M.S. THESIS

Explicit reconstruction of the entanglement wedge

얽힘 쐐기의 직접적 재구성

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김정욱

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이 논문을 이학석사 학위논문으로 제출함

2017 년 2 월

서울대학교 대학원

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Abstract

AdS/CFT is a duality between the gravitational theory on asymptotically AdS space and the conformal theory defined on its boundary. Although many non-trivial checks were made and much is known about the dictionary that interpolates both side of the duality, compilation of the dictionary is still far from being complete. One of the major issues that needs more elucidation is the problem of subregion duality; what information of the bulk a given boundary subregion encodes. Although the proof given by Dong, Harlow, and Wall [28] states that the entanglement wedge of the bulk should be encoded in boundary subregions, explicit procedure for reconstructing the entanglement wedge remained a mystery for some time. Based on [1], a generalisation of the mode sum approach to smearing functions that incorporates reconstruction on the entanglement wedge is described. Contrary to naïve expectations, there is no need to solve the wave equation on a complicated coordinate patch. This hard problem is transferred to a less hard but tractable problem of matrix inversion.

Keywords: AdS/CFT correspondence, Bulk locality

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Chapter 1

Introduction

1.1 AdS/CFT correspondence

How many qubits can a cubic hold if its dimensions are 1 m $\times$ 1 m $\times$ 1 m? A naïve guess is that the number of qubits a container can contain depends on the volume of the container. However, 't Hooft has argued [2] that the number should depend on the *surface area* of the container in a theory with gravity. Also known as the holographic principle [3], this argument is based on the observation by Bekenstein [4] that black holes have entropy proportional to their surface area($S_{\mathrm{BH}} = \frac{1}{4}A$). 't Hooft's argument [2] goes as follows: The most entropic state of matter confined to a limited volume —neglecting gravitational effects —would be a gaseous state. Suppose that this volume is a box of dimension L. The energy and entropy of the gas inside this box would be approximately given as follows[1].

$$E_{\text{gas}} \sim C_1 L^3 T^4 \tag{1.1}$$

$$S_{\text{gas}} \sim C_2 L^3 T^3 \tag{1.2}$$

This approximation is valid unless a gravitational collapse occurs. Gravitational collapse occurs when too much matter is crammed into a volume smaller than its Schwarzschild limit $R_{\text{Sch}} = 2E$. Disallowing gravitational collapse $R_{\text{Sch}} < L$ gives an upper bound on the energy and the temperture of the confined gas.

$$E_{\text{gas}} < L/2 \tag{1.3}$$

$$T < C_3 L^{-1/2} \tag{1.4}$$

Thus, the maximum entropy that a confined gas can hold is approximately

$$S_{\text{gas}} < C_4 L^{3/2}\,, \tag{1.5}$$

while the entropy of a single black hole that can fit inside the same container is given as follows.

$$S_{\text{BH}} = C_5 L^2 \tag{1.6}$$

The coefficients C_i are $\mathcal{O}(1)$ constants in natural units. If L is sufficiently large, the single black hole state is the state with the most entropy. Therefore, its entropy—proportional to its surface area—gives the upper limit of degrees of freedom that can be enclosed in some container. Coming back to the problem raised in the beginning of this paragraph, the surface area of this cubic box is 6 m^2, corresponding to 2.3×10^{70} square planck lengths. This surface area corresponds to entropy of 5.9×10^{69} in units of Boltzmann's constant according to Bekenstein-Hawking formula, and to 8.5×10^{69} Boolean degrees of freedom

[1]Natural units $c = \hbar = k = G = 1$ are used.

according to 't Hooft's conjecture [2]. In sum, this cubic box can contain up to $\sim 10^{69}$ qubits, although the contents of the box would be extremely massive[2].

An explicit realisation of the holographic principle is the celebrated AdS/CFT correspondence. The AdS/CFT correspondence, first conjectured by Maldacena [5], states that the gravitational theory on a asymptotically anti-de Sitter spacetime (AdS) is dual to the conformal field theory (CFT) defined on its boundary. Maldacena's idea is based on the observation that dynamics of N stacked D3 branes in type IIB superstring theory has effective descriptions that look radically different. The relevant parameters for this physical system are string coupling g_s, string length $l_s = \sqrt{\alpha'}$, 10d Newton constant $16\pi G_N^{10} = 2\kappa_{10}^2 = g_s(g_s l_s)(2\pi l_s)^7$, and D3 brane tension $T_3 = (g_s l_s)^{-1}(2\pi l_s)^{-3}$. The different descriptions can be parametrised by defining $\lambda \equiv 2\pi g_s N$.

The gravitational effects of D3 branes are characterised by the number $N G_N^{10} T_3 = N g_s (2\pi l_s)^4/16\pi = (\pi^2/2)\lambda l_s^4$. Gravitational interactions cause the stacked D3 branes to deform the geometry into an extremal black brane geometry. The metric is obtained by solving the supergravity equations, which is given as [6, 7]

$$ds^2 = f(r)^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + f(r)^{1/2}\left(dr^2 + r^2 d\Omega_5^2\right), \quad f(r) = 1 + \frac{2\lambda l_s^4}{r^4}, \quad (1.7)$$

where x^μ refer to worldvolume coordinates of D3 branes and $d\Omega_5^2$ is the metric of unit 5-sphere S^5. This solution is reliable when ten-dimensional Planck length $l_P = g_s^{1/4} l_s$ is small, compared to the string length l_s and the characteristic length scale of the solution $r_0 = (2\lambda l_s^4)^{1/4}$. The near-horizon geometry is obtained by taking the limit $r \ll r_0$.

$$ds^2 = \frac{r^2}{r_0^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{r_0^2}{r^2}dr^2 + r_0^2 d\Omega_5^2 \quad (1.8)$$

[2]The mass of a Schwarzschild black hole with radius 0.5 m is 3.4×10^{26} kg. For comparison, the mass of Saturn is 5.7×10^{26} kg and the mass of Neptune is 1.0×10^{26} kg.

This is just a product spacetime of five-dimensional anti-de Sitter space (AdS_5) of radius r_0 and five-dimensional sphere (S^5), so the theory becomes a full string theory on $\mathrm{AdS}_5 \times S^5$ background. To sum up, the physics of stacked D3 branes is described by type IIB string theory on $\mathrm{AdS}_5 \times S^5$ background. Note that classical Einsteinian supergravity description is reliable when AdS radius r_0 is large compared to the string length l_s.

On the other hand, the effective worldvolume theory of stacked D3 branes turns out to be a supersymmetric Yang-Mills theory (SYM) with 16 supercharges ($\mathcal{N} = 4$) and gauge group $U(N)$[3], characterised by the number $g_{YM}^{-2} = (2\pi\alpha')^2 T_3 = (2\pi g_s)^{-1}$. The scaling of the Yang-Mills coupling constant can be heuristically derived by studying the effective action of a single D3 brane system, which contains electromagnetism as its low energy effective theory. The Dirac-Born-Infeld term in the single D3 brane effective action is responsible for the emergence of this Maxwell term.

$$S_3 = -T_3 \int d^4\xi e^{-\Phi} \sqrt{-\det\left(\hat{G}_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta}\right)} + \cdots \tag{1.9}$$

The indices α, β are indicate coordinates parametrising D3 branes and $\hat{G}$ is the induced metric of the D3 brane. Expanding the square root to quadratic order of field strength $F_{\alpha\beta}$ gives the following formula.

$$S_3 = -T_3 \int d^4\xi e^{-\Phi} \sqrt{-\det\hat{G}_{\alpha\beta}} \left(1 + \pi^2\alpha'^2 F_{\alpha\beta}F^{\alpha\beta}\right) + \cdots \tag{1.10}$$

The multiplicative coefficient in front of the Maxwell term is $1/4e^2 = \pi^2\alpha'^2 T_3$, the coupling constant of low energy effective electromagnetism. For a stack of D3 branes this coupling constant becomes the Yang-Mills coupling constant $e^2 \to g_{YM}^2$ from which the Yang-Mills coupling $g_{YM}^2 = 2\pi g_s = \lambda/N$ can be readily seen. Yang-Mills theory is well-defined for any value of λ and N, but

[3] $SU(N)$ when decoupling centre of mass degrees of freedom.

not all regimes are subject to perturbative computations. In the special limit of large N with finite $\lambda = g_{YM}^2 N$, 't Hooft showed that it is possible to do perturbative computations in expansions of λ and $1/N$ [8]. The name 't Hooft coupling for the coupling constant λ originates from this work, and reliable perturbative expansions are possible for small λ and large N.

Table 1.1 Meaning of parameters in AdS/CFT correspondence. "perturbative" refers to parameter regimes where perturbative loop expansion computations are reliable for $\mathcal{N} = 4$ SYM and classical Einsteinian supergravity computations are reliable for AdS gravity.

Variables	**4d $\mathcal{N} = 4$ SYM**	**AdS$_5 \times S^5$ gravity**
λ	't Hooft coupling($g_{YM}^2 N$)	stringy effects($\lambda \sim (r_0/l_s)^4$)
N	number of colours	quantum effects($N \sim (r_0/l_P)^4$)
perturbative	$N \gg 1 \gg \lambda$	$N \gg \lambda \gg 1$

Thus, a stack of D3 branes have two radically different descriptions. One is type IIB string theory on AdS$_5 \times S^5$, which is a theory containing gravity in (ignoring compact S^5) $4+1$ spacetime dimensions. Another is $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ in $3+1$ spacetime dimensions. What Maldacena has conjectured is that these seemingly different theories are actually opposite sides of a same coin; the string theory on AdS$_5 \times S^5$ is equivalent to SYM defined on the asymptotic boundary of AdS spacetime. Because 4d $\mathcal{N} = 4$ SYM is known to possess conformal symmetry, this conjecture is now known as AdS/CFT correspondence. Table 1.1 shows how variables λ and N are related to dimensionless quantities of each disparate description. The parameter space of this duality is given by figure 1.1.

One of the early tests of this correspondence compared the operator spectrum of each side. Using (1.16), the test compared operator spectrum obtained from Kaluza-Klein reduction of AdS$_5 \times S^5$ supergravity fields to the operator

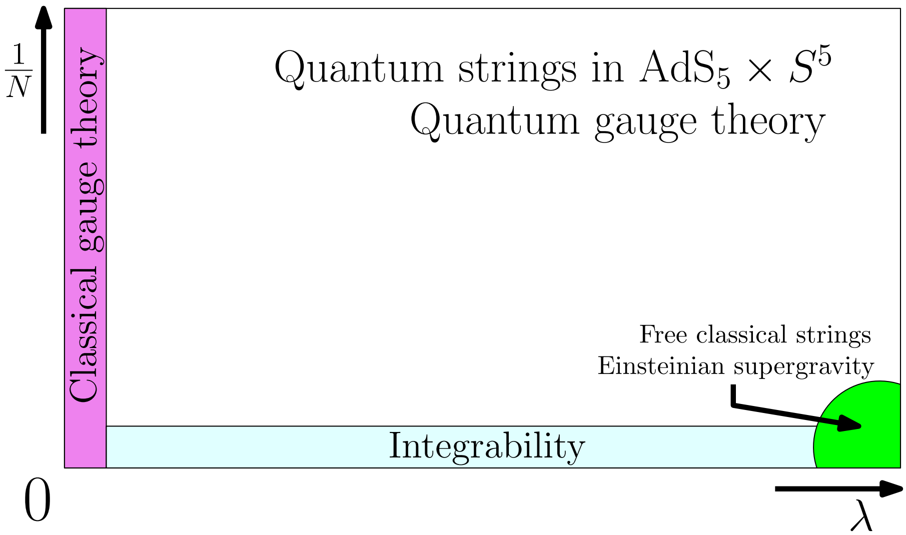

Figure 1.1 The parameter space of 4d $\mathcal{N} = 4$ SYM / $\mathrm{AdS}_5 \times S^5$ duality. The original version of this figure appears in [9].

spectrum of 4d $\mathcal{N} = 4$ SYM. Due to possible quantum corrections, it is in general hard to compute the scaling dimensions of the operators of CFT side at large λ, the regime reliably described by supergravity. However, existence of supersymmetry allows exact computation of scaling dimensions for some restricted cases. Also known as short multiplets or BPS operators, chiral primary operators are operators that are annihilated by some combination of supercharges Q. For such operators, the scaling dimension is uniquely fixed by R charges due to superconformal algebra constraints, and that relation cannot change unless supersymmetry is broken. Thus, scaling dimensions for chiral primary operators can be computed at $\lambda = 0$ limit and then the computation can be reliably extrapolated to $\lambda \gg 1$ regimes. Comparison shows that a corresponding dual CFT operator always exists for each KK fields of the supergravity side [6, 58]. There are other nontrivial tests such as comparison of correlation functions and anomalies. Consult [6, 58, 59] for further information.

The correspondence is not limited to the specific example of 4d $\mathcal{N} = 4$ SYM and $\mathrm{AdS}_5 \times S^5$ correspondence, and a general version of the AdS/CFT correspondence states that a theory containing gravity on an asymptotically AdS spacetime and the conformal field theory defined on the boundary this spacetime is equivalent. A wealth of reviews has been written on the subject [6, 7, 31, 58, 59], so an interested reader can consult these references. How the physical observables are matched on each side of the correspondence will be the subject of the following section.

1.2 Duality dictionary

The description of a correspondence is not complete unless a full dictionary is given that converts physical observables on one side of the correspondence to the other. Two kinds of the dictionaries are mainly known in the literature: One is based on the equivalence of generating functionals, and the other is based on equivalence of operators. These two different kinds of dictionaries are known to be equivalent, at least for an interacting scalar [12].

The first kind is known as the GKP-Witten dictionary [10, 11]. This dictionary states that generating functional of the coformal theory of the boundary and the gravitational theory of the bulk are equivalent.

$$Z_{\mathrm{CFT}}[\phi_0] = Z_{\mathrm{bulk}}[\phi_0] \tag{1.11}$$

To be more specific, the bulk side generating functional is written as[4]

$$Z_{\mathrm{bulk}}[\phi_0] = \int \mathcal{D}[\mathrm{fields}] e^{-S[\phi\ :\ \phi \to \phi_0 \text{ at boundary}]} \tag{1.12}$$

while the boundary side generating functional is written as

$$Z_{\mathrm{CFT}}[\phi_0] = \left\langle e^{\int \mathcal{O}\phi_0} \right\rangle \tag{1.13}$$

[4]Euclidean signature is used as the dictionary was originally formulated for Euclidean AdS.

In practice, this dictionary can be used to compute the boundary theory correlation functions in the regime where bulk theory becomes semi-classical. In this regime the bulk side can be approximated by a classical supergravity action subject to the boundary condition constraints.

$$\int \mathcal{D}[\text{fields}] e^{-S[\phi\,:\,\phi\to\phi_0 \text{ at boundary}]} \simeq e^{-S_{\text{SUGRA}}[\phi]}\Big|_{\phi\to\phi_0 \text{ at boundary}} \tag{1.14}$$

The field configuration is chosen to obey equation of motion and boundary conditions when computing the action.

The second kind is known as the extrapolate dictionary, also referred to as BDHM dictionary in [12]. The name BDHM dictionary follows from the work [15] where this dictionary was explicitly stated. This dictionary states that boundary operators are equal to appropriately scaled boundary limits of the dual bulk operators.

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = \lim_{z\to 0} z^{-\Delta_1}\cdots z^{-\Delta_n}\,\langle\phi_1(x_1,z)\cdots\phi_n(x_n,z)\rangle \tag{1.15}$$

The asymptotic boundary of AdS is situated at $z = 0$, and z is the "radial" coordinate of AdS; points deeper in the bulk are probed by larger values of z. The exponent Δ_i of the scaling factor $z^{-\Delta_i}$ is the scaling dimension of the CFT operator $\mathcal{O}_i$. An appropriate variant of this dictionary was used in the work [13, 14], where a method of reconstructing the bulk quasi-local operator from boundary data was proposed. For a p-form field of mass m on the gravity side, the scaling dimension of the dual CFT operator is given by the following relation. R is the AdS radius in the following equation.

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2R^2 - p(p-d)} \tag{1.16}$$

Still, the dictionary is not complete. One of the facets of this duality that remains elusive is emergence of a (quasi-)local point in the bulk when the dual boundary CFT is given. There is a vast amount of literature on the subject, some of which probe the bulk by examining the entanglement entropy and related quantities of the boundary.[5] For example, there are approaches utilising differential entropies[43, 44], inverting the Ryu-Takayanagi formula to obtain bulk local data[48, 45, 50], and using entanglement entropy to derive linearised Einstein equations in the bulk[46, 47, 48, 49]. A more complete list of references in related directions can be found in the thesis by Lin[50]. Another line of research focuses on the causal structure of the bulk, using null geodesics[18] or light-cones[53] that extend into the bulk. Other approaches utilise constraints imposed by CFT to probe bulk data[51, 52]. These approaches attempt to study reconstruction of the bulk metric[43, 44, 53], the bulk matter fields[18, 52], or bulk dynamics[46, 47, 48, 49, 51].

A different approach is pursued here, however, as these approaches are rather unwieldy for answering the question 'How is the bulk local operator ϕ_P at bulk point P represented by operators of the boundary?'. The approach explained in this thesis is based on the construction of the dual bulk operator using smearing functions proposed by Hamilton, Kabat, Lifschytz, and Lowe(HKLL) [13, 14], employing the second kind of the dictionaries explained above. Smearing functions are integral kernels integrated against the dual boundary operator, which can be schematically written as follows.

$$\phi(x) = \int K(x|Y)\mathcal{O}(Y)dY \tag{1.17}$$

Smearing functions depend on the patch used to reconstruct the bulk operator. For example, in global patch reconstruction the integration domain is taken to be boundary points that are spacelike separated from the bulk point being

[5] Although emergence of the dual bulk theory is in principle independent of entanglement in the boundary theory, the insight gained by Van Raamsdonk[54] has tempted many researchers to equate the two.

constructed, while in AdS-Rindler reconstruction the integration domain is the whole boundary covered by the coordinate patch.

1.3 Subregion duality

The original formulation of AdS/CFT correspondence is a duality between the whole gravitational bulk theory and the whole conformal boundary theory. Since it is possible to split the whole system into subsystems, there should be a correspondence between the subsystems on the both sides of the duality. This is known in the literature as subregion-subregion duality [18, 19, 20].

One of the main motivations for studying subregion-subregion duality is the Ryu-Takayanagi formula [21]. This formula relates the entanglement entropy S_A of a given subregion A of the boundary to the area of certain codimension two hypersurface χ_A of the bulk.

$$S_A = \frac{\text{Area of } \chi_A}{4G_N} \tag{1.18}$$

G_N is the Newton constant of the bulk theory. The hypersurface χ_A, known as RT surface or HRT surface [22], is the surface with minimal area constrained by the condition that it is homologous to A, i.e. $\partial\chi_A = \partial A$. The surface χ_A is also relevant to the definition of the entanglement wedge. Consider a time-slice of the bulk which contains χ_A in the bulk and A of the boundary. There is a subregion of this time-slice having χ_A and A as its boundary, and the entanglement wedge is defined as its bulk domain of dependence. The entanglement wedge will turn out to play a key role in bulk reconstruction of subregion-subregion duality.

Just like the duality of the full system, the following question can be asked in subregion-subregion duality; how is the bulk local operator reconstructed from given boundary data? Some answer is already given in [14] as AdS-Rindler reconstruction. This reconstruction has an interesting property that while the bulk point X can be constructed from subregion A or subregion B, it may not be possible to construct the same bulk point from the intersection $A \cap B$. This observation had led to the conjecture that the AdS/CFT correspondence has a quantum error correcting code-like structure, which was proposed by Almheiri, Dong, and Harlow(ADH) [25]. Since the boundary of the AdS-Rindler wedge is connected, these discussions were mostly based on AdS/CFT correspondence for a *connected* subregion of the boundary.

The most straightforward way of constructing the bulk from *disconnected* boundary subregions is to decompose the disconnected set into connected parts and construct the bulk from each connected parts. This can be called the *causal wedge reconstruction* as all bulk points lie in the causal wedge[6] of the boundary subregion[17, 18, 20, 38, 53]. Nevertheless, a recent conjecture in the literature states that the bulk dual larger than the causal wedge can be constructed from the boundary subregion[27, 28, 29], and this conjecture has been claimed prooved by Dong, Harlow, and Wall(DHW)[28]. How this construction can be explicitly done in a way similar to the HKLL procedure remained as a mystery for some time [28, 30], but it has been finally clarified [1].

1.4 Outline of the thesis

This thesis is based on author's work [1], which attempts to reconstruct the entanglement wedge in a way similar to that of HKLL. The procedure works provided the global patch dictionary of the duality is already given in terms of mode operators and mode functions. The key tactic is just a small variant of mode sum approach, a widely used method to construct the bulk from the boundary [13, 14, 15, 16, 18]. The key feature that distinguishes this approach

[6]The *causal wedge* $\mathcal{W}_C[A]$ of the given subregion A is the set of bulk points that can receive and transmit light signals to its boundary domain of dependence $\mathcal{D}[A]$[25].

from others found in the literature is that it is based on appropriate rearrangement of mode functions, rather than obtaining a new mode function in a new coordinate patch and using it to construct smearing functions [14, 18, 25]. While new mode functions obtained by solving the wave equation on a new coordinate patch are orthogonal to each other, this is not the case for the mode functions obtained from the procedure that author has proposed. The problem simplifies to the problem of obtaining coefficients in a non-orthogonal basis, where basis vectors are simply the mode functions at the boundary.

Author's work [1] is mainly concerned with linearised perturbations around pure AdS, because it is the regime where many things can be said without much computational difficulty. This does not mean that the proposal is restricted to pure AdS; the methods employed to extend HKLL reconstruction to other interesting cases can be equally applied to this proposal. Some examples of such extentions are incorporation of $1/N$ corrections and interactions[34, 35, 36], and construction on a black hole background geometry[37, 32, 38, 39].

Preliminaries to understanding the proposal of [1] will be given in chapter 2. Section 2.1 builds a dictionary of scalar perturbations around the vacuum. This is an explicit construction of the dictionary of the second kind introduced in 1.2 in the limited regime of $N \to \infty$ and $\lambda \to \infty$, so readers unfamiliar with AdS/CFT correspondence can have a feeling of what is going on. Section 2.2 provides the details of HKLL reconstruction, which is the starting point of the author's proposal. Section 2.3.2 outlines the proof of DHW [28] that entanglement wedge can be reconstructed from the given boundary subregion. Since understanding the ADH viewpoint on AdS/CFT correspondence—AdS/CFT is endowed with a quantum error correcting code-like structure—is needed, the section will give more detailed explanations on this viewpoint.

The main ideas of author's proposal will be given in chapter 3. Section 3.1 is mainly focused on constructing the necessary mathematical structure that captures the physics inside the entanglement wedge. In particular, it will be shown that it is not necessary to solve wave equations on the coordinate patch that covers the entanglement wedge. Section 3.2 explains how HKLL reconstruction can be generalised to entanglement wedge reconstruction. An appropriate transformation rule for the smearing function be given as well, so that covariance under AdS isometries is recovered. Conclusion will be given in chapter 4, where subtleties of this method and future prospects of the proposal is discussed.

All fields in this paper will be considered as operator *distributions*, i.e. fields will acquire meaning as operators only when integrated against a suitable test function. For example, the local bulk operator ϕ_P mentioned earlier in this introduction is defined as $\phi_P = \int \eta_P(x)\phi(x)dx$, where $\eta_P(x)$ is a $\mathcal{C}^\infty$ function having compact support on a neighbourhood of bulk point P. This allows manipulations not allowed in functions and circumvention of miscellaneous complications. For those who are not familiar with distribution theory, [64] is a good introduction to the subject. [61] also has a short introduction to the theory of distributions.

Chapter 2

Preliminaries

2.1 AdS/CFT at semiclassical gravity limit

2.1.1 Free scalar on pure AdS

Much of this subsection resembles the materials of chapter 4 and 5 of [55]. Consider a free scalar field in AdS_{d+1} of AdS radius 1.

$$ds^2 = \frac{1}{\cos^2(\rho)}\left(-dt^2 + d\rho^2 + \sin^2(\rho)d\Omega_{d-1}^2\right) \tag{2.1}$$

$$S = \int \sqrt{|g|}d^{d+1}x\left(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}m^2\phi^2\right) \tag{2.2}$$

The classical equation of motion for this action is the Klein-Gordon equation. The following mode decomposition provides an orthogonal set of solutions that correspond to normalisable modes[1][2] which do not diverge at $\rho = 0$ [14, 18, 55,

[1]Normalisable modes correspond to Dirichlet boundary conditions for the spatial coordinate ρ. Since normalisable modes relate the bulk operator to the boundary operator and non-normalisable modes act as sources, normalisable modes are needed.

[2]For tachyonic fields above the Breitenlohner-Freedman bound, non-normalisable modes could be needed. Such cases are not considered in this paper.

58].[3]

$$\phi(t,\rho,\Omega) = \sum_{n,l,m} \tilde{a}_{nlm}\; e^{-i(\Delta+2n+l)t} Y_{lm}(\Omega) \sin^l \rho \cos^\Delta \rho \; P_n^{(l+d/2-1,\Delta-d/2)}(\cos(2\rho)) \tag{2.3}$$

$n \in \mathbb{N} = \{0,1,2,\cdots\}$. l is the total angular momentum of the corresponding mode and m collectively refers to quantum numbers that are needed to specify a mode. Δ is the conformal dimension of the dual CFT$_d$ operator and is given by $\Delta = d/2 + \sqrt{m^2 + d^2/4}$, the greater solution to $m^2 = \Delta(\Delta - d)$. Quantisation leads to the following mode decomposition.

$$\phi(t,\rho,\Omega) = \sum_{n,l,m} a_{nlm} f_{nlm}(t,\rho,\Omega) + a^\dagger_{nlm} f^*_{nlm}(t,\rho,\Omega) \tag{2.4}$$

$$[a_{nlm}, a^\dagger_{n'l'm'}] = \delta_{nn'}\delta_{ll'}\delta_{mm'} \tag{2.5}$$

$$[a_{nlm}, a_{n'l'm'}] = [a^\dagger_{nlm}, a^\dagger_{n'l'm'}] = 0 \tag{2.6}$$

Each $a^\dagger_{nlm}$ raises the energy of the state by $\Delta + 2n + l$. Note that the number of distinct states with energy $\Delta + k$, $k \in \mathbb{N}$, is given by $(k+n-1)!/[(n-1)!k!]$.

The inner product between mode functions [55] is defined as follows, and is used to determine the normalisation of the mode functions.

$$(f|g)_\Sigma \equiv i \int_\Sigma d^d y \sqrt{|h|} n^\mu \left(f^* \nabla_\mu g - g \nabla_\mu f^*\right) \tag{2.7}$$

$$(f_{nlm}|f_{n'l'm'})_\Sigma = -(f^*_{nlm}|f^*_{n'l'm'})_\Sigma = \delta_{nn'}\delta_{ll'}\delta_{mm'} \tag{2.8}$$

$$(f_{nlm}|f^*_{n'l'm'})_\Sigma = 0 \tag{2.9}$$

This is a slightly unconventional notation employed to exploit the full power of Dirac's bra-ket notation. Σ denotes the spacelike surface on which the inner

[3] Hypergeometric functions are better for examining the wave equation, while Jacobi polynomials are more suited for calculations because of their orthogonality. Their mutual relations can be found in [62] or [63].

product is evaluated, y^a is the coordinate system that covers the submanifold Σ, n^μ is the unit future-directed normal($n^\mu n_\mu = -1$) to Σ, and h is the determinant of the induced metric $h_{ab} = \frac{\partial x^\mu}{\partial y^a}\frac{\partial x^\nu}{\partial y^b} g_{\mu\nu}$. Note that the inner product satisfies the following relations.

$$[(f|g)_\Sigma]^* = (g|f)_\Sigma = -(f^*|g^*)_\Sigma \tag{2.10}$$

For reference, an explicit expression for $f_{nlm}(t,\rho,\theta)$ is given.

$$f_{nlm}(t,\rho,\Omega) = \frac{1}{N_{nlm}} e^{-i(\Delta+2n+l)t} Y_{lm}(\Omega) \sin^l \rho \cos^\Delta \rho \; P_n^{(l+d/2-1,\Delta-d/2)}(\cos(2\rho)) \tag{2.11}$$

$$N_{nlm} = \sqrt{\frac{\Gamma(n+l+d/2)\Gamma(n+\Delta-d/2+1)}{n!\Gamma(n+l+\Delta)}} \tag{2.12}$$

2.1.2 Generalised free theory

Generalised free fields(GFF) are defined as operators whose correlators factorise into a sum of products of two-point functions.

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle \propto \frac{1}{(x_1-x_2)^{2\Delta}} \tag{2.13}$$

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\cdots\mathcal{O}(x_{2n})\rangle = \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle \cdots \langle \mathcal{O}(x_{2n-1})\mathcal{O}(x_{2n})\rangle + \cdots \tag{2.14}$$

The symbol $\cdots$ on the second line denotes permutations of two-point contractions. This is a sort of an infinite N limit of a CFT [31]. Generalised free theory(GFT) is a theory purely consisting of GFFs. The reason it is called generalised *free* is because its Hilbert space has a Fock space like structure, which is the Hilbert space of a free theory. A GFT is *not* a CFT because it does not have a stress tensor in its spectrum[4], but shares many properties that a CFT

[4]This is because existence of $T_{\mu\nu}$ spoils the factorisation of correlators. Nevertheless, a GFT can be considered as an effective description of operators with small scaling dimensions in a CFT in the infinite c limit [31].

has [31, 55]. In a Euclidean theory with radial quantisation, for example, an operator can be inserted at the origin to produce a state with the same conformal dimension; operator-state correspondence exists just as in a CFT.

$$\mathcal{O}(0)\left|0\right\rangle \to \left|\mathcal{O}\right\rangle \tag{2.15}$$

$$(P^2)^n(P_\mu)^l\mathcal{O}(0)\left|0\right\rangle \to \left|(P^2)^n(P_\mu)^l\mathcal{O}\right\rangle \tag{2.16}$$

The expression $(P_\mu)^l$ denotes a traceless symmetric combination of l P_μ operators that corresponds to some generalised spherical harmonic of angular momentum l. The state $\left|\mathcal{O}\right\rangle$ has conformal dimension Δ, while the state $\left|(P^2)^n(P_\mu)^l\mathcal{O}\right\rangle$ has conformal dimension $\Delta+2n+l$ and spin l. Note that this spectrum matches exactly with the spectrum formed by a single mode excitation in AdS free scalar theory. Imposing bootstrap constraints requires product operators such as $:\mathcal{O}\mathcal{O}:$, which are devoid of anomalous dimensions and can be matched to multiple mode excitations in AdS free scalar theory, to exist in a GFT [31].

To compare a GFF of a GFT to a free field of a free theory on AdS, it is useful to have a mode expansion of a GFF. The operator-state correspondence relations (2.15) and (2.16) can be used to expand a GFF in terms of mode operators.

$$\mathcal{O}(x) = \sum \mathcal{O}^\dagger_{i_1,\cdots,i_d}\frac{1}{M_{i_1,\cdots,i_d}}\left[\left(x^1\right)^{i_1}\times\cdots\times\left(x^d\right)^{i_d}\right] + \text{h.c.} \tag{2.17}$$

The mode operator $\mathcal{O}^\dagger_{i_1,\cdots,i_d}$ acts on the vacuum to create a state corresponding to $\left|(P_1)^{i_1}\cdots(P_d)^{i_d}\mathcal{O}\right\rangle$, while Hermitian conjugate part annihilates the vacuum. The state created by the product of operators $:\mathcal{O}_1\mathcal{O}_2:$, where $\mathcal{O}_i$s represent primary and descendent states collectively, can be obtained from the vacuum by acting mode operators successively.

$$\left|\mathcal{O}_1\right\rangle = \mathcal{O}_1^\dagger\left|0\right\rangle, \left|\mathcal{O}_2\right\rangle = \mathcal{O}_2^\dagger\left|0\right\rangle \Longrightarrow \left|:\mathcal{O}_1\mathcal{O}_2:\right\rangle = \mathcal{O}_1^\dagger\mathcal{O}_2^\dagger\left|0\right\rangle \tag{2.18}$$

It can be shown that the only algebra that can be satisfied by mode operators is the following algebra up to normalisation, which is absorbed by normalisation factors $M_{i_1,\cdots,i_d}$[5].

$$[\mathcal{O}_{i_1,\cdots,i_d}, \mathcal{O}^\dagger_{i'_1,\cdots,i'_d}] = \delta_{i_1 i'_1} \cdots \delta_{i_d i'_d} \tag{2.19}$$

$$[\mathcal{O}_{i_1,\cdots,i_d}, \mathcal{O}_{i'_1,\cdots,i'_d}] = [\mathcal{O}^\dagger_{i_1,\cdots,i_d}, \mathcal{O}^\dagger_{i'_1,\cdots,i'_d}] = 0 \tag{2.20}$$

Using polar coordinates and rearranging the modes, this mode expansion can be tidied up into a form involving spherical harmonics[6].

$$\mathcal{O}(r,\Omega) = \sum \mathcal{O}^\dagger_{nlm} \frac{r^{2n+l}\; Y^*_{lm}(\Omega)}{M_{nlm}} + \text{h.c.} \tag{2.21}$$

The algebra of mode operators is not very different from (2.19) and (2.20). Up to now, the GFT being considered is defined on the manifold $\mathbb{R}^d$. To set up a GFT on the manifold $\mathbb{R} \times S^{d-1}$, the exponential map $r = e^\tau$ and Weyl rescaling $g \to g' = e^{-2\tau} g$ is introduced. This means the GFF $\mathcal{O}$ undergoes a conformal rescaling $\mathcal{O} \to \mathcal{O}' = (e^\tau)^\Delta \mathcal{O}$ as well[7]. In sum, the mode expansion of $\mathcal{O}$ in coordinates (t, Ω) takes the following form.

$$\mathcal{O}(\tau,\Omega) = \sum \mathcal{O}^\dagger_{nlm} \frac{e^{(\Delta+2n+l)\tau}\; Y^*_{lm}(\Omega)}{M_{nlm}} + \text{h.c.} \tag{2.22}$$

Wick rotating to Lorentzian manifold by the substitution $\tau = i(1 - i\epsilon)t$ [32], the mode expansion of $\mathcal{O}$ on Lorentzian $\mathbb{R} \times S^{d-1}$ that will be used in later

[5]The first line is established by examining orthogonality of states. The second line follows from commutativity of scalars. Because radial ordering is implicitly assumed in radial quantisation, scalars always commute.

[6]The reason for taking the complex conjugate of spherical harmonics will become clear soon.

[7]Introduction of conformal rescaling induced by Weyl rescaling guarantees that the vacuum state on the original manifold is mapped to the vacuum state of the new manifold. In other words, the two-point correlation function remains in a form expected for a CFT only when conformal rescaling is done.

sections is obtained.

$$\mathcal{O}(t,\Omega) = \sum_{n,l,m} \mathcal{O}_{nlm} g_{nlm}(t,\Omega) + \mathcal{O}^{\dagger}_{nlm} g^{*}_{nlm}(t,\Omega) \tag{2.23}$$

$$g_{nlm}(t,\Omega) = \frac{1}{M_{nlm}} e^{-i(\Delta+2n+l)t}\, Y_{lm}(\Omega) \tag{2.24}$$

Two-point functions in Lorentzian signature can be obtained by the same procedure[8].

$$\langle T\mathcal{O}(x)\mathcal{O}(x')\rangle \propto \frac{1}{(\cos(t-t') - \Omega\cdot\Omega' + i\epsilon)^{\Delta}} \tag{2.25}$$

2.1.3 One-to-one correspondence between free bulk theory and GFT

It is tempting to conjecture the equivalence of free scalar theory in AdS and GFT defined on its boundary based on the spectrum of energy (conformal dimension for GFT) eigenstates. To claim equivalence, however, requires more nontrivial checks. One of the checks that can be used is to see how correlators of one theory can be related to the correlators of the other. The time-ordered two-point functions $iG(x,x') = \langle T\phi(x)\phi(x')\rangle$ of AdS free scalar turns out to be [58]

$$iG(x,x') = \frac{C}{(\cosh^2 s)^{\frac{\Delta}{2}}}\ {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta+1}{2}; \Delta - \frac{d}{2} + 1; \frac{1}{\cosh^2 s} - i\epsilon\right) \tag{2.26}$$

where C is an appropriate normalisation constant and s is the geodesic distance, which satisfies the following relation.

$$\cosh s = \frac{\cos(t-t') - \sin\rho \sin\rho' \Omega\cdot\Omega'}{\cos\rho\cos\rho'} \tag{2.27}$$

[8]The correlation function for the Euclidean signature implicitly contains radial ordering operator, so time ordering operator appears when Wick rotated to Lorentzian signature. The precise procedure is not very different from the procedure outlined in [32].

Taking both points to the boundary by the extrapolate map

$$\mathcal{O}(t,\Omega) = \lim_{\rho\to\pi/2} \frac{\phi(t,\rho,\Omega)}{\cos^{\Delta}\rho} \tag{2.28}$$

the boundary two-point functions become

$$\langle T\mathcal{O}(x)\mathcal{O}(x')\rangle \propto \frac{1}{\left[(\cos(t-t')-\Omega\cdot\Omega')^2+i\epsilon\right]^{\frac{\Delta}{2}}} \tag{2.29}$$

which is equal to (2.25). Comparing mode expansions (2.4) and (2.23), the natural identification seems to be the following.

$$\mathcal{O}_{nlm} \leftrightarrow a_{nlm} \tag{2.30}$$

$$\mathcal{O}^{\dagger}_{nlm} \leftrightarrow a^{\dagger}_{nlm} \tag{2.31}$$

$$g_{nlm}(t,\Omega) \leftrightarrow \lim_{\rho\to\pi/2}\left(\frac{f_{nlm}(t,\rho,\Omega)}{\cos^{\Delta}\rho}\right) \tag{2.32}$$

The above dictionary reproduces the algebra of mode operators and all correlators of the boundary theory. The normalisation factor M_{nlm} is given by

$$M_{nlm} = \frac{N_{nlm}}{P_n^{(l+d/2-1,\Delta-d/2)}(-1)} \tag{2.33}$$

2.2 HKLL reconstruction

2.2.1 Review of HKLL reconstruction

The papers by HKLL attempt to reconstruct the operators of the bulk gravitational theory from the oprators of the boundary for the vacuum state dual to pure AdS in the semiclassical limit(corresponding to infinite N, infinite 't Hooft coupling λ limit), which is the limit where quasi-local bulk spacetime emerges [13, 14]. The main idea behind HKLL bulk reconstruction [14] is quite simple; find an integration kernel that reads out mode operators of the CFT[9] operator when integrated against the operator $\mathcal{O}$ over some boundary integration

[9] To be more precise, GFT.

domain, use the dictionary (2.30) and (2.31) to substitute the mode operators of the bulk mode expansion (2.4) into integrals over the boundary, and change the order of summation and integration to write a smearing function. Some of the constructions elaborated in [14] will be briefly reviewed to demonstrate the procedure and its subtleties.

The global reconstruction of the bulk in [14] attempted to reconstruct the bulk operator at the origin. After the smearing function that reconstructs the origin has been constructed, the smearing function for an arbitrary bulk point can be found by using AdS isometries to bring the point to the origin. Restricting the reconstruction to the origin has one notable advantage; since only s-wave components contribute to the bulk operator at the origin, the summation simplifies drastically. The construction starts by dividing the bulk and boundary operators into positive and negative frequency parts.

$$\phi(t,\rho,\Omega) = \phi_+(t,\rho,\Omega) + \phi_-(t,\rho,\Omega) \tag{2.34}$$

$$\phi_+(t,\rho,\Omega) = \sum_{n,l,m} a_{nlm} f_{nlm}(t,\rho,\Omega) = [\phi_-(t,\rho,\Omega)]^\dagger \tag{2.35}$$

$$\mathcal{O}(t,\Omega) = \mathcal{O}_+(t,\Omega) + \mathcal{O}_-(t,\Omega) \tag{2.36}$$

$$\mathcal{O}_+(t,\Omega) = \sum_{n,l,m} \mathcal{O}_{nlm} g_{nlm}(t,\Omega) = [\mathcal{O}_-(t,\Omega)]^\dagger \tag{2.37}$$

The mode functions f_{nlm} for nonzero l vanish at the origin, so the bulk mode sum simplifies.

$$\phi(0) = \sum_n a_{n00} f_{n00}(0) + \text{h.c.} \tag{2.38}$$

$$= \sum_n a_{n00} \frac{Y_{00}\ P_n^{(d/2-1,\Delta-d/2)}(1)}{N_{n00}} + \text{h.c.} \tag{2.39}$$

On the other hand, the mode expansion for the positive frequency part of the boundary operator can be rewritten as follows.

$$\mathcal{O}_+(t,\Omega) = \sum_n \mathcal{O}_{n00} \frac{Y_{00}\ e^{-i(\Delta+2n)t}}{M_{n00}} + \sum_{n,l\neq 0,m} \mathcal{O}_{nlm} g_{nlm}(t,\Omega) \tag{2.40}$$

Using the dictionary (2.30), the mode operators a_{n00} can be read out from the positive frequency part of the boundary operator by the following integral.

$$a_{n00} = \mathcal{O}_{n00} = \frac{M_{n00}}{\pi} \int_{-\pi/2}^{\pi/2} dt\ e^{i(\Delta+2n)t} \int \sqrt{g_\Omega} d\Omega\ Y_{00}^*(\Omega)\mathcal{O}_+(t,\Omega) \tag{2.41}$$

$$= \int_{-\pi/2}^{\pi/2} dt \int \sqrt{g_\Omega} d\Omega\ P_{n00}^+(t,\Omega)\mathcal{O}_+(t,\Omega) \tag{2.42}$$

$$P_{n00}^+(t,\Omega) = \frac{M_{n00}}{\pi} e^{i(\Delta+2n)t}\ Y_{00}^*(\Omega) \tag{2.43}$$

Note that the projection operator P_{n00}^+ can be always changed to $P'^+_{n00} = P_{n00}^+ + \delta P$, where δP integrated against $\mathcal{O}(t,\Omega)$ always yields zero[10]. This residual freedom in choosing P_{n00}^+ simplified the summation in [14]. Substitution of the above formula into (2.39) and changing the order of summation and integration gives the smearing function K.

$$\phi(0) = \sum_n \frac{Y_{00}\ P_n^{(d/2-1,\Delta-d/2)}(1)}{N_{n00}} \int_{-\pi/2}^{\pi/2} dt \int \sqrt{g_\Omega} d\Omega\ P_{n00}^+(t,\Omega)\mathcal{O}_+(t,\Omega) + \text{h.c.} \tag{2.44}$$

$$= \left[\int_{-\pi/2}^{\pi/2} dt \int \sqrt{g_\Omega} d\Omega\ K_+(0|t,\Omega)\mathcal{O}_+(t,\Omega)\right] + \text{h.c.} \tag{2.45}$$

In [14], the freedom to change the projection operator $P_{n00}^+ \to P_{n00}^+ + \delta P$ is used to make K_+ real, thereby making the smearing function for the negative part K_- equal to K_+. The integration kernel for the positive frequency part and the negative frequency part becomes the same, so the final outcome simplifies to an integration kernel for $\mathcal{O}(t,\Omega)$. As the domain of integration at the boundary is the set of points space-like separated from the origin, the global reconstruction

[10]The meaning of this residual freedom is explained in footnote 12 of section 2.2.3.

uses the boundary data on the points space-like separated from the bulk point of interest. The results of [14] are given for future reference.

$$K = \begin{cases} \dfrac{\Gamma(\Delta - \frac{d}{2} + 1)\Gamma(1 - d/2)}{\pi \mathrm{vol}(S^{d-1})\Gamma(\Delta - d + 1)} \lim\limits_{\rho \to \pi/2} (2\sigma \cos\rho)^{\Delta - d} & \text{even} \\ \dfrac{2(-1)^{(d/2-1)}\Gamma(\Delta - d/2 + 1)}{\pi \mathrm{vol}(S^{d-1})\Gamma(\Delta - d + 1)\Gamma(d/2)} \lim\limits_{\rho \to \pi/2} (2\sigma \cos\rho)^{\Delta - d} \log(\sigma \cos\rho) & \text{odd} \end{cases} \tag{2.46}$$

Even means even-dimensional AdS_{d+1} and odd means odd-dimensional AdS_{d+1}. The invariant distance σ is given by the formula (2.27), where $\sigma = \cosh s$. As already explained, the smearing function K has support on boundary spacetime regions spacelike separated from the bulk point of interest[11]. When the bulk point of interest is the origin, the limit $\lim \sigma \cos\rho$ reduces to $\cos t$.

The construction is not very different for AdS-Rindler reconstruction of the bulk in [14]. The only caveat is that the summation diverges when order of integration and summation is changed. The divergent sum is made convergent by analytic continuation of coordinates in [14], but as commented by [24] this does not seem to be an appropriate way of working with the divergent sum. Rather, it seems more appropriate to give an interpretation of the divergent sum in the context of distribution theory [24]. This point of view will be explained in more detail in the following subsection. Another property that is frequently neglected is that the smearing function for the Rindler wedge of boundary domain of dependence $\mathcal{D}[\mathcal{A}]$ vanishes completely on the boundary domain of dependence of the complementary boundary subregion $\mathcal{D}[\mathcal{A}^c]$, which is obvious from the fact that the domain of smearing function for AdS-Rindler reconstruction is $\mathcal{D}[\mathcal{A}]$.

[11]There is a small caveat to this statement. The *domain* of the smearing function K is different from its *support* for a specific bulk point.

2.2.2 Meaning of bulk reconstruction

Bulk reconstruction aims to obtain bulk data from available boundary data. What is the exact meaning of this statement? Bulk data is encoded in the bulk field ϕ, but this field itself is not used as an observable; observables are constructed from the bulk field ϕ through smearing by integration against a suitable test function η [61].

$$\phi[\eta] = \int d^{d+1}x\sqrt{|g|}\eta(x)\phi(x) \tag{2.47}$$

The same applies to observables of the boundary theory [24].

$$\mathcal{O}[\zeta] = \int d^dY\sqrt{|\gamma|}\zeta(Y)\mathcal{O}(Y) \tag{2.48}$$

In both theories, finite sums of finite products of the above smeared fields define the algebra of local observables. When smearing function K is introduced to interpolate the formulas, the following relation is obtained.

$$\phi[\eta] = \int d^{d+1}x\sqrt{|g|}\eta(x)\phi(x) \tag{2.49}$$

$$= \int d^{d+1}x\sqrt{|g|}\eta(x)\int d^dY\sqrt{|\gamma|}K(x|Y)\mathcal{O}(Y) \tag{2.50}$$

$$= \int d^dY\sqrt{|\gamma|}\left[\int d^{d+1}x\sqrt{|g|}\eta(x)K(x|Y)\right]\mathcal{O}(Y) \tag{2.51}$$

$$= \mathcal{O}[\eta^\partial] \tag{2.52}$$

The test function for the boundary η^∂ is defined as follows.

$$\eta^\partial(Y) = \int d^{d+1}x\sqrt{|g|}\eta(x)K(x|Y) \tag{2.53}$$

This is how [24] argued that the problematic divergent behaviour of mode summation in AdS-Rindler reconstruction is actually not a problem. This relation suggests that the smearing function K should rather be considered as a rule to assign bulk test functions to their boundary counterparts, not as a rule that relates bulk field values to their boundary counterparts.

2.2.3 Distilling mode operators and constructing the smearing function

The gist of bulk reconstruction lies in obtaining the mode operators at the boundary. The role of the smearing function K is to automatise the procedure of obtaining a mode operator through a simple integral from the boundary, attaching the corresponding mode function to the mode operator, and summing the result over all mode operators. The problem is that in some cases, it is hard to distill the wanted mode operators by a simple integral. When mode functions of the boundary field are orthogonal, the wanted mode operator can be regained by exploiting this orthogonality. Unfortunately, such a miracle will not happen generally; this is why [14] needed to separate positive frequency modes and negative frequency modes in the beginning. This subsection is devoted to working around this nonorthogonality problem.

Suppose that a region Ξ of boundary spacetime is given as the domain for the wanted smearing function K. In the case of HKLL reconstruction [14], the domain of K is taken to be the whole boundary spacetime for the global reconstruction and boundary of the AdS-Rindler wedge for the AdS-Rindler reconstruction. The mode functions g_{nlm} and g^*_{nlm} constitute a *complete* set of basis[12], while it is hard to find an inner product on the boundary that makes them orthonormal. Nevertheless, it is formally possible to discern contributions from different mode functions. Define the following inner product on the bound-

[12]A complete set of basis for functions satisfying the boundary equation of motion defined by boundary Hamiltonian, which is the dilatation operator. The space of such functions only covers a subspace V_1 of the full boundary function space V of L^2 norm. The residual freedom of shifting the projection operator $P^+ \to P^+ + \delta P$ is equivalent to adding a projection operator for a vector in $V_1^\perp$, the orthogonal complement of V_1. This freedom is interpreted as a kind of gauge freedom in [26] and [33].

ary spacetime region Ξ.

$$(f, g)_\Xi \equiv \int_\Xi f^* g \; dV \tag{2.54}$$

$dV = \sqrt{g_\Omega} dt d\Omega$ is the standard spacetime volume measure of the boundary. The mode functions form a countably infinite basis, which is schematically refered to as g_m. Construct the Gram matrix $\mathbf{g} = g_{mn}$ by the given inner product.

$$g_{mn} \equiv (g_m, g_n)_\Xi \tag{2.55}$$

The Gram matrix is Hermitian, i.e. $\mathbf{g}^\dagger = \mathbf{g}$. Suppose that the inverse of $\mathbf{g}$, $\mathbf{g}^{-1} = g^{mn}$, exists. The inverse is defined by the relation $g^{mn} g_{nl} = g_{ln} g^{nm} = \delta^m_l$. The inverse can be used to define projection operator P^m.

$$P^m(f) \equiv \int_\Xi g^{mn} g^*_n f \; dV \tag{2.56}$$

$$P^m(g_n) = g^{ml} \int_\Xi g^*_l g_n \; dV = g^{ml} g_{ln} = \delta^m_n \tag{2.57}$$

The mode operators $\mathcal{O}_{nlm}$ can be extracted from the given boundary data on Σ by this projection operator P^{nlm}. For convenience, schematic mode index m is used below.

$$\mathcal{O}_{nlm} = N_{nlm} P^{nlm}(\tilde{\mathcal{O}}(t, \Omega)) = N_{nlm} g^{nlm,m'} \int_\Xi g^*_{m'} \tilde{\mathcal{O}}(t, \Omega) \; dV \tag{2.58}$$

$$\mathcal{O}^\dagger_{nlm} = N_{nlm} P^{nlm*}(\tilde{\mathcal{O}}(t, \Omega)) = N_{nlm} g^{nlm*,m'} \int_\Xi g^*_{m'} \tilde{\mathcal{O}}(t, \Omega) \; dV \tag{2.59}$$

The asterisk on P^{nlm*} is intended as a reminder that this operator projects onto g^*_{nlm}. Although this is an unfortunate notation that may cause confusion, the integration kernel for extracting the mode operator will be simplified to $P^\pm_{nlm}$, P^+_{nlm} for $\mathcal{O}_{nlm}$ and P^-_{nlm} for $\mathcal{O}^\dagger_{nlm}$. Given projection operators, obtaining the smearing function is a trivial task.

$$\begin{aligned}\tilde{\phi}(t,\rho,\Omega) &= \sum a_{nlm} f_{nlm}(t,\rho,\Omega) + a^{\dagger}_{nlm} f^{*}_{nlm}(t,\rho,\Omega) \qquad (2.60)\\ &= \sum f_{nlm}(t,\rho,\Omega) \int_{\Xi} P^{+}_{nlm}(t',\Omega')\tilde{\mathcal{O}}(t',\Omega')\ dV' \\ &\quad + \sum f^{*}_{nlm}(t,\rho,\Omega) \int_{\Xi} P^{-}_{nlm}(t',\Omega')\tilde{\mathcal{O}}(t',\Omega')\ dV' \\ &= \sum f_{nlm}(t,\rho,\Omega) N_{nlm} g^{nlm,m'} \int_{\Xi} g^{*}_{m'} \tilde{\mathcal{O}}(t',\Omega')\ dV' \\ &\quad + \sum f^{*}_{nlm}(t,\rho,\Omega) N_{nlm} g^{nlm*,m'} \int_{\Xi} g^{*}_{m'} \tilde{\mathcal{O}}(t',\Omega')\ dV' \\ &= \int_{\Xi} K(t,\rho,\Omega|t',\Omega')\tilde{\mathcal{O}}(t',\Omega') dV' \qquad (2.61)\end{aligned}$$

The explicit expression for the smearing function $K(t,\rho,\Omega|t',\Omega')$ is as follows. Note that this smearing function has support on the whole domain, where domain is taken to be the region Ξ of the boundary spacetime.

$$K(t,\rho,\Omega|t',\Omega') = \sum_{n,l,m} f_{nlm}(t,\rho,\Omega) P^{+}_{nlm}(t',\Omega') + f^{*}_{nlm}(t,\rho,\Omega) P^{-}_{nlm}(t',\Omega') \tag{2.62}$$

2.2.4 Covariance under mode function choices

A natural question is to ask whether the smearing function constructed above depends on the choice of mode functions, which form a basis of the solution space of the wave equation. Some algebra shows that it does not depend on any specific choice: An example is given in [14], where it is shown that smearing functions constructed from global patch mode functions and Poincaré patch mode functions are equivalent up to some irrelevant factor. Having a schematic form of the smearing function is helpful for showing this independence. Schematically, the smearing function constructed can be written as follows.

$$K(x|Y) = \sum f_k(x) g^{kl} g_l^*(Y) \tag{2.63}$$

$$g_{ij} = \int_\Xi dY g_i^*(Y) g_j(Y), \quad g^{ki} g_{ij} = \delta_j^k \tag{2.64}$$

The labels refer to schematic mode indices, i.e. the indices run over mode functions and their complex conjugates collectively. Suppose that another set of mode functions, ${f'}_k(x)$, is given. The smearing function constructed from this set is written in the following way.

$$K'(x|Y) = \sum {f'}_k(x) {g'}^{kl} {g'}_l^*(Y) \tag{2.65}$$

$${g'}_{ij} = \int_\Xi dY {g'}_i^*(Y) {g'}_j(Y), \quad {g'}^{ki} {g'}_{ij} = \delta_j^k \tag{2.66}$$

The mode functions form a basis, so there exists a matrix that relates the different bases.

$$f_k(x) = \sum_l \alpha(k|l) {f'}_l \tag{2.67}$$

$${f'}_k(x) = \sum_l \beta(k|l) f_l \tag{2.68}$$

The matrices α and β are inverses of each other; $\sum \alpha(i|j)\beta(j|k) = \delta_{ik}$ and $\sum \beta(i|j)\alpha(j|k) = \delta_{ik}$. The sets of boundary mode functions g_k and ${g'}_k$ follow the same relations, since boundary mode functions are obtained as some limit of the bulk mode functions. A bit of algebra can show that the relation $K(x|Y) = K'(x|Y)$ holds.

$$K'(x|Y) = \sum {f'}_k(x) {g'}^{kl} {g'}_l^*(Y) \tag{2.69}$$

$$= \sum {f'}_k(x) [\beta^*(i|l) g_{ij} \beta(j|k)]^{-1} \beta^*(l|m) g_m^*(Y)$$

$$= \sum {f'}_k(x) \alpha(j|k) g^{ji} \alpha^*(i|l) \beta^*(l|m) g_m^*(Y)$$

$$= \sum \alpha(j|k) {f'}_k(x) g^{jl} g_l^*(Y)$$

$$= \sum f_k(x) g^{kl} g_l^*(Y) = K(x|Y) \tag{2.70}$$

This relation implies that the construction of the smearing function is independent of the choice of mode functions.

2.2.5 Feasibility of distillation

Since the Gram matrix $\mathbf{g}$ is an infinite dimensional matrix, it is not clear whether it is possible to calculate the inverse matrix elements or not. However, it is possible to write an formal expression that corresponds to $\mathbf{g}^{-1}$. Decompose $\mathbf{g}$ into its diagonal and off-diagonal parts $\mathbf{D}$ and $\mathbf{h}$. $\mathbf{D}^{-1}$ is easily calculated since none of the diagonal components of $\mathbf{g}$ are zero. Then $\mathbf{g}^{-1}$ has the following formal expression.

$$\mathbf{g} = \mathbf{D} - \mathbf{h} \tag{2.71}$$

$$\mathbf{g}^{-1} = \mathbf{D}^{-1}(\mathbb{1} + \bar{\mathbf{h}} + \bar{\mathbf{h}}^2 + \bar{\mathbf{h}}^3 + \cdots) \tag{2.72}$$

$$\bar{\mathbf{h}} = \mathbf{h}\mathbf{D}^{-1} \tag{2.73}$$

If the spectrum λ_m of $\bar{\mathbf{h}}$ satisfies the criteria $\forall m\{|\lambda_m| < 1\}$ then this formal expression is exact. Unfortunately, there is no good criteria for determining the upper bound for the spectrum of $\bar{\mathbf{h}}$, so invertibility of $\mathbf{g}$ is not guaranteed.

On the other hand, noninvertibility of $\mathbf{g}$ seems natural for reasons given in section 3.2.2. A possible resolution is to consider the theory on both sides of the correspondence to be an effective description. Introducing a cut-off to n and l restricts the dimension of the vector space generated by mode functions to be finite and $\mathbf{g}$ becomes a finite dimensional matrix, which is always invertible. Considering that GFT is only an effective description of a CFT [31], this seems to be a natural resolution to the noninvertibility problem.

2.3 Reconstructibility of the entanglement wedge

2.3.1 Quantum error correction

To understand the proof given by DHW [28] that entanglement wedge can be reconstructed, it is necessary to understand the viewpoint of ADH [25]. The work of ADH [25] put forward the viewpoint that AdS/CFT correspondence has a structure similar to quantum error correcting codes. Quantum error correction refers to protective measures aimed to prevent quantum data from being spoiled by external noise, and quantum error correcting codes are explicit encoding mechanisms to implement quantum error correction. Just like its classical counterpart, redundant data are used to compensate for loss in any quantum error correcting codes. Since the purpose of reviewing quantum error correction is to provide a better understanding of ADH proposal, the presentation of the topic will be limited to materials relevant for understanding the new viewpoint. For reader's information, the standard references to quantum error correction are [56, 57].

A quantum error correcting code relevant for understanding the ADH proposal is the three qutrit code [25]. A qutrit is a three-state quantum system, and qutrits can encode a message written in three alphabets 0,1, and 2. In three qutrit code, one qutrit data is encoded by three qutrits in the following way.

$$\begin{aligned} |0\rangle \to \left|\tilde{0}\right\rangle &= \frac{1}{\sqrt{3}}\left(|000\rangle + |111\rangle + |222\rangle\right) \\ |1\rangle \to \left|\tilde{1}\right\rangle &= \frac{1}{\sqrt{3}}\left(|012\rangle + |120\rangle + |201\rangle\right) \\ |2\rangle \to \left|\tilde{2}\right\rangle &= \frac{1}{\sqrt{3}}\left(|021\rangle + |102\rangle + |210\rangle\right) \end{aligned} \tag{2.74}$$

In any quantum error correcting code, only a subspace of the full Hilbert space is used; this subspace is called the *code subspace*. A notable feature of this code is that operations on only two qutrits is enough to manipulate data stored in the set, i.e. it is resilient to erasure of one of the qutrits. For example, there exists U_{12} such that

$$(U_{12} \otimes \mathbb{1}_3) \left| \tilde{\psi} \right\rangle = |\psi\rangle \otimes \frac{1}{\sqrt{3}} \left(|00\rangle + |11\rangle + |22\rangle \right) \tag{2.75}$$

so any intended manipulation on the stored data—also called *logical operations*—can be implemented by manipulating the first two qutrits. Another feature of this code is that the stored data can never be accessed by a single qutrit. This can be checked by computing the reduced density matrix of any state in the code subspace, which always gives a maximally mixed state.

$$\rho_1 = \mathrm{Tr}_{23} \left| \tilde{\psi} \right\rangle \left\langle \tilde{\psi} \right| = \frac{1}{3} \left(|0\rangle \langle 0| + |1\rangle \langle 1| + |2\rangle \langle 2| \right) \tag{2.76}$$

Similar properties emerge in bulk reconstruction of AdS/CFT correspondence and leads to ADH proposal [25] that AdS/CFT has a quantum error correcting code-like structure.

2.3.2 AdS/CFT as quantum error correction

Bulk reconstruction in subregion-subregion duality is pertinent for ADH proposal, and the starting point for bulk reconstruction of subregions is the AdS-Rindler reconstruction [14]. Just like its Minkowski space counterpart, a time-slice of AdS-Rindler coordinates covers half of the time-slice of global AdS coordinates. The boundary time-slice of AdS-Rindler coordinates is a hemisphere, which covers half of the time-slice of global AdS coordinates boundary. The bulk spacetime covered by AdS-Rindler coordinates is also called the AdS-Rindler wedge, and this is the bulk region that is reconstructed in AdS-Rindler reconstruction. In what follows, it is implied that statements about any subregion refer to a time-slice, unless stated otherwise.

In AdS-Rindler reconstruction the data in the boundary of AdS-Rindler wedge is used to reconstruct bulk data [14]. Calling the boundary hemisphere as H, the boundary of AdS-Rindler wedge turns out to be boundary domain of dependence $\mathcal{D}[H]$ of H. The boundary domain of dependence $\mathcal{D}[H]$ is defined as the set of boundary points p with the property that any fully extended causal curve passing through p must intersect H. In theory, all data in $\mathcal{D}[H]$ must be reconstructible from data in H by time evolution. Therefore, the whole AdS-Rindler wedge is theoretically reconstructible from data in H [25]. Note that the AdS-Rindler wedge is also the causal wedge $\mathcal{W}_C[H]$ of boundary subregion H defined in footnote **??** of section 1.3. An alternate definition of $\mathcal{W}_C[H]$ is the intersection $\mathcal{J}^+[\mathcal{D}[H]] \cap \mathcal{J}^-[\mathcal{D}[H]]$ of the bulk future $\mathcal{J}^+[\mathcal{D}[H]]$ of $\mathcal{D}[H]$ and the bulk past $\mathcal{J}^-[\mathcal{D}[H]]$ of $\mathcal{D}[H]$.

Since not all boundary subregions are a hemisphere, it is necessary to generalise bulk reconstruction to boundary subregions other than a hemisphere. Isometries of AdS bulk, which corresponds to conformal transformations of the boundary, can be used to map AdS-Rindler reconstruction to reconstruction from a boundary ball. From boundary—which is $(d-1)$-dimensional sphere S^{d-1}—point of view, the hemisphere covered by the boundary of the AdS-Rindler wedge is a ball $\mathcal{B}^{d-1}$ in S^{d-1} with the largest surface area, so a ball-shaped boundary subregion can be mapped to the hemisphere of the AdS-Rindler wedge by a conformal transformation. Denoting this boundary ball as A, it follows that the bulk reconstructible from A by this conformal transformation is the causal wedge $\mathcal{W}_C[A]$. Thus, this method of reconstructing the bulk from a given boundary ball will be called *causal wedge reconstruction.*

Figure 2.1 schematically shows how causal wedge reconstruction from a single boundary subregion A works. The green region is the boundary domain of dependence $\mathcal{D}[A]$, and the bulk spacetime reconstructible from data in A is the wedge formed by $\mathcal{D}[A]$ and χ_A, which is nothing but the causal wedge

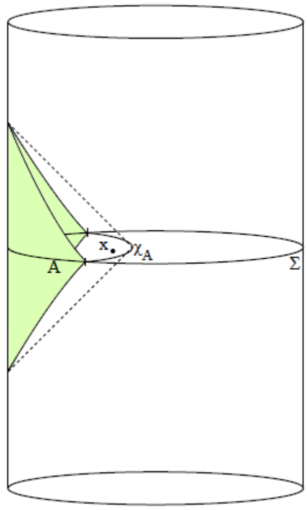

Figure 2.1 The causal wedge reconstruction. The green diamond is the boundary domain of dependence of A. Bulk is reconstructed from data in this green region, and data in this region is determined by data in A. Thus, the causal wedge can be reconstructed from data in A. This picture is taken from [25].

$\mathcal{W}_C[A]$. The "rim" χ_A of $\mathcal{W}_C[A]$ is called the causal surface of A, which is the intersection of past and future horizons of $\mathcal{D}[A]$ [25]. In pure AdS_3, the causal surface χ_A is equal to the geodesic that connects the endpoints of A.

Since any connected interval in S^1 is a ball, bulk reconstruction from subregion is particularly simple in AdS_3. This simplifies a lot of discussions, so AdS_3 was used to motivate the new viewpoint in [25]. Consider the following set-up as in figure 2.2. In the figure, the bulk that can be reconstructed from data in A is coloured blue, while the bulk that can be reconstructed from data in B is coloured yellow. The bulk point X lies inside the intersection of the blue region and yellow region, so the bulk point X can be reconstructed from either the data in A or data in B. A naïve expectation is that since the full data of X is contained in A and in B, the full data of X should also be contained in the intersection $A \cap B$. However, the bulk point X lies outside the causal surface

$\chi_{A\cap B}$ which is the bulk geodesic connecting the endpoints of $A\cap B$, so it should not be reconstructible by data in $A \cap B$.

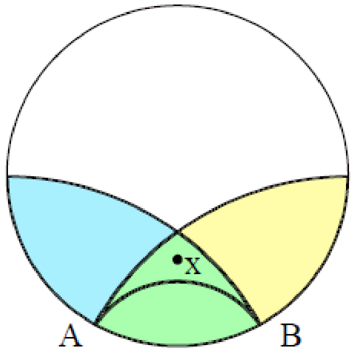

Figure 2.2 The bulk subregion that can be reconstructed from boundary subregion A is coloured blue, while the bulk subregion that can be reconstructed from boundary subregion B is coloured yellow. The bulk point X can be reconstructed from boundary subregion A and boundary subregion B, but it cannot be reconstructed from their intersection $A \cap B$. This picture is taken from [25].

This is reminiscent of the properties of quantum error correcting codes briefly reviewed in section 2.3.1. In analogy with three qutrit code reviewed in 2.3.1, the boundary subregion A can be thought as the first and second qutrits, the boundary subregion B can be thought as the second and third qutrits, the intersection $A \cap B$ can be thought as the second qutrit, and the bulk point X can be thought as the data $\left|\tilde{\psi}\right\rangle$ stored in the three-qutrit set. This data can be read whenever two qutrits of the whole set is accessible, but it cannot be read if only one qutrit of the whole set is accessible. This is the example that motivated ADH viewpoint of understanding AdS/CFT as having quantum error correcting code-like structure [25].

In ADH viewpoint, the bulk Hilbert space $\mathcal{H}_{bulk}$ is viewed as a code subspace $\mathcal{H}_{code}$ of the full boundary Hilbert space $\mathcal{H}_{CFT}$. A bulk operator $\mathcal{O}_b : \mathcal{H}_{bulk} \to \mathcal{H}_{bulk}$ can be realised by a boundary operator $\mathcal{O} : \mathcal{H}_{CFT} \to \mathcal{H}_{CFT}$ such that the following relation holds for any state $|\psi\rangle \in \mathcal{H}_{bulk} = \mathcal{H}_{code}$.

$$\mathcal{O}_b |\psi\rangle = \mathcal{O} |\psi\rangle \tag{2.77}$$

$$\mathcal{O}_b^\dagger |\psi\rangle = \mathcal{O}^\dagger |\psi\rangle \tag{2.78}$$

The error correcting code-like structure appears when operators that acts only on a subspace of the full Hilbert space are considered. Suppose that the bulk and boundary Hilbert space factorises into $\mathcal{H}_{bulk} = \mathcal{H}_a \otimes \mathcal{H}_{\bar{a}}$ and $\mathcal{H}_{CFT} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$. Consider a bulk operator that acts on the factorised part of the bulk Hilbert space $O_b : \mathcal{H}_a \to \mathcal{H}_a$ that can be realised by a boundary operator $O : \mathcal{H}_A \to \mathcal{H}_A$, meaning that the analogues of equations (2.77) and (2.78) hold for O_b and O. This is equivalent to the statement that for any operator $X_{\bar{A}} : \mathcal{H}_{\bar{A}} \to \mathcal{H}_{\bar{A}}$ and any states $|\psi\rangle, |\phi\rangle \in \mathcal{H}_{bulk}$ the following relation holds [25].

$$\langle\psi| [O_b, X_{\bar{A}}] |\phi\rangle = 0 \tag{2.79}$$

There is some arbitrariness in the choice of the boundary factorisation $\mathcal{H}_{CFT} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$, which depends on the choice of the bulk factorisation $\mathcal{H}_{bulk} = \mathcal{H}_a \otimes \mathcal{H}_{\bar{a}}$[13]. This arbitrariness can be exploited to realise the operator O_b by O which does not act on the inaccessible part $\mathcal{H}_{\bar{A}}$ of the full Hilbert space, so that any error that occurred on $\mathcal{H}_{\bar{A}}$ does not affect the action of the operator O_b. When a different error occurred so that a different part $\mathcal{H}_{\bar{A}'}$ of the full Hilbert space $\mathcal{H}_{CFT} = \mathcal{H}_{A'} \otimes \mathcal{H}_{\bar{A}'}$ is inaccessible, a different realisation $O' : \mathcal{H}_{A'} \to \mathcal{H}_{A'}$ can be used to realise the same operator O_b. This resiliency in arbitrary erasure of the boundary is the property ADH proposal has sought to capture [25].

[13] This arbitrariness is greatest when the factor Hilbert space of the bulk $\mathcal{H}_a$ is taken to be the Hilbert space generated by states obtained from the vacuum by acting operators localised near the centre of the AdS. This is why [25] claims that this choice is the most robust code subspace against arbitrary boundary erasures.

2.3.3 Proof of entanglement wedge reconstruction

It was already known in the literature that the bulk reconstructible from the given boundary subregion is larger than the causal wedge, and it was conjectured to be the entanglement wedge [25, 27]. However, there was no direct proof of its reconstructibility until recently; [27] has only shown that the entanglement wedge satisfies the necessary conditions that the bulk spacetime reconstructible from the given boundary subregion should satisfy.

The proof that entanglement wedge can be reconstructed was given by DHW [28][14]. The DHW argument [28] relies on the observation of JLMS that relative entropies computed in the bulk and the boundary coincide [30]. Decompose the boundary Hilbert space by $\mathcal{H}_{CFT} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$, and decompose the bulk Hilbert space by $\mathcal{H}_{bulk} = \mathcal{H}_a \otimes \mathcal{H}_{\bar{a}}$ where $\mathcal{H}_a$ denotes the Hilbert space of bulk excitations in the entanglement wedge $\mathcal{E}[A]$ of boundary region A. Likewise, $\mathcal{H}_{\bar{a}}$ denotes the Hilbert space of bulk excitations in $\mathcal{E}[\bar{A}]$. The JLMS result [30] can be formulated as follows, which is the form required by the DHW argument.

$$S(\rho_{\bar{A}}|\sigma_{\bar{A}}) = S(\rho_{\bar{a}}|\sigma_{\bar{a}}) \tag{2.80}$$

The left hand side gives the relative entropy of the boundary subregion theory, while the right hand side gives the relative entropy of the bulk theory *inside the entanglement wedge* of the given boundary subregion. Since relative entropy $S(\rho|\sigma) = 0$ if and only if $\rho = \sigma$, the following conclusion can be derived from the JLMS result.

$$\rho_{\bar{a}} = \sigma_{\bar{a}} \Longrightarrow \rho_{\bar{A}} = \sigma_{\bar{A}} \tag{2.81}$$

What has been shown by DHW [28] is that (2.81) implies (2.79) for density matrices formed by partial trace over pure states. To be specific, the density

[14] [29] gives another approach to the subject, albeit restricted to pure AdS space.

matrices $\rho_{\bar{A}}$, $\sigma_{\bar{A}}$, $\rho_{\bar{a}}$, and $\sigma_{\bar{a}}$ are assumed to be obtained by the following partial traces.

$$
\begin{aligned}
\rho_{\bar{A}} \equiv \mathrm{Tr}_A \left|\phi\right\rangle \left\langle\phi\right|, \quad \sigma_{\bar{A}} \equiv \mathrm{Tr}_A \left|\psi\right\rangle \left\langle\psi\right| \\
\rho_{\bar{a}} \equiv \mathrm{Tr}_a \left|\phi\right\rangle \left\langle\phi\right|, \quad \sigma_{\bar{a}} \equiv \mathrm{Tr}_a \left|\psi\right\rangle \left\langle\psi\right|
\end{aligned}
\tag{2.82}
$$

This in turn implies that equivalents of (2.77) and (2.78) hold for $O_b : \mathcal{H}_a \to \mathcal{H}_a$ and $O : \mathcal{H}_A \to \mathcal{H}_A$. The interpretation is clear; *an operator O_b that acts on $\mathcal{E}[A]$ can be realised by an operator O that acts on the boundary subregion A*. In other words, the bulk dual of boundary subregion is its entanglement wedge. Note that this constructive proof does not give an explicit procedure for reconstructing the entanglement wedge [28].

Chapter 3

Entanglement wedge reconstruction

3.1 Subregion decomposition of the bulk

3.1.1 Separation of Hilbert space into subregion Hilbert spaces

Conventionally, the Hilbert space of a QFT is constructed as the Fock space obtained from creation and annihilation operators. Creation and annihilation operators are obtained from a mode function decomposition of the corresponding operator, and the inner product (2.7) defined between functions can be used to explicitly decompose the operator.

$$\phi(t,x) = \sum_k a_k f_k(t,x) + a_k^\dagger f_k^*(t,x) \tag{3.1}$$

$$a_k = (f_k|\phi)_\Sigma \tag{3.2}$$

$$a_k^\dagger = -(f_k^*|\phi)_\Sigma \tag{3.3}$$

k denotes the set of indices, discrete or continuous, that characterises each distict mode of a complete set. The mode functions are assumed to satisfy the conventional normalisation $(f_k|f_{k'})_\Sigma = -(f_k^*|f_{k'}^*)_\Sigma = \delta_{kk'}$.

A similar construction can be used to build the Hilbert spaces for subregions. A state in the Hilbert space $\mathcal{H}$ of a QFT is determined by the field configuration on a time slice Σ. If the time slice Σ is decomposed into a discrete set of subregions Σ_α, the field configuration on the subregion Σ_α determines a state in the subregion Hilbert space $\mathcal{H}_\alpha$. The tensor product of subregion Hilbert spaces becomes the total Hilbert space, i.e. $\mathcal{H} = \prod \mathcal{H}_\alpha$. The complete set of basis for a subregion Hilbert space $\mathcal{H}_\alpha$ can be constructed in an analogous way to the full Hilbert space $\mathcal{H}$; find the normalised subregion mode functions $f_{\alpha|k}$ and $f^*_{\alpha|k}$, use the inner product modified for subregions $(f|g)_{\Sigma_\alpha}$ to find the creation and annihilation operators for subregion Σ_α, and use the mode operators to construct the Fock states for subregion Σ_α.

Having a new coordinate patch simplifies finding the normalised subregion mode functions, at least conceptually. Given a subregion Σ_α of the time slice Σ, it is possible to find a coordinate patch U_α that only covers the bulk domain of dependence $\mathcal{D}_B[\Sigma_\alpha]$[1]. The subregion mode functions are solutions to the wave equation that only has support on the coordinate patch U_α and regions of the full spacetime causally connected to Σ_α; in other words, subregion mode functions for the subregion Σ_α are solutions to the wave equation that does not have any support on coordinate patch(es) U_β, $\beta \neq \alpha$. Solving the wave equation on U_α and finding a complete set of solutions with Dirichlet boundary conditions[2] give the subregion mode functions $f_{\alpha|k}$ and $f^*_{\alpha|k}$.

[1] $\mathcal{D}_B[\Sigma_\alpha]$ is a set of bulk points which any fully extended causal curve passing through the point must intersect Σ_α.

[2] Treating the fields as operator valued *distributions* play a subtle role here as Dirichlet boundary conditions to subregion mode functions implies that $\phi(x)$ at $x \in \partial\Sigma_\alpha$ for some α

The inner product between subregion mode functions can be defined in an analogous way.

$$(f|g)_{\Sigma_\alpha} \equiv i \int_{\Sigma_\alpha} \sqrt{h} dy \; n^\mu \left(f^* \nabla_\mu g - g \nabla_\mu f^* \right) \tag{3.4}$$

Σ_α denotes the subregion of the time slice, the spacelike surface on which the inner product is evaluated. Other symbols are defined in the same way as in (2.7). The field ϕ has an expansion in subregion mode functions and subregion mode operators, and subregion mode operators admit an expression in terms of subregion inner products.

$$\phi(t,x) = \sum_{\alpha,k} a_{\alpha|k} f_{\alpha|k}(t_\alpha, x_\alpha) + a^\dagger_{\alpha|k} f^*_{\alpha|k}(t_\alpha, x_\alpha) \tag{3.5}$$

$$a_{\alpha|k} = (f_{\alpha|k}|\phi)_{\Sigma_\alpha} \tag{3.6}$$

$$a^\dagger_{\alpha|k} = -(f^*_{\alpha|k}|\phi)_{\Sigma_\alpha} \tag{3.7}$$

The coordinates (t_α, x_α) refers to the coordinate labels of the coordinate patch U_α; the coordinate patch that covers $\mathcal{D}_B[\Sigma_\alpha]$.

The Fock states or the single particle states that are used as the basis for constructing the subregion Hilbert space can be constructed in the same way as how Fock states for the full Hilbert space in conventional QFT is constructed, so it will not be elaborated here. Having constructed the Fock states for each subregion Hilbert space $\mathcal{H}_\alpha$, the bases for the full Hilbert space $\mathcal{H}$ can be constructed as tensor products of subregion Fock states.

3.1.2 Relationships between mode operators

It turns out that the mode operators a_k and $a^\dagger_k$ of the full Hilbert space can be expressed as a linear combination of subregion mode operators $a_{\alpha|k}$ and $a^\dagger_{\alpha|k}$,

is always zero, where $\partial\Sigma_\alpha$ denotes the boundary of Σ_α. In the context of distribution theory, however, this does not matter since the set $\partial\Sigma_\alpha$ constitutes a measure zero set and operators are obtained by integrating the field ϕ with respect to the corresponding test functions.

and vice versa. The relation can be obtained from two distinct but equivalent mode decomposition of the field $\phi(x,t)$.

$$\phi(t,x) = \sum_k a_k f_k(t,x) + a_k^\dagger f_k^*(t,x) \tag{3.8}$$

$$= \sum_{\alpha,k} a_{\alpha|k} f_{\alpha|k}(t_\alpha, x_\alpha) + a_{\alpha|k}^\dagger f_{\alpha|k}^*(t_\alpha, x_\alpha) \tag{3.9}$$

It is more instructive to write the mode expansion in the following form, which is motivated by the fact that mode functions can be considered as vectors of a vector space.

$$\phi(t,x) = \sum_k a_k |f_k) + a_k^\dagger |f_k^*) \tag{3.10}$$

$$= \sum_{\alpha,k} a_{\alpha|k} |f_{\alpha|k}) + a_{\alpha|k}^\dagger |f_{\alpha|k}^*) \tag{3.11}$$

Dual vectors $(f|$ are defined as functionals that map functions that satisfy the wave equation $|g)$ to numbers $(f|g)$. Generalised Bogoliubov relations relating a_k and $a_k^\dagger$ to $a_{\alpha|k}$ and $a_{\alpha|k}^\dagger$ can be read out from the above expression using the inner product between mode functions.

$$a_k = (f_k|\phi)_\Sigma = \sum_{\alpha,q} (f_k|f_{\alpha|q})_{\Sigma_\alpha} a_{\alpha|q} + (f_k|f_{\alpha|q}^*)_{\Sigma_\alpha} a_{\alpha|q}^\dagger \tag{3.12}$$

$$a_k^\dagger = -(f_k^*|\phi)_\Sigma = \sum_{\alpha,q} -(f_k^*|f_{\alpha|q})_{\Sigma_\alpha} a_{\alpha|q} - (f_k^*|f_{\alpha|q}^*)_{\Sigma_\alpha} a_{\alpha|q}^\dagger \tag{3.13}$$

$$a_{\alpha|k} = (f_{\alpha|k}|\phi)_{\Sigma_\alpha} = \sum_q (f_{\alpha|k}|f_q)_{\Sigma_\alpha} a_q + (f_{\alpha|k}|f_q^*)_{\Sigma_\alpha} a_q^\dagger \tag{3.14}$$

$$a_{\alpha|k}^\dagger = -(f_{\alpha|k}^*|\phi)_{\Sigma_\alpha} = \sum_q -(f_{\alpha|k}^*|f_q)_{\Sigma_\alpha} a_q - (f_{\alpha|k}^*|f_q^*)_{\Sigma_\alpha} a_q^\dagger \tag{3.15}$$

Using inner product relations, the above expression can be recast in a more familiar form.

$$a_k = \sum_{\alpha,q} (f_k|f_{\alpha|q})_{\Sigma_\alpha} a_{\alpha|q} + (f_k|f^*_{\alpha|q})_{\Sigma_\alpha} a^\dagger_{\alpha|q} \tag{3.16}$$

$$a^\dagger_k = \sum_{\alpha,q} \left[(f_k|f_{\alpha|q})_{\Sigma_\alpha}\right]^* a^\dagger_{\alpha|q} + \left[(f_k|f^*_{\alpha|q})_{\Sigma_\alpha}\right]^* a_{\alpha|q} \tag{3.17}$$

$$a_{\alpha|k} = \sum_{q} \left[(f_q|f_{\alpha|k})_{\Sigma_\alpha}\right]^* a_q - (f_q|f^*_{\alpha|k})_{\Sigma_\alpha} a^\dagger_q \tag{3.18}$$

$$a^\dagger_{\alpha|k} = \sum_{q} (f_q|f_{\alpha|k})_{\Sigma_\alpha} a^\dagger_q - \left[(f_q|f^*_{\alpha|k})_{\Sigma_\alpha}\right]^* a_q \tag{3.19}$$

In terms of the usual generalised Bogoliubov coefficients α and β, the coefficient $(f_q|f_{\alpha|k})_{\Sigma_\alpha}$ corresponds to the coefficient $\alpha(\alpha, k|q)$ and the coefficient $(f_q|f^*_{\alpha|k})_{\Sigma_\alpha}$ corresponds to the coefficient $\beta^*(\alpha, k|q)$. The analogues of the unitarity condition $|\alpha|^2 - |\beta|^2 = 1$ are the following two relations.

$$\delta_{kk'} = \sum_{\alpha,q} \alpha(\alpha, q|k)\alpha^*(\alpha, q|k') - \beta^*(\alpha, q|k)\beta(\alpha, q|k') \tag{3.20}$$

$$\delta_{\alpha\alpha'}\delta_{kk'} = \sum_{q} \alpha^*(\alpha, k|q)\alpha(\alpha', k'|q) - \beta^*(\alpha, k|q)\beta(\alpha', k'|q) \tag{3.21}$$

The analogues of the unitarity condition $\alpha\beta - \beta\alpha = 0$ are the following two relations.

$$0 = \sum_{\alpha,q} \alpha(\alpha, q|k)\beta^*(\alpha, q|k') - \beta^*(\alpha, q|k)\alpha(\alpha, q|k') \tag{3.22}$$

$$0 = \sum_{q} \alpha(\alpha, k|q)\beta(\alpha', k'|q) - \beta(\alpha, k|q)\alpha(\alpha', k'|q) \tag{3.23}$$

After a bit of algebra, the bra-ket notation for mode functions yields equivalent but more intuitive expressions.

$$\delta_{kk'} = -(f_k^*| \left[\sum_{\alpha,q} |f_{\alpha|q})(f_{\alpha|q}| - |f_{\alpha|q}^*)(f_{\alpha|q}^*| \right] |f_{k'}^*) \tag{3.24}$$

$$\delta_{\alpha\alpha'}\delta_{kk'} = -(f_{\alpha|k}^*| \left[\sum_{q} |f_q)(f_q| - |f_q^*)(f_q^*| \right] |f_{\alpha'|k'}^*) \tag{3.25}$$

$$0 = -(f_k| \left[\sum_{\alpha,q} |f_{\alpha|q})(f_{\alpha|q}| - |f_{\alpha|q}^*)(f_{\alpha|q}^*| \right] |f_{k'}^*) \tag{3.26}$$

$$0 = -(f_{\alpha|k}^*| \left[\sum_{q} |f_q)(f_q| - |f_q^*)(f_q^*| \right] |f_{\alpha'|k'}) \tag{3.27}$$

The meaning is clear; the sum $\sum |f_k)(f_k| - |f_k^*)(f_k^*|$ over an index of a complete set k is nothing but the identity.

3.1.3 Operator truncation and mode function resummation

Suppose that subregion Σ_β is not available. The state is now better described by the reduced density matrix $\rho = \mathrm{Tr}_{\mathcal{H}_\beta} |\psi\rangle \langle\psi|$ and only the operators that do not act on $\mathcal{H}_\beta$ are considered. In applications to QFT, however, obtaining the coefficients for reduced density matrices is a daunting task if not impossible. Therefore it is more desirable to work with the state $|\psi\rangle$ rather than the reduced density matrix. This is the motivation for developing *operator truncation.*

For simplicity, assume that the total Hilbert space $\mathcal{H}$ is a tensor product of two subsystem Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Also, assume that all operators of interest admit the following decomposition,

$$O = O_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes O_2 \tag{3.28}$$

where $\mathbb{1}_1$ and $\mathbb{1}_2$ are identity operators of $\mathcal{H}_1$ and $\mathcal{H}_2$, $O_1 : \mathcal{H}_1 \to \mathcal{H}_1$ is an operator of $\mathcal{H}_1$, and $O_2 : \mathcal{H}_2 \to \mathcal{H}_2$ is an operator of $\mathcal{H}_2$, respectively. The class of operators having a decomposition of the form (3.28) consists of linear combination of local operators.[3]

Suppose subsystem 2 is inaccessible. The information that can be withdrawn is encoded in the reduced density matrix $\rho_1 = \mathrm{Tr}_{\mathcal{H}_2} |\psi\rangle \langle\psi|$. In this case, the natural modification to an operator $O : \mathcal{H} \to \mathcal{H}$ would be to trace it over $\mathcal{H}_2$ and divide by $N_2 = \dim \mathcal{H}_2$.

$$|\psi\rangle \quad \to \quad \rho_1 = \mathrm{Tr}_{\mathcal{H}_2} |\psi\rangle \langle\psi| \tag{3.29}$$

$$O \quad \to \quad O_{\mathrm{red}} = \frac{1}{N_2} \mathrm{Tr}_{\mathcal{H}_2} O \tag{3.30}$$

If the operator O is of the form (3.28) and $\mathrm{Tr}_{\mathcal{H}} O = 0$[4] holds, something more interesting can be said. Define $\tilde{O}$, the *truncation of* O, as follows.

$$O = O_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes O_2 \quad \to \quad \tilde{O} = O_1 \otimes \mathbb{1}_2 \tag{3.31}$$

$\tilde{O} = O_{\mathrm{red}} \otimes \mathbb{1}_2$ up to a multiple of identity because $\mathrm{Tr}_{\mathcal{H}} O = 0$. If any of $\mathrm{Tr}_{\mathcal{H}_1} O_1 = 0$ or $\mathrm{Tr}_{\mathcal{H}_2} O_2 = 0$ holds together with $\mathrm{Tr}_{\mathcal{H}} O = 0$, then the equality is exact. Ignoring the multiple of identity, the following relation holds as well.

$$\mathrm{Tr}_{\mathcal{H}_1} [O_{\mathrm{red}} \rho_1] = \mathrm{Tr}_{\mathcal{H}_1} [O_1 \rho_1] = \langle\psi| \tilde{O} |\psi\rangle \tag{3.32}$$

The formula (3.32) suggests that through appropriate truncation of operators, it is possible to work with the ground state instead of the reduced density matrix.

Truncation defined above can be readily generalised to free scalar theory in

[3]In quantum information language, the class of operators having the form $O_1 \otimes \mathbb{1}_2$ or $\mathbb{1}_1 \otimes O_2$ are called local operations [60].

[4]This condition can be always met by substracting an appropriate multiple of identity from O.

AdS, since the full operator is already in a form similar to (3.28).

$$\phi(t,x) = \sum_{\alpha,k} a_{\alpha|k}|f_{\alpha|k}) + a^{\dagger}_{\alpha|k}|f^{*}_{\alpha|k}) \tag{3.33}$$

$$= \sum_{\alpha\in A,k} \left[a_{\alpha|k}|f_{\alpha|k}) + a^{\dagger}_{\alpha|k}|f^{*}_{\alpha|k})\right] + \sum_{\alpha\notin A,k} \left[a_{\alpha|k}|f_{\alpha|k}) + a^{\dagger}_{\alpha|k}|f^{*}_{\alpha|k})\right] \tag{3.34}$$

Considering A as the set of accessible subregions, the truncated operator $\tilde{\phi}$ has the following mode expansion.

$$\tilde{\phi}(t,x) = \sum_{\alpha\in A,k} a_{\alpha|k}\ |f_{\alpha|k}) + a^{\dagger}_{\alpha|k}\ |f^{*}_{\alpha|k}) \tag{3.35}$$

A little algebra using generalised Bogoliubov transforms reveals that $\tilde{\phi}$ can be expanded by full Hilbert space mode operators a_k and $a^{\dagger}_k$.

$$\tilde{\phi}(t,x) = \sum_{\alpha\in A,k} a_k\left[\mathbb{1}_{\alpha}|f_k)\right] + a^{\dagger}_k\left[\mathbb{1}_{\alpha}|f^{*}_k)\right] \tag{3.36}$$

$$= \sum_{k} a_k\ |\tilde{f}_k) + a^{\dagger}_k\ |\tilde{f}^{*}_k) \tag{3.37}$$

$$\mathbb{1}_{\alpha} \equiv \sum_{k} |f_{\alpha|k})(f_{\alpha|k}| - |f^{*}_{\alpha|k})(f^{*}_{\alpha|k}| \tag{3.38}$$

$$|\tilde{f}_k) \equiv \sum_{\alpha\in A} \mathbb{1}_{\alpha}|f_k) = \sum_{\alpha\in A,k'} |f_{k'})(f_{k'}|f_k)_{\Sigma_\alpha} - |f^{*}_{k'})(f^{*}_{k'}|f_k)_{\Sigma_\alpha} \tag{3.39}$$

It is worthy of note that $\mathbb{1}_{\alpha}$ acts as an projection operator, since the mode functions $f_{\alpha|k}$ and $f^{*}_{\alpha|k}$ are assumed to constitute a complete basis for the solution space of the wave equation in domain of dependence of subregion α. When this operator is sandwiched in between the inner product, it restricts the region of integration to Σ_{α}.

$$(f|g)_{\Sigma} = i\int_{\Sigma}\sqrt{h}dy\ n^{\mu}\left(f^{*}\nabla_{\mu}g - g\nabla_{\mu}f^{*}\right) \tag{3.40}$$

$$(f|\mathbb{1}_{\alpha}|g)_{\Sigma} = i\int_{\Sigma_{\alpha}}\sqrt{h}dy\ n^{\mu}\left(f^{*}\nabla_{\mu}g - g\nabla_{\mu}f^{*}\right) \tag{3.41}$$

This means resummed mode functions $\tilde{f}_k$ and $\tilde{f}_k^*$ can be evaluated without directly working out the solutions to the equation of motion for subregion coordinate patches. Note that in $\mathcal{D}_B[\cup_{\alpha\in A}\Sigma_\alpha]$ the resummed mode function $\tilde{f}_k$ and the original mode function f_k have the same value. This property will play a critical role in the following section.

3.2 Building smearing functions for the entanglement wedge

3.2.1 Restriction to subregions of the boundary

The Hilbert space of the whole boundary theory was given as span of states created from the vacuum by applying mode operators. Naïve expectation for the boundary subregion Hilbert space is that it is given as the span of states created from the vacuum by applying boundary subregion mode operators; this is what happened for the bulk.

Unfortunately, it is in general not easy to find a procedure that can construct mode operators for the subregions at the boundary. Suppose that a boundary subregion A of a time slice Σ is given. The subregion Hilbert space $\mathcal{H}_A$ should describe physics on the boundary domain of dependence $\mathcal{D}[A]$ of A. To construct the mode operators following the procedures of section 2.1.2, construct a coordinate patch that covers $\mathcal{D}[A]$ and Wick rotate the time coordinate to Euclidean signature. Then use a conformal transformation including Weyl rescaling that maps the "past infinity" of $\mathcal{D}[A]$, defined as the most causally past point of $\mathcal{D}[A]$, to the origin. The states in $\mathcal{H}_A$ is then given by states generated by inserting operators at the origin via operator-state correspondence. In theory everything should work. In practice it is almost impossible to find such a map for a general subregion A.

On the other hand, it is possible to impose the following conditions on the truncated operators.

$$\tilde{\mathcal{O}}(x) = 0 \qquad \text{for } x \in \mathcal{D}[A^c] \tag{3.42}$$

$$\langle \tilde{\mathcal{O}}(x_1)\tilde{\mathcal{O}}(x_2)\cdots\tilde{\mathcal{O}}(x_n)\rangle = \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\cdots\mathcal{O}(x_n)\rangle \quad \text{for } x_1, x_2, \cdots, x_n \in \mathcal{D}[A] \tag{3.43}$$

The first condition requires the truncated operators to be identically zero in the domain of dependence $\mathcal{D}[A^c]$ of the complementary subregion $A^c = \Sigma - A$. The second condition requires the truncated operators to be indistinguishable from the original operators in the domain of dependence $\mathcal{D}[A]$ of the given subregion A. One possible way to satisfy both criteria is to write the truncated field $\tilde{\mathcal{O}}(t, \Omega)$ of the boundary as the following mode expansion.

$$\tilde{\mathcal{O}}(t, \Omega) = \sum \mathcal{O}_{nlm}\tilde{g}_{nlm}(t, \Omega) + \text{h.c} \tag{3.44}$$

$$\tilde{g}_{nlm}(t, \Omega) = \begin{cases} g_{nlm}(t, \Omega) & (t, \Omega) \in \mathcal{D}[A] \\ 0 & (t, \Omega) \in \mathcal{D}[A^c] \end{cases} \tag{3.45}$$

$\mathcal{O}_{nlm}$ and g_{nlm} refers to the original boundary mode operators and mode functions. Note the similarity of this expansion to bulk mode expansion (3.37). In view of this similarity, the mode function $\tilde{g}_{nlm}$ will be called the *resummed mode function for the boundary*, although which resummation it originates from is totally vague. Since any analogue of wave equation does not exist at the boundary, there is no good criteria for determining the resummed boundary mode function at boundary spacetime outside $\mathcal{D}[A]$ and $\mathcal{D}[A^c]$.

While the mode expansion (3.44) obscures the structure of the Hilbert space $\mathcal{H}_A$ corresponding to the given boundary subregion A, it does capture the relevant physics in the given subregion. Therefore, the mode expansion (3.44) will be treated as a valid mode decomposition of the truncated operator $\tilde{\mathcal{O}}$.

3.2.2 Building smearing functions for the entanglement wedge

To construct the bulk, it remains to find resummed bulk mode functions $\tilde{f}_{nlm}$ that reduce to resummed boundary mode functions $\tilde{g}_{nlm}$ at the boundary by a slight variant of the last dictionary (2.32).

$$\tilde{g}_{nlm}(t,\Omega) \leftrightarrow \lim_{\rho\to\pi/2}\left(\frac{\tilde{f}_{nlm}(t,\rho,\Omega)}{\cos^{\Delta}\rho}\right) \tag{3.46}$$

As resummed boundary mode functions are well-defined only inside $\mathcal{D}[A]$ and $\mathcal{D}[A^c]$ of the boundary spacetime, it remains to find the bulk subregion Σ_A which gives the right boundary behaviour of the resummed bulk mode functions evaluated as $|\tilde{f}_{nlm}) = \mathbb{1}_A|f_{nlm})$. Define Σ as the time slice which includes causal surfaces χ_A and χ_{A^c}. Also define $\Sigma_{cw}[A]$ as the intersection of Σ and the causal wedge $\mathcal{W}_C[A]$ of A, i.e. $\Sigma_{cw}[A] = \mathcal{W}_C[A]\cap\Sigma$. Then, any subregion $\Sigma_{cw}[A] \subseteq \Sigma_A \subseteq \Sigma - \Sigma_{cw}[A^c]$ satisfies the resummed mode function condition (3.46) at $\mathcal{D}[A]$ and $\mathcal{D}[A^c]$. This ambiguity in choice of Σ_A is resolved by the fact that the bulk reconstructible from the given boundary subregions corresponds to the entanglement wedge $\mathcal{E}[A]$ of boundary subregion A [28]. This means the following requirement is imposed on the resummed bulk mode function $\tilde{f}_{nlm}$.

$$\tilde{f}_{nlm}(t,\rho,\Omega) = \begin{cases} f_{nlm}(t,\rho,\Omega) & (t,r,\Omega)\in\mathcal{E}[A] \\ 0 & (t,\rho,\Omega)\in\mathcal{E}[A^c] \end{cases} \tag{3.47}$$

Having determined the appropriate mode functions of the bulk, determining the smearing function can be proceeded in the manner described in section 2.2. Picking the region of boundary spacetime Ξ to be equal to $\mathcal{D}[A]$ in 2.2.3, the smearing function K^A which reconstructs the bulk based on the boundary data in $\mathcal{D}[A]$ can be written schematically as follows.

$$K^A(x|Y) = \sum_k |\tilde{f}_k(x))P^+_{k,\mathcal{D}[A]}(Y) + |\tilde{f}^*_k(x))P^-_{k,\mathcal{D}[A]}(Y) \tag{3.48}$$

The vectors $|\tilde{f}_k)$ and $|\tilde{f}^*_k)$ denote resummed mode functions of the bulk, and $P^{+/-}_{k,\mathcal{D}[A]}$ denotes the integral kernel corresponding to the mode projection operator that projects out the positive/negative frequency mode operator that corresponds to the (set of) quantum number(s) k. Since details of this reconstruction are only a mere repetition of the procedure outlined in section 2.2, it will be omitted. Note that this process only requires existence of the dictionary between mode operators and resummed mode functions; if the dictionary can be built for other background spacetime, then it is possible to apply the same reconstruction method to that spacetime.

This construction sheds a new light onto the noninvertibility problem of the gram matrix $\mathbf{g}$ considered in section 2.2.5. If $\mathbf{g}$ is indeed invertible in general, it means that any state can be built solely from arbitrary subregions *without* any cost of arbitrarily high precision[5], which seems to contradict the assumption that only the entanglement wedge is reconstructed; existence of $\mathbf{g}^{-1}$ means contribution of higher (in the sense of quantum numbers n and l) modes to a lower mode projection operator diminishes sufficiently fast. This is another supporting evidence of the conclusion given in section 2.2.5 that the construction should be viewed as an effective description rather than an exact one.

An interesting feature of this reconstruction mechanism is that when accessible boundary spacetime $\Xi = \mathcal{D}[A]$ becomes smaller, lower modes become harder to discern. Since lower modes in the global patch of AdS are weighted to the centre of the AdS, this property is consistent with the expectation that a larger portion of the boundary is needed to probe deeper into the bulk. In [25] this expectation was cast as the statement that operators of the deeper bulk

[5]It is known that in QFT any state can be approximated by acting on the vacuum with local observables confined to a subregion of the total spacetime(Reeh-Schlieder property), which indicates presence of an enormous amount of entanglement [24].

are more robust against local erasures of the boundary. Another implication this observation poses is that in some cases disregarding the modes lower than a certain cut-off is needed for an efficient bulk reconstruction.

3.2.3 Covariance under AdS isometries

A relativistic system must obey covariance, but there is some subtlety regarding the construction proposed in this paper. It has been argued in section 2.2.5 that non-invertibility of infinite dimensional matrices implies the need to introduce a cut-off. This cut-off is a UV cut-off at the boundary, which corresponds to an IR cut-off in the bulk. This fact raises doubts on covariance of the construction with respect to AdS isometries; the IR cut-off, the confining box of the system, will move around with the flow generated by the Killing vectors of AdS. Nevertheless, there are reasons to believe that covariance must remain. First of all, IR cut-off should not affect the local, microscopic details of the system and local causal structure is one of those details. Since relativity is about consistent description of causal structures, the subsystem corresponding to the entanglement wedge must be relativistic because the full system is relativistic. How is covariance under AdS isometries realised in this construction?

The smearing function obtained in [14], reproduced in (2.46), has an explicit dependence on the radial coordinate ρ which is a frame-dependent quantity. This means the smearing function does not change as a scalar under AdS isometries.

$$K(x|Y) \to K'(x'|Y') = K\left(x(x')|Y(Y')\right) \lim_{\rho\to\pi/2}\left(\frac{\cos\rho'}{\cos\rho}\right)^{\Delta-d} \tag{3.49}$$

$$= K\left(x(x')|Y(Y')\right)\left|\frac{\partial Y'}{\partial Y}\right|^{(\Delta/d)-1} \tag{3.50}$$

The last line follows from the following identity which relates limiting values of cosines to Jacobian factors of boundary coordinates, which is proved in appendix 3.2.4.

$$\lim_{\rho\to\pi/2}\left(\frac{\cos\rho'}{\cos\rho}\right)^{d}=\left|\frac{\partial Y'}{\partial Y}\right| \tag{3.51}$$

The covariant transformation rule (3.50) fits nicely with the fact that primary fields on the CFT(or GFT) side needs to be rescaled under coordinate transformations if two-point correlation function structure is required to be preserved in the new coordinates.

$$\mathcal{O}_Y(Y)\to\mathcal{O}_{Y'}(Y')=\left|\frac{\partial Y}{\partial Y'}\right|^{\Delta/d}\mathcal{O}_Y\left(Y(Y')\right) \tag{3.52}$$

The subscript under GFF $\mathcal{O}$ is there to serve as a reminder that this field has the correct correlation function structure for the subscript coordinate system; $\mathcal{O}_Y$ has the correct correlators when coordinate system Y is used, and $\mathcal{O}_{Y'}$ when Y' is used. The transformation rule for the bulk scalar $\phi(x)$ is given by

$$\phi(x)\to\phi'(x')=\phi\left(x(x')\right) \tag{3.53}$$

and representation of the bulk scalar by boundary GFF is subject to the following transformation rule.

$$\begin{aligned}\phi'(x')&=\int K'(x'|Y')\mathcal{O}_{Y'}dY' &(3.54)\\ &=\phi\left(x(x')\right)=\int K\left(x(x')|Y\right)\mathcal{O}_Y dY\\ &=\int K\left(x(x')|Y(Y')\right)\left|\frac{\partial Y'}{\partial Y}\right|^{\Delta/d}\mathcal{O}_{Y'}\left|\frac{\partial Y}{\partial Y'}\right|dY'\\ &=\int K\left(x(x')|Y(Y')\right)\left|\frac{\partial Y'}{\partial Y}\right|^{(\Delta/d)-1}\mathcal{O}_{Y'}dY' &(3.55)\end{aligned}$$

Therefore imposing the transformation rule (3.50) to the smearing function constructed by the procedure proposed in this paper will guarantee covariance under AdS isometries, since integration over the new coordinates will be naturally recast into an integration over the coordinates that the smearing function was defined by. Note that in general the smearing function constructed in one reference frame would be different from the one constructed in another, as boundary mode functions g_k and the gram matrix g_{ij} constructed from them will be different in general. This is an explicit realisation of the ambiguities in projection operator P^m mentioned in footnote 12 of section 2.2.3.

3.2.4 Appendix: Computation of boundary Jacobian factor

The goal of this appendix is to prove the identity (3.51).

$$\lim_{\rho\to\pi/2}\left(\frac{\cos\rho'}{\cos\rho}\right)^d = \left|\frac{\partial Y'}{\partial Y}\right| \tag{3.56}$$

This relation is based on the fact that bulk measure remains invariant under isometries of AdS_{d+1}.

$$dx = \sec^{d+1}\rho\ d\rho dY \to dx' = \sec^{d+1}\rho'\ d\rho' dY' = dx \tag{3.57}$$

dY refers to the spacetime measure of the boundary; $dY = \sqrt{\left|g_{\mathbb{R}\times S^{d-1}}\right|}\ dtd\Omega$. Using Dirac delta and integrating over the radial coordinate ρ gives the following relation.

$$\int \delta\left[\sec\rho(\rho - \pi/2 + \epsilon)\right]\sec^{d+1}\rho\ d\rho dY = \sec^d\rho\Big|_{\rho=\pi/2-\epsilon}\ dY \tag{3.58}$$

The factor of $\sec\rho$ is included inside the argument of Dirac delta, since the *proper length element* along the radial direction should be written as $\sec\rho\ d\rho$. The wanted identity is proved by combining the above relation with invariance of bulk measure and taking the limit $\epsilon \to 0$.

$$\lim_{\rho\to\pi/2}\sec^d\rho'\ dY' = \lim_{\rho\to\pi/2}\sec^d\rho\ dY \tag{3.59}$$

$$dY' = \lim_{\rho\to\pi/2}\left(\frac{\cos\rho'}{\cos\rho}\right)^d\ dY = \left|\frac{\partial Y'}{\partial Y}\right| dY \tag{3.60}$$

As an example, consider AdS_3. Using the embedding space coordinates u^μ, AdS_3 of radius 1 can be given by the following equation.

$$-1 = -\left(u^{-1}\right)^2 - \left(u^0\right)^2 + \left(u^1\right)^2 + \left(u^2\right)^2 \tag{3.61}$$

The following parametrisation of the hypersurface gives the global patch of AdS_3.

$$\begin{aligned} u^{-1} &= \sec\rho\cos t \\ u^0 &= \sec\rho\sin t \\ u^1 &= \tan\rho\cos\theta \\ u^2 &= \tan\rho\sin\theta \end{aligned} \tag{3.62}$$

The global patch of AdS_3 has the following metric.

$$ds^2 = \sec^2\rho(-dt^2 + d\rho^2 + \sin^2\rho\ d\theta^2) \tag{3.63}$$

Consider the following $u^{-1} - u^1$ plane rotation, which is an isometry of AdS_3. The primed coordinates are parametrised in the same way as in (3.62).

$$\begin{aligned} u'^{-1} &= \cosh\lambda\ u^{-1} + \sinh\lambda\ u^1 \\ u'^1 &= \sinh\lambda\ u^{-1} + \cosh\lambda\ u^1 \end{aligned} \tag{3.64}$$

After some manipulation, the following relation between primed and unprimed coordinates of AdS_3 can be obtained.

$$\begin{aligned} \sec^2\rho' &= \sec^2\rho + 2\cosh\lambda\sinh\lambda\sec\rho\tan\rho\cos\theta\cos t \\ &\qquad + \sinh^2\lambda(\sec^2\rho\cos^2 t + \tan^2\rho\cos^2\theta) \\ \tan t' &= \frac{\sin t}{\cosh\lambda\cos t + \sinh\lambda\sin\rho\cos\theta} \\ \tan\theta' &= \frac{\sin\theta}{\cosh\lambda\cos\theta + \sinh\lambda\csc\rho\cos t} \end{aligned} \tag{3.65}$$

The limit of the ratio of cosines can be directly evaluated by the first relation.

$$\lim_{\rho\to\pi/2}\left(\frac{\cos\rho}{\cos\rho'}\right)^2 = 1 + 2\cosh\lambda\sinh\lambda\cos\theta\cos t + \sinh^2\lambda(\cos^2 t + \cos^2\theta) \tag{3.66}$$

The boundary is parametrised by t, θ or t', θ'. The volume measure of the boundary is given by $dY = dtd\theta$ and $dY' = dt'd\theta'$. The relation between boundary coordinates can be deduced from the other two relations by taking the limit $\rho \to \pi/2$.

$$\begin{aligned} \tan t' &= \frac{\sin t}{\cosh\lambda\cos t + \sinh\lambda\cos\theta} \\ \tan\theta' &= \frac{\sin\theta}{\cosh\lambda\cos\theta + \sinh\lambda\cos t} \end{aligned} \tag{3.67}$$

Direct evaluation of the Jacobian gives

$$\left|\frac{\partial(t',\theta')}{\partial(t,\theta)}\right| = \frac{1}{1 + 2\cosh\lambda\sinh\lambda\cos\theta\cos t + \sinh^2\lambda(\cos^2 t + \cos^2\theta)} \tag{3.68}$$

or

$$\lim_{\rho\to\pi/2}\left(\frac{\cos\rho'}{\cos\rho}\right)^2 = \left|\frac{\partial(t',\theta')}{\partial(t,\theta)}\right| \tag{3.69}$$

which is consistent with (3.51).

Chapter 4

Discussion

A procedure has been proposed in this paper which explicitly reconstructs a bulk scalar ϕ from its boundary dual operator $\mathcal{O}$ when only a portion of the boundary data is given. The key idea is the observation that smearing function K is an automation of reading mode operators at the boundary and assigning the corresponding bulk mode functions, so that it is possible to reconstruct the bulk whenever such a dictionary is given. The dictionary between mode operators and mode functions of the bulk and boundary for pure AdS perturbations was built to establish a working example for the construction. The bulk dual to subregions of the boundary was chosen to be the entanglement wedge, based on the recent proof [28] of bulk reconstruction. Resummed mode functions were used to obtain the mode functions for the entanglement wedge without explicitly solving the wave equation on a complicated coordinate patch that covers the entanglement wedge. Non-orthogonality of resummed mode functions was resolved by introducing a Gram matrix and its inverse, although an UV cut-off had to be introduced to resolve complications induced by infinite dimensionality of the Gram matrix. Because of the introduction of an UV cut-off, the smearing function constructed by this process is at best an effective reconstruction. In effect, the hard problem of solving the wave equation on a complicated coordinate patch was transferred to the problem of inverting a very large matrix, which is still hard but relatively tractable. Covariance under choice of mode functions guarantees that the construction is unique for a chosen reference frame.

There exist some subtleties that has not been adressed in the main article, one of which is the ambiguity of mode projection operators. As explained in footnote 12 of section 2.2.3, there is some ambiguity in defining mode projection operators, which was essential for equivalence of smearing functions constructed in different reference frames. A criterion that seems to work that can kill this ambiguity is to require the L^2 norm of the mode projection operator, $\int_{\Xi} |P^m|^2 dV$, to be minimised. On the other hand, it is rather obscure how this criterion can be imposed in practice; while it is possible to project out the orthogonal complement $V_1^{\perp}$ of vector space V_1 spanned by mode functions when some arbitrary cut-off is imposed, obtaining basis vectors of $V_1^{\perp}$ without introducing any cut-off is not readily available. Another subtlety that remains unresolved is consistent implementation of UV cut-off in the boundary and the bulk. The cut-off was introduced to regulate infinite dimensionality of the Gram matrix, so it is imposed in an ad hoc fashion. Had the cut-off been introduced in the beginning, there should be a consistent way of imposing the cut-off to the subregions so that dimensionality of the full Hilbert space factorise consistently. Although these subtleties do not seem to invalidate the procedure outlined in the main article, they do seem to provide some food for thought.

As the proposal to understand AdS/CFT as an error-correcting structure

[25] was one of the main motivations to study the problem of entanglement wedge reconstruction, it would be interesting to see how the structure manifests itself in the proposal of this thesis. Reviewing the procedure of reconstruction, the only place that error-correcting structure can reside in seems to be the mode operator retrieval process. Note that there is nothing quantum inherent in this process; this is a classical signal retrieval process from data given by a non-orthogonal basis in absence of noise. A similar structure exists in the bulk theory; the mode expansion (3.37) shows that an arbitrary subregion of the bulk spacetime can encode mode operators of the global Fock states. This seems to suggest that such an error-correcting structure could be a generic feature of quantum field thoeries, which has its roots in Reeh-Schlieder property. In view of holographic code models [40, 41], this observation could be an indication that error correcting codes can also be built from non-hyperbolic tensor networks as well, a possibility already conjectured in [41].

Just like the original reconstructions of HKLL [13, 14], the bulk reconstruction suggested in this paper does not include $1/N$ corrections. Adding $1/N$ corrections implies interactions kick in, meaning that signal retrieval process described in section 2.2.3 must include tolerance from errors induced by interactions. Whether such incorporation of errors can be used to understand the "phase transition" behaviour of the entanglement wedge [25] would be an intersting problem to ponder on. Another facet of $1/N$ corrections is that non-perturbative objects such as solitons or instantons may appear at finite N. In presence of instantons, the total Hilbert space splits into a sum of different sectors. In such a case, it may not be possible to write the total Hilbert space as a product of subregion Hilbert spaces. While it was implicitly assumed that the total Hilbert space of the bulk and boundary can be decomposed into product of subregion Hilbert spaces, whether it is possible at all is a potential problem that this manuscript has avoided to answer. An indication of this problem already existed in [25] when considering tripartitioning of the boundary to show that the bulk centre cannot be reconstructed in any of the subregions alone. This problem has also been raised in a somewhat different context in [42].

As a final remark, the causal structure of the entanglement wedge resembles the causal structure of long Einstein-Rosen bridges. It would be an interesting exercise to probe the implications of the reconstruction algorithm elaborated here to the problem of reconstructing the interior of long Einstein-Rosen bridges from boundary data. This problem will be left for future work.

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초록

AdS/CFT는 점근적 반 더 시터르 공간(asymptotically anti-de Sitter) 위의 중력을 포함하는 이론과 그 등각 경계에 정의된 등각장론 사이의 이중성이다. 비록 이 이중성은 자명하지 않은 다양한 방법으로 확인되었고 이중성의 양변을 잇는 사전에 대해 많은 것이 알려졌음에도 불구하고, 이 사전은 완전히 집필되었다고는 할 수 없다. 불완전한 사례의 하나로 주어진 등각경계의 일부분이 내부(bulk)의 어떤 정보를 담는지를 묻는 소구역 이중성(subregion duality) 문제가 있다. Dong, Harlow, Wall의 증명[28]에 따르면 주어진 등각경계의 일부분은 그 얽힘 쐐기(entanglement wedge)에 담긴 정보를 담고 있으나, 얽힘 쐐기에 담긴 정보를 어떻게 직접적으로 재구성할 것인가에 대한 대답은 한동안 불분명했다. 여기에서는 [1]의 논문을 바탕으로 모드 합(mode sum)을 이용해 퍼트리기 함수(smearing function)를 얻는 과정을 확장하여 주어진 등각경계의 얽힘 쐐기를 재구성하는 과정을 포함시키는 방법을 설명한다. 소박한 기대와는 달리 복잡한 좌표계에서 파동방정식의 해를 구해야 할 필요는 없으며, 이 어려운 문제는 보다 쉬운 역행렬을 찾는 문제로 변환된다.

주요어: AdS/CFT 대응성, 내부 국소성

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