이학석사 학위논문

# Properties of obstacle problem and free boundary problem

## (장애물문제와 자유경계문제의 특성들)

**2014년 8월**

서울대학교 대학원
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이 논문을 이학석사 학위논문으로 제출함

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# Properties of obstacle problem and free boundary problem

**by**

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A DISSERTATION

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# Abstract

This paper is a paper which is written based on the contents of [1] and introduction of obstacle problem for nonlinear second-order parabolic operator. In chapter 1, we introduce classical obstacle problem and we deal with existence, uniqueness and $C^{1,1}$ regularity of solution of the problem. In chapter 2, we show $C^{1,1}$ regularity of solution of Obstacle-type problem. In chapter 3, we prove some elementary properties of free boundary. In chapter 4, We reference [2] to show the continuity of solution of obstacle problem for nonlinear second-order parabolic operator.

**Key words** : Obstacle, Obstacle problem, classical obstacle problem, Obstacle-type problem, free boundary, $C^{1,1}$ regularity, nonlinear second order parabolic operator.
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# Contents

# 1 The classical obstacle problem

## 1.1 The obstacle problems

It is well-known fact that the solution of the boundary value problem

$$\Delta u = 0 \text{ in } D, \quad u = g \text{ on } \partial D,$$

can be found as the minimizer of the functional

$$J_0(u) = \int_D |\nabla u|^2 dx,$$

for all $u$ such that $u = g$ on $\partial D$. It is *the Dirichlet principle* and the functional is *the Dirichlet functional.* More generally, for a bounded open set $D$ in $\mathbb{R}^n$, $g \in W^{1,2}(D)$ and $f \in L^\infty(D)$, the minimizer of

$$J(u) = \int_D (|\nabla u|^2 + 2fu) dx$$

over the set

$$K_g = \{u \in W^{1,2}(D) : u - g \in W_0^{1,2}(D)\}$$

solves the equation

$$-\Delta u + f = 0 \text{ in } D, u = g \text{ on } \partial D,$$

in the sense of distributions, i.e.

$$\int_D (\nabla u \nabla \eta + f\eta) dx = 0,$$

for all $\eta \in C_c^\infty(D)$.

Now, let a function $\psi \in C^2(D)$, obstacle, satisfying $\psi \leq g$ on $\partial D, (\psi - g)_+ \in W_0^{1,2}(D)$ be given. Consider the minimizing problem of the functional $J(\cdot)$, over the set

$$K_{g,\psi} = \{u \in W^{1,2}(D) : u - g \in W_0^{1,2}(D), u \geq \psi \text{ a.e. in } D\}$$

The set

$$\Lambda = \{u = \psi\},$$

is *the coincisence set* and $\Omega = D \setminus \Lambda$. The boundary

$$\Gamma = \partial\Lambda \cap D = \partial\Omega \cap D$$

is *the free boundary*, since it is unknown before. In this rest of the section we will show that the minimizer $u$ of $J(\cdot)$ satisfy

$$\Delta u = f \text{ in } \Omega, \quad \Delta u = \Delta\psi \text{ a.e on } \Lambda. \tag{1}$$

It is *the classical obstacle problem.*

**Theorem 1.1.** *Let $D$ be a bounded open subset in $\mathbb{R}^n, g \in W^{1,2}(D)$ and $f \in L^\infty(D), \psi \in C^2(D), \psi \le g$ on $\partial D, (\psi - g)_+ \in W_0^{1,2}(D), J(u) = \int_D (|\nabla u|^2 + 2fu)dx$ over the set $K_{g,\psi} = \{u \in W^{1,2}(D) : u - g \in W_0^{1,2}(D), u \ge \psi$ a.e. in $D\}$. Let*

$$J_1(v) = \int_D (|\nabla v|^2 + 2f_1 v)dx$$

*be a functional over the set*

$$K_{g_1,0} = \{u \in W^{1,2}(D) : u - g_1 \in W_0^{1,2}(D), u \ge 0 \text{ a.e. in } D\},$$

*where $f_1 = f - \Delta\psi, g_1 = g - \psi$. Then $u$ is the minimizer of $J$ if and only if $v$ is the minimizer of $J_1$ where $v = u - \psi$.*

*Proof.* For $u \in K_{g,\psi}, v = u - \psi \in K_{g_1,0}$, and for $v \in K_{g_1,0}, v + \psi \in K_{g,\psi}$.

$$\begin{aligned}
J_1(v) &= \int_D |\nabla u - \nabla\psi|^2 + 2(f - \Delta\psi)(u - \psi)dx \\
&= \int_D |\nabla u|^2 - 2\nabla u \cdot \nabla\psi + |\nabla\psi|^2 + 2(fu - f\psi - u\Delta\psi + (\Delta\psi)\psi)dx \\
&= J(u) + \int_D -2\nabla u \cdot \nabla\psi - 2u\Delta\psi + |\nabla\psi|^2 - 2f\psi + 2(\Delta\psi)\psi dx \\
&= J(u) - 2\int_D (\nabla u - \nabla g) \cdot \nabla\psi + (u - g)\Delta\psi dx + C \\
&= J(u) + C,
\end{aligned}$$

where constant $C = \int_D -2\nabla g \cdot \nabla\psi - 2g\Delta\psi + |\nabla\psi|^2 - 2f\psi + 2(\nabla\psi)\psi dx$. $u - g = 0$ on $\partial D$. The last equation holds, by the integration by part. □

If we show

$$\Delta v = f_1 \text{ a.e. in } \{v > 0\}, \quad \Delta v = 0 \text{ a.e on } \{v = 0\},$$

(1) is obtained, consequently. We have reduced the problem to the case of zero obstacle. Thus we cover the case of zero obstacle, only, in the rest of this section.

**Theorem 1.2.** *Let $D$ be a bounded open subset in $\mathbb{R}^n, g \in W^{1,2}(D)$ and $f \in L^\infty(D), 0 \le g$ on $\partial D, (-g)_+ \in W_0^{1,2}(D)$. Let $\tilde{J}(u) = \int_D (|\nabla u|^2 + 2fu_+)dx$ over the set $K_g = \{u \in W^{1,2}(D) : u - g \in W_0^{1,2}(D)\}$. Then $u$ is the minimizer of $J$ over $K_{g,0}$ if and only if $u$ is the minimizer of $\tilde{J}$ over $K_g$.*

*Proof.* For $u \in K_g, u_+ \in K_{g,0}$, and we know that

$$\nabla u_+ = \begin{cases} \nabla u & \text{a.e. on } \{u > 0\} \\ 0 & \text{a.e. on } \{u \le 0\}. \end{cases}$$

Thus we have

$$\tilde{J}(u_+) = \int_D (|\nabla u_+|^2 + 2fu_+)dx \le \int_D (|\nabla u|^2 + 2fu_+)dx = \tilde{J}(u).$$

On the other hand,

$$\tilde{J}(u_+) = \tilde{J}(u) \Leftrightarrow \int_D |\nabla u| dx = \int_D |\nabla u_+| dx \Leftrightarrow \nabla u_- = 0 \text{ a.e in } D.$$

Thus $u_-$ is locally constant and since $u_- \in W_0^{1,2}(D)$, we have $u_- = 0$. Therefore

$$\tilde{J}(u_+) = \tilde{J}(u) \text{ for any } u \in K_g \text{ if and only if } u = u_+.$$

Then if $u \in K_g$ is the minimizer of $\tilde{J}(\cdot)$, then $\tilde{J}(u) \leq \tilde{J}(u_+)$. Thus $\tilde{J}(u) = \tilde{J}(u_+)$. By the uniqueness of the minimizer $u = u_+$. Hence $u \in K_{g,0}$. That means $\tilde{J}(\cdot)$ has it minimum on $K_{g,0}$. Since $\tilde{J}(\cdot) = J(\cdot)$ on $K_{g,0}$, the sets of minimizers of $J(\cdot)$ and $\tilde{J}(\cdot)$ are coincide. □

**Theorem 1.3.** *Let $D$ be a bounded open subset in $\mathbb{R}^n, g \in W^{1,2}(D)$ and $f \in L^\infty(D), 0 \leq g$ on $\partial D, (-g)_+ \in W_0^{1,2}(D)$. Let $0 < \epsilon < 1, \chi_\epsilon(s)$ be a $C^\infty$ function on $\mathbb{R}$ such that*

$$\chi_\epsilon(s) = 0 \quad \text{for } s \leq -\epsilon, \quad \chi_\epsilon(s) = 1 \quad \text{for } s \geq \epsilon, \quad \chi'_\epsilon \geq 0.$$

*Let*

$$\Phi_\epsilon(s) = \int_{-\infty}^{s} \chi_\epsilon(t) dt, \quad J_\epsilon(u) = \int_D (|\nabla u|^2 + 2f(x)\Phi_\epsilon(u(x))) dx$$

*over $K_g$ and $u_\epsilon$ is the minimizer of $J_\epsilon$. Then*

$$\int_D (\nabla u_\epsilon \nabla \eta + f\chi_\epsilon(u_\epsilon)\eta) dx = 0,$$

*for $\eta \in W_0^{1,2}(D)$.*

*Proof.* Let $\eta \in W_0^{1,2}(\Omega)$ and $t \in \mathbb{R}$. Then $u_\epsilon + t\eta \in K_g$. Set $h(t) = J_\epsilon(u_\epsilon + t\eta)$. Since $u_\epsilon$ is the minimizer and $u_\epsilon + t\eta \in K_{g,0}, h(t) \geq h(0) = J(u_\epsilon)$. Thus $h'(0) = 0$.

$$\begin{aligned} h(t) =& J_\epsilon(u_\epsilon + t\eta) = \int_D |\nabla(u_\epsilon + t\eta)|^2 + 2f(x)\Phi_\epsilon(u_\epsilon(x) + t\eta(x)) dx \\ =& \int_D |\nabla u_\epsilon|^2 dx + 2t \int_D \nabla u_\epsilon \cdot \nabla \eta + t^2 \int_D |\nabla \eta|^2 dx + \int_D 2f\Phi_\epsilon(u_\epsilon + t\eta) dx \end{aligned}$$

Therefore

$$\begin{aligned} h'(t) &= 2\int_D \nabla u_\epsilon \cdot \nabla \eta dx + 2t \int_D |\nabla \eta|^2 dx + 2\int_D f\Phi_\epsilon(u_\epsilon + t\eta)' dx \\ &= 2\int_D \nabla u_\epsilon \cdot \nabla \eta dx + 2t \int_D |\nabla \eta|^2 dx + 2\int_D f\chi_\epsilon(u_\epsilon(x) + t\eta(x))\eta(x) dx. \end{aligned}$$

Therefore

$$h'(0) = 2\int_D \nabla u_\epsilon \cdot \nabla \eta dx + 2\int_D f\chi_\epsilon(u_\epsilon)\eta dx = 0.$$

□

## 1.2 Existense and uniqueness of the solution of the obstacle problems

**Lemma 1.4.** *Let $\mathcal{A}$ be a subset of a reflexive Banach space $X$. Let a functional $J(\cdot)$ over $\mathcal{A}$. If*
($a$) *$\mathcal{A}$ is weakly closed in $X$,*
($b$) *There exists $u_0 \in \mathcal{A}$ such that $J(u_0) < +\infty$,*
($c$) *$J(u) > -C_0 > -\infty$ for all $u \in \mathcal{A}$,*
($d$) *$J(\cdot)$ is coercive, i.e.$J(u_k) \to +\infty$, provided $\|u_k\|_X \to \infty$,*
($e$) *$J(\cdot)$ is weakly lower semi-continuous on $\mathcal{A}$, i.e. if $u_k \rightharpoonup u$(weakly), then $J(u) \le \underline{\lim}_{k\to\infty} J(u_k)$, then there exists minmizer $u \in \mathcal{A}$ ,i.e. $J(u) = \inf_{v\in\mathcal{A}} J(v)$.*

*Proof.* Set $J_* = \inf_{v\in\mathcal{A}} J(v)$. By $(b), (c), -C_0 \le J_* \le J(u_0) < +\infty$. Then there exists $u_k \in \mathcal{A}$ such that $J(u_k) \searrow J_*$ and hence there exists $N \in \mathbb{N}$ such that $J(u_k) < J_* + 1$ for $k \ge N$. By coercivity there exists $M > 0$ such that $\|u_k\|_X < M$, for all $k \ge N$. By the weak-compactness of $X$, there exists $u \in X$ such that $u_k \rightharpoonup u$ (up to subsequence). Since $\mathcal{A}$ is weakly closed, $u \in \mathcal{A}$ and from the weakly lower semi-contiuity of $J(\cdot)$, $J(u) \le \underline{\lim}_{k\to\infty} J(u_k) = J_*$. Therefore $J(u) = J_*$, and $u$ is a minimizer. □

**Theorem 1.5.** *Let $D$ be a bounded open subset in $\mathbb{R}^n, g \in W^{1,2}(D)$ and $f \in L^\infty(D), 0 \le g$ on $\partial D, (-g)_+ \in W_0^{1,2}(D)$. Let $J(u) = \int_D(|\nabla u|^2 + 2fu)dx$ be a functional over the set $K_{g,0} = \{u \in W^{1,2}(D) : u-g \in W_0^{1,2}(D), u \ge 0$ a.e. in $D\}$. Then $J(\cdot)$ has a unique minimizer in $K_{g,0} \subset W^{1,2}(D)$.*

*Proof.* ($a$) Let $u_k \rightharpoonup u$ in $W^{1,2}(D), u_k \in K_{g,0}$. Since $W^{1,2}(D) \hookrightarrow L^2(D)$, we know that $u_k \to u$ in $L^2(D)$. Thus $u_k \to u$ a.e in $D$, up to subsequence. Hence $u \ge 0$ a.e. in $D$. Since $W_0^{1,2}(D)$ is weakly closed, $u - g \in W_0^{1,2}(D)$. Thus $u \in K_{g,0}$.
($b$) Since $g \ge 0$ on $\partial D, g_+ = g$ on $\partial D$. Thus $g_+ - g \in W_0^{1,2}$. Therefore $g_+ \in K_{g,0}$, and we have

$$J(g_+) = \int_D |\nabla g_+|^2 + 2fg_+dx \le \|\nabla g\|^2_{L^2(D)} + 2\|f\|_{L^2(D)}\|g\|_{L^2(D)} < +\infty,$$

since $f \in L^\infty(D), g \in W^{1,2}(D)$.
($c$) Let $u \in K_{g,0}$, then $u - g \in W_0^{1,2}(D)$.

$$\begin{aligned}
J(u) &= \int_D |\nabla u|^2 + 2fudx \\
&\ge \|\nabla u\|^2_{L^2(D)} - 2\|f\|_{L^2(D)}\|u\|_{L^2(D)} \\
&\ge \|\nabla u\|^2_{L^2(D)} - 2\|f\|_{L^2(D)}(\|u-g\|_{L^2(D)} + \|g\|_{L^2(D)}) \\
&\ge \|\nabla u\|^2_{L^2(D)} - 2\|f\|_{L^2(D)}(C\|\nabla(u-g)\|_{L^2(D)} + \|g\|_{L^2(D)}) \\
&= \|\nabla u\|^2_{L^2(D)} - 2C\|f\|_{L^2(D)}\|\nabla(u-g)\|_{L^2(D)} + -2\|f\|_{L^2(D)}\|g\|_{L^2(D)} \\
&\ge \|\nabla u\|^2_{L^2(D)} - \frac{1}{4}\|\nabla(u-g)\|^2_{L^2(D)} - C'\|f\|^2_{L^2(D)} - 2\|f\|_{L^2(D)}\|g\|_{L^2(D)} \\
&\ge \|\nabla u\|^2_{L^2(D)} - \frac{1}{4}(\|\nabla u\|_{L^2(D)} + \|\nabla g\|_{L^2(D)})^2 - C'\|f\|^2_{L^2(D)} - 2\|f\|_{L^2(D)}\|g\|_{L^2(D)} \\
&\ge \frac{1}{2}\|\nabla u\|^2_{L^2(D)} - \frac{1}{2}\|\nabla g\|^2_{L^2(D)} - C'\|f\|^2_{L^2(D)} - 2\|f\|_{L^2(D)}\|g\|_{L^2(D)} \\
&\ge -\frac{1}{2}\|\nabla g\|^2_{L^2(D)} - C'\|f\|^2_{L^2(D)} - 2\|f\|_{L^2(D)}\|g\|_{L^2(D)} = -\tilde{C},
\end{aligned}$$

by Poincar$\acute{e}$'s inequality, Young's inequality, and $(a+b)^2 \le 2(a^2+b^2)$, where $a, b \in \mathbb{R}$.
($d$) Since

$$J(u_k) \ge \frac{1}{2}\|\nabla u_k\|^2_{L^2(D)} - \frac{1}{2}\|\nabla g\|^2_{L^2(D)} - C'\|f\|^2_{L^2(D)} - 2\|f\|_{L^2(D)}\|g\|_{L^2(D)},$$

$J(u_k) \to \infty$ as $\|\nabla u_k\|_{L^2(D)} \to \infty$.

$$\begin{aligned}\|u_k\|_{L^2} &\le \|u_k - g\|_{L^2} + \|g\|_{L^2} \le C\|\nabla(u_k - g)\|_{L^2} + \|g\|_{L^2}\\ &\le C\|\nabla u_k\|_{L^2} + C\|\nabla g\|_{L^2} + \|g\|_{L^2}.\end{aligned}$$

Thus $J(u_k) \to \infty$ as $\|u_k\|_{L^2(D)} \to \infty$. If $\|u_k\|_{w^{1,2}(D)} \to \infty$, then $\|u_k\|_{L^2(D)} \to \infty$ or $\|\nabla u_k\|_{L^2(D)} \to \infty$. Therefore the coercivity condition for $J(\cdot)$ holds.
($e$) Let $u_k \in K_{g,0}$ such that $u_k \rightharpoonup u$ in $W^{1,2}(D)$ as $k \to \infty$. Since $W^{1,2}(D) \hookrightarrow L^2(D)$,

$$\nabla u_k \rightharpoonup \nabla u \text{ in } L^2(D), \quad u_k \to u \text{ in } L^2(D),$$

as $k \to \infty$, up to subsequence. $\nabla u_k \rightharpoonup \nabla u$ in $L^2(D)$ gives

$$\int_D |\nabla u|^2 dx \le \varliminf_{k\to\infty} \int_D |\nabla u_k|^2 dx.$$

Since

$$\left|\int_D f u_k - f u dx\right| \le \|f\|_{L^2(D)}\|u_k - u\|_{L^2(D)} \text{ as } k \to \infty,$$

we have

$$\lim_{k\to\infty} \int_D f u_k dx = \int_D f u dx.$$

Thus $J(\cdot)$ is weakly lower semi-continuous.

To show the uniqueness, we assume $u, \hat{u} \in K_{g,0}$ are two minimizers of the problem such that $u \ne \hat{u}$. Then $v = (u + \hat{u})/2 \in K_{g,0}$, by the convexity of $K_{g,0}$.

$$\begin{aligned}J(v) &= \int_D \left|\frac{\nabla(u+\hat{u})}{2}\right|^2 + 2f\left(\frac{u+\hat{u}}{2}\right) dx\\ &= \int_D \frac{1}{4}(|\nabla u|^2 + 2\nabla u \cdot \nabla \hat{u} + |\nabla \hat{u}|^2) + f(u+\hat{u}) dx\\ &= \int_D \frac{1}{4}(2|\nabla u|^2 + 2|\nabla \hat{u}|^2 - |\nabla u - \nabla \hat{u}|^2) + f(u+\hat{u}) dx\\ &< \frac{J(u) + J(\hat{u})}{2}.\end{aligned}$$

The last inequality holds, since $u \ne \hat{u}$. therefore it is a contradiction and we have the uniqueness of the minimizer. □

**Theorem 1.6.** *Let $D$ be a bounded open subset in $\mathbb{R}^n$, $g \in W^{1,2}(D)$ and $f \in L^\infty(D)$, $0 \le g$ on $\partial D$, $(-g)_+ \in W_0^{1,2}(D)$. Let $J_\epsilon(u) = \int_D (|\nabla u|^2 + 2f\Phi_\epsilon(u))dx$ be a functional over $K_g$. Then $J_\epsilon(\cdot)$ has a unique minimizer in $K_g \subset W^{1,2}$.*

*Proof.* (*a*) Clear
(*b*) With out loss of generality, we may assume that $\Phi_\epsilon(u(x)) = \int_{-\infty}^{s}\chi_\epsilon(t)dt \le u(x)_+$. Thus $J_\epsilon(g) \le \|\nabla g\|^2_{L^2(D)} + 2\|f\|_{L^2(D)}\|g\|_{L^2(D)} < +\infty$
(*c*), (*d*) Since $\Phi_\epsilon(u(x)) \le u(x)$, we also obtain

$$J_\epsilon(u) \ge \|\nabla u\|^2_{L^2(D)} - 2\|f\|_{L^2(D)}\|u\|_{L^2(D)}.$$

As the same manner in the proof of Theorem 1.5, we have (*c*), (*d*).
(*e*) Let $u_k \in K_g$ such that $u_k \rightharpoonup u$ in $W^{1,2}(D)$ as $k \to \infty$. Then we have $\int_D |\nabla u|^2 dx \le \underline{\lim}_{k\to\infty}\int_D |\nabla u_k|^2 dx$.

$$\begin{aligned}\left|\int_D f\Phi_\epsilon(u_k) - f\Phi_\epsilon(u)dx\right| &\le \int_D |f(x)||\chi_\epsilon(t_x)||u_k(x) - u(x)|dx \text{ for } t_x \text{ between } u_k(x), u(x)\\ &\le \|f\|_{L^2(D)}\|u_k - u\|_{L^2(D)} \to 0 \text{ as } k \to \infty.\end{aligned}$$

Thus $J_\epsilon(\cdot)$ is weakly lower semicontinuous.
the convexity of $K_g$ give the uniqueness of the minimizer. □

## 1.3 $W^{2,p}$ regularity of the solution of the classical obstacle problem

**Lemma 1.7.** *(Calderón-zygmund estimates) Let $u \in L^1(D)$, $f \in L^p(D)$, $1 < p < \infty$, and $\Delta u = f$ in D in the sense of distributions. Then $u \in W^{2,p}_{loc}(D)$ and*

$$\|u\|_{W^{2,p}(K)} \le C(\|u\|_{L^1(D)} + \|f\|_{L^p(D)}),$$

*for any $K \Subset D$ with $C = C(p, n, K, D)$.*

**Theorem 1.8.** *Let D be a bounded open subset in $\mathbb{R}^n$, $g \in W^{1,2}(D)$ and $f \in L^\infty(D)$, $0 \le g$ on $\partial D$, $(-g)_+ \in W^{1,2}_0(D)$. Let $J_\epsilon(u) = \int_D(|\nabla u|^2 + 2f\Phi_\epsilon(u))dx$ be a functional over $K_g$. Let $u_\epsilon$ be the minimizer of $J_\epsilon$ over $K_g$. Then the family $\{u_\epsilon\}$ is unformly bounded in $W^{1,2}(D)$ and in $W^{2,p}(K)$ for any $K \Subset D$, $1 < p < \infty$ where $0 < \epsilon < 1$*

*Proof.* By Theorem 1.3, we know that

$$\int_D (\nabla u_\epsilon \nabla\eta + f\chi_\epsilon(u_\epsilon)\eta)dx = 0,$$

for $\eta \in W^{1,2}_0(D)$. Take $\eta = u_\epsilon - g$.

$$\begin{aligned}0 &= \int_D \nabla u_\epsilon\nabla(u_\epsilon - g) + f\chi_\epsilon(u_\epsilon)(u_\epsilon - g)dx\\ &= \int_D \nabla(u_\epsilon - g)\cdot\nabla(u_\epsilon - g) + \nabla g\cdot\nabla(u_\epsilon - g) + f\chi_\epsilon(u_\epsilon)(u_\epsilon - g)dx\\ &\ge \|\nabla(u_\epsilon - g)\|^2_{L^2(D)} - \|\nabla g\|_{L^2(D)}\|\nabla(u_\epsilon - g)\|_{L^2(D)} - \|f\|_{L^2(D)}\|u_\epsilon - g\|_{L^2(D)}\\ &\ge \|\nabla(u_\epsilon - g)\|^2_{L^2(D)} - (\|\nabla g\|_{L^2(D)} + C\|f\|_{L^2(D)})\|\nabla(u_\epsilon - g)\|_{L^2(D)}\\ &\ge \frac{1}{2}\|\nabla(u_\epsilon - g)\|^2_{L^2(D)} - C'(\|\nabla g\|_{L^2(D)} + C\|f\|_{L^2(D)}),\end{aligned}$$

by Poincar$\acute{e}$'s inequality and Young's inequality. Then

$$\|\nabla(u_\epsilon - g)\|^2_{L^2(D)} \leq C(f, g).$$

Applying Calder$\acute{o}$n-zygmund estimates and Poincar$\acute{e}$'s inequality,

$$\begin{aligned}\|u_\epsilon\|_{W^{2,p}(K)} &\leq C(p, n, K, D)(\|u_\epsilon\|_{L^1(D)} + \|f\chi_\epsilon(u_\epsilon)\|_{L^p(D)})\\ &\leq C(p, n, K, D, f, g),\end{aligned}$$

for any $K \Subset D$, $1 < p < \infty$. $\square$

**Theorem 1.9.** *Let $D$ be a bounded open subset in $\mathbb{R}^n, g \in W^{1,2}(D)$ and $f \in L^\infty(D), 0 \leq g$ on $\partial D, (-g)_+ \in W^{1,2}_0(D)$. Let $u$ be the minimizer for the functional $J(u) = \int_D(|\nabla u|^2 + 2fu)dx$ over the set $K_{g,0} = \{u \in W^{1,2}(D) : u - g \in W^{1,2}_0(D), u \geq 0$ a.e. in $D\}$, then $u \in W^{2,p}_{loc}(D)$ for any $1 < p < \infty$.*

*Proof.* Since $u_\epsilon$ is unformly bounded for $0 < \epsilon < 1$ in $W^{1,2}(D)$, then there exists $u \in W^{1,2}(D)$, such that $u_\epsilon \rightharpoonup u$ in $W^{1,2}(D)$ and since $W^{1,2}(D) \hookrightarrow L^2(D)$,

$$\nabla u_\epsilon \rightharpoonup \nabla u \text{ in } L^2(D), \quad u_\epsilon \to u \text{ in } L^2(D),$$

as $\epsilon = \epsilon_k \to 0$. Since $W^{1,2}_0(D)$ is weakly closed, $u_\epsilon - g \in W^{1,2}_0(D)$, then $u - g \in W^{1,2}_0(D)$, i.e. $u \in K_g$.

By Theorem 1.8, we know that $u_\epsilon \in W^{2,p}_{loc}(D)$ and

$$\|u_\epsilon\|_{W^{2,p}(K)} \leq C(p, n, K, D, f, g),$$

for any $K \Subset D$, $1 < p < \infty$. Thus $u_\epsilon \rightharpoonup u$ in $W^{2,p}_{loc}(D)$, as $\epsilon \to 0$ for any $1 < p < \infty$. Hence $u \in W^{2,p}_{loc}(D)$ for any $1 < p < \infty$.

$$\begin{aligned}\left|\int_D f\Phi_\epsilon(u_\epsilon) - fu_+dx\right| &\leq \left|\int_D f\Phi_\epsilon(u_\epsilon) - f\Phi_\epsilon(u)dx + \int_D f\Phi_\epsilon(u) - fu_+dx\right|\\ &\leq \|f\|_{L^2(D)}\|u_\epsilon - u\|_{L^2(D)} + \|f\|_{L^2(D)}\|\Phi_\epsilon(u) - u_+\|_{L^2(D)}\\ &\leq \|f\|_{L^2(D)}(\|u_\epsilon - u\|_{L^2(D)} + 4\epsilon^2|D|) \to 0 \text{ as } \epsilon \to 0,\end{aligned}$$

by the same computation in the proof of Theorem 1.6, and we know that $\|\Phi_\epsilon - u_+\|_{L^\infty(D)} \leq 2\epsilon$. Thus we have

$$\int_D fu_+dx = \lim_{\epsilon\to 0}\int_D f\Phi_\epsilon(u_\epsilon)dx.$$

$\nabla u_\epsilon \rightharpoonup \nabla u$ in $L^2(D)$ gives

$$\int_D |\nabla u|^2dx \leq \varliminf_{\epsilon\to 0}\int_D |\nabla u_\epsilon|^2dx.$$

Therefore

$$\tilde{J}(u) = \int_D |\nabla u|^2 + fu_+dx \leq \varliminf_{\epsilon\to 0} J_\epsilon(u_\epsilon) \leq \varliminf_{\epsilon\to 0} J_\epsilon(v) = \varliminf_{\epsilon\to 0}\int_D |\nabla v|^2 + f\Phi_\epsilon(v)dx = \tilde{J}(v),$$

for any $v \in K_g$. by Theorem 1.2, $u$ is the minimizer of $J(\cdot)$ over $K_{g,0}$ and $u \in W^{2,p}_{loc}(D)$, for any $1 < p < \infty$. $\square$

**Theorem 1.10.** *Let $D$ be a bounded open subset in $\mathbb{R}^n, g \in W^{1,2}(D)$ and $f \in L^\infty(D), 0 \leq g$ on $\partial D, (-g)_+ \in W_0^{1,2}(D)$. Let $u$ be the minimizer for the functional $J(\cdot)$ over the set $K_{g,0}$, then $\Delta u = f\chi_{\{u>0\}}$ a.e. in $D$, equivalently, $\Delta u = f$ a.e in $\Omega = \{u > 0\}$ and $\Delta u = 0$ a.e. on $\Lambda = \{u = 0\}$.*

*Proof.* Since $u_\epsilon \in W^{2,p}_{loc}(D), \Delta u_\epsilon = f\chi_\epsilon(u_\epsilon)$ for a.e. in $D$. For $p > n$

$$u_\epsilon \to u \text{ in } C^{1,\alpha}_{loc}(D),$$

by the Sobolev embedding theorem with $\alpha = 1 - \frac{n}{p}$. Then $\Delta u = f$ a.e. in $\{u > 0\}$, by the locally unform convergence. Since $u \in W^{2,p}_{loc}(D)$, $\Delta u = 0$ a.e on $\{u = 0\}$. □

## 1.4 $C^{1,1}$ regularity of the solution of the classical obstacle problem

**Theorem 1.11.** *Let $u, f \in L^\infty(D), u \geq 0$*

$$\Delta u = f\chi_{\{u>0\}} \text{ in } D.$$

*Choose $x_0 \in \Gamma(u) = \partial\Omega \cap D$ such that $B_{2R}(x_0) \subset D$. Then*

$$\sup_{B_R(x_0)} u \leq C\|f\|_{L^\infty(D)}R^2,$$

*where $C = C(n)$.*

*Proof.* Let $u = u_1 + u_2$ such that

$$\begin{cases} \Delta u_1 = \Delta u, & \Delta u_2 = 0 \quad \text{in } B_{2R}(x_0), \\ u_1 = 0, & \quad u_2 = u \quad \text{on } \partial B_{2R}(x_0). \end{cases}$$

Let $\psi(x) = (4R^2 - |x - x_0|^2)/2n$, then $\Delta\psi = -1$ in $B_{2R}(x_0)$, $\psi = 0$ on $\partial B_{2R}(x_0)$. Consider $u_1 + M\psi$, where $M = \|f\|_{L^\infty(D)}$ then

$$\begin{cases} \Delta(u_1 + M\psi) \leq 0 & \text{in } B_{2R}(x_0), \\ u_1 + M\psi = 0 & \text{on } \partial B_{2R}(x_0). \end{cases}$$

This implies $u_1 + M\psi \geq 0, u_1 \geq -M\psi$ in $B_{2R}(x_0)$. In the similar way, we know that $-M\psi \leq u_1 \leq M\psi$ in $B_{2R}(x_0)$. Thus

$$|u_1| \leq \frac{2MR^2}{n} \text{ in } B_{2R}(x_0). \tag{2}$$

Since $\Delta u_2 = 0$ in $B_{2R}(x_0), u_2 = u \geq 0$ on $\partial B_{2R}(x_0)$, $u_2 \geq 0$ in $B_{2R}(x_0)$. since $u(x_0) = u_1(x_0) + u_2(x_0) = 0, u_2(x_0) = -u_1(x_0) \leq 2MR^2/n$. By the Harnack inequality,

$$u_2(x) \leq Cu_2(x_0) \leq CMR^2, \text{ for any } x \in B_R(x_0), \tag{3}$$

where $C = C(n)$. Using (2),(3) we have the inequality. □

**Lemma 1.12.** *Let $\Delta v = f$ in $B_{2R}(x_0) \Subset D$ and $f$ has a $C^{1,1}-$*regular potential, *i.e.* $f = \Delta\phi$ *in $D$, where $\phi \in C^{1,1}(D)$. Then*

$$\|D^2 v\|_{L^\infty(B_R(x_0))} \leq C(n)\Big(\frac{\|v\|_{L^\infty(B_{2R}(x_0))}}{R^2} + \|D^2\phi\|_{L^\infty(B_{2R}(x_0))}\Big).$$

*Proof.* We may assume that $\phi(x_0) = |\nabla\phi(x_0)| = 0$. Let $w = v - \phi$. By using the mollification, we have

$$\|D^2 w\|_{L^\infty(B_R(x_0))} \leq \frac{C(n)}{R^2}\|w\|_{L^\infty(B_{2R}(x_0))},$$

and

$$\|D^2 v\|_{L^\infty(B_R(x_0))} \leq C(n)\Big(\frac{\|v\|_{L^\infty(B_{2R}(x_0))} + \|\phi\|_{L^\infty(B_{2R}(x_0))}}{R^2} + \|D^2\phi\|_{L^\infty(B_{2R}(x_0))}\Big).$$

By the Taylor expansion,

$$\phi(x_0 + h) = \frac{1}{2}\sum_{i,j} h_i h_j \frac{\partial^2\phi}{\partial x_i x_j}(\theta h_1, ..., \theta h_n) \leq R^2 C(n)\|D^2\phi\|_{L^\infty(B_{2R}(x_0))},$$

where $|h| < 2R, 0 < \theta < 1$. Thus we obtain

$$\|\phi\|_{L^\infty(B_{2R}(x_0))} \leq R^2 C(n)|D^2\phi\|_{L^\infty(B_{2R}(x_0))}$$

and the desired inequrity. □

**Theorem 1.13.** *Let $u \in L^\infty(D), u \geq 0, \Delta u = f\chi_{\{u>0\}}$ in $D$ for $f \in L^\infty(D)$ such that $f = \Delta\phi$ in $D$, where $\phi \in C^{1,1}(D)$. Then $u \in C^{1,1}_{loc}(D)$ and*

$$\|u\|_{C^{1,1}(K)} \leq C(\|u\|_{L^\infty(D)} + \|D^2\phi\|_{L^\infty(D)}),$$

*for $K \Subset D$, where $C = C(n, dist(K, \partial D))$.*

*Proof.* Let $K \Subset D$. We know that $u \in W^{2,p}_{loc}(D)$ for any $1 < p < \infty$ and $D^2 u = 0$ a.e on $\Omega^c(u)$. Thus it suffice to show that $\|D^2(u)\|_{L^\infty(\Omega(u)\cap K)} < +\infty$. Let $x_0 \in \Omega(u) \cap K$, $d = dist(x_0, \Omega^c(u))$, $\delta = dist(K.\partial D)$.

Case 1) $d < \delta/5$. Let $y_0 \in \partial B_d(x_0) \cap \partial\Omega$, then $B_{4d}(y_0) \subset B_{5d}(x_0) \Subset D$. By Theorem 1.11 we obtain

$$\|u\|_{L^\infty(B_{2d}(y_0))} \leq C(n)\|f\|_{L^\infty(D)} d^2.$$

We know that $B_d(x_0) \subset B_{2d}(y_0)$ and $\Delta u = f$ in $B_d(x_0)$. By Lemma 1.12, and $\|f\|_{L^\infty(D)} \leq \|D^2\phi\|_{L^\infty(D)}$,

$$\begin{aligned}
\|D^2 u\|_{L^\infty(B_{d/2}(x_0))} &\leq C(n)\Big(\frac{\|u\|_{L^\infty(B_d(x_0))}}{d^2} + \|D^2\phi\|_{L^\infty(B_d(x_0))}\Big) \\
&\leq C(n)\Big(\frac{\|u\|_{L^\infty(B_{2d}(y_0))}}{d^2} + \|D^2\phi\|_{L^\infty(B_d(x_0))}\Big) \\
&\leq C(n)(\|f\|_{L^\infty(D)} + \|D^2\phi\|_{L^\infty(D)}) \leq C(n)(\|D^2\phi\|_{L^\infty(D)}).
\end{aligned}$$

Case 2) $d \geq \delta/5$. In this case, the interior derivative estimate for $u$ in $B_{\delta/5}(x_0)$ gives

$$\left\|D^2 u\right\|_{L^\infty(B_{\delta/10}(x_0))} \leq C(n)\left(\frac{\|u\|_{L^\infty(D)}}{\delta^2} + \left\|D^2\phi\right\|_{L^\infty(D)}\right).$$

Combining cases above, we obtain

$$\|u\|_{C^{1,1}(K)} \leq C(n)\left(\frac{\|u\|_{L^\infty(D)}}{\delta^2} + \left\|D^2\phi\right\|_{L^\infty(D)}\right).$$

□

# 2 Optimal regularity of solutions of obstacle problems

## 2.1 Model problems $A, B, C$ and $OT_1 - OT_2$

**Definition 2.1.** (Problem $A$, No-sign obstacle problem)
Let $D$ be a open set in $\mathbb{R}^n$. Let a problem finding a function $u$ in $D$ such that

$$\Delta u = \chi_{\Omega(u)} \text{ in } D, \text{ where } \Omega(u) = D \setminus \{u = |\nabla u| = 0\}$$

be a Problem $A$. The free boundary in this case is $\Gamma(u) = \partial\Omega(u) \cap D$.

**Definition 2.2.** (Problem $B$, superconductivity problem)
Let $D$ be a open set in $\mathbb{R}^n$. Let a problem finding a function $u$ in $D$ such that

$$\Delta u = \chi_{\Omega(u)} \text{ in } D, \text{ where } \Omega(u) = \{|\nabla u| > 0\}$$

be a Problem $B$. The free boundary in this case is $\Gamma(u) = \partial\Omega(u) \cap D$.

**Definition 2.3.** (Problem $C$, Two-phase membrane problem)
Let $D$ be a open set in $\mathbb{R}^n$. Let a problem finding a function $u$ in $D$ such that

$$\Delta u = \lambda_+\chi_{\Omega_+(u)} - \lambda_-\chi_{\Omega_-(u)} \text{ in } D, \text{ where } \Omega_\pm(u) = \{u_\pm > 0\}$$

be a Problem $C$, where $\lambda_\pm > 0$. In this case $\Omega(u) = \Omega_+(u) \cup \Omega_-(u)$ and the free boundary is $\Gamma(u) = \partial\Omega(u) \cap D = \Gamma_+(u) \cup \Gamma_-(u)$ where $\Gamma_\pm(u) = \partial\Omega_\pm(u) \cap D$.

**Definition 2.4.** (Obstacle-type problems, $OT_1 - OT_2$)
Let $D$ be a open set in $\mathbb{R}^n$. Let a problem finding $u \in L^\infty_{loc}(D)$ satisfies $(OT_1)$,

$$\Delta u = f(x, u)\chi_{G(u)} \text{ in } D, \quad |\nabla u| = 0 \text{ on } D \setminus G(u),$$

where $G(u) \subset D$ is open and $f : D \times \mathbb{R} \to \mathbb{R}$ satisfies $(OT_2)$,

$$\begin{cases} |f(x,t) - f(y,t)| \leq M_1|x - y|, & x, y \in D, t \in \mathbb{R}, \\ f(x,s) - f(x,t) \geq -M_2(s - t), & x \in D, \text{ such that } \in R, s \geq t \end{cases}$$

for $M_1, M_2 \geq 0$, be a Problem $OT_1 - OT_2$. The free boudary is $\partial G(u) \cap D$and/or the set of discontinous points of $f(x, u)$. It depends on the problem.

In the case of Problems $A, B$, $G = \Omega(u), f(x,t) = 1$, and in the case of Problem $C$, $G = D, f(x,t) = \lambda_+\chi_{\Omega_+(t)} - \lambda_-\chi_{\Omega_-(t)}$, so the condition $|\nabla u| = 0$ on $D \setminus G$ is eliminated. For any cases, we can assign 0 for $M_1$ and $M_2$ and then Problems $A, B, C$ fit into Problem $OT_1 - OT_2$.

## 2.2 ACF monotonicity formula and generalizations

**Theorem 2.1.** *Let $u$ be a harmonic function in $B_1$ and*

$$J(r,u) = \frac{1}{r^2}\int_{B_r} \frac{|\nabla u|^2}{|x|^{n-2}} dx, \quad 0 < r < 1,$$

*then $r \mapsto J(r,u)$ is monotone nondecreasing and $|\nabla u(0)| \le C(n)\|u\|_{L^2(B_1)}$.*

*Proof.* $u$ can be represented as a locally unformly convergent series $u(x) = \sum_{k=0}^{\infty} f_k(x)$, where $f_k(x)$ are homogeneous harmonic polynimial of degree $k$, $f_k(tx) = t^k f(x)$, and $f_k, f_l$ are orthogonal, when $k \neq l$. Then

$$\begin{aligned}
J(r,u) &= \frac{1}{r^2}\int_0^r \int_{\partial B_1} |\nabla u(\rho\theta)|^2 \rho d\theta d\rho \\
&= \frac{1}{r^2}\int_0^r \int_{\partial B_1} \rho \sum_{k=1}^{\infty} |\nabla f_k(\rho\theta)|^2 d\theta d\rho \\
&= \frac{1}{r^2}\int_0^r \int_{\partial B_1} \sum_{k=1}^{\infty} \rho^{2k-1} |\nabla f_k(\theta)|^2 d\theta d\rho \\
&= \sum_{k=1}^{\infty} a_k r^{2(k-1)},
\end{aligned}$$

where $a_k = (1/2k)\int_{\partial B_1} |\nabla f_k(\theta)|^2 d\theta \ge 0$. Thus $r \mapsto J(r,u)$ is monotone nondecreasing. Let $r \to 0+$, then $J(0+,u) \le J(1/2,u)$. since $u$ is $C^1$ near the orgin, for given $\epsilon > 0$, there exists $r > 0$ such that $|x| < r$ implies $||\nabla u(x)|^2 - |\nabla u(0)|^2| \le \epsilon$. Let $c(n) = (1/r^2)\int_0^r \int_{\partial B_1} \rho d\theta d\rho$, then

$$\begin{aligned}
&\left| J(r,u) - c(n)|\nabla u(0)|^2 \right| \\
&= \left| \frac{1}{r^2}\int_0^r \int_{\partial B_1} |\nabla u(\rho\theta)|^2 \rho d\theta d\rho - \frac{1}{r^2}\int_0^r \int_{\partial B_1} |\nabla u(0)|^2 \rho d\theta d\rho \right| \\
&\le \frac{1}{r^2}\int_0^r \int_{\partial B_1} \left| |\nabla u(x)|^2 - |\nabla u(0)|^2 \right| \rho d\theta d\rho \\
&\le \frac{1}{r^2}\int_0^r \int_{\partial B_1} \epsilon \rho d\theta d\rho = |\nabla u(0)|^2 \epsilon
\end{aligned}$$

Therefore $J(0+,u) = c(n)|\nabla u(0)|^2$, for $c(n) > 0$. Hence

$$c(n)|\nabla u(0)|^2 \le J(\frac{1}{2}, u).$$

We will prove $J(1/2,u) \le C_n\|u\|^2_{L^2(B_1)}$. Let $V$ be a smooth extention of $|x|^{2-n}$ from $B_{1/2}$ to $B_1$ such that $V(x) \ge 0$ and $V = 0$ near $\partial B_1$. This implies $\nabla V = 0$ on $\partial B_1$, and let $\tilde{V} = \min(V, (1/\delta^{n-2})$, for a small $\delta > 0$. Since $\Delta u = 0$, $\Delta((1/2)u^2) = u\Delta u + |\nabla u|^2 = |\nabla u|^2$.

$$
\begin{aligned}
\int_{B_{1/2}\setminus B_\delta} \frac{|\nabla u|^2}{|x|^{n-2}}dx &\leq \int_{B_1} \big(\Delta\frac{u^2}{2}\big)\tilde{V}dx = -\int_{B_1} \nabla\frac{u^2}{2}\cdot\nabla\tilde{V}dx \\
&= -\int_{B_1\setminus B_\delta} \nabla\frac{u^2}{2}\cdot\nabla V dx \\
&= -\int_{\partial(B_1\setminus B_\delta)} \frac{u^2}{2}(\nabla V\cdot\nu)d\sigma_x + \int_{B_1\setminus B_\delta}\frac{u^2}{2}\Delta V dx \qquad (4)\\
&= -\int_{\partial B_\delta} \frac{u^2}{2}(\nabla V\cdot -x)d\sigma_x + \int_{B_1\setminus B_{1/2}}\frac{u^2}{2}\Delta V dx \\
&= -\int_{\partial B_\delta} \frac{(n-2)u^2}{2\delta^{n-2}}d\sigma_x + \int_{B_1\setminus B_{1/2}}\frac{u^2}{2}\Delta V dx \\
&\leq \int_{B_1\setminus B_{1/2}}\frac{u^2}{2}\Delta V dx.
\end{aligned}
$$

letting $\delta\to 0$, we have $J(1/2,u)\leq C(n)\|u\|^2_{L^2(B_1)}$ Thus we have the desired inequality. □

**Theorem 2.2.** *(Alt- Caffarelli-Friedman (ACF) monotonicity formula) Let $u_\pm$ be a pair of continuous functions such that*

$$u_\pm \geq 0, \quad \Delta u_\pm \geq 0, \quad u_+\cdot u_- = 0 \text{ in } B_1,$$

*then*

$$r\mapsto \Phi(r) = \Phi(r,u_+,u_-) = J(r,u_+)J(r,u_-) = \frac{1}{r^4}\int_{B_r}\frac{|\nabla u_+|^2}{|x|^{n-2}}dx\int_{B_r}\frac{|\nabla u_-|^2}{|x|^{n-2}}dx$$

*is nondecreasing for* $0<r<1$.

**Example.** (Friedland-Hayman inequalty) Let $C=\{r\theta : r>0,\theta\in\Sigma_0\}$, whrere $\Sigma_0\subset\partial B_1$. Let $h$ be a homogeneous harmonic function in $C$ such that $h(r\theta)=r^\alpha f(\theta),\alpha>0$, and $h(x)=0$ for $x\in\partial C$.

$$
\begin{aligned}
\Delta h &= \partial_{rr}h + \frac{n-1}{r}\partial_r h + \frac{1}{r^2}\Delta_\theta h \\
&= r^{\alpha-2}[(\alpha(\alpha-1)+(n-1)\alpha)f(\theta)+\Delta_\theta f(\theta)],
\end{aligned}
$$

where $\Delta_\theta$ is the spherical Laplacian. Therefore $h$ is harmonic in $C$ if and only if $-\Delta_\theta f(\theta)=\lambda f(\theta)$ in $\Sigma_0$ where $\lambda=\alpha(n-2+\alpha)$. If $h>0$ in $\Sigma_0$, then $\lambda$ will be the principal eigenvalue, and we denote $\alpha=\alpha(\Sigma_0)$ and call it the characteristic harmonic function. Let $\Sigma_\pm$ be open subsets on $B_1$, $\lambda_\pm$ be the principal eigenvalues of $\Sigma_\pm$ and $f_\pm$ be the corresponding eigenfunctions, $u_\pm$ be homogeneous harmonic functions, such that

$$u_\pm = r^{\alpha_\pm}f_\pm(\theta), \text{ in } C_\pm=\{r\theta : r>0,\theta\in\Sigma_\pm\},$$

where $\alpha_\pm=\alpha(\Sigma_\pm)>0$ are the characteristic constant of $\Sigma_\pm$. Then $u_\pm$ is harmonic in $C_\pm$. we extend $u_\pm$ to $\mathbb{R}$ by zero in the complements of $C_\pm$, repectively. Then $\Delta u_\pm\geq 0$. (see Lemma

(2.11)). Thus $u_\pm$ satisfies the assumptions of the ACF formula. Let the pair $(u, f, \alpha, C)$ be either $(u_+, f_+, \alpha_+, C_+)$ or $(u_-, f_-, \alpha_-, C_-)$.

$$J(r,u) = \frac{1}{r^2}\int_0^r \int_{\partial B_1\cap C} |\nabla u(\rho\theta)|^2 \rho d\theta d\rho = \frac{1}{r^2}\int_0^r \int_{\partial B_1\cap C} \rho^{2\alpha-1}|\nabla u(\theta)|^2 d\theta d\rho$$
$$= \frac{1}{r^2}\int_0^r \rho^{2\alpha-1}d\rho \cdot \int_{\partial B_1\cap C} |\nabla f(\theta)|^2 d\theta = \frac{1}{2\alpha}C(n,f)r^{2(\alpha-1)}.$$

Thus

$$\Phi(r,u_+,u_-) = J(r,u_+)J(r,u_-) = \frac{C(n,f_\pm)}{4\alpha_+\alpha_-}r^{2(\alpha_++\alpha_--2)}, \text{ with } \frac{C(n,f_\pm)}{4\alpha_+\alpha_-} > 0.$$

In this case, the ACF monotonicity formula is equivalent to $\alpha_+ + \alpha_- - 2 \geq 0$.

**Lemma 2.3.** *Let $v \in C(D)$ be a nonnegative subharmonic function in an open set $D$ of $\mathbb{R}^n$, then $v \in W^{1,2}_{loc}(D)$.*

*Proof.* Let $v_\epsilon$ be mollifications of $v$, such that $v_\epsilon \leq 0, \Delta v_\epsilon \geq 0$. Let $K \Subset D, \delta = dist(K, \partial D)$ and let $\psi \in C_c^\infty(D)$, such that $\psi = 1$ on $K, |\nabla\psi| \leq 2/\delta$ on $D$, $\operatorname{supp}\psi \Subset D$. Let $\phi = v_\epsilon\psi^2$, then we have

$$\int_D \nabla v_\epsilon \cdot \nabla\phi dx = \int_D \psi^2|\nabla v_\epsilon|^2 + 2v_\epsilon\psi\nabla v_\epsilon \cdot \nabla\psi dx \leq 0.$$

Consequently,

$$\int_D \psi^2|\nabla v_\epsilon|^2 dx \leq -2\int_D v_\epsilon\psi\nabla v_\epsilon\cdot\nabla\psi dx \leq 2\int_D v_\epsilon\psi|\nabla v_\epsilon||\nabla\psi|dx \leq \int_D \frac{1}{2}\psi^2|\nabla v_\epsilon|^2 + 2v_\epsilon^2|\nabla\psi|^2 dx.$$

Therefore

$$\int_D \psi^2|\nabla v_\epsilon|^2 dx \leq 4\int_D v_\epsilon^2|\nabla\psi|^2 dx.$$

Letting $\epsilon \to 0+$ gives

$$\int_K |\nabla v|^2 dx \leq \frac{4^2}{\delta^2}\int_{supp\psi} v^2 dx < +\infty,$$

by the properties of $\psi$. Thus the proof is complete. □

**Example.** (Reduction of ACF monotonicity formula to Friedland-Hayman inequality) Let $u_\lambda(x) = (1/\lambda)u(\lambda x)$, then

$$J(r/\lambda, u_\lambda) = J(r,u), \quad \Phi(r,u_+,u_-) = \Phi(r/\lambda, u_{+\lambda}, u_{-\lambda}).$$

Let $u$ be either $u_+$ or $u_-$ in $B_1$, fix $r < 1$, then $u_r(x) = (1/r)u(rx)$ for $x \in B_{1/r}, (1/r) > 1$. Since

$$\frac{\Phi(1+h, u_{+r}, u_{-r}) - \Phi(1, u_{+r}, u_{-r})}{h} = r\frac{\Phi(r(1+h), u_+, u_-) - \Phi(r, u_+, u_-)}{rh},$$

we have $\Phi'(1, u_{+r}, u_{-r}) = r\Phi'(r, u_+, u_-)$. Therefore it suffice to show that $\Phi'(1) \geq 0$ for any pair of function that satisfies the condition of ACF formula for $B_R, R > 1$.
Let $u$ be either $u_+$ or $u_-$ in $B_R$. Let

$$I(r, u) = \int_{B_r} \frac{|\nabla u|^2}{|x|^{n-2}} dx.$$

i.e. $I(r, u) = r^2 J(r, u)$. Then $\Phi(r, u_+, u_-) = \dfrac{1}{r^4} I(r, u_+) I(r, u_-)$.

$$\Phi'(r, u_+, u_-) = \frac{1}{r^4} I'(r, u_+) I(r, u_-) + \frac{1}{r^4} I(r, u_+) I'(r, u_-) - \frac{4}{r^5} I(r, u_+) I(r, u_-),$$

then

$$\Phi'(1, u_+, u_-) = I'(1, u_+) I(1, u_-) + I(1, u_+) I'(1, u_-) - 4I(1, u_+) I(1, u_-).$$

Thus we need to show that

$$\frac{I'(1, u_+)}{I(1, u_+)} + \frac{I'(1, u_-)}{I(1, u_-)} \geq 4.$$

Let $u_\epsilon$ be a mollification of $u$, such that $\Delta u_\epsilon \geq 0, u_\epsilon \geq 0$.

$$\begin{aligned}
\int_{B_1 \setminus B_\rho} \frac{\Delta(u_\epsilon^2/2)}{|x|^{n-2}} dx &= \int_{\partial(B_1 \setminus B_\rho)} (\nabla \frac{u_\epsilon^2}{2} \cdot \nu) \frac{1}{|x|^{n-2}} - (\nabla \frac{1}{|x|^{n-2}} \cdot \nu) \frac{u_\epsilon^2}{2} d\sigma_x \\
&= \int_{\partial(B_1 \setminus B_\rho)} (u_\epsilon \nabla u_\epsilon \cdot \nu) \frac{1}{|x|^{n-2}} + \frac{n-2}{2} \frac{u_\epsilon^2}{r^n} (x \cdot \nu) d\sigma_x \\
&= \int_{\partial B_1} u_\epsilon \partial_r u_\epsilon + \frac{n-2}{2} u_\epsilon^2 d\theta - \int_{\partial B_\rho} (u_\epsilon \partial_r u_\epsilon + \frac{n-2}{2} u_\epsilon^2) \frac{1}{\rho^{n-2}} d\sigma_x \\
&= \int_{\partial B_1} u_\epsilon \partial_r u_\epsilon + \frac{n-2}{2} u_\epsilon^2 d\theta - \int_{\partial B_\rho} (u_\epsilon \partial_r u_\epsilon + \frac{n-2}{2} u_\epsilon^2) \rho d\theta
\end{aligned}$$

Letting $\rho \to 0$, we have

$$\int_{B_1} \frac{\Delta(u_\epsilon^2/2)}{|x|^{n-2}} dx = \int_{\partial B_1} u_\epsilon \partial_r u_\epsilon + \frac{n-2}{2} u_\epsilon^2 d\theta.$$

And using $|\nabla u_\epsilon|^2 \leq \Delta(u_\epsilon^2/2)$, we obtain

$$I(1, u_\epsilon) = \int_{B_1} \frac{|\nabla u_\epsilon|^2}{|x|^{n-2}} dx \leq \int_{B_1} \frac{\Delta(u_\epsilon^2/2)}{|x|^{n-2}} dx = \int_{\partial B_1} u_\epsilon \partial_r u_\epsilon + \frac{n-2}{2} u_\epsilon^2 d\theta.$$

Letting $\epsilon \to 0+$, $I(1, u) \leq \int_\Sigma (u\partial_r u + (n - 2/2)u^2) d\theta$, where $\Sigma = \{u > 0\} \cap \partial B_1$, and we know that $I'(1, u) = \int_\Sigma |\nabla u|^2 d\theta$. Hence

$$\frac{I'(1, u)}{I(1, u)} \geq \frac{\int_\Sigma (\partial_r u)^2 + |\nabla_\theta u|^2 d\theta}{\int_\Sigma u\partial_r u + \dfrac{n-2}{2} u^2 d\theta}.$$

For the pricipal eigenvalue $\lambda = \lambda(\Sigma)$ of the spherical Laplacian $\Delta_\theta$ in $\Sigma$,

$$\frac{\int_\Sigma |\nabla_\theta u|^2}{\int_\Sigma u^2} \geq \lambda.$$

By the Young's inequality $\int_\Sigma u\partial_r u \leq \frac{1}{2}\Big[\alpha \int_\Sigma u^2 + \frac{1}{\alpha}\int_\Sigma(\partial_r u)^2\Big]$, for $\alpha > 0$. Hence

$$\frac{I'(1,u)}{I(1,u)} \geq 2\frac{\int_\Sigma(\partial_r u)^2 + \lambda u^2}{(1/\alpha)\int_\Sigma(\partial_r u)^2 + (\alpha + n - 2)\int_\Sigma u^2}.$$

Let's choose $\alpha$ such that $1/\alpha = \alpha + n - 2/\lambda$, i.e. $\alpha = \alpha(\Sigma)$ is the characteristic constant of $\Sigma$. Then

$$\frac{I'(1,u)}{I(1,u)} \geq 2\alpha,$$

consequently,

$$\frac{I'(1,u_+)}{I(1,u_+)} + \frac{I'(1,u_-)}{I(1,u_-)} - 4 \geq 2(\alpha_+ + \alpha_- - 2),$$

where $\Sigma_\pm = \{u_\pm > 0\} \cap \partial B_1$ and $\alpha_\pm = \alpha(\Sigma_\pm)$. By the Friedland-Hayman inequality $\alpha_+ + \alpha_- - 2 \geq 0$, we have the desired inequality.

**Theorem 2.4.** *(ACF estimate) Let $u_\pm$ be a pair of continuous functions such that*

$$u_\pm \geq 0, \quad \Delta u_\pm \geq 0, \quad u_+ \cdot u_- = 0 \textit{ in } B_1,$$

*then $\Phi(r, u_+, u_-) \leq C(n)\|u_+\|^2_{L^2(B_1)}\|u_-\|^2_{L^2(B_1)}$, for $0 < r \leq 1/2$.*

*Proof.* Since $\Delta u_\pm \geq 0, |\nabla u_\pm|^2 \leq \Delta(u^2/2), J(1/2,u) \leq C(n)\|u\|^2_{L^2(B_1)}$, by the same argument at (4) in Theorem 2.1. Since $\Phi(r)$ is nondecreasing,

$$\Phi(r, u_+, u_-) \leq C(n)\|u_+\|^2_{L^2(B_1)}\|u_-\|^2_{L^2(B_1)}, \text{ for } 0 < r \leq 1/2.$$

□

**Theorem 2.5.** *(Case of equality in ACF monotonicity formula) Let $u_\pm$ be as in above theorem and suppose that $\Phi(r_1) = \Phi(r_2)$ for some $0 < r_1 < r_2 < 1$. Then either of the following holds:*

*(a) $u_+ = 0$ in $B_{r_2}$ or $u_- = 0$ in $B_{r_2}$,*

*(b) There exists unit vector $e$ and constants $k_\pm > 0$ such that*

$$u_+(x) = k_+(x\cdot e)_+, \quad u_-(x) = k_-(x\cdot e)_- \textit{ in } B_{r_2}.$$

**Theorem 2.6.** *(Caffarelli-Jerison-Kenig (CJK) estimate) Let $u_\pm$ be a pair of continuous functions in $B_1$ such that*

$$u_\pm \geq 0, \quad \Delta u_\pm \geq -1, \quad u_+ \cdot u_- = 0 \textit{ in } B_1,$$

*then*

$$\Phi(r, u_+, u_-) \leq C(n)(1 + J(1,u_+) + J(1,u_-))^2, \quad 0 < r < 1.$$

**Theorem 2.7.** *(scaled version) Let $u_\pm$ be a pair of continuous functions in $B_R$ such that*

$$u_\pm \geq 0, \quad \Delta u_\pm \geq -L, \quad u_+ \cdot u_- = 0 \text{ in } B_R,$$

*then*

$$\Phi(r, u_+, u_-) \leq C(n)(R^2L^2 + J(R, u_+) + J(R, u_-))^2, \quad 0 < r < R.$$

**Theorem 2.8.** *Let $u_\pm$ be a pair of continuous functions in $B_1$ such that*

$$u_\pm \geq 0, \quad \Delta u_\pm \geq -1, \quad u_+ \cdot u_- = 0 \text{ in } B_1,$$

*then*

$$\Phi(r, u_+, u_-) \leq C(n)(1 + \|u_+\|^2_{L^2(B_1)} + \|u_-\|^2_{L^2(B_1)})^2, \text{ for } 0 < r \leq 1/2.$$

*Proof.* Since $u_\pm$ are nonnegative and $\Delta u_\pm \geq -1$ in $B_1$, $|\nabla u_\pm|^2 \leq \Delta(u_\pm^2/2) + u_\pm$. Using this inequality, we have $J(1/2, u_\pm) \leq C(n)(1 + \|u_\pm\|^2_{L^2(B_1)})$, by the same argument at (4) in Theorem 2.1.

Consider $u_\pm$ as function in $B_{1/2}$ then

$$\Phi(r, u_+, u_-) \leq C(n)(1/4 + J(1/2, u_+) + J(1/2, u_-))^2, \quad 0 < r < 1/2,$$

by the scaled CJK estimate. For $r = 1/2, \Phi(1/2, u_+, u_-) = J(1/2, u_+)J(1/2, u_-) \leq (1/4 + J(1/2, u_+) + J(1/2, u_-))^2$. □

**Theorem 2.9.** *(scaled version) Let $u_\pm$ be a pair of continuous functions in $B_R$ such that*

$$u_\pm \geq 0, \quad \Delta u_\pm \geq -L, \quad u_+ \cdot u_- = 0 \text{ in } B_R,$$

*then*

$$\Phi(r, u_+, u_-) \leq C(n)\left(R^2L^2 + \frac{\|u_+\|^2_{L^2(B_R)} + \|u_-\|^2_{L^2(B_R)}}{R^{n+2}}\right)^2, \text{ for } 0 < r \leq R/2.$$

**Theorem 2.10.** *Let $u_\pm$ be a pair of continuous functions in $B_1$ such that*

$$u_\pm \geq 0, \quad \Delta u_\pm \geq -1, \quad u_+ \cdot u_- = 0 \text{ in } B_1,$$

*and assume that $u_\pm(x) \leq C_0|x|^\epsilon$ in $B_1$ for some $\epsilon > 0$. Then for $0 < r_1 \leq r_2 < 1$, we have*

$$\Phi(r_1) \leq (1 + r_2^\epsilon)\Phi(r_2) + C_1 r_2^{2\epsilon},$$

*where $C_1 = C_1(C_0, n, \epsilon)$. In particular, the limit $\Phi(0_+)$ exists.*

## 2.3 Optimal regularity in $OT_1 - OT_2$

**Lemma 2.11.** *Let $u \in W^{1,2}_{loc}(D) \cap C(D)$ such that $u \geq 0$ in open set $D \in \mathbb{R}^n$. If $\Delta u \geq -a$ in the sense of distribution on $\{u > 0\}$ for some $a \geq 0$, then $\Delta u \geq -a$ in $D$.*

*Proof.* Let $\psi_\epsilon \in C^\infty(\mathbb{R})$ such that $0 \le \psi_\epsilon \le 1, \psi'_\epsilon \ge 0, \psi_\epsilon(t) = 0$ for $t \le \epsilon/2, \psi_\epsilon(t) = 1$ for $t \ge \epsilon$. Let $\phi \in C_c^\infty(D), \phi \ge 0$ and $\eta = \psi_\epsilon(u)\phi$, then $\eta \ge 0, \eta \in W_0^{1,2}(E)$, where $E = \{u > 0\}$. Thus

$$\int_E \nabla u \cdot \nabla \eta dx \le a \int_E \eta.$$

Note that

$$\begin{aligned}\int_E \psi_\epsilon(u)\nabla u \cdot \nabla\phi dx &\le \int_E \psi_\epsilon(u)\nabla u \cdot \nabla\phi + \psi'_\epsilon(u)\phi|\nabla u|^2 dx = \int_E \nabla u \cdot \nabla(\psi_\epsilon(u)\phi)dx \\ &\le a\int_E \psi_\epsilon(u)\phi dx \le a\int_E \phi dx.\end{aligned}$$

Letting $\epsilon \to 0+$ gives

$$\int_D \nabla u \cdot \nabla\phi dx \le a \int_D \phi dx,$$

since on $\{u = 0\}$, $\nabla u = 0$ a.e. We have $\Delta u \ge -a$ in the sense of distribution in $D$. □

**Lemma 2.12.** *Let $u \in C^1(D)$ satisfy $OT_1 - OT_2$, $e$ be a unit vector, and D is bounded then*

$$\Delta(\partial_e u)_\pm \ge -L \text{ in } D,$$

*where $L = M_1 + M_2\|\nabla u\|_{L^\infty(D)}$.*

*Proof.* Fix $e$ and let $v = \partial_e u, E = \{v > 0\}$. Since $|\nabla u| = 0$ on $D \setminus G(u), E \subset G$. We will show that $\Delta v \ge -L$ in the sense of distributions in $E$. Let $\eta \in C_c^\infty(D), \eta \ge 0$ such that $supp(\eta(x)) \subset \{v > \delta\}$ for $\delta > 0$. Then $supp(\eta(x - he)) \subset \{v > 0\} \subset G$, for sufficiently small $h > 0$. For brevity, we will use $\eta$ to denote either $\eta(x)$ or $\eta(x - he)$. Then

$$-\int_D \nabla u \cdot \nabla\eta dx = \int_D f\chi_G \eta dx = \int_D f\eta dx,$$

since $\Delta u = f(x,u)\chi_G$ in $D$ and $supp(\eta) \subset G$. Thus we obtain

$$-\int_D \nabla v_h \cdot \nabla\eta(x)dx = \frac{1}{h}\int_D [f(x + he, u(x + he)) - f(x, u(x))]\eta(x)dx,$$

where $v_h(x) = \dfrac{u(x + he) - u(x)}{h}$. we know that $u(x + he) > u(x)$ on $supp(\eta) \subset \{v > \delta\}$ and by $OT_2$,

$$\begin{cases}|f(x,t) - f(y,t)| \le M_1|x - y|, & x, y \in D, t \in \mathbb{R},\\ f(x,s) - f(x,t) \ge -M_2(s - t), & x \in D, suchthat \in R, s \ge t,\end{cases}$$

we obtain

$$\begin{aligned}f(x + he, u(x + he)) - f(x, u(x)) &= f(x + he, u(x + he)) - f(x + he, u(x)) \\ &\quad + f(x + he, u(x)) - f(x, u(x)) \\ &\ge -M_2(u(x + he) - u(x)) - M_1 h.\end{aligned}$$

Thus

$$-\int_D \nabla v_h \cdot \nabla \eta dx \geq -\int_D (M_2 v_h + M_1)\,\eta dx.$$

Letting $h \to 0$ and then $\delta \to 0$ we have

$$-\int_D \nabla v \cdot \nabla \eta dx \geq -\int_D (M_1 + M_2 v)\eta dx \geq -L \int_D \eta dx,$$

for $\eta \in C_c^\infty(D), \eta \geq 0$ with $supp(\eta) \Subset \{v > 0\}$. This gives $\Delta v_+ \geq -L$ in the sense of distribution on $\{v_+ > 0\}$. Apply Lemma 2.11, we have $\Delta v_+ = \Delta(\partial_e u)^+ \geq -L$ in $D$. Since $\partial_e u = -\partial_{-e} u$, then $(\partial_e u)_- = (\partial_{-e} u)_+$. Thus we have the same inequality for $(\partial_e u)_-$. □

**Theorem 2.13.** *Let $u \in L^\infty(D)$ satisfy $OT_1 - OT_2$, then $u \in C^{1,1}_{loc}(D)$ and*

$$\|u\|_{C^{1,1}(K)} \leq CM(1 + \|u\|_{L^\infty}(D) + \|f\|_{L^\infty}(D)),$$

*for $K \Subset D, C = C(n, dist(K, \partial D)), M = max\{1, M_1, M_2\}$.*

*Proof.* By the Calderón-Zygmund estimates, $u \in W^{2,p}_{loc}(D)$ with $p > n$. Thus $u$ is twice differentiable at Lebesque point of $D^2 u$. Therefore $u$ is twice differntiable a.e. By the Sobolev embedding $W^{2,p}_{loc} \hookrightarrow C^{1,\alpha}_{loc}, u \in C^{1,\alpha}_{loc}(D)$. Define

$$v(x) = \partial_e u(x),$$

where

$$e = \begin{cases} arbitrary & \text{if } \nabla u(x_0) = 0 \\ e \perp \nabla u(x_0) & \text{if } \nabla u(x_0) \neq 0. \end{cases}$$

With out loss of generality, we assume $x_0 = 0$. we will show that there is a uniform estimate for $\partial_{x_j e} u(0) = \partial_{x_j} v(0)$, for $1 \leq j \leq n$. We may assume $v(0) = 0$, $v$ is diferentiable at 0, then we have

$$v(x) = \zeta \cdot x + o(|x|), \quad \zeta = \nabla v(0).$$

If $\zeta = 0$, we have $\partial_{x_j} v(0) = 0$ for $1 \leq j \leq n$. Thus we have done.
If $\zeta \neq 0$, let the cone $C = \{x \in \mathbb{R} : \zeta \cdot x \geq |\zeta||x|/2\}$, then for sufficiently small $r > 0$,

$$C \cap B_r \subset \{v > 0\}, \quad -C \cap B_r \subset \{v < 0\}.$$

Let $v_r(x) = v(rx)/r, x \in B_1$ and let $v(x) = \zeta \cdot x + h(x)$ where $\lim_{|x| \to 0}(h(x)/|x|) = 0$, then

$$v_r(x) = \frac{v(rx)}{r} = \zeta \cdot x + \frac{h(rx)}{r} \to \zeta \cdot x \text{ as } r \to 0,$$

i.e. $v_r(x) \to v_0(x) := \zeta \cdot x$ uniformly as $r \to 0$ in $B_1$.

$$\int_{B_1} |\nabla v_r(x) - \zeta|^p dx = \frac{1}{r^n} \int_{B_r} |\nabla v(x) - \nabla v(0)|^p dx \to 0, \text{ as } r \to 0,$$

since $x_0 = 0$ is a Lebesgue point for $\nabla v$. i.e. we have $\|\nabla v_r - \zeta\|_{L^p(B_1)} = \|\nabla v_r - \nabla v_0\|_{L^p(B_1)} \to 0$, as $r \to 0$ with $p > n$. We may assume that $p \geq 2(n-1)$. Then

$$\left\| |\nabla v_r - \nabla v_0|^2 / |x|^{n-2} \right\|_{L^1(B_1)} \leq \|\nabla v_r - \nabla v_0\|^{1/(n-1)}_{L^{2(n-1)}(B_1)} \left\| 1/|x|^{n-1} \right\|^{(n-2)/(n-1)}_{L^1(B_1)} \to 0, \text{ as } r \to 0.$$

Therefore we obtain

$$\lim_{r\to 0}\int_{B_1}\frac{|\nabla v_r|^2}{|x|^{n-2}}dx=\int_{B_1}\frac{|\nabla v_0|^2}{|x|^{n-2}}dx,$$

and the same equality holds for $C\cap B_1$ and $-C\cap B_1$. Thus we have

$$\begin{aligned}C(n)^2|\zeta|^4&=\int_{C\cap B_1}\frac{|\nabla v_0(x)|^2}{|x|^{n-2}}dx\int_{-C\cap B_1}\frac{|\nabla v_0(x)|^2}{|x|^{n-2}}dx\\&=\lim_{r\to 0}\int_{C\cap B_1}\frac{|\nabla v_r(x)|^2}{|x|^{n-2}}dx\int_{-C\cap B_1}\frac{|\nabla v_r(x)|^2}{|x|^{n-2}}dx\\&=\lim_{r\to 0}\frac{1}{r^4}\int_{C\cap B_r}\frac{|\nabla v(x)|^2}{|x|^{n-2}}dx\int_{-C\cap B_r}\frac{|\nabla v(x)|^2}{|x|^{n-2}}dx\\&\leq\underline{\lim}_{r\to 0}\Phi(r,v_+,v_-),\end{aligned}$$

where $C(n)=|C\cap B_1|\cdot|-C\cap B_1|>0$.
Let $\delta=(1/2)dist(K,\partial D)$ and $K_\delta=\{x:dist(x,K)<\delta\}$. By Lemma 2.12 , $\Delta v_\pm\geq -L_\delta$ in $K_\delta$, where $L_\delta=M(1+\|\nabla u\|_{L^\infty(K_\delta)})$ and $M=max\{1,M_1,M_2\}$. Apply T.h 2.9 , we have

$$\begin{aligned}C(n)^2|\zeta|^4\leq\underline{\lim}_{r\to 0}\Phi(r,v_+,v_-)&\leq C(n)\left(L_\delta^2\delta^2+\frac{\|v_+\|^2_{L^2(B_\delta)}+\|v_-\|^2_{L^2(B_\delta)}}{\delta^{n+2}}\right)^2\\&\leq C(n)\left(L_\delta^2\delta^2+\frac{\|\nabla u\|^2_{L^\infty(K_\delta)}}{\delta^{n+2}}\right)^2\leq C(n,\delta)L_\delta^4.\end{aligned}$$

Thus we have $|\zeta|\leq C(n,\delta)L_\delta$.
By the Calderón-Zygmund estimates and the Sobolev embedding $W^{2,p}_{loc}\hookrightarrow C^{1,\alpha}_{loc}$, we have $\|\nabla u\|_{L^\infty(K_\delta)}\leq\|u\|_{C^{1,\alpha}(K_\delta)}\leq\|u\|_{W^{2,p}(K_\delta)}\leq C(n)(\|u\|_{L^\infty(D)}+\|f\|_{L^\infty(D)})$. Hence we have

$$L_\delta=M(1+\|\nabla u\|_{L^\infty(K_\delta)})\leq C(n.\delta)N,$$

where $N=M(1+\|u\|_{L^\infty(D)}+\|f\|_{L^\infty(D)})$. Since $\zeta=\nabla_e u(x_0)$,

$$|\nabla\partial_e u(x_0)|\leq C(n,\delta)N. \tag{5}$$

since

$$e=\begin{cases}arbitrary & \text{if } \nabla u(x_0)=0\\ e\perp\nabla u(x_0) & \text{if } \nabla u(x_0)\neq 0,\end{cases}$$

(5) gives the desered estimate on $|D^2u|$ where $\nabla u(x_0)=0$. If $\nabla u(x_0)\neq 0$ and $e_n$ be a unit vector such that $e\parallel\nabla u(x_0)$, then choose the coordinate system which contains $e_n$. Apply (5) for $e=e_1,...,e_{n-1}$, we have

$$|\partial_{x_ix_j}u(x_0)|\leq C(n,\delta)N, i\in\{1,...,n-1\}, j\in\{1,...,n\}.$$

Since $\Delta u(x_0)=f(x_0,u(x_0))\chi_{G(u(x_0))}=f(x_0,u(x_0))$,

$$\begin{aligned}|\partial_{x_nx_n}u(x_0)|&\leq|\Delta u(x_0)|+|\partial_{x_1x_1}u(x_0)|+...+|\partial_{x_{n-1}x_{n-1}}u(x_0)|\\&\leq\|f\|_{L^\infty(D)}+C(n,\delta)N\leq C(n,\delta)N.\end{aligned}$$

and the proof is complete. □

# 3 Preliminary analysis of the free boundary

## 3.1 Nondegeneracy

**Lemma 3.1.** *Let* $\Delta u = 1$ *in the ball* $B_R$. *Then*

$$\sup_{\partial B_r} u \geq u(0) + \frac{r^2}{2n}, \quad 0 < r < R. \tag{6}$$

*Proof.* Let $w(x) = u(x) - |x|^2/2n, x \in B_R$ then $\Delta w = 0$. By the maximum principle $w(0) \leq \sup_{\partial B_r} w = (\sup_{\partial B_r} u) - r^2/2n$. Thus we have the inequality. □

**Lemma 3.2.** *(Nondegeneracy: Problem A). Let u be a soultion of Problem A in D. If* $B_r(x_0) \Subset D$, *then*

$$\sup_{\partial B_r(x_0)} u \geq u(x_0) + \frac{r^2}{8n}, \text{ for } x_0 \in \overline{\Omega(u)}.$$

*Proof.* Note that

$$\sup_{B_r(x_0)} u = \sup_{\partial B_r(x_0)} u, \tag{7}$$

since $\Delta u \geq 0$ and the maximum principle.

Let $x_0 \in \Omega(u)$ and $u(x_0) > 0$,

$$w(x) = u(x) - u(x_0) - \frac{|x - x_0|^2}{2n}.$$

Then $\Delta w = 0$ in $B_r(x_0) \cap \Omega(u)$. By the maximum principle and $w(x_0) = 0$,

$$\sup_{\partial(B_r(x_0) \cap \Omega)} w \geq 0.$$

Since $u = 0$ on $\partial\Omega(u) \subset \Omega(u)^c$, we know that $w(x) = -u(x_0) - |x - x_0|^2/2n < 0$ on $\partial\Omega(u)$. Thus we have

$$\sup_{\partial B_r(x_0) \cap \Omega(u)} w \geq 0, \text{ that means } \sup_{\partial B_r(x_0) \cap \Omega(u)} u \geq u(x_0) + \frac{r^2}{2n} > 0.$$

Therefore $\sup_{\partial B_r(x_0)} u = \sup_{\partial B_r(x_0) \cap \Omega(u)} u \geq u(x_0) + r^2/2n$. we have the desired inequality in this case.

Let $x_0 \in \Omega(u)$ and $u(x_0) \leq 0$. If there exists $x_1 \in B_{r/2}(x_0)$ such that $u(x_1) > 0$, then

$$\sup_{B_r(x_0)} u \geq \sup_{B_r/2(x_1)} u \geq u(x_1) + \frac{(r/2)^2}{2n} \geq u(x_0) + \frac{r^2}{8n},$$

by the above case, and we have the inequality. Let $u \leq 0$ in $B_{r/2}(x_0)$. By the strong maximum principle for subharmonic function $u, u \equiv 0$ in $B_{r/2}(x_0)$ or $u < 0$ in $B_{r/2}(x_0)$. Since $x_0 \in \Omega(u)$, $u < 0$ in $B_{r/2}(x_0)$. Then $B_{r/2}(x_0) \subset \Omega(u)$. This implies $\Delta u = 1$ in $B_{r/2}(x_0)$. By Lemma 3.1 ,

$$\sup_{B_r(x_0)} u \geq \sup_{B_r/2(x_0)} u \geq u(x_0) + \frac{r^2}{8n}.$$

Let $x_0 \in \overline{\Omega(u)}, \{x_i\} \subset \Omega(u)$ such that $x_i \to x_0$ as $i \to \infty$. Passing to the limit in the inequality for $x_i$ gives the desired inequality. □

**Lemma 3.3.** *(Nondegeneracy: Problem B). Let u be a solution of Problem B in D. If $B_r(x_0) \Subset D$, then*

$$\sup_{\partial B_r(x_0)} u \geq u(x_0) + \frac{r^2}{2n}, \textit{ for } x_0 \in \overline{\Omega(u)}.$$

*Proof.* It is enough to show the inequality for $x_0 \in \Omega(u) = \{|\nabla u| > 0\}$, by the continuity of $u$. Let $w(x) = u(x) - u(x_0) - |x - x_0|^2/2n$. We will show that

$$\sup_{B_r(x_0)} w = \sup_{\partial B_r(x_0)} w.$$

Suppose there exists $y \in B_r(x_0)$ such that $y = \sup_{B_r(x_0)} w$, then $|\nabla w(y)| = 0$. It is equivalent to $|\nabla u(y)| = |y - x_0|/n$. Since $|\nabla u(x_0)| > 0$, $y \neq x_0$. Thus $|\nabla u(y)| > 0$, therefore $y \in \Omega(u)$. Since $\Delta w = 0$ in $\Omega(u)$, the strong maximum principle for $w$ implies $w$ is constant in some neighborhood of $y$. Hence the set of maxima is relatively open and closed in $B_r(x_0)$. Thus $w$ is constant in $B_r(x_0)$. Therefore we have

$$\sup_{B_r(x_0)} w = \sup_{\partial B_r(x_0)} w.$$

and this implies

$$0 = w(x_0) \leq \sup_{\partial B_r(x_0)} w = \sup_{\partial B_r(x_0)} u - \frac{r^2}{2n} - u(x_0).$$

□

**Lemma 3.4.** *(Nondegeneracy: Problem C). Let u is a solution of Problem C in D. If $B_r(x_0) \Subset D$, then*

$$\sup_{\partial B_r(x_0)} u \geq u(x_0) + \lambda_+ \frac{r^2}{2n}, \textit{ for } x_0 \in \overline{\Omega_+(u)},$$

$$\inf_{\partial B_r(x_0)} u \leq u(x_0) - \lambda_- \frac{r^2}{2n}, \textit{ for } x_0 \in \overline{\Omega_-(u)}.$$

*Proof.* The inequalities are obtained using

$$w(x) = u(x) - u(x_0) \mp \lambda_\pm \frac{|x - x_0|^2}{2n}$$

and the similar argument in first part of Lemma 3. . we will prove the infimum case, only. Let $x_0 \in \Omega_-(u)$, i.e. $u(x_0) < 0$. Let

$$w(x) = u(x) - u(x_0) + \lambda_- \frac{|x - x_0|^2}{2n}.$$

Then $\Delta w = 0$ in $B_r(x_0) \cap \Omega_-(u)$. By the maximum principle and $w(x_0) = 0$,

$$\inf_{\partial(B_r(x_0) \cap \Omega)} w \leq 0.$$

We know that $w(x) = -u(x_0) + \lambda_- |x - x_0|^2/2n > 0$ on $\partial\Omega_-(u)$. Thus we have

$$\inf_{\partial B_r(x_0) \cap \Omega_-(u)} w \leq 0, \text{ that means } \inf_{\partial B_r(x_0) \cap \Omega_-(u)} u \leq u(x_0) - \lambda_- \frac{r^2}{2n} < 0.$$

Since $u \geq 0$ in $\Omega_-(u)$, $\inf_{\partial B_r(x_0) \cap \Omega_-(u)} u = \inf_{\partial B_r(x_0)} u$, we have the inequality. □

**Corollary 3.5.** *Under the conditions of either Lemmas 3.2 , 3.3 , or 3.4. the following inequality holds:*

$$\sup_{B_r(x_0)} |\nabla u| \geq Cr,$$

*for $C > 0, C = C(n)$ in Problems $A, B$ and $C = C(n, \lambda_\pm)$ in Problem $C$.*

*Proof.* □

## 3.2 Lebesgue and Hausdoff measures of the free boundary

**Definition 3.1.** A measurable set $E \subset \mathbb{R}^n$ is porous with porosity constant $0 < \delta < 1$ if every ball $B = B_r(x)$ contains a smaller ball $B' = B_{\delta r(y)}$ such that

$$B_{\delta r(y)} \subset B_r(x) \setminus E.$$

$E$ is locally porous in an open set $D$ if $E \cap K$ is porous ( with possibly different porosity constants) for $K \Subset D$.

**Proposition 3.6.** *If $E \subset \mathbb{R}^n$ is porous then $|E| = 0$. If $E$ is locally porous in D, then $|E \cap D| = 0$.*

*Proof.* Let $E$ be a porous subset in $\mathbb{R}$. We know that

$$\chi_E(x) = \lim_{r\to 0} \frac{\int_{B_r(x)} \chi_E(y)dy}{|B_r(x)|} = \lim_{r\to 0} \frac{|B_r(x) \cap E|}{|B_r(x)|} \text{ a.e. in } \mathbb{R}.$$

That means the metric densty, $\lim_{r\to 0} |B_r(x) \cap E|/|B_r(x)| = 1$ a.e. on $E$. On the other hand, for $x_0 \in E, |B_r(x_0)| = |B_r(x_0) \cap E| + |B_r(x_0) \cap E^c|, r^n \geq |B_r(x_0) \cap E| + \delta^n r^n$. Thus

$$\overline{\lim_{r\to 0}} \frac{|E \cap B_r(x_0)|}{|B_r|} \leq 1 - \delta^n < 1.$$

Hece $|E| = 0$.

Let $E$ be a locally porous subset in $D$, Then we have $|E \cap K| = 0$, for any $K \Subset D$. Since $E$ is a coutable union of compact subset of $E$, $|E \cap D| = 0$. □

**Lemma 3.7.** *Let $E$ be a bounded measurable set in $\mathbb{R}$. If for every ball $B = B_r(x)$ centered at $x \in E$ there exists a ball $B' = B_{\delta r}(y)$ such that $B' \subset B \setminus E$, then $E$ is $C(n)\delta$ porous.*

**Lemma 3.8.** *Let $u$ be a solution of Problem A or B in an open set $D \subset \mathbb{R}^n$. Then $\Gamma(u)$ is locally porous in D. Let $u$ be a solution of Problem C, then $\Gamma^0(u) = \Gamma(u) \cap \{|\nabla u| = 0\}$ is locally porous.*

*Proof.* Case 1) Problem $A, B$.
Let $K \Subset D, x_0 \in \Gamma(u)$ and $B_r(x_0) \subset K$, then by Corollary 3.5, there exists $y \in \overline{B_{r/2}(x_0)}$ such that $|\nabla u(y)| \geq (C/2)r$. Thus we have

$$\inf_{B_\delta r(y)} |\nabla u| \geq \left(\frac{C}{2} - M\delta\right) r \geq \frac{C}{4} r,$$

where $\delta = C/4M, M = ||D^2u||_{L^\infty(K)} < \infty$. Thus

$$B_{\hat{\delta}r}(y) \subset B_r(x_0) \cap \Omega(u) \subset B_r \setminus \Gamma,$$

where $\hat{\delta} = min\{\delta, 1/2\}$. By Lemma 3.7 , $\Gamma(u)$ is locally porous.

Case2) Problem $C$.

Note that $\Omega(u) = \Omega_\pm(u)$. Let $K \Subset D, x_0 \in \Gamma(u)^0$ and $B_r(x_0) \subset K$. Let $y \in \overline{B_{r/2}(x_0)}$ such that $\inf_{B_{\delta r}(y)} |\nabla u| \geq \dfrac{C}{4}r$, as in case 1. If we show that $B_{\hat{\delta}r}(y) \subset \Omega(u) \cup \Gamma^*(u)$, where $\hat{\delta} = min\{\delta, 1/2\}$, then

$$B_{\hat{\delta}r}(y) \subset B_r(x_0) \cap [\Omega(u) \cup \Gamma^*(u)] \subset B_r(x_0) \setminus \Gamma^0(u),$$

and we have the local porosity of $\Gamma^0$. Suppose $B_{\hat{\delta}r}(y) \not\subseteq \Omega(u) \cup \Gamma^*(u)$, then there exists $z \in B_{\hat{\delta}r}(y)$ such that $z \in [\Omega(u) \cup \Gamma^*(u)]^c$. Since $z \in B_{\hat{\delta}r}(y)$, $z \in (\Gamma^0(u))^c$. Thus $z \in [\Omega(u) \cup \Gamma(u)]^c = [\overline{\Omega(u)} \cap D]^c$. We may assume that $z \in D$. Thus $z \in \overline{\Omega(u)}^c \cap D$. Since $\overline{\Omega(u)}^c \cap D$ is open subset of $\{u = 0\}$, we have $\nabla u(z) = 0$. It is a contradiction. □

**Corollary 3.9.** *Let u be a solution of Problem $A, B$ or $C$ in $D$. Then $\Gamma(u)$ has a Lebesgue measure zero.*

*Proof.* In case of Problems $A, B$, It is a consequnce of Proposition 3.6 and Lemma 3.8. In case of Problem $C$, we know $|\Gamma^0| = 0$. Since $\Gamma^*(u)$ is locally a $C^{1,\alpha}$ surface, $|\Gamma^*| = 0$. □

**Lemma 3.10.** *Let u be a solution of Problem $A, B$, or $C$ in $D$ and $x_0 \in \Gamma(u)$. Then*

$$\frac{|B_r(x_0) \cap \Omega(u)|}{|B_r|} \geq \beta$$

*if $B_r(x_0) \subset D$, where $\beta = \beta(||D^2u||_{L^\infty}, n)$ in case of $A, B$ and $\beta = \beta(||D^2u||_{L^\infty}, n, \lambda_\pm)$ in case of $C$.*

*Proof.* In case of $A, B$, by the porosity and the argument in proof of Lemma 3.6 , we have

$$\frac{|B_r(x_0) \cap \Omega(u)|}{|B_r|} \geq \frac{(r\hat{\delta})^n}{r^n} = \hat{\delta}^n,$$

and $\hat{\delta}$ depends only on $||D^2u||_{L^\infty}$ and $n$. In case of $C$,

$$\frac{|B_r(x_0) \cap \Omega(u)|}{|B_r|} \geq \frac{|B_r(x_0) \cap [\Omega(u) \cup \Gamma^*(u)]|}{|B_r|} \geq \hat{\delta}^n,$$

since $|\Gamma^*(u)| = 0$. and this case $\hat{\delta} = \hat{\delta}(||D^2u||_{L^\infty}, n, \lambda_\pm)$. □

**Lemma 3.11.** *If u is a $C^{1,1}$ solution of Problem $A, B$ or $C$ in a bounded open set $D \subset \mathbb{R}^n$, then $\Gamma(u)$ is a set of finite (n-1)-dimensional Hausdorff measure locally in D.*

*Proof.* Let

$$v_i = \partial_{x_i}u, \quad i \in \{1, ..., n\}, \quad E_\epsilon = \{|\nabla u| < \epsilon\} \cap \Omega(u).$$

Since

$$(\Delta u)^2 = (\sum_{i=1}^n u_{ii})^2 \leq C(n) \sum_{i=1}^n u_{ii}^2 \leq C(n) \sum_{i,j=1}^n u_{i,j}^2 = C(n) \sum_{i=1}^n |\nabla v_i|^2,$$

we have

$$C_0 \le (\Delta u)^2 \le C(n) \sum_{i=1}^{n} |\nabla v_i|^2 \text{ in } \Omega,$$

where $C_0 = 1$ in the case of Problems $A, B$ and $C_0 = min\{\lambda_+^2, \lambda_-^2\}$ in the case of Problem $C$. Let $K \Subset D$, then

$$C_0|K \cap E_\epsilon| \le C(n) \int_{K \cap E_\epsilon} \sum_{i=1}^{n} |\nabla v_i|^2 dx \le C(n) \sum_{i=1}^{n} \int_{K \cap \{|v_i| < \epsilon\} \cap \Omega(u)} |\nabla v_i|^2 dx. \tag{8}$$

In Lemma 2.12, we can take $M_1 = M_2 = 0$ for solutions of Problems $A, B$ and $C$. Hence we have

$$\int_D \nabla v_{i\pm} \cdot \nabla \eta dx \le 0, \text{ for } i \in \{1, ..., n\},$$

for $\eta \in W_0^{1,2}(D), \eta \ge 0$. since Let $\phi \in C_c^\infty(D), \phi = 1$ on $K$ and

$$\psi_\epsilon(t) := \begin{cases} 0, & t \le 0 \\ \epsilon^{-1} t, & 0 \le t \le \epsilon \\ 1 & t \ge 0, \end{cases}$$

then $\eta := \psi_\epsilon(v_{i\pm})\phi$ is in $W_0^{1,2}(D)$. Thus we have

$$\begin{aligned} \int_D \nabla v_{i\pm} \cdot \nabla(\psi_\epsilon(v_{i\pm})\phi) dx &= \int_{\{0 < v_{i\pm} < \epsilon\}} \epsilon^{-1} \phi |\nabla v_{i\pm}|^2 dx + \int_D \psi_\epsilon(v_{i\pm}) \nabla v_{i\pm} \cdot \nabla \phi dx \\ &\le 0. \end{aligned}$$

Therefore

$$\epsilon^{-1} \int_{K \cap \{0 < v_{i\pm} < \epsilon\} \cap \Omega(u)} |\nabla v_{i\pm}|^2 dx \le - \int_D \psi_\epsilon(v_{i\pm}) \nabla v_{i\pm} \cdot \nabla \phi dx \le \int_D |\nabla v_{i\pm}| |\nabla \phi| dx.$$

Hence we have

$$\epsilon^{-1} \int_{K \cap \{0 < |v_i| < \epsilon\} \cap \Omega(u)} |\nabla v_i|^2 dx \le \int_D |\nabla v_i| |\nabla \phi| dx \le C(n) M \int_D |\nabla \phi| dx, \tag{9}$$

where $M = ||D^2 u||_{L^\infty(D)}$. Combining (8), (9) gives

$$C_0 |K \cap E_\epsilon| \le C \epsilon M, \tag{10}$$

where $C = C(n, K, D)$, since $\phi$ depends on $K, D$.

By the Besicovich covering lemma, $\Gamma \cap K$ has a covering $\{B^i\}_{i \in I}$ which is finite family of closed balls of radius $\epsilon$ centered on $\Gamma \cap K$ the number of overlaped balls no more than $N(n)$, and it does not depend on $\epsilon$. Take $\epsilon$ by $B^i \subset K'$ where $K'$ is a compact set such that $K \Subset Int(K') \Subset D$.

Case 1) Problems $A$ and $B$.

We know that $|\nabla u| < M\epsilon$ in each $B^i$ and it implies that $B^i \cap \Omega \subset E_{M\epsilon}$. By Lemma 3.10 ,and (10), we have

$$\sum_{i\in I} |B^i| \le \frac{1}{\beta}\sum_{i\in I} |B^i \cap \Omega| \le \frac{1}{\beta}\sum_{i\in I} |B^i \cap E_{M\epsilon}| \le \frac{N}{\beta}|K' \cap E_{M\epsilon}| \le \frac{CNM^2\epsilon}{C_0\beta}.$$

Therefore we obtain

$$\sum_{i\in I} diam(B^i)^{n-1} \le C(n, M, K', D)$$

and letting $\epsilon \to 0$ gives

$$H^{n-1}(\Gamma(u) \cap K) \le C(n, M, K', D).$$

Case 2) Ploblem $C$.

The estimation for $\Gamma^0(u)$ is obtained by the same proof as above. Thus it suffice to obtain the estimation for $H^{n-1}(\Gamma^*(u))$.

Let $v = \partial_e u, \eta \in W_0^{1,2}(D)$, and $\eta = 0$ a.e. on $\Gamma^0(u)$, then

$$\begin{aligned}
\int_D \nabla v \cdot \nabla \eta &= \int_D \Delta u \partial_e \eta = \int_D (\lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}})\partial_e \eta = \lambda_+ \int_{\{u>0\}} \partial_e \eta - \lambda_- \int_{\{u<0\}} \partial_e \eta \\
&= \lambda_+ \int_{\partial\{u>0\}\cap\Gamma^*(u)} (e \cdot (-\omega))\eta dH^{n-1} - \lambda_- \int_{\partial\{u<0\}\cap\Gamma^*(u)} (e \cdot \omega)\eta dH^{n-1} \\
&= (-\lambda_+ - \lambda_-) \int_{\Gamma^*(u)} (e \cdot \omega)\eta dH^{n-1}, \qquad (11)
\end{aligned}$$

where $\omega = (\nabla u(x))/(|\nabla u(x)|)$. Take $\eta = \psi_\epsilon(v)\phi$ where $\psi_\epsilon(v), \phi$ are defined above, then $\eta \in W_0^{1,2}(D)$ and $\eta = 0$ on $\Gamma^0(u)$. Since $\psi_\epsilon \neq 0$ implies $v = \partial_e u > 0, e \cdot \omega > 0$ and by using (11), we have

$$\begin{aligned}
&\epsilon^{-1} \int_{K\cap\{0<|v|<\epsilon\}} |\nabla v|^2 dx + (\lambda_+ + \lambda_-) \int_{\Gamma^*(u)\cap K} (e \cdot \nu)\psi_\epsilon(v) dH^{n-1} \\
&\le \epsilon^{-1} \int_{\{0<|v|<\epsilon\}} \phi |\nabla v|^2 dx + (\lambda_+ + \lambda_-) \int_{\Gamma^*(u)} (e \cdot \nu)\psi_\epsilon(v)\phi dH^{n-1} \\
&\le \int_{\{0<|v|<\epsilon\}} \epsilon^{-1}\phi |\nabla v|^2 dx - \int_D \nabla v \cdot \nabla(\psi_\epsilon(v)\phi) dx \\
&= \int_D \psi_\epsilon(v) \nabla v \cdot \nabla \phi dx \le C(n)M \int_D |\nabla \phi| dx.
\end{aligned}$$

Therefore we have

$$(\lambda_+ + \lambda_-) \int_{\Gamma^*(u)\cap K} (e \cdot \nu)\psi_\epsilon(v) dH^{n-1} \le CM.$$

Letting $\epsilon \to 0$ gives

$$(\lambda_+ + \lambda_-) \int_{\Gamma^*(u)\cap K} (e \cdot \nu)_+ dH^{n-1} \le CM,$$

for any normal vector $e$. For fixed $x \in \Gamma^*(u) \cap K$, there exists $e \in \{\pm e_1, ..., \pm e_n\}$ such that $e \cdot \nu \ge 1/\sqrt{n}$. This gives

$$H^{n-1}(\Gamma^*(u) \cap K) \le \frac{CM}{\lambda_+ + \lambda_-}.$$

□

## 3.3 Classes of solutions, rescalings, and blowups

**Definition 3.2.** (Local solutions). Let $P_R(x_0, M)$ be the class of $C^{1,1}$ solutions $u$ of Problems $A, B$, or $C$ in $B_R(x^0)$ such that

$$\|D^2u\|_{L^\infty(B_R(x_0))} \leq M,$$

where $x_0 \in \Gamma(u)$ in Problems $A, B$ and $x_0 \in \Gamma^0(u)$ in Problem $C$ for given $R, M > 0$.

**Definition 3.3.** (Global solutions). Let $P_\infty(x_0, M)$ be the class of $C^{1,1}$ solutions $u$ of Problems $A, B$, or $C$ in $\mathbb{R}^n$ such that

$$\|D^2u\|_{L^\infty(\mathbb{R}^n)} \leq M,$$

where $x_0 \in \Gamma(u)$ in Problems $A, B$ and $x_0 \in \Gamma^0(u)$ in Problem $C$ for given $M > 0$.

We Denote $P_R(M), P_\infty(M)$ by $P_R(0, M), P_\infty(0, M)$, respectively.
Let $u \in P_R(x_0, M)$ and $\lambda > 0$ and the rescaling of $u$ at $x_0$

$$u_\lambda(x) = u_{x_0,\lambda}(x) := \frac{u(x_0 + \lambda x) - u(x_0)}{\lambda^2}, x \in B_{R/\lambda},$$

then by simple computatuion we know that $u_\lambda \in P_{R/\lambda}(M)$.
For $u \in P_R(M)$ for any $\lambda > 0$ the rescaling $u_\lambda$ satisfy $|D^2u_\lambda(x)| \leq M$ in $B_{R/\lambda}$. Hence we obtain

$$|\nabla u_\lambda(x)| \leq M|x|, \quad |u_\lambda(x)| \leq \frac{1}{2}M|x|^x, \quad \text{for } x \in B_{R/\lambda}.$$

Therefore there exists a sequence $\lambda = \lambda_j \to 0$ such that

$$u_\lambda \to u_0 \text{ in } C^{1,\alpha}_{loc}(\mathbb{R}^n) \text{ for any } 0 < \alpha < 1,$$

where $u_0 \in C^{1,1}_{loc}(\mathbb{R}^n)$.

**Proposition 3.12.** *(Limit of solutions). Let $\{u_j\}_{j=1}^\infty$ be a sequence of solutions of Problems $A, B$ or $C$ in an open set $D$, such that*

$$u_j \to u_0 \textit{ in } C^{1,\alpha}_{loc}(D),$$

*for some $0 < \alpha < 1$. Then we have the followings:*

*(a) For $x_0 \in D$, we have the implications*

$$u_0(x_0) > 0 \Rightarrow u_j > 0 \quad u_0(x_0) > 0 \Rightarrow u_j > 0 \quad |\nabla u_0(x_0)| > 0 \Rightarrow |\nabla u_j| > 0$$

*on $B_\delta(x_0)$, $j \geq j_0$, for some $\delta > 0$ and suficiently large $j_0$.*

*(b) For $B_\delta(x_0) \subset D$, we have*

$$|\nabla u_0| = 0 \textit{ on } B_\delta(x_0) \Rightarrow |\nabla u_j| = 0 \textit{ on } B_{\delta/2}(x_0),$$

*$j \geq j_0$, for sufficiently large $j_0$.*

*(c) $u_0$ is a solution of the same Problem $A, B$ or $C$, as $u_j$, $j = 1, 2, ....$*

*(d) For some* $j_k \to \infty$, *and* $x_{j_k} \to x_0 \in D, x_{j_k} \in \Gamma(u_{j_k})$ *implies* $x_0 \in \Gamma(u_0)$.

*(e)* $u_j \to u_0$ *in* $W^{2,p}_{loc}(D)$ *for any* $1 < p < \infty$.

*Proof.* (a) $u_j \to u_0$ in $C^{1,\alpha}_{loc}(D)$ implies the implications.

(b) Suppose it is not, then there exists $j_k \to \infty, y_k \in B_{\delta/2}(x_0)$ such that $|\nabla u_{j_k}(y_k)| > 0$ and $|\nabla u_0| = 0$ in $B_\delta(x_0)$. By Corollary 3. , at $y_k$ and $B_{\delta/4}(y_k) \subset B_{(3\delta/4)}(x_0)$, we have

$$\sup_{B_{3\delta/4}(x_0)} |\nabla u_{j_k}| \geq C\delta.$$

By the $C^{1,\alpha}$ convergence, passing to the limit gives

$$\sup_{B_{3\delta/4}(x_0)} |\nabla u_0| \geq C\delta.$$

This is a contradiction to the fact that $|\nabla u_0| = 0$ on $B_\delta(x_0)$.

(c) With out loss of generality, we may assume that $\{u_j\}$ is uniformly bounded in $W^{2,p}(K)$, $1 < p \leq \infty$ for any $K \Subset D$ and hence $u_0 \in W^{2,p}_{loc}(D)$. therefore it is enought to show that the equation for $u_0$ is satisfied a.e. in $D$.

Case 1) Problems $A, B$.
Since $\nabla u_0 = 0$ on $\Omega^c(u_0), \Delta u_0 = 0$ a.e. on $\Omega^c(u_0)$. Let $x_0 \in \Omega(u_0)$, then by (a) we have that $B_\delta(x_0) \subset \Omega(u_j)$ for some $\delta > 0$ and $j \geq j_0$. Therefore

$$\Delta u_j = 1 \text{ in } B_\delta(x_0), \quad j \geq j_0,$$

and this implies

$$\Delta u_0 = 1 \text{ in } \Omega(u_0).$$

Thus we obtain

$$\nabla u_0 = \chi_{\Omega(u_0)} \quad \text{a.e. in } D.$$

Case 2) Problem $C$.
Since $|\Gamma^*(u_0)| = 0$, the same argument give the desired equation.

(d) Let $x_{j_k} \in \Gamma(u_{j_k}) \subset \Omega^c(u_{j_k})$, then we obtain $x_0 \in \Omega^c(u_0)$, by $(a)$. Therefore If we assume $x_0 \notin \Gamma(u_0)$, then there exists a ball $B_\delta(x_0) \subset \Omega^c(u_0)$. For any Problems $A, B$, or $C$, in this ball, $|\nabla u_0| = 0$. Thus we have $B_{\delta/2}(x_{j_k}) \subset \Omega^c(u_{j_k})$, by (b), and this is a contradiction to the fact that $x_{j_k} \in \Gamma(u_{j_k})$.

(e) Since $D^2 u_j$ are uniformly bounded on any $K \Subset D$, It suffice to show that

$$D^2 u_j \to D^2 u_0 \text{ a.e. in } D,$$

to prove (e). Fix a point $x_0$ in $\Omega(u_0)$. For some $\delta > 0$, $j_0$, we have

$$\Delta u_j = 1 \text{ in } B_\delta(x_0), \quad j \geq j_0, \quad \Delta u_0 = 1 \text{ in } B_\delta(x_0).$$

Thus these functions are in $C^\infty(B_\delta(x_0))$ and we have the pointwise convergence for $x_0$. Let $x_0 \in Int(\Omega(u_0)^c)$. By (b), we know that there exists $\delta > 0$, $j_0$ such that

$$|\nabla u_0| = 0 \text{ on } B_\delta(x_0) \text{ on}, \quad |\nabla u_j| = 0 \text{ on } B_{\delta/2}(x_0), \quad j \geq j_0$$

It also give the regularity and we have the convergence of second derivative at $x_0$. Since the free boundary has a Lebesque measure zero, we have a.e. second dervative convergence. □

# 4 Obstacle problem for nonlinear second-order parabolic operator

## 4.1 Viscosity solution of parabolic equations

We deal with the space $\mathbb{R}^{n+1}$, denote the points in $\mathbb{R}^{n+1}$ by $(x,t)$ where $x = (x_1, x_2, ..., x_n)$ is the n-dimentional space variable and $t$ is the time variable.
The *parabolic distance* from $P_1 = (x_1, t_1)$ to $P_2 = (x_2, t_2)$ is defined by

$$d(P_1, P_2) = \begin{cases} (|x_1 - x_2|^2 + |t_1 - t_2|)^{1/2} & t_1 \leq t_2, \\ \infty & t_1 > t_2. \end{cases}$$

For a point $(x_0, t_0) \in \mathbb{R}^{n+1}$, the *$\epsilon$-neighborhood* of $(x_0, t_0)$ is the set

$$\{(x,t) : d((x,t),(x_0,t_0)) < \epsilon\}.$$

This $\epsilon$-neighborhoods give a topology in $\mathbb{R}^n$ and we call it the *parabolic topology*.
Let $\Omega$ be a domain in $\mathbb{R}^{n+1}$, i.e. a open set in the parabolic topology. The boundary of a domain $\Omega$ under the parabolic topology is called the *parabolic boundary* and denoted by $\partial_p\Omega$. Let $Q_r := \{|x| < r\} \times (-r^2, 0]$, $Q_r(x,t) := Q_r + (x,t)$. these are typical open set in the parabolic topology.

**Definition 4.1.** $F(M, P, v, x, t)$ is uniformly elliptic if there are $\lambda, \Lambda > 0$ such that

$$\lambda|N| \leq F(M + N, P, v, x, t) - F(M, P, v, x, t) \leq \Lambda|N|$$

holds for arbitrary postive definite matrix $N$.

**Lemma 4.1.** *the following are equivalent:*

1. *$F$ is uniformly elliptic.*
2. *$F(M + N) \leq F(M) + \Lambda|N^+| - \lambda|N^-|$, for any $M, N$.*

**Definition 4.2.** a function $u$ has interior minimum in a neighborhood $\Omega$, if we have

$$\min_{\Omega} u < \min_{\partial_p\Omega} u.$$

**Definition 4.3.** We say $u$ is a supersolution of $u_t - F(D^2u(x), Du(x), u(x), x, t) = 0$ if

$$u_t - F(D^2\psi, D\psi, \psi, x_0, t) \geq 0$$

whenever $\psi$ is $C^2$ and $u \leq \psi$ for some neighborhood of $(x_0, t_0)$, and $u(x_0, t_0) = \psi(x_0, t_0)$.

The notions of subsolutions and solutions are then obvious.

## 4.2 The existence and the continuity theory

**Definition 4.4.** Let $u \in LSC(\overline{\Omega} \times [0.T))$ be a *supersolution* of the following obstacle ploblem on $\overline{\Omega} \times [0.T)$ if

$$\begin{cases} (E) \quad u_t - F(D^2u, x) \geq 0, & \text{in } \Omega \times (0, T) = Q_T, \\ (O) \quad u(x,t) \geq \phi(x,t) & \text{in } \Omega \times (0, T), \\ (BC) \quad u(x,t) \geq 0 & \text{for } x \in \partial\Omega \text{ and } 0 \leq t \leq T, \\ (IC) \quad u(x, 0) \geq g(x) & \text{for } x \in \overline{\Omega}, \end{cases} \tag{12}$$

where $\Omega \subset \mathbb{R}^n$ is open and $T > 0$, $g \in C(\overline{\Omega})$ and $\phi \in C^2(Q_T)$ are given and $F(M, x)$ in $(E)$ is a uniformly ellipic operator and $F(0, x) = 0$.

The notions of subsolutions and solutions are then obvious. Let $\Omega(u) = \{(x,t) \mid u(x,t) > \phi(x,t)\}$, $\Lambda(u) = \{(x,t) \mid u(x,t) = \phi(x,t)\}$, $\Gamma(u) = \partial\overline{\Omega(u)} \cap \partial\Lambda(u) \cap Q_T$, $\Omega_t(u) = \{x \mid (x,t) \in \Omega(u)\}$, $\Lambda_t(u) = \{x \mid (x,t) \in \Lambda(u)\}$, and $\Gamma_t(u) = \{x \mid (x,t) \in \Gamma(u)\}$.

**Theorem 4.2.** *There exists a lower semicontinuous viscosity supersoltion $u$ which satiesfies* (12) *and $u$ satisfies $u_t - F(D^2u, x) = 0$ in $\Omega(u)$.*

**Theorem 4.3.** *(Weak Harnack Inequality) Let $u$ be a non-negative and $u_t - F(D^2u, x) \geq 0$ in $Q_{2r}$. Then*

$$\left( \fint_{Q^-} u^p \right)^{1/p} \leq C \left( \inf_{Q^+} u \right),$$

*where $Q^+ = Q_r$ and $Q^- = Q_r + (0, -2r^2)$.*

**Theorem 4.4.** *(Harnack Inequality) Let $u$ be a non-negative and $u_t - F(D^2u, x) = 0$ in $Q_{2r}$. Then*

$$\sup_{Q^-} u \leq C \left( \inf_{Q^+} u \right),$$

*where $Q^+ = Q_r$ and $Q^- = Q_r + (0, -2r^2)$.*

**Definition 4.5.** We say $u$ satisfy *the subquadratic free boundary condition* at $(x_0, t_0) \in \Gamma(u)$ if for given $M > 0$,

$$\Gamma_t(u) \cap \{x \mid M(x_0 - x)^2 < t_0 - t\} \neq \emptyset, \text{where } t < t_0 \text{ and}$$

$$\Gamma_t(u) \cap \{x \mid (5/4)M(x_0 - x)^2 < t - t_0\} \neq \emptyset, \text{where } t > t_0.$$

The constant 5/4 is just a technical number to prove the following theorem.

**Lemma 4.5.** *Let $u$ be as in Theorem 4.2. $Q_r(y, s) \subset Q_T$. If the condition satisfied by $u$ in $Q_r(y, s)$ uniformly with constant $M > 0$, $u$ is continuous on $Q_{r/2}(y, s)$.*

*Proof.* The only possible problem is on $\Gamma(u) \cap Q_{r/2}(y, s)$. Assume $u$ is discontinuous at some point $(x_0, t_0)$ on $\Gamma(u) \cap Q_{r/2}(y, s)$. There exists a sequence $(x_k, t_k)$ in $\Omega(u)$ converging to $(x_0, t_0)$ such that $u(x_k, t_k)$ converges to $\mu$ (possibly $\infty$) with $\mu > \liminf_{x \to x_0, t \to t_0^-} u + \delta \geq u(x_0, t_0) + \delta$, for sufficently small $\delta > 0$. Without loss of generality, we may assume $\liminf_{x \to x_0, t \to t_0^-} u \geq u(x_0, t_0) = 0$.

1. $M = 16/5$ and $(27/40)(x_0 - x_k)^2 \geq t_0 - t_k$. and $Q_{2r_k}(x_k, t'_k) \subset \Omega(u)$ where $r_k = |x_0 - x_k|/4$, $t'_k = t_k + 2r_k^2$.
Let $\hat{x}_k = x_0 + (3/2)r(x_k - x_0)/|x_k - x_0|$ and $\hat{t} = t_0 - (4/5)r_k^2$. Since $u$ is upersemicontinuous and $u(x_0, t_0) = 0$, for any $\delta > 0$, there is a neighborhood of $(x_0, t_0)$, with $u(x, t) \geq -\delta$. The neighborhood is as large as it contains $Q_{2r_k}(x_k, t'_k)$ and $Q_{4r_k}(\hat{x}_k, \hat{t}_k)$ for large $k$. For $(x, t)$ in our neighborhood, $u(x, t) + \delta \geq 0$ and $u(x_k, t_k) + \delta \geq \mu > 0$ for large $k$. By the Harnack inequality, $u(x, t) + \delta \geq C\mu$ in $Q_{r_k}(x_k, t'_k)$.
Choose small $\delta > 0$ such that $u(x, t) \geq C\mu - \delta \geq (C/2)\mu$ in $Q_{r_k}(x_k, t'_k)$. Let $(y_k, s_k) \in \Gamma(u) \cap Q_{2r_k}(\hat{x}_k, \hat{t}_k)$. Now by the weak Harnack inequality,

$$
\begin{aligned}
u(y_k, s_k) + \delta &\geq C \left[ \fint_{Q_{2r_k}(\hat{x}_k, t'_k)} (u + \delta)^p \right]^{1/p} \\
&\geq C \left[ \fint_{Q_{2r_k}(\hat{x}_k, t'_k) \cap Q_r(x_k, t'_k)} (u + \delta)^p \right]^{1/p} \\
&\geq C\mu + C\delta.
\end{aligned}
$$

Since $\delta$ is arbitrary, $u(y_k, s_k) \geq C\mu > 0$. Since $(y_k, s_k)$ converge to $(x_0, t_0)$, we have a contradiction.

2.$M = 16/5$ and $(27/40)(x_0 - x_k)^2 \geq t_0 - t_k$.
Choose $r_k$ as large as possible such that $Q_{2r_k}(x_k, t'_k)$ is in $\Omega(u)$. We may assume that $r_k < |x_0 - x_k|/4$, since the other case is 1. The same argume in 1 and the subquadratic free boundary condition for future time implies a contradiction.

3.$M = 16/5$ and $N(x_0 - x_k)^2 \geq t_0 - t_k$ for $N < (27/40)$ and 4.$M$ is arbitrary and $N(x_0 - x_k)^2 \geq t_0 - t_k$ for some $N > 0$.
Some general version of weak Harnack and Harnack inequarity for another open set may operate the machiney.

4.$M$ is arbitrary.

□

**Corollary 4.6.** *(a generalization of Evans theorem) Let u be as in Theorem 4.2. If for any* $Q_r(y, s) \subset Q_T$, *u satisfies the subquadratic free boundary condition with unform constant* $M = M(Q_r(y, s)) > 0$, *then u is continuous in* $(0, T) \times \Omega$.

# References


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# 국문초록

이 논문은 [1]의 내용을 요약하고 비선형 2차 포물 연산자의 장애물문제를 소개한 논문이다. 1장에서는 전형장애물문제(classical obstacle problem)를 소개하고 이 문제의 해의 존재성과 유일성 $C^{1,1}$ 정칙성을 다루었다. 2장에서는 장애물-종류문제(Obstacle-type problem)의 해의 $C^{1,1}$ 정칙성을 보였다. 3장에서는 자유경계의 기본적인 성질들에 대하여 증명하였다. 4장에서는 비선형 2차 포물 연산자의 장애물문제를 소개하고 해의 연속성을 보이기 위해 [2]의 방법을 참고하였다.

**주요 어휘 :** 장애물, 장애물문제, 전형장애물문제(classical obstacle problem), 장애물-종류문제(Obstacle-type problem), 자유경계, $C^{1,1}$ 정칙성, 비선형 2차 포물 연산자.
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