_4	a_5	a_6
20.25561	50.13782	-89.3968	79.13316	-928.159	29.02624	33.87474
a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}
-1169.24	4198.771	1949.21	25.6433	-785.634	2271.219	-5903.02
a_{14}						
-4271.21						

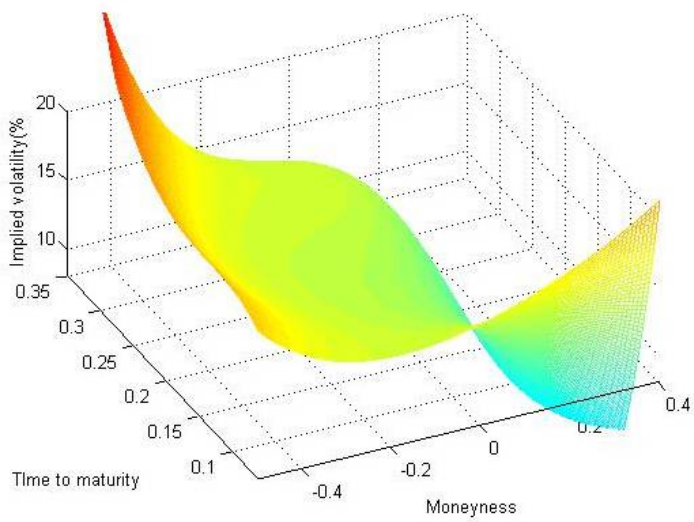

Figure 3.1: Approximated Implied Volatility Surface in L^1 Metric Sense.(data:2012.11.20)

Optimal Coefficients(data: 2012.11.30)

a_0	a_1	a_2	a_3	a_4	a_5	a_6
20.25164	62.73613	-100.687	149.7709	-1214.36	-100.773	58.06626
a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}
-1771.49	7066.11	4161.205	-106.293	-980.574	4640.737	12385.4
a_{14}						
-9862.84						

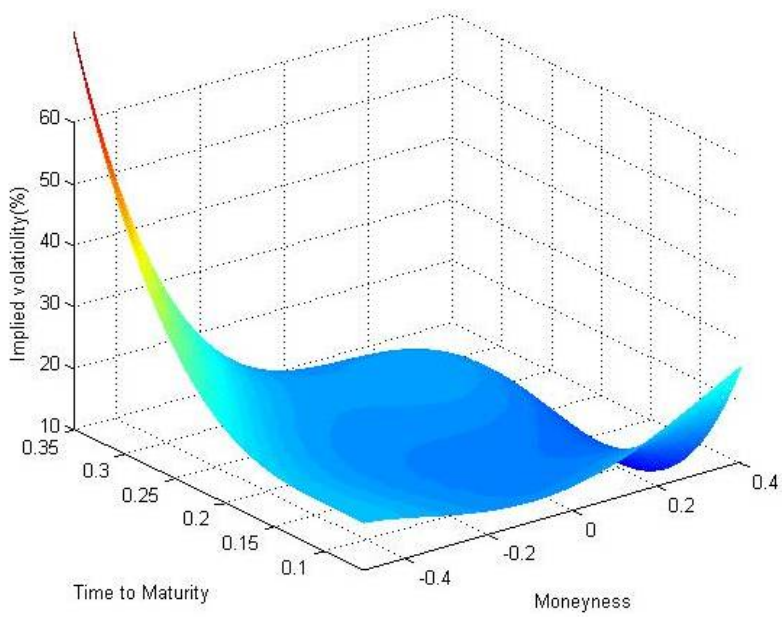

Figure 3.2: Approximated Implied Volatility Surface in L^1 Metric Sense.(data:2012.11.30)

3.2 L^2 Approximation

For methodology of L^2 approximation, there are so many general methods. The method I adopted this study entirely relys on the theorem below in the Stephen H.Friedberg [5]

(Theorem). *Let $A \in M_{m\times n}(F)$ and $y \in F^m$. Then there exists $x_0 \in F^n$ such that $(A^*A)x_0 = A^*y$ and $\| Ax_0 - y \| \leq \| Ax - y \|$ for all $x \in F^n$. Furthermore, if rank(A) = n, then $x_0 = (A^*A)^{-1}A^*y$.*

Practically, for the 'sufficiently' large m relative to n , the full rank condition '$rank(A) = n$' in above theorem is satisfied. If then, We can directly calculate x_0 by calculating $(A^*A)^{-1}A^*y$.

For my study, $n = 15$(the number of coefficient) and $m =$ 'the number of data'; for 2012-11-20 data, $m = 60$ and for 2012-11-30 data, $m = 65$. So, in this study, m(the number of data) is 'sufficiently' large relative to n(the number of coefficient) in each case.

Therefore, by performing the above calculation, $(A^*A)^{-1}A^*y$, we can get the following numerical results.

Optimal Coefficients(data: 2012.11.20)

a_0	a_1	a_2	a_3	a_4	a_5	a_6
22.8069	51.40858	-163.127	82.01927	-960.643	734.7318	63.87123
a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}
-1225.28	4481.395	-751.866	31.86624	-1225.89	2323.384	-6472.6969
a_{14}						
-709.809763						

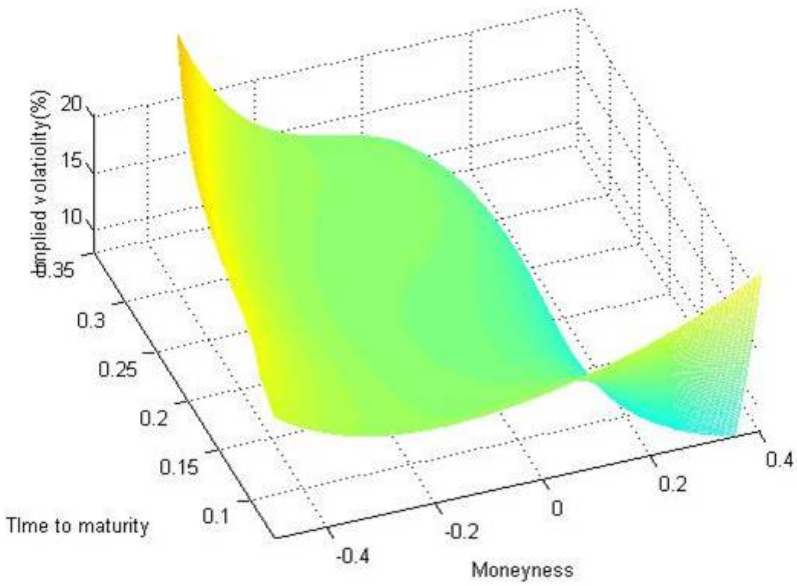

Figure 3.3: Approximated Implied Volatility Surface in L^2 Metric Sense.(data:2012.11.20)

Optimal Coefficients(data: 2012.11.30)

a_0	a_1	a_2	a_3	a_4	a_5	a_6
21.70247	56.90459	-187.473	138.7565	-1086.48	1066.032	59.29455
a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}
-1598.5	6205.973	-1433.44	-84.3477	-1049.48	4141.915	-10684.6
a_{14}						
-1087.8						

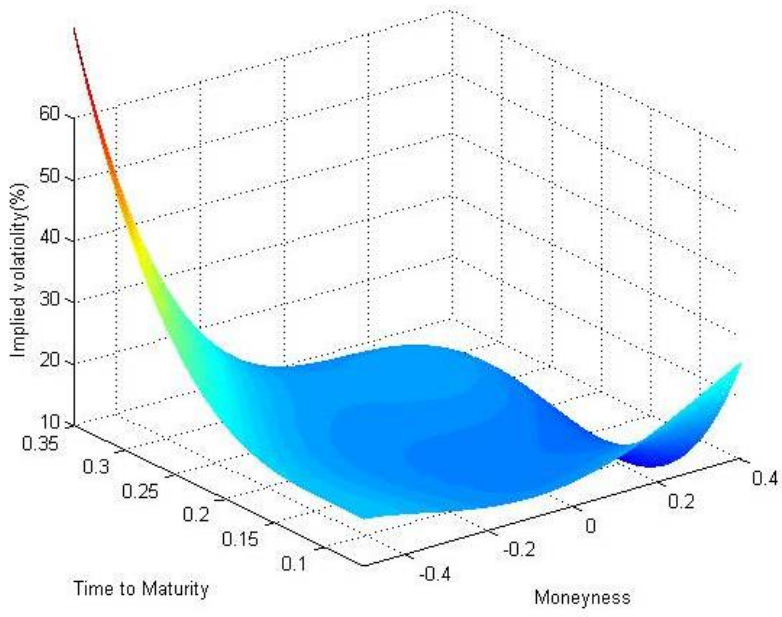

Figure 3.4: Approximated Implied Volatility Surface in L^2 Metric Sense.(data:2012.11.30)

Chapter 4

Comparison of the Results of L^1 & L^2 Approximation

In this chapter, I will compare the numerical results of chapter 3. To each numerical result to data of 2012-11-20, 2012-11-30, for the comparison, I draw 'the approximation error graph' for L^1 and L^2 approximation;x-axis is the number of a data and y-axis is the approximation error that is represented in percentage. After then, I make the distribution tables of the approximation error against the two different approximation method, L^1 and L^2 approximation ,and the time of data.

By doing this, I compare the extent of accuracy of L^1 and L^2 approximation graphically and numerically.

4.1 Data of 2012-11-20

For each L^1 and L^2 approximation, If we draw the graph of approximation errors, we can get the below graphs.

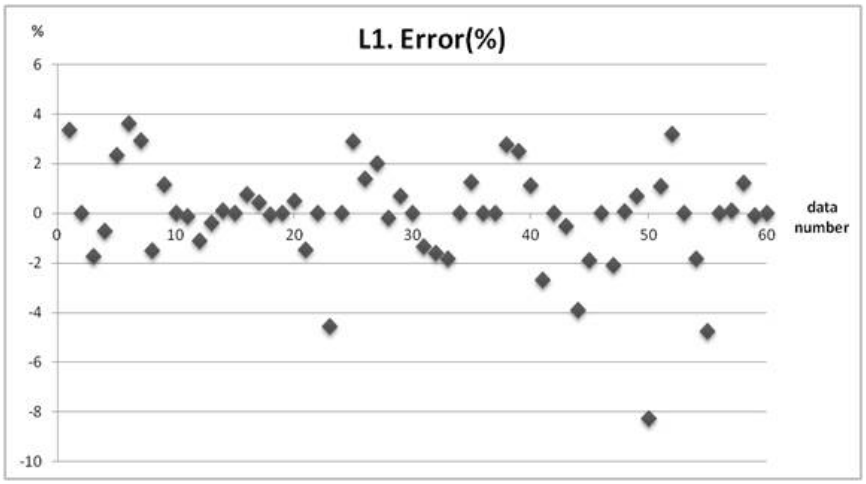

Figure 4.1: Approximation Error of L^1 Approximation.(2012.11.20)

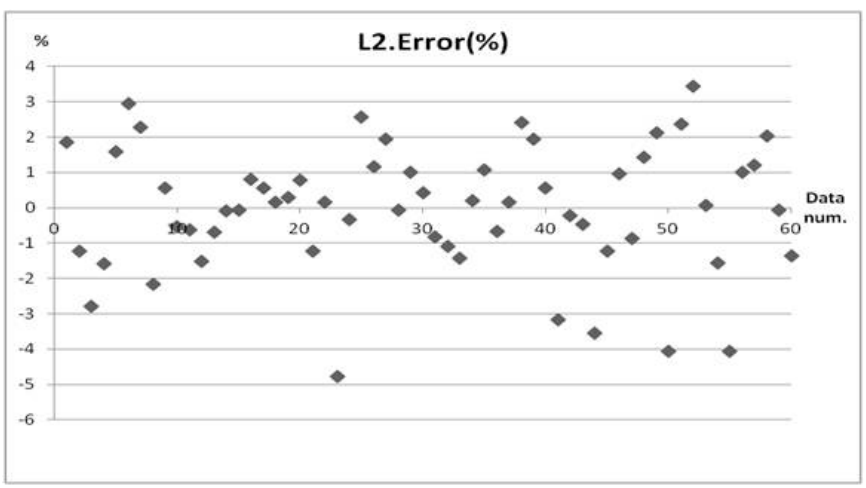

Figure 4.2: Approximation Error of L^2 Approximation.(2012.11.20)

Based on the same data used in the above graphs, we can get the below tables of distribution of error.

Distribution of Error(data: 2012.11.20)

Error 범위	데이터 갯수	
	L^1	L^2
$\|x\| < 0.5$	24	14
$0.5 \leq \|x\| < 1$	6	13
$1 \leq \|x\| < 1.5$	9	11
$1.5 \leq \|x\| < 2$	7	7
$2 \leq \|x\| < 2.5$	2	6
$2.5 \leq \|x\|$	12	9
합계	60	60

Error 범위	데이터 분포율(%)	
	L^1	L^2
$\|x\| < 0.5$	40	23.3
$0.5 \leq \|x\| < 1$	10	21.7
$1 \leq \|x\| < 1.5$	15	18.3
$1.5 \leq \|x\| < 2$	11.7	11.7
$2 \leq \|x\| < 2.5$	3.3	10
$2.5 \leq \|x\|$	20	15
합계	100	100

As shown in first row of the above tables, we can see that the error of L^1 approximation is distributed more closely to zero line than that of L^2 approximation.

4.2 Data of 2012-11-30

Like in the previous section, for each L^1 and L^2 approximation, If we also draw the graph of approximation errors, we can get the below graphs.

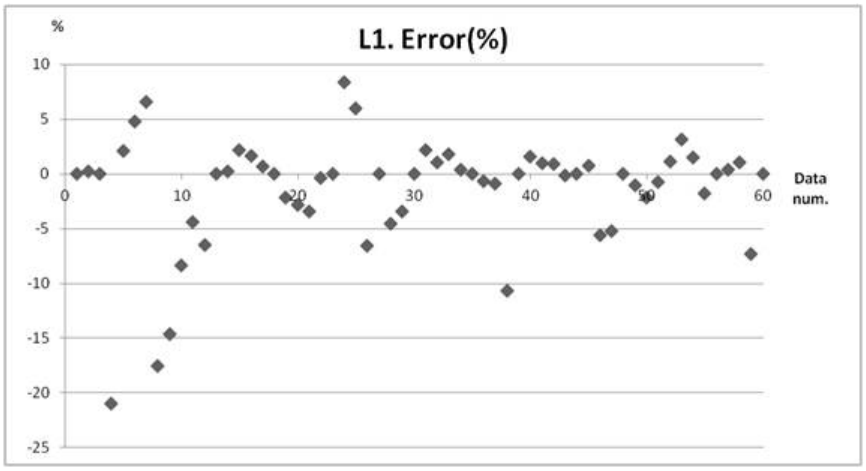

Figure 4.3: Approximation Error of L^2 Approximation.(2012.11.30)

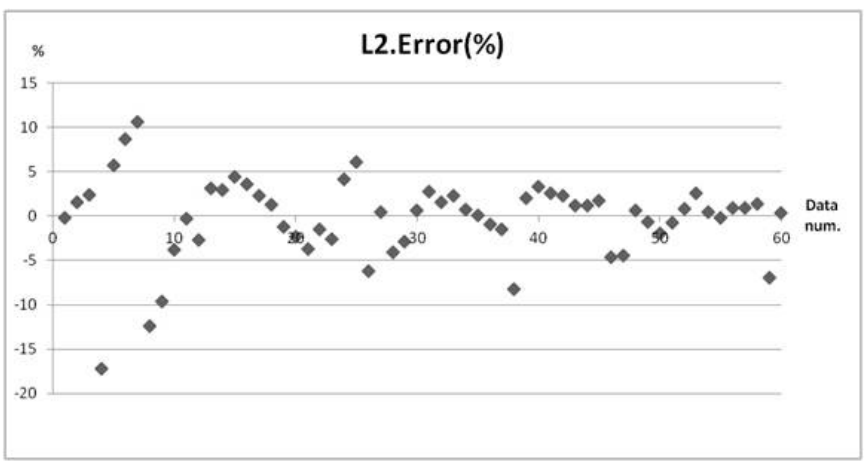

Figure 4.4: Approximation Error of L^2 Approximation.(2012.11.30)

Like in the previous section, based on the same data used in the above graphs, we can get the below tables of distribution of error.

Distribution of Error(data: 2012.11.30)

Error 범위	데이터 갯수	
	L^1	L^2
$\|x\| < 0.5$	20	8
$0.5 \leq \|x\| < 1$	9	10
$1 \leq \|x\| < 1.5$	5	7
$1.5 \leq \|x\| < 2$	4	6
$2 \leq \|x\| < 2.5$	5	6
$2.5 \leq \|x\|$	22	28
합계	65	65

Error 범위	데이터 분포율(%)	
	L^1	L^2
$\|x\| < 0.5$	30.8	12.4
$0.5 \leq \|x\| < 1$	13.9	15.4
$1 \leq \|x\| < 1.5$	7.7	10.8
$1.5 \leq \|x\| < 2$	6.1	9.2
$2 \leq \|x\| < 2.5$	7.7	9.2
$2.5 \leq \|x\|$	33.8	43
합계	100	100

Like the preceding section, as shown in first row of the above tables, we can see that the error of L^1 approximation is distributed more closely to zero line than that of L^2 approximation.

Chapter 5

Conclusion

In this paper, I estimated the implied volatility surface using the two different methods, L^1 and L^2 approximation. As the result of the estimating, we have found that L^1 approximation produces much more accurate approximation than that of L^2 approximation if the range of error is sufficiently small. I can guess that this phenomenon may come from the fact that L^2 metric put more weight on a point of absolute value larger than 1 than L^1 metric does.

So, our study suggests that if the approximating model is well made that it reflects the original date nicely, then L^1 approximation would produce the much better approximation.

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국문초록

본 논문에서는 코스피 200 콜 옵션의 내재 변동성 표면을 4차 2변수 다항식을 이용하여 추정한다. 추정방법에 있어서는 'L^1 근사', 'L^2 근사' 라는 두 가지 추정방법을 채택한다.

마지막으로 두 추정방법으로부터 도출된 수치적 결과들을 비교함으로써 두 추정법의 정확성에 대해 다룬다.

핵심 어휘: 내재 변동성 표면 추정, L^1 근사, L^2 근사

학번: 2011-20264

감사의 글

낯선 분야에 대한 두려움에 쌓여 있을 때, 좋은 연구자가 되기 위해서는 '사상누각'의 마음가짐이 필요함을 일깨워 주시며, 저에게 첫 발을 내 딛을 용기를 주신 최형인 지도교수님께 우선 감사의 말씀을 드립니다. 또한 다른 전공으로부터 온 저에게 수학이라는 학문의 아름다움을 일깨워 주신 많은 수리과학부 교수님께도 감사 드립니다.

저의 이 부족한 논문의 완성에 있어서 처음부터 끝까지 정말 큰 도움을 주신 준열이형에게 진심으로 깊은, 글로는 다 하지 못할 감사의 말씀을 드리고 싶습니다. 힘든 연구실 생활과 논문 작업에 있어, 그리고 또한 제 대학원 생활에 있어, 등대같은 역할을 해 주신 준열이형이 있었기 때문에 지금 이 논문이 존재할 수 있으며, 연구실 이 자리에 제가 앉아 있을 수 있었습니다. 정말 어떻게 감사의 말씀을 드려야 할지 모를만큼의 감사의 말씀을 드립니다.

연구실 졸업하시는 마지막 순간까지 저희에게 부담주지 않으시려, 저희 대신에 온갖 힘든일을 도맡아주신 정용이형, 논문 작업에 힘들어 지쳐 있을 때 항상 응원의 메세지를 보내주신 지현누나에게 진심으로 감사의 말씀을 드립니다.

처음 MTL에 들어와 낯설어 하고 있던 저를 따뜻하게 맞아주신 태형이형 철홍이형 그리고 준행이형과 힘들 때 항상 서로에게 힘이 되었던 연구실 동기 영순이 은희, 그리고 연구실에 새로 들어온 남규 도영 시연 시형 아란 건우 에게 고마움을 전합니다.

인생의 반을 함께한, 그리고 남은 인생을 함께할 저의 가장 소중한 친구들 병헌,운봉,진환(가나다 순임)에게 또한 고마음을 전하고 싶습니다. 학업과 논문에 지쳐 있을때, 항상 따뜻한 진심어린 말로 위로해 주었던 병헌이, 힘든 순간 연락을 하면 슈퍼맨처럼 만사를 제쳐두고 항상 달려와 주었던 운

봉이, 항상 밝음과 긍정의 에너지로 우리에게 힘을 주던 진환이에게 이 자리를 빌어 진심으로 감사와 사랑의 말을 전합니다.

힘든 수리과학부 대학원 생활동안 학문적으로 인간적으로 큰 힘이 되어준 기현이를 비롯한 많은 수리과학부 대학원 동기 및 선후배들에게 또한 감사의 말씀을 드립니다.

마지막으로 힘든 환경 속에서도 저를 여기까지 키워주신 어머니와 이번 논문의 코딩에 있어 정말 큰 도움을 준 형에게 진심어린 감사의 말은 전합니다. 항상 건강하고 행복한 가족이 되도록 노력하겠습니다.