M.S. THESIS

Survey of Quantum entanglement and Geometry

양자얽힘과 기하에 대한 조사

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Jiwon Kim

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지도교수 Cheolhyeon Cho

이 논문을 이학석사 학위논문으로 제출함

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서울대학교 대학원

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Abstract

The aim of this paper is a understanding quantum entanglement states using algebraic geometry First, this survey introduce a concept of the quantum entanglement and recent observation on equivalent class of stochastic local operations and classical communication(SLOCC). We also survey Moduli stacks that birational to the SLOCC classes of quantum entanglement states.

Keywords: quantum entanglement,qubit,moduli stack,SLOCC classes

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Contents

Chapter 1

Introduction

Following papers [1][2][3], Understanding quantum entanglement structure is recently researched by many people. So this survey paper first introduce a conclusion of recent study of quantum states and entanglement mathematically

In quantum mechanics, quantum states is represented by vectors of Hilbert space. If bipartite quantum system exist, then states of bipartite quantum system is vectors of tensor product of Hilbert spaces. In pure state, Quantum states that can be represented by tensor product of two state are called separated state. The other states called by Quantum entanglement.

It is important to study quantum entanglement that classification of entanglement states. Especially, for classifying entanglement structure, using stochastic local operations and classical communication -called SLOCC- that is tensor product of invertible operator of each Hilbert space. For example, In [2] three qubit has two inequivalent class of states by SLOCC. And in [3] four qubit has nine inequivalent class of states by SLOCC. We will study that qubit states classification using stochastic local operation and classical communication in next section.

Algebraic geometry also used to understanding quantum entanglement.[4][5] Moduli stack of equivalent SLOCC class of n-qudit is birational to the moduli stack of pairs of a Calabi-Yau manifold and line bundle. Especially stochastic local operations and classical communication classes of three qutrit case and four qubit states one to one correspond with moduli stack of elliptic curve and line bundles.[4] We will cover some moduli stacks and finally calculate correspondence between three qutrit and four qubit SLOCC class and moduli stacks of elliptic curve.

Chapter 2

Quantum Entanglement

In this chapter, we study basic structure of quantum entanglement very briefly and we observe SLOCC classes(Stochastic local operations and classical communication) of quantum entanglement about bipartite quantum structure, three qubit structure[2] and four qubit structure[3].

2.1 Qudit system

Qudit states are elements of a d-level Quantum mechanical system such as $\mathbb{C}^d$ space. N-qudit systems are tensor product of each qudits. for example, 4-qubit system can be denoted by 16-dimensional $\mathbb{C}$-vector spaces $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ and 3-qutrit system can be denoted by 27-dimensional $\mathbb{C}$-vector spaces $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. Generally we can understand n-qudit space as d^ndimensional $\mathbb{C}$-vector space.

2.1.1 SLOCC classes

In quantum system, suppose quantum system $\mathcal{H}$ exist such as $\mathcal{H}$=$\otimes_{i=1}^{n}\mathcal{H}_i$ SLOCC-operations of $\mathcal{H}$ are products of invertible operator for each Hilbert spaces $\mathcal{H}_i$. We can define SLOCC-equivalent of two quantum states. Assume that $|\phi\rangle$ and $|\psi\rangle$ are quantum state. If SLOCC-operation $\mathcal{A}$ exist such as $\mathcal{A}|\phi\rangle$=$|\psi\rangle$, then we called $|\phi\rangle$ and $|\psi\rangle$ are SLOCC-equivalent.

2.2 Separated and Entanglement states

Assume a quantum system $\mathcal{H}$ can be denoted by tensor product of two Hilbert space $\mathcal{H}_a \otimes \mathcal{H}_b$. Then a quantum state ϕ in $\mathcal{H}$ is called separated state for $\phi \in \mathcal{H}_a \otimes \mathcal{H}_b$, $\phi = \phi_1 \otimes \phi_2$ such that $\phi_i \in \mathcal{H}_i$ In other word, ϕ is called an entanglement state if that state can not be tensor product of two states.

2.2.1 Bipartite quantum structure

Suppose quantum state $|\Phi\rangle$is a element of $\mathbb{C}^n \otimes \mathbb{C}^m$ such as $n \geq m$ then we can apply Schmidt decomposition to $|\Phi\rangle$ We obtain $|\Phi\rangle = \Sigma_{i=1}^{m} |j\rangle \langle j|$
Consequently, 2-qudit state have exactly d-SLOCC equivalent classes. Especially 2-qubit state exactly have 2-equivalent classes. One is separated, the other is entanglemented

2.2.2 Schmidt decomposition

Schmidt decomposition is a decomposition of tensor product of finite dimensional Hilbert spaces. If elements of Hilbert spaces $w \in \mathcal{H}_a \otimes \mathcal{H}_B$ exist, such that $deg(\mathcal{H}_a) = n$, $deg(\mathcal{H}_b) = m$, $n \geq m$ then there exist orthonormal basis of Hilbert spaces (u_i),(j_i) satisfying $\Sigma_{k=1}^{m}$= $\alpha_i u_i \otimes j_i$ when α_i is nonnegative real number. We can consider that this decomposition is another version of singular value decomposition

2.3 Three, Four Qubit structure

3-qubit system can be entangled by two different ways [2] following that paper, Using rank of reduced density matrices, make invariant to prove that 2 entanglement states can not equivalent. 3-qubit system also have 4-separated class.

- GHZ -class can be reduced to $(1/\sqrt{2})(|000\rangle \oplus |111\rangle)$
- W class can be reduced to $(1/\sqrt{3})(|100\rangle \oplus |010\rangle \oplus |001\rangle$

In generally,

- GHZ-class represents qubit states that can be reduced $(1/\sqrt{2})(|00...0\rangle \oplus |11...1\rangle)$,
- W-class represents qubit states that can be reduce d$(1/\sqrt{n})(|10...0\rangle \oplus |01...0\rangle \oplus ... \oplus |0...01\rangle$

4-qubit entanglement have 9-inequivalent class [3] using some fact of lie group theory, $SU(2) \otimes SU(2) \cong SO(4)$,decompose $SO(4)$ to two matrices and use Jordan decomposition.

Chapter 3

Algebraic Geometry

In this chapter, we will define moduli stacks[7] of Calabi-Yau manifold. And artin stack of general n-qudit system[5], sequentially prove birationality of two artin stacks. We will follow [5]'s proof

3.1 Fibered Category

- Assume that functor Φ of categories $\Phi : \mathcal{F} \longrightarrow \mathcal{E}$ exist
- S is object of $\mathcal{E}$
- let $\mathcal{F}_S$ be Subcategory of $\mathcal{F}$ containing object x such that $\Phi(x) = S$ and morphism m such that $\Phi(m) = id_S$

Then we called subcategory $\mathcal{F}_s$ Fibered category

3.2 $\mathcal{M}_{d,n}$ category

Definition.

A category $\mathcal{M}_{d,n}$ is defined by requiring to follow conditions

An object $\phi : (\mathcal{Y} \longrightarrow S,\ (\mathcal{L}_i)_{1 \le i \le n-1})$ includes

- a smooth morphism $\phi : (\mathcal{Y} \longrightarrow S$ of schemes
- a collection$(\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_{n-1})$ of line bundles on $\mathcal{Y}$
- the relative canonical sheaf is trivial
- $\phi_*(\mathcal{L}_i)$ for $1 \le i \le n-1$ are locally free sheaves of rank d
- the composition morphism
  $\mu :\ \otimes_{i=1}^{n-1} \phi_*(\mathcal{L}_i) \longrightarrow \phi_*(\otimes_{i=1}^{n-1} \mathcal{L}_i)$ is surjective with kernal of rank d
- the natural morphism
  $\phi : \mathcal{Y} \longrightarrow \mathbb{P}(\phi_*(\mathcal{L}_1)) \times_S ... \times_S \mathbb{P}(\phi_*(\mathcal{L}_{n-1}))$ is closed embedding, image of morphism is a complete intersection over S

Define morphism $\phi, \bar{\phi})$ satisfying to following conditions

- a morphism $\bar{\phi} :\ S \longrightarrow S'$of schemes
- ϕ induces an isomorphism $\mathcal{Y} \simeq S \times_{S'} \mathcal{Y}'$ schemes
- $\phi^* \mathcal{L'}_i \cong \mathcal{L}_i$

3.3 Artin stack

Definition.

Artin stack is defined by imposing following conditions as a category.

An object (S,$\mathcal{P}, \psi$) includes

- a scheme S
- a prinipal G-bundles $\mathcal{P}$ over S
- a G-equivalent morphism $\psi : \mathcal{P} \longrightarrow \mathcal{R}$

Define morphism $(\phi, \bar{\phi})$ satisfying to following conditions

- a morphism $\bar{\phi} :\ S \longrightarrow S'$of schemes
- an isomorphism $\phi : \mathcal{P} \simeq S \times_{S'} \mathcal{P}$ of principal G-bundles satisfying $\psi = \pi' \circ \pi_2 \circ \phi$ such that π_2 is a natural projection map from $S \times_{S'} \mathcal{P}$ to $\mathcal{P}'$

3.4 Quotient stack

- Define $V_1, ..., V_n$ are d- dimensional complex vector space
- $\mathcal{R} = V_1 \otimes ... \otimes V_n$
- The group G = $GL(V_1) \times ... \times GL(V_n)$ act naturally on $\mathcal{R}$
- Define quotient stack $\mathcal{Q}_{d,n} = \mathcal{R}/G$

Then, Quotient stack $\mathcal{Q}_{d,n}$ is Artin stack.

3.5 Main Theorem

The category $\mathcal{M}_{d,n}$ is an Artin stack which is birational to $\mathcal{Q}$ We will prove this theorem following idea of [5]

3.5.1 Proof of Main theorem

Step 1.
Main idea is find open substack of $\mathcal{Q}$ that equivalent to $\mathcal{M}_{d,n}$ through fibered category in groupoid

Using isomorphism about $\otimes_{i=1}^{n} V_i \cong Hom(V_n^{\perp}, \otimes_{i=1}^{n-1} V_i)$, element $\zeta \in \mathcal{R}$ can be regarded linear operator.

- Define $Im(\zeta)$ to $V_\zeta \subset V_1 \otimes ... \otimes V_{n-1}$
- Define $I_\zeta \subset S(V_1) \otimes ... \otimes S(V_{n-1})$
- $S(V_{n-1})$Then subscheme Y_ζ exist
  such that $Y_\zeta \subset \mathbb{P}(V_1) \times ... \times \mathbb{P}(V_{n-1})$
- Let $\mathcal{R}^{sm}$ denoted subscheme of $\mathcal{R}$ including ζ
  such that $rk(V_\zeta) = n$ and Y_ζ is smooth morphism

Step 2. Make functor $\Psi : \mathcal{M}_{d,n} \longrightarrow \mathcal{Q}^{sm}$

Choose open cover S_λ of S following conditions.

- $\phi_*(\mathcal{L}_i) \mid_{S_\lambda}$ is trivial for any i
- trivialization $\psi_{i,\lambda} : \phi_*(\mathcal{L}_i) \mid_{S_\lambda} \simeq V_i \otimes \mathcal{O}_{S_\lambda}$ exist
- Subsheaf $Ker(\mu) \mid_{S_\lambda}$ of $V_1 \otimes ... \otimes V_{n-1} \otimes \mathcal{O}_{S_\lambda}$ is also free sheaf We can choose trivialization $\psi_n : Ker(\mu) \mid_{S_\lambda} \simeq V_n^{\perp} \otimes \mathcal{O}_{S_\lambda}$ Then we obtain morphism $\bar{\psi_\lambda} : S_\lambda \longrightarrow \mathcal{R}$ depending trivialization of ϕ_i, $1 \leq i \leq n$

  Trivialization is given by action of G
  We can lift $(\bar{\psi_\lambda} : S \to \mathcal{R})$ to G-equivalent morphism $(\psi : \mathcal{P} \to \mathcal{R})$
  finally, using smoothness of scheme morphism ϕ, image of ψ in $\mathcal{R}^{sm}$.

Step 3. Make functor $\Phi : \mathcal{Q}^{sm} \longrightarrow \mathcal{M}_{d,n}$

- $\mathcal{Y}_{\mathcal{R}^{sm}}$ be a subscheme of $\mathcal{R}^{sm} \times \mathbb{P}(V_1) \times ... \times \mathbb{P}(V_{n-1})$ such as $\mathcal{Y}_\zeta$ fiber over $\zeta \in \mathcal{R}^{sm}$ $Y_\zeta \subset \mathbb{P}(V_1) \times ... \times \mathbb{P}(V_{n-1})$

- Similrary, we can define $\mathcal{Y}_{\mathcal{P}}$ subscheme of $\mathcal{P}\times\mathbb{P}(V_1)\times...\times\mathbb{P}(V_{n-1})$ naturally have G action and G equivalent morphism $\mathcal{Y}_{\mathcal{P}} \longrightarrow \mathcal{P}$

- applying G-equivalence condition, $\mathcal{Y}_{\mathcal{P}} \subset \mathcal{P} \times \mathbb{P}(V_1) \times ... \times \mathbb{P}(V_{n-1})$ make well defineness of $\mathcal{Y}$ such as $\mathcal{Y} \subset \mathcal{S} \times \mathbb{P}(V_1) \times ... \times \mathbb{P}(V_{n-1})$

- Smooth morphism $\phi : \mathcal{Y} \longrightarrow S$ is projection map,
  $\mathcal{L}_i$ is naturally defined by pullback of tautological line bundle over $\mathbb{P}(V_i)$

Step 4. Check equivalence

- $\Psi \circ \Phi$ is isomorphic $Id_{\mathcal{Q}^{sm}}$ following definition of functor

- the other case, image of $\Phi \circ \Psi$ is obtained by choosing $Ker\mu$. It is subscheme of $\mathbb{P}(\phi_*(\mathcal{L}_1)) \times_S ... \times_S \mathbb{P}(\phi_*(\mathcal{L}_{n-1}))$ it is a complete intersection over S ,it obtained pullback of tautological bundle

- $\mathcal{Y}$ must have trivial canonical bundle. Then using adjunction formula[6][V.1.5] it is bigger than image, otherwise image of functor contain it.

Chapter 4

Noncommutative algebra

4.1 $\mathcal{N}_3$ stack

4.1.1 Definition

A category $\mathcal{N}_3$ is defined by requiring to following conditions

An object $\phi : (\mathcal{Y} \longrightarrow S,\ (\mathcal{L}_i)_{1 \leq i \leq 2})$ includes

- a flat morphism $\phi : \mathcal{Y} \longrightarrow S$ of schemes
- a collection$(\mathcal{L}_1, \mathcal{L}_2)$of line bundles on $\mathcal{Y}$ satisfying following condition
- the curve C is embedded as a divisor of degree 3 in $\mathbb{P}^2$ by global sections of both $\mathcal{L}_1$ and $\mathcal{L}_2$
- $\deg(\mathcal{L}_i \mid_D)$ is same for all i's and every irreducible component D of C
- $\mathcal{L}_1$ is not isomorphic to $\mathcal{L}_2$

4.2 $\mathcal{N}_4$ stack

4.2.1 Definition

A category $\mathcal{N}_4$ is defined by requiring to follow conditions
An object $\phi : (\mathcal{Y} \longrightarrow S, (\mathcal{L}_i)_{1\leq i\leq 3})$ includes

- a flat morphism $\phi : \mathcal{Y} \longrightarrow S$ of schemes
- a collection$(\mathcal{L}_1, \mathcal{L}_2)$of line bundles on $\mathcal{Y}$ satisfying following condition
- the curve C is embedded as a divisor of degree (2,2) in $\mathbb{P} \times \mathbb{P}$ by global sections of both ($\mathcal{L}_1$, $\mathcal{L}_2$) and ($\mathcal{L}_2$, $\mathcal{L}_3$)
- deg($\mathcal{L}_1 \mid_D$)= deg($\mathcal{L}_3 \mid_D$) for every irreducible component D of C
- $\mathcal{L}_1$ is not isomorphic to $\mathcal{L}_3$

morphism is similarly defined to $\mathcal{M}_{d,n}$

4.2.2 Theorem

The category $\mathcal{M}_{2,4}$ is an Moduli stack which is birational to $\mathcal{N}_4$[5]

4.2.3 Theorem

The category $\mathcal{M}_{3,3}$ is an Moduli stack which is birational to $\mathcal{N}_3$[5]

4.3 Conclusion

Quantum entanglement has many geometric structure, this survey paper observe relation of moduli stacks a little.
Moduli stack of Calabi-Yau manifold $\mathcal{M}_{d,n}$ is birational to stack of SLOCC class $\mathcal{Q}_{d,n}$

if (d,n)=(3,3) or(2,4) then Moduli stacks $\mathcal{M}_{d,n}$ and$\mathcal{Q}_{d,n}$ also birational with moduli stacks of elliptic curve $\mathcal{N}$

We did not cover AS-regular algebra, but following [8], moduli stacks also related cubic and quadratic AS-Z algebra.

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초록

이 논문의 목적은 양자얽힘상태를 대수기하를 사용하여 이해하려는 것이다. 먼저, 이 조사 논문은 양자 얽힘에 관한 기본적인 개념과 최근의 SLOCC-클래스(확률적 국소연산과 고전통신)에 대한 연구들을 알아보았다. 또한, 모듈러스 스택과 양자 얽힘 상태의 SLOCC-클래스 사이의 쌍유리관계에 대해 조사했다.

주요어: 양자 얽힘, 큐빗, 모듈러스 스택, SLOCC 클래스
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