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TOWARD COHERENCE OF ANALYTIC ADJOINT IDEAL SHEAVES

(수반아이디얼 층의 일관성에 관하여)

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TOWARD COHERENCE OF ANALYTIC ADJOINT IDEAL SHEAVES

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Abstract

First, we introduce the notion of multiplier ideal sheaves which measure singularities of plurisubharmonic functions. And then we look into Hörmander's estimates which enable to prove coherence of multiplier ideal sheaves. Secondly, we introduce the analogue of multiplier ideal sheaves namely, adjoint ideal sheaves and suggest analytic approaches to proof of coherence of adjoint ideal sheaves.

Key words: Mutiplier ideal sheaves, Adjoint ideal sheaves
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Chapter 0

Introduction

Given an $\mathbb{Q}$-divisor D on smooth complex manifold, the multiplier ideal sheaf associated to D is defined by $\mu_* \mathcal{O}_{X'}(K_{X'} - \mu^* K_X - [\mu^* D])$, where $\mu : X' \to X$ is a log resolution of D. If $\sum a_i D_i$ be a $\mathbb{Q}$-divisor and if g_i be a local defining function of D_i, respectively, then we can define a plurisubharmonic (PSH) function $\varphi_D := \sum a_i \log |g_i|$.

On the other hand, given a PSH function φ on a complex manifold, the multiplier dieal sheaf associated to φ is defined by germs of holomorphic functions $f \in \mathcal{O}_{X,x}$ such that $|f| e^{-\varphi} \in L^2_{loc}(Leb)$. Thus for one divisor, we have two multiplier ideal sheaves which are $\mathcal{I}(D)$ and $\mathcal{I}(\varphi_D)$. In the first part of this paper, we study the theorem which states these two sheaves are the same, and then using Hörmander estimates, we prove that for any PSH function φ, $\mathcal{I}(\varphi)$ is coherent.

Guenancia defines an analytic analogue, adjoint ideal sheaves $Adj_H(\varphi)$ attached to a PSH function φ with respect to a hypersurface H. In the second part of this paper, we intoduce the definition of adjoint ideal sheaves, and give analytic approach to proof of coherence of adjoint ideal sheaves. More percisely, in order to prove cohrence of ajoint ideal sheaves, using the same argument for multiplier ideal sheaves, we need a version of Hörmander estimates for a weight of the form $e^{\psi - \varphi}$. And then we define another sheaves $Adj^{\alpha}_H(\varphi)$ depending on a real number $\alpha > 1$, but for a PSH function φ_D associated to a divisor D, $Adj^{\alpha}_H(\varphi_D)$ is the same as $Adj_H(\varphi_D)$.

Chapter 1

Multiplier ideal sheaves

1.1 Algebraic and Analytic Definition

We will introduce the concept of multiplier ideal sheaves. In the following definition, a $\mathbb{Q}$-divisor means simply a finite formal linear combination of divisors with rational number coefficients. The following definitions are cited from [7].

Definition 1.1.1. Let $D = \sum D_i$ be a divisor. Then D is said to have *simple normal crossings* if each D_i is smooth, and if for any point $x \in X$, we may choose local holomorphic coordinates $(z_1, \cdots, z_n)$ in a neighborhood U of x such that $D \cap U = \{z_1 \cdots z_k = 0\}$ for some $k \leq n$. A $\mathbb{Q}$-divisor $\sum a_i D_i$is said to have *simple normal crossing support* if $\sum D_i$ has simple normal crossing.

In order to define multiplier ideal sheaves, we need to know log resolutions.

Definition 1.1.2. Let $D = \sum a_i D_i$ be a $\mathbb{Q}$-divisor on an irreducible variety X. A *log resolution* of the pair (X, D) is a projective birational mapping

$$\mu : X' \to X,$$

with X' non-singular, such that the divisor $\mu^* D$+except(μ) has simple normal crossing support, where except(μ) denotes the sum of exceptional divisors of μ.

Our focus is the case when X is smooth.

Definition 1.1.3. Let D be an effective $\mathbb{Q}$-divisor on a smooth complex variety X, and fix a log resolution $\mu : X' \to X$ of D. Then the algebraic *multiplier ideal sheaf*

$$\mathcal{I}(D) = \mathcal{I}(X, D) \subset \mathcal{O}_X$$

associated to D is defined to be

$$\mathcal{I}(D) = \mu_* \mathcal{O}_{X'} (K_{X'/X} - [\mu^* D]).$$

,where $K_{X'/X} = K_{X'} - \mu^* K_X$ is the relative canonical divisor and $[\mu^* D]$ is the integral part of $\mu^* D$.

Remark 1.1.4. Since $\mu_* \mathcal{O}_{X'}(K_{X'}) = \mathcal{O}_X(K_X)$, $\mu_* \mathcal{O}_{X'}(K_{X'/X}) = \mathcal{O}_X$. Thus, for an effective integral divisor N, $\mu_* \mathcal{O}_{X'} (K_{X'/X} - N) \subset \mathcal{O}_X$ is naturally an ideal sheaf of $\mathcal{O}_X$.
Thus $\mathcal{I}(D) \subset \mathcal{O}_X$ is indeed a sheaf of ideals. For more detail, see [7].

Let us introduce multiplier ideal sheaves in the analytic approach. At this point, multiplier ideal sheaves are attached to a plurisubharmonic function which is

Definition 1.1.5. A function $\varphi : \Omega \to \mathbb{R} \cup \{-\infty\}$ defined on an open set $\Omega \subset \mathbb{C}^n$ is said to be *Plurisubharmonic* (abbreviated by PSH) if

a) φ is upper semicontinuous;

b) for all $a \in \Omega$, $|\xi| < d(a, \complement\Omega)$,

$$\varphi(a) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(a + e^{i\theta}\xi) d\theta.$$

On a complex manifold X, if $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is a PSH function on every coordinate, we say that φ is PSH on X.

Remark 1.1.6. If $\varphi : X \to \mathbb{R}$ is C^2 function, then f is PSH if and only if $i\partial\bar{\partial} f$ is a semi-positive $(1, 1)$-form. (see [2] p.40)

Definition 1.1.7. Let X be a complex manifold and let φ be a PSH function on X. Then the analytic *multiplier ideal sheaf* $\mathcal{I}(\varphi)$ associated to φ is the sheaf of germs of holomorphic functions $f \in \mathcal{O}_{X,x}$ such that $|f|^2 e^{-2\varphi}$ is integrable with respect to the Lebesgue measure in some local coordinates near x.

We shall define the notion of an equivalence singularities.

Example 1.1.8. Let g be a holomorphic function on $\mathbb{C}$ and let $\varphi_1 = \frac{a_1}{2}\log|g|^2$ and let $\varphi_2 = \frac{a_2}{2}\log|g|^2$, for some positive number a_1 and a_2. Since $\log|x|^2$ is PSH (i.e. subharmonic), φ_1 and φ_2 are PSH. If $a_1 \geq a_2$,

$$\frac{e^{-2\varphi_1}}{e^{-2\varphi_2}} = \frac{|g|^{a_2}}{|g|^{a_1}}$$

is locally bounded above. In that case, we can say that φ_1 is more singular than φ_2.

Definition 1.1.9. A PSH function φ is said to *have analytic singularities* if φ can be written locally as

$$\varphi = \frac{\alpha}{2}\log(|f_1|^2 + \cdots + |f_N|^2) + \nu,$$

where α is a positive number, each f_i is holomorphic and ν is locally bounded.

Definition 1.1.10. Let φ_1, φ_2 be PSH functions having analytic singularities. Then φ_1 and φ_2 are said to have *equivalence singularities*, written by $\varphi_1 \sim \varphi_2$, if $\frac{e^{-2\varphi_1}}{e^{-2\varphi_2}}$ and $\frac{e^{-2\varphi_2}}{e^{-2\varphi_1}}$ are locally bounded.

Remark 1.1.11. Let φ_1, φ_2 be PSH functions such that $\varphi_1 \sim \varphi_2$. Then local integrablities of $e^{-2\varphi_1}$ and $e^{-2\varphi_2}$ are equivalent. So we get $\mathcal{I}(\varphi_1) = \mathcal{I}(\varphi_2)$. So we can think that multiplier ideal sheaves measure singularities of PSH functions.

Remark 1.1.12. Let $\sum a_i D_i$ be a $\mathbb{Q}$-divisor with simple normal crossing support on a complex manifold X of dimension n and let g_i be a local defining function of D_i, respectively. Let $\varphi_D = \sum a_i \log|g_i|$. $\mathcal{I}(\varphi_D)$ is the sheaf of holomorphic functions f on an open set $U \subset X$ such that $\frac{|f|^2}{\prod_i |g_i|^{2a_i}}$ is integrable on U. By the definition of the simple normal crossing, each g_i can be written by a coordinate function. In other words, $\frac{|f|^2}{\prod_i |g_i|^{2a_i}} =_{\text{locally}} \frac{|f|^2}{\prod_i |z_i|^{2a_i}}$, for some $i \leq n$. Since f is analytic, f is locally representable by a power series. By Parseval's identity, the integrability of $\frac{|f|^2}{\prod |z_i|^{2a_i}}$ is equivalent to one of all monomials of $\frac{|f|^2}{\prod |z_i|^{2a_i}}$. Here, we have a lemma.

Lemma 1.1.13. *The function* $\dfrac{1}{|z|^{2t}}$ *is integrable near* 0 *if and only if* $t < 1$

Proof of lemma 1.1.13. Using the polar coordinates (ρ, θ),

$$\int \frac{1}{|z|^{2t}} dV = \int_0^{\epsilon} \int_0^{2\pi} \frac{1}{\rho^{2t-1}} = \int_0^{\epsilon} \frac{2\pi}{\rho^{2t-1}}$$

which is integrable if and only if $t < 1$. □

By the lemma, the integrability of $\dfrac{|z|^{2m}}{|z|^{2a_i}}$ near 0 is equivalent to $a_i - m < 1$ or $m \geq [a_i]$. Thus, $\dfrac{|f|^2}{\prod |z_i|^{2a_i}}$ is local integrable at the origin if and only if f is divisible by $\prod |z_i|^{[a_i]}$. Therefore, $f \in \mathcal{I}(\varphi_D)$ if and only if f is divisible by $\prod |g_i|^{[a_i]}$. In conclusion,

Proposition 1.1.14. *In the case when* D *has simple normal crossing support,*

$$\mathcal{I}(\varphi_D) = \mathcal{O}(-\sum [a_i] D_i).$$

Notation. In the above example, we construct PSH function the φ_D associated to D. For arbitrary effective $\mathbb{Q}$-divisor D, we can do this work. We denote this PSH function by φ_D from now on.

In general, we have the following theorem.

Theorem 1.1.15 ([2])**.**
For a complex manifold X *of dimension* n *and* $\mathbb{Q}$*-divisor* $D = \sum a_i D_i$,

$$\mathcal{I}(\varphi_D) = \mathcal{I}(D).$$

Proof. Let S and S' be sets such that $\mu : X' \setminus S' \to X \setminus S$ is biholomorphic, where $\mu : X' \to X$ is the log resolution. Let us compute $\mathcal{O}(K_X) \otimes \mathcal{I}(\varphi_D)$. If f is an element of $\mathcal{O}(K_X) \otimes \mathcal{I}(\varphi_D)$ on an open set U, it can be written by n-form f such that $(i)^{n^2} f \wedge \overline{f} e^{-2\varphi_D} \in L^2_{\text{loc}}(U)$. By Hartogs theorem, if f is defined on $U \setminus S$, then f will be extended to U. Thus we think that f is defined on $U \setminus S$. On the complement of S, μ is biholomorphic, so we can apply the change of variables. Therefore,

$$\int_U (i)^{n^2} f \wedge \overline{f} e^{-2\varphi_D} = \int_{\mu^{-1}(U)} (i)^{n^2} \mu^* f \wedge \overline{\mu^* f} e^{-2\varphi_D \circ \mu}.$$

Thus, $\mu^* f \in \Gamma\big(\mu^{-1}(U), \mathcal{O}(K_{X'}) \otimes \mathcal{I}(\varphi_D \circ \mu)\big)$ Therefore, $\mathcal{O}(K_X) \otimes \mathcal{I}(\varphi_D) = \mu_* \mathcal{O}(K_{X'}) \otimes \mathcal{I}(\varphi_D \circ \mu)$. Then, by the projection formula,

$$\begin{aligned} \mathcal{I}(\varphi_D) &= \mathcal{O}(-K_X) \otimes \mu_* \mathcal{O}(K_{X'}) \otimes \mathcal{I}(\varphi_D \circ \mu) \\ &= \mu_*\big(\mathcal{O}(K_{X'} - \mu^* K_X) \otimes \mathcal{I}(\varphi_D \circ \mu)\big). \end{aligned} \tag{*}$$

Recall that φ_D is locally $\sum a_i \log |g_i|$, for a local defining function g_i of D_i. Then $\varphi_D \circ \mu = \sum a_i \log |g_i \circ \mu|$ and each $g_i \circ \mu$ is a local defining function of $\mu^* D_i$. By the proposition 1.1.14 with the fact $\mu^* D$ has simple normal crossing support, $\mathcal{I}(\varphi_D \circ \mu) = \mathcal{O}(-[\mu^* D])$. By *, $\mathcal{I}(\varphi_D) = \mu_*\big(\mathcal{O}_{X'}(K_{X'/X} - [\mu^* D])\big) = \mathcal{I}(D)$. $\square$

1.2 Hörmander Estimates

In the next section, we are going to prove coherence of multiplier ideal sheaves. In order to do this, we need Hörmander estimates for singular weights. We first introduce some notations and definitions.

Notation.
Let (X, ω) be a complex n-dimensional Hermitian manifold. A smooth (Hermitian) metric of a holomorphic line bundle L associated with a holomorphic function φ is a norm defined by

$$|\eta|^2_\varphi = |\theta(\eta)|^2 e^{-2\varphi(x)}, \ x \in \Omega, \eta \in E_x,$$

where $\theta : E_\Omega \simeq \Omega \times \mathbb{C}$ is a local trivialization on a open set $\Omega \subset M$ and φ is a smooth real valued function. With a local trivialization $\theta : E_\Omega \simeq \Omega \times \mathbb{C}$, for any section $s : X \to L$, $\theta \circ s$ is a holomorphic function. That is, any section s can be given by a collection of scalar functions (s_α). Then the norm associated with φ of a section s is defined by $|s|^2_\varphi = |s_\alpha|^2 e^{-2\varphi}$. Likewise, we define the norm of a L-valued (p,q)-forms by $|s|^2_{\omega,\varphi} = |s_\alpha|^2_\omega e^{-2\varphi}$
$C^\infty(X, L)$. denotes the set of all global sections of E with smooth coefficients. $L^2(X, L)$ denotes the set of all global sections s of E satisfying

$$\|s\|^2 := \int |s(x)|^2_\varphi dV_\omega(x) < +\infty.$$

$C^\infty_{p,q}(X,L)$ denotes $C^\infty(M,\Lambda^{p,q}T^*_M\otimes E)$.
$L^2_{p,q}(X,L)$ denotes set of L-valued (p,q)-forms such that

$$\|s\|^2 := \int |s(x)|^2_{\omega,\varphi} dV_\omega(x) < +\infty.$$

Defining an inner product

$$\langle\langle s,t\rangle\rangle := \int \langle s,t\rangle_{\omega,\varphi} dV,\ \forall s,t\in L^2_{p,q}(X,L)$$

make $L^2_{p,q}(X,L)$ Hilbert space.
$D_{p,q}(X,L)$ denotes the set of compactly supported (p,q)-forms in $C^\infty_{p,q}(X,L)$.

The following inequality is essential for the proof of Hörmander estimates theorem.

1.2.1 (**Bochner-Kodaira-NakanoInequality**). *[6]*
Let (X,ω) be a Kähler manifold and L be a holomorphic line bundle with a smooth metric φ. Assume that $i\partial\bar\partial\varphi \geq \epsilon\omega$. Then, for all $s\in D_{n,q}(X,L)$,

$$\int |\overline{\partial}s|^2_{\omega,\varphi} + |\overline{\partial}^* s|^2_{\omega.\varphi} \geq \epsilon q \int |s|^2_{\omega,\varphi} dV_\omega.$$

For $s\in L^2_{(p,q)}(X,L)$, we can compute $\bar\partial s$ as a distribution. The mapping $\bar\partial : L^2_{(p,q)}(X,L)\to L^2_{(p,q+1)}(X,L)$ between two Hilbert spaces are non-bounded operator.

Definition 1.2.2. Let $T : H_1 \to H_2$ is a operator between two Hilbert spaces. Then T is said to be *closed densely defined* if $Dom(T)$ is dense in H_1 and if $Graph(T)$ is closed a subspace of $H_1\times H_2$.

Proposition 1.2.3. *If $T : H_1 \to H_2$ is a closed desely defined operator between two Hilbert spaces, then we can define the adjoint operator $T^* : H_2 \to H_1$ and it is also closed desely defined operator.*

Proof. Let $Dom(T^*)$ be the set of all $x\in H_2$ such that $x\mapsto \langle Tx,y\rangle$ is continuous linear functional on $Dom(T)$. After extending this functional to $H_1 = \overline{Dom(T)}$ via Hahn-Banach theorem, then by Riesz Representation theorem, we can find a $z\in H_1$ such that for all $x\in Dom(T)$,

$$\langle Tx,y\rangle = \langle x,z\rangle.$$

So we define T^*y by such z. Let $F : H_1 \times H_2 \to H_2 \times H_1$ be a mapping given by

$$F(x, y) = (y, -x).$$

Then for $(y, x) \in H_2 \times H_1$, $(y, x) \perp F(Graph(T))$ means that for all $z \in H_1$,

$$\langle (y, x), (Tz, -z) \rangle_{1,2} = 0$$

or

$$\langle Tz, y \rangle_2 = \langle z, x \rangle_1.$$

Since $z \mapsto \langle z, x \rangle_1$ is continuous, $y \in Dom(T^*)$ and $T^*y = x$. Therefore,

$$F(Graph(T)^{\perp}) = Graph(T^*). \tag{*}$$

So $Graph(T^*)$ is closed. For the proof of the desity of $Dom(T^*)$, assume $y \in Dom(T^*)^{\perp}$. Then for all $z \in Dom(T^*)$, $\langle y, z \rangle = 0$. Then $\langle (y, 0), (z, T^*z) \rangle_{1,2} = 0$. In other words, $(y, 0) \perp Graph(T^*)$. By (*) and closedness of T, $Graph(T^*)^{\perp} = F(\overline{Graph(T)}) = F(Graph(T))$. Therefore, $y = T0 = 0$, that is, $Dom(T^*)^{\perp} = 0$. Thus $Dom(T^*)$ is dense. □

Let $Dom(\overline{\partial}) := \{s \in L^2_{p,q}(X, L) : \overline{\partial}s \in L^2_{p,q+1}(X, L)\}$. Then $Dom(\overline{\partial})$ contains $D_{p.q}(X, L)$ which is dense in $L^2_{p,q}(X, L)$. And suppose (s_n) converges to s in $L^2_{p,q}(X, L)$ and if $(\overline{\partial}s_n)$ converges to t in $L^2_{p,q+1}(X, L)$, then $(\overline{\partial}s_n)$ converges to $\overline{\partial}s$ with respect to weak-topology, so $\overline{\partial}s = \overline{\partial}t$ which means $Graph(\overline{\partial})$ is closed. Thus the operator $\overline{\partial} : L^2_{p,q}(X, L) \to L^2_{p,q+1}(X, L)$ is closed densely defined. Therefore, by the above proposition, we get the densely defined operator $\overline{\partial}^*$.

When we prove Hörmander estimates, the following analysis lemma make it easier.

Lemma 1.2.4. *Let $T : H_1 \to H_2$ be a closed densely defined operator between two Hilbert spaces. Suppose that for some $\alpha \in H_2$, there exist a constant $C > 0$ such that for all $\beta \in Dom(T^*)$,*

$$|\langle \alpha, \beta \rangle|^2 \leq C \|T^*\beta\|^2. \tag{*}$$

Then there exist $u \in H_1$ such that

$$Tu = \alpha$$

and

$$\|u\|^2 \leq C.$$

Proof. Consider the functional

$$F : Image(T^*) \ni T^*\beta \to \langle \beta, \alpha \rangle.$$

This functional is clearly linear. By (*), if $T^*\beta = 0$, then $\langle \beta, \alpha \rangle = 0$, hence the functional is well-defined. Again by (*), the functional is continuous. By Hahn-Banach theorem and the Riesz Representation theorem, this functional can be extended to a linear functional on H_1 and there exist $u \in H_1$ such that

$$F(v) = \langle v, u \rangle$$

and

$$\|u\|^2 \leq C.$$

On $Image(T^*)$,

$$\langle T^*\beta, u \rangle = \langle \beta, \alpha \rangle.$$

Since $Dom(T)$ is dense, $Tu = \alpha$. □

From now on, we will prove Bochner-Kodaira-Nakano inequality on bigger space than $D_{n,q}(X, L)$.

Definition 1.2.5. Let X be a complex manifold with a Kähler metric ω.

- A function $\chi : X \to [0, \infty)$ is said to be *exhaustion* if for all $c \in \mathbb{R}$, $\{z \in X | \chi(z) < c\}$ is relative compact in X.
- A C^2 function $\psi : X \to \mathbb{R} \cup \{-\infty\}$ is said to be *strictly plurisubharmonic* if $i\partial\bar{\partial}\psi > 0$.
- (X, ω) is said to be *complete* if there is a smooth exhaustion function $\chi : X \to [0, \infty)$ such that $|d\chi|_\omega$ is bounded.

Lemma 1.2.6. *Let (X, ω) be a complete Kähler manifold of complex dimension n and let L be a holomorphic line bundle. For all $s \in Dom(\bar{\partial}) \cap Dom(\bar{\partial}^*)$, there is a sequence $(s_m) \subset D_{p,q}(X, L)$ such that*

$$\|s_m - s\|,$$

$$\|\bar{\partial} s_m - \bar{\partial} s\|$$

and

$$\|\bar{\partial}^* s_m - \bar{\partial}^* s\|$$

tend to 0.

Proof. Let ψ be a partition of unity on a neighborhood of $Supp(s)$ with a small support. Then $s = \sum \psi_j s$, so it is enough to approximate each $\psi_j s$. Thus we may assume that $Supp(s)$ is contained in a coordinate neighborhood, where L is trivial. Let χ be a smooth exhaustion function with $|d\chi| \leq C$ and let $f(t)$ be a smooth function whose values 1 for $t \leq 0$, 0 for $t \geq 1$ and $|df| \leq 2$. For all $m \in \mathbb{N}$, let $\chi_m(x) = f(C^{-1}2^{-m-1}\chi(x))$, then $|d\chi_m| \leq 2^{-m}$. Let $s_m = \chi_m s$. Clearly, $s_m \to s$. Also,

$$\bar{\partial} s_m = \chi_m \bar{\partial} s + \bar{\partial}\chi_m \wedge s.$$

But,

$$|\bar{\partial} s_m - \chi_m \bar{\partial} s| = |\bar{\partial}\chi_m \wedge s| \leq |\bar{\partial}\chi_m||s| \leq 2^{-m}|s|.$$

By Lebesgue dominated convergence theorem,

$$\|\bar{\partial} s_m - \chi_m \bar{\partial} s\| \to 0 \text{ or } \|\bar{\partial} s_m - \bar{\partial} s\| \to 0.$$

Furthermore,

$$|\langle \bar{\partial}^*(\chi_m s) - \chi_m \bar{\partial}^* s, t\rangle| = |\langle s, \chi_m \bar{\partial} t - \bar{\partial}(\chi_m t)\rangle| = |\langle s, -\bar{\partial}\chi_m \wedge t\rangle| \leq |s||t|2^{-m}.$$

Thus,

$$|\bar{\partial}^*(\chi_m s) - \chi_m \bar{\partial}^* s| \leq |s|2^{-m}.$$

As above, this implies $\|\bar{\partial}^* s_m - \bar{\partial}^* s\| \to 0$. Since s_m has compact support, we may assume that s has a compact support. Therefore, let $s \in Dom(\bar{\partial}) \cap Dom(\bar{\partial}^*)$ with compact support which is contained in a neighborhood where L is trivial. Let ρ be a smooth function with compact support such that the integral is equal to 1 and let

$$\rho_m(z) = m^{2n}\rho(mz).$$

Let $s_m = \rho_m * s$ be the convolution. Then (s_m) is smooth and converges to s in L^2. Since $\bar{\partial}$ commute with convolutions, $(\bar{\partial} s_m)$ converges to $\bar{\partial} s$. The remaining part is complicated because $\bar{\partial}^*$ is not a order 1 operator with constant. Let us compute $\bar{\partial}^*$. Let $s \in Dom(\bar{\partial})$ and let $t \in D_{p,q}(X, L)$. Locally, we can write

$$s = \sum_{i,j} s_{i,j} dz_i \wedge d\overline{z}_j$$

and

$$t = \sum_{i,k} t_{i,k} dz_i \wedge d\overline{z}_k$$

, where $|i| = p, |j| = q + 1, |k| = q$. Now

$$\bar{\partial} t = \partial t_{i,k}/\partial \overline{z}_m \, d\overline{z}_m \wedge dz_i \wedge d\overline{z}_k = (-1)^p \partial t_{i,k}/\partial \overline{z}_m \, dz_i \wedge d\overline{z}_m \wedge d\overline{z}_k.$$

Then

$$\begin{aligned}
\int \sum_{i,k} (\bar{\partial}^* s)_{i,k} \bar{t}_{i,k} e^{-2\phi} dV &= \langle\langle \bar{\partial}^* s, t \rangle\rangle \\
&= \langle\langle s, \bar{\partial} t \rangle\rangle \\
&= (-1)^p \int \sum_{i,j} s_{i,j} \overline{\partial t_{i,k}/\partial \overline{z}_m} e^{-2\phi} dV \\
&= (-1)^p \int \sum_{i,j} \sum_m e^{-2\phi} s_{i,j} \, \partial \bar{t}_{i,k}/\partial z_m dV \\
&= (-1)^{p+1} \int \sum_{i,j} \sum_m \partial(e^{-2\phi} s_{i,j})/\partial z_m \, \bar{t}_{i,k} dV.
\end{aligned}$$

Therefore,

$$\bar{\partial}^* s = (-1)^{p+1} \sum_{i,j} \sum_m e^{2\phi} \partial(e^{-2\phi} s_{i,j})/\partial z_m dz_i \wedge dz_k.$$

We conclude that

Proposition 1.2.7. *$\bar{\partial}^*$ operator is of the form*

$$\sum_m (\partial s)/\partial z_m + as, \;\; s \in Dom(\bar{\partial}^*),$$

where a is smooth.

we cite the following lemma which complete the proof from [2].

Lemma 1.2.8. *Let $Pf = \sum a_k \partial f/\partial x_k + bf$ be a differential operator of order 1 on an open set $\Omega \subset \mathbb{R}^n$, with coefficients $a_k \in C^1(\Omega),\ b \in C^0(\Omega)$. Then for any $\nu \in L^2(\mathbb{R}^\ltimes)$ with compact support in Ω, we have*

$$\lim_{\epsilon \to 0} \| P(\nu * \rho_\epsilon) - (P\nu) * \rho_\epsilon \|_{L^2} = 0.$$

□

Proposition 1.2.9. *If (X, ω) is complete, then by the lemma 1.2.6, Bochner-Kodaira-Nakano inequality holds, for all $s \in Dom(\bar{\partial}) \cap Dom(\bar{\partial}^*)$.*

The following theorem due to [5].

1.2.10 (Hörmander Estimates).
Let (X, ω) be a complete Kähler manifold of complex dimension n and let L be a holomorphic line bundle with a smooth metric φ. Suppose that $i\Theta(L) = i\partial\bar{\partial}\varphi \geq \epsilon\omega$. Then for all $f \in L^2_{n,q}(X, L)$ satisfying $\bar{\partial} f = 0$, there exists $g \in L^2_{n,q-1}(X, L)$ such that $\bar{\partial} g = f$ and

$$\int |g|^2_{\omega,\varphi} dV \leq \frac{1}{\epsilon q} \int |f|^2_{\omega,\varphi} dV.$$

Proof. By the proposition 1.2.9, we have for all $s \in Dom(\bar{\partial}) \cap Dom(\bar{\partial}^*)$,

$$\|\bar{\partial}^* s\|^2 \geq \epsilon q \|s\|^2.$$

Thus, for all $s \in Dom(\bar{\partial}) \cap Dom(\bar{\partial}^*)$,

$$|\langle\langle f, s \rangle\rangle|^2 \leq \|f\|^2 \|s\|^2 \leq \|f\|^2 \frac{1}{\epsilon q} \|\bar{\partial}^* s\|^2.$$

By the Lemma 1.2.4 with $C = \frac{\|f\|^2}{\epsilon q}$, there is a $g \in L^2_{n,q-1}(X, L)$ such that

$$\|g\|^2 \leq C.$$

The proof is complete. □

Definition 1.2.11. Let X be a complex manifold. X is said to be *Stein* if there is a smooth strictly PSH exhaustion $\psi : X \to [0, +\infty)$.

Property 1.2.12. *If X is a Stein manifold, then X has a complete Kähler metric.*

Proof. Let ψ be a smooth exhaustion such that $i\partial\bar{\partial}\psi > 0$ and let $\omega = i\partial\bar{\partial}e^{\psi}$. Then this form is Kähler. Take $\chi = e^{\psi/2}$. Then

$$\partial\chi \wedge \bar{\partial}\chi = \frac{1}{4}(i\partial\psi \wedge \bar{\partial}\psi)e^{\psi}$$

and

$$\omega = i\partial\bar{\partial}e^{\psi} = (i\partial\psi \wedge \bar{\partial}\psi + i\partial\bar{\partial}\psi)e^{\psi} \geq \frac{1}{4}(i\partial\psi \wedge \bar{\partial}\psi)e^{\psi}.$$

Thus $|\partial\chi|^2_{\omega} \leq 1$. In other words, $|\partial\chi|_{\omega}$ is bounded, so is $|d\chi|_{\omega}$. □

Remark 1.2.13. In the above proof, we can know easily the followings. If ω is complete, then $c\omega$ is also complete for some constant $c > 0$. Furthermore, if ω, ω' are Kähler metrics and if ω is complete, then $\omega + \omega'$ is complete.

1.2.14 (Hörmander Estimates on Stein manifolds).
Let (X, ω) be a Kähler manifold of complex dimension n and let L be a holomorphic line bundle with a smooth metric φ. Suppose that X is a Stein manifold and $i\Theta(L) = i\partial\bar{\partial}\varphi \geq \epsilon\omega$. Then for all $f \in L^2_{n,q}(X, L)$ satisfying $\bar{\partial}f = 0$, there exists $g \in L^2_{n,q-1}(X, L)$ such that $\bar{\partial}g = f$ and

$$\int |g|^2_{\omega,\varphi} dV \leq \frac{1}{\epsilon q} \int |f|^2_{\omega,\varphi} dV.$$

Proof. By the assumption, we have a smooth exhaustion ψ such that $i\partial\bar{\partial}\psi > 0$. Also, in the proof of 1.2.12, $\omega' = i\partial\bar{\partial}e^{\psi}$ is complete metric. Then for all $c > 0$, the Kähler metric $\omega_c = \omega + c\omega'$ is complete. Let L_c be a holomorphic line bundle L endowed with the metric $\varphi_c = \varphi + \epsilon c e^{\psi}$ so that $i\partial\bar{\partial}\varphi_c > \epsilon\omega_c$. We denote $|\cdot|_c$ by the norm with respect to ω_c, φ_c.

Lemma 1.2.15. *Let ω, γ be two Kähler forms on X such that $\omega \leq \gamma$. For every (n, q)-form f,*

$$|f|^2_{\gamma} dV_{\gamma} \leq |f|^2_{\omega} dV_{\omega}.$$

Proof. Let $x \in X$ be a point. On the coordinate $(z_1, \ldots, z_n)$ near x,

$$\omega = i \sum_{1 \leq i \leq n} dz_j \wedge d\bar{z}_j$$

and

$$\gamma = i \sum_{1 \le j \le n} \gamma_j dz_j \wedge d\bar{z}_j,$$

where $\gamma_1 \le \cdots \gamma_n$ are eigenvalues of γ with respect to ω. So $\sqrt{\gamma_j} dz_j$'s are orthonormal for γ. By the assumption, $\gamma_1 \ge 1$. For every $f = \sum f_K dz_N \wedge d\bar{z}_K$ and $|K| = q$,

$$|f|_\gamma^2 = \sum (\gamma_1 \cdots \gamma_n)^{-1} \gamma_K^{-1} |f_K|^2,$$

where $\gamma_K = \prod_{j \in K} \gamma_j$. Furthermore,

$$dV_\gamma = \gamma_1 \cdots \gamma_n dV_\omega.$$

Therefore,

$$|f|_\gamma^2 dV_\gamma \le |f|_\omega^2 dV_\omega.$$

□

Now $|f|_{\omega,\phi}^2 < +\infty$. By the Lemma, $|f|_{\omega_c,\phi}^2 dV_{\omega_c} \le |f|_{\omega,\phi}^2 dV_\omega$. But $e^{-2\phi_c} \le e^{-2\phi}$, thus $|f|_c^2 < +\infty$. Applying 1.2.14 to (X, E_c, ω_c) gives a solution g_c to $\bar{\partial} g_c = f$ with

$$\epsilon q \|g_c\|_c^2 \le \|f\|_c^2 \le \|f\|^2.$$

Therefore, the family $(g_c e^{\phi_c})$ is bounded in L^2, so we get a weakly convergent subsequence (g_{cj}) in L_{loc}^2. The weak limit g is a solution what we need. □

A *singular* Hermitian metric φ on L is a Hermitian metric on L except that φ is not necessarily smooth, but PSH.

1.2.16 (Hörmander Estimates for singular weights).
Let Ω be a Stein domain in $\mathbb{C}^n$, let φ be a PSH function defined on Ω and let ω be a Kähler form on Ω. Suppose that $i\partial\bar{\partial}\varphi \ge \epsilon\omega$. Then for all (n,q)-form f such that

$$\int_\Omega |f|_\omega^2 e^{-2\varphi} dV < +\infty, \ \bar{\partial} f = 0,$$

there exists a $(n, q-1)$-form g such that $\bar{\partial} g = f$ and

$$\int_\Omega |g|_\omega^2 e^{-2\varphi} dV \le \frac{1}{\epsilon q} \int_\Omega |f|_\omega^2 e^{-2\varphi} dV.$$

Proof. Using the standard regularization $\varphi_\nu = \varphi * \rho_\nu$, we can get a decreasing sequence of smooth PSH functions $\varphi_\nu = \varphi * \rho_\nu$ defined on Ω_ν such that φ_ν converges to φ. Apply 1.2.10 to smooth metric φ_ν, we then get g_ν such that $\bar{\partial} g_\nu = f$ and,

$$\int_{\Omega_\nu} |g_\nu|^2_\omega e^{-2\varphi_\nu} dV \le \frac{1}{\epsilon q} \int_{\Omega_\nu} |f|^2_\omega e^{-2\varphi_\nu} dV \le \frac{1}{\epsilon q} \int_{\Omega} |f|^2_\omega e^{-2\varphi} dV.$$

Since φ_ν is decreasing to φ, the family $(g_\nu e^{\varphi_\nu})$ is bounded, so there is a subsequence converging weakly in L^2_{loc}. Therefore the weak limit g satisfies

$$\int_{\Omega} |g|^2_\omega e^{-2\varphi} dV \le \frac{1}{\epsilon q} \int_{\Omega} |f|^2_\omega e^{-2\varphi} dV.$$

Since $\bar{\partial}$ is continuous in distribution sense,

$$\bar{\partial} g = f \quad \text{on } \Omega.$$

□

1.3 Coherence of Multiplier Ideal Sheaves

In this section, we will prove that multiplier ideal sheaves are coherent. Following definitions and properties are cited from [2].

Definition 1.3.1. Let $\mathscr{A}$ be a sheaf of rings on a topological space X and let $\mathscr{S}$ be a $\mathscr{A}$-module. Then $\mathscr{S}$ is said to be *locally finitely generated* if for every point $x_0 \in X$, one can find a neighborhood Ω and sections $F_1, \cdots, F_q \in \mathscr{S}(\Omega)$ such that for every $x \in \Omega$, the stalk $\mathscr{S}_x$ is generated by germs $F_{1,x}, \cdots, F_{q,x}$ as a $\mathscr{A}_x$-module.

Definition 1.3.2. Let $\mathcal{O}_X$ be a sheaf of rings on a topological space X and let $\mathcal{S}$ be a sheaf of modules over $\mathcal{O}_X$. $\mathcal{S}$ is said to be *coherent* if:

a) $\mathcal{S}$ is locally finitely generated;

b) for any open subset U of X, any $n \in \mathbb{N}$ and any morphism $F : \mathcal{O}_X^n|_U \to \mathcal{S}|_U$ of $\mathcal{O}_X$-modules, the kernel of F is locally finitely generated.

1.3.3 (**Strong Noetherian Property**).
Let $\mathscr{F}$ be a coherent analytic sheaf on a complex manifold M and let $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \cdots$ be an increasing sequence of coherent subsheaves of $\mathscr{F}$. Then the sequence $(\mathscr{F}_k)$ is stationary on every compact subset of M.

1.3.4 (**Krull lemma**).
Let F be a finitely generated R-module and let E be a submodule. Then

a) $\cap_{k\geq 0}\mathfrak{m}^k F = \{0\}$*;*

b) $\cap_{k\geq 0}(E + \mathfrak{m}^k F) = E$*.*

Lemma 1.3.5 ([3]).
Let φ be a PSH function on a domain $\Omega \subset \mathbb{C}^n$ and $x \in \Omega$. If

$$\liminf_{z\to x} \frac{\varphi(x)}{\log|z-x|} \geq n + s$$

for some integer $s \geq 0$, then

$$\mathcal{I}(\varphi)_x \subset \mathfrak{m}_{\Omega,x}^{s+1},$$

where $\mathfrak{m}_{\Omega,x}^{s+1}$ is the maximal ideal of $\mathcal{O}_{\Omega,x}$.

Proposition 1.3.6 ([8, 3]).
Let Ω be a open set in a Kähler manifold (X,ω). For any PSH function φ on Ω, the multiplier ideal sheaf $\mathcal{I}(\varphi)$ is coherent over Ω.

Proof. Since coherence is a local property, we may assume that Ω is a bounded Stein domain in $\mathbb{C}^n$. Thus Since $\mathcal{O}_X$ is coherent, by the strong noetherian property, the family of sheaves generated by finite subsets of $H^2(\Omega,\varphi) := \{f \in \mathcal{O}(\Omega) : \int_\Omega |f|^2 e^{-\varphi} dV < +\infty\}$ has a maximal element on every compact subset of Ω. Thus $H^2(\Omega,\varphi)$ generates a coherent ideal sheaf $\mathcal{J}$. Clearly, $\mathcal{J} \subset \mathcal{I}(\varphi)$.

Let us prove $\mathcal{I}(\varphi) \subset \mathcal{J}$. By Krull lemma, it suffices to show that for all $x \in \Omega$,

$$\mathcal{I}(\varphi)_x \subset \mathcal{J}_x + \mathcal{I}(\varphi)_x \cap \mathfrak{m}_x^{k+1},$$

for all integer $k \geq 0$, where $\mathfrak{m}$ is the maximal ideal at x. Let $h \in \mathcal{I}(\varphi)_x$ such that h is holomorphic on a neighborhood U and let $V \subset U$ be a neighborhood of x. Let χ be a cutoff function such that $Supp(\chi) \subset U$ and $\chi|_V \equiv 1$. we choose V sufficiently small so that $|\bar{\partial}\chi| \leq 1$ Let $f = \bar{\partial}(\chi h)$. Let L be a trivial line bundle equipped with the singular metric

$$\varphi_k(z) = \varphi(z) + (n+k)\log|z-x| + |z|^2.$$

On $V \cup U^c$, $f \equiv 0$. Also, on $V^c \cap U$, $|z-x|^{-2(n+k)}e^{-2|z|^2}$ is bounded.

Therefore,

$$\int_\Omega |f|^2 e^{-2\varphi_k} dV \leq C \int_U |f|^2 e^{-2\varphi} dV \leq C \int_U |h|^2 e^{-2\varphi} dV < +\infty.$$

The second inequality follows because $|i\partial\bar{\partial}f| = |h\bar{\partial}\chi| \leq |f|$.

Since

$$i\partial\bar{\partial}(\sum |z_i|^2) = i\sum dz_i \wedge d\overline{z}_i,$$

$$i\partial\bar{\partial}\varphi_k > i\sum dz_i \wedge d\overline{z}_i.$$

Apply theorem 1.2.16 to $(\Omega, i\sum dz_i \wedge d\overline{z}_i, L)$, we then get a solution g_k such that $\bar{\partial}g_k = f = \bar{\partial}\chi h$ and

$$\int_\Omega |g_k|^2 e^{-2\varphi}|z-x|^{-2(n+k)} dV \leq C' \int_\Omega |g_k|^2 e^{-2\varphi}|z-x|^{-2(n+k)}e^{-2|z|^2} dV$$

$$\leq CC' \int_\Omega |f|^2 e^{-2\varphi} dV < +\infty.$$

Therefore, $g_k \in L^2(\Omega, \varphi)$, so $G_k := \chi h - g_k \in L^2(\Omega, \varphi)$. Since $\bar{\partial}G_k = 0$, G_k is holomorphic and then $G_k \in \mathcal{J}$. Furthermore, by the lemma 1.3.5, $h_x - G_{k_x} = g_{k_x} \in \mathcal{I}(\varphi)_x \cap \mathfrak{m}^{k+1}$ thus $h_x \in \mathcal{J}_x + \mathcal{I}(\varphi)_x \cap \mathfrak{m}_x^{k+1}$. The proof is done. $\square$

Chapter 2

Adjoint Ideal Sheaves

2.1 Algebraic and Analytic Definition

Definition 2.1.1. Let H be a smooth hypersurface in a complex manifold X and let D be a $\mathbb{Q}$-divisor on X such that H is not contained in $Supp(D)$. We fix a log resolution $\mu : X' \to X$ of D such that $\mu^*D + \mu^*H + K_{X'/X} + Exc(\mu)$ has SNC support. The *adjoint ideal sheaf* $Adj_H(D)$ of D with respect to H is defined by

$$Adj_H(D) = \mu_*\mathcal{O}_{X'}(K_{X'/X} - [\mu^*D] - \mu^*H + H'),$$

where H' is the strict transformation of H.

Definition 2.1.2. Let φ be a PSH function on a complex manifold X and let H be a smooth hypersurface. Then $Adj^0_H(\varphi)$ is defined by the sheaf of germs $u \in \mathcal{O}_{X,x}$ such that $|u|^2e^{-2\varphi}$ is integrable with respect to $\frac{1}{|h|^2 \log^2 |h|}Leb$, where h is a local equation for H and Leb is the Lebesgue measure.

Unlike the multiplier case, in general, $Adj^0_H(\varphi)$ is not the same the algebraic one.

Definition 2.1.3 ([4])**.**
The analytic adjoint ideal sheaf $Adj_H(\phi)$ attached to a PSH function φ with respect to a hypersurface H defined by

$$Adj_H(\varphi) := \bigcup_{\epsilon>0} Adj^0_H\big((1+\epsilon)\varphi\big).$$

Theorem 2.1.4 ([4]).
Let H be a smooth hypersurface in a smooth complex variety X and let D be a $\mathbb{Q}$-divisor on X such that H is not contained in Supp(D). Let φ_D be a corresponding PSH function to D. Then

$$Adj_H(\varphi_D) = Adj_H(D).$$

Naturally, we have the following question.

Q. Are adjoint ideal sheaves coherent?

H. guenancia proves the coherence of $Adj_H(\varphi)$ using the method, namely fundamental adjuction exact sequence, under the additional hypothesis that e^{φ} is locally Hölder continuous.

Definition 2.1.5. A function $f : \mathbb{C}^n \to \mathbb{C}$ is said to be Hölder continuous if there are nonnegative constants C and α such that for all $x, y \in \mathbb{C}^n$,

$$|f(x) - f(y)| \leq C|x - y|^{\alpha}.$$

We then can define Hölder continuity of a function on a complex manifold X through coordinate charts.

Theorem 2.1.6 ([4]). *Let X be a complex manifold, let H be a smooth hypersurface and let φ be a PSH function on X, $\varphi|_H \not\equiv -\infty$, such that e^{φ} is locally Hölder continuous. Then $Adj_H(\varphi)$ is a coherent ideal sheaf on X.*

2.2 Modified L^2 Estimates

In order to prove coherence of adjoint ideal sheaves using the same method for multiplier ideal sheaves, we need a version of Hörmander estimates for a weight of the form $e^{-\varphi+\psi}$, where φ and ψ are PSH functions. In this section, we introduce the Blocki's L^2 estimates.
Let Ω be a Stein open set in $\mathbb{C}^n$ and let $f = \sum_j f_j d\bar{z}$ be a $(0,1)$-form on Ω. If φ is smooth and strictly PSH, then $i\partial\bar{\partial}\varphi$ determines a Hermitian metric. Then

$$|f|^2_{i\partial\bar{\partial}\varphi} = \sum \varphi^{j\bar{k}} \overline{f}_j f_k \qquad (*)$$

, where $(\varphi^{j\bar{k}})$ is the inverse transposed of $(\partial^2\varphi/\partial z_j\partial\overline{z}_k)$.
For a PSH function φ not strictly, $(\varphi^{j\bar{k}})$ does not make sense. However, if φ is strictly PSH, we have; For all $h \in L^\infty_{loc}(\Omega)$,

$$|f|^2_{i\partial\bar{\partial}\varphi} \leq h \text{ if and only if } i\bar{f} \wedge f \leq h \cdot i\partial\bar{\partial}\varphi.$$

Therefore, for a PSH function φ, we take any $h \in L^\infty_{loc}(\Omega)$ satisfying $i\bar{f} \wedge f \leq h \cdot i\partial\bar{\partial}\varphi$ as $|f|^2_{i\partial\bar{\partial}\varphi}$. Thus, the inequality $|f|^2_{i\partial\bar{\partial}\varphi} \leq C$ means that there is a $h \in L^\infty_{loc}(\Omega)$ such that

$$i\bar{f} \wedge f \leq h \cdot i\partial\bar{\partial}\varphi$$

and

$$h \leq C.$$

Theorem 2.2.1 ([1]).
Assume that Ω is a pseudoconvex domain in $\mathbb{C}^n$ and take $\alpha \in L^2_{loc,(0,1)}(\Omega)$ with $\bar{\partial}\alpha = 0$. Let φ, ψ be PSH function in Ω such that $|\bar{\partial}\psi|^2_{i\partial\bar{\partial}\psi} \leq 1$ in Ω and $|\bar{\partial}\psi|^2_{i\partial\bar{\partial}\psi} \leq \delta < 1$ on $Supp(\alpha)$. Then there exists $f \in L^2_{loc}(\Omega)$ solving $\bar{\partial}f = \alpha$ and such that

$$\int_\Omega (1 - |\bar{\partial}\psi|^2_{i\partial\bar{\partial}\psi})|f|^2 e^{\psi-\varphi} D\lambda \leq \frac{1}{(1-\sqrt{\delta})^2}\int_\Omega |\alpha|^2_{i\partial\bar{\partial}\psi} e^{\psi-\varphi} d\lambda.$$

Proof. Let f be the minimal solution to $\bar{\partial}f = \alpha$ in the $L^2(\Omega, e^{-\varphi})$ norm so that f is perpendicular to $Ker(\bar{\partial})$. Then for all holomorphic function u,

$$\int f e^{\psi}\overline{u}e^{-\varphi-\psi} = \int f\overline{u} = 0$$

which means that $g := fe^\psi$ is perpendicular to $Ker(\bar{\partial})$ in the $L^2(\Omega, e^{-\varphi-\psi})$ norm. Thus, g is the minimal solution to $\bar{\partial}g = \beta$ in the norm $L^2(\Omega, e^{-\varphi-\psi})$, where $\beta = (\alpha + f\bar{\partial}\psi)e^\psi$. By Hörmander estimates for singular weights,

$$\begin{aligned}
\int_\Omega |f|^2 e^{\psi-\varphi} d\lambda &= \int_\Omega |g|^2 e^{-\varphi-\psi} d\lambda \\
&\leq \int_\Omega |\beta|^2_{i\partial\bar{\partial}(\varphi+\psi)} e^{-\varphi-\psi} d\lambda \\
&\leq \int_\Omega |\alpha + f\bar{\partial}\psi|^2_{i\partial\bar{\partial}\psi} e^{\psi-\varphi}.
\end{aligned}$$

The last inequality follows since $i\partial\bar{\partial}\psi \leq i\partial\bar{\partial}(\varphi+\psi)$. Now, consider $(0,1)$-froms P and Q. For any $t>0$,

$$\begin{aligned}
&i(P+Q)\wedge(\overline{P}+\overline{Q}) \\
&\quad = (1+t^{-1})iP\wedge\overline{P} + (1+t)iQ\wedge\overline{Q} - t^{-1}i(P-tQ)\wedge(\overline{P}-t\overline{Q}) \\
&\quad \leq (1+t^{-1})iP\wedge\overline{P} + (1+t)iQ\wedge\overline{Q}.
\end{aligned}$$

Thus,

$$\begin{aligned}
i\bar{\beta}\wedge\beta &\leq e^{2\psi}[(1+t^{-1})i\alpha\wedge\bar{\alpha} + (1+t)|f|^2 i\partial\psi\wedge\bar{\partial}\psi] \\
&\leq e^{2\psi}[(1+t^{-1})|\alpha|^2_{i\partial\bar{\partial}\psi} + (1+t)|f|^2|\bar{\partial}\psi|^2_{i\partial\bar{\partial}\psi}].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_\Omega |\alpha + f\bar{\partial}\psi|^2_{i\partial\bar{\partial}\psi} e^{\psi-\varphi} d\lambda \\
&\leq (1+t^{-1})\int_\Omega |\alpha|^2_{i\partial\bar{\partial}\psi} e^{\psi-\varphi} d\lambda + (1+t)\int_\Omega |f|^2|\bar{\partial}\psi|^2_{i\partial\bar{\partial}\psi} e^{\psi-\varphi} d\lambda.
\end{aligned}$$

By the assumption,

$$\begin{aligned}
&(1+t)\int_\Omega |f|^2|\bar{\partial}\psi|^2_{i\partial\bar{\partial}\psi} e^{\psi-\varphi} d\lambda \\
&\leq (1+t)\int_{Supp(\alpha)} |f|^2|\bar{\partial}\psi|^2_{i\partial\bar{\partial}\psi} e^{\psi-\varphi} d\lambda + \int_{\Omega\setminus Supp(\alpha)} |f|^2|\bar{\partial}\psi|^2_{i\partial\bar{\partial}\psi} e^{\psi-\varphi} d\lambda \\
&\leq \delta(1+t)\int_{Supp(\alpha)} |f|^2 e^{\psi-\varphi} d\lambda + \int_{\Omega\setminus Supp(\alpha)} |f|^2|\bar{\partial}\psi|^2_{i\partial\bar{\partial}\psi} e^{\psi-\varphi} d\lambda.
\end{aligned}$$

Combine above three inequalities, then

$$\begin{aligned}
&(1+t^{-1})\int_\Omega |\alpha|^2_{i\partial\bar{\partial}\psi} e^{\psi-\varphi} d\lambda. \\
&\geq \int_{\Omega\setminus Supp(\alpha)} (1-|\bar{\partial}\psi|^2_{i\partial\bar{\partial}\psi})|f|^2 e^{\psi-\varphi} d\lambda + (1-\delta(t+1))\int_{Supp(\alpha)} |f|^2 e^{\psi-\varphi} d\lambda \\
&\geq (1-\delta(t+1))\int_\Omega (1-|\bar{\partial}\psi|^2_{i\partial\bar{\partial}\psi})|f|^2 e^{\psi-\varphi} d\lambda.
\end{aligned}$$

The last inequality follows since $1-\delta(t+1)\leq 1$ and since $1-|\bar{\partial}\psi|^2_{i\partial\bar{\partial}\psi}\leq 1$. Take $t=\delta^{-1/2}-1$, then proof is done. $\square$

But this is not enough to show the coherence of adjooint ideal sheaves for any PSH function φ. Because our definition does not hold the assumption $|\bar{\partial}\psi|^2_{i\partial\bar{\partial}\psi} \leq 1$. Indeed, for a smooth hypersurface H, we choose local coordinates such that $H = (z = 0)$, locally. To apply the Berndsson's inequality, $\psi = -\log(\log^2|z|)$. Then

$$i\overline{\bar{\partial}\psi} \wedge \bar{\partial}\psi = i\frac{4}{|z|^2\log^2|z|}d\bar{z} \wedge dz$$

and

$$i\partial\bar{\partial}\psi \quad = \quad i\frac{1}{|z|^2\log^2|z|}d\bar{z} \wedge dz$$

Therefore,

$$i\bar{\partial}\psi \wedge \partial\psi \leq 4i\partial\bar{\partial}\psi.$$

Thus, $|\bar{\partial}\psi|^2_{i\partial\bar{\partial}\psi} > 1$, so our ψ does not satisfy the hypothesis.

Finally, we define analytic sheaves eased a condition.

Definition 2.2.2. Let φ be a PSH function on a complex manifold X and let H be a smooth hypersurface. Then $Adj_H^{*;\alpha}(\varphi)$ is defined by the sheaf of germs $u \in \mathcal{O}_{X,x}$ such that $|u|^2e^{-2\varphi}$ is integrable with respect to $\frac{1}{|h|^2(-\log|h|)^\alpha}Leb$, where h is a local equation for H and Leb is the Lebesgue measure. Then $Adj_H^\alpha(\phi)$ is defined by

$$Adj_H^\alpha(\varphi) := \bigcup_{\epsilon>0} Adj_H^{*;\alpha}\big((1+\epsilon)\varphi\big).$$

Theorem 2.2.3. *Let H be a smooth hypersurface in a smooth complex variety X and let D be a $\mathbb{Q}$-divisor on X such that H is not contained in Supp(D). Let $\varphi := \varphi_D$ be a corresponding PSH function to D. Then if $2 \geq \alpha > 1$,*

$$Adj_H^\alpha(\varphi) = Adj_H(D).$$

Proof. Let $x \in X$. We may assume that x is 0 in our chart. We fix a log resolution $\mu : X' \to X$ with exceptional divisors $E_1, \ldots, E_m$ and E_{m+1} such that $\mu^*D+\mu^*H+Exc(\mu)$ has SNC support. For convenience, let $H'=E_{m+1}$. We use the following notation;

$$\mu^*D=\sum_{j=1}^{m}a_jE_j$$

$$\mu^*H=H'+\sum_{j=1}^{m}b_jE_j$$

$$K_{X'}=\mu^*K_X+\sum_{j=1}^{m+1}c_jE_j.$$

Take a local generator x_{m+1} of H. Let $f\in\mathcal{O}_{X,x}$ defined on a sufficiently small neighborhood U of 0. Then, using the change of variables,

$$\int_U\frac{|f|^2e^{-2(1+\epsilon)\varphi}}{|x_{m+1}|^2(-\log|x_{m+1}|}dV=\int_{U'}\frac{|f\circ\mu|^2e^{-2(1+\epsilon)\varphi\circ\mu}}{|x_{m+1}\circ\mu|^2(-\log|x_{m+1}\circ\mu|)^{\alpha}}|J_\mu|^2dV',$$

where $U'=\mu^{-1}(U)$. Using the same argument for multiplier ideal sheaves, we only check the case when f is a monomial. Thus we may assume that $f\circ\mu=\prod z_j^{d_j}$. Then the right hand side is

$$\begin{aligned}&\int_{U'}\frac{\prod_{k=1}^{m+1}|z_k|^{2(c_k+d_k-(1+\epsilon)a_k)}}{|z_{m+1}|^2\prod_{k=1}^{m}|z_k|^{2b_k}(-\log(|z_{m+1}|\prod_{k=1}^{m}|z_k|^{b_k}))^{\alpha}}dV'\\=&\int_{U'}\frac{\prod_{k=1}^{m+1}|z_k|^{2\lambda_k(\epsilon)}}{(-\log(|z_{m+1}|\prod_{k=1}^{m}|z_k|^{b_k}))^{\alpha}}dV',\end{aligned}$$

where we set $a_{m+1}=0, b_{m+1}=1$ and $\lambda_k(\epsilon)=c_k+d_k-b_k-(1+\epsilon)a_k$. Changing to polar coordinates gives the integral is

$$\int_V\frac{\prod_{k=1}^{m+1}w^{2\lambda_k(\epsilon)+1}}{(-\log(w_{m+1}\prod_{k=1}^{m}w_k^{b_k}))^{\alpha}}dw_1\cdots dw_{m+1},$$

where V is a neighborhood of 0 in $\mathbb{R}_+^{m+1}$. We may assume that $V\subset B(0,r)$, for some $r<1$. Using the usual criterion which determines integrability near 0 of $x^\beta(-\log x)^\alpha$, the integrability implies that $2\lambda_k(\epsilon)+1\geq -1$. For the converse, if $2\lambda_k(\epsilon)+1 \geq -1$,

$$\begin{aligned}
&\int_V \frac{\prod_{k=1}^{m+1} w^{2\lambda_k(\epsilon)+1}}{(-\log(w_{m+1}\prod_{k=1}^m w_k^{b_k}))^\alpha} dw_1\cdots dw_{m+1}\\
\leq &\int_V \frac{w_{m+1}^{-1}\prod_{k=1}^{m} w^{2\lambda_k(\epsilon)+1}}{(-\log(w_{m+1}\prod_{k=1}^m w_k^{b_k}))^\alpha} dw_1\cdots dw_{m+1}\\
\leq &\, C\int_{V'} \frac{\prod_{k=1}^{m} w^{2\lambda_k(\epsilon)+1}}{(-\log(\prod_{k=1}^m w_k^{b_k}))^{\alpha-1}} dw_1\cdots dw_{m}\\
< &+\infty,
\end{aligned}$$

where V' is a neighborhood of 0 in $\mathbb{R}^m_+$. Therefore the convergence is equivalent to $2\lambda_k(\epsilon)+1 \geq -1$, that is, $\lambda_k(\epsilon) \geq -1$. It is equivalent to

$$d_k \geq -c_k b_k + [(1+\epsilon)a_k] = -(c_k - b_k - [a_k]),$$

for sufficiently small $\epsilon > 0$. Therefore the integrability is equivalent to $f \in \mu^*\mathcal{O}_{X'}(K_{X'/X} - [\mu^*D] - \mu^*H + H')$. The proof is done. $\square$

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국문초록

먼저, 우리는 다수조화함수의 특이점을 측정하는 승수아이디얼층의 개념에 대해 공부하고 승수아이디얼층의 일관성을 증명하게 해주는 회르만더의 추정에 대해 살펴본다. 그리고 승수아이디얼의 유사체인 수반아이디얼층에대해 소개하고 그것의 일관성 증명에 대한 접근법을 제안한다.

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