The Pricing of Option on Bond Forwards

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We derive closed-form solutions for the equilibrium prices of bonds, bond forwards, bond futures, options on the bond forwards and options on the bond futures when the interest rate is stochastic. The prices of options on the bond forwards are shown to be greater than the prices of options on the corresponding bond futures.

Keywords: Bond, Equilibrium, Forwards, Futures, Option

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I. Introduction

This paper derives closed-form solutions for the equilibrium prices of bonds, bond forwards, bond futures, options on the bond forwards and options on the bond futures when the interest rate is stochastic.

Cox, Ingersoll, and Ross (1981), Richard and Sundaresan (1981) and Jarrow and Oldfield (1981), among others, provide the relationship between the futures price and the forward price when the underlying asset is an equity. Ramaswamy and Sundaresan (1985) and Turnbull and Milne (1991) also examine options on the futures in an environment of stochastic interest rates when the underlying asset is an equity. Dybvig (1996) investigates pricing of bonds and options on bonds based on the term structure at a point in time.

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Jamshidian (1991) examines options on futures and options on forwards when the underlying asset is an equity.

Turnbull and Milne (1991) derive closed-form solutions for various options by a discrete-time general equilibrium model with the assumption of the distribution of the marginal rate of substitution. Ahn (1996a) derives a closed-form solution of a foreign currency futures option under stochastic interest rates. Our article is similar to Turnbull and Milne (1991) and Ahn (1996a) in using a martingale probability measure. But it is different from Turnbull and Milne (1991) in several respects. First, our article is based on a continuous-time framework. Second, it does not require the distribution of the marginal rate of substitution. The marginal rate of substitution may be more difficult to estimate than the interest rate. Also, it is different from Turnbull and Milne (1991) and Ahn (1996a) in that our article uniquely provides closed-form solutions for options on bond futures and options on bond forwards.

Chen (1992) derives closed-form solutions for options on futures of a default-free discount bond under the normal process. Feldman (1993) derives closed-form solutions for options on futures of a default-free discount bond under the square root process. Chen (1992) and Feldman (1993) do not use a martingale probability measure. They do not derive closed-form solutions for options on forwards of a default-free discount bond and do not compare options on bond futures with options on bond forwards.

Section II derives closed-form solutions for the futures price and the forward price of a default-free discount bond under the stochastic interest rate. Then, futures prices and forward prices of a bond are compared. Section III derives closed-form solutions for options on futures and options on forwards of a default-free discount bond. Then, options on bond futures and options on bond forwards are compared. Finally, section IV provides a summary and conclusion.

II. Bond Forwards and Bond Futures

In this section, we examine the equilibrium prices of forwards and futures of a default-free discount bond under the stochastic interest rate. We assume that the risk-neutral process of the short-term interest rate is given by

$$dr = [\kappa(r_0 - r) - \lambda_r]dt + \sigma dz \tag{1}$$

where r is the instantaneous interest rate, κ, r_0, λ_r, and σ are positive constants, and z is a standard Wiener process. The instantaneous interest rate is expected to drift toward the long-run average level, r_0, with the speed of adjustment κ and the standard deviation σ. The risk premium of the interest rate is denoted by λ_r. Equation (1) is called an Ornstein-Uhlenbeck process. This process was originally used by Vasicek (1977) to model the term structure of interest rates.[1] It has been employed by Jamishidian (1989) for the process of the spot interest rates and by Heath, Jarrow, and Morton (1992) for the process of the forward interest rates.

Let t denote the current date, s denote the delivery date of a futures (or forward) contract, T denote the maturity date of a default-free bond, and τ denote the expiration date of an option. Let $B[r(t), t, T]$ denote the equilibrium price at time t of a default-free discount bond yielding \$1 at time T. Let $F[r(t), t, s, T]$ and

[1] Even though this model has the disadvantage of a small probability of a negative interest rate, it has a surprisingly good fit to observed term structure movements (see Dybvig (1996)). The probability of having a negative interest rate from this model during the period of time t through T is given by

$$N\left[\frac{-\mu_r(t,T)}{\sigma_r(t,T)}\right] +$$

$$\exp\left[\frac{-2r(t)\bar{r}e^{-\kappa(T-t)}(1-e^{-\kappa(T-t)})}{\sigma_r^2(t,T)}\right] N\left[\frac{\mu_r(t,T)-(r(t)+\bar{r})e^{-\kappa(T-t)}}{\sigma_r(t,T)}\right]$$

where N is a standard normal cumulative distribution function and $\mu_r(t,T)$ and $\sigma_r^2(t,T)$ are defined as

$$\mu_r(t,T) = E_t[r(T)] = \bar{r} + e^{-\kappa(T-t)}(r(t) - \bar{r})$$

$$\sigma_r^2(t,T) = Var_t[r(T)] = (\sigma^2/2\kappa^2[1 - e^{-2\kappa(T-t)}].$$

The probability of a negative interest rate is small with plausible parameter values. For example, suppose that $\bar{r}=0.5$, $\kappa=0.2$ and $\sigma=0.02$ (which approximately imply the long-term interest rate of 5 percent and standard deviation of the risk-free interest rate of 2 percent per annum). If the current interest rate is 6 percent, then the probability of having a negative interest rate within a year is approximately 1.1 percent.

$f[r(t), t, s, T]$ denote the forward and futures prices of a default-free discount bond at time t with the delivery date s and the maturity date T, respectively. The following theorem provides closed-form solutions for the current price, the forward price and the futures price of a default-free discount bond.

Theorem 1: The equilibrium price at time t of a default-free discount bond yielding \$1 at time T is

$$B[r(t), t, T] = \exp[-\mu_R(r(t), t, T) + (1/2)\sigma_R^2(t, T)] \tag{2}$$

where [2]

$$\mu_R(r(t), t, T) = E_t[\int_t^T r(u)du] = \bar{r}(T-t) + \frac{r(t)-\bar{r}}{\kappa}[1 - e^{-\kappa(T-t)}] \tag{2a}$$

$$\sigma_R^2(t, T) = Var_t[\int_t^T r(u)du] = \frac{\sigma^2}{2\kappa^3}[2\kappa(T-t) + 4e^{-\kappa(T-t)} - 3 - e^{-2\kappa(T-t)}]. \tag{2b}$$

The forward price at time t of a default-free discount bond yielding \$1 at time T is

$$F(r(t), t, s, T) = \frac{B(r(t), t, T)}{B(r(t), t, s)} \tag{3}$$

and the futures price at time t of a default-free discount bond yielding \$1 at time T is

$$\begin{aligned} f(r(t), t, s, T) &= \frac{B(r(t), t, T)}{B(r(t), t, s)} \exp[-\frac{q(t, s)}{\kappa}(1 - e^{-\kappa(T-s)})] \\ &= F(r(t), t, s, T)\exp[-\frac{q(t, s)}{\kappa}(1 - e^{-\kappa(T-s)})] \end{aligned} \tag{4}$$

[2] Note that $\bar{r}$ is $r_0 - \lambda_r/\kappa$. Also note that E_t, Var_t and Cov_t are respectively an expectation operator, a variance operator and a covariance operator. For example, we can calculate $\sigma_R^2(t, T)$ in the following way:

$$\sigma_R^2(t, T) = Cov(\sigma\int_t^T\int_t^w e^{-\kappa(w-u)}dz(u)dw,\ \sigma\int_t^T\int_t^w e^{-\kappa(w-u)}dz(u)dw)$$

$$= \frac{\sigma^2}{2\kappa^3}[2\kappa(T-t) + 4e^{-\kappa(T-t)} - 3 - e^{-2\kappa(T-t)}]$$

where $R(t, T) \equiv \int_t^T r(u)du$ denotes a long-term interest rate and r denotes a short-term interest rate.

where

$$q(t,s)=Cov_t[r(s),\int_t^s r(u)du]=\frac{\sigma^2}{2\kappa^2}[1-e^{-\kappa(s-t)}]^2. \tag{4a}$$

***Proof*:** See the Appendix.

Even though a closed-form solution for a default-free discount bond, (2), has been derived by Vasicek (1977) under a mean-reverting diffusion process of interest rates, the forward price and the futures price of a default-free discount bond have not been compared. It follows from (3) and (4) that the ratio of the futures price to the forward price of a default-free discount bond yielding $1 at time T is

$$\begin{aligned}\frac{f(r(t),t,s,T)}{F(r(t),t,s,T)}&=\exp[-\frac{q(t,s)}{\kappa}(1-e^{-\kappa(T-s)})]\\&=\exp[-\frac{\sigma^2}{2\kappa^3}(1-e^{-\kappa(s-t)})^2(1-\exp^{-\kappa(T-s)})]\leq 1.\end{aligned} \tag{5}$$

Generally, the forward price of a default-free discount bond is greater than or equal to the futures price of the default-free discount bond.[3] As the interest rate rises, the prices of bond and bond futures decrease and the investor who has the long position has to deposit the money into the account and borrow the money with higher interest rates. If the interest rate decreases, the price of bond and bond futures increase and the investor who has the long position can withdraw the money from the account and invest the money with the lower interest rates. Thus 'marking to market' from the futures contract lowers the value compared to the forward contract.

This ratio depends on the three factors: i) the variance of the interest rate, ii) the length of the period between the delivery date and the current date, and iii) the length of the period between the delivery date and the maturity date of the default-free discount

[3] This result is consistent with Cox, Ingersoll, and Ross (1981) who have shown that the forward price is greater than the futures price when the undelying asset is a Treasury bill, since the long-term bond has greater volatility than the short-term bond so that the long-term bond does not provide a hedge against bond price fluctuations.

bond. As the uncertainty of the interest rate increases, the ratio gets smaller. As the length of the period between the delivery date and the current date increases, the ratio gets smaller. As the length of the period between the delivery date and the maturity date of the default-free discount bond increases, the ratio gets smaller. However, the forward price is the same as the futures price if there is no uncertainty regarding to the interest rate (*i.e.*, with the constant interest rate).

The same result is also obtained by the following corollary.

Corollary 1: The forward price at time t of a default-free discount bond yielding \$1 at time T is

$$F(r(t), t, s, T) = \frac{1}{\sqrt{2\pi}\,\sigma_r(t, s)} \int_{-\infty}^{\infty} B[r(s), T] \exp\left[-\frac{[r(s) - \{\mu_r(t, s) - q(t, s)\}]^2}{2\sigma_r^2(t, s)}\right] dr(s) \tag{6}$$

and the futures price at time t of a default-free discount bond yielding \$1 at time T is

$$f(r(t), t, s, T) = \frac{1}{\sqrt{2\pi}\,\sigma_r(t,s)} \int_{-\infty}^{\infty} B[r(s), T] \exp\left[-\frac{[r(s) - \mu_r(t, s)]^2}{2\sigma_r^2(t, s)}\right] dr(s). \tag{7}$$

Even though the forward price and the futures price of a default-free discount bond are the expected bond prices, the probability distribution for the forward price is different from the probability distribution for the futures price. The expected value of $r(s)$ for the forward price is $\mu_r(t, s) - q(t, s)$ while the expected value of $r(s)$ for the futures price is $\mu_r(t, s)$. The forward price, $F(r(t), s, T)$, is $\alpha \exp[-(\{\mu_r(t, s) - q(t, s)\}/\kappa)(1 - e^{-\kappa(T-s)})] + (\sigma_r^2(t, T)/2\kappa^2)(1 - e^{-\kappa(T-s)})^2]$ where α is $\exp\{(1/2)\sigma_R^2(s, T) - \bar{r}(T-s) + (\bar{r}/\kappa)[1 - e^{-\kappa(T-s)}]\}$. The futures price, $f(r(t), s, T)$, is $\alpha \exp[-\{\mu_r(t, s)/\kappa\}(1 - e^{-\kappa(T-s)})] + (\sigma_r^2(t, T)/2\kappa^2)(1 - e^{-\kappa(T-s)})^2]$. Thus, Equation (5) is obtained.

III. Options on Bond, Bond Futures, and Bond Forwards

In this section, we examine options on bond forwards and bond futures when there is an uncertainty on the interest rate. We derive closed-form solutions for European call options on bonds, bond forwards, and bond futures.

Let $C[r(t), K, t, \tau, T]$ denote the European call option price on a bond at time t with a boundary condition $\text{Max}[0, B(r(\tau), \tau, T)-K]$, where τ is the expiration date and K is the exercise price. Next, let $\bar{C}[f(t), r(t), K, t, \tau, s, T]$ denote the European call option price on bond futures at time t with a boundary condition $\text{Max}[0, f(\tau)-K]$, where $f(\tau)=f(r(\tau), \tau, s, T)$, τ is the expiration date of the option, s is the delivery date of the futures contract and T is the maturity date of the discount bond. Let $\hat{C}[F(t), r(t), K, t, \tau, s, T]$ denote the European call option price on bond forwards at time t with a boundary condition $\text{Max}[0, F(\tau)-K]$, where $F(\tau)=F(r(\tau), \tau, s, T)$. The following theorem provides closed-form solutions of European call option prices on a bond, bond futures and bond forwards.

Theorem 2: The European call option price on a bond at time t is [4]

$$C[r(t), K, t, \tau, T]=B(r(t), t, T)N(d_1)-KB(r(t), t, \tau)N(d_2) \tag{8}$$

where

$$d_1=[\ln \frac{B(r(t), t, T)}{KB(r(t), t, \tau)}+(1/2)\sigma_B^2]/\sigma_B \tag{8a}$$

$$d_2=d_1-\sigma_B \tag{8b}$$

$$\sigma_B^2=\frac{\sigma^2}{2\kappa^3}[1-e^{-2\kappa(\tau-t)}][1-\exp^{-\kappa(T-\tau)}]^2. \tag{8c}$$

The European call option price on bond futures at time t is [5]

$$\bar{C}[f(t), r(t), K, t, \tau, s, T]=PV_t[f(r(\tau), \tau, s, T)]N(d_1)-KB(r(t), t, \tau)N(d_2) \tag{9}$$

[4] The European call option pricing formula on a bond, (8), has been derived by Jamishidian (1989).

[5] $PV_t[\cdot]$ denotes the present value operator.

where

$$d_1 = \{ \ln \frac{PV_t [f(r(\tau), \tau, s, T)]}{KB(r(t), t, \tau)} + (1/2)\sigma_f^2 \} / \sigma_f \tag{9a}$$

$$d_2 = d_1 - \sigma_f \tag{9b}$$

$$h(\tau, s, T) = \frac{[\exp^{-\kappa(T-\tau)} - \exp^{-\kappa(s-\tau)}]}{\kappa} \tag{9c}$$

$$q(t, \tau) = \frac{\sigma^2}{2\kappa^2} [1 - e^{-\kappa(\tau - t)}]^2 \tag{9d}$$

$$PV_t [f(r(\tau), \tau, s, T)] = f(r(t), t, s, T) B(r(t), t, \tau) e^{-h(\tau, s, T) q(t, \tau)} \tag{9e}$$

$$= f(r(t), t, s, T) B(r(t), t, \tau) \exp\{ -[\frac{e^{-\kappa(T-\tau)} - e^{-\kappa(s-\tau)}}{\kappa}] \frac{\sigma^2}{2\kappa^2} [1 - e^{-\kappa(\tau - t)}]^2 \}$$

$$Var_t [r(\tau)] = \frac{\sigma^2}{2\kappa} [1 - e^{-2\kappa(\tau - t)}] \tag{9f}$$

$$\sigma_F^2 = Var_t [r(\tau)] [h(\tau, s, T)]^2$$
$$= \frac{\sigma^2}{2\kappa^3} [1 - e^{-2\kappa(\tau - t)}] [e^{-\kappa(T-\tau)} - e^{-\kappa(s-\tau)}]^2. \tag{9g}$$

The European call option price on bond forwards at time t is

$$\hat{C} [F(t), r(t), K, t, \tau, s, T] = PV_t [F(r(\tau), \tau, s, T)] N(d_1) - KB(r(t), t, \tau) N(d_2) \tag{10}$$

where

$$d_1 = \{ \ln \frac{PV_t [F(r(\tau), \tau, s, T)]}{KB(r(t), t, \tau)} + (1/2)\sigma_F^2 \} / \sigma_F \tag{10a}$$

$$d_2 = d_1 - \sigma_F \tag{10b}$$

$$PV_t [F(r(\tau), \tau, s, T)] = PV_t [f(r(\tau), \tau, s, T)] \exp[\frac{q(\tau, s)}{\kappa} (1 - e^{-\kappa(T-s)})] \tag{10c}$$

$$= PV_t [f(r(\tau), \tau, s, T)] \exp[\frac{\sigma^2}{2\kappa^3} (1 - e^{-\kappa(s-\tau)})^2 (1 - e^{-\kappa(T-s)})]$$

$$\sigma_F = \sigma_f . \tag{10d}$$

Proof: See the Appendix

By visual inspection of these option formulas, we can see that the present value operator makes these formulas look different from the Black-Scholes option formula. However, these formulas and the Black-Scholes formula are consistent with each other in that they include the present value of the spot price, futures price, and forward price at the expiration date, depending on the type of options. In the case of a spot stock option, the present value of the future spot price is simplified to the current spot price which appears in the Black-Scholes formula. In the case of the spot bond option, the present value of the future spot bond price is simplified to the current bond price which appears in (8). In the case of an option on the bond futures and an option on the bond forwards, the present values of the bond futures price and the bond forward price at the expiration date cannot be simplified to the current bond futures price and the current bond forward price. These formulas and the Black-Scholes formula are also consistent with each other in that the present values of the spot price, futures price, and forward price at the expiration date, depending on the type of options, are multiplied with the probability of being in the money respectively.[6]

Since the logarithmic bond futures price and the logarithmic bond forward price have the same volatility at the expiration date, option formula on the bond futures and option formula on the bond forwards have the same volatility ($\sigma_f = \sigma_F$). The difference between a call option on the bond futures and a call option on the bond forwards comes from the present value of the bond futures price and the present value of the bond forward prices at the expiration date. It follows from (10c) that the ratio of the present value of the bond futures price to the present value of the bond

[6] The probability of being in the money, when the short-term interest rate process follows an Ornstein-Uhlenbeck process, is given by the standard normal distribution function. On the other hand, the probability of being in the money, when the short-term interest rate process follows a square root process as in Cox, Ingersoll, and Ross (1985), is given by the noncentral *chi-square* distribution function as in Cox, Ingersoll, and Ross (1985) and Feldman (1993).

forward price is

$$\frac{PV_t\,[f(r(\tau),\tau,s,T)]}{PV_t\,[F(r(\tau),\tau,s,T)]}=\exp[-\frac{q(\tau,s)}{\kappa}(1-e^{-\kappa(T-s)})]$$

$$=\frac{f(r(\tau),\tau,s,T)}{F(r(\tau),\tau,s,T)}=\exp[-\frac{\sigma^2(1-e^{-\kappa(s-\tau)})^2}{2\kappa^3}(1-e^{-\kappa(T-s)})]\leq 1. \tag{11}$$

The present value of the bond futures price at τ is less than or equal to the present value of the bond forward price at τ. This implies that a call option price on the bond futures is less than or equal to a call option price on the bond forwards. If the interest rate becomes deterministic ($\sigma=0$), the call option price on bond futures is identical to the call option price on bond forwards.

Using put-call parity, the European put option price on a bond at time t can be written as

$$P[r(t),K,t,\tau,T]=C[r(t),K,t,\tau,T]+KB(r(t),t,\tau)-B(r(t),t,T)$$

and the European put option price on bond futures at time t can be written as

$$\bar{P}[f(t),r(t),K,t,\tau,s,T]=\bar{C}[f(t),r(t),K,t,\tau,s,T]$$
$$+KB(r(t),t,\tau)-PV_t\,[f(r(\tau),\tau,s,T)]$$

and the European put option price on bond forwards at time t can be written as

$$\hat{P}[F(t),r(t),K,t,\tau,s,T]=\hat{C}[F(t),r(t),K,t,\tau,s,T]$$
$$+KB(r(t),t,\tau)-PV_t\,[F(r(\tau),\tau,s,T)]$$

respectively.

To supplement the theoretical analysis, a numerical analysis is performed to generate option prices on bond, bond futures and bond forwards under the stochastic interest rate by using equations (8), (9) and (10). The values of parameters used in the theoretical models are assumed in Table 1.[7] In Table 1, option prices are

[7] The values of parameters used here are reported in Ahn (1996b) who

TABLE 1

OPTION PRICES ON BOND, BOND FUTURES, AND BOND FORWARDS

(Units: U.S. Cents)

Interest Rate	Option Price on Bond	Option Price on Bond Futures	Option Price on Bond Forwards
7%	8.078	9.902	10.051
7.5%	7.919	9.772	9.921
8%	7.763	9.644	9.791
8.5%	7.608	9.518	9.663
9%	7.456	9.392	9.536
9.5%	7.303	9.268	9.411
10%	7.155	9.144	9.286
10.5%	7.009	9.019	9.163
11%	6.865	8.899	9.038
11.5%	6.724	8.779	8.917
12%	6.584	8.661	8.797

Note: $\bar{r}=0.08$, $r=0.09$, $\kappa=0.9$, $\sigma=0.6$, Time to Delivery $=s-t=5$ Months, Time to Expiration $=\tau-t=3$ Months, Time to Maturity $=15$ Months, Exercise Interest Rate $=0.08$

stated in terms of cents. The option prices on bond futures and bond forwards are greater than the corresponding option prices on bond since the bond futures price and the bond forward price are greater than the bond price. The option price on bond forwards is greater than the corresponding option price on bond futures since the bond forward price is greater than the bond futures price as shown in Equation (5).

Ahn (1996a) shows that the currency futures option price is greater than the corresponding currency forward option price if the domestic long-term interest rate is negatively correlated with the foreign long-term interest rate and positively correlated with the

obtained them through the maximum likelihood estimation using the Korean daily CD interest rates and the Korean daily corporate bond interest rates during the period between April 1, 1994 and July 31, 1995. The sensitivities of the differences between the option price on the bond futures and the option price on the bond forwards with respect to the exercise interest rate, the bond maturity date, the bond delivery date, and the option expiration date turn out to be insignificant according to our numerical analysis.

foreign exchange rate. The futures exchange rate is greater than the forward exchange rate if the condition is satisfied. Ahn (1996a) gives the numerical example where the currency futures option price is greater than the currency forward option price.

Our contrasting result follows from that the forward price of a default-free discount bond is greater than the futures price of the default-free discount bond from 'marking to market'.

IV. Conclusion

This article derives closed-form solutions for bond futures price, bond forward price, call option price on bond futures, and call option price on bond forwards, when the instantaneous interest rate follows a mean-reverting diffusion process.

The resulting pricing formulas are consistent with the Black-Scholes formula. These formulas include the present value operator which makes them look different from the Black-Scholes formula.

An important conclusion is that the bond forward price is greater than the bond futures price, and consequently the call option price on the bond forwards is greater than the call option price on the bond futures. This result is contrasting to the case where the foreign currency futures option price can be greater than the corresponding forward option price.

Future research can extend the current model of one-factor setting to multi-factor setting and improve our understanding in this field.

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Appendix

Proof of Theorem 1

It is known that $B(r(t), t, T)$ is $E_t[\exp(-R(T))]$ where $R(T) \equiv \int_t^T r(u)du$ is normally distributed and E_t is an expectation operator. Thus, we have

$$B(r(t), t, T) = \exp[-\mu_R(r(t), t, T) + (1/2)\sigma_R^2(t, T)]. \tag{A.1}$$

Since the cost of a forward contract is zero,

$$0 = E_t\,[\exp(-R(s))\,(B(r(s), s, T) - F(r(s), s, s, T))]$$
$$= B\,(r(t), t, T) - F(r(t), t, s, T)\,B(r(t), t, s). \quad \text{(A.2)}$$

Solving (A.2) gives (3).

The futures price of a default-free discount bond is given by

$$E_t\,[B\,(r(s), s, T)]$$
$$= \int_{-\infty}^{\infty} B\,(r(s), s, T) \frac{1}{\sqrt{2\pi\, Var_t\,[r(s)]}} \exp[-\frac{(r(s) - E_t\,[r(s)])^2}{2Var_t\,[r(s)]}]dr(s). \quad \text{(A.3)}$$

Solving (A.3) gives (4)-(4a). This completes the proof.

Proof of Theorem 2

The European call option price on a bond is given by

$$E_t\,[\,\exp\,[-R(\tau)]\,[\text{Max}\,[0, B(r(\tau), \tau, T) - K]]]$$
$$= E_t\,[\,\exp\,[-R(\tau)]\,B(r(\tau), \tau, T)\,|\,B\,(r(\tau), \tau, T) \geq K] \quad \text{(A.4)}$$
$$- KE_t\,[\exp[-R(\tau)]\,|\,B(r(\tau), \tau, T) \geq K].$$

Evaluating (A.4) gives (8)-(8c).

The European call option price on bond futures is given by

$$E_t\,[\,\exp\,[-R(\tau)]\,[\text{Max}\,[0, f(\tau) - K]]]$$
$$= E_t\,[\,\exp\,[-R(\tau)]f(\tau)\,|\,f(\tau) \geq K] - KE_t\,[\,\exp(-R(\tau))\,|\,f(\tau) \geq K]. \quad \text{(A.5)}$$

Evaluating (A.5) gives (9)-(9g).

The European call option price on bond forwards is given by

$$E_t\,[\,\exp\,[-R(\tau)]\,[\text{Max}\,[0, F(\tau) - K]]]$$
$$= E_t\,[\,\exp\,[-R(\tau)]f(\tau)\,|\,F(\tau) \geq K] - KE_t\,[\,\exp\,(-R(\tau))\,|\,F(\tau) \geq K]. \quad \text{(A.6)}$$

Evaluating (A.6) gives (10)-(10d). This completes the proof.

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