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이학석사 학위논문

An Analysis of the Simplest Mixed Finite Element Method for the Elastic Wave Equation

(탄성 파동 방정식을 위한 가장 간단한 혼합 유한 요소 방법에 대한 분석)

2015년 2월

서울대학교 대학원

수리과학부

박 소 윤

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An Analysis of the Simplest Mixed Finite Element Method for the Elastic Wave Equation

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Master of Science
to the faculty of the Graduate School of
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by

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Abstract

Although it is well-known that there are various mixed finite elements for solving elastic wave equation, in this paper, we will approach the elastic wave equation with 2D, the simplest mixed finite element method. The superiority of the family of elements over the existing elements is its simplicity and high accuracy. It satisfies the discrete inf-sup condition for the stability analysis and has convergence property of the consistency error. In this paper, by using this mixed finite element method, we will get the approximated solution of the elastic wave equation, and also prove that this approximated solution stably converges to real solution through elliptic projection.

Key words: mixed finite element, elastic wave equation
Student Number: 2013-20232

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Chapter 1

Introduction

In this paper, we analyze a family of mixed finite elements for the elastic wave equation. Developing the mixed formulation involving both velocity and pressure, not the traditional displacement formulation, we change the variation form of the elastic wave eqution into the first order system, which is very familiar to us. This method can be also applied to the more complicated elastic wave equation with a free boundary condition, but for the simplicity of the implementation and proof, we assumed the dirichlet boundary condition.
The initial condition of the system is determined by the elliptic projection, which is inherited from the Becache, Joly, Tsogka, [5]. They devolop the mixed formulation come from the variation formula for solving elliptic problem. This method shows that the stationary problem associated to the evolution problem gives the mixed approximation.

The mixed finite element they analyze in [5] is $Q_{k+1}^{div} - Q_k$. However, since the classical analysis does not fit for this mixed finite element(especially coercivity condition does not satisfied), they had to develop the new, nonclassical error estimates.

To overcome this difficulty, we use here the minimal, any-space dimensional, symmetric, nonconforming mixed finite element studied in 2013, by Jun Hu, Hongying Man, and Shangyou Zhang [4]. The mixed finite element they proposed is the simplest rectangular element with 7 stress and 2 displacement degrees of freedom on each rectangle.

This element was motivated by a simple fact that the derivative on a normal stress component σ_{ii} is only in x_i direction; while those on σ_{ij} are only in x_i and x_j directions. The minimal finite element space for σ_{ii} would be span$\{1, x_i\}$ on each n-dimensional rectangular element; while the minimal finite element space for σ_{ij} would be span$\{1, x_i, x_j\}$ on each n-dimensional rectangular element. For the displacement, there is no derivative and the minimal finite element space is the constant space, span$\{1\}$. In [4], they prove that this minimal finite element spaces can actually form a family of stable and convergent methods for the pure displacement problem.

The obstacle they had to overcome was the symmetry constraint on the stress tensor, i.e., $\sigma_{ij} = \sigma_{ji}$. Because of this symmetry property, it is extremely difficult to construct stable conforming finite elements for the elliptic problem, even for 2D and 3D. So that, related this difficulty, efforts for developing composite elements or enforcing the symmetry condition weakly existed. In [4], they overcome this by giving an explicit constructive proof for the discrete inf-sup condition. So with this proof, we know the existance and uniquseness of solution of the elliptic problem. Doubtless, the superioriy of the family of elements over the existing elements is its simplicity and high accuracy.

In this paper, we are going to describe several aspects of the pure displacement problem and elastic wave equation. In Chaper2, we introduce the variation formula of the model problem and present some terminologies related to norms and inner products of these finite element spaces. And also we shall explain the basic notion of the simplest, two dimensional mixed finite element spaces we will use and related interpotion operator and its interpolation estimates, which plays an important role in convergence of cosistency error. In Chapter3, the well-posedness of the finite element problem, i.e., the discrete inf-sup condition, and another elliptic projection operator is presented. The rest of section is devoted to the error analysis of the pure displacement and the elastic wave eqauiton. Numerical results in 2D, including results for the elliptic projcetion, are provided in Chapter4, which show a convergence of the minimal element herein.

Chapter 2

Preliminaries

2.1 The model problem

Let Ω be a bounded rectangle of $\mathbb{R}^2$(it is straightforward that results can be extended to domains which can be covered by rectangles), which is subdivided by a family of rectangular grids $\mathcal{T}_h$(with grid size h).
We consider the system of equations governing the motion of a homogeneous, isotropic, liniearly elastic body consists of the stress equations of motion, Hooke's law and the strain-displacement relations :

$$\begin{aligned}
&\sigma_{ij,j} + f_i = \rho \ddot{u}_i, \\
&\sigma_{ij} = \lambda \epsilon_{kk} \delta ij + 2\mu \epsilon_{ij}, \\
&\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}).
\end{aligned} \tag{2.1.1}$$

If the strain-displacement relations are subsituted into Hooke's law and the expressions for the stresses are subsequently subsituted in the stress-equations of motion, we obtain the displacement equations of motion

$$\rho \ddot{u}_i - \{\mu u_{i,jj} + (\lambda + \mu) u_{j,ji}\} \ = \ f_i, \quad i = 1, 2$$

Alternatively, we can formulate the two–dimensional elastic wave equations in vector form:

$$\begin{cases} \text{Find } \mathbf{u} \ : \ [0,T] \mapsto (H_0^1(\Omega))^2 \text{ such that} \\ \rho \dfrac{\partial^2 \mathbf{u}}{\partial t^2} \ - \ \mathrm{div}(\mathbb{A}^{-1}(x)\boldsymbol{\epsilon}(\mathbf{u})) \ = \ \mathbf{f}, \qquad \mathbf{f} \in C^0(0,T;(L^2(\Omega))^2), \\ \text{where } \mathbb{A}^{-1}(x)\boldsymbol{\epsilon}(\mathbf{u}) = \lambda \ \mathrm{tr}(\boldsymbol{\epsilon}(\mathbf{u})) \ \mathbb{I} + 2\mu\boldsymbol{\epsilon}(\mathbf{u}) \end{cases} \tag{2.1.2}$$

subject to the initial conditions

$$\mathbf{u}(t=0) = \mathbf{u}_0 \in (H_0^1(\Omega))^2 \ ; \qquad \frac{\partial \mathbf{u}}{\partial t}(t=0) = \mathbf{u}_1 \in (L^2(\Omega))^2.$$

The homogeneous Dirichlet condition on $\partial\Omega$ has been considered for simplicity only.
Now, to present mixed variational Formula, let

$$\boldsymbol{\sigma} = \mathbb{A}^{-1}(x)\epsilon(\mathbf{u}) \text{ and } \mathbf{v} = \frac{\partial \mathbf{u}}{\partial t}.$$

Subsituting into (2.1.2) yields

$$\begin{cases} \rho \dfrac{\partial \mathbf{v}}{\partial t} \ - \ \mathrm{div}\boldsymbol{\sigma} = f, \\ \mathbb{A}\dfrac{\partial \boldsymbol{\sigma}}{\partial t} \ - \ \epsilon(\mathbf{v}) = 0, \end{cases} \tag{2.1.3}$$

with initial conditions

$$\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 = \mathbb{A}^{-1}(x)\epsilon(\mathbf{u}_0) \ ; \ \mathbf{v}(0) = \mathbf{v}_0 = \mathbf{u}_1. \tag{2.1.4}$$

A mixed weak formulation associated to (2.1.3) is given by the following problem:

$$\begin{cases} \text{Find } (\boldsymbol{\sigma}, \mathbf{v}) : [0,T] \ \mapsto \ X \times M \equiv H(div, \Omega, \mathbb{S}) \times \L^2(\Omega, \mathbb{R}^2) \text{ such that} \\ \\ \dfrac{d}{dt} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\mathbf{v}, \boldsymbol{\sigma}) = 0 \qquad \forall \boldsymbol{\tau} \in X, \\ \\ \dfrac{d}{dt} (\mathbf{v}, \mathbf{w}) - b(\mathbf{w}, \boldsymbol{\sigma}) = \langle \mathbf{f}, \mathbf{w} \rangle \qquad \forall \mathbf{w} \in M, \end{cases} \tag{2.1.5}$$

where

$$\begin{cases} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \displaystyle\int_\Omega \mathbb{A}(x) \ \boldsymbol{\sigma} : \boldsymbol{\tau} \ dx \quad \forall (\boldsymbol{\sigma}, \boldsymbol{\tau}) \in X \ \times X, \\ \\ b(\mathbf{w}, \boldsymbol{\tau}) = \displaystyle\int_\Omega \mathbf{w} \cdot \operatorname{div} \boldsymbol{\tau} \ dx \qquad \forall (\mathbf{w}, \boldsymbol{\tau}) \in M \ \times X, \\ \\ \langle \mathbf{f}, \mathbf{w} \rangle = \displaystyle\int_\Omega f \cdot \mathbf{w} \ dx \qquad\qquad \forall \mathbf{w} \in M. \end{cases} \tag{2.1.6}$$

and

$$\begin{cases} X = H(\operatorname{div}, \Omega, \mathbb{S}) = \{ (\sigma_{ij})_{n \times n} \in H(\operatorname{div}, \Omega) \mid \sigma_{ij} = \sigma_{ji} \} \\ \\ M = L^2(\Omega, \mathbb{R}^2) = \left\{ (u_1, u_2)^T \mid u_i \in L^2(\Omega) \right\} \end{cases} \tag{2.1.7}$$

The bilinear form $a(\cdot,\cdot)$ (resp., $b(\cdot,\cdot)$) is continuous on $H \times H (H = (L^2(\Omega))^2)$ (resp., on $X \times M$). The bilinear form $a(\cdot,\cdot)$ (resp., $b(\cdot,\cdot)$) thus defines a linear continuous operator $\mathbb{A} : H \mapsto H'$ by $\langle \mathbb{A}\boldsymbol{\sigma}, \boldsymbol{\tau} \rangle_{H' \times H} = a(\boldsymbol{\sigma}, \boldsymbol{\tau})$ (resp., $\mathcal{B} : X \mapsto M'$ *by* $\langle \mathcal{B}\boldsymbol{\tau}, \mathbf{w} \rangle_{M' \times M} = b(\mathbf{w}, \boldsymbol{\tau})$). They satisfy the following properties(see, for instance, [3]) :

$$
\left|
\begin{array}{l}
(i) \quad \text{The continuous inf-sup condition holds :} \\[2ex]
\exists c > 0 \ / \ \forall \mathbf{w} \in M, \exists \boldsymbol{\tau} \in X \ / \ b(\mathbf{w}, \boldsymbol{\tau}) \geq c \|\mathbf{w}\|_M \|\boldsymbol{\tau}\|_X . \\[2ex]
(ii) \quad \text{The coercivity of the form } a(\cdot,\cdot) \textit{ on } V \equiv \mathrm{Ker}B : \\[2ex]
\exists \alpha > 0 \ / \ \forall \boldsymbol{\sigma} \in V, a(\boldsymbol{\sigma}, \boldsymbol{\sigma}) \geq \alpha \|\boldsymbol{\sigma}\|_X^2 .
\end{array}
\right.
\tag{2.1.8}
$$

Hence the discrete problem associated to (2.1.6) and (2.1.4) is

$$
\begin{cases}
\text{Find } (\boldsymbol{\sigma}_h, \mathbf{v}_h) : [0,T] \ \mapsto \ X_h \times M_h \text{ such that} \\[2ex]
\dfrac{d}{dt} a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\mathbf{v}_h, \boldsymbol{\sigma}_h) = 0 \qquad \forall \boldsymbol{\tau}_h \in X_h, \\[2ex]
\dfrac{d}{dt} (\mathbf{v}_h, \mathbf{w}_h) - b(\mathbf{w}_h, \boldsymbol{\sigma}_h) = \langle \mathbf{f}, \mathbf{w}_h \rangle \qquad \forall \mathbf{w}_h \in M_h,
\end{cases}
\tag{2.1.9}
$$

subject to the initial condition

$$
\boldsymbol{\sigma}_h(0) = \boldsymbol{\sigma}_{0,h} \ ; \ \mathbf{v}_h(0) = \mathbf{v}_{1,h} \tag{2.1.10}
$$

In the following, we consider a pair of finite element spaces to solve (2.1.5).

2.2 A minimal element in 2D

The set of all edges in $\mathcal{T}_h$ is denoted by $\mathcal{E}_h$, which is divided into two sets, the set $\mathcal{E}_{h,H}$ of horizontal edge and the set $\mathcal{E}_{h,V}$ of vertical edges. Given any edge $e \in \mathcal{E}_h$, one fixed unit normal vector n with the components $(n1, n2)$ is assigned. For each $K \in \mathcal{T}_h$, define the affine invertible transformation

$$
F_k \ : \qquad \hat{K} \mapsto K,
$$

$$
\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{h_{x,K}}{2} \hat{x} + x_{0,K} \\ \frac{h_{y,K}}{2} \hat{y} + y_{0,K} \end{pmatrix},
$$

with the center $(x_{0,K}, y_{0,K})$ of K, the horizontal lengh $h_{x,K}$, and the vertical length $h_{y,K}$, and the reference element $\hat{K} = [-1,1]^2$.
On each element $K \in \mathcal{T}_h$, a constant finite element space for the displacement is defined by

$$M_K = \mathcal{P}_0(K, \mathbb{R}^2) = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mid v_1, v_2 \in P_0(K) \right\}; \tag{2.2.1}$$

while the symmetric linear finite element space for the stress is defined by

$$X_K = \left\{ \sigma \in \begin{pmatrix} P_{1,1}(K) & \mathcal{P}_1(K) \\ \mathcal{P}_1(K) & P_{1,2}(K) \end{pmatrix}_{\mathbb{S}} \right\}, \tag{2.2.2}$$

where subscript $\mathbb{S}$ indicates a symmetric matrix stress, and

$$\begin{aligned} &P_{1,1}(K) = \text{span}\{1, x\}, \\ &\mathcal{P}_1(K) = \text{span}\{1, x, y\} \\ &P_{1,2}(K) = \text{span}\{1, y\}. \end{aligned}$$

The dimenstion of the space M_K is 2, and that of X_K is 7. Locally $\mathcal{P}_1(K)$ is the space of linear polynomials. Globally, let W_h be the P_1-nonconforming space on $\mathcal{T}_h$, which is first introduced in [1] as a nonconforming approximation space to $H^1(\Omega)$ on the quadrilateral mesh; To be exact, $\mathbf{W}_h$ is the space of piecewise linear polynomials, which are continuous at all mid -edge points of triangulation $\mathcal{T}_h$. $\mathbf{W}_h$ is the finite element space approximating function σ_{12}.
The global space X_h and M_h are defined by

$$\begin{aligned} X_h = \{\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \in L^2(\Omega, \mathbb{S}) \mid \sigma|_K \in X_K \text{ for all } K \in \mathcal{T}_h, \quad & (2.2.3) \\ \sigma_{11} \text{ is continuous on all vertical interior edges,} \\ \sigma_{22} \text{ is continuous on all horizontal interior edges,} \\ \sigma_{12} \text{ is continuous at all mid-points of interior edges,} \quad \} \end{aligned}$$

$$M_h = \{\mathbf{v} \in L^2(\Omega, \mathbb{R}^2) \mid \mathbf{v}|_K \in V(K) \text{ for all } K \in \mathcal{T}_h\}. \tag{2.2.4}$$

The discrete stress space X_h is a nonconforming approximation to $H(div, \Omega, \mathcal{S})$. And the discrete divergence operator div_h is defined elementwise with respect to $\mathcal{T}_h$.

$$\mathrm{div}_h \boldsymbol{\sigma}|_K = \ \mathrm{div}(\boldsymbol{\sigma}|_K) \quad \forall \boldsymbol{\sigma} \in X_h.$$

Finally, the last part of this section is devoted to the interpolation operator $\Pi_h \sigma$ for any $\sigma \in H(\mathrm{div}, \Omega, \mathbb{S}) \ \cap \ H^2(\Omega, \mathbb{S})$ in order to analyze the approximation error in Chaper3. Define an interpolation(see [4])

$$\Pi_h \boldsymbol{\sigma} = \begin{pmatrix} \Pi_{11}\sigma_{11} & \Pi_{12}\sigma_{12} \\ \Pi_{12}\sigma_{12} & \Pi_{22}\sigma_{22} \end{pmatrix} \in X_h, \tag{2.2.5}$$

where Π_{11} and Π_{22} are standard, satisfying, respectively,

$$\int_e \Pi_{11}\sigma_{11}\, ds = \int_e \sigma_{11}\, ds \text{ for any vertical edge } e \in \mathcal{E}_h,$$

$$\int_e \Pi_{22}\sigma_{22}\, ds = \int_e \sigma_{22}\, ds \text{ for any horizontal edge } e \in \mathcal{E}_h.$$

Π_{12} is the interpolation operator from space $H^2(\Omega)$ to nonconforming finite element space.

$$\Pi_{12}\sigma_{12}(e_m) = \frac{1}{2}(\sigma_{12}(e_1) + \sigma_{12}(e_2)),$$

where for an edge $e \in \mathcal{E}_h$, e_1 and e_2 are two endpoints of e, and e_m is the mid-point of these two endpoints. It is shown by Park and Sheen [1] that

$$|\mathbf{v} - \Pi_{12}\mathbf{v}|_{m,K} \leq Ch^{2-m}|\mathbf{v}|_{2,K}, \quad m = 0, 1 \quad K \in \mathcal{T}_h \tag{2.2.6}$$

The convergence result of elastic wave equation will follow from below Lemma.

Lemma 2.2.1. ([4], Theorem 4.1) *For any* $\sigma \in H^2(\Omega, \mathbb{S})$*, it hold that*

$$\|(\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma})\|_0 \leq Ch\|\boldsymbol{\sigma}\|_2,$$
$$\|div_h(\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma})\|_0 \leq Ch\|\boldsymbol{\sigma}\|_2.$$

2.3 The ellptic problem

In order to analyze apporximation of the evolution problem, we need to use an abstract result for a class of ellptic problems posed in a more general framework. In this section, the mixed approximation of the elliptic problem which is in fact the stationary problem associated to the evolution problem will be presented.
Before presenting the new equation, we need to introduce Hibert space H. X and M are same as (2.1.7). Then $H = L^2(\Omega, \mathbb{R}^2)$ satisfies the following :

$$X \subset H, \quad |\cdot|_H \leq \|\cdot\|_X$$

since

$$\|\boldsymbol{\sigma}\|_X^2 = |\boldsymbol{\sigma}|_H^2 + |\text{div } \boldsymbol{\sigma}|_M^2 \tag{2.3.1}$$

Let $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ be two continuous bilinear forms in $H \times H$ and $M \times X$. In the same way as in (2.1.6), $a(\cdot,\cdot)$ defines an operator $\mathbb{A}$ in $\mathcal{L}(H)$, such that $a(\boldsymbol{\sigma},\boldsymbol{\tau}) = (\mathbb{A}\boldsymbol{\sigma},\boldsymbol{\tau})_H$ $\forall(\boldsymbol{\sigma},\boldsymbol{\tau}) \in H \times H$ and the bilinear form $b(\cdot,\cdot)$ defines an operatoer $B : X \mapsto M'$ (and its transpose $B^t : M \mapsto X'$) such that

$$\langle B\boldsymbol{\sigma}, \mathbf{w}\rangle_{M' \times M} = \left\langle \boldsymbol{\sigma}, B^t w\right\rangle_{X \times X'} = b(\mathbf{w}, \boldsymbol{\sigma}) \qquad \forall(\boldsymbol{\sigma}.\mathbf{w}) \in X \times M.$$

The kernels of B and B^t are defined as follows :

$$\begin{cases} V \equiv \mathrm{Ker}B = \{\boldsymbol{\sigma} \in X \ / \ b(\mathbf{w}, \boldsymbol{\sigma}) = 0 \ \forall \mathbf{w} \in M\} \\ \\ \mathrm{Ker}B^t = \{\mathbf{w} \in M \ / \ b(\mathbf{w}, \boldsymbol{\sigma}) = 0 \ \forall \boldsymbol{\sigma} \in X\} \end{cases}$$

This function spaces X, M, H satisfy the following two inequalities (2.1.8). Using (2.1.2), we know that

$$\mathbb{A}\boldsymbol{\sigma} = \frac{1}{2\mu}\boldsymbol{\sigma} - \frac{\lambda}{4\mu(\lambda+\mu)} \ \mathrm{tr}(\boldsymbol{\sigma}) \ I,$$

we can easily check that $\mathbb{A}(x)$ be a positive difinite symmetric operator satisfying

$$\mathbb{A}(x) \ \xi \cdot \xi \geq \left(\frac{1}{2\mu} - \frac{\lambda}{2\mu(\lambda+\mu)} \right) \cdot |\xi|_H^2, \quad \xi \in \mathbb{R}^4 \tag{2.3.2}$$

We can also check that operator $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ is bounded.

$$a(\boldsymbol{\sigma},\boldsymbol{\sigma}) = \frac{1}{2\mu}(\boldsymbol{\sigma},\boldsymbol{\sigma}) - \frac{\lambda}{4\mu(\lambda+\mu)}(\mathrm{tr}(\boldsymbol{\sigma}) \ I, \boldsymbol{\sigma}) \leq \left(\frac{1}{2\mu} + \frac{\lambda}{2\mu(\lambda+\mu)} \right) |\boldsymbol{\sigma}|_H^2, \tag{2.3.3}$$

$$b(\mathbf{w},\boldsymbol{\tau}) = (\mathbf{w}, \mathrm{div}\boldsymbol{\tau}) \leq \|\mathbf{w}\|_M \ \|\mathrm{div}\boldsymbol{\tau}\|_M \leq \|\mathbf{w}\|_M \|\boldsymbol{\tau}\|_M. \tag{2.3.4}$$

In this circumstance, the ellptic problem we consider here is pure displacement problem :

$$\begin{cases} \text{Find } \mathbf{u} \in H_0^1(\Omega, \mathbb{R}^2) \text{ such that} \\ \\ -\mathrm{div}(\mathbb{A}^{-1}(x)\epsilon(\mathbf{u})) \ = \ f, \qquad f \in L^2(\Omega, \mathbb{R}^2). \end{cases} \tag{2.3.5}$$

As for the time dependent problem, we set

$$\boldsymbol{\sigma} = \mathbb{A}^{-1}(x)\epsilon(\mathbf{u}),$$

and this gives

$$-\mathrm{div} \ \boldsymbol{\sigma} = f.$$

We are interested in the numerical approximation of the solution $(\boldsymbol{\sigma}, \mathbf{u})$ to the following problem :

$$\left\{\begin{array}{lll} \text{Find } (\boldsymbol{\sigma}, \mathbf{u}) \in X \times M \text{ such that} & & \\ a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\mathbf{u}, \boldsymbol{\tau}) = 0 & \forall \boldsymbol{\tau} \in X, & \\ b(\mathbf{w}, \boldsymbol{\sigma}) = -\langle \mathbf{f}, \mathbf{w} \rangle & \forall \mathbf{w} \in M. & \end{array}\right. \tag{2.3.6}$$

with $f \in M'$ the dual space of M.
Suppose $X_h \subset X$ and $M_h \subset M$ are finite dimensional approximation spaces. We consider then the apporximate problem

$$\left\{\begin{array}{ll} \text{Find } (\boldsymbol{\sigma}_h, \mathbf{u}_h) \in X_h \times M_h \text{ such that} & \\ a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\mathbf{u}_h, \boldsymbol{\tau}_h) = 0 & \forall \boldsymbol{\tau}_h \in X_h, \\ b(\mathbf{w}_h, \boldsymbol{\sigma}_h) = -\langle \mathbf{f}, \mathbf{w}_h \rangle & \forall \mathbf{w}_h \in M_h. \end{array}\right. \tag{2.3.7}$$

And finially we set

$$\left\{\begin{array}{l} V_h(f) = \{\boldsymbol{\tau}_h \in X_h \;/\; b(\mathbf{w}_h, \boldsymbol{\tau}_h) = -\langle \mathbf{f}, \mathbf{w}_h \rangle \;\; \forall \mathbf{w}_h \in M_h\}, \\ V_h = V_h(0) = \text{Ker } B_h = \{\boldsymbol{\tau}_h \in X_h \;/\; b(\mathbf{w}_h, \boldsymbol{\tau}_h) = 0 \; \forall \mathbf{w}_h \in M_h\} \end{array}\right. \tag{2.3.8}$$

Under these assumptions and the well-posedness of the problem (2.1.8), we have the following classical result (see [3]).

Theorem 2.3.1. *For all $f \in Im\ B$, problem (2.3.6) has a unique solution $(\boldsymbol{\sigma}, u)$ in $X \times M$. Moreover, $\|u\|_M + \|\boldsymbol{\sigma}\|_X \leq C\|f\|_{M'}$*

We need to check similar Theorem hold for (2.3.7). For this, inf-sup condition and the coercivity of discrete version should be presented, which will be continue in Chapter3.

Chapter 3

Analysis of the Elastic Wave Equation

3.1 Problem setting

In this chapter, we construct a system to solve the elastic wave equation using the mixed finite elements described in $Section 2.2$. We introduce some notations: $B_{N_1} = \{\boldsymbol{\sigma}_i\}_{i=1}^{N_1}$, $B_{N_2} = \{\phi_i\}_{i=1}^{N_2}$ to denote the bases of X_h and M_h, respectively, where $N_1 = \dim X_h$ and $N_2 = \dim M_h$. We denote by $[\boldsymbol{\Sigma}] = (\boldsymbol{\Sigma}_1, \cdots, \boldsymbol{\Sigma}_{N_1})$ and $[V] = (\mathbf{V}_1, \cdots, \mathbf{V}_{N_2})$ the coordinates of $\boldsymbol{\sigma}_h$ and $\mathbf{v}_h$ with respect to the bases B_{N_1} and B_{N_2}. Based on these bases, Problem (2.1.9) can be written in the following form:

$$\begin{cases} \text{Find } (\boldsymbol{\Sigma}, \mathbf{V}) \ : \ [0,T] \mapsto \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \text{ such that} \\ \\ \quad M_{\boldsymbol{\sigma}} \dfrac{d\boldsymbol{\Sigma}}{dt} + C^T \mathbf{V} = 0, \\ \\ \quad \rho M_v \dfrac{d\mathbf{V}}{dt} - C\boldsymbol{\Sigma} = F, \\ \\ \quad + \qquad \text{initial conditions} \end{cases} \tag{3.1.1}$$

with

$$\begin{cases} (i) \quad (M_{\boldsymbol{\sigma}})_{i,j} = (\mathbb{A}\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j) & 1 \le i, j \le N_1, \\ \\ (ii) \quad (M_v)_{i,j} = (\phi_i, \phi_j) & 1 \le i, j \le N_2, \\ \\ (iii) \quad (C)_{i,j} = (\phi_i, \mathrm{div}\ \boldsymbol{\sigma}_j) & 1 \le i \le N_2,\ 1 \le j \le N_1, \\ \\ (iv) \quad (F)_{i,j} = (f, \phi_j) & 1 \le j \le N_2. \end{cases} \tag{3.1.2}$$

C^T denotes the transpose of C. To solve this system, we use another new vector $[X] = (\mathbf{\Sigma}_1, \cdots, \mathbf{\Sigma}_{N_1}, \mathbf{V}_1, \cdots, \mathbf{V}_{N_2})$. Using this vector X, (3.1.1) becomes a system of ODE problem since matrix $M_{\boldsymbol{\sigma}},\ M_v,\ C$ are independant of time :

$$\begin{cases} \text{Find } X \ : \ [0,T] \mapsto \mathbb{R}^{N_1+N_2} \text{ such that} \\ \\ \begin{pmatrix} M_{\boldsymbol{\sigma}} & O \\ O & \rho M_v \end{pmatrix} \dot{X} = \begin{pmatrix} O & -C^T \\ C & O \end{pmatrix} X + \begin{pmatrix} O \\ F \end{pmatrix} \\ \\ \quad + \qquad \text{initial conditions} \end{cases} \tag{3.1.3}$$

Thus, we can use Euler method or Runge Kutta method to solve this systemf of ODE. From the properties of our mixed finite element space, M_v is identity matrix, and invese of matrix $M_{\boldsymbol{\sigma}}$ can be obtained by Congugate Gradient method.

3.2 Analysis of mixed finite element for an elliptic problem

In this section, we will see that the mixed approximation of the elliptic problem which is in fact the stationary problem associated to the evloution problem (2.1.2). Acutually, we gave in section 2.3 an preliminaries to anaylze ellpctic problem. To develope things in section 2.3, we need to check that our mixed finite element space satisfies some hypothises :

(H0) $\forall f \in \mathrm{Im}\ B$, $V_h(f) \neq \varnothing$.

(H1) Strong discrete uniform inf-sup condition :

- there exists a constant $c > 0$, independent of h, such that $\forall \mathbf{w}_h \in M_h$, $\exists \boldsymbol{\tau}_h \in X_h$ / $b(\mathbf{w}_h, \boldsymbol{\tau}_h) \geq c\|\mathbf{w}_h\|_M \|\boldsymbol{\tau}_h\|_X$,

(H2) "Weak" coercivity :

- there exists a constant $\alpha > 0$, independent of h, such that $\forall \boldsymbol{\sigma}_h \in \mathbf{V}_h$, $a(\boldsymbol{\sigma}_h, \boldsymbol{\sigma}_h) \geq \alpha \|\boldsymbol{\sigma}_h\|_X^2$.

Remark. For further study, We need to know the hypothesis (H2) is equivalent to

$$\begin{cases} \text{there exists a constant C} > 0\text{, independent of } h\text{, such that} \\ \\ \forall \boldsymbol{\tau}_h \in X_h,\ \sup\limits_{\mathbf{w}_h \in M_h} \dfrac{b(\mathbf{w}_h, \boldsymbol{\tau}_h)}{\|\mathbf{w}_h\|_M} \geq C \|\boldsymbol{\tau}_h\|_X. \end{cases}$$

Under these hypotheses we can get the following result :

Theorem 3.2.1. *Under the hypothesis (H0)-(H2), problem (2.3.7) admits a unique solution such that* $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in X_h \times M_h$*, and the following convergence result holds :* $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \to (\boldsymbol{\sigma}, \mathbf{u})$ *in* $X \times M$*. More precisely,*

$$\begin{aligned} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_X + \|\mathbf{u} - \mathbf{u}_h\|_M &\leq C \left(\inf_{\boldsymbol{\tau}_h \in X_h} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_X + \inf_{\mathbf{w}_h \in M_h} \|\mathbf{u} - \mathbf{w}_h\|_M \right) \\ &\leq Ch \left(\|\boldsymbol{\sigma}\|_2 + \|\mathbf{u}\|_2 \right) \end{aligned} \tag{3.2.1}$$

Let us begin by checking hypothesis (H0). Hypothesis (H0) is equivalent to the following statement :

$$\mathrm{Ker} B_h^t \cap M_h \subset \mathrm{Ker} B^t$$

Since $\mathrm{Ker}B_h^t = \mathrm{Ker}B^t = 0$ we can easily check (H0). For (H1), a constructive proof is adopted.(see [4]) For convenience, suppose that the domain Ω is a unit square $[0,1]^2$ which is triagulated evenly into N^2 elements, K_{ij}. For any $\mathbf{v} \in M_h$, it can be decomposed as a sum,

$$\mathbf{v}_h = \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{V}_{ij} \phi_{ij}(x,y),$$

where $\phi_{ij}(x)$ is the characteristic function on the element K_{ij}, and $\mathbf{V}_{ij} = (\mathbf{V}_{1,ij}, \mathbf{V}_{2,ij})$. we can construct a discrete stress function $\boldsymbol{\sigma}_h \in X_h$ with

$$\mathrm{div}_h \boldsymbol{\sigma}_h = \mathbf{v}_h \text{ and } \|\boldsymbol{\sigma}_h\|_X \leq C \|\mathbf{v}_h\|_M$$

The key of this construction is the shear stress $\boldsymbol{\sigma}_{12}$ can be taken zero, i.e., $\boldsymbol{\sigma}_{12} \equiv 0$; Only by using the degree of normal stress $\boldsymbol{\sigma}_{11}$(resp.,$\boldsymbol{\sigma}_{12}$) of $\boldsymbol{\sigma}_h$, we can make any $\mathbf{v}_h$ in M_h. More precisely, $\boldsymbol{\sigma}_{11}$(resp.,$\boldsymbol{\sigma}_{12}$) is a continuous piecewise linear function of the variable x (resp.,y) and a piecewise constant function of y (resp.,x). Therefore, in the following form :

$$\boldsymbol{\sigma}_{11}(x,y) = h \sum_{m=1}^{i-1} V_{1,mj} + V_{1,ij}(x - x_{i-1}),$$

$$\boldsymbol{\sigma}_{22}(x,y) = h \sum_{k=1}^{j-1} V_{2,kj} + V_{2,ij}(y - y_{j-1}),$$

for $x_{i-1} \leq x \leq x_i$ and $y_{j-1} \leq y \leq y_j$ ((x_i, y_j) is the upper-right corner vertex of square K_{ij}.) Thus, define

$$\boldsymbol{\sigma}_h = \begin{pmatrix} \boldsymbol{\sigma}_{11} & O \\ O & \boldsymbol{\sigma}_{22} \end{pmatrix} \in X_h$$

By this construction, $\boldsymbol{\sigma}_x \boldsymbol{\sigma}_{11} = (\mathbf{v}_h)_1$ and
$bsig_y \boldsymbol{\sigma}_{22} = (\mathbf{v}_h)_2$. This gives

$$\mathrm{div}_h = \mathbf{v}_h.$$

For the important inequality, an elementary calculation gives,

$$\|\mathbf{v}_h\|_M^2 = \sum_{i,j=1}^{N} \|V_{ij}\phi_{ij}\|_{M,K_{ij}}^2 = \sum_{i,j=1}^{N} \int_{K_{ij}} |V_{ij}\phi_{ij}|^2 dxdy$$

$$= \sum_{i,j=1}^{N} ((V_{1,ij})^2 + (V_{2,ij})^2)h^2$$

By the Schwarz inequality,

$$\|\boldsymbol{\sigma}_{11}\|_0^2 = \textstyle\sum_{i,j=1}^{N} \displaystyle\int_{K_{ij}} \left(\sum_{m=1}^{i-1} V_{1,mj} + V_{1,ij}(x - x_{i-1}) \right)^2 dxdy$$

$$\leq \textstyle\sum_{i,j=1}^{N} \displaystyle\int_{K_{ij}} \left(h^2 \sum_{m=1}^{i-1} V_{1,mj}^2 + V_{1,ij}^2 (x - x_{i-1})^2 \right) \cdot i \; dxdy$$

since, $N = 1/h$, $x - x_{i-1} \leq h$ and $\int_{K_{ij}} = h^2$,

$$\|\boldsymbol{\sigma}_{11}\|_0^2 \leq \sum_{i,j=1}^{N} \left(h^2 \sum_{m=1}^{i} (V_{1,mj})^2 \right) \cdot Nh^2 \leq \sum_{j=1}^{N} \left(h^2 \sum_{m=1}^{N} (V_{1,mj})^2 \right) \cdot N^2 h^2$$

$$= \sum_{i,j=1}^{N} h^2 (V_{1,ij})^2.$$

Similarly,

$$\|\boldsymbol{\sigma}_{22}\|_0^2 \leq \sum_{i,j=1}^{N} h^2 (V_{2,ij})^2$$

The combination of the aforementioned two identities and two inequalities yields

$$\|\boldsymbol{\sigma}_h\|_X^2 = \|\boldsymbol{\sigma}_h\|_0^2 + \|\mathrm{div}_h \boldsymbol{\sigma}_h\|_0^2$$

$$= \|\boldsymbol{\sigma}_{11}\|_0^2 + \|\boldsymbol{\sigma}_{22}\|_0^2 + \|\mathbf{v}_h\|_0^2 \leq 2\|\mathbf{v}_h\|_0^2$$

Hence, for any $\mathbf{v}_h \in \mathbf{V}_h$, hypothesis (H1) holds with $C = 1/\sqrt{2}$

$$\inf_{\mathbf{v}_h \in \mathbf{V}_h} \sup_{\boldsymbol{\sigma}_h \in X_h} \frac{(\mathrm{div}_h \boldsymbol{\sigma}_h, \mathbf{v}_h)}{\|\boldsymbol{\sigma}_h\|_X \|\mathbf{v}_h\|_M} \geq \inf_{\mathbf{v}_h \in \mathbf{V}_h} \frac{\|\mathbf{v}_h\|_M^2}{\sqrt{2}\|\mathbf{v}_h\|_M^2} = \frac{1}{\sqrt{2}}.$$

this completes the proof.
Now, for checking (H1), from (2.3.8) we immediately know that $\mathbf{V}_h$ is a strong discrete divergence-free space from definition. i.e.,:

$$\mathbf{V}_h = \{\boldsymbol{\tau}_h \in X_h \ / \ b(\mathbf{w}_h, \boldsymbol{\tau}_h) = 0 \ \forall \mathbf{w}_h \in M_h\}$$

$$= \{\boldsymbol{\tau}_h \in X_h \ / \ \mathrm{div}_h \boldsymbol{\tau}_h = 0 \text{ pointwise}\}.$$

So that if $\boldsymbol{\sigma}_h \in \mathbf{V}_h$, $\|\boldsymbol{\sigma}_h\|_X = |\boldsymbol{\sigma}_h|_H$, and from ellipticity of bilinear form $a(\cdot,\cdot)$ in (2.3.2), $a(\boldsymbol{\sigma}_h, \boldsymbol{\sigma}_h) \geq \alpha \|\boldsymbol{\sigma}_h\|_X^2$ holds for all $\boldsymbol{\sigma}_h$ in $\mathbf{V}_h$. Now our mixed finite element satisfied all hypothesis from (H0) to (H2).
Before presenting the proof of Theorem 3.2.1, we need a Lemma below.

Lemma 3.2.2. *For $\boldsymbol{\sigma}$, the solution of elliptic problem (2.3.6), the following inequality holds :*

$$\inf_{\boldsymbol{\tau}_h \in \mathbf{V}_h(f)} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_X \leq C \inf_{\boldsymbol{\tau}_h \in X_h} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_X$$

Proof. suppose that $\gamma_h \in X_h$ satisfies $\|\boldsymbol{\sigma} - \gamma_h\|_X = \inf_{\boldsymbol{\tau}_h \in X_h} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_X$. Then, for given $\gamma_h \in X_h$, there exists $\boldsymbol{\sigma}_h \in X_h$ such that (cf.[3])

$$b(\mathbf{v}_h, \gamma_h) = b(\mathbf{v}_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \quad \forall \mathbf{v}_h \in M_h,$$

which means that $\gamma_h + \boldsymbol{\sigma}_h \in \mathbf{V}_h(f)$ and,

$$\inf_{\boldsymbol{\tau}_h \in \mathbf{V}_h(f)} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_X \leq \|\boldsymbol{\sigma} - (\gamma_h + \boldsymbol{\sigma}_h)\|_X,$$

From the equivalent form of hypothesis (H1) in Remark 3.2,

$$\left\{ \begin{array}{l} \|\boldsymbol{\sigma}_h\|_X \leq C \sup\limits_{\mathbf{w}_h \in M_h} \dfrac{b(\mathbf{w}_h, \boldsymbol{\sigma}_h)}{\|\mathbf{w}_h\|_M} = C \sup\limits_{\mathbf{w}_h \in M_h} \dfrac{b(\mathbf{w}_h, \boldsymbol{\sigma} - \gamma_h)}{\|\mathbf{w}_h\|_M} \\ \\ \qquad \leq C \ \|\mathrm{div}_h(\boldsymbol{\sigma} - \gamma_h)\|_M \leq C \ \|\boldsymbol{\sigma} - \gamma_h\|_X \end{array} \right.$$

Finally from the triagular inequality, the lemma is proved. □

Now we are ready to prove Theorem 3.2.1.

Proof. The hypothesis (H0), and ellpticity (H1), and descrete inf-sup condition (H2) ensures that the existence and uniqueness of the solution $(\boldsymbol{\sigma}_h, \mathbf{u}_h)$ in $X_h \times M_h$.
The second equation of (2.3.7) means that $\boldsymbol{\sigma}_h \in \mathbf{V}_h(f)$. If we also take $\boldsymbol{\tau}_h \in \mathbf{V}_h(f)$, the difference is the in kernel, i.e., $\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h \in \mathbf{V}_h$. We will divide $\boldsymbol{\sigma} - \boldsymbol{\sigma}_h$ into two parts using $\boldsymbol{\tau}_h \in \mathbf{V}_h(f)$

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_X \leq \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_X + \|\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h\|_X$$

Then, the second part of right side is effectively bounded. To show this, observe that

$$a(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h, \boldsymbol{\sigma}_h - \boldsymbol{\tau}_h) = a(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}, \boldsymbol{\sigma}_h - \boldsymbol{\tau}_h) + a(\boldsymbol{\sigma} - \boldsymbol{\tau}_h, \boldsymbol{\sigma}_h - \boldsymbol{\tau}_h) \quad (3.2.2)$$

substraction First equation of (2.3.7) from (2.3.6) yields,

$$a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\sigma}_h - \boldsymbol{\tau}_h) + b(\mathbf{u} - \mathbf{u}_h, \boldsymbol{\sigma}_h - \boldsymbol{\tau}_h) = 0 \quad (3.2.3)$$

and the difference between the second equation of (2.3.7) and (2.3.6) yields,

$$\begin{aligned} b(\mathbf{u} - \mathbf{u}_h, \boldsymbol{\sigma}_h - \boldsymbol{\tau}_h) &= b(\mathbf{u} - \mathbf{w}_h, \boldsymbol{\sigma}_h - \boldsymbol{\tau}_h) + b(\mathbf{w}_h - \mathbf{u}_h, \boldsymbol{\sigma}_h - \boldsymbol{\tau}_h) \quad \forall \mathbf{w}_h \in M_h \\ &= b(\mathbf{u} - \mathbf{w}_h, \boldsymbol{\sigma}_h - \boldsymbol{\tau}_h) \ (\because \boldsymbol{\sigma}_h - \boldsymbol{\tau}_h \in \mathbf{V}_h) \end{aligned} \quad (3.2.4)$$

so that by combining above equations we obtain,

$$a(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h, \boldsymbol{\sigma}_h - \boldsymbol{\tau}_h) = b(\mathbf{u} - \mathbf{w}_h, \boldsymbol{\sigma}_h - \boldsymbol{\tau}_h) + a(\boldsymbol{\sigma} - \boldsymbol{\tau}_h, \boldsymbol{\sigma}_h - \boldsymbol{\tau}_h) \quad (3.2.5)$$

By the V-ellipticity of $a(\cdot, \cdot)$ in hypothesis (H2), (3.2.5) leads to

$$\alpha \|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\|_X^2 \leq (\|b\| \|\mathbf{u} - \mathbf{w}_h\|_M |\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h|_H + \|a\| |\boldsymbol{\sigma} - \boldsymbol{\tau}_h|_H |\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h|_H)$$

At this point, we have a difficulty related to $\boldsymbol{\tau}_h \in \mathbf{V}_h(f)$ since we want to take any $\boldsymbol{\tau}_h \in X_h$. To solve this problem, we need to recall in Lemma 3.2.2, the inf-sup condition (H1) implies

$$\inf_{\boldsymbol{\tau}_h \in \mathbf{V}_h(f)} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_X \leq C \inf_{\boldsymbol{\tau}_h \in X_h} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_X \tag{3.2.6}$$

And this gives using (2.3.1) ,

$$\|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\|_X \leq C \left(\inf_{\mathbf{w}_h \in M_h} \|\mathbf{u} - \mathbf{w}_h\|_M + \inf_{\boldsymbol{\tau}_h \in X_h} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_X \right)$$

Finally, it remains to prove estimates for $\|\mathbf{u} - \mathbf{u}_h\|_M$. Similar to the case of stress part error estimate, divide $\mathbf{u} - \mathbf{u}_h$ into two parts using $\mathbf{w}_h \in M_h$

$$\|\mathbf{u} - \mathbf{u}_h\|_M \leq \|\mathbf{u} - \mathbf{w}_h\|_M + \|\mathbf{w}_h - \mathbf{u}_h\|_M$$

Let us subtract the first equation of (2.3.7) from that of (2.3.6). We get

$$a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\mathbf{u} - \mathbf{u}_h, \boldsymbol{\tau}_h) = 0 \quad \forall \boldsymbol{\tau}_h \in X_h. \tag{3.2.7}$$

so that, for any $\mathbf{w}_h \in M_h$,

$$b(\mathbf{u}_h - \mathbf{w}_h, \boldsymbol{\tau}_h) = a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\mathbf{u} - \mathbf{w}_h, \boldsymbol{\tau}_h) \quad \forall \boldsymbol{\tau}_h \in X_h. \tag{3.2.8}$$

Using this and the inf-sup condition in Remark 3.2,

$$\begin{aligned}
\|\mathbf{w}_h - \mathbf{u}_h\|_M &\leq C \sup_{\boldsymbol{\tau}_h \in X_h} \frac{b(\mathbf{u}_h - \mathbf{w}_h, \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_X} \\
&\leq C \sup_{\boldsymbol{\tau}_h \in X_h} \frac{a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\mathbf{u} - \mathbf{w}_h, \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_X} \\
&\leq C \, (\|a\| |\boldsymbol{\sigma} - \boldsymbol{\sigma}_h|_H + \|b\| \|\mathbf{u} - \mathbf{w}_h\|_M).
\end{aligned}$$

It follows from triangle inequality,

$$\|\mathbf{u}-\mathbf{u}_h\|_M \leq C\left\{\inf_{\mathbf{w}_h\in M_h}\|\mathbf{u}-\mathbf{w}_h\|_M + |\boldsymbol{\sigma}-\boldsymbol{\sigma}_h|_H\right\}$$

If we choose $\mathbf{w}_h = P_h\mathbf{u}$ (P_h is the L^2 projection into piecewise constant spaces), from Lemma 2.2.1 and the fact that

$$\|\mathbf{u}-P_h\mathbf{u}\|_M \leq Ch|\mathbf{u}|_2,$$

We obtain the last inequality of the theorem. □

3.3 Application to the elliptic projection operator

In this section, we present another interpolation of $(\boldsymbol{\sigma},\mathbf{u}) \in X\times M$ i.e.,finding $(\hat{\boldsymbol{\sigma}}_h,\hat{\mathbf{u}}_h) \equiv \Pi_h(\boldsymbol{\sigma},\mathbf{u})$ such that

$$\begin{cases} a(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}_h,\boldsymbol{\tau}_h) + b(\mathbf{u}-\hat{\mathbf{u}}_h,\boldsymbol{\tau}_h) = 0 & \forall \boldsymbol{\tau}_h \in X_h \\ \\ b(\mathbf{w}_h,\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}_h) = 0 & \forall \mathbf{w}_h \in M_h \end{cases} \tag{3.3.1}$$

Note that, especially, if

$$a(\boldsymbol{\sigma},\boldsymbol{\tau}_h) + b(\mathbf{u},\boldsymbol{\tau}_h) = 0 \quad \forall \boldsymbol{\tau}_h \in X_h$$

$(\hat{\boldsymbol{\sigma}}_h,\hat{\mathbf{u}}_h)$is a solution of elliptic problem. From the discussion in Section 3.2, problem (3.3.1) admits an unique solution $(\hat{\boldsymbol{\sigma}}_h,\hat{\mathbf{u}}_h)$ and error estimate of Theorem 3.2.1 exactly holds for this interporlation. Then Let us introduce the notation here,

$$\|(\boldsymbol{\sigma}, \mathbf{u}) - \Pi_h(\boldsymbol{\sigma}, \mathbf{u})\|_C = \|\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}_h}\|_X + \|\mathbf{u} - \hat{\mathbf{u}}_h\|_M$$

$$\mathcal{E}_h(\boldsymbol{\sigma}, \mathbf{u}) = \inf_{\boldsymbol{\tau}_h \in X_h} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_X + \inf_{\mathbf{w}_h \in M_h} \|\mathbf{u} - \mathbf{w}_h\|_M$$

With these notations, we can conclude that

$$\|(\boldsymbol{\sigma}, \mathbf{u}) - \Pi_h(\boldsymbol{\sigma}, \mathbf{u})\|_C \leq C\ \mathcal{E}_h(\boldsymbol{\sigma}, \mathbf{u})$$

3.4 Analysis of mixed finite element for an elastic wave equation

Let us come back to the initial elastic wave eqution (2.1.5) (2.1.9). In this section we will see how we can relate the error estimates to the one obtained for the elliptic problem (2.3.6) (2.3.7). In this part, we use the same notation and hypothesis as in Section 2.1 and we use new notation here $\mathcal{C}^{m,r} = C^m(0,T;H) \cap C^r(0,T;X)$. Recall the approximation problem :

$$\begin{cases} \text{Find } (\boldsymbol{\sigma}, \mathbf{v}) : [0,T] \ \mapsto \ X_h \times M_h \text{ such that} \\ \\ \dfrac{d}{dt} a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\mathbf{v}_h, \boldsymbol{\sigma}_h) = 0 \qquad \forall \boldsymbol{\tau}_h \in X_h, \\ \\ \dfrac{d}{dt}(\mathbf{v}_h, \mathbf{w}_h) - b(\mathbf{w}_h, \boldsymbol{\sigma}_h) = (f, \mathbf{w}_h) \qquad \forall \mathbf{w}_h \in M_h, \end{cases} \tag{3.4.1}$$

As in Section3.3, finite element dimensional spaces satisfying hypothesis (H0) tho (H2). From the classical theory of ODE, we have the following result.

Theorem 3.4.1. *If $f \in C^0(0,T;M_h)$, then problem (3.4.1) has a unique solution $(\boldsymbol{\sigma}_h, \mathbf{v}_h) \in C^1(0,T;X_h) \cap C^1(0,T;M_h)$.*

By application of elliptic projection operator, we get the following results.

Lemma 3.4.2. *Let* $(\boldsymbol{\sigma}, \mathbf{v})$ *be the solution of* (2.1.5) *and assume that* $(\boldsymbol{\sigma}, \mathbf{v}) \in \mathcal{C}^{1,0} \times C^1(0,T;M)$ *Then we have the following :*
(i) There exists a primitive of $\mathbf{v}$, $\mathbf{u} \in C^1(0,T;M)$, *satisfying*

$$\frac{d\mathbf{u}}{dt} = \mathbf{v}, \quad a(\boldsymbol{\sigma}_0, \boldsymbol{\tau}) + b(\mathbf{u}(0), \boldsymbol{\tau}) = 0 \qquad \forall \boldsymbol{\tau} \in X \tag{3.4.2}$$

(ii) $\forall t \in [0,T]$, *problem (3.4.1) admits an unique solution* $\Pi_h(\boldsymbol{\sigma}, \mathbf{u})(t) = (\hat{\boldsymbol{\sigma}}_h, \hat{\mathbf{u}}_h)(t) \in X_h \times M_h$ *and there exists a constant* C *independent of* h *such that*

$$\|(\boldsymbol{\sigma}, \mathbf{u}) - \Pi_h(\boldsymbol{\sigma}, \mathbf{u})\|_C(t) \leq C\ \mathcal{E}_h(\boldsymbol{\sigma}, \mathbf{u})(t) \tag{3.4.3}$$

(iii) In the same way, if $(\boldsymbol{\sigma}, \mathbf{u}) \in C^k(0,T;X) \times C^k(0,T;M), k \geq 1$, *there exists a constant* C *indepedent of* h *such that*

$$\|(\partial_t^k \boldsymbol{\sigma}, \partial_t^k \mathbf{u}) - \Pi_h(\partial_t^k \boldsymbol{\sigma}, \partial_t^k \mathbf{u})\|_C(t) \leq C\ \mathcal{E}_h(\partial_t^k \boldsymbol{\sigma}, \partial_t^k \mathbf{u})(t) \tag{3.4.4}$$

Remark. Operators Π_h and ∂_t^k commute, and we set

$$\hat{\mathbf{v}}_h = \partial_t(\hat{\mathbf{u}}_h) = (\widehat{\partial_h \mathbf{u}})_h$$

Proof. (i)We set $f_0 = -B\boldsymbol{\sigma}_0 \in Im\ B$. From hypothesis (H0)-(H2), we know that there is a unique $(\boldsymbol{\sigma}_0, \mathbf{u}_0) \in X \times M$ such that

$$\left\{ \begin{array}{ll} a(\boldsymbol{\sigma}_0, \boldsymbol{\tau}) + b(\mathbf{u}, \boldsymbol{\tau}) = 0 & \forall \boldsymbol{\tau} \in X \\ \\ b(\mathbf{w}, \boldsymbol{\sigma}) = -(f_0, \mathbf{w}) & \forall \mathbf{w} \in M, \end{array} \right. \tag{3.4.5}$$

which means that, $\boldsymbol{\sigma}_0$ being fixed, there is a unique $\mathbf{u}_0 \in M$ such that $a(\boldsymbol{\sigma}_0, \boldsymbol{\tau}) + b(\mathbf{u}_0, \boldsymbol{\tau}) = 0 \quad \forall \boldsymbol{\tau} \in X$. Now we define $\mathbf{u}$ as

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}(s)\, ds$$

It is clear that $\mathbf{u} \in C^1(0,T;M)$ and is the unique solution.
(ii) Let $\mathbf{u} \in C^1(0,T;M)$ be the primitive of $\mathbf{v}$; substituting this into the first equation of model problem gives

$$\frac{d}{dt}(a(\boldsymbol{\sigma},\boldsymbol{\tau}) + b(\mathbf{u},\boldsymbol{\tau})) = 0$$

$$\therefore\ a(\boldsymbol{\sigma}(t),\boldsymbol{\tau}) + b(\mathbf{u}(t),\boldsymbol{\tau})) = a(\boldsymbol{\sigma}_0,\boldsymbol{\tau}) + b(\mathbf{u}(0),\boldsymbol{\tau})) = 0$$

thus $(\boldsymbol{\sigma},\mathbf{u}) \in C^1(0,T;X) \times C^1(0,T;M)$ satisfies

$$a(\boldsymbol{\sigma},\boldsymbol{\tau}) + b(\mathbf{u},\boldsymbol{\tau}) = 0 \quad \forall \boldsymbol{\tau} \in X,$$

we get the existence and uniqueness of the elliptic problem, for t fixed, and also the error estimate (3.4.3).
(iii)If $(\boldsymbol{\sigma},\mathbf{u})$ is sufficiently regular in time, we can defferentiate with respect to t and get

$$a(\partial_t^k\boldsymbol{\sigma},\boldsymbol{\tau}) + b(\partial_t^k\mathbf{u},\boldsymbol{\tau}) = 0 \quad \forall \boldsymbol{\tau} \in X,$$

Similar to the proof of (i), (ii) above, we get the error estimate (3.4.6). □

Now we give the main result.

Theorem 3.4.3. *Assume (H0)-(H2), let $(\boldsymbol{\sigma},\mathbf{v})$ be the solution of the model problem. and $(\boldsymbol{\sigma}_h,\mathbf{v}_h)$ the solution of the approximation problem (3.4.1) with the initial conditions*

$$(\boldsymbol{\sigma}_0,h,\mathbf{v}_0,h) = \Pi_h(\boldsymbol{\sigma}_0,\mathbf{v}_0).$$

If $(\boldsymbol{\sigma},\mathbf{v}) \in C^2(0,T;X) \times C^1(0,T;M)$, then, $\forall t \in [0,T]$,

$$|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h|_H(t) \to 0; \quad \|\mathbf{v} - \mathbf{v}_h\|_M \to 0.$$

$$\begin{cases} |\boldsymbol{\sigma} - \boldsymbol{\sigma}_h|_H(t) \leq C \left(\mathcal{E}_h(\boldsymbol{\sigma}, \mathbf{u})(t) + \int_0^t \mathcal{E}_h(\partial_t^2 \boldsymbol{\sigma}, \partial_t \mathbf{v})(t)\; ds \right) \\ \\ \|\mathbf{v} - \mathbf{v}_h\|_M(t) \leq C \left(\mathcal{E}_h(\partial_t \boldsymbol{\sigma}, \mathbf{v})(t) + \int_0^t \mathcal{E}_h(\partial_t^2 \boldsymbol{\sigma}, \partial_t \mathbf{v})(s)\; ds \right) \end{cases} \tag{3.4.6}$$

If, in addition, $(\boldsymbol{\sigma}, \mathbf{v}) \in C^3(0,T;X) \times C^2(0,T;M)$ *and* $(\boldsymbol{\sigma}_h, \mathbf{v}_h) \in C^2(0,T;X_h) \times C^2(0,T;M_h)$

$$\forall t \in [0,T],\; \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_X(t) \to 0;$$

More precisely,

$$\begin{cases} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_X(t) \leq C\; (\mathcal{E}_h(\boldsymbol{\sigma}, \mathbf{u})(t) + \mathcal{E}_h(\partial_t^2 \boldsymbol{\sigma}, \partial_t \mathbf{v})(t) \\ \\ \qquad\qquad + \int_0^t \mathcal{E}_h(\partial_t^2 \boldsymbol{\sigma}, \partial_t \mathbf{v})(s) + \mathcal{E}_h(\partial_t^3 \boldsymbol{\sigma}, \partial_t^2 \mathbf{v})(s)\; ds). \end{cases} \tag{3.4.7}$$

Proof. Now here shows that why we defined elliptic operator here. Divide error into two part :

$$\begin{aligned} |\boldsymbol{\sigma} - \boldsymbol{\sigma}_h|_H(t) &\leq |\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h|_H(t) + |\hat{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h|_H(t) \\ \|\mathbf{v} - \mathbf{v}_h\|_M(t) &\leq \|\mathbf{v} - \hat{\mathbf{v}}_h\|_M(t) + \|\hat{\mathbf{v}}_h - \mathbf{v}_h\|_M(t) \end{aligned} \tag{3.4.8}$$

As we see in the Section 3.3, the first parts of error is bounded by applying elliptic interpolation error estimate.

$$\|\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h\|_X(t) + \|\mathbf{v} - \hat{\mathbf{v}}_h\|_M(t) \leq C\; \mathcal{E}_h(\boldsymbol{\sigma}, \mathbf{u})(t); \tag{3.4.9}$$

For the second parts, we start to observe the followings, substracting (3.4.1) from original variation formula (2.1.5), (for the simplicity of proof, let $\rho = 1$)

$$\begin{cases} \dfrac{d}{dt}a(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h)+b(\mathbf{v}-\mathbf{v}_h,\boldsymbol{\tau}_h)=0 \qquad \forall\boldsymbol{\tau}_h\in X_h, \\ \dfrac{d}{dt}(\mathbf{v}-\mathbf{v}_h,\mathbf{w}_h)-b(\mathbf{w}_h,\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)=0 \qquad \forall\mathbf{w}_h\in M_h, \\ (\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)(0)=\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0,h; \qquad (\mathbf{v}-\mathbf{v}_h)(0)=\mathbf{v}_0-\mathbf{v}_0,h; \end{cases} \tag{3.4.10}$$

Note that since we choose as approximate initial conditions the elliptic projection of the exact initial condition, so that at $t=0$,

$$(\hat{\boldsymbol{\sigma}}_h-\boldsymbol{\sigma}_h)(0)=0; \qquad (\hat{\mathbf{v}}_h-\mathbf{v}_h)(0)=0; \tag{3.4.11}$$

By (3.4.10), for any $(\boldsymbol{\tau}_h,\boldsymbol{\sigma}_h)\in X_h\times M_h$,

$$\begin{cases} a(\partial_t(\hat{\boldsymbol{\sigma}}_h-\boldsymbol{\sigma}_h),\boldsymbol{\tau}_h)+b(\hat{\mathbf{v}}_h-\mathbf{v}_h,\boldsymbol{\tau}_h)=-a(\partial_t(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}_h,\boldsymbol{\tau}_h)-b(\mathbf{v}-\hat{\mathbf{v}}_h,\boldsymbol{\tau}_h) \\ (\partial_t(\hat{\mathbf{v}}_h-\mathbf{v}_h),\mathbf{w}_h)-b(\mathbf{w}_h,\hat{\boldsymbol{\sigma}}_h-\boldsymbol{\sigma}_h)=-(\partial_t(\mathbf{v}-\hat{\mathbf{v}}_h,\mathbf{w}_h),\mathbf{w}_h)+b(\mathbf{w}_h,\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}_h) \end{cases} \tag{3.4.12}$$

By differentiating the first equation of (3.3.1), we see that

$$\begin{cases} a(\partial_t(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}_h),\boldsymbol{\tau}_h)+b(\mathbf{v}-\hat{\mathbf{v}}_h,\boldsymbol{\tau}_h)=0 \quad \forall\boldsymbol{\tau}_h\in X_h \\ b(\mathbf{w}_h,\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}_h)=0 \quad \forall\mathbf{w}_h\in M_h \end{cases} \tag{3.4.13}$$

Substituting into (3.4.12) gives, for any $(\boldsymbol{\tau}_h,\boldsymbol{\sigma}_h)\in X_h\times M_h$,

$$\begin{cases} a(\partial_t(\hat{\boldsymbol{\sigma}}_h-\boldsymbol{\sigma}_h),\boldsymbol{\tau}_h)+b(\hat{\mathbf{v}}_h-\mathbf{v}_h,\boldsymbol{\tau}_h)=0 \\ (\partial_t(\hat{\mathbf{v}}_h-\mathbf{v}_h),\mathbf{w}_h)-b(\mathbf{w}_h,\hat{\boldsymbol{\sigma}}_h-\boldsymbol{\sigma}_h)=-(\partial_t(\mathbf{v}-\hat{\mathbf{v}}_h),\mathbf{w}_h) \end{cases} \tag{3.4.14}$$

Furthermore, by taking $\boldsymbol{\tau}_h=\hat{\boldsymbol{\sigma}}_h-\boldsymbol{\sigma}_h$ and $\mathbf{w}_h=\hat{\mathbf{v}}_h-\mathbf{v}_h$ in (3.4.14) and by adding the two equations, we get

$$a(\partial_t(\hat{\boldsymbol{\sigma}}_h-\boldsymbol{\sigma}_h),\hat{\boldsymbol{\sigma}}_h-\boldsymbol{\sigma}_h))+(\partial_t(\hat{\mathbf{v}}_h-\mathbf{v}_h),\hat{\mathbf{v}}_h-\mathbf{v}_h)=-(\partial_t(\mathbf{v}-\hat{\mathbf{v}}_h,\hat{\mathbf{v}}_h-\mathbf{v}_h) \tag{3.4.15}$$

Next, set

$$E_h(t) = \frac{1}{2}(a(\hat{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h, \hat{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h)) + (\hat{\mathbf{v}}_h - \mathbf{v}_h, \hat{\mathbf{v}}_h - \mathbf{v}_h))(t).$$

Since for some constant $C > 0$,

$$E_h^{1/2}(t) \geq C\ (|\hat{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h|_H^2(t) + \|\hat{\mathbf{v}}_h - \mathbf{v}_h\|_M^2(t))^{1/2}.$$

and

$$\frac{dE_h^{1/2}}{dt}(t) \leq C\|\partial_t(\mathbf{v} - \hat{\mathbf{v}}_h)\|_M(t)$$

So that we obtain, from $E_h(0) = 0$

$$|\hat{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h|_H(t) + \|\hat{\mathbf{v}}_h - \mathbf{v}_h\|_M(t) \leq C\int_0^t \|\partial_t(\mathbf{v} - \hat{\mathbf{v}}_h)\|_M(t)\ ds. \tag{3.4.16}$$

Here we use Lemma 3.4.2 (iii) for $k = 2$, which requires $(\boldsymbol{\sigma}, \mathbf{v}) \in C^2(0, T; X) \times C^1(0, T; M)$. We get

$$\|\partial_t(\mathbf{v} - \hat{\mathbf{v}}_h)\|_M(t) \leq C\ \mathcal{E}_h(\partial_t^2\boldsymbol{\sigma}, \partial_t\mathbf{v})(t); \tag{3.4.17}$$

Hence, from (3.4.9), (3.4.16) and (3.4.17) , the first inequality of (3.4.6) is proved.
Now for $\mathbf{v}$, we apply Lemma 3.4.2 (iii) for $k = 1$ and get

$$\|\mathbf{v} - \hat{\mathbf{v}}_h\|_M(t) \leq C\ \mathcal{E}_h(\partial_t\boldsymbol{\sigma}, \mathbf{v})(t); \tag{3.4.18}$$

Now, from (3.4.9), (3.4.16) and (3.4.18), the second inequality of (3.4.6) is proved.

To obtain the inequality of (3.4.7), we start by recalling that inf-sup condition is equivalent to Remark 3.2 :

$$\begin{cases} \text{there exists a constant C} > 0, \text{ independent of } h, \text{ such that} \\ \\ \forall \boldsymbol{\tau}_h \in X_h, \ \sup\limits_{\mathbf{w}_h \in M_h} \dfrac{b(\mathbf{w}_h, \boldsymbol{\tau}_h)}{\|\mathbf{w}_h\|_M} \geq C\|\boldsymbol{\tau}_h\|_X \end{cases}$$

Set $\boldsymbol{\tau}_h = \hat{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h$. Then using above equivalent inequality and the second equation of (3.4.15) we obtain,

$$\|\hat{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h\|_X \leq C \left\{\|\partial_t(\mathbf{v} - \hat{\mathbf{v}}_h)\|_M + \|\partial_t(\hat{\mathbf{v}}_h - \mathbf{v}_h)\|_M\right\} \tag{3.4.19}$$

In order to bound $\|\partial_t(\hat{\mathbf{v}}_h - \mathbf{v}_h)\|_M$, we need C^2. Indeed we want to apply Lemma 3.4.2 (iii) for k=2, and do same things similar to prodedure for (3.4.16) with $\mathbf{v}_h$ replaced by $\partial_t \mathbf{v}_h$, $\hat{\mathbf{v}}_h$ by $\partial_t \hat{\mathbf{v}}_h$, and so on. More precisely, we have

$$\|\partial_t(\hat{\mathbf{v}}_h - \mathbf{v}_h)\|_M(t) \leq C \int_0^t \|\partial_t^2(\mathbf{v} - \hat{\mathbf{v}}_h)\|_M(s)\, ds \tag{3.4.20}$$

Finially, combining (3.4.19) and (3.4.20), we get

$$\|\hat{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h\|_X \leq C \left\{\|\partial_t(\mathbf{v} - \hat{\mathbf{v}}_h)\|_M + \int_0^t \|\partial_t^2(\mathbf{v} - \hat{\mathbf{v}}_h)\|_M(s)\, ds\right\} \tag{3.4.21}$$

The rest of proof need to show the bound of second derivative of $\mathbf{v} - \hat{\mathbf{v}}_h$. We thus use estimate Lemma 3.4.2 (iii) for k=3, which requires $(\boldsymbol{\sigma}, \mathbf{v}) \in C^3(0, T; X) \times C^2(0, T; M)$ and we get (3.4.7). $\square$

Chapter 4

Numerical Result

4.1 Numerical Result for elliptic problem

The followings are two examples in 2D elliptic problem in Section 2 (2.3.5). Since the material is isotropic in the sense that

$$\mathbb{A}\sigma = \frac{1}{2\mu}\left(\sigma - \frac{\lambda}{2\mu+2\lambda}\mathrm{tr}(\sigma)\,\delta\right)$$

where μ and λ are the Lamé constants such that $0 < \mu_1 < \mu < \mu_2$ and $0 < \lambda < \infty$.
Let the solution on the unit square $[0,1]^2$ be

$$\mathbf{u} = \begin{pmatrix} e^{x-y}x(1-x)y(1-y) \\ \sin(\pi x)\sin(\pi y) \end{pmatrix} \tag{4.1.1}$$

and

$$\mathbf{u} = \begin{pmatrix} x^2(1-x)^2y^2(1-y)^2 \\ -x^2(1-x)^2y^2(1-y)^2 \end{pmatrix} \tag{4.1.2}$$

the parameters μ and λ are chosen as

$$\mu = \frac{1}{2}, \quad \lambda = 1$$

	$\lvert\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\rvert_H$	h^n	$\lVert\mathrm{div}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\rVert_M$	h^n	$\lVert\mathbf{u}-\mathbf{u}_h\rVert_M$	h^n
1	1.72600	0.0	8.51300	0.0	0.30430	0.0
2	0.82350	1.1	4.56700	0.9	0.15900	0.9
3	0.40450	1.0	2.32500	1.0	0.08020	1.0
4	0.20130	1.0	1.16800	1.0	0.04017	1.0
5	0.10050	1.0	0.05846	1.0	0.02010	1.0

Table 4.1: The error and the order of the convergence, for (4.1.1)

	$\lvert\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\rvert_H$	h^n	$\lVert\mathrm{div}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\rVert_M$	h^n	$\lVert\mathbf{u}-\mathbf{u}_h\rVert_M$	h^n
1	0.01758	0.0	0.11290	0.0	0.02264	0.0
2	0.06918	1.3	0.06891	0.7	0.00947	1.3
3	0.00290	1.3	0.03613	0.9	0.00419	1.2
4	0.00136	1.1	0.01829	1.0	0.00201	1.1
5	0.00067	1.0	0.00917	1.0	0.00100	1.0

Table 4.2: The error and the order of the convergence, for (4.1.2)

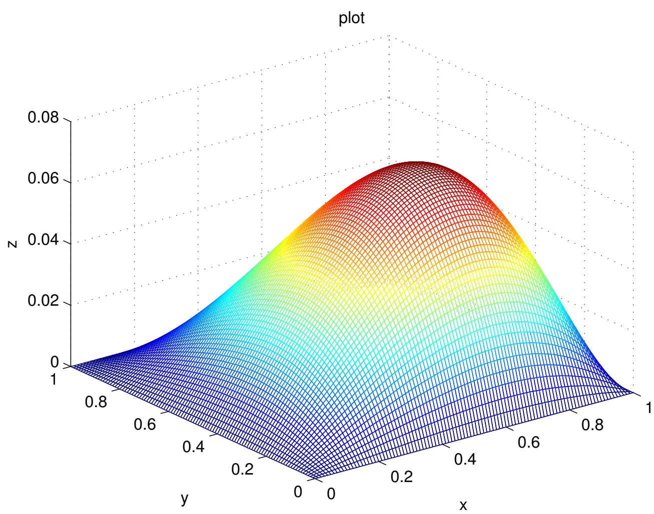
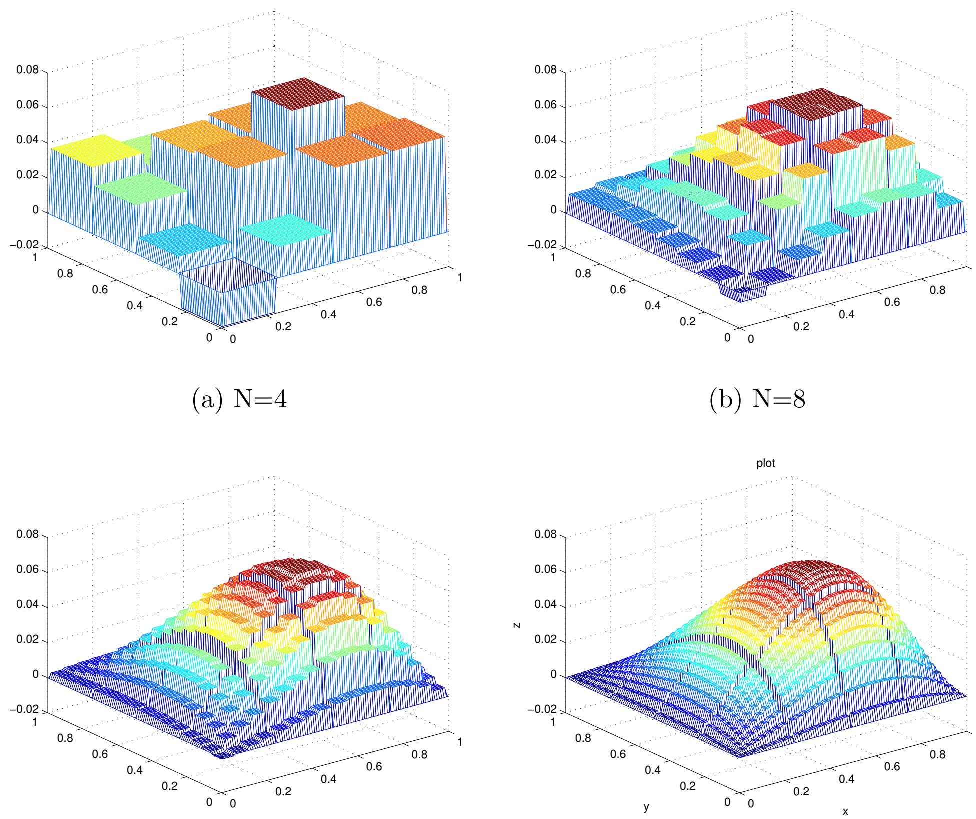


Figure 4.1.1: The first component of displacement, for (4.1.1)



(a) N=4

(b) N=8

(c) N=16

(d) N=32

Figure 4.1.2: The first component of displacement, for (4.1.1), N=4,8,16,32

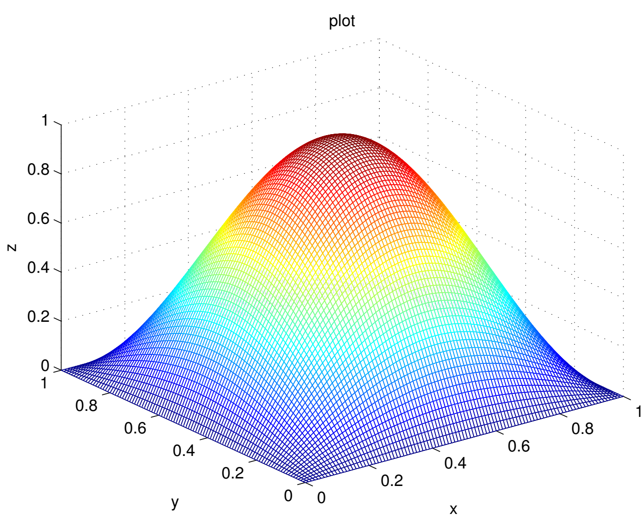
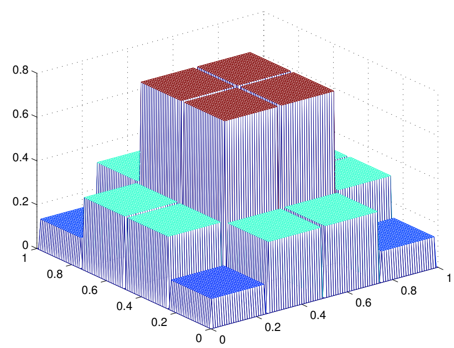
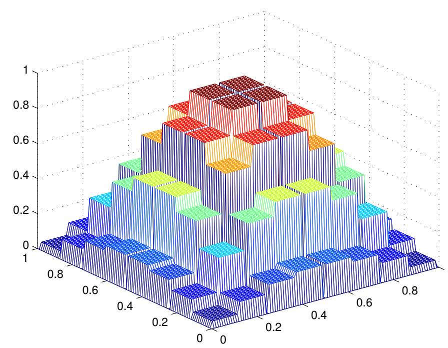


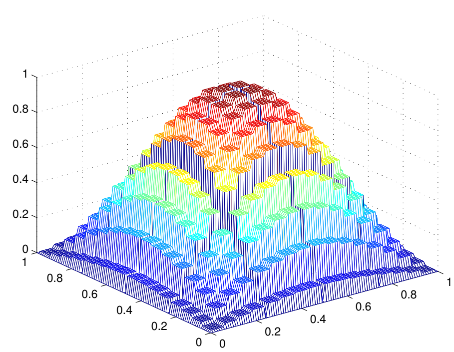
Figure 4.1.3: The second component of displacement, for (4.1.1)



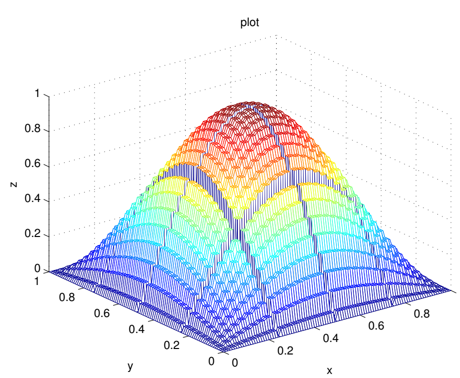
(a) N=4



(b) N=8



(c) N=16



(d) N=32

Figure 4.1.4: The second component of displacement, for (4.1.1), N=4,8,16,32

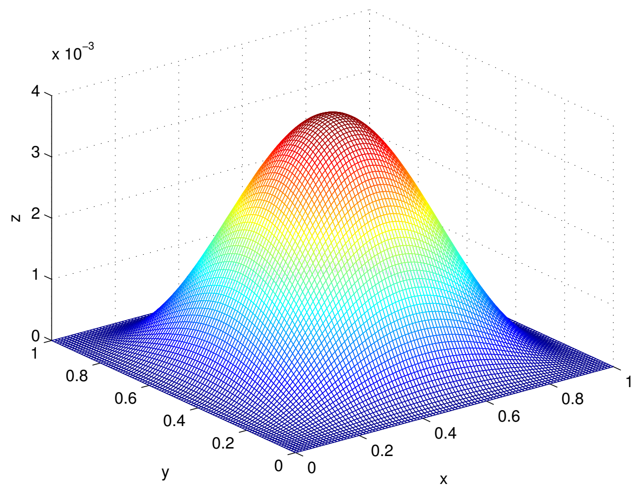
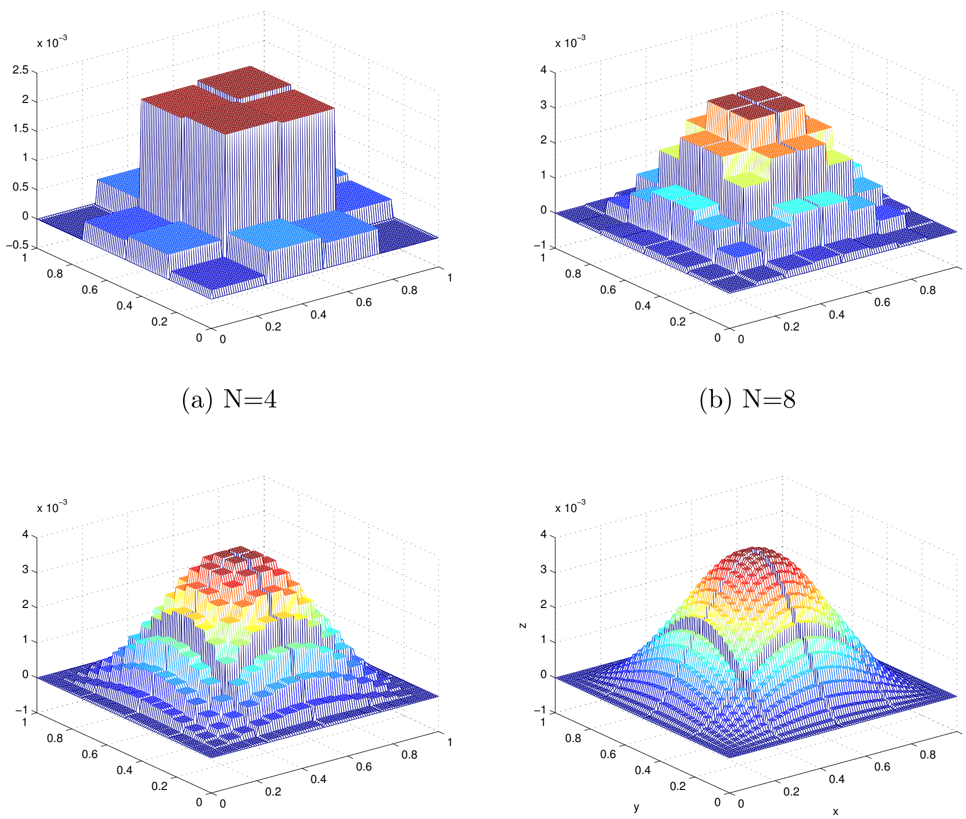


Figure 4.1.5: The first component of displacement, for (4.1.2)



(a) N=4

(b) N=8

(c) N=16

(d) N=32

Figure 4.1.6: The first component of displacement, for (4.1.2), N=4,8,16,32

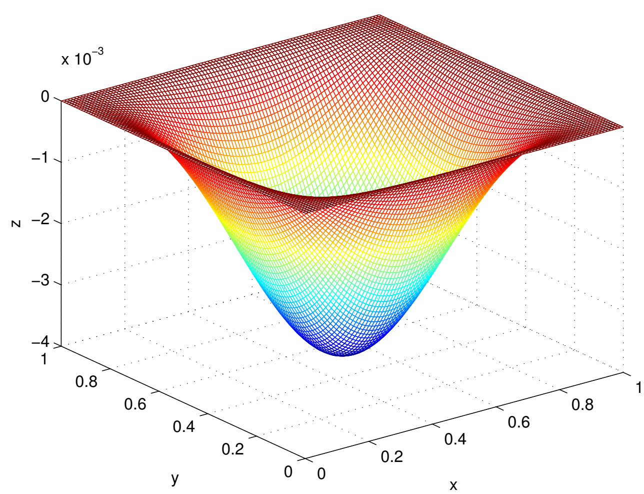
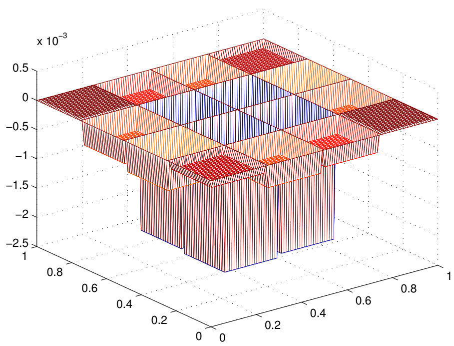
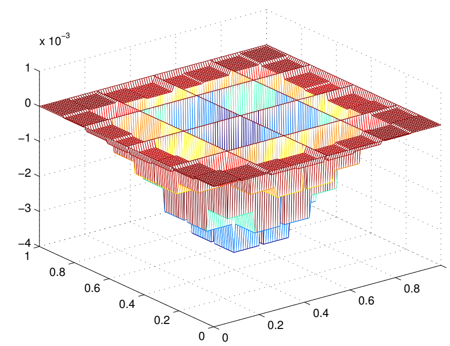


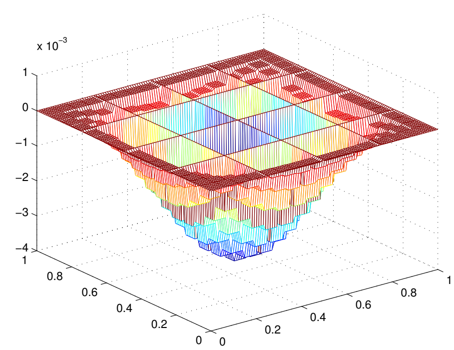
Figure 4.1.7: The second component of displacement, for (4.1.2)



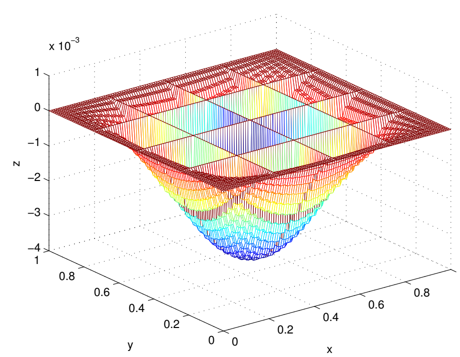
(a) N=4



(b) N=8



(c) N=16



(d) N=32

Figure 4.1.8: The second component of displacement, for (4.1.2), N=4,8,16,32

	$\lvert\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\rvert_H+\lVert\mathbf{v}-\mathbf{v}_h\rVert_M$	h^n	$\lVert\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\rVert_X$	h^n
2	0.7825	0.0	0.8021	0.0
3	0.5387E-02	7.2	0.1078	2.9
4	0.2266E-03	1.3	0.5650E-01	0.9
5	0.1065E-02	1.0	0.2860E-01	1.0
6	0.5231E-03	1.0	0.1434E-01	1.0

Table 4.3: The error and the order of the convergence, for (4.2.1)

4.2 Numerical Result for elastic wave equation

In this Section, we shall present the numerical result for elastic wave equation (2.1.5). The parameters ρ, μ and λ are chosen as

$$\rho = 1, \quad \mu = \frac{1}{2}, \quad \lambda = 1$$

and the exact solutions we use here are,

$$\mathbf{u} = \begin{pmatrix} 100(t^2+1)(x-0.25)^2(x-0.75)^2(y-0.25)^2(y-0.75)^2 \\ -100(t^2+1)(x-0.25)^2(x-0.75)^2(y-0.25)^2(y-0.75)^2 \end{pmatrix} \tag{4.2.1}$$

$$\text{for } 0.25 \le x \le 0.75,\ 0.25 \le y \le 0.75,\ \text{else } u = 0,$$

$$\text{where } 0 \le t \le 1$$

$$\mathbf{u} = \begin{pmatrix} 4\sin(t)x(1-x)y(1-y) \\ -4\sin(t)x(1-x)y(1-y) \end{pmatrix} \tag{4.2.2}$$

$$\text{where } 0 \le t \le 1$$

	$\lvert\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\rvert_H+\lVert\mathbf{v}-\mathbf{v}_h\rVert_M$	h^n	$\lVert\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\rVert_X$	h^n
2	0.8233	0.0	1.4140	0.0
3	0.1302	2.7	0.5984	1.2
4	0.6520E-01	1.0	0.3003	1.0
5	0.3260E-01	1.0	0.1526	1.0
6	0.1632E-01	1.0	0.7540E-01	1.0

Table 4.4: The error and the order of the convergence, for (4.2.2)

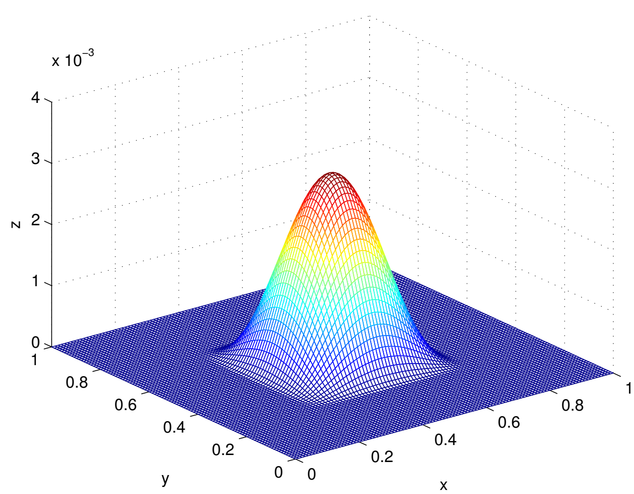
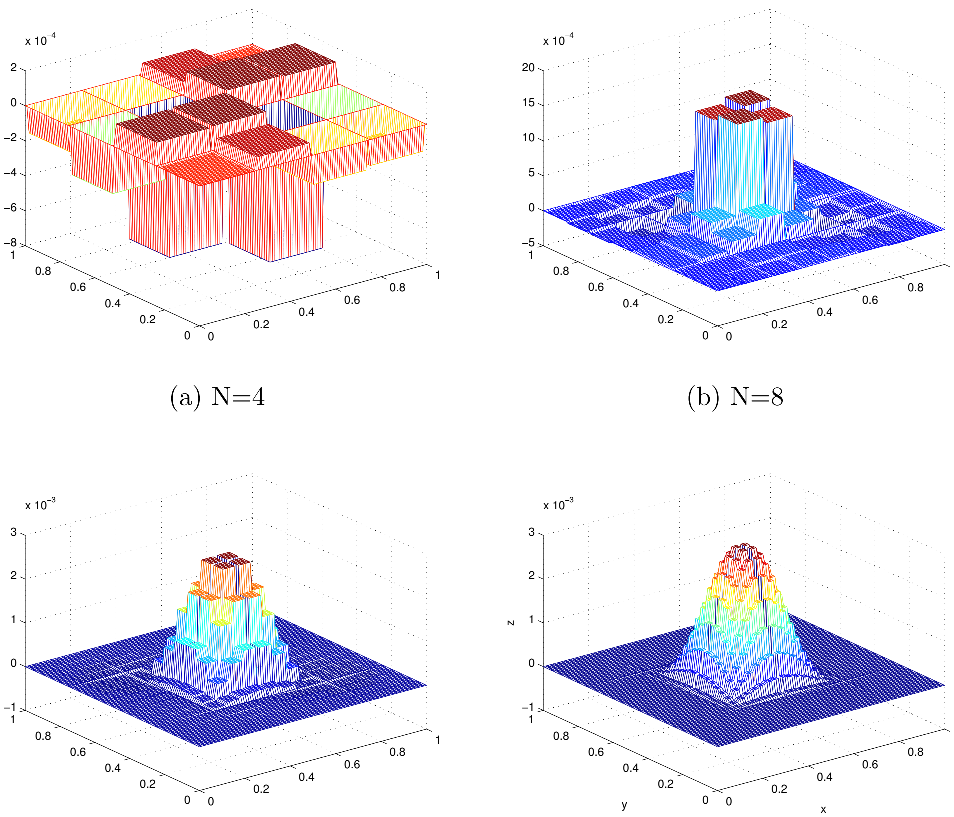


Figure 4.2.9: The first component of t-derivative of displacement, for (4.2.1)



(a) N=4

(b) N=8

(c) N=16

(d) N=32

Figure 4.2.10: The first component of t-derivative of displacement, for (4.2.1), N=4,8,16,32

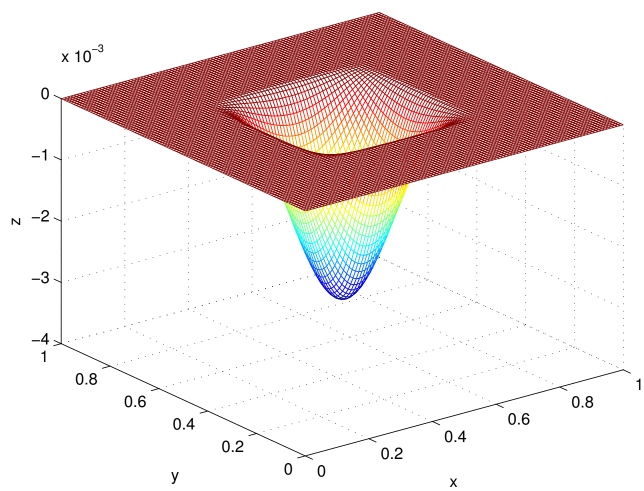
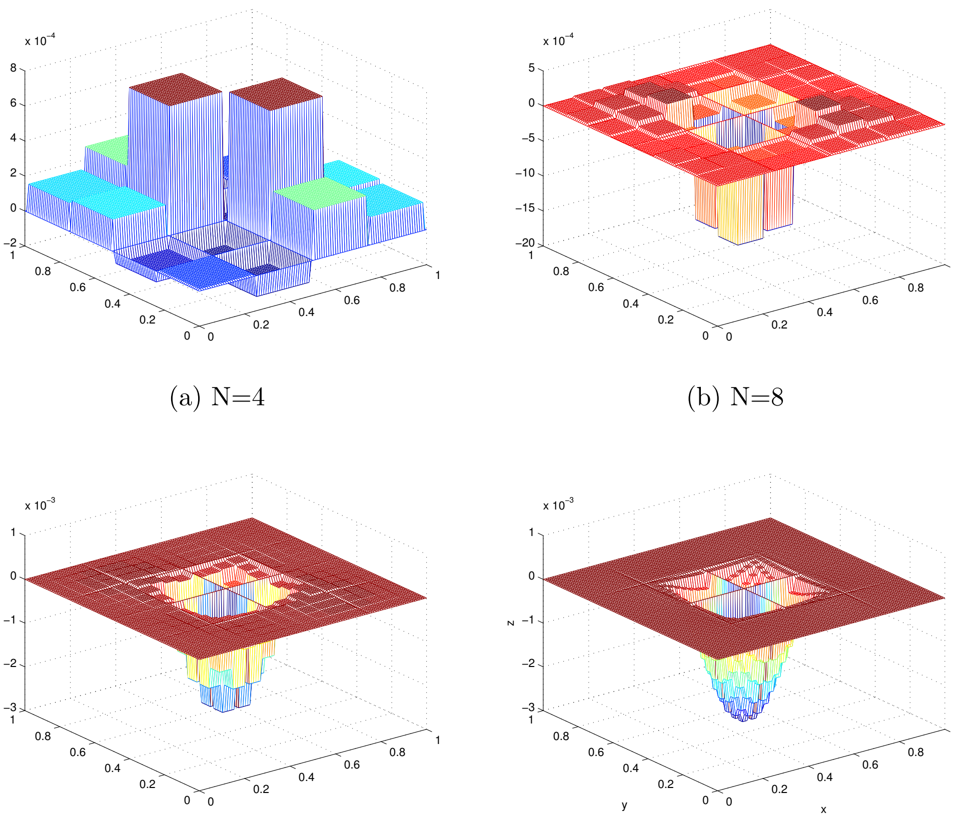


Figure 4.2.11: The second component of t-derivative of displacement, for (4.2.1)



(a) N=4

(b) N=8

(c) N=16

(d) N=32

Figure 4.2.12: The second component of t-derivative of displacement, for (4.2.1), N=4,8,16,32

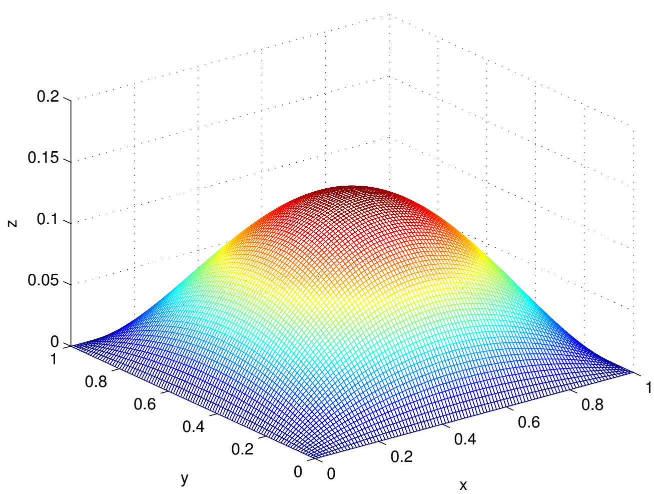
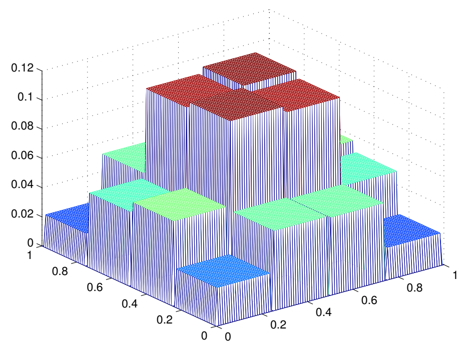
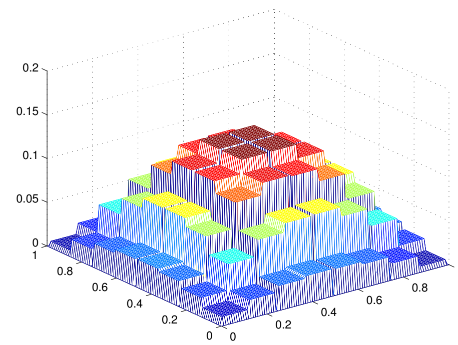


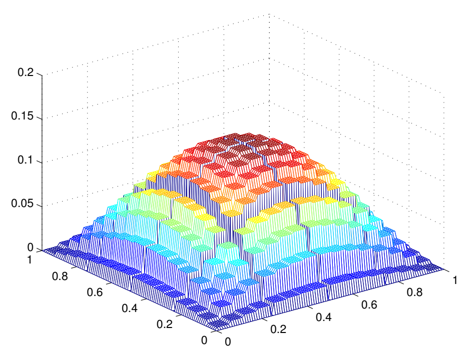
Figure 4.2.13: The first component of t-derivative of displacement, for (4.2.2)



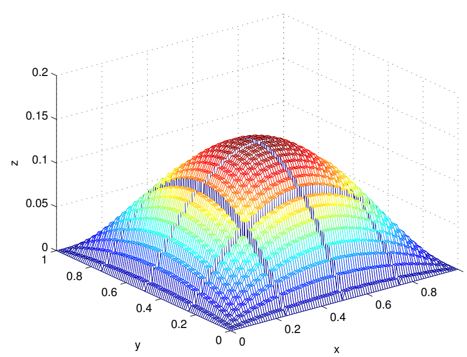
(a) N=4



(b) N=8



(c) N=16



(d) N=32

Figure 4.2.14: The first component of t-derivative of displacement, for (4.2.2), N=4,8,16,32

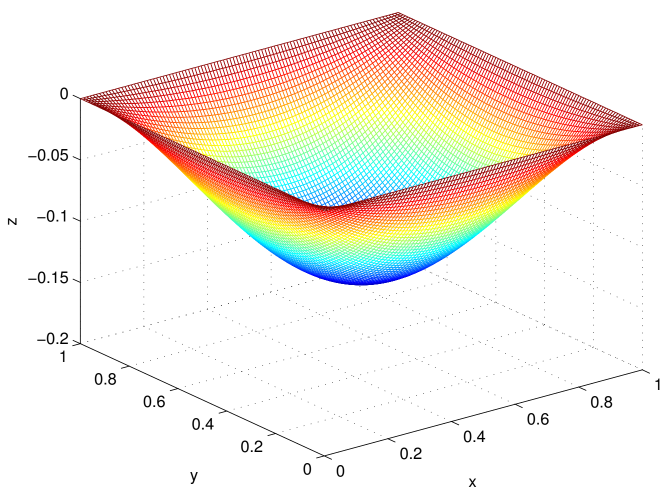
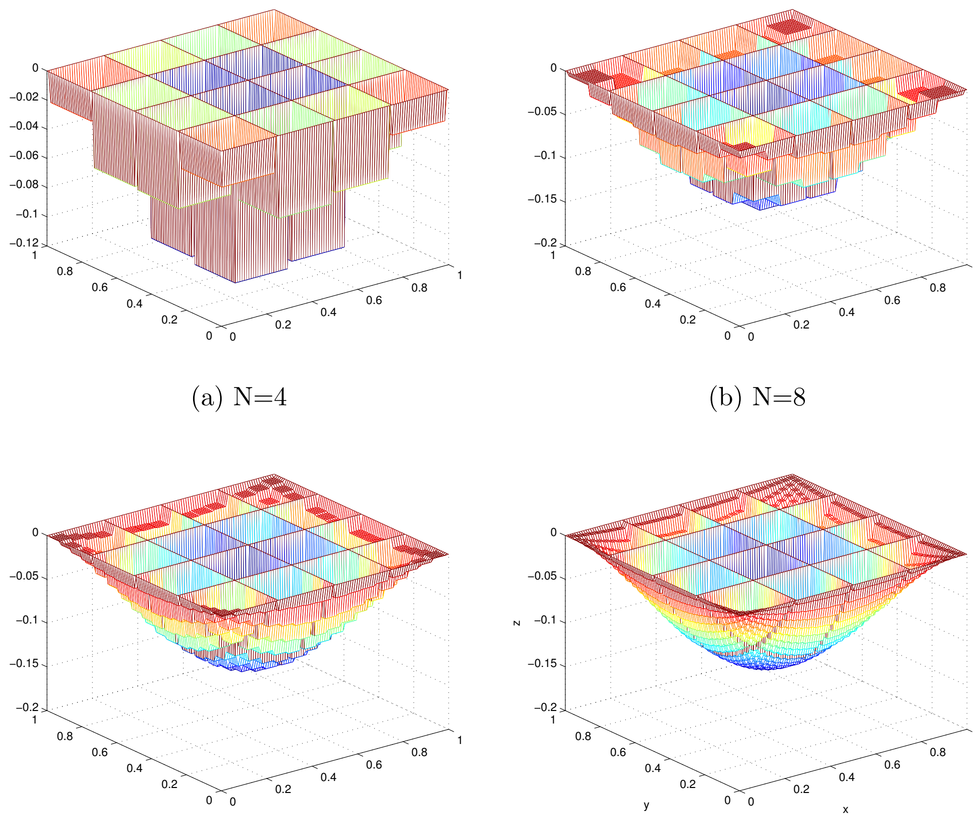


Figure 4.2.15: The second component of t-derivative of displacement, for (4.2.2)



(a) N=4

(b) N=8

(c) N=16

(d) N=32

Figure 4.2.16: The second component of t-derivative of displacement, for (4.2.2), N=4,8,16,32

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국문초록

탄성 편미분 방정식을 풀기 위한 혼합 유한 요소에 대한 많은 연구 결과들이 알려져 있지만, 이 논문에서는 가장 간단한 2차원 혼합 유한 요소로 탄성 파동 방정식에 접근한다. 이 혼합 유한 요소는 가장 작은 자유도를 가지면서도, 해의 유일성과 존재성을 위한 안정 조건을 만족하고 해와의 오차가 안정적으로 감소하는 것으로 알려져 있다. 이 논문에서는, 이 혼합 유한 요소를 이용하여 시간의 변화에 따라 진행하는 탄성 파동 방정식의 근사치를 구하고, 순수 변위 문제를 통한 탄성 투사가 투사 전의 함수에 근사함을 관찰하여 탄성 파동 방정식 해의 근사치가 안정적으로 해에 수렴함을 보였다.

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