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이 논문을 이학석사 학위논문으로 제출함

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Pseudodifferential Operators

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A DISSERTATION

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Abstract

This master thesis presents a survey of theory of pseudodifferential operators.

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Contents

CHAPTER 1

Introduction

In this thesis we discuss the basic theory of pseudodifferential operators as it has been developed to treat problems in PDE. In Chapter 2, we review the distribution theory which is also essential in pseudodifferential operators. In addition, we define Schwartz kernels and state Schwartz kernel theorem. In Chapter 3, we define classical symbols and standard symbols. In Chapter 4, we define pseudodifferential operators with classical symbols. In Chapter 5, we define conormal distributions, using this, see pseudodifferential operators in different perspectives. In Chapter 6, we define transposes and adjoints of pseudodifferential operators. In Chapter 7, we define compositions of pseudodifferential operators. In Chapter 8, we define the Sobolev space and consider how pseudodifferential operators affect its regularity. In Chapter 9, we give an important theorem about a relation between ellipticity and a parametrix. Moreover, we prove elliptic regularity theorem by using pseudodifferential operators. In Chapter 10, we generalize pseudodifferential operators to manifolds.

CHAPTER 2

Preliminaries: Distribution Theory

Unless otherwise stated U is an open subset of $\mathbb{R}^n$ throughout the thesis.

1. Differential Operators

A (linear) differential operator on U is a linear operator $P : C_c^\infty(U) \to C^\infty(U)$ of the form, given by

$$P = \sum_{|\alpha| \le N} a_\alpha(x) \partial_x^\alpha \quad \text{ where } a_\alpha \in C^\infty(U).$$

It will be convenient to use the following notation:

$$D_{x_j} := \frac{1}{i} \frac{\partial}{\partial x_j} \quad \text{for } j = 1, ..., n, \qquad D_x^\alpha := D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}.$$

Then $D_x^\alpha = i^{-|\alpha|} \partial_x^\alpha$ for all $\alpha = (\alpha_1, ..., \alpha_n)$ in $\mathbb{N}_0^n$, where $|\alpha| = \sum_{j=1}^{n} \alpha_j$. Therefore, we can rewrite P as

$$P = \sum_{|\alpha| \le N} a_\alpha(x) D_x^\alpha \quad \text{ where } a_\alpha \in C^\infty(U).$$

Now we define the *symbol* of P to be the function on $U \times \mathbb{R}^n$ given by

$$p(x, \xi) := \sum_{|\alpha| \le N} a_\alpha(x) \xi^\alpha \quad \forall (x, \xi) \in U \times \mathbb{R}^n.$$

P can be written formally as $P = p(x, D)$. We use the Fourier transform of f defined by

$$\hat{f}(\xi) = (2\pi)^{-n} \int f(x) e^{-ix\cdot\xi} dx.$$

Let $u \in C_c^\infty(U)$. Then $(D_x^\alpha u)^\wedge(\xi) = \xi^\alpha \hat{u}(\xi)$ and

$$D_x^\alpha u(x) = ((D_x^\alpha u)^\wedge)^\vee(x) = (\xi^\alpha \hat{u})^\vee(x) = \int e^{ix\cdot\xi} \xi^\alpha \hat{u}(\xi) d\xi. \tag{2.1}$$

By using (2.1), we get

$$Pu(x) = \sum_{|\alpha| \le N} a_\alpha(x) D_x^\alpha u(x) = \sum_{|\alpha| \le N} a_\alpha(x) \int e^{ix\cdot\xi} \xi^\alpha \hat{u}(\xi) d\xi.$$

Therefore, in terms of the symbol $p(x, \xi) = \sum_{|\alpha| \le N} a_\alpha(x) \xi^\alpha$ of P, we have

$$Pu(x) = \int e^{ix\cdot\xi} p(x, \xi) \hat{u}(\xi) d\xi \qquad \forall u \in C_c^\infty(U) \ \forall x \in U. \tag{2.2}$$

The above formula is the main motivation for defining pseudodifferential operators by extending that formula to classes of symbols $p(x, \xi)$ which are not required to be polynomial in ξ anymore, but satisfy suitable growth conditions with respect to the ξ-variable. The upshot will be a larger class of operators containing inverses(or rather parametrices) of elliptic operators together with a corresponding calculus for symbols which allows us to explicitly construct these inverses.

2. Review of a topological vector space and Distribution Theory

A topological vector space (TVS) is a vector space E over $\mathbb{C}$ which is endowed with a topology such that the maps $(x, y) \to x + y$ and $(\lambda, x) \to \lambda x$ are continuous from $E \times E$ and $\mathbb{C} \times E$ to E.

A locally convex topological vector space (LCTVS) is a TVS if there is a base for the topology consisting of convex sets. Then we obtain the following proposition.

Proposition 2.1. *Let E be a TVS. Then the following are equivalent.*
(1) E is locally convex.
(2) The topology of E is defined by a family of semi-norms.

Definition 2.1. *A Fréchet space is a complete metrizable LCTVS.*

Definition 2.2. *A Montel space is a LCTVS whose bounded subsets are precompact. A LCTVS which is both a Fréchet space and a Montel space is called a Fréchet-Montel space.*

We endow $C^\infty(U)$ with the LCTVS topology defined by the semi-norms,

$$p_{N,K}(u) = \sup_{|\alpha|\leq N} \sup_{x\in K} |\partial_x^\alpha u(x)| \quad \forall u \in C^\infty(U),$$

where N ranges over all non-negative integers and K ranges over all compact subsets of U. Then $C^\infty(U)$ is a Fréchet-Montel space.

Let us denote by $C_K^\infty(U)$ the space of all smooth functions on U whose supports are contained in K. This is a closed subspace of $C^\infty(U)$, and hence $C_K^\infty(U)$ is a Fréchet-Montel space with respect to the induced topology. Furthermore, the induced topology agrees with the LCTVS topology defined by the semi-norms,

$$p_N(u) = \sup_{|\alpha|\leq N} \sup_{x\in U} |\partial_x^\alpha u(x)| \quad \forall u \in C_K^\infty(U),$$

where N ranges over all non-negative integers.

Now we endow $C_c^\infty(U)$ with the weakest LCTVS topology with respect to which, for all compact subsets K of U, the inclusion $C_K^\infty(U) \hookrightarrow C_c^\infty(U)$ is continuous.

Let $\mathcal{D}'(U)$ denote the space of distributions on U, that is, the topological dual of $C_c^\infty(U)$ endowed with its weak-$*$ topology. In addition, let $\mathcal{E}'(U)$ denote the topological dual of $C^\infty(U)$ endowed with its weak-$*$ topology.

The topology of $\mathcal{S}(\mathbb{R}^n)$ is the LCTVS topology defined by the semi-norms,

$$q_N(u) = \sup_{|\alpha|+k\leq N} \sup_{x\in\mathbb{R}^n} (1 + |x|)^k |\partial_x^\alpha u(x)| \quad \forall u \in \mathcal{S}(\mathbb{R}^n),$$

where N ranges over all non-negative integers. With respect to this topology $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet-Montel space. Let $\mathcal{S}'(\mathbb{R}^n)$ be the topological dual of $\mathcal{S}(\mathbb{R}^n)$ endowed with its weak-$*$ topology.

Remark 2.1. Note that $C_c^\infty(U)$ is not a Fréchet space.

Proposition 2.2. *Let E and F be LCTVS's defined by two families of semi-norms $(p_i)_{i\in I}$ and $(q_j)_{j\in J}$ respectively. Then a linear operator $T : E \to F$ is continuous if and only if, for all $j \in J$, there exists $i_1, ..., i_N \in I$ and $C > 0$ such that*

$$q_j(Tx) \leq C \max\{p_{i_1}(x), ..., p_{i_N}(x)\} \qquad \forall x \in E.$$

In particular, a linear form $T : E \to \mathbb{C}$ is continuous if and only if there exists $i_1, ..., i_N \in I$ and $C > 0$ such that

$$|\langle T, x\rangle| \leq C \max\{p_{i_1}(x), ..., p_{i_N}(x)\} \qquad \forall x \in E.$$

Definition 2.3 (Uniform Boundedness Principle)**.** *Let E be a TVS and we denote by E' its topological dual endowed with its weak-$*$ topology. Then E' satisfies the uniform boundedness principle in the following form. If a family $(T_j)_{j\in J} \subset E'$ is weakly bounded, that is,*

$$\sup_{j\in J} |\langle T_j, x\rangle| < \infty \quad \forall x \in E,$$

then the family $(T_j)_{j\in J}$ is equicontinuous, that is, there exists $i_1, ..., i_N \in I$ and $C > 0$ such that

$$|\langle T_j, x\rangle| \leq C \max\{p_{i_1}(x), ..., p_{i_N}(x)\} \qquad \forall x \in E \;\; \forall j \in J.$$

Proposition 2.3. *If E is a Fréchet space, E' satisfies the uniform boundedness principle.*

Let E be a TVS and we denote by E' its topological dual endowed with its weak-$*$ topology. Moreover, we shall denote by $C^\infty(U, E)$ the space of smooth families with values in E. Assume that E' satisfies the uniform boundedness principle. It ensures that any pointwise partial derivative of a family of elements of E' is automatically an element of E'. It follows from this fact that a family $(T_x)_{x\in U} \subset E'$ is smooth if and only if, for all $\xi \in E$, the function $x \to \langle T_x, \xi\rangle$ is smooth.

Proposition 2.4. *Let E be a TVS and $(\xi_x)_{x\in U} \in C^\infty(U, E)$.*

(1) *Let $T : E \to F$ be a sequentially continuous linear operator with values in a TVS F. Then $(T(\xi_x))_{x\in U} \in C^\infty(U, F)$.*
(2) *Let $(T_x)_{x\in U}$ be a smooth family with values in E'. If we assume that E' satisfies the uniform boundedness principle, then $x \to \langle T_x, \xi_x\rangle$ is a smooth function on U.*

3. Kernel Theorems

Theorem 2.1 (Schwartz)**.** *A linear operator $P : C_c^\infty(U) \to \mathcal{D}'(U)$ is continuous if and only if there exists a distribution $k_P(x, y) \in \mathcal{D}'(U \times U)$ such that*

$$\langle Pv(x), u(x)\rangle = \langle k_P(x, y), u(x)v(y)\rangle \quad \forall u \in C_c^\infty(U) \;\; \forall v \in C_c^\infty(U).$$

Lemma 2.1. *$\mathcal{D}'(U)$ satisfies the uniform boundedness principle.*

Since $D'(V)$ satisfies the uniform boundedness principle, $(k_x)_{x\in U} \in C^\infty(U, \mathcal{D}'(V))$

if and only if, for all $v \in C_c^\infty(V)$, the function $x \to \langle k_x(y), v(y)\rangle_y$ is smooth.

We will be interested in opeators whose Schwartz kernels lie in $C^\infty(U, \mathcal{D}'(U))$. Also we shall regard $C^\infty(U, \mathcal{D}'(U))$ as a subspace of $\mathcal{D}'(U \times U)$ when it is needed.

Proposition 2.5. *A linear operator $P : C_c^\infty(U) \to C^\infty(U)$ is continuous if and only if there exists a family $k_P(x, y) \in C^\infty(U, \mathcal{D}'(U))$ such that*

$$Pu(x) = \langle k_P(x, y), u(y)\rangle_y \qquad \forall u \in C_c^\infty(U) \quad \forall x \in U. \tag{2.3}$$

Proposition 2.6. *A linear operator $P : \mathcal{E}'(U) \to C^\infty(U)$ is sequentially continuous if and only if there exists a family $k_P(x, y) \in C^\infty(U \times U)$ such that*

$$Pu(x) = \langle u(y), k_P(x, y)\rangle_y \qquad \forall u \in \mathcal{E}'(U) \quad \forall x \in U. \tag{2.4}$$

If $u \in C_c^\infty(U)$, then (2.3) becomes

$$Pu(x) = \int_U k_P(x, y)u(y)dy \qquad \forall x \in U. \tag{2.5}$$

Remark 2.2. A continuous linear operator $P : C_c^\infty(U) \to C^\infty(U)$ extends to a sequentially continuous linear operator from $\mathcal{E}'(U)$ to $C^\infty(U)$ if and only if it is given by the Schwartz kernel in $C^\infty(U \times U)$. Such operators are called smoothing operators.

CHAPTER 3

Classes of Symbols

In this section, we introduce the classes of symbols that will be used to define pseudodifferential operators.

Definition 3.1 (Homogeneous Symbols)**.** $S_m(U \times \mathbb{R}^n), m \in \mathbb{C}$, *consists of functions* $p(x,\xi) \in C^\infty(U \times (\mathbb{R}^n \backslash 0))$ *such that*

$$p(x,\lambda\xi) = \lambda^m p(x,\xi) \qquad \forall (x,\xi) \in U \times (\mathbb{R}^n \backslash 0) \ \ \forall \lambda > 0.$$

Definition 3.2 (Classical Symbols)**.** $S^m(U \times \mathbb{R}^n), m \in \mathbb{C}$, *consists of functions* $p(x,\xi) \in C^\infty(U \times \mathbb{R}^n)$ *that admit an asymptotic expansion*

$$p(x,\xi) \sim \sum_{j \geq 0} p_{m-j}(x,\xi), \quad \textit{where } p_{m-j} \in S_{m-j}(U \times \mathbb{R}^n),$$

in the sense that, for any $N \in \mathbb{N}_0$*, any compact subset* $K \subset U$ *and any* α *and* β *in* $\mathbb{N}_0^n$ *, there exists* $C_{NK\alpha\beta} > 0$ *such that, for all* $x \in K$ *and* $\xi \in \mathbb{R}^n$ *with* $|\xi| \geq 1$*, we have*

$$\left| \partial_x^\alpha \partial_\xi^\beta (p - \sum_{0 \leq j < N} p_{m-j}(x,\xi)) \right| \leq C_{NK\alpha\beta} |\xi|^{\Re m - N - |\beta|}. \tag{3.1}$$

Proposition 3.1. *A classical symbol* $p(x,\xi) \in S^m(U \times \mathbb{R}^n)$ *admits a unique asymptotic expansion.*

Proof. Fix $(x,\xi) \in U \times (\mathbb{R}^n \backslash 0)$ and $j \in \mathbb{N}_0$. There exists $C_j > 0$ such that for any large $\lambda > 0$ with $|\lambda\xi| \geq 1$,

$$\left| p(x,\lambda\xi) - \sum_{0 \leq l < j} \lambda^{m-l} p_{m-l}(x,\xi) \right| \leq C_j \lambda^{\Re m - j} |\xi|^{\Re m - j}.$$

Then we can deduce that

$$p_{m-j}(x,\xi) = \lim_{\lambda \to \infty} \lambda^{-(m-j)} \left\{ p(x,\lambda\xi) - \sum_{0 \leq l < j} \lambda^{m-l} p_{m-l}(x,\xi) \right\}.$$

Therefore the homogeneous symbols are determined inductively. The proof is complete. □

Now we let X be an open subset of $\mathbb{R}^l$, where l may be different from n.

Definition 3.3 (Standard Symbols)**.** $\mathbb{S}^m(X \times \mathbb{R}^n), m \in \mathbb{R}$, *consists of functions* $p(x, \xi) \in C^\infty(X \times \mathbb{R}^n)$ *such that for any compact* $K \subset X$ *and multi-indicies* α *and* β*, a constant* $C_{K\alpha\beta} > 0$ *exists for which*

$$\left|\partial_x^\alpha \partial_\xi^\beta p(x, \xi)\right| \leq C_{K\alpha\beta}(1 + |\xi|)^{m-|\beta|} \quad \forall (x, \xi) \in K \times \mathbb{R}^n.$$

Remark 3.1. If $m \in \mathbb{C}$, then we can show that the inclusion,

$$S^m(U \times \mathbb{R}^n) \subset \mathbb{S}^{\Re m}(U \times \mathbb{R}^n).$$

From this, we can consider any classical of order m as a standard symbol of order $Re(m)$ when it is useful for developing arguments.

Definition 3.4. $\mathbb{S}^{-\infty}(X \times \mathbb{R}^n) := \bigcap_{m \in \mathbb{R}} \mathbb{S}^m(X \times \mathbb{R}^n)$.

Remark 3.2. $\mathbb{S}^{-\infty}(X \times \mathbb{R}^n)$ consists of functions $p(x, \xi) \in C^\infty(X \times \mathbb{R}^n)$ such that, for any integer N, any compact $K \subset X$ and any multi-indicies α and β, there exists $C_{NK\alpha\beta} > 0$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta p(x, \xi)\right| \leq C_{NK\alpha\beta}(1 + |\xi|)^{-N} \quad \forall (x, \xi) \in K \times \mathbb{R}^n.$$

Proposition 3.2. *A classical symbol* $p(x, \xi) \in S^m(U \times \mathbb{R}^n)$ *is an element of* $\mathbb{S}^{-\infty}(U \times \mathbb{R}^n)$ *if and only if* $p(x, \xi) \sim 0$.

It follows from this that two classical symbols of the same order p, q have the identical homogeneous symbols if and only if $p = q \mod \mathbb{S}^{-\infty}(U \times \mathbb{R}^n)$.

As $\mathbb{S}^{m_1}(X \times \mathbb{R}^n) \subset \mathbb{S}^{m_2}(X \times \mathbb{R}^n)$ for $m_1 < m_2$, we can also make a natural notion of an asympotic expansion of a standard symbol.

Definition 3.5. *Let* $(m_j)_{j \geq 0} \subset \mathbb{R}$ *be a decreasing sequence converging to* $-\infty$, $p(x, \xi) \in C^\infty(X \times \mathbb{R}^n)$ *and* $p_j(x, \xi) \in \mathbb{S}^{m_j}(X \times \mathbb{R}^n)$ *for* $\forall j \in \mathbb{N}_0$*. Then we write*

$$p(x, \xi) \sim \sum_{j \geq 0} p_j(x, \xi)$$

in the sense that, for any $N \in \mathbb{N}_0$,

$$p(x, \xi) - \sum_{0 \leq j < N} p_j(x, \xi) \in \mathbb{S}^{m_N}(X \times \mathbb{R}^n). \tag{3.2}$$

Remark 3.3. Applying (3.2) for $N = 0$ shows immediately $p(x, \xi) \in \mathbb{S}^{m_0}(X \times \mathbb{R}^n)$.

Remark 3.4. When $X = U$ and $m_j = m - j$, (3.1) and (3.2) are equivalent since what only matters is the behavior of ξ as $|\xi|$ goes to ∞. Therefore, the above notion for standard symbols is defined in a natural way.

A question about the possibility of finding a symbol having given asymptotic expansion can arise at this point. We state two theorems for this problem, the first one is for standard symbols and the second one is for classical symbols. It is a crucial property, for instance, which plays an important role for constructing parametrices of elliptic operators which we will discuss.

Lemma 3.1. *Let $(m_j)_{j\geq 0}\subset\mathbb{R}$ be a decreasing sequence converging to $-\infty$ and $p_j(x,\xi)\in\mathbb{S}^{m_j}(X\times\mathbb{R}^n)$ for all $j\in\mathbb{N}_0$. Then there exists a symbol $p(x,\xi)\in\mathbb{S}^{m_0}(X\times\mathbb{R}^n)$ such that*

$$p(x,\xi)\sim\sum_{j\geq 0}p_j(x,\xi).$$

Lemma 3.2. *Let $m\in\mathbb{C}$ and $p_{m-j}(x,\xi)\in S_{m-j}(U\times\mathbb{R}^n)$ for all $j\in\mathbb{N}_0$. Then there exists a symbol $p(x,\xi)\in S^m(U\times\mathbb{R}^n)$ such that, in the sense of (3.1),*

$$p(x,\xi)\sim\sum_{j\geq 0}p_{m-j}(x,\xi).$$

Now we state and prove a lemma that gives a useful criterion for showing that a function is a classical symbol.

Lemma 3.3. *Let $p(x,\xi)\in C^\infty(U\times\mathbb{R}^n)$ and $m\in\mathbb{C}$. Assume that there exist $q_k(x,\xi)\in S^{m-k}(U\times\mathbb{R}^n)$ for all $k\in\mathbb{N}_0$ such that, in the sense of (3.2),*

$$p(x,\xi)\sim\sum_{k\geq 0}q_k(x,\xi).$$

Then $p(x,\xi)$ is a classical symbol in $S^m(U\times\mathbb{R}^n)$ and for all $j\in\mathbb{N}_0$, its homogeneous symbol $p_{m-j}(x,\xi)$ of degreee $m-j$ is given by

$$p_{m-j}(x,\xi)=\sum_{0\leq k\leq j}p_{k,m-j}(x,\xi),$$

where $p_{k,m-j}(x,\xi)$ is the homogeneous symbol of degree $m-j$ of $q_k(x,\xi)$.

Proof. Let K be a compact subset of U. Let $N\in\mathbb{N}_0$ and let α and β in $\mathbb{N}_0^n$. Then there exists $C_{K\alpha\beta}>0$ such that

$$\left|\partial_x^\alpha\partial_\xi^\beta(p-\sum_{0\leq k<N}q_k(x,\xi))\right|\leq C_{K\alpha\beta}(1+|\xi|)^{\Re m-N-|\beta|}\qquad\forall(x,\xi)\in K\times\mathbb{R}^n.$$

Also we have $q_k(x,\xi)\sim\sum_{j\geq 0}p_{k,m-j}(x,\xi)$ in the sense of (3.1). This means that there exists $C_{k,N,K\alpha\beta}>0$ such that, for all $x\in K$ and $\xi\in\mathbb{R}^n$ with $|\xi|\geq 1$, we have

$$\left|\partial_x^\alpha\partial_\xi^\beta(q_k-\sum_{0\leq j<N}p_{k,m-k-j}(x,\xi))\right|\leq C_{k,N,K\alpha\beta}|\xi|^{\Re m-k-N-|\beta|}.$$

We can change and rewrite the above inequality by putting $N-k$ instead of N as

$$\left|\partial_x^\alpha\partial_\xi^\beta(q_k-\sum_{k\leq j<N}p_{k,m-j}(x,\xi))\right|\leq C_{k,N-k,K\alpha\beta}|\xi|^{\Re m-N-|\beta|}.$$

By using the triangle inequality, we obtain that there exists $C_{NK\alpha\beta}>0$ such that, for all $x\in K$ and $\xi\in\mathbb{R}^n$ with $|\xi|\geq 1$, we have

$$\left|\partial_x^\alpha\partial_\xi^\beta(p-\sum_{0\leq k<N}\sum_{k\leq j<N}p_{k,m-j}(x,\xi))\right|\leq C_{NK\alpha\beta}|\xi|^{\Re m-N-|\beta|}.$$

Then it is easily deduced that $\sum_{0\le k<N}\sum_{k\le j<N} p_{k,m-j}(x,\xi) = \sum_{0\le j<N}\sum_{0\le k\le j} p_{k,m-j}(x,\xi)$ by changing the order of summation. For $j \in \mathbb{N}_0$, set $p_{m-j}(x,\xi) := \sum_{0\le k\le j} p_{k,m-j}(x,\xi)$. For each j, this is a homogeneous symbols of degree $m-j$obviously and the last inequality implies that $p(x,\xi) \sim \sum_{j\ge 0} p_{m-j}(x,\xi)$. The proof is complete. $\square$

CHAPTER 4

Pseudodifferential Operators

In this section we define the pseudodifferential operators from symbols which we discussed in the previous section.

Definition 4.1. *For $p(x,\xi) \in \mathbb{S}^m(U \times \mathbb{R}^n)$, we define a linear operator $p(x,D) : C_c^\infty(U) \to C^\infty(U)$ as*

$$p(x,D)u(x) := \int e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)d\xi \quad \forall u \in C_c^\infty(U).$$

Proposition 4.1. *The operator $p(x,D)$ is continuous.*

Proof. Let K be a compact subset of U and let $\alpha \in \mathbb{N}_0^n$. We denote by $\mathfrak{q}_N$, $N \in \mathbb{N}_0$, the continuous semi-norm on $\mathcal{S}(\mathbb{R}^n)$ defined by

$$\mathfrak{q}_N(u) := \sup_{x\in\mathbb{R}^n} (1+|x|)^N |u(x)| \quad \forall u \in \mathcal{S}(\mathbb{R}^n).$$

Then there exists $C_{K\alpha} > 0$ such that

$$\left|\partial_x^\alpha(e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi))\right| \le C_{K\alpha}\mathfrak{q}_{m+n+1+|\alpha|}(\hat{u})(1+|\xi|)^{-(n+1)}$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$ and $(x,\xi) \in K \times \mathbb{R}^n$. From this estimate, we obtain that $p(x,D)$ is well-defined. Also we can easily deduce from this estimate that for any $N \in \mathbb{N}_0$, there exists $C_{KN} > 0$ such that

$$\sup_{|\alpha\le N}\sup_{x\in K}\left|\partial_x^\alpha(p(x,D)u(x))\right| \le C_{KN}\mathfrak{q}_{m+n+1+N}(\hat{u}) \quad \forall u \in C_c^\infty(U).$$

Since the Fourier transform is a continuous linear isomorphism from $\mathcal{S}(\mathbb{R}^n)$ onto itself, the functional $u \to q_{m+n+1+N}(\hat{u})$ is a continuous semi-norm on $\mathcal{S}(\mathbb{R}^n)$. This is also a continuous semi-norm on $C_c^\infty(U)$ because of the continuity of the inclusion from $C_c^\infty(U)$ to $\mathcal{S}(\mathbb{R}^n)$. Therefore we can conclude that $p(x,D)$ is continuous from the last inequality. □

Definition 4.2. $\Psi^{-\infty}(U)$ *is the space of smoothing operators, that is, the space of linear operators $R : C_c^\infty(U) \to C^\infty(U)$ with smooth Schwartz kernel.*

Remark 4.1. Recall that the smoothing operators are exactly the continuous linear operators that can be uniquely extend to the sequentially continuous linear operators from $\mathcal{E}'(U)$ to $C^\infty(U)$.

We define pseudodifferential operators (in short ΨDOs) as follows.

Definition 4.3. *$\Psi^m(U)$, $m \in \mathbb{C}$, consists of linear operators $P : C_c^\infty(U) \to C^\infty(U)$ of the form*

$$P = p(x, D) + R, \tag{4.1}$$

with $p(x, \xi) \in S^m(U \times \mathbb{R}^n)$ and with $R \in \Psi^{-\infty}(U)$.

Remark 4.2. From Remark 3.1, we know that $p(x, D)$ can be defined by using classical symbols. Of course, the ΨDOs can also can be defined similarly by way of standard symbols. However, we will work through this definition. Therefore, unless otherwise stated, ΨDOs means ΨDOs associated with classical symbols.

Remark 4.3. Since in (4.1) both $p(x, D)$ and R are continuous linear operators from $C_c^\infty(U)$ to $C^\infty(U)$, we see that any ΨDO is a continuous linear operator from $C_c^\infty(U)$ to $C^\infty(U)$.

Now we discuss whether the expression of P in (4.1) is unique or not. Let $q(x, \xi)$ be another symbol for expressing ΨDO in (4.1). Then $p(x, D) - q(x, D)$ is a smoothing operator. We will prove the fact that $p(x, \xi) - q(x, \xi) \in \mathbb{S}^{-\infty}(U \times \mathbb{R}^n)$ is equivalent to $p(x, D) - q(x, D) \in \Psi^{-\infty}(U)$. According to this, even if the expression of P in (4.1) is not the only one so that there is no representative symbol of P , we can say that the symbol $p(x, \xi)$ up to modulo $\mathbb{S}^{-\infty}(U \times \mathbb{R}^n)$ is unique. For this reason, it is customary to consider and call $p(x, \xi)$ the *symbol* of P.

Also from Proposition 3.1 and 3.2, homogeneous symbols $p_{m-j}(x, \xi)$ of degree $m - j$ of the symbol $p(x, \xi)$ of P is uniquely determined whatever the symbol is. Especially, $p_m(x, \xi)$ is called the *principal symbol* of P.

For technical purposes, it would sometimes be useful to deal with ΨDOs that are *properly supported.*

Definition 4.4. *A linear operator $P : C_c^\infty(U) \to C^\infty(U)$ is said to be properly supported if, for any compact $K \subset U$, there exist compact subsets K_1 and K_2 of U such that, for all $u \in C_c^\infty(U)$,*

$$\text{supp}\, u \subset K \Rightarrow \text{supp}\, Pu \subset K_1, \tag{4.2}$$

$$K_2 \cap \text{supp}\, u = \varnothing \Rightarrow K \cap \text{supp}\, Pu = \varnothing. \tag{4.3}$$

Example 4.1. Let $P : C_c^\infty(U) \to C^\infty(U)$ be a continuous linear operator and assume that its Schwartz kernel $k_P(x, y)$ has compact support. Then there exists a compact subset $L \subset U$ such that $k_P(x, y) \subset L \times L$. Recall that by definition of the Schwartz kernel, for any u and v in $C_c^\infty(U)$, we have

$$\langle Pv(x), u(x)\rangle = \langle k_P(x, y), u(x)v(y)\rangle_y.$$

In particular, if $\text{supp}\, v \cap L = \varnothing$, then $Pv = 0$. In addition, for any $u \in C_c^\infty(U)$ such that $\text{supp}\, u \cap L = \varnothing$, we have $\langle Pv(x), u(x)\rangle = \langle k_P(x, y), u(x)v(y)\rangle_y = 0$, which show that $\text{supp}\, Pv$ is contained in L. Thus, the properties (4.2) and (4.3) are satisfied for any compact $K \subset U$ with $K_1 = K_2 = L$, proving that P is properly supported.

The main motivation for the above definition stems from the following result, *properly supported.*

Proposition 4.2. *Let $P : C_c^\infty(U) \to C^\infty(U)$ be a continuous linear operator which is properly supported. Then*

(1) *P induces a continuous linear mapping $P : C_c^\infty(U) \to C_c^\infty(U)$.*

(2) *P extends uniquely to a continuous linear mapping $P : C^\infty(U) \to C^\infty(U)$.*

Proof. Let K be a compact subset of U. Since the inclusion $C_K^\infty(U) \hookrightarrow C_c^\infty(U)$ is continuous, P induces a continuous linear mapping from $C_K^\infty(U)$ to $C^\infty(U)$. From (4.2) and Chapter 2, we can obtain that there exists a compact subset K_1 of U such that P induces a continuous linear mapping from $C_K^\infty(U)$ to $C_{K_1}^\infty(U)$. Combining this with the continuity of the inclusion $C_{K_1}^\infty(U) \hookrightarrow C_c^\infty(U)$, then we deduce that P induces a continuous linear mapping from $C_K^\infty(U)$ to $C_c^\infty(U)$. Since K was arbitrary, we proved (1).

We can choose an open subset V of U such that $K \subset V \subset \overline{V} \subset U$ and $\overline{V}$ is compact. Applying (4.3) to $\overline{V}$, there exists a compact subset K_2 of U such that if $K_2 \cap \operatorname{supp} u = \varnothing$, then $\overline{V} \cap \operatorname{supp} Pu = \varnothing$. Choose $\psi \in C_c^\infty(U)$ be such that $\psi \equiv 1$ near K_2 and L denote the support of ψ. The multiplication by ψ, that is, the map $u \longmapsto \psi u$ is a continuous linear map from $C^\infty(U)$ to $C_L^\infty(U)$. Hence we can consider this map as a continuous linear map from $C^\infty(U)$ to $C_c^\infty(U)$ due to the continuity of the inclusion $C_L^\infty(U) \hookrightarrow C_c^\infty(U)$. From this, the operator $P\psi$ is a continuous linear opeartor from $C^\infty(U)$ to itself. Thus, for any $N \in \mathbb{N}_0$, there exists $N' \in \mathbb{N}_0$, a compact subset K' of U and a constant $C_{KK'NN'} > 0$ such that

$$\sup_{|\alpha|\le N} \sup_{x\in K} \left|\partial_x^\alpha (P(\psi u)(x))\right| \le C_{KK'NN'} \sup_{|\alpha|\le N'} \sup_{x\in K'} \left|\partial_x^\alpha u(x)\right| \quad \forall u \in C^\infty(U).$$

Let $u \in C_c^\infty(U)$. The function $(1-\psi)u \in C_c^\infty(U)$ vanishes on K_2 by construction. This means $K_2 \cap \operatorname{supp}(1-\psi)u = \varnothing$. From our previous construction, we obtain that $P((1-\psi)u)$ vanishes on $\overline{V}$, in other words, $Pu \equiv P(\psi u)$ on $\overline{V}$. Since V contains K, the last inequality can be changed as

$$\sup_{|\alpha|\le N} \sup_{x\in K} \left|\partial_x^\alpha (Pu(x))\right| \le C_{KK'NN'} \sup_{|\alpha|\le N'} \sup_{x\in K'} \left|\partial_x^\alpha u(x)\right| \quad \forall u \in C_c^\infty(U).$$

This inequailty means the semi-norm estimates required for the continuity of a linear operator from $C^\infty(U)$ to itself. Since $C^\infty(U)$ is a Fréchet space and $C_c^\infty(U)$ is dense in $C^\infty(U)$, the extension linear operator of P is well-defined and it is continuous. We proved (2). □

Proposition 4.3. *Let $R : C_c^\infty(U) \to C^\infty(U)$ be a smoothing operator which is properly supported. Then*

(1) *R induces a sequentially continuous linear mapping $R : \mathcal{D}'(U) \to C^\infty(U)$.*

(2) *R induces a sequentially continuous linear mapping $R : \mathcal{E}'(U) \to C_c^\infty(U)$.*

CHAPTER 5

Conormal Distributions and Schwartz Kernels

In this section, we define conormal distributions and then present a different point of view on ΨDOs, considered as operators whose Schwartz kernel are conormal distributions.

Definition 5.1 (Standard Amplitudes). *$\mathbb{S}^m(U \times U \times \mathbb{R}^n)$, $m \in \mathbb{R}$, consists of functions $a(x, y, \xi) \in C^\infty(U \times U \times \mathbb{R}^n)$ such that for any compact subset $K \subset U$ and multi-indicies α, β, and γ, a constant $C_{K\alpha\beta\gamma} > 0$ exists for which*

$$\left|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a(x, y, \xi)\right| \le C_{K\alpha\beta\gamma}(1 + |\xi|)^{m-|\gamma|} \quad \forall (x, y, \xi) \in K \times K \times \mathbb{R}^n.$$

Definition 5.2. $\mathbb{S}^{-\infty}(U \times U \times \mathbb{R}^n) := \bigcap_{m\in\mathbb{R}} \mathbb{S}^m(U \times U \times \mathbb{R}^n)$.

Let $a(x, y, \xi) \in \mathbb{S}^m(U \times U \times \mathbb{R}^n)$ for $m < -n$. $\check{a}_{\xi\to z}(x, y, z)$ denote the inverse Fourier transformation of $a(x, y, \xi)$ with respect to the variable ξ, which is the function defined as

$$\check{a}_{\xi\to z}(x, y, z) := \int e^{iz\cdot\xi} a(x, y, \xi) d\xi \quad \forall (x, y, z) \in U \times U \times \mathbb{R}^n.$$

Definition 5.3. $I_a(x, y) := (2\pi)^{-n} \check{a}_{\xi\to z}(x, y, x - y)$ *for all* $(x, y) \in U \times U$.

Proposition 5.1. *$I_a(x, y)$ is a well-defined continuous function on $U \times U$.*

Proof. For any compact subset $K \subset U$, there exists $C_K > 0$ such that

$$\left|a(x, y, \xi)\right| \le C_K(1 + |\xi|)^m \quad \forall (x, y, \xi) \in K \times K \times \mathbb{R}^n.$$

Since $m < -n$, this fact satisfies the integrability so that the function is well-defined and continuous. □

Now our plan is that $I_a(x, y)$ makes sense even if $m \ge -n$ by considering as an element of $C^\infty(U, \mathcal{D}'(U))$. First, let $m < -n$.

Fix $x \in U$ and regard $I_a(x, y)$ as a function of y. For $u \in C_c^\infty(U)$, we can obtain that

$$\int I_a(x, y)u(y)dy = (2\pi)^{-n} \int e^{ix\cdot\xi} \left(\int e^{-iy\cdot\xi} a(x, y, \xi)u(y)dy\right) d\xi$$

by applying Fubini's theorem.

Notice that $\int e^{-iy\cdot\xi} a(x, y, \xi)u(y)dy$ is a well-defined smooth function of x and ξ even if we don't need to assume $m < -n$ anymore. Furthermore, we have a proposition which plays an important role in our plan.

Proposition 5.2. *Let $a \in \mathbb{S}^m(U \times U \times \mathbb{R}^n)$. For any compact subsets K and L of U, any $N \in \mathbb{N}_0$ and any $\alpha \in \mathbb{N}_0^n$, there exists a constant $C_{KLN\alpha} > 0$ such that, for all $u \in C_K^\infty(U)$ and $(x,\xi) \in L \times \mathbb{R}^n$,*

$$\left|\partial_x^\alpha \left(\int e^{-iy\cdot\xi} a(x,y,\xi)u(y)dy\right)\right| \le C_{KLN\alpha}(1+|\xi|)^{m-2N} \sup_{|\beta|\le 2N} \sup_{y\in U} |\partial_y^\beta u(y)|.$$

PROOF. Let $u \in C_K^\infty(U)$. By using induction, we have

$$(1+|\xi|^2)^N \partial_x^\alpha \left(\int e^{-iy\cdot\xi} a(x,y,\xi)u(y)dy\right) = \int [(1-\Delta_y)^N(e^{-iy\cdot\xi})]\partial_x^\alpha a(x,y,\xi)u(y)dy.$$

Since $(1-\Delta_y)^N$ is a sum of differential operators of even degree , we can also obtain from integration by parts formula

$$\int [(1-\Delta_y)^N(e^{-iy\cdot\xi})]\partial_x^\alpha a(x,y,\xi)u(y)dy = \int e^{-iy\cdot\xi}(1-\Delta_y)^N[\partial_x^\alpha a(x,y,\xi)u(y)]dy.$$

From the above facts, there exists a constant $C_K > 0$ such that, for all $u \in C_K^\infty(U)$ and $(x,\xi) \in U \times \mathbb{R}^n$,

$$\left|\partial_x^\alpha \left(\int e^{-iy\cdot\xi} a(x,y,\xi)u(y)dy\right)\right| \le C_K(1+|\xi|^2)^{-N} \sup_{y\in K} \left|(1-\Delta_y)^N[\partial_x^\alpha a(x,y,\xi)u(y)]\right|.$$

In addition, Leibniz's theorem shows $(1-\Delta_y)^N[\partial_x^\alpha a(x,y,\xi)u(y)]$ is a linear combination of terms of the form $[\partial_x^\alpha \partial_y^\beta a(x,y,\xi)]\partial_y^\gamma u(y)$ with $|\beta|+|\gamma| \le 2N$. From the definition of a, there exists $C_{K\cup L,\alpha\beta} > 0$ such that $\left|\partial_x^\alpha \partial_y^\beta a(x,y,\xi)\right| \le C_{K\cup L,\alpha\beta}(1+|\xi|)^m$ for all $(x,y,\xi) \in (K\cup L) \times (K \cup L) \times \mathbb{R}^n$.

If we combine the above results, then we can deduce that there exists a constant $C'_{KLN\alpha} > 0$ such that

$$\sup_{y\in K} \left|(1-\Delta_y)^N[\partial_x^\alpha a(x,y,\xi)u(y)]\right| \le C'_{KLN\alpha}(1+|\xi|)^m \sup_{|\beta|\le 2N} \sup_{y\in U} |\partial_y^\beta u(y)|$$

for all $u \in C_K^\infty(U)$ and $(x,\xi) \in L \times \mathbb{R}^n$. Also $(1+|\xi|^2)^{-N} \le 2^N(1+|\xi|)^{-2N}$ implies $(1+|\xi|^2)^{-N}(1+|\xi|)^m \le 2^N(1+|\xi|)^{m-2N}$. This proves the proposition. □

For each fixed $x \in U$, we can regard $I_a(x,y)$ as a distribution as follows,

$$(5.1)\quad \langle I_a(x,y), u(y)\rangle := (2\pi)^{-n} \int e^{ix\cdot\xi} \left(\int e^{-iy\cdot\xi} a(x,y,\xi)u(y)dy\right) d\xi \;\; \forall u \in C_c^\infty(U).$$

Since we apply the last proposition when $\alpha = 0$, $L = \{x\}$ and large N such that $m - 2N < -n$, then we obtain the semi-norm estimate required for the continuity of a linear operator from $C_c^\infty(U)$ to $\mathbb{C}$.

Also from Proposition 5.2, we can consider $I_a(x,y)$ as an element of $C^\infty(U, \mathcal{D}'(U))$ without the condition $m < -n$. After using the embedding of $C^\infty(U, \mathcal{D}'(U))$ into $\mathcal{D}'(U \times U)$, we can also think $I_a(x,y)$ as an element of $\mathcal{D}'(U \times U)$.

Definition 5.4. *The distribution $I_a(x,y)$ is called the conormal distribution with amplititude $a(x,y,\xi)$.*

Remark 5.1. As we already saw in the proof of the above proposition, if we choose the integer N so that $m - 2N < -n$ and use Fubini's theorem, then we can rewrite (5.1) as

$$\langle I_a(x,y), u(y)\rangle = (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi}(1+|\xi|^2)^{-N}(1-\Delta_y)^N[a(x,y,\xi)u(y)]d\xi dy.$$

Let $p(x,\xi) \in \mathbb{S}^m(U \times \mathbb{R}^n)$. Then the definition of standard symbols implies that, for any compact subset $L \subset U$ and $\alpha \in \mathbb{N}_0^n$, there exists a constant $C_{L\alpha} > 0$ such that, for all $u \in \mathcal{S}(\mathbb{R}^n)$ and $(x,\xi) \in L \times \mathbb{R}^n$,

$$\left|\partial_x^\alpha\Big(p(x,\xi)u(\xi)\Big)\right| \leq C_{L\alpha}\mathfrak{q}_{m+n+1}(u)(1+|\xi|)^{-(n+1)}.$$

It follows that $p(x,\xi)$ can be seen as an element of $C^\infty(U, \mathcal{S}'(\mathbb{R}^n))$ by similar arguments which we did right before and hence we can define the inverse Fourier transform with respect to ξ as the element $\check{p}_{\xi\to y}(x,y)$ of $C^\infty(U,\mathcal{S}'(\mathbb{R}^n))$ given by,

$$\langle \check{p}_{\xi\to y}(x,y), u(y)\rangle := \langle p(x,\xi), \check{u}(\xi)\rangle = \int p(x,\xi)\check{u}(\xi)d\xi \quad \forall x \in U \;\; \forall u \in \mathcal{S}(\mathbb{R}^n).$$

In addition, let us denote by $\check{p}_{\xi\to y}(x, x-y)$ the element of $C^\infty(U,\mathcal{S}'(\mathbb{R}^n))$ obtained by pushing-forward $\check{p}_{\xi\to y}(x,y)$ under the affine transformation $y \to x - y$, that is,

$$\langle \check{p}_{\xi\to y}(x, x-y), u(y)\rangle_y := \langle \check{p}_{\xi\to y}(x,y), u(x-y)\rangle_y \quad \forall x \in U \;\; \forall u \in \mathcal{S}(\mathbb{R}^n).$$

Proposition 5.3. *Let $p(x,\xi) \in \mathbb{S}^m(U\times\mathbb{R}^n)$ and let us regard $p(x,\xi)$ as an amplitude in $\mathbb{S}^m(U \times U \times \mathbb{R}^n)$ not depending on the middle variable y. Then the conormal distribution $I_p(x,y)$ makes sense as an element of $C^\infty(U,\mathcal{S}'(\mathbb{R}^n))$ and we have*

$$I_p(x,y) = (2\pi)^{-n}\check{p}_{\xi\to y}(x, x-y).$$

Lemma 5.1. *Let $a \in \mathbb{S}^m(U \times U \times \mathbb{R}^n)$. Then, for any $\alpha \in \mathbb{N}_0^n$,*

$$(y-x)^\alpha I_a(x,y) = I_{D_\xi^\alpha a}(x,y).$$

Now we denote by Γ the diagonal of $U \times U$, that is,

$$\Gamma := \{(x,x)|x \in U\} \subset U \times U.$$

Proposition 5.4. *Let $a \in \mathbb{S}^m(U\times U\times\mathbb{R}^n)$ and let us regard $I_a(x,y)$ as a distribution on $U \times U$.*

(1) *If $m < -n$, then*

$$I_a(x,y) \in C^k(U\times U) \quad \textit{where } k := -([m]+n+1).$$

(2) *$I_a(x,y)$ is smooth on $U \times U\backslash\Gamma$.*

Proof. First assume $m < -n$. Then $I_a(x,y)$ is given by

$$I_a(x,y) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,y,\xi)d\xi.$$

from Definition 5.3, and set $k := -([m]+n+1)$ and $\delta := -(m+n+k)$.

Let α and β be in $\mathbb{N}_0^n$ such that $|\alpha|+|\beta| \le k$. Then, $\partial_x^\alpha \partial_y^\beta (e^{i(x-y)\cdot\xi} a(x,y,\xi))$ is a linear combination of terms of the form,

$$\xi^{\alpha'+\beta'} e^{i(x-y)\cdot\xi} \partial_x^{\alpha''} \partial_y^{\beta''} a(x,y,\xi) \quad \text{where } \alpha'+\alpha''=\alpha \text{ and } \beta'+\beta''=\beta.$$

Combining this fact and the property of a, we deduce that for any compact subset K of U, there exists $C_{K\alpha\beta} > 0$ such that, for all $(x,y,\xi) \in K \times K \times \mathbb{R}^n$,

$$\left|\partial_x^\alpha \partial_y^\beta (e^{i(x-y)\cdot\xi} a(x,y,\xi))\right| \le C_{K\alpha\beta}(1+|\xi|)^{m+|\alpha|+|\beta|} \le C_{K\alpha\beta}(1+|\xi|)^{-n-\delta}.$$

Since $\delta \in (0,1]$, the last inequalities proves (1).

Now we do not assume $m < -n$. For any $N \in \mathbb{N}_0$, we get

$$|y-x|^{2N} I_a(x,y) = I_{\Delta_\xi^N a}(x,y) \tag{5.2}$$

from Lemma 5.1 where $\Delta_\xi = -\sum_{j=1}^n \partial_{\xi_j}^2$. Also notice that $\Delta_\xi^N a(x,y,\xi)$ is an element of $\mathbb{S}^{m-2N}(U \times U \times \mathbb{R}^n)$.

Fix $l \in \mathbb{N}$ and choose $N \in \mathbb{N}_0$ so that $m-2N < -n-l$. Then $l \le -([m-2N]+n+1)$ and for any amplitude $b \in \mathbb{S}^{m-2N}(U \times U \times \mathbb{R}^n)$, the associated conormal distribution $I_b(x,y)$ is contained in $C^l(U \times U)$ from (1). On $U \times U \backslash \Gamma$, we can modify (5.2) as

$$I_a(x,y) = |y-x|^{-2N} I_{\Delta_\xi^N a}(x,y).$$

Since $I_{\Delta_\xi^N a(x,y)} \in C^l(U \times U)$ and $|y-x|^{-2N}$ is smooth on $U \times U \backslash \Gamma$, it follows that $I_a(x,y)$ is contained in $C^l(U \times U \backslash \Gamma)$. As l was arbitrary, now we proved (2). ☐

Remark 5.1. If $p(x,\xi)$ is a symbol in $\mathbb{S}^m(U \times \mathbb{R}^n)$ and we regard as an amplitude in $\mathbb{S}^m(U \times U \times \mathbb{R}^n)$ not depending on the variable y, then it can be shown in a similary way as before that $I_p(x,y)$ is smooth on $U \times \mathbb{R}^n \backslash \Gamma$ and is in $C^k(U \times \mathbb{R}^n)$ if $m < -n$ and $k = -([m]+n+1)$.

We shall now use conormal distributions to define ΨDOs in terms of amplitudes. As $a \in \mathbb{S}^m(U \times U \times \mathbb{R}^n)$, the conormal distribution $I_a(x,y)$ is an element of $C^\infty(U, \mathcal{D}'(U))$. Therefore $I_a(x,y)$ is the Schwartz kernel of the continuous linear operator $A : C_c^\infty(U) \to C^\infty(U)$ defined by

$$Au(x) := \langle I_a(x,y), u(y)\rangle_y \quad \forall x \in U \;\; \forall u \in C_c^\infty(U).$$

Definition 5.5. *Let $a \in \mathbb{S}^m(U \times U \times \mathbb{R}^n)$. Then the operator with Schwartz kernel $I_a(x,y)$ is called the ΨDO with amplitude $a(x,y,\xi)$.*

As we shall see this definition of ΨDOs is essentially equivalent to the previous definition of standard ΨDOs in terms of symbols and smoothing operators.

Proposition 5.5. *Let $p(x,\xi) \in \mathbb{S}^m(U \times \mathbb{R}^n)$ and let us regard $p(x,\xi)$ as an amplitude in $\mathbb{S}^m(U \times U \times \mathbb{R}^n)$ not depending on the middle variable y. Then $p(x,D)$ has Schwartz kernel*

$$k_{p(x,D)}(x,y) = I_p(x,y) = (2\pi)^{-n} \check{p}_{\xi\to y}(x, x-y).$$

In particular, in the sense of Definition 5.5, $p(x,D)$ is the ΨDO with amplitude $p(x,\xi)$.

Proposition 5.6. *Any $R \in \Psi^{-\infty}(U)$ is the ΨDO associated to an amplitude in $\mathbb{S}^{-\infty}(U \times U \times \mathbb{R}^n)$.*

Proposition 5.7. *Let $P \in \Psi^m(U)$. Also let $p(x,\xi) \in S^m(U \times \mathbb{R}^n)$ be the symbol of P and $k_P(x,y) \in C^\infty(U, \mathcal{D}'(U))$ be the Schwartz kernel, which we shall regard as a distribution on $U \times U$. Then $k_P(x,y)$ is smooth on $U \times U \backslash \Gamma$ and we have*

$$k_P(x,y) = I_p(x,y) \mod C^\infty(U \times U).$$

For $\xi \in \mathbb{R}^n$, let us denote by e_ξ the smooth function on U defined by

$$e_\xi(x) := e^{ix\cdot\xi} \qquad \forall x \in U. \tag{5.3}$$

Theorem 5.1. *Let A be a properly supported ΨDO defined by means of an amplitude $a(x,y,\xi) \in \mathbb{S}^m(U \times U \times \mathbb{R}^n)$. For $(x,\xi) \in U \times \mathbb{R}^n$, set*

$$p(x,\xi) = e^{-ix\cdot\xi}(Ae_\xi)(x)$$

where the function e_ξ is defined as in (5.3). Then $p(x,\xi)$ is a symbol in $\mathbb{S}^m(U \times \mathbb{R}^n)$ such that

$$A = p(x,D) \quad \text{and} \quad p(x,\xi) \sim \sum \frac{1}{\alpha!} \partial_\xi^\alpha D_y^\alpha a(x,y,\xi)\Big|_{y=x} \tag{5.4}$$

where the asymtotic expansion is taken in the sense of (3.2).

Remark 5.2. Since A is properly supported, we can consider A as a continuous linear mapping from $C^\infty(U)$ to itself by Proposition 4.2. Therefore Ae_ξ is well-defined and also each function $\partial_\xi^\alpha D_y^\alpha a(x,y,\xi)\Big|_{y=x}$ is a symbol in $\mathbb{S}^{m-|\alpha|}(U \times \mathbb{R}^n)$ so that the asymptotic expansion (5.4) makes sense.

Proof. Let K be a compact subset of U. In the sense of (4.3), we can choose K_2 and let $\psi \in C_c^\infty(U)$ so that $\psi \equiv 1$ near K_2. Then for any $u \in C^\infty(U)$, $Au \equiv A(\psi u)$ near K from the proof of Proposition 4.2. Using affine transformations, we obtain that

$$\begin{aligned}
p(x,\xi) &= e^{-ix\cdot\xi}(Ae_\xi)(x) = e^{-ix\cdot\xi}(A(\psi e_\xi))(x) \\
&= (2\pi)^{-n} e^{-ix\cdot\xi} \int e^{ix\cdot\eta} \left(\int e^{-iy\cdot\eta} a(x,y,\eta) e^{iy\cdot\xi} \psi(y) dy \right) d\eta \\
&= (2\pi)^{-n} \int e^{ix\cdot\eta} \left(\int e^{-iy\cdot\eta} a(x,y,\xi+\eta) \psi(y) dy \right) d\eta \\
&= (2\pi)^{-n} \int e^{ix\cdot\eta} \left(\int e^{-iy\cdot\eta} a(x,y,\xi+\eta) dy \right) d\eta
\end{aligned}$$

for all x near K and $\xi \in \mathbb{R}^n$. Let supp $\psi = L$. Then by modifying the proof of Proposition 5.2 slightly, we can deduce that for any $M \in \mathbb{N}_0$ and α and β in $\mathbb{N}_0^n$, there exists $C_{KM\alpha\beta} > 0$ such that, for all $x \in K$, ξ and $\eta \in \mathbb{R}^n$,

$$\begin{aligned}
\left| \partial_x^\alpha \left(\int e^{i(x-y)\cdot\eta} \partial_\xi^\beta a(x,y,\xi) \psi(y) dy \right) \right| &\le C_{KM\alpha\beta}(1+|\eta|^2)^{-M}(1+|\xi|)^{m-|\beta|}(1+|\eta|)^{|\alpha|} \\
&\le 2^M C_{KM\alpha\beta}(1+|\xi|)^{m-|\beta|}(1+|\eta|)^{|\alpha|-2M}.
\end{aligned}$$

Let $b(x,y,\xi)=a(x,x+y,\xi)$ and $\hat{b}(x,\eta,\xi)=(2\pi)^{-n}\displaystyle\int e^{-iy\cdot\eta}b(x,y,\xi)dy$. Then

$$p(x,\xi)=(2\pi)^{-n}\int\left(\int e^{-iy\cdot\eta}b(x,y,\xi+\eta)dy\right)d\eta=\int\hat{b}(x,\eta,\xi+\eta)d\eta.$$

In addition, from the above inequality, there exists $C'_{KM\alpha\beta}>0$ such that

$$\left|\partial_x^\alpha\partial_\xi^\beta\hat{b}(x,\eta,\xi)\right|\le C'_{KM\alpha\beta}(1+|\xi|)^{m-|\beta|}(1+|\eta|)^{|\alpha|-2M}$$

for $\forall x\in K$, ξ and $\eta\in\mathbb{R}^n$. This proves that $p(x,\xi)$ is an element of $\mathbb{S}^m(U\times\mathbb{R}^n)$.

Using Fubini's theorem, for $x\in K$ and $u\in C_c^\infty(U)$, we have

$$\begin{aligned}\int e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)d\xi &= (2\pi)^{-n}\int\hat{u}(\xi)\left(\int e^{ix\cdot\eta}\left(\int e^{-iy\cdot\eta}a(x,y,\eta)e^{iy\cdot\xi}dy\right)d\eta\right)d\xi\\ &= (2\pi)^{-n}\int e^{ix\cdot\eta}\left(\int e^{-iy\cdot\eta}a(x,y,\eta)u(y)dy\right)d\eta\\ &= Au(x).\end{aligned}$$

Since K was arbitrary, so we can conclude that $A=p(x,D)$.

If we consider the Taylor series of $\hat{b}(x,\eta,\xi+\eta)$ about the point ξ, then we can obtain that there exists a constant $C_{KMN}>0$ such that, $\forall x\in K$, $\forall M,N\in\mathbb{N}_0$, and $\forall\xi,\eta\in\mathbb{R}^n$,

$$\begin{aligned}\left|\partial_x^\alpha\partial_\xi^\beta\hat{b}(x,\eta,\xi+\eta)-\sum_{|\gamma|<N}\frac{1}{\gamma!}\partial_x^\alpha\partial_\xi^{\beta+\gamma}\hat{b}(x,\eta,\xi)\eta^\gamma\right| &\le \tilde{C}_{Nn}|\eta|^N\sup_{\substack{0\le t\le1\\|\gamma|=N}}\left|\partial_x^\alpha\partial_\xi^{\beta+\gamma}\hat{b}(x,\eta,\xi+t\eta)\right|\\ &\le C_{KMN}|\eta|^N(1+|\eta|)^{|\alpha|-2M}\sup_{0\le t\le1}(1+|\xi+t\eta|)^{m-|\beta|-N}\\ &\le C_{KMN}(1+|\eta|)^{|\alpha|+N-2M}(1+|\xi|)^{m-|\beta|-N}(1+|\eta|)^{|m-|\beta|-N|},\end{aligned}$$

where $\tilde{C}_{Nn}=\begin{pmatrix}N+n-1\\n-1\end{pmatrix}$.

By choosing M such that $|\alpha|+N+|m+|\beta|-N|+n<2M$ and using Fourier inverse transformation, we deduce that

$$\begin{aligned}\int\partial_x^\alpha\partial_\xi^{\beta+\gamma}\hat{b}(x,\eta,\xi)\eta^\gamma d\eta &= \int e^{ix\cdot\eta}\left(\int\eta^\gamma e^{-iy\cdot\eta}\partial_x^\alpha\partial_\xi^{\beta+\gamma}a(x,y,\xi)dy\right)d\eta\\ &= D_y^\gamma\partial_x^\alpha\partial_\xi^{\beta+\gamma}a(x,y,\xi)\Big|_{y=x}\end{aligned}$$

and

$$\left|\partial_x^\alpha\partial_\xi^\beta p(x,\xi)-\sum_{|\gamma|<N}\frac{1}{\gamma!}D_y^\gamma\partial_x^\alpha\partial_\xi^{\beta+\gamma}a(x,y,\xi)\Big|_{y=x}\right|\le C(1+|\xi|)^{m-|\beta|-N}$$

for some constant $C>0$. So this implies the last part of this theorem. □

As we shall see throughout the rest of the chapter, Theorem 5.1. is a very useful tool to prove properties of ΨDOs. The first application of this result we shall see is to characterize smoothing operators among ΨDOs.

Theorem 5.2. *Let $p\in\mathbb{S}^m(U\times\mathbb{R}^n)$. Then $p(x,D)$ is smoothing if and only if $p(x,\xi)$ is contained in $\mathbb{S}^{-\infty}(U\times\mathbb{R}^n)$.*

Proof. Let $P = p(x, D)$. First suppose that $p \in \mathbb{S}^{-\infty}(U \times \mathbb{R}^n)$. Then it follows from Proposition 5.4 that $I_p(x, y)$ is of class C^k for all $k \in \mathbb{N}$, that is, $I_p(x, y)$ is smooth. Also from Proposition 5.5, the Schwartz kernel of P is $I_p(x, y)$ so that P is a smoothing operator.

Conversely, suppose that P is a smoothing operator. Let φ and ψ be in $C_c^\infty(U)$ such that $\psi \equiv 1$ near supp φ. For $(x, \xi) \in U \times \mathbb{R}^n$, we define $p_{\phi,\psi}(x, \xi)$ as

$$p_{\varphi,\psi}(x, \xi) = e^{-ix\cdot\xi}\varphi(x)P(\psi e_\xi)(x).$$

Let $Q = \varphi P \psi$. Then $Qu(x) = \langle \varphi(x)k_P(x, y)\psi(y), u(y)\rangle_y$ for all $u \in C_c^\infty(U)$. Also from Proposition 5.5 and 5.6, we obtain that P is the ΨDO associated to an amplitude $p(x, \xi)$ in $\mathbb{S}^{-\infty}(U \times U \times \mathbb{R}^n)$. Therefore Q has a compactly supported Schwartz kernel so that it is properly supported from Example 4.1 and Q is the ΨDO associated to the amplitude $\varphi(x)p(x, \xi)\psi(y)$ because P is the ΨDO associated to the amplitude $p(x, \xi)$ from Proposition 5.6. Applying Theorem 5.1 to Q and using the fact $\psi \equiv 1$ near supp φ, it follows that $p_{\varphi,\psi}(x, \xi)$ is a symbol in $\mathbb{S}^{-\infty}(U \times \mathbb{R}^n)$ and we get

$$p_{\varphi,\psi}(x, \xi) \sim \frac{1}{\alpha!}\partial_\xi^\alpha D_y^\alpha \varphi(x)p(x, \xi)\psi(y)\Big|_{y=x} = \varphi(x)p(x, \xi).$$

In addition, $\varphi(x)p(x, \xi)$ is $\mathbb{S}^{-\infty}(U \times \mathbb{R}^n)$ by Proposition 3.2. For any compact subset $K \subset U$, we can choose $\varphi \in C_c^\infty(U)$ such that $\varphi \equiv 1$ near K. Then $p(x, \xi)$ satisfies the estimate in Remark 3.2 on $K \times \mathbb{R}^n$. This shows that $p(x, \xi)$ is a symbol in $\mathbb{S}^{-\infty}(U \times \mathbb{R}^n)$. □

Theorem 5.3. *Let* $P \in \Psi^m(U)$.

(1) *There exists a properly supported operator* $Q \in \Psi^m(U)$ *such that*
$$P = Q \mod \Psi^{-\infty}(U).$$
(2) *If* P *is properly supported, then there exists a symbol* $p(x, \xi) \in S^m(U \times \mathbb{R}^n)$ *such that*
$$P = p(x, D).$$
(3) *We can choose a symbol* $p(x, \xi)$ *of* P *so that* $p(x, D)$ *is properly supported.*

Proof. Let $(\varphi_i)_{i\in I}$ be a partition of unity subordinate to a locally finite open covering $(U_i)_{i\in I}$ of U. For each $i \in I$, choose $\psi_i \in C_c^\infty(U_i)$ such that $\psi_i \equiv 1$ near supp φ_i. Also we define a linear operator $Q : C_c^\infty(U) \to C^\infty(U)$ as

$$Qu := \sum_{i\in I} \varphi_i P(\psi_i u) \quad \forall u \in C_c^\infty(U).$$

It can be shown that Q is properly supported. Let $P = p(x, D) + R$ where R is smoothing. Then we already showed that $\sum_{i\in I} \varphi_i p(x, D)\psi_i$ is a ΨDO of order m in the proof of Theorem 5.2. It is clear that $\sum_{i\in I} \varphi_i R\psi_i$ is also smoothing. From these facts, we know that Q is a ΨDO of order m.

Set $\tilde{R} := P - Q$. Then the Schwartz kernel of $\tilde{R}$ is

$$k_{\tilde{R}}(x, y) = \sum_{i\in I} \varphi_i(x)(1 - \psi_i(y))k_P(x, y).$$

From our construction of φ_i and ψ_i, we know that $\sum_{i\in I}\varphi_i(x)(1-\psi_i(y))$ is a smooth function on $U\times U$ which vanishes near the diagonal Γ. In addition, $k_P(x,y)$ is smooth on $U\times U\backslash\Gamma$ by Proposition 5.4. We deduce that $k_{\tilde{R}}$ is a smooth Schwartz kernel, that is, $\tilde{R}$ is smoothing so that $P = Q \mod \Psi^{-\infty}(U)$. We proved the first part of the proposition.

Now for proving the second part, assume that P is properly supported. Let $p(x,\xi)$ be the symbol of P and then $p(x,D)$ is the ΨDO with an amplitude in $\mathbb{S}^{\Re m}(U\times U\times\mathbb{R}^n)$ by Proposition 5.5. Also by Proposition 5.6, R is the ΨDO associated to $\mathbb{S}^{-\infty}(U\times U\times\mathbb{R}^n)$. So we can obtain the fact that P is the the ΨDO with some amplitude in $\mathbb{S}^{\Re m}(U\times U\times\mathbb{R}^n)$. Using Theorem 5.1 in case of P and have a symbol $\tilde{p}(x,\xi)$ in $\mathbb{S}^{\Re m}(U\times\mathbb{R}^n)$ such that $P=\tilde{p}(x,D)$. Finally, using Theorem 5.2, $\tilde{p}(x,\xi)-p(x,\xi)$ is contained in $\mathbb{S}^{-\infty}(U\times\mathbb{R}^n)$, that is, $\tilde{p}(x,\xi)\sim p(x,\xi)$. Thus $\tilde{p}(x,\xi)$ is a classical symbol in $S^m(U\times\mathbb{R}^n)$. We proved the second part.

Suppose that P is not properly supported. By the first part of our proposition, we know that there exist a properly supported operator $Q\in\Psi^m(U)$ and a smoothing operator R such that $P=Q+R$. Also by the second part, let $Q=q(x,D)$ where $q(x,\xi)\in S^m(U\times\mathbb{R}^n)$. This proves the third part of our proposition. □

We define the notion of asymptotic expansion of conormal distributions that is an interpretation of the notion of asymptotic expansion of symbols in the sense of (3.2) as follows.

Definition 5.6. *Let $k(x,y)\in\mathcal{D}'(U\times U)$ and $(a_j)_{j\geq 0}$ be a sequence of amplitudes such that $a_j\in\mathbb{S}^{m_j}(U\times U\times\mathbb{R}^n)$ with $m_j\downarrow-\infty$. We say that $k(x,y)$ is asymptotic to the $I_{a_j}(x,y)$, and we write*

$$k(x,y)\sim\sum_{j\geq 0}I_{a_j}(x,y),$$

when, for any $N\in\mathbb{N}_0$, there exists $J\in\mathbb{N}$ such that

$$k(x,y)-\sum_{j<J}I_{a_j}(x,y)\in C^N(U\times U). \tag{5.5}$$

Proposition 5.8. *Let $p(x,\xi)\in\mathbb{S}^m(U\times\mathbb{R}^n)$ and $(p_j)_{j\geq 0}$ be a sequence of symbols on $U\times\mathbb{R}^n$ such that $p_j(x,\xi)\in\mathbb{S}^{m_j}(U\times\mathbb{R}^n)$ with $m_j\downarrow-\infty$. Then the following asymptotic expansions are equivalent*

$$p(x,\xi)\sim\sum_{j\geq 0}p_j(x,\xi)\quad\text{and}\quad I_p(x,y)\sim\sum_{j\geq 0}I_{p_j}(x,y), \tag{5.6}$$

where the former asymptotic expansion is taken in the sense of (3.2) and the latter is taken in the sense of (5.5).

Proof. Suppose that $p(x,\xi)\sim\sum_{j\geq 0}p_j(x,\xi)$ in the sense of (3.2). Let $N\in\mathbb{N}_0$. Then we can choose $J\in\mathbb{N}$ such that $m_J<-n$ and $N<-(m_J+n+1)$. We define $r_j\in\mathbb{S}^{m_J}(U\times\mathbb{R}^n)$ as

$$r_j(x,\xi):=p(x,\xi)-\sum_{j<J}p_j(x,\xi).$$

Furthermore, Proposition 5.4 ensures us that

$$I_p(x,y) - \sum_{j<J} I_{p_j}(x,y) = I_{r_j}(x,y) \in C^N(U \times U).$$

Therefore it follows that $I_p(x,y) \sim \sum_{j\geq 0} I_{p_j}(x,y)$ in the sense of (5.5).

Conversely, suppose that $I_p(x,y) \sim \sum_{j\geq 0} I_{p_j}(x,y)$ in the sense of (5.5). For each $N \in \mathbb{N}_0$, there exists a least element $J(N) \in \mathbb{N}$ such that $I_p(x,y) - \sum_{j<J(N)} I_{p_j}(x,y) \in C^N(U \times U)$. If $\{J(N)\}_{N\in\mathbb{N}_0}$ is bounded, then we are done. So we may assume that $\{J(N)\}_{N\in\mathbb{N}_0}$ is not bounded, that is, $\lim_{N\to\infty} J(N) = \infty$. By Lemma 3.1, we can find a symbol $\tilde{p}(x,\xi) \in \mathbb{S}^{m_0}(U \times \mathbb{R}^n)$ such that $\tilde{p}(x,\xi) \sim \sum_{j\geq 0} p_j(x,\xi)$.

Fix $N \in \mathbb{N}_0$. Then it follows from the first part of the proof that $I_{\tilde{p}}(x,y) - \sum_{j<J'} I_{p_j}(x,y) \in C^N(U \times U)$ for all large J'. We can obtain that

$$(I_p - I_{\tilde{p}})(x,y) - \sum_{J(N)\leq j<J'} I_{p_j}(x,y) \in C^N(U \times U).$$

It follows that

$$(I_p - I_{\tilde{p}})(x,y) \in C^{\min\{N,-([m_{J(N)}]+n+1)\}}(U \times U).$$

This means that $I_{p-\tilde{p}}(x,y)$ is smooth. By Proposition 5.5, $(p-\tilde{p})(x,D)$ is smoothing. In addition, by Theorem 5.2, $p(x,\xi) - \tilde{p}(x,\xi)$ is contained in $\mathbb{S}^{-\infty}(U \times \mathbb{R}^n)$. Thus,

$$p(x,\xi) \sim \sum_{j\geq 0} p_j(x,\xi).$$

The proof is complete. □

CHAPTER 6

Transposes and Adjoints of ΨDOs

Definition 6.1. *Let $P \in \Psi^m(U)$. We define its transpose P^t as the operator $\mathcal{E}'(U) \to \mathcal{D}'(U)$ such that*

$$\langle P^t u, v\rangle = \langle u, Pv\rangle \qquad \forall u \in \mathcal{E}'(U) \;\; \forall v \in C_c^\infty(U).$$

Lemma 6.1. *If P is properly supported, then P^t is properly supported, that is, for any compact $K \subset U$, there exist compact subsets K_1 and K_2 of U such that, for all $u \in \mathcal{E}'(U)$.*

$$\operatorname{supp} u \subset K \Rightarrow \operatorname{supp} P^t u \subset K_1 \;\; and \;\; K_2 \cap \operatorname{supp} u = \varnothing \Rightarrow K \cap \operatorname{supp} P^t u = \varnothing.$$

Proposition 6.1. *Let $P \in \Psi^m(U)$ have the symbol $p(x,\xi) \sim \sum p_{m-j}(x,\xi)$.*

(1) *The transpose P^t is a ΨDO of order m on U.*

(2) *Let $p^t(x,\xi) \sim \sum p^t_{m-j}(x,\xi)$ be the symbol of P^t. Then*

$$p^t(x,\xi) \sim \sum \frac{(-1)^{|\alpha|}}{\alpha!}(\partial_\xi^\alpha D_x^\alpha) p(x,-\xi). \tag{6.1}$$

(3) *The homogeneous components of the symbol $p^t(x,\xi)$ are given by*

$$p^t_m(x,\xi) = p_m(x,-\xi), \tag{6.2}$$

$$p^t_{m-j}(x,\xi) = \sum_{|\alpha|+k=j} \frac{(-1)^{|\alpha|}}{\alpha!}(\partial_\xi^\alpha D_x^\alpha) p_{m-k}(x,-\xi) \quad \forall j \geq 1. \tag{6.3}$$

Proof. By Theorem 5.3, we can choose the symbol $p(x,\xi)$ of P so that $p(x,D)$ is properly supported. Let $P = p(x,D) + R$ where R is a smoothing operator. Let u and v be in $C_c^\infty(U)$. Since

$$\begin{aligned}
\langle R^t u, v\rangle &= \langle u, Rv\rangle = \langle Rv, u\rangle \\
&= \langle k_{R^t}(x,y), u(x)v(y)\rangle \\
&= \langle k_R(y,x), v(y)u(x)\rangle,
\end{aligned}$$

it follows that $k_{R^t}(x,y) = k_R(y,x)$ so that we can also consider R^t as a smoothing operator. So, without loss of generality, we may assume that $P = p(x,D)$. By our construction of $p(x,D)$, P is properly supported and P^t is also by Lemma 6.1.

Choose $N \in \mathbb{N}$ such that $m + n < 2N$ and then we get from the proof of Proposition 5.2 that

$$\begin{aligned}
\langle P^t v(x), u(x)\rangle &= \langle v(y), Pu(y)\rangle = \langle v(y), \langle I_p(y,x), u(x)\rangle_x\rangle = \langle I_p(y,x), v(y)u(x)\rangle \\
&= (2\pi)^{-n} \iiint e^{i(y-x)\cdot\xi}(1+|\xi|^2)^{-N}(1-\Delta_y)^N[p(y,\xi)v(y)]u(x)\,dx\,d\xi\,dy.
\end{aligned}$$

In addition, we define $a(x,y,\xi) \in \mathbb{S}^{\Re m}(U \times U \times \mathbb{R}^n)$ as

$$a(x,y,\xi) := p(y,-\xi) \qquad \forall (x,y,\xi) \in U \times U \times \mathbb{R}^n.$$

By changing of variable $\xi \to -\xi$ and using Fubini's theorem, we obtain that

$$\begin{aligned} \langle P^t v(x), u(x)\rangle &= (2\pi)^{-n} \int u(x) \left(\iint e^{i(x-y)\cdot\xi}(1+|\xi|^2)^{-N}(1-\Delta_y)^N[p(y,-\xi)v(y)]d\xi dy \right) dx \\ &= (2\pi)^{-n} \int u(x) \langle I_a(x,y), v(y)\rangle_y dx = (2\pi)^{-n} \langle \langle I_a(x,y), v(y)\rangle_y, u(x)\rangle. \end{aligned}$$

Therefore P^t is a ΨDO with amplitude $a(x,y,\xi)$.

Now we can apply Theorem 5.1 in case of P^t so that we deduce that there exists $p^t(x,\xi) \in \mathbb{S}^{\Re m}(U \times \mathbb{R}^n)$ such that

$$p^t(x,\xi) \sim \sum \frac{1}{\alpha!} \partial_y^\alpha D_\xi^\alpha a(x,y,\xi)\Big|_{y=x} = \sum \frac{(-1)^{|\alpha|}}{\alpha!} \partial_x^\alpha D_\xi^\alpha p(x,-\xi),$$

where $P^t = p^t(x,D)$ since P^t is properly supported.

Also let $q_k(x,\xi) = \displaystyle\sum_{|\alpha|=k} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_x^\alpha D_\xi^\alpha p(x,-\xi)$. Then $q_k(x,\xi) \in S^{m-k}(U \times \mathbb{R}^n)$. Using Lemma 3.3, it follows that $p^t(x,\xi) \in S^m(U \times \mathbb{R}^n)$ and

$$\begin{aligned} p^t_{m-j}(x,\xi) &= \sum_{k \le j} \sum_{|\alpha|=k} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_x^\alpha D_\xi^\alpha p_{m-j+k}(x,-\xi) \\ &= \sum_{|\alpha|=j-l} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_x^\alpha D_\xi^\alpha p_{m-l}(x,-\xi). \end{aligned}$$

So we proved all of parts of this proposition. □

Remark 6.1. From the proof of Proposition 6.1, we get

$$P^t v(x) = (2\pi)^{-n} \int e^{ix\cdot\xi}(1+|\xi|^2)^{-N} \left(\int e^{-iy\cdot\xi}(1-\Delta_y)^N [p(y,-\xi)v(y)]dy \right) d\xi,$$

that is, $P^t v(x)$ is the inverse Fourier transform of

$$\xi \longrightarrow (2\pi)^{-n}(1+|\xi|^2)^{-N} \left(\int e^{-iy\cdot\xi}(1-\Delta_y)^N [p(y,-\xi)v(y)]dy \right) d\xi.$$

Proposition 6.2. *Let $P \in \Psi^m(U)$. Then P uniquely extends to a continuous linear operator*

$$P : \mathcal{E}'(U) \to \mathcal{D}'(U).$$

Proof. By Proposition 6.1, the transpose P^t is a ΨDO of order m, and hence induces a continuous linear mapping $P^t : C_c^\infty(U) \to C^\infty(U)$. Then its transpose is a continuous linear mapping

$$(P^t)^t : \mathcal{E}'(U) \to \mathcal{D}'(U).$$

For all $u, v \in C_c^\infty(U)$, we have

$$\langle (P^t)^t u, v\rangle = \langle u, P^t v\rangle = \langle P^t v, u\rangle = \langle v, Pu\rangle = \langle Pu, v\rangle,$$

that is, P, seen as a linear mapping from $C_c^\infty(U)$ to $\mathcal{D}'(U)$, is the restriction to $C_c^\infty(U)$ of $(P^t)^t$. Since $C_c^\infty(U)$ is dense in $\mathcal{E}'(U)$, this shows that $(P^t)^t$ is the unique continuous extension of P to $\mathcal{E}'(U)$. The proof is complete. □

Proposition 6.3. *Let $P \in \Psi^m(U)$. If P is properly supported, then it uniquely extends to continuous linear operators*

$$P : \mathcal{E}'(U) \longrightarrow \mathcal{E}'(U) \quad \text{and} \quad P : \mathcal{D}'(U) \longrightarrow \mathcal{D}'(U).$$

If P is a continuous linear operator from $C_c^\infty(U)$ to $C^\infty(U)$, we denote by $\overline{P}$ its complex conjugate, that is, the continuous linear operator $\overline{P} : C_c^\infty(U) \to C^\infty(U)$ defined as

$$\overline{P}u = \overline{P(\overline{u})} \quad \forall u \in C_c^\infty(U).$$

Proposition 6.4. *Let $P \in \Psi^m(U)$ have the symbol $p \sim \sum p_{m-j}$. Then $\overline{P}$ is contained in $\Psi^{\overline{m}}(U)$ and has symbol $\overline{p(x,-\xi)} \sim \sum \overline{p_{m-j}(x,-\xi)}$.*

Definition 6.2. The (formal) adjoint is the operator $P^* : C_c^\infty(U) \to C^\infty(U)$ such that, for all $u, v \in C_c^\infty(U)$,

$$\begin{aligned} \langle P^* u, v\rangle &= \int \overline{u(x)} Pv(x)dx = \langle \overline{u}, Pv\rangle = \langle P^t\overline{u}, v\rangle \\ &= \int P^t\overline{u}(x)v(x)dx = \langle \overline{P^t}u, v\rangle. \end{aligned}$$

Therefore we define P^* as $\overline{P^t}$.

Proposition 6.5. *Let $P \in \Psi^m(U)$ have the symbol $p(x,\xi) \sim \sum p_{m-j}(x,\xi)$.*

(1) *The adjoint P^* is a ΨDO of order $\overline{m}$ on U.*

(2) *Let $p^*(x,\xi) \sim \sum p^*_{\overline{m}-j}(x,\xi)$ be the symbol of P^*. Then*

$$p^*(x,\xi) \sim \sum \frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha(\overline{p(x,\xi)}). \tag{6.4}$$

(3) *The homogeneous components of the symbol $p^*(x,\xi)$ are given by*

$$p^*_{\overline{m}}(x,\xi) = \overline{p_m(x,\xi)}, \tag{6.5}$$

$$p^*_{\overline{m}-j}(x,\xi) = \sum_{|\alpha|+k=j} \frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha(\overline{p_{m-k}(x,\xi)}) \quad \forall j \geq 1. \tag{6.6}$$

CHAPTER 7

Composition of ΨDOs

Let $P \in \Psi^{m_1}(U)$ and $Q \in \Psi^{m_2}(U)$. If P is properly supported, then P can be extended to a continuous linear operator from $C^\infty(U)$ to $C^\infty(U)$ by Proposition 4.2. This defines PQ as the composition,

$$PQ : C_c^\infty(U) \xrightarrow{Q} C^\infty(U) \xrightarrow{P} C^\infty(U).$$

Simiarly, if Q is properly supported, then Q gives rise to a continuous linear operator from $C_c^\infty(U)$ to $C_c^\infty(U)$ by Proposition 4.2. This defines PQ as the composition,

$$PQ : C_c^\infty(U) \xrightarrow{Q} C_c^\infty(U) \xrightarrow{P} C^\infty(U).$$

Proposition 7.1. *Let $P \in \Psi^{m_1}(U)$ with the symbol $p(x,\xi) \sim \sum p_{m_1-j}(x,\xi)$ and $Q \in \Psi^{m_2}(U)$ with the symbol $q(x,\xi) \sim \sum q_{m_2-j}(x,\xi)$. In addition, assume that P or Q is properly supported.*

(1) *PQ is contained in $\Psi^{m_1+m_2}(U)$.*

(2) *Let $r \sim \sum r_{m_1+m_2-j}$ be the symbol of PQ. Then*

$$r(x,\xi) \sim \sum \frac{1}{\alpha!}\partial_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi). \tag{7.1}$$

(3) The homogeneous components of the symbol $r(x,\xi)$ are given by

$$r_{m_1+m_2}(x,\xi) = p_{m_1}(x,\xi) q_{m_2}(x,\xi), \tag{7.2}$$

$$r_{m_1+m_2-j}(x,\xi) = \sum_{|\alpha|+k+l=j} \frac{1}{\alpha!}\partial_\xi^\alpha p_{m_1-k}(x,\xi) D_x^\alpha q_{m_2-l}(x,\xi) \quad \forall j \geq 1. \tag{7.3}$$

Proof. Assume that P is properly supported and Q is not. By Theorem 5.3, we can choose a properly supported operator $\tilde{Q} \in \Psi^{m_2}(U)$ such that $Q = \tilde{Q} + R$ for some $R \in \Psi^{-\infty}(U)$. Then PR is a smoothing operator so that it is enough to prove the proposition for $P\tilde{Q}$. Similarly, Assume that Q is properly supported and P is not. By Theorem 5.3, we can choose a properly supported operator $\tilde{P} \in \Psi^{m_1}(U)$ such that $P = \tilde{P} + S$ for some $S \in \Psi^{-\infty}(U)$. Then SQ is a smoothing operator so that it is enough to prove the proposition for $\tilde{P}Q$. In any case we may assume that P and Q are both properly supported.

By Theorem 5.3, we can choose the symbols $p(x,\xi) \in S^{m_1}(U \times \mathbb{R}^n)$ and $q^t(x,\xi) \in S^{m_2}(U \times \mathbb{R}^n)$ such that $P = p(x,D)$ and $Q^t = q^t(x,D)$. In addition, we know $Q = (Q^t)^t$. Using this fact and Remark 6.1, we have

$$(Qu)^\wedge(\xi) = (2\pi)^{-n}(1+|\xi|^2)^{-N} \int e^{-iy\cdot\xi}(1-\Delta_y)^N [q^t(y,-\xi)u(y)]dy$$

for any $u \in C_c^\infty(U)$, $\xi \in \mathbb{R}^n$ and $N \in \mathbb{N}$ with $\Re m_2 - 2N < -n$.

Define $a(x,y,\xi) \in \mathbb{S}^{\Re(m_1+m_2)}(U \times U \times \mathbb{R}^n)$ as

$$a(x,y,\xi) := p(x,\xi)q^t(y,-\xi) \quad \forall (x,y,\xi) \in U \times U \times \mathbb{R}^n.$$

Choose $N \in \mathbb{N}_0$ such that both $\Re m_2 - 2N$ and $\Re m_1 + \Re m_2 - 2N$ are less than $-n$. Then

$$\begin{aligned} PQu(x) &= p(x,D)(Qu)(x) = \int e^{ix\cdot\xi} p(x,\xi)(Qu)^\wedge(\xi)d\xi \\ &= (2\pi)^{-n} \int e^{ix\cdot\xi} p(x,\xi)(1+|\xi|^2)^{-N} \left(\int e^{-iy\cdot\xi}(1-\Delta_y)^N[q^t(y,-\xi)u(y)]dy \right) d\xi \\ &= (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi}(1+|\xi|^2)^{-N}(1-\Delta_y)^N[a(x,y,\xi)u(y)]dyd\xi \\ &= (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} a(x,y,\xi)u(y)dyd\xi. \end{aligned}$$

This proves that PQ is the ΨDO with amplitude $a(x,y,\xi)$.

Since P and Q are both properly supported, it is obvious that PQ is also properly supported. So we can apply Theorem 5.1 in case of PQ, and we deduce that there exists a symbol $r(x,\xi) \in \mathbb{S}^{\Re(m_1+m_2)}(U \times \mathbb{R}^n)$ such that $PQ = r(x,D)$ and

$$r(x,\xi) \sim \sum \frac{1}{\alpha!}\partial_\xi^\alpha D_y^\alpha a(x,y,\xi)\Big|_{y=x} = \sum \frac{1}{\alpha!} D_\xi^\alpha[p(x,\xi)\partial_x^\alpha q^t(x,-\xi)]. \tag{7.4}$$

Also by Proposition 6.1, we have

$$q^t(x,-\xi) \sim \sum \frac{(-1)^{|\alpha|}}{\alpha!}(D_\xi^\alpha \partial_x^\alpha)q(x,\xi).$$

Combining this with (7.4) and doing some computations, we obtain

$$\begin{aligned} r(x,\xi) &\sim \sum \frac{(-1)^{|\beta|}}{\alpha!\beta!} D_\xi^\alpha \Big(p(x,\xi)\partial_x^{\alpha+\beta} D_\xi^\beta q(x,\xi) \Big) \\ &= \sum \frac{1}{\alpha!}\partial_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi). \end{aligned}$$

Likewise with the proof of (6.3), we can deduce that $r(x,\xi) \in S^{m_1+m_2}(U)$ and

$$r_{m_1+m_2-j}(x,\xi) = \sum_{|\alpha|+k+l=j} \frac{1}{\alpha!}\partial_\xi^\alpha p_{m_1-k}(x,\xi) D_x^\alpha q_{m_2-l}(x,\xi) \quad \forall j \geq 0.$$

The proof is complete. □

CHAPTER 8

Sobolev space regularity of ΨDOs

We define the Sobolev spaces on $\mathbb{R}^n$ as follows.

Let $s \in \mathbb{R}$. The Sobolev space $L^2_s(\mathbb{R}^n)$ is defined by

$$L^2_s(\mathbb{R}^n) := \{u \in \mathcal{S}'(\mathbb{R}^n); (1+|\xi|^2)^{\frac{s}{2}}\hat{u} \in L^2(\mathbb{R}^n)\}.$$

This is a Hilbert space equipped with the inner product given by

$$\langle u, v\rangle_{L^2_s} := (2\pi)^n \int (1+|\xi|^2)^s \overline{\hat{u}(\xi)}\hat{v}(\xi)d\xi \qquad \forall u, v \in L^2_s(\mathbb{R}^n).$$

For $s = 0$, we recover the Hilbert space $L^2(\mathbb{R}^n)$ with its usual inner product.

Lemma 8.1. *Let $\varphi \in C_c^\infty(U)$. Then for every $s \in \mathbb{R}$, the multiplication by φ induces a continuous linear mapping from $L^2_s(\mathbb{R}^n)$ onto itself.*

As a consequence of Lemma 8.1, we can localize the Sobolev spaces $L^2_s(\mathbb{R}^n)$. For $s \in \mathbb{R}$, we define

$$L^2_{s,\,\mathrm{loc}}(U) := \{u \in \mathcal{D}'(U); \varphi u \in L^2_s(\mathbb{R}^n) \text{ for } \forall \varphi \in C_c^\infty(U)\}.$$

We endow $L^2_{s,\,\mathrm{loc}}(U)$ with the LCTVS defined by the semi-norms,

$$u \longrightarrow \|\varphi u\|_{L^2_s},$$

where φ ranges over all functions in $C_c^\infty(U)$. Then $L^2_{s,\,\mathrm{loc}}(U)$ becomes a Fréchet space.

Let K be a compact subset of U. For $s \in \mathbb{R}^n$ and K, we define

$$L^2_{s,\,K}(U) := \{u \in \mathcal{E}'(U);\ \operatorname{supp} u \subset K \text{ and } u \in L^2_s(\mathbb{R}^n)\}.$$

where we have identified $\mathcal{E}'(U)$ with a subspace of $\mathcal{S}'(\mathbb{R}^n)$. The space $L^2_{s,\,K}(U)$ can be seen as a closed subspace of both $L^2_s(\mathbb{R}^n)$ and $L^2_{s,\,\mathrm{loc}}(U)$, but the two corresponding induced topologies agree. In particular, $L^2_{s,\,K}(U)$ is a Hilbert space with respect to the L^2_s-norm.

Proposition 8.1. *Let s_1 and s_2 be real numbers with $s_1 < s_2$, and K be a compact subset of U. Then the inclusion of $L^2_{s_2,\,K}(U)$ into $L^2_{s_1,\,K}(U)$ is compact.*

Definition 8.2. $L^2_{s,\,c}(U) := \{u \in \mathcal{E}'(U);\ u \in L^2_s(\mathbb{R}^n)\}$.

We endow $L^2_{s,\,c}(U)$ with the weakest locally convex topological vector space topology such that, for any compact $K \subset U$, the inclusion $L^2_{s,\,K}(U) \subset L^2_{s,\,c}(U)$ is continuous.

Proposition 8.2. *Let $s \in \mathbb{R}$. There are continuous inclusions,*

$$C_c^\infty(U) \subset L^2_{s,\,c}(U) \quad \textit{and} \quad C^\infty(U) \subset L^2_{s,\,\mathrm{loc}}(U),$$

$$L^2_{s,\,c}(U) \subset \mathcal{E}'(U) \quad \textit{and} \quad L^2_{s,\,\mathrm{loc}}(U) \subset \mathcal{D}'(U).$$

Lemma 8.2. *Let $u \in \mathcal{E}'(U)$. Then there is $s \in \mathbb{R}$ such that $u \in L^2_{s,\,c}(U)$.*

Proof. We have $\hat{u}(\xi) = (2\pi)^{-n}\langle u, e_{-\xi}\rangle$. Then there exists a compact subset K of U, $N \in \mathbb{N}_0$ and $C, C' > 0$ such that

$$|\hat{u}(\xi)| \leq C \sup_{|\alpha|\leq N} \sup_{x\in K} |\partial_x^\alpha e_{-\xi}(x)| \leq C'(1+|\xi|)^N \quad \forall \xi \in \mathbb{R}^n.$$

Thus, for s with $2s + 2N < -n$,

$$(1+|\xi|^2)^s|\hat{u}(\xi)|^2 \leq C'^2(1+|\xi|^2)^s(1+|\xi|)^{2N} \leq 2^{-s}C'^2(1+|\xi|)^{2s+2N}.$$

The proof is complete. □

Now we are in a position to state the main result of this section as follows.

Theorem 8.1. *Let $P \in \Psi^m(U)$ and $a = \Re m$. For any $s \in \mathbb{R}$, P induces a continuous linear operator,*

$$P : L^2_{s+a,\,c}(U) \to L^2_{s,\,\mathrm{loc}}(U).$$

If in addition P is properly supported, then it also induces continuous linear operators,

$$P : L^2_{s+a,\,c}(U) \to L^2_{s,\,c}(U) \quad \textit{and} \quad P : L^2_{s+a,\,c}(U) \to L^2_{s,\,\mathrm{loc}}(U).$$

CHAPTER 9

Parametrices and Ellipticity

Definition 9.1. *Let $P \in \Psi^m(U)$ have the symbol $p \sim \sum p_{m-j}$. Then P is said to be elliptic when*

$$p_m(x,\xi) \neq 0 \qquad \forall (x,\xi) \in U \times (\mathbb{R}^n \backslash 0).$$

Example 9.1. The Laplace operator $\Delta = -\sum_{j=1}^n \partial_{x_j}^2$ on $\mathbb{R}^n$ has the symbol $p(x,\xi) = \sum_{j=1}^n \xi_j^2 = |\xi|^2$. As this is homogeneous of degree 2, so $p_2(x,\xi) = p(x,\xi) = |\xi|^2$. Therefore Δ is elliptic.

Theorem 9.1. *Let $P \in \Psi^m(U)$. Then the following are equivalent.*

(1) *P is elliptic.*

(2) *There exists a properly supported operator $Q \in \Psi^{-m}(U)$ which is a parametrix for P, that is,*

$$PQ = QP = 1 \mod \Psi^{-\infty}(U).$$

Proof. First, let P with the symbol $p \sim \sum p_{m-j}$ and $Q \in \Psi^{-m}(U)$ with the symbol $q \sim \sum q_{-m-j}$ in the sense of (2). By Proposition 7.1, PQ is a ΨDO of order 0 with the symbol $r \sim \sum r_{-j}$ such that

$$r_{-j} = \sum_{|\alpha|+k+l=j} \frac{1}{\alpha!} \partial_\xi^\alpha p_{m-k}(x,\xi) D_x^\alpha q_{-m-l}(x,\xi) \qquad \forall j \geq 0. \tag{9.1}$$

By Proposition 3.2 and Theorem 5.2, $PQ = 1 \mod \Psi^{-\infty}(U)$ is equivalent to $r \sim 1$, that is, $r_0 = 1$ and $r_j = 0$ for all $j \geq 1$. So we obtain $p_m(x,\xi)q_{-m}(x,\xi) = 1$. It follows that $p(x,\xi)$ is elliptic.

Conversely, suppose that P is elliptic. Then we can define q_{-m} as

$$q_{-m}(x,\xi) := p_m(x,\xi)^{-1} \quad \forall (x,\xi) \in U \times (\mathbb{R}^n \backslash \mathbf{0}). \tag{9.2}$$

Moreover, we can define a sequence $\{q_{-m-j}\}_{j\geq 1} \subset C^\infty(U \times (\mathbb{R}^n \backslash \mathbf{0}))$ inductively as

$$q_{-m-j}(x,\xi) := -p_m(x,\xi)^{-1} \sum_{\substack{|\alpha|+k+l=j \\ l<j}} \frac{1}{\alpha!} \partial_\xi^\alpha p_{m-k}(x,\xi) D_x^\alpha q_{-m-l}(x,\xi) \quad \forall j \geq 1. \tag{9.3}$$

It is obvious that q_{-m-j} is the homogeneous symbol of degree $-m-j$ for all $j \geq 0$.

By Lemma 3.2, there exists $\tilde{Q} \in \Psi^{-m}(U)$ with the symbol $\tilde{q} \in S^{-m}(U \times \mathbb{R}^n)$ such that $\tilde{q} \sim \sum q_{-m-j}$. Also there exists $Q \in \Psi^{-m}(U)$ with the symbol $q \in S^{-m}(U \times \mathbb{R}^n)$ which is properly supported and $Q = \tilde{Q} \mod \Psi^{-\infty}(U)$. So we obtain $q \sim \sum q_{-m-j}$. Therefore $PQ = 1 \mod \Psi^{-\infty}(U)$ by (9.1),(9.2) and (9.3).
Similary, we can choose Q' properly supported such that $Q'P = 1 \mod \Psi^{-\infty}(U)$. Then

$$Q' = Q'(PQ) = (Q'P)Q = Q \mod \Psi^{-\infty}(U).$$

So we deduce that

$$Q'P = QP = 1 \mod \Psi^{-\infty}(U).$$

The proof is complete. □

As a consequence of the previous result, we obtain the following regularity properties for elliptic ΨDOs.

Theorem 9.2 (Elliptic Regularity Theorem). *Let $P \in \Psi^m(U)$ be elliptic and set $a = \Re m$. Let $s \in \mathbb{R}$. Then, for all $u \in \mathcal{E}'(U)$,*

$$Pu \in L^2_{s,\,\mathrm{loc}}(U) \Longrightarrow u \in L^2_{s+a,\,c}(U).$$

Proof. By Theorem 9.1, there exists $Q \in \Psi^{-m}(U)$ which is a properly supported parametrix for P. Let $R \in \Psi^{-\infty}(U)$ such that $QP = 1-R$. Also let $u \in \mathcal{E}'(U)$ and assume that $Pu \in L^2_{s,\,\mathrm{loc}}(U)$. Since Q is a properly supported ΨDO of order $-m$, it induces a continuous linear operator from $L^2_{s,\,\mathrm{loc}}(U)$ to $L^2_{s+a,\,\mathrm{loc}}(U)$ by Theorem 8.1. This implies that QPu is contained in $L^2_{s+a,\,\mathrm{loc}}(U)$. In addition, Ru is also contained in $L^2_{s+a,\,\mathrm{loc}}(U)$ since Ru is an element of $C^\infty(U)$. Thus $u = QPu + Ru$ shows that u is contained in $L^2_{s+a,\,\mathrm{loc}}(U)$ too. Since u has compact support, u lies in $L^2_{s+a,\,c}(U)$. □

CHAPTER 10

ΨDOs on a Manifold

Let $\phi : V \to U$ be a diffeomorphism from an open set $V \subset \mathbb{R}^n$ onto U. If P is a linear operator from $C_c^\infty(U)$ to $C^\infty(U)$, we denote by $\phi^* P$ its pullback by ϕ, that is, $\phi^* P$ is the linear operator from $C_c^\infty(V)$ to $C^\infty(V)$ defined by

$$(\phi^* P)v(x) = P(v \circ \phi^{-1})(\phi(x)) \quad \forall v \in C_c^\infty(V). \tag{10.1}$$

Similary, if Q is a linear operator from $C_c^\infty(V)$ to $C^\infty(V)$, we denote by $\phi_* Q$ its push-forward by ϕ, that is, $\phi_* Q$ is the linear operator from $C_c^\infty(U)$ to $C^\infty(U)$ defined by

$$(\phi_* Q)u(x) = Q(u \circ \phi)(\phi^{-1}(x)) \quad \forall u \in C_c^\infty(U). \tag{10.2}$$

For all $(x, y, \xi) \in V \times V \times \mathbb{R}^n$, we set

$$\phi(x, y, \xi) = i\langle \phi(x) - \phi(y) - \phi(x)' \cdot (x - y), \xi\rangle + \mathrm{Tr}\Big[\phi'(y) - \phi'(x)\Big].$$

Lemma 10.1. *Let $\alpha \in \mathbb{N}_0^n$. Then*

$$\frac{\partial^\alpha}{\partial y^\alpha} e^{\phi(x,y,\xi)}\Big|_{y=x} = \sum_{2|\beta| \le |\alpha|} a_{\alpha\beta}(x)\xi^\beta, \quad a_{\alpha\beta}(x) \in C^\infty(U).$$

Proposition 10.1. *Let $P \in \Psi^m(U)$ with the symbol $p(x, \xi) \sim \sum p_{m-j}(x, \xi)$.*

(1) *The pullback $\phi^* P$ is a ΨDO of order m on V, that is, it is contained in $\Psi^m(V)$.*

(2) *Let $p^\phi(x, \xi) \in S^m(V \times \mathbb{R}^n)$ be the symbol of $\phi^* P$. Then*

$$p^\phi(x, \xi) \sim \sum_{2|\beta| \le |\alpha|} \frac{1}{\alpha!} a_{\alpha\beta}(x)\xi^\beta (D_\xi^\alpha p)(\phi(x), (\phi'(x)^{-1})^t \xi). \tag{10.3}$$

(3) *For $j \in \mathbb{N}_0$, let $p_{m-j}^\phi(x, \xi)$ be the homogeneous symbol of degree $m - j$ of P. Then*

$$p_{m-j}^\phi(x, \xi) = \sum_{\substack{k+|\alpha|-|\beta|=j \\ 2|\beta| \le |\alpha|}} \frac{1}{\alpha!} a_{\alpha\beta}(x)\xi^\beta (D_\xi^\alpha p_{m-k})(\phi(x), (\phi'(x)^{-1})^t \xi). \tag{10.4}$$

In particular,

$$p_m^\phi(x, \xi) = p_m(\phi(x), (\phi'(x)^{-1})^t \xi).$$

Let V be an open subset of U and P be a continuous linear operator from $C_c^\infty(U)$ to $C^\infty(U)$. We define the *restriction* of P to V, denoted by $P_{|V}$, as

$$P_{|V}v = (Pv)_{|V} \qquad \forall v \in C_c^\infty(V).$$

Then it follows that $P_{|V}$ is a continuous linear operator from $C_c^\infty(V)$ to $C^\infty(V)$. Moreover, we see that if P is a ΨDO with the symbol $p(x,\xi) \in S^m(U \times \mathbb{R}^n)$, then $P_{|V}$ is a ΨDO with the symbol $p_{|V\times\mathbb{R}^n} \in S^m(V \times \mathbb{R}^n)$.

Combining this with Proposition 5.4 and 10.1, we deduce that $\Psi^m(U)$ consists of continuous linear operators $P : C_c^\infty(U) \to C^\infty(U)$ such that

(1) The Schwartz kernel of P is smooth on $\{(x,y) \in U \times U : x \neq y\}$.

(2) For any diffeomorphism ϕ from an open $V \subset U$ onto an open $W \subset \mathbb{R}^n$, the operator $\phi_*(P_{|V})$ is contained in $\Psi^m(W)$.

This characterization will give an idea of generalizing ΨDO to manifolds.

From now on, we let M be a (smooth) manifold of dimension of n. We endow $C^\infty(M)$ with a Fréchet-Montel space topology as follows. If $\kappa : V \to U$ is a local chart, from an open $V \subset M$ onto an open $U \subset \mathbb{R}^n$, then we have a linear bijection $\kappa_* : C^\infty(V) \to C^\infty(U)$ defined by

$$\kappa_* u = u \circ \kappa^{-1} \qquad \forall u \in C^\infty(V).$$

This gives rise to a linear map,

$$C^\infty(M) \ni u \longrightarrow \kappa_*(u|_V) \in C^\infty(U).$$

Then we endow $C^\infty(M)$ with the weakest LCTVS that makes these linear maps being continuous as κ ranges over all local charts of M. Thus, $C^\infty(M)$ becomes a Fréchet-Montel Space. Also we define and give each topology on $C_c^\infty(M)$, $\mathcal{D}'(M)$, $\mathcal{E}'(M)$ in a similar way to we did when M was U.

Let us denote by $|\Lambda|(M)$ the line bundle of densities on M (see Appendix). Then we have natural inclusions as follows.

$$C^\infty(M, |\Lambda|(M)) \subset \mathcal{D}'(M) \quad \text{and} \quad C^\infty(M) \subset \mathcal{D}'(M, |\Lambda|(M))$$

All these inclusions are continuous with respect to the topologies described as above.

Proposition 10.2. *A linear operator $P : C_c^\infty(M) \to \mathcal{D}'(M, |\Lambda|(M))$ is continuous if and only if there exists $k_P(x,y) \in \mathcal{D}'(M \times M, |\Lambda|(M) \boxtimes 1)$ such that, for all $u \in C_c^\infty(M, |\Lambda|(M))$ and $v \in C_c^\infty(M)$,*

$$\langle Pv(x), u(x)\rangle = \langle k_P(x,y), u(x)v(y)\rangle.$$

Remark 10.1. Let P be a continuous linear operator from $C_c^\infty(M)$ to $C^\infty(M)$. Then we can regard P as a continuous linear operator from $C_c^\infty(M)$ to $\mathcal{D}'(M, |\Lambda|(M))$ since there is a continuous inclusion of $C^\infty(M)$ into $\mathcal{D}'(M, |\Lambda|(M))$.

Definition 10.1. *The space $\Psi^m(M)$ consists of all continuous linear operators $P : C_c^\infty(M) \to C^\infty(M)$ such that*

(1) *For any pair of smooth functions φ and ψ on M, the operator $\varphi P \psi$ is smoothing.*

(2) *For any local chart κ from an open $V \subset M$ onto an open $U \subset \mathbb{R}^n$, the operator $\kappa_*(P_{|V})$ is a ΨDO of order m on U, that is, it is contained in $\Psi^m(U)$.*

Let $P \in \Psi^m(M)$ and κ be a local chart from an open $V \subset M$ onto an open $U \subset \mathbb{R}^n$. Set $P_\kappa := \kappa_*(P_{|V}) \in \Psi^m(U)$. Then from (2) in Definition 10.1, we see that

$$Pu(\kappa^{-1}(x)) = P_\kappa(u \circ \kappa^{-1})(x) \qquad \forall x \in U.$$

This means that P is given by the ΨDO P_κ in the local coordinates defined by κ.

Let $p_\kappa(x,\xi) \sim p_{\kappa,m-j}(x,\xi)$ be the symbol of P. We shall call $p_\kappa(x,\xi)$ the symbol of P in the local coordinates defined by κ. Then the principal symbol $p_{\kappa,m}(x,\xi)$ has an intrinsic meaning as follows.

Let $\kappa_1 : V_1 \to U_1$ be another local chart of M and $\phi := \kappa \circ \kappa_1^{-1}$ be the associate coordinate change, mapping $W_1 := \kappa_1(V \cap V_1)$ onto $W := \kappa(V \cap V_1)$. Then we have

$$P_{\kappa_1|W_1} = (\phi^{-1} \circ \kappa)_*(P_{|V \cap V_1}) = \phi^*(\kappa_*(P_{|V\cap V_1})) = \phi^*(P_{\kappa|W}).$$

Let $p_{\kappa_1,m}(x,\xi)$ be the principal symbol of P_{κ_1}. Then by Proposition 10.1,

$$p_{\kappa_1,m}(x,\xi) = p^\phi_{\kappa,m}(x,\xi) = p_{\kappa,m}(\phi(x),(\phi'(x)^{-1})^t\xi) \quad \forall (x,\xi) \in W_1 \times (\mathbb{R}^n \backslash \mathbf{0}). \tag{10.5}$$

Upon indentifying T^*W with $W \times \mathbb{R}^n$, we can regard $p_{\kappa,m}(x,\xi)$ as a function on $T^*W\backslash 0$, where 0 is the zero-section. Simiarly, we can regard $p_{\kappa_1,m}(x,\xi)$ as a function on $T^*W_1\backslash 0$. By duality, the tangent map $\phi' : TW_1 \to TW$ gives rise to a cotangent map $\phi'^t : T^*W \to T^*W_1$ whose inverse is given by

$$\phi'^{t^{-1}}(x,\xi) = (\phi(x),(\phi'(x)^{-1})^t\xi) \quad \forall (x,\xi) \in T^*W_1.$$

Then we can rewrite (10.5) as

$$p_{\kappa_1,m}(x,\xi) = p_{\kappa,m}(x,\xi)\left[\phi'^{t^{-1}}(x,\xi)\right] = (\phi'^t_* p_{\kappa,m})(x,\xi) \quad \forall (x,\xi) \in T^*W_1\backslash 0.$$

If we pull-back everything to $T^*(V \cap V_1)\backslash 0$, then we have

$$(\kappa'^t_{1*} p_{\kappa_1,m})(x,\xi) = (\kappa'^t_{1*}(\phi'^t_* p_{\kappa,m}))(x,\xi) = (\kappa'^t_* p_{\kappa_1,m})(x,\xi) \quad \forall (x,\xi) \in T^*(V \cap V_1)\backslash 0.$$

This shows that the value of $(\kappa'^t_* p_{\kappa_1,m})(x,\xi)$ at any point $(x,\xi) \in T^*V\backslash 0$ depends only on (x,ξ), not on the choice of the local chart κ near x.

Definition 10.2. *$S_m(T^*M)$ consists of smooth functions $p(x,\xi)$ on $T^*M\backslash 0$ such that*

$$p(x,\lambda\xi) = \lambda^m p(x,\xi) \qquad \forall (x,\xi) \in T^*M\backslash 0 \quad \forall \lambda > 0.$$

Proposition 10.3. *There is a unique symbol $p_m(x,\xi) \in S_m(T^*M)$ such that, for any local chart $\kappa : V \to U$,*

$$p_m(x,\xi) = (\kappa^t_* p_{\kappa,m})(x,\xi) \qquad \forall (x,\xi) \in T^*V\backslash 0.$$

Appendix

Definition: The line bundle densities

Let M be an n-dimensional manifold and let $\{(\kappa, U_\kappa) : \kappa \text{ is a local chart of } M\}$ be an atlas.

We define

$$|\Lambda_x|(M) := \{\mu : \Lambda^n T_x M \to \mathbb{C};\ \ \mu(\lambda v) = |\lambda|\mu(v)\ \ \forall v \in \Lambda^n T_x M\ \ \forall \lambda \in \mathbb{C}\}.$$

Moreover, we define $|\Lambda|(M) := \bigsqcup_{x\in M} |\Lambda_x|(M)$. Then $|\Lambda|(M)$ is a trivializable line bundle. There is one-to-one correspondence between the datum of $C^\infty(M, |\Lambda|(M))$ and the datum on each local chart $\kappa : V \to U$ of a function $\rho_\kappa \in C^\infty(V)$ such that, for any local charts κ_1, κ_2 with transition map $\phi = \kappa_1 \circ \kappa_2^{-1}$, we have

$$\rho_{\kappa_2}(x) = |\det \phi'(x)| \rho_{\kappa_1}(\phi(x))$$

on the domain of ϕ.

Let $\rho \in C^\infty(M, |\Lambda|(M))$. If $\kappa : V \to U$ is a local chart and $v \in C_c^\infty(V)$, then

$$\int v(x)\rho(x) := \int v(\kappa^{-1}(x))\rho_\kappa(x)dx$$

does not depend on the choice of the local chart since

$$\begin{aligned}\int v(\kappa_2^{-1}(x))\rho_{\kappa_2}(x)dx &= \int v(\kappa_1^{-1}(\phi(x)))\rho_{\kappa_1}(\phi(x))|\det \phi'(x)|dx \\ &= \int v(\kappa_1^{-1}(x))\rho_{\kappa_1}(x)dx.\end{aligned}$$

Let (ϕ_i) be a smooth partition of unity subordinate to a covering (V_i) of domains of local charts $\kappa_i : V_i \to U_i$. Then for $u \in C_c^\infty(M)$, we define

$$\begin{aligned}\int_M u(x)\rho(x) &:= \sum_i \int \phi_i(x)u(x)\rho(x) \\ &= \sum_i \int \phi_i(\kappa_i^{-1}(x))u(\kappa_i^{-1}(x))\rho_{\kappa_i}(x)dx.\end{aligned}$$

Fix $\rho \in C^\infty(M, |\Lambda|(M))$. For $u \in C_c^\infty(M)$, $u \longrightarrow \int_M u(x)\rho(x)$ is an element of $\mathcal{D}'(M)$. This yields a natural inclusion

$$C^\infty(M, |\Lambda|(M)) \subset \mathcal{D}'(M).$$

Similary, fix $u \in C^\infty(M)$. For $\rho \in C_c^\infty(M, |\Lambda|(M))$, $\rho \longrightarrow \int_M u(x)\rho(x)$ is an element of $\mathcal{D}'(M, |\Lambda|(M))$. This yields a natural inclusion

$$C^\infty(M) \subset \mathcal{D}'(M, |\Lambda|(M)).$$

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