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이 논문을 경제학 석사학위 논문으로 제출함.

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Abstract

How Korean SAT Scores Differ Between Coed and Single-sex Schools

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Whether student academic achievement is better in single-sex schools has been controversial. Despite much research on this issue, it hardly seems to reach any consensus on the effect of single-sex schooling due to student selection into certain school types. I take advantage of a unique setting in the Republic of Korea, where students had been randomly assigned to high schools until 2009. Using the Korean data of students entering high schools in 2005, I observe that students' socioeconomic backgrounds are well-balanced between coed and single-sex schools. This supports the randomness of student assignment rule in Korea. Regarding the assignment to coed or single-sex schools as treatment, I estimate the treatment effect with parametric and semiparametric approach. Since household income and student studying time also affect Korean SAT (college entrance exam) scores, I primarily control for these factors.

For male students, attending single-sex schools rather than coed schools is shown to be significantly associated with higher KSAT scores. To further examine distributional consequences of this educational treatment, I conduct nonparametric hypotheses testing regarding conditional average treatment effect and first-order stochastic dominance across control variables. I consider univariate control, multivariate control, and a fixed control among two covariates. Male students in boys' high schools are shown to get higher scores than those in coed schools across all income level or study time, whereas girls' scores show no significant difference between two school types. However, fixing study time, hard-working male students seem not to be negatively affected by coed schooling.

Keywords: single-sex/coed schooling, treatment effect, nonparametric hypothesis testing, stochastic dominance

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1 Introduction

Whether student academic achievement is better in single-sex or coed schools has been controversial in many countries. Despite much research, it hardly seems to reach any consensus regarding the effect of single-sex schooling. Riordian (1990) and Trickett and Trickett (1982) argue that single-sex schools are better for increasing educational attainment since the presence of the opposite sex can distract students. Pomerantz et al. (2002) and Killgore and Yurgelun-Todd (2004) mention that boys and girls learn differently because of their dissimilar socialization and biological traits. This can indirectly support single-sex schooling since teachers can tailor instruction by taking into account some sex-specific characteristics. In contrast, Hoxby (2000b) and Lavy and Schlosser (2009) argue that larger shares of female students in coed classrooms seem to be linked to higher educational performance.

These studies, however, do not compare single-sex and coed schools. Rather, they study cohort-to-cohort variation in gender makeup in order to identify the effect of larger shares of girls in classrooms. Jackson (2011) use ruled-based student assignments in Trinidad and Tobago which show pseudo-randomization feature. This research argues that only those with strong preferences for single-sex schools achieve better educational output in single-sex schools. Park et al. (2012) uses Korean data on college entrance exam scores to examine causal effect of single-sex schools. It takes advantage of the randomized procedure of Korean student assignment rule. However, due to the unavailability of individual-level background data, it has a limitation to merely link test scores with school level data for analyzing causal effects of single-sex schooling.

The merits of single-sex schools have been fiercely debated in Korea where single-sex schools are much more prevalent than in other countries. The case of Korea is perfectly suited for analyzing the influence of single-sex and coed schools since students are quite randomly assigned to high schools. This feature is suitable for studying the effect of single-sex schooling in contrast to other countries' cases where most single-sex schools are private or religious, thereby resulting in selection issues and many confounding factors. Using Korean data on secondary education, I attempt to analyze how students' academic achievement represented by college entrance exam scores, the so-called Korean SAT scores, differs between coed and single-sex schools. From the data, I observe comparable socioeconomic backgrounds among students attending single-sex and coed schools, which strengthens the claim that students are randomly assigned to schools regardless of schools' gender composition.

I regard the assignment to coed or single-sex schools as treatment and try to estimate the treatment effect. First of all, I use parametric and semiparametric methods to study the treatment effect. Since individuals' background characteristics such as household income and study time also affect test scores, I try to control for various factors. To examine distributional consequences of this educational treatment, I also conduct nonparametric hypotheses testing regarding conditional average treatment effect (CATE) and first-order stochastic dominance across control variables. I conduct empirical analysis adopting the hypothesis testing technique suggested in Lee, Song, and Whang (Journal of Econometrics, 2012). The null of nonpositive average treatment effect of single-sex schooling uniformly across a control variable is significantly rejected only in males' case. From this, it is conjectured that male students in boys' high schools get higher scores than those in coed schools across all income level or studying time, whereas girls' scores show no significant difference between two school types. Likewise, the null hypothesis that test score of single-sex school group is stochastically dominated by that of coed school group conditional on income or studying time is rejected only in case of males.

Furthermore, I propose an extended version of the test in Lee, Song, and Whang (2012) to incorporate a case with a fixed control among two covariates. Using asymptotic theory of the proposed test statistic, I conduct Monte Carlo experiments to see finite sample performance of the statistic. Then, I apply the suggested test with a fixed control to a single-sex schooling context. I set the null of nonpositive CATE uniformly over income and a fixed level of study time. Comparing 0.5-quantile and 0.75-quantile of study time, it is shown that hard-working male students are not negatively affected in coed schools, whereas moderate workers get lower scores in coed schools.

2 Student Assignment Rule and Data

Since 1974 when *Equalization Policy* was first enacted, students living in regions under this policy had been randomly assigned to high schools. This policy was designed to hinder excessive competition to get into renowned high schools which caused overspending on private tutors and institutes. As a notable example to clarify how this policy had worked, the capital of Korea, Seoul, was separated into 11 education-blocks in 1974. This separation had been maintained until 2009. Students in Seoul were randomly assigned to high schools in the specific education-block where their residence was. They could not apply for high schools in different education-blocks. Since 2010, the student assignment policy has been modified to allow students to choose which high school to apply for to some extent. In Figure 1 of the year 2008, blue and red dots stand for single-sex schools whereas green dots represent coed high schools. The distribution of single-sex and coed schools looks quite random. The ratio of coed schools in Seoul is 0.43.

I use the Korean Education and Employment Panel (KEEP) data constructed by the Korea Research Institute for Vocational Education and Training (KRIVET). The KEEP data was first collected in 2004 in order to analyze Korean adolescents' educational experience and transition from school to workplace. The original sample is categorized into two subsamples which consist of high school seniors or middle school seniors in 2004. Since 2004, it has tracked these subsamples, updating the data once a year. In this analysis, I use the data for the middle school senior cohort who lived in regions affected by *Equalization Policy* and entered high school in 2005. These students received their college entrance exam scores, that is, Korean SAT (KSAT, hereafter) scores in 2008.

For KSAT, most high school seniors take tests at least in three sections: Korean, Math, and English. Depending on one's choice, he/she can take 3 selective tests in the field of social sciences or natural sciences and also a second language test. Most tests are paper-based and multiple choice. Before and after the year 2008, students can see both their raw scores and rank from 1 to 9. For instance, students in the first rank are those who achieve KSAT score in the top 4th percentile. However, in 2008, KSAT score was reported not in raw value but only in rank scale due to the temporary educational policy. Therefore, in this analysis, I use the weighted average of test scores as KSAT score variable giving the same weight to three primary tests (Korean, Math, English)[1] and to average of all selective tests a student took. For convenience, I subtract these weighted average scores from 9 in order to represent better educational performance with higher values.

To analyze whether students' KSAT scores are differently influenced under single-sex or coed schools, we need to control for factors which can also affect the test scores. First of all, household income should be taken into account since

[1]Some observations do not have Math or English test scores. If a student shows only Korean and Math test scores, I take average with the two results only.

it reflects a student's background and his access to educational resources. Student self-study time can also be perceived as a control variable since it reflects one's effort level and passion for studying.[2] To be specific, income is monthly household income and its unit is KRW/1000000 (1USD=1100KRW). Study time is a student's self-studying time per week other than regular class hours in school.

There are other variables which would affect KSAT score such as education level of a student's parents, whether he/she lives in Seoul, and household spending on private tutors or institutes. I try to control for these variables in parametric and semiparametric analysis. Since females tend to score higher than males in KSAT on average, I separate the whole sample into two gender groups and analyze them separately. This is also because I conjecture that girls and boys would be differently affected in single-sex or coed schools.

Table 1 shows the unconditional mean of KSAT scores between different school types and genders. From this table, it is noted that students in single-sex schools get higher KSAT scores on average. However, the gap between single-sex and coed schools is greater among males than among females. Therefore, it can be justified that I analyze the effect of coed and single-sex schools on educational output separately between males and females. Table 2 shows descriptive statistics of dependent variable (KSAT scores) and control variables in the empirical analysis. I conduct two-sample group mean t-test to see whether the variables

[2] I primarily consider these two control variables in the following nonparametric tests.

differ significantly between coed and single-sex schools. The result shows that the average KSAT score differs significantly between two school types among male students whereas that of females displays no significant difference (under 5%−significance level). In contrast, when I conduct the same t-test on control variables, no variable is shown to differ significantly between coed and single-sex schools. This supports the argument that there hardly exists any selection issue under the random student assignment policy in Korea.

3 Empirical Analysis I: Univariate/Multivariate Control

3.1 Parametric and Semiparametric Estimation

3.1.1 Results from OLS Estimation

In this section, I adopt parametric and semiparametric approach to estimate the effect of coed schooling on educational outcome. First of all, Table 3 displays the results from standard OLS estimation. If it is true that Korean students are randomly assigned to high schools and there is any effect of coed school on students' test scores, it is likely to observe some treatment effect even with the very simple specification.

When I control nothing and regress KSAT scores on coed dummy variable, the coefficient is shown to be negative and significant in males' case whereas that of females is not significant. In addition, the absolute value of coefficient is much larger in case of male students. This also holds throughout specifications (2)-(7) in case I add some control variables. In addition to income and study time, I control the following variables: Parent edu variable is formulated by the maximum level of education of a student's parents, i.e. $max(father\ edu, mother\ edu)$. Each education variable is categorical in the sense that 5 means high school graduate and 6 means college graduate (2-year or vocational). Private edu variable is a dummy variable giving 1 if a student reports positive monthly expenditure or weekly time spent on private education. This form of dummy variable is adopted in order to secure a fair number of data.[3]

In all specifications of males, the coefficient on coed dummy is significant under 5%-significance level.[4] Income, study time, and parent edu variables have significant coefficient in all cases considered whereas Seoul dummy has insignificant coefficient both in (5) and (7) under 5%−significance level. This is plausible taking into account the fact that the average KSAT score of students in Seoul is no higher than that of students in other regions.

[3]Since private education is quite a controversial topic in Korea, some people might feel reluctant to report the amount of money they spend for private education. It does not seem consistent that a person reports not private education expenditure but only hours spent on it. That is the reasoning behind the formulation of private education dummy.

[4]After conducting regressions for specification (7), 447 (males) and 496 (females) observations are left. When I perform OLS estimation with these final data on specification (1)-(3) again, the significant coefficients for coed appear only in males' case. Since the results are not much different from those in Table 3, they are omitted.

Using all the control variables as in specification (7), I also perform quantile regression with respect to several quantiles (0.05, 0.25, 0.5, 0.75, 0.95). The results in Table 4 are in a similar vein of those from OLS estimation. Focusing on males, the coefficient of coed dummy is significant in case of 0.25-quantile and 0.75-quantile with $5\%-$significance level. Also, when performing median regression, the coefficient on coed is significant with $10\%-$significance level. In case of 0.95-quantile regression, the coefficient on coed dummy is insignificant, which suggests that coed schooling might not influence test scores of quantiles in the tail part. In contrast, in every quantile regression, females' test scores are shown to have no significant differences between coed and girls' high schools.

Moreover, I conduct OLS estimation with different dependent variables. Instead of KSAT, Korean, Math, and English scores are separately adopted as a dependent variable. Table 5 shows the results from conducting OLS estimation with specification (7) on these new dependent variables. When Math score is dependent variable, the coefficient on coed is significant under $5\%-$significance level. What is interesting in this case is that the private education dummy is also significant. When Korean is dependent variable, p-value for coed is 0.072. When English is dependent variable, the coefficient on coed is not significant. In case of females, throughout all dependent variables considered, the coefficient on coed is not significant. Therefore, the result from the previous OLS estimation is similarly maintained even when different dependent variables are adopted.

3.1.2 Results from Matching Estimator

I adopt matching estimator suggested in Abadie and Imbens (2002).[5] Since a student is assigned to one of two different school types, there comes the problem of counter-factual outcomes. That is, we can only observe one of the two outcomes: KSAT scores under coed schools or those under single-sex schools. To analyze the average treatment effect in the sample (SATE), I use matching estimator to estimate the unobserved outcome by averaging the outcomes of individuals exposed to the opposite treatment but have similar covariates. I set up $\widehat{KSAT}_i$(coed), $\widehat{KSAT}_i$(single-sex) and get SATE by

$$\frac{1}{N}\sum_{i=1}^{N}\left(\widehat{KSAT}_i(coed) - \widehat{KSAT}_i(single - sex)\right)$$

In Table 6, SATE is shown to be negative in all cases for males. From this analysis, it can be said that attending coed schools rather than single-sex schools is related to lower KSAT scores of males. All the matching variables are the same ones from the previous OLS analysis. If we focus on the estimated SATE results using four matches, these values are observed to lie between -0.3042 and -0.2509 among males. Similar to the OLS result, the absolute value of treatment effect is

[5]Even though the result is omitted in this paper, I also try to analyze with propensity score matching. I use the same matching variables as in Table 6. As the matching methods, I adopt the nearest neighbors upto 4 matches or kernel matching or matching with local linear regression. The results of estimated SATE are quite similar to those achieved from Abadie and Imbens (2002) matching estimator using four matches. However, in the case with matching on the estimated propensity score, it is hardly known how the variance is computed. It is also not justifiable to use bootstrap as suggested in Abadie and Imbens (2006).

much greater among males than females. In all specifications with four matches, male students' KSAT scores are shown to decrease under coed schools under 10%−significance level. With some combinations of matching variables, females are shown to have positive treatment effect though it is highly insignificant. We can also conjecture that coed schooling affects males and females differently and males' scores are more likely to be influenced negatively by coed schooling.

3.2 Nonparametric Hypothesis Testing

3.2.1 Results from No Control

In this section, I perform unconditional dominance test before considering hypotheses on conditional average treatment effect or stochastic dominance. Test on unconditional dominance, so-called residual dominance, can also be an empirically important issue. As suggested in Linton et al. (2005), I test unconditional dominance between distributions of intercept-adjusted regression error of coed and single-sex schools. The difference between means of adjusted regression error is the estimated coefficient on coed dummy. As in the OLS analysis, 7 specifications are used to get regression errors. Figure 2, 3, and 4 show unconditional cumulative distribution functions (CDF) of KSAT residuals. In most specifications of males, residuals of single-sex schools seem to first-order stochastically dominate those of coed schools. However, females' residuals show no such dominance relationship. The gap between two empirical distributions is very small and the two lines are crossing more than once.

It is necessary to perform hypothesis test to conclude that there indeed exists first-order or second-order dominance relationship between two residual distributions. Table 7 shows the results from nonparametric test on unconditional dominance. McFadden-type test statistics are adopted: $min\, max \mid F_N(x) - G_N(x) \mid$ for first-order stochastic dominance, and $min\, max \mid \int_{-\infty}^{t} F_N(x)dx - \int_{-\infty}^{t} G_N(x)dx \mid$ where $F_N(x), G_N(x)$ are empirical distributions. p-values are computed from pair bootstrapping for 1000 times.

For males, the null that residuals of coed schools first-order stochastically dominate those of single-sex schools is rejected in specification (1), (4), (5), (6) under $5\%-$significance level. Under (2) and (3), the null is rejected under $10\%-$significance level. After controlling all covariates available, the null cannot be rejected. The same null cannot be rejected in all specifications among females. If we consider the null that residuals of single-sex schools first-order stochastically dominate those of coed schools, this null cannot be rejected among both males and females. Second-order dominance means that the dominating one is weakly preferred by a risk-averse decision maker. In case of both males and females, the null of second-order stochastically dominating residuals of coed schools is rejected except for two females' specifications. Therefore, it can be inferred that risk-averse students tend to prefer single-sex schools in terms of getting higher (residual) scores.

3.2.2 Results from Univariate Control

I control for one of the two control variables, income and study time, then conduct nonparametric hypothesis testing on conditional mean or first-order stochastic dominance over the univariate control. Figure 5 and 6 show nonparametric estimates of conditional means of KSAT scores as functions of a control variable for coed and single-sex schools. I constrain the support from 0.1-quantile to 0.9-quantile of the control variable used. The kernel function in this estimation is $K(u) = 3/2(1 - (2u)^2)I\{\mid u \mid \leq 0.5\}$. The dotted line is for single-sex and the other line is for coed schools.

In Figure 5, there seems to be negative average treatment effect for male students when assignment to coed schools is regarded as treatment and income is adopted as control variable.[6] However, the lines drawn from the estimates of conditional means are crossing and the gap between them is not substantial among females. In Figure 6, it is more conspicuous among males that coed schooling is associated with lower KSAT scores across all study time. For females, however, the estimates of conditional mean of test scores are not showing the similar relationship as observed among males. Two lines are crossing and they do not show any substantial gap.

[6] Coed schooling seems to be linked with negative treatment effect except for income interval above 4.5. However, when bandwidth from Silverman's rule-of-thumb method is chosen, coed schools' estimates of conditional means of KSAT scores lie below those of boys' schools across the whole support. For girls, the two lines are crossing in the middle of the support regardless of bandwidth selection methods.

I adopt the testing technique suggested in Lee, Song, and Whang (2012) in this section. I denote KSAT scores as Y and $\mathcal{W}_{\mathcal{X}}$ is the support of a control variable which I am interested in. The combinations of hypotheses on conditional average treatment effect (CATE, hereafter) and first-order stochastic dominance (F denotes cumulative distribution function) are as follows when assignment to single-sex school is treatment:[7]

$$(1)\quad \begin{cases} H_0 : E[Y_{single-sex} - Y_{coed} \mid X = x] \leq 0 \ for\ all\ x \in \mathcal{W}_{\mathcal{X}} \\ H_1 : E[Y_{single-sex} - Y_{coed} \mid X = x] > 0 \ for\ some\ x \in \mathcal{W}_{\mathcal{X}} \end{cases}$$

$$(2)\quad \begin{cases} H_0 : F[Y_{coed} \mid X = x] \leq F[Y_{single-sex} \mid X = x] \ for\ all\ x \in \mathcal{W}_{\mathcal{X}} \\ H_1 : F[Y_{coed} \mid X = x] > F[Y_{single-sex} \mid X = x] \ for\ some\ x \in \mathcal{W}_{\mathcal{X}} \end{cases}$$

I use three weight functions in Lee, Song, and Whang (2012): uniform, inverse SE, and density weight. I consider 0.10 and 0.90-quantiles of a control variable in both tests. When I test for first-order stochastic dominance, I use the entire support of Y for the uniform weight and density weight functions and the interval between 0.10 and 0.90-quantiles of Y for the inverse standard-error weight function. Instead of developing a bandwidth based on high-order asymptotic theories, I report testing results under various bandwidths. Bandwidths

[7] At first, I test the zero CATE against nonzero CATE for some income or study time, i.e., $H_0 : E[Y_{single-sex} - Y_{coed} \mid X = x] = 0 \ for\ all\ x \in \mathcal{W}_{\mathcal{X}},\ H_1 : E[Y_{single-sex} - Y_{coed} \mid X = x] \neq 0 \ for\ some\ x \in \mathcal{W}_{\mathcal{X}}$. I cannot reject the null in both cases of males and females. Getting some hint from Figure 5, I instead report results from testing the null of nonpositive CATE against positive CATE, since the two-sided test is usually less powerful than one-sided test.

are in the form $h = C_h \cdot \hat{s}_X \cdot n^{-2/7}$, where $\hat{s}_X$ is the sample standard deviation of the control variable, and C_h is a constant from a set $\{4, 4.5, 5, 5.5, 6, 6.5\}$. I intend to display the overall results of hypotheses testing and reliability of the test I perform.

Table 8 shows the results when household income is adopted as control variable. If single-sex schooling is treatment, it is shown that the null hypothesis of nonpositive CATE is rejected in case of males at the nominal level 5% across all weight functions and bandwidths. This result is quite consistent with Figure 5. For females, the null hypothesis of test (1) is not rejected regardless of treatment variables. The result of testing conditional stochastic dominance shows that there is significant evidence against the null hypothesis that the test score of boys' school group is stochastically dominated by that of coed school group across all income levels. The null is rejected at the 5% nominal level among males whereas the null is not rejected among females. This result is consistent with the result of nonpositive CATE test.

Table 9 shows the results when conducting the same hypotheses test with study time as control variable. As expected in Figure 6, with single-sex schooling as treatment, the null hypothesis of nonpositive CATE is rejected in case of males at the nominal level 5% across all weight functions and bandwidths except for only one case (p-value 0.059). However, I cannot reject the same null hypothesis in case of females. Likewise, the result of nonparametric test (2) is shown to have a similar pattern as in Table 8. There is also significant evidence against the null hypothesis of stochastically dominated test scores of boys' schools by those of coed schools. The null is rejected at the 5% nominal level for male students, but not for females.

Setting coed-schooling as treatment, the null hypothesis of nonpositive CATE cannot be rejected among both males and females regardless of what control variable I use. When I set the null hypothesis that test score of coed school is stochastically dominated by that of single-sex school, the null cannot be rejected regardless of genders and control variables adopted.

3.2.3 Results from Multivariate Control

Additional tests with multivariate covariate are conducted. I consider two outcome variables, KSAT and Math.[8] The null and alternative hypotheses considered are in the same form of test (1) in the previous univariate control case. The only difference is that the control X is multi-dimensional. First, I consider income and study time as control variables. Then, I additionally control parent edu with these two control variables.[9] Since all covariates are scalar, the interval between the 10 and 90 percentiles of X is the domain of interest $\mathcal{W}_{\mathcal{X}}$.

[8] In the OLS estimation on Math as dependent variable, the coefficient of coed is highly significant with males. That is why Math is specifically considered as outcome variable in this section.

[9] I also consider other pairwise combinations of three controls. The results are omitted because they are not much different from Table 10.

The product kernel with the kernel in the previous section is adopted with a bandwidth $h = C_h(1/12)^{1/2}n^{-2/(7d)}$, where $C_h \in \{5, 5.5, 6, 6.5, 7, 7.5\}$ and d is number of controls. Three kinds of weight functions are also considered.

Table 10 shows the hypothesis test results when income and study time are used as controls. From the results, it is shown that males students get significantly higher KSAT or Math scores in single-sex schools uniformly over income and study time. When I use other combinations of two-dimensional controls like X =(income, parent edu) or X = (study time, parent edu), the results are similar to Table 10. The null of nonpositive CATE uniformly over covariates is rejected significantly among males, when assignment to single-sex school is regarded as treatment.

Table 11 shows the test results when income, study time and parent edu are adopted as multivariate control. Tests are performed for two outcomes, KSAT and Math respectively. When the assignment to boys' school is treatment and KSAT is outcome variable, the null of nonpositive CATE cannot be rejected under 5%−significance level except for bandwidth h=2.796. Regarding Math, the same null cannot be rejected under 5%−significance level for all bandwidths and weights. It seems to be harder to reject the null due to the curse of dimensionality when we use several controls.

4 Empirical Analysis II: a Fixed Control among Two Covariates

4.1 Monte Carlo Experiment

I would like to extend the previous nonparametric test to use in case one of the control variables is fixed. Then, it is necessary to verify how the distribution of test statistic under the null would be changed. I am interested in the case that there are two control variables X and Z, and Z is fixed as a certain value z. In the theoretical appendix, I show that the asymptotic distribution of the proposed test statistic is the standard normal under least favorable case.[10] In this section, some Monte Carlo experiments are conducted to examine the finite sample performance of this test. The null hypothesis of nonpositive CATE for every $x \in \mathcal{W}_{\mathcal{X}}$ and a fixed z is tested against the alternative of positive CATE for some $x \in \mathcal{W}_{\mathcal{X}}$ and a fixed z.

The covariates X and Z are drawn from $Unif[0, 1]$. The treatment indicator variable D is drawn independently from a Bernoulli distribution with probability 0.5. The outcome variable Y is generated by

[10] Andrews and Shi (2011) also studies testing conditional moment inequalities for X and $Z = z$. They construct confidence sets to test hypothesis. Their strategy is to transform the conditional moment inequalities for X and $Z = z$ into equivalent ones given only $Z = z$ by adopting suitable $X-$instruments. On the other hand, I use the Lee, Song, and Whang (2012) approach with Poissonization to deal with the nonparametric test with a fixed control.

$$Y = [g_1(X, Z) + U_1] \times D + [g_0(X, Z) + U_0] \times (1 - D),$$

where U_1 and U_0 are independent normals with $N(0, 0.1^2)$. The function $g_0(x, z) \equiv 0$ for all specifications. $g_1(x, z)$ in the analysis has the form of $g_1(x, z) \equiv z - x^2(1 - \beta)$. I consider five forms of $g_1(x, z)$ with different fixed z values and β. The DGPs considered are as follows: (z,β)=(0,0) under DGP1, (z,β)=(0,0.5) under DGP2, (z,β)=(0,1) under DGP3, (z,β)=(0.3,0) under DGP4, (z,β)=(0.5,0) under DGP5. If CATE is denoted as $\tau_0(x, z)$, $\tau_0(x, z) = -x^2$ for $x \in [0, 1]$ under DGP1. Under DGP2, $\tau_0(x, z) = -0.5x^2$ for $x \in [0, 1]$. Under DGP3, $\tau_0(x, z) = 0$ for $x \in [0, 1]$, which denotes the least favorable case of H_0. On the other hand, the null hypothesis is false under DGP4 and DGP5.

For each DGP, 1000 repeated samples are generated. Tests are performed twice with sample size (n) 200 and 500, respectively. $\mathcal{W}_\mathcal{X}$ is set to $[0.1, 0.9]$ which is an interval between the 10 and 90 percentiles of X. Three weight functions as in the previous analysis are used. I use the kernel $K(u) = 3/2(1 - (2u)^2)I\{| u |\leq 0.5\}$ with a bandwidth $h = C_h \hat{s}_X n^{-1/7}$, where $\hat{s}_X$ is the sample standard deviation of X and $C_h \in \{3, 4, 5\}$.

Table 12 (uniformly weighted), Table 13 (inverse SE weighted), and Table 14 (density weighted) show the results from Monte Carlo experiments using sample size 200 and 500. All tables contain rejection probabilities of testing the null of nonpositive CATE for every $x \in \mathcal{W}_\mathcal{X}$ and a fixed value of z. Rejection probabilities with different nominal levels are reported.

Under DGP1 which is interior case of H_0, rejection probabilities are almost zero in both n=200 and n=500. Under DGP2, rejection probabilities are quite close to zero. Under DGP3, the least favorable case, the rejection probabilities are similar to nominal levels. However, the rejection probabilities seem to fall below the nominal levels when bandwidth is smaller. Under DGP4 and DGP5 where the null is clearly false, the rejection probabilities are quite large as expected. The probabilities are much larger with larger value of fixed z (DGP4: z=0.3, DGP5: z=0.5). The proposed test seems to be more powerful with larger sample size. Under DGP4 and DGP5, the rejection probabilities are much larger with sample size 500. Across all weight functions, similar results are observed supporting the robustness of test. The results from Monte Carlo experiments show that finite sample performance of the suggested test is consistent with its asymptotic property shown in theory.

4.2 An Empirical Example with Fixed Study Time

In this section, I try to extend the previous empirical analysis on single-sex and coed schooling by incorporating the case one of two control variables is fixed. I consider two covariates, income (say X) and study time (Z). From the previous results, it is shown that males students get significantly higher KSAT scores in single-sex schools, uniformly over income or study time or both. On the other hand, females show no conspicuous difference between the two school forms. It is also an interesting issue to see whether this pattern appears even in case we fix one of the two control variables. For instance, if we fix study time to a large number, which means focus only on hard-working students, those in coed schools might not be affected that much negatively by school system since they work hard anyway.

By using the test proposed in this paper, I test the null of nonpositive CATE uniformly over income level with a fixed study time. I consider two fixed levels of study time: moderate work (0.5-quantile) and hard work (0.75-quantile). For males, moderate work is 5 hours and hard work is 12 hours. For females, moderate work is 5 hours and hard work is 14 hours. Figure 7 and 8 show conditional means of KSAT scores across income levels with fixed study time. In males' case, there seems to be a significant score gap between single-sex and coed schools among moderate workers. In addition, what is interesting is that the gap is much wider among students with low household income level. However, among hard-working male students, two lines are drawn to cross and have very small gap. In females' case, the lines are crossing among both moderate workers and hard-workers.

When we regard single-sex schooling as treatment, the hypotheses are $H_0 : E[Y_{single-sex} - Y_{coed} \mid X = x, Z = z] \leq 0$ for all $x \in \mathcal{W}_{\mathcal{X}}$ and fixed z and $H_1 : E[Y_{single-sex} - Y_{coed} \mid X = x, Z = z] > 0$ for some $x \in \mathcal{W}_{\mathcal{X}}$ and fixed

z. I use three weight functions as in the previous analysis. The 0.10 and 0.90-quantiles of income is considered as $\mathcal{W}_{\mathcal{X}}$. I report testing results under various bandwidths. Bandwidths are in the form $h = C_h \cdot \hat{s}_X \cdot n^{-1/7}$, where $\hat{s}_X$ is the sample standard deviation of the control variable chosen, and C_h is a constant from a set $\{3.5, 4, 4.5, 5, 5.5\}$.

Table 15 and 16 show the test results with two fixed study times. Table 15 displays p-values of testing hypotheses for males and Table 16 does for females. In case of males with moderate study time, the null of nonpositive CATE is rejected significantly with single-sex schooling as treatment. The p-values are all zero regardless of bandwidths and weights. This result is consistent with the shape of Figure 7. For hard-working male students, the null of nonpositive CATE cannot be rejected regardless of treatment variables. Therefore, the negative treatment effect of coed schooling doesn't seem to exist among hard-working male students.

On the other hand, even with a fixed control, the conditional means of KSAT scores seem to cross each other among females. Hence, the results from hypothesis testing can be expected not to reject the null of nonpositive CATE uniformly over income level. Table 16 shows that the null cannot be rejected regardless of fixed z level, bandwidth, weight, and treatment variables. The proposed nonparametric hypothesis test with a fixed control seem to work fairly well with the real-world data.

5 Conclusion

To study whether single-sex schooling is better for improving student academic performance is meaningful in that it is easy to implement sex-segregation in order to increase overall educational level with relatively less cost. The case of Korea is adequate to analyze this issue since students are randomly assigned to high schools in contrast to other countries. In this paper, I study how student achievement represented by Korean SAT scores differs across coed and single-sex schools.

In the parametric and semiparametric analysis, I regard the assignment to coed schools as treatment and try to estimate the treatment effect. Since other factors such as household income and student study time also affect test scores, I attempt to control them. The treatment effect is shown to be negative and significant in males' case whereas that of females is not significant. This pattern is similarly observed when I use the matching estimator to compare test scores of students in coed and single-sex schools who are similar along key features.

Then, to examine distributional consequences of this educational treatment, I conduct nonparametric hypotheses test regarding nonpositive conditional average treatment effect or first-order conditional stochastic dominance. This is important to assess the overall impact of a certain treatment. I conduct empirical analysis adopting the hypothesis testing technique suggested in Lee, Song, and Whang (2012). In this analysis, the null of nonpositive CATE of single-sex schooling uniformly across a control variable is significantly rejected only in males' case. Females show no significant results from hypothesis testing. Likewise, the null hypothesis that test score of single-sex school group is stochastically dominated by that of coed school group conditional on income level or studying time is rejected in case of males only.

Lastly, I suggest a nonparametric hypothesis test to incorporate a case with a fixed control among two covariates. I conduct Monte Carlo experiments to see finite sample performance of the suggested test. Then, I apply it to test hypotheses in a single-sex schooling context. I set the null of nonpositive CATE uniformly over income and a fixed level of study time. Comparing 0.5 and 0.75-quantile of study time, test scores of hard-working male students are not negatively affected in coed schools, whereas moderate workers get lower scores in coed schools.

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6 Appendix A: Tables

(Males)	Obs	Mean	Std. Err.	(Females)	Obs	Mean	Std. Err.
Single-sex	294	3.885	0.090	Single-sex	271	4.046	0.090
Coed	216	3.497	0.107	Coed	285	3.941	0.095
Difference		0.388	0.139	Difference		0.105	0.132

Table 1: Unconditional Mean of KSAT Scores

Group	Variable	Obs	Mean	Std. Dev.	Min	Max
Single-Sex/ Males	KSAT	252	3.876	1.530	0.313	7.75
	Income	252	310.992	199.268	0	2000
	Study Time	252	8.626	8.658	0	46
	Parent Edu	252	5.623	1.203	3	9
	Seoul	252	0.242	0.429	0	1
	Private Edu	252	0.837	0.429	0	1
Coed/ Males	KSAT	195	3.546	1.544	0.188	7.375
	Income	195	306.508	183.883	0	1000
	Study Time	195	7.964	8.085	0	63
	Parent Edu	195	5.605	1.190	3	9
	Seoul	195	0.190	0.393	0	1
	Private Edu	195	0.836	0.393	0	1
Single-sex/ Females	KSAT	246	3.978	1.458	0	7.5
	Income	246	292.846	181.420	0	1200
	Study Time	246	8.772	9.015	0	60
	Parent Edu	246	5.630	1.166	2	9
	Seoul	246	0.187	0.391	0	1
	Private Edu	246	0.780	0.415	0	1
Coed/ Females	KSAT	250	3.984	1.616	0.688	7.875
	Income	250	301.6	207.451	0	1630
	Study Time	250	9.512	9.375	0	43
	Parent Edu	250	5.58	1.243	3	9
	Seoul	250	0.136	0.343	0	1
	Private Edu	250	0.784	0.412	0	1

Table 2: Descriptive Statistics

Note (for Table 1-2): KSAT is the weighted average of Korean SAT scores giving the same weight to the three primary tests (Korean, Math, English) and to the average of all selective tests (related to social sciences, natural sciences, and second languages) a student took. The highest possible value for KSAT is 8. Income is monthly household income with unit KRW/1000000 (1 USD=1100 KRW). Study time is one's self-studying time per week other than regular class hours in school. For the other variables, please refer to Note for Table 3.

Group	Variables	(1)	(2)	(3)	(4)	(5)	(6)	(7)
	constant	3.8852**	3.4176**	3.0398**	2.0607**	2.0536**	1.8689**	1.8397**
		(0.0905)	(0.1492)	(0.1611)	(0.3373)	(0.3367)	(0.3644)	(0.3639)
	coed	-0.3883**	-0.3219**	-0.2907**	-0.2951**	-0.3092**	-0.2954**	-0.3107**
		(0.1391)	(0.1436)	(0.1395)	(0.1384)	(0.1384)	(0.1383)	(0.1382)
	income		0.0015**	0.0015**	0.0009**	0.0010**	0.0009**	0.0009**
			(0.0004)	(0.0004)	(0.0004)	(0.0004)	(0.0004)	(0.0004)
Males	study time			0.0440**	0.0397**	0.0395**	0.0392**	0.0390**
				(0.0082)	(0.0083)	(0.0083)	(0.0083)	(0.0082)
	parent edu				0.2104**	0.2218**	0.2110**	0.2233**
					(0.0639)	(0.0641)	(0.0638)	(0.0640)
	seoul					-0.2741		-0.2967*
						(0.1678)		(0.1682)
	private edu						0.2596	0.2887
							(0.1879)	(0.1882)
	obs.	510	451	451	447	447	447	447
	R^2	0.0151	0.0446	0.1023	0.1243	0.1296	0.1281	0.1342
Group	Variables	(1)	(2)	(3)	(4)	(5)	(6)	(7)
	constant	4.0459**	3.5353**	3.1217**	2.3778**	2.3766**	2.2794**	2.2776**
		(0.0943)	(0.1406)	(0.1457)	(0.3154)	(0.3159)	(0.3233)	(0.3239)
	coed	-0.1046	-0.0062	-0.0336	-0.0297	-0.0308	-0.0302	-0.0316
		(0.1318)	(0.1357)	(0.1296)	(0.1288)	(0.1293)	(0.1287)	(0.1292)
	income		0.0015**	0.0014**	0.0010**	0.0010**	0.0009**	0.0009**
			(0.0003)	(0.0003)	(0.0004)	(0.0004)	(0.0004)	(0.0004)
Females	study time			0.0512**	0.0477**	0.0476**	0.0476**	0.0476**
				(0.0071)	(0.0071)	(0.0072)	(0.0071)	(0.0072)
	parent edu				0.1595**	0.1601**	0.1509**	0.1516**
					(0.0594)	(0.0596)	(0.0597)	(0.0599)
	seoul					-0.0218		-0.0279
						(0.1826)		(0.1825)
	private edu						0.2181	0.2187
							(0.1603)	(0.1605)
	obs.	556	499	497	496	496	496	496
	R^2	0.0011	0.0347	0.1274	0.1409	0.1410	0.1442	0.1442

Table 3: Results from OLS Estimation

Note: ** p-value $<$ 0.05 * p-value $<$ 0.10. Standard errors are in parenthesis. The dependent variable is KSAT as described in Table 2. Coed is a dummy variable giving 1 to coed schools and 0 to single-sex schools. Parent edu stands for the maximum level of education of a student's parents, i.e. $max(father\, edu, mother\, edu)$. Seoul is a dummy variable giving 1 in case a student lives in Seoul, the capital of Korea. Private edu is a dummy variable giving 1 if a student reports positive monthly expenditure or weekly time spent on private education. Private education includes private tutors and institutes.

Group	Variables	OLS(7)	q(0.1)	q(0.25)	q(0.50)	q(0.75)	q(0.9)
Males	constant	1.8397**	0.4671	0.7450	1.3881**	2.4331**	3.9059**
		(0.3639)	(0.6039)	(0.5898)	(0.5637)	(0.4191)	(0.8173)
	coed	-0.3107**	-0.3924*	-0.4261**	-0.3722*	-0.3281**	-0.0816
		(0.1382)	(0.2281)	(0.2176)	(0.2171)	(0.1593)	(0.2435)
	income	0.0009**	0.0013*	0.0015**	0.0005	0.0009*	0.0008
		(0.0004)	(0.0007)	(0.0006)	(0.0005)	(0.0006)	(0.0008)
	study time	0.0390**	0.0590**	0.0461**	0.0415**	0.0314**	0.0374**
		(0.0082)	(0.0198)	(0.0129)	(0.0113)	(0.0109)	(0.0112)
	parent edu	0.2233**	0.0905**	0.2183**	0.3062**	0.2771**	0.2062
		(0.0640)	(0.1052)	(0.0952)	(0.0818)	(0.0852)	(0.1377)
	seoul	-0.2967*	-0.3192	-0.4906**	-0.3240	-0.1779	0.2069
		(0.1682)	(0.3092)	(0.2322)	(0.3034)	(0.2272)	(0.3282)
	private edu	0.2887	0.3995	0.1466	0.5568**	0.3776	-0.1160
		(0.1882)	(0.2593)	(0.3377)	(0.2539)	(0.2602)	(0.5250)
	R^2	0.1342	0.0779	0.0894	0.0719	0.0938	0.0817
Group	Variables	OLS(7)	q(0.1)	q(0.25)	q(0.50)	q(0.75)	q(0.9)
Females	constant	2.2776**	0.1155	0.7915	2.8040**	3.1702**	3.7373**
		(0.3239)	(0.4137)	(0.4518)	(0.3790)	(0.5100)	(0.5121)
	coed	-0.0316	-0.0798	-0.1448	-0.0145	0.1466	-0.0206
		(0.1292)	(0.1739)	(0.1994)	(0.1534)	(0.1793)	(0.2251)
	income	0.0009**	0.0010**	0.0009**	0.0008*	0.0005	0.0021**
		(0.0004)	(0.0005)	(0.0004)	(0.0004)	(0.0007)	(0.0009)
	study time	0.0476**	0.0352**	0.0577**	0.0585**	0.0412**	0.0514**
		(0.0072)	(0.0127)	(0.0130)	(0.0083)	(0.0100)	(0.0151)
	parent edu	0.1516**	0.2535**	0.2128**	0.0384	0.2172*	0.1539
		(0.0599)	(0.0848)	(0.0752)	(0.0687)	(0.1135)	(0.1107)
	seoul	-0.0279	0.2636	0.1354	-0.1452	-0.3678	-0.0397
		(0.1825)	(0.2300)	(0.2228)	(0.1534)	(0.2883)	(0.3630)
	private edu	0.2187	0.0349	0.2046	0.3303*	0.0495	0.3429
		(0.1605)	(0.1634)	(0.2342)	(0.1940)	(0.2211)	(0.2708)
	R^2	0.1442	0.0882	0.0771	0.0743	0.0847	0.1131

Table 4: Results from Quantile Regression

Note: ** p-value < 0.05 * p-value < 0.10. Standard errors are in parenthesis. They are achieved from bootstrapping for 1000 times. The number of observations for males is 447, and that for females is 496.

Group	Variables	KSAT	Korean	Math	English
Males	constant	1.8397**	1.8156**	2.0163**	1.1606**
		(0.3639)	(0.4407)	(0.4456)	(0.4424)
	coed	-0.3107**	-0.3023*	-0.3349**	-0.2525
		(0.1382)	(0.1674)	(0.1695)	(0.1673)
	income	0.0009**	0.0008	0.0008*	0.0012**
		(0.0004)	(0.0005)	(0.0005)	(0.0005)
	study time	0.0390**	0.0391**	0.0402**	0.0459**
		(0.0082)	(0.0010)	(0.0101)	(0.0099)
	parent edu	0.2233**	0.2604**	0.1599**	0.3030**
		(0.0640)	(0.0775)	(0.0785)	(0.0781)
	seoul	-0.2967*	-0.2579	-0.1979	-0.2501
		(0.1682)	(0.2037)	(0.2060)	(0.2045)
	private edu	0.2887	0.0400	0.5422**	0.2880
		(0.1882)	(0.2279)	(0.2319)	(0.2335)
	obs.	447	447	446	431
	R^2	0.1342	0.0959	0.0913	0.1430
Group	Variables	KSAT	Korean	Math	English
Females	constant	2.2776**	1.9686**	2.7525**	1.3918**
		(0.3239)	(0.3873)	(0.3951)	(0.3973)
	coed	-0.0316	-0.0234	-0.1451	0.0476
		(0.1292)	(0.1545)	(0.1576)	(0.1573)
	income	0.0009**	0.0009**	0.0006	0.0013**
		(0.0004)	(0.0004)	(0.0005)	(0.0005)
	study time	0.0476**	0.0483**	0.0461**	0.0490**
		(0.0072)	(0.0086)	(0.0087)	(0.0087)
	parent edu	0.1516**	0.2265**	0.0265	0.2998**
		(0.0599)	(0.0716)	(0.0732)	(0.0743)
	seoul	-0.0279	-0.4134*	0.2886	0.1583
		(0.1825)	(0.2182)	(0.2221)	(0.1953)
	private edu	0.2187	0.2874	0.4333**	0.1768
		(0.1605)	(0.1920)	(0.1953)	(0.1953)
	obs.	496	496	492	465
	R^2	0.1442	0.1290	0.0853	0.1737

Table 5: OLS Results from Different Dependent Variables

Note: ** p-value < 0.05 * p-value < 0.10. Standard errors are in parenthesis. Dependent variables are Korean, Math, and English scores, respectively. The original rank data is subtracted from 9 so that larger values can represent better scores. The highest possible value for all dependent variables is 8. In every case, OLS estimation is conducted with all the control variables available (specification (7)).

Group	Matching Variables (Number of Matches)	SATE	Std.Err.	z	P>| z |
Males	Income(1)	-0.3127	0.1518	-2.06	0.039
	Income(4)	-0.3042	0.1498	-2.03	0.042
	Income, Study time(1)	-0.1159	0.1693	-0.68	0.493
	Income, Study time(4)	-0.2509	0.1506	-1.67	0.096
	Income, Study time, Parent edu(1)	-0.2709	0.1689	-1.60	0.109
	Income, Study time, Parent edu(4)	-0.2841	0.1516	-1.87	0.061
	Income, Study time, Parent edu, Seoul(1)	-0.3181	0.1653	-1.92	0.054
	Income, Study time, Parent edu, Seoul(4)	-0.2950	0.1501	-1.97	0.049
	Income, Study time, Parent edu, Seoul, Private edu(1)	-0.3179	0.1667	-1.91	0.056
	Income, Study time, Parent edu, Seoul, Private edu(4)	-0.2775	0.1466	-1.89	0.058
Group	Matching Variables (Number of Matches)	SATE	Std.Err.	z	P>| z |
Females	Income(1)	0.0084	0.1401	0.06	0.952
	Income(4)	0.0056	0.1384	0.04	0.967
	Income, Study time(1)	-0.0091	0.1415	-0.06	0.949
	Income, Study time(4)	0.0211	0.1310	0.16	0.872
	Income, Study time, Parent edu(1)	0.0397	0.1479	0.27	0.789
	Income, Study time, Parent edu(4)	0.0418	0.1348	0.31	0.756
	Income, Study time, Parent edu, Seoul(1)	0.0508	0.1501	0.34	0.735
	Income, Study time, Parent edu, Seoul(4)	0.0125	0.1335	0.09	0.926
	Income, Study time, Parent edu, Seoul, Private edu(1)	0.0496	0.1478	0.34	0.737
	Income, Study time, Parent edu, Seoul, Private edu(4)	-0.0348	0.1333	-0.26	0.794

Table 6: Results using Abadie and Imbens (2002) Matching

Note: The table shows the average treatment effect in the sample (SATE). The matching estimator suggested by Abadie and Imbens (2002) is adopted to estimate the unobserved outcome by averaging the outcomes of individuals exposed to the opposite treatment but have similar covariates.

Test (Dominating)	Control Variables (Specification)	Males p-value	Females p-value
	None(1)	0.027	0.225
	Income(2)	0.092	0.678
FOSD	Income, Study Time(3)	0.074	0.683
(u_{coed})	Income, Study Time, Parent Edu(4)	0.001	0.133
	Income, Study Time, Parent Edu, Seoul(5)	0.004	0.136
	Income, Study Time, Parent Edu, Private Edu(6)	0.030	0.237
	Income, Study Time, Parent Edu, Seoul, Private Edu(7)	0.140	0.288
	None(1)	0.661	0.730
	Income(2)	0.749	0.608
FOSD	Income, Study Time(3)	0.704	0.752
($u_{single-sex}$)	Income, Study Time, Parent Edu(4)	0.416	0.794
	Income, Study Time, Parent Edu, Seoul(5)	0.553	0.716
	Income, Study Time, Parent Edu, Private Edu(6)	0.566	0.801
	Income, Study Time, Parent Edu, Seoul, Private Edu(7)	0.670	0.839
	None(1)	0.002	0.094
	Income(2)	0.007	0.512
SOSD	Income, Study Time(3)	0.032	0.288
(u_{coed})	Income, Study Time, Parent Edu(4)	0.002	0.003
	Income, Study Time, Parent Edu, Seoul(5)	0.000	0.003
	Income, Study Time, Parent Edu, Private Edu(6)	0.001	0.026
	Income, Study Time, Parent Edu, Seoul, Private Edu(7)	0.000	0.026
	None(1)	0.170	0.524
	Income(2)	0.550	0.426
SOSD	Income, Study Time(3)	0.498	0.533
($u_{single-sex}$)	Income, Study Time, Parent Edu(4)	0.385	0.483
	Income, Study Time, Parent Edu, Seoul(5)	0.297	0.484
	Income, Study Time, Parent Edu, Private Edu(6)	0.319	0.494
	Income, Study Time, Parent Edu, Seoul, Private Edu(7)	0.110	0.491

Table 7: Unconditional Dominance Test (Residual Dominance)

Note: The table shows the results from nonparametric test on unconditional dominance (so-called residual dominance). FOSD means first-order stochastic dominance and SOSD stands for second-order stochastic dominance test. McFadden-type test statistics are used: for FOSD, $min\,max \mid F_N(x) - G_N(x) \mid$ and for SDSD, $min\,max \mid \int_{-\infty}^{t} F_N(x)dx - \int_{-\infty}^{t} G_N(x)dx \mid$ where $F_N(x), G_N(x)$ are empirical distributions. p-values are computed from pair bootstrapping for 1000 times.

Test (1)	Treatment Weight/ Bandwidth	Single-sex Uniform	Single-sex Inverse SE p-values	Single-sex Density	Coed Uniform	Coed Inverse SE p-values	Coed Density
Males	1.338	0.038	0.024	0.012	0.842	0.870	0.876
	1.506	0.034	0.021	0.011	0.845	0.869	0.872
	1.673	0.031	0.018	0.009	0.846	0.866	0.865
	1.840	0.028	0.016	0.008	0.847	0.862	0.857
	2.007	0.026	0.014	0.007	0.848	0.858	0.850
	2.175	0.023	0.013	0.006	0.848	0.854	0.843
Females	1.321	0.726	0.656	0.573	0.377	0.475	0.669
	1.486	0.707	0.642	0.559	0.352	0.447	0.641
	1.651	0.691	0.628	0.545	0.330	0.423	0.618
	1.816	0.677	0.617	0.534	0.313	0.402	0.595
	1.981	0.666	0.609	0.528	0.300	0.385	0.576
	2.147	0.658	0.604	0.523	0.291	0.374	0.561

Test (2)	Treatment Weight/ Bandwidth	Single-sex Uniform	Single-sex Inverse SE p-values	Single-sex Density	Coed Uniform	Coed Inverse SE p-values	Coed Density
Males	1.338	0.022	0.020	0.008	0.870	0.895	0.911
	1.506	0.021	0.017	0.007	0.875	0.894	0.913
	1.673	0.020	0.016	0.006	0.885	0.895	0.914
	1.840	0.020	0.016	0.006	0.895	0.898	0.914
	2.007	0.019	0.014	0.006	0.898	0.897	0.913
	2.175	0.019	0.014	0.006	0.904	0.898	0.913
Females	1.321	0.795	0.600	0.612	0.346	0.578	0.629
	1.486	0.792	0.600	0.615	0.337	0.570	0.618
	1.651	0.786	0.596	0.611	0.325	0.558	0.599
	1.816	0.777	0.593	0.604	0.316	0.551	0.583
	1.981	0.769	0.588	0.597	0.298	0.533	0.560
	2.147	0.763	0.586	0.597	0.299	0.530	0.550

Table 8: Nonparametric Test (Control: Income)

Note: Each table shows p-values for two different combinations of null and alternative hypotheses when income is used as a control variable X. With single-sex schooling as treatment, the hypotheses for test (1) are: $H_0 : E[Y_{single-sex} - Y_{coed} \mid X = x] \leq 0$ for all $x \in \mathcal{W}_{\mathcal{X}}$ and $H_1 : E[Y_{single-sex} - Y_{coed} \mid X = x] > 0$ for some $x \in \mathcal{W}_{\mathcal{X}}$. For test (2), $H_0 : F[Y_{coed} \mid X = x] \leq F[Y_{single-sex} \mid X = x]$ for all $x \in \mathcal{W}_{\mathcal{X}}$ and $H_1 : F[Y_{coed} \mid X = x] > F[Y_{single-sex} \mid X = x]$ for some $x \in \mathcal{W}_{\mathcal{X}}$, where $F[\cdot]$ is cumulative distribution function.

Test (1)	Treatment Weight/ Bandwidth	Single-sex Uniform	Single-sex Inverse SE p-values	Single-sex Density	Coed Uniform	Coed Inverse SE p-values	Coed Density
Males	5.882	0.013	0.017	0.059	0.910	0.919	0.834
	6.617	0.012	0.016	0.048	0.901	0.910	0.825
	7.352	0.011	0.014	0.040	0.889	0.898	0.817
	8.087	0.010	0.012	0.034	0.878	0.887	0.810
	8.822	0.009	0.011	0.029	0.868	0.877	0.804
	9.557	0.009	0.010	0.026	0.859	0.867	0.798
Females	6.236	0.448	0.386	0.305	0.572	0.606	0.657
	7.015	0.449	0.393	0.338	0.561	0.605	0.659
	7.794	0.449	0.404	0.372	0.562	0.610	0.664
	8.574	0.457	0.423	0.406	0.571	0.621	0.673
	9.353	0.472	0.447	0.438	0.587	0.637	0.684
	10.133	0.495	0.476	0.466	0.607	0.655	0.697

Test (2)	Treatment Weight/ Bandwidth	Single-sex Uniform	Single-sex Inverse SE p-values	Single-sex Density	Coed Uniform	Coed Inverse SE p-values	Coed Density
Males	5.882	0.006	0.010	0.040	0.928	0.914	0.889
	6.617	0.006	0.010	0.032	0.924	0.907	0.881
	7.352	0.007	0.011	0.027	0.923	0.905	0.876
	8.087	0.007	0.011	0.023	0.926	0.905	0.876
	8.822	0.008	0.011	0.021	0.926	0.904	0.874
	9.557	0.009	0.012	0.019	0.927	0.905	0.874
Females	6.236	0.512	0.297	0.330	0.572	0.772	0.654
	7.015	0.515	0.298	0.366	0.561	0.770	0.648
	7.794	0.496	0.292	0.384	0.545	0.760	0.636
	8.574	0.506	0.300	0.416	0.556	0.764	0.636
	9.353	0.509	0.305	0.438	0.560	0.765	0.635
	10.133	0.513	0.309	0.454	0.563	0.762	0.635

Table 9: Nonparametric Test (Control: Study Time)

Note: Each table shows p-values for two different combinations of null and alternative hypotheses when study time is used as a control variable X. With single-sex schooling as treatment, the hypotheses for test (1) are: $H_0 : E[Y_{single-sex} - Y_{coed} \mid X = x] \leq 0$ for all $x \in \mathcal{W}_{\mathcal{X}}$ and $H_1 : E[Y_{single-sex} - Y_{coed} \mid X = x] > 0$ for some $x \in \mathcal{W}_{\mathcal{X}}$. For test (2), $H_0 : F[Y_{coed} \mid X = x] \leq F[Y_{single-sex} \mid X = x]$ for all $x \in \mathcal{W}_{\mathcal{X}}$ and $H_1 : F[Y_{coed} \mid X = x] > F[Y_{single-sex} \mid X = x]$ for some $x \in \mathcal{W}_{\mathcal{X}}$, where $F[\cdot]$ is cumulative distribution function.

with KSAT	Treatment Weight/ Bandwidth	Single-sex Uniform	Single-sex Inverse SE p-values	Single-sex Density	Coed Uniform	Coed Inverse SE p-values	Coed Density
Males	2.091	0.000	0.000	0.000	0.621	0.622	0.617
	2.300	0.000	0.000	0.000	0.610	0.611	0.608
	2.509	0.001	0.001	0.001	0.602	0.602	0.600
	2.718	0.001	0.001	0.001	0.594	0.594	0.593
	2.927	0.002	0.002	0.002	0.587	0.588	0.587
	3.137	0.004	0.003	0.004	0.582	0.582	0.581
Females	2.060	0.618	0.618	0.612	0.552	0.557	0.568
	2.266	0.610	0.610	0.607	0.545	0.548	0.556
	2.472	0.602	0.602	0.600	0.538	0.540	0.545
	2.678	0.594	0.594	0.593	0.532	0.534	0.537
	2.884	0.588	0.588	0.587	0.528	0.529	0.531
	3.090	0.582	0.582	0.582	0.525	0.525	0.527

with Math	Treatment Weight/ Bandwidth	Single-sex Uniform	Single-sex Inverse SE p-values	Single-sex Density	Coed Uniform	Coed Inverse SE p-values	Coed Density
Males	2.092	0.000	0.000	0.000	0.621	0.622	0.618
	2.301	0.001	0.001	0.001	0.610	0.611	0.609
	2.510	0.001	0.001	0.001	0.602	0.602	0.600
	2.719	0.002	0.002	0.002	0.594	0.594	0.593
	2.928	0.003	0.003	0.003	0.587	0.588	0.587
	3.138	0.004	0.004	0.004	0.582	0.582	0.581
Females	2.063	0.548	0.544	0.531	0.619	0.620	0.617
	2.269	0.557	0.554	0.546	0.609	0.609	0.608
	2.475	0.561	0.559	0.554	0.600	0.600	0.559
	2.681	0.562	0.561	0.558	0.592	0.593	0.592
	2.888	0.562	0.561	0.559	0.586	0.586	0.586
	3.094	0.561	0.561	0.559	0.581	0.581	0.581

Table 10: Nonparametric Test (Control: Income, Study Time)

Note: Each table shows p-values regarding a nonparametric test with multivariate control. Income and study time are used as control X. With single-sex schooling as treatment, the hypotheses for test are: $H_0 : E[Y_{single-sex} - Y_{coed} \mid X = x] \leq 0$ for all $x \in \mathcal{W}_{\mathcal{X}}$ and $H_1 : E[Y_{single-sex} - Y_{coed} \mid X = x] > 0$ for some $x \in \mathcal{W}_{\mathcal{X}}$.

with KSAT	Treatment Weight/ Bandwidth	Single-sex Uniform	Single-sex Inverse SE p-values	Single-sex Density	Coed Uniform	Coed Inverse SE p-values	Coed Density
Males	2.796	0.049	0.048	0.049	0.545	0.545	0.545
	3.076	0.068	0.067	0.068	0.539	0.539	0.539
	3.355	0.089	0.088	0.089	0.534	0.535	0.534
	3.635	0.111	0.110	0.111	0.531	0.531	0.530
	3.915	0.133	0.133	0.133	0.527	0.527	0.527
	4.194	0.155	0.155	0.155	0.525	0.525	0.525
Females	2.769	0.553	0.553	0.552	0.513	0.514	0.517
	3.045	0.546	0.546	0.546	0.509	0.510	0.512
	3.322	0.540	0.540	0.540	0.507	0.507	0.508
	3.599	0.536	0.536	0.536	0.505	0.505	0.506
	3.876	0.532	0.532	0.532	0.504	0.504	0.504
	4.153	0.529	0.529	0.529	0.503	0.503	0.503

with Math	Treatment Weight/ Bandwidth	Single-sex Uniform	Single-sex Inverse SE p-values	Single-sex Density	Coed Uniform	Coed Inverse SE p-values	Coed Density
Males	2.797	0.054	0.053	0.053	0.545	0.545	0.545
	3.076	0.073	0.072	0.072	0.539	0.539	0.539
	3.356	0.094	0.093	0.093	0.534	0.535	0.534
	3.636	0.116	0.115	0.116	0.531	0.531	0.530
	3.915	0.138	0.138	0.138	0.527	0.527	0.527
	4.195	0.160	0.159	0.159	0.525	0.525	0.525
Females	2.771	0.526	0.525	0.522	0.552	0.552	0.552
	3.048	0.527	0.526	0.524	0.545	0.545	0.545
	3.325	0.526	0.526	0.527	0.540	0.540	0.540
	3.602	0.525	0.525	0.524	0.535	0.535	0.535
	3.879	0.524	0.524	0.523	0.532	0.532	0.532
	4.156	0.522	0.522	0.522	0.529	0.529	0.529

Table 11: Nonparametric Test (Control: Income, Study Time, Parent Edu)

Note: Each table shows p-values regarding a nonparametric test with multivariate control. Income, study time, and parent edu are used as control X. With single-sex schooling as treatment, the hypotheses for test are: $H_0 : E[Y_{single-sex} - Y_{coed} \mid X = x] \leq 0$ for all $x \in \mathcal{W}_{\mathcal{X}}$ and $H_1 : E[Y_{single-sex} - Y_{coed} \mid X = x] > 0$ for some $x \in \mathcal{W}_{\mathcal{X}}$.

DGP	Bandwidth (n=200)	prob. 0.10	0.05	0.01	Bandwidth (n=500)	prob. 0.10	0.05	0.01
1	0.405	0.001	0.000	0.000	0.356	0.001	0.000	0.000
	0.540	0.000	0.000	0.000	0.475	0.000	0.000	0.000
	0.676	0.000	0.000	0.000	0.593	0.000	0.000	0.000
2	0.405	0.010	0.004	0.000	0.356	0.002	0.001	0.001
	0.540	0.007	0.000	0.000	0.475	0.002	0.001	0.000
	0.676	0.004	0.001	0.000	0.593	0.001	0.001	0.000
3	0.405	0.096	0.056	0.011	0.356	0.067	0.027	0.006
	0.540	0.091	0.056	0.010	0.475	0.076	0.037	0.013
	0.676	0.109	0.062	0.023	0.593	0.086	0.048	0.020
4	0.405	0.288	0.164	0.034	0.356	0.523	0.346	0.101
	0.540	0.424	0.266	0.090	0.475	0.730	0.572	0.266
	0.676	0.493	0.352	0.144	0.593	0.846	0.722	0.427
5	0.405	0.937	0.870	0.646	0.356	0.999	0.995	0.952
	0.540	0.985	0.965	0.871	0.475	1.000	1.000	0.998
	0.676	0.998	0.994	0.956	0.593	1.000	1.000	1.000

Table 12: Results of Monte Carlo Experiments (Uniformly Weighted)

Note: The table shows rejection probabilities of testing the null hypothesis of nonpositive CATE for every $x \in \mathcal{W}_{\mathcal{X}}$ and a fixed value of z against the alternative hypothesis of positive CATE for some $x \in \mathcal{W}_{\mathcal{X}}$ and a fixed value of z (depending on DGP). The uniform weight function is used. 1000 repeated samples are generated with sample size 200 and 500.

DGP	Bandwidth (n=200)	0.10	prob. 0.05	0.01	Bandwidth (n=500)	0.10	prob. 0.05	0.01
1	0.405	0.006	0.000	0.000	0.356	0.001	0.001	0.000
	0.540	0.001	0.000	0.000	0.475	0.001	0.001	0.000
	0.676	0.000	0.000	0.000	0.593	0.001	0.000	0.000
2	0.405	0.016	0.008	0.003	0.356	0.003	0.001	0.001
	0.540	0.006	0.001	0.000	0.475	0.002	0.001	0.001
	0.676	0.003	0.002	0.000	0.593	0.001	0.001	0.001
3	0.405	0.113	0.076	0.023	0.356	0.075	0.036	0.008
	0.540	0.096	0.062	0.015	0.475	0.088	0.041	0.013
	0.676	0.110	0.070	0.025	0.593	0.087	0.049	0.021
4	0.405	0.324	0.197	0.065	0.356	0.548	0.373	0.137
	0.540	0.445	0.310	0.112	0.475	0.740	0.591	0.301
	0.676	0.517	0.395	0.179	0.593	0.845	0.736	0.467
5	0.405	0.940	0.884	0.695	0.356	0.998	0.995	0.957
	0.540	0.985	0.967	0.875	0.475	1.000	1.000	0.995
	0.676	0.997	0.992	0.956	0.593	1.000	1.000	1.000

Table 13: Results of Monte Carlo Experiments (Inverse SE Weighted)

Note: The table shows rejection probabilities of testing the null hypothesis of nonpositive CATE for every $x \in \mathcal{W}_{\mathcal{X}}$ and a fixed value of z against the alternative hypothesis of positive CATE for some $x \in \mathcal{W}_{\mathcal{X}}$ and a fixed value of z (depending on DGP). The inverse SE weight function is used. 1000 repeated samples are generated with sample size 200 and 500.

DGP	Bandwidth (n=200)	prob. 0.10	0.05	0.01	Bandwidth (n=500)	prob. 0.10	0.05	0.01
1	0.405	0.002	0.001	0.000	0.356	0.003	0.002	0.000
	0.540	0.001	0.000	0.000	0.475	0.002	0.001	0.000
	0.676	0.000	0.000	0.000	0.593	0.001	0.000	0.000
2	0.405	0.015	0.006	0.002	0.356	0.003	0.003	0.001
	0.540	0.007	0.002	0.000	0.475	0.004	0.002	0.001
	0.676	0.004	0.002	0.000	0.593	0.002	0.001	0.001
3	0.405	0.083	0.051	0.017	0.356	0.067	0.033	0.009
	0.540	0.091	0.061	0.015	0.475	0.081	0.041	0.017
	0.676	0.110	0.065	0.023	0.593	0.091	0.047	0.025
4	0.405	0.289	0.190	0.065	0.356	0.523	0.364	0.138
	0.540	0.393	0.262	0.103	0.475	0.692	0.536	0.280
	0.676	0.455	0.317	0.136	0.593	0.796	0.666	0.379
5	0.405	0.890	0.818	0.602	0.356	0.992	0.983	0.928
	0.540	0.974	0.939	0.819	0.475	0.999	0.998	0.989
	0.676	0.995	0.981	0.920	0.593	1.000	1.000	0.998

Table 14: Results of Monte Carlo Experiments (Density Weighted)

Note: The table shows rejection probabilities of testing the null hypothesis of nonpositive CATE for every $x \in \mathcal{W}_{\mathcal{X}}$ and a fixed value of z against the alternative hypothesis of positive CATE for some $x \in \mathcal{W}_{\mathcal{X}}$ and a fixed value of z (depending on DGP). The density weight function is used. 1000 repeated samples are generated with sample size 200 and 500.

Fixed z	Treatment Weight/ Bandwidth	Single-sex Uniform	Single-sex Inverse SE p-values	Single-sex Density	Coed Uniform	Coed Inverse SE p-values	Coed Density
q(0.5)	2.804	0.000	0.000	0.000	0.773	0.838	0.909
	3.204	0.000	0.000	0.000	0.781	0.842	0.910
	3.605	0.000	0.000	0.000	0.817	0.863	0.913
	4.005	0.000	0.000	0.000	0.867	0.894	0.919
	4.406	0.000	0.000	0.000	0.899	0.912	0.923
q(0.75)	2.804	0.240	0.133	0.826	0.647	0.632	0.432
	3.204	0.401	0.416	0.845	0.578	0.524	0.324
	3.605	0.596	0.704	0.862	0.507	0.414	0.257
	4.005	0.739	0.813	0.880	0.440	0.338	0.225
	4.406	0.874	0.899	0.912	0.726	0.684	0.608

Table 15: Nonparametric Test with a Fixed Control (Males)

Fixed z	Treatment Weight/ Bandwidth	Single-sex Uniform	Single-sex Inverse SE p-values	Single-sex Density	Coed Uniform	Coed Inverse SE p-values	Coed Density
q(0.5)	2.806	0.873	0.876	0.834	0.155	0.234	0.500
	3.207	0.874	0.877	0.843	0.121	0.188	0.469
	3.608	0.876	0.879	0.852	0.098	0.158	0.443
	4.009	0.879	0.881	0.859	0.090	0.148	0.427
	4.409	0.883	0.880	0.853	0.291	0.408	0.669
q(0.75)	2.806	0.767	0.680	0.516	0.218	0.261	0.754
	3.207	0.770	0.686	0.562	0.198	0.260	0.723
	3.608	0.777	0.701	0.620	0.222	0.309	0.698
	4.009	0.786	0.722	0.673	0.285	0.398	0.686
	4.409	0.454	0.341	0.224	0.684	0.757	0.874

Table 16: Nonparametric Test with a Fixed Control (Females)

Note (for Table 15-16): Each table shows p-values regarding a nonparametric test with a fixed control. Regarding two covariates, income is denoted as X and study time is as Z. For fixed value of study time, 0.5-quantile and 0.75-quantile are chosen. With single-sex schooling as treatment, the hypotheses for test are: $H_0 : E[Y_{single-sex} - Y_{coed} \mid X = x, Z = z] \leq 0$ for all $x \in \mathcal{W}_{\mathcal{X}}$ and a fixed z and $H_1 : E[Y_{single-sex} - Y_{coed} \mid X = x, Z = z] > 0$ for some $x \in \mathcal{W}_{\mathcal{X}}$ and a fixed z.

7 Appendix B: Figures

Figure 1: High Schools in Seoul, the Capital of Korea

Note: The reference for this figure is Korea Institute for Curriculum and Evaluation (KICE). It shows high schools in Seoul, Korea in the year 2008. There were 11 education-blocks in Seoul and each student was randomly assigned to high schools only in the education-block where his residence was. Blue and red dots stand for boys' and girls' high schools respectively. Green dots represent coed high schools. The ratio of coed schools in Seoul is 0.43.

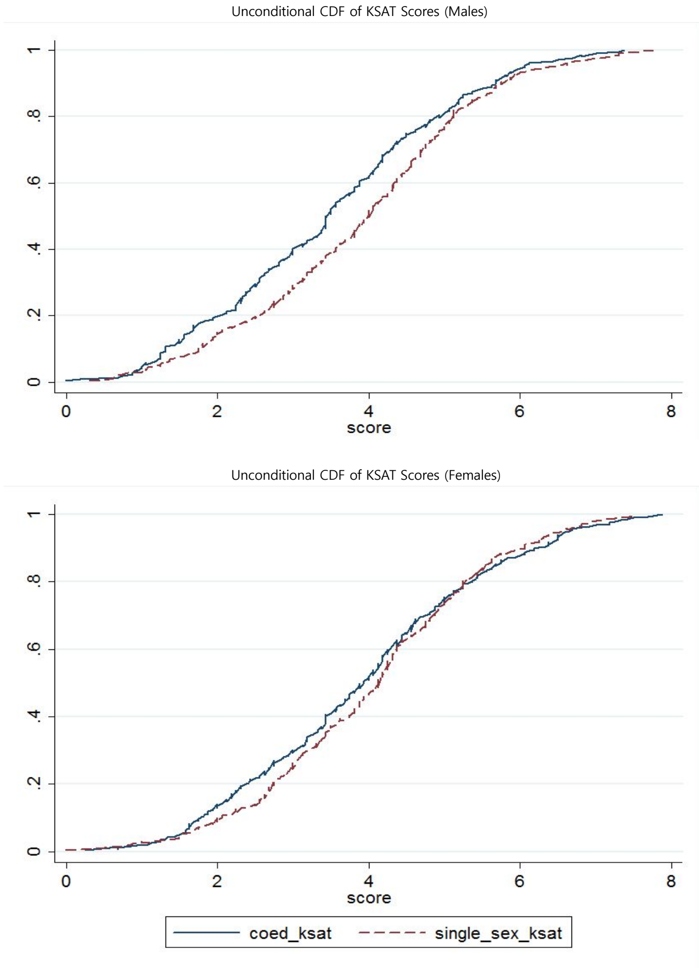

Figure 2: Unconditional CDF of KSAT Scores

Note: The figure shows the unconditional cumulative distribution function (CDF) of KSAT scores. The dotted line is for single-sex schools and the other line is for coed schools.

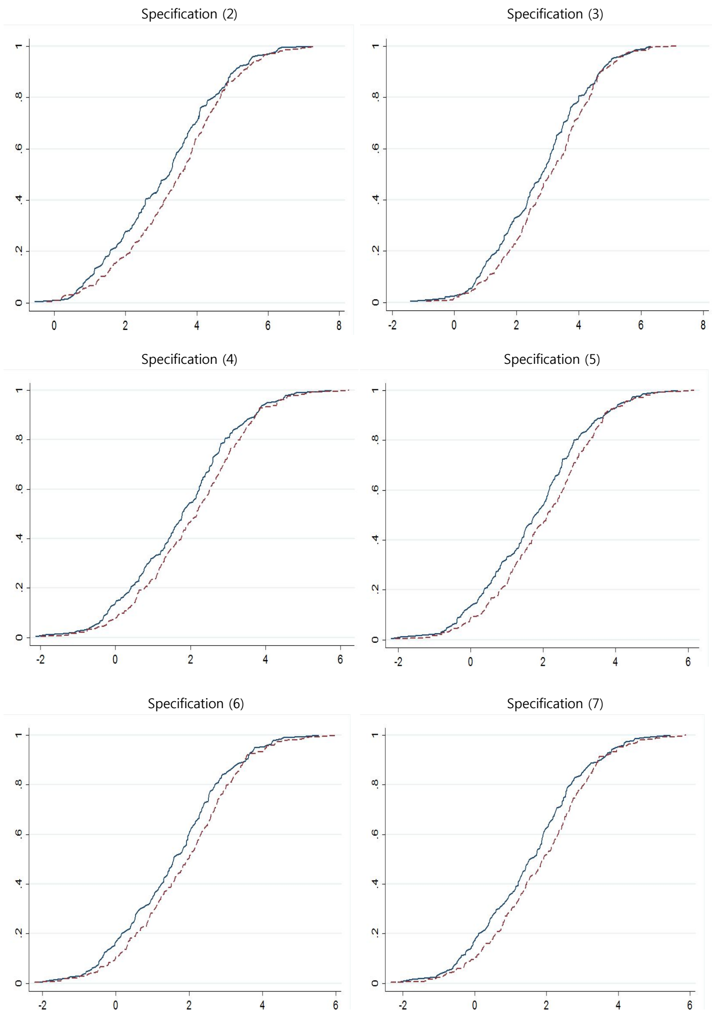

Figure 3: Unconditional CDF of KSAT Residuals (Males)

Note: The figure shows the unconditional cumulative distribution function (CDF) of residuals. They are intercept-adjusted regression errors using covariates as in OLS specifications. The dotted line is for single-sex schools and the other line is for coed schools.

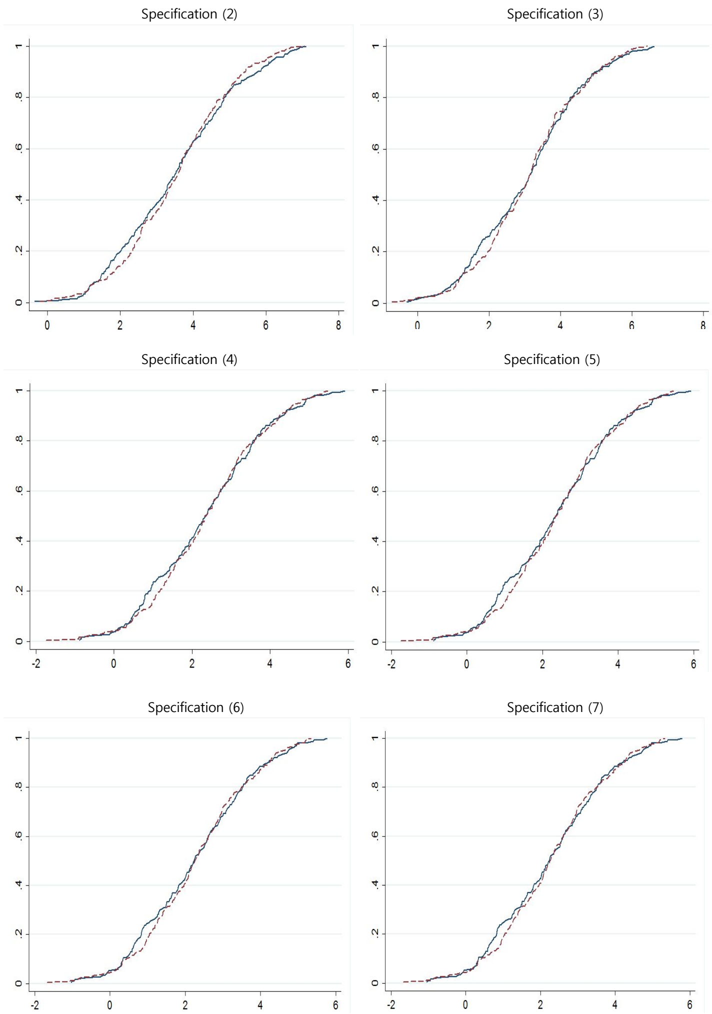

Figure 4: Unconditional CDF of KSAT Residuals (Females)

Note: The figure shows the unconditional cumulative distribution function (CDF) of residuals. They are intercept-adjusted regression errors using covariates as in OLS specifications. The dotted line is for single-sex schools and the other line is for coed schools.

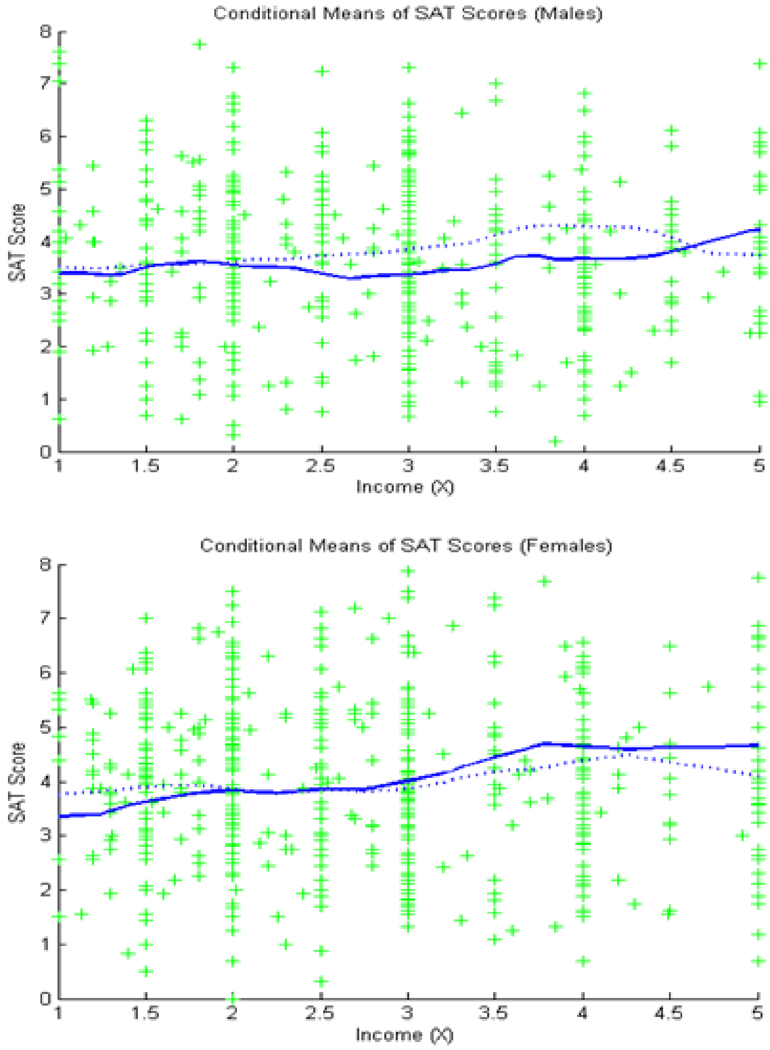

Figure 5: Conditional Mean of KSAT Scores (Control: Income)

Note: The figure shows the nonparametric estimates of conditional mean of Korean SAT scores as function of monthly household income. The support is constrained from 0.1-quantile to 0.9-quantile of income. The kernel function used is $K(u) = 3/2(1-(2u)^2)I\{|\ u\ | \leq 0.5\}$. Bandwidth h is computed to be 1.3403 for males, and be 1.4925 for females from cross-validation. The dotted line is for single-sex schools and the other line is for coed schools.

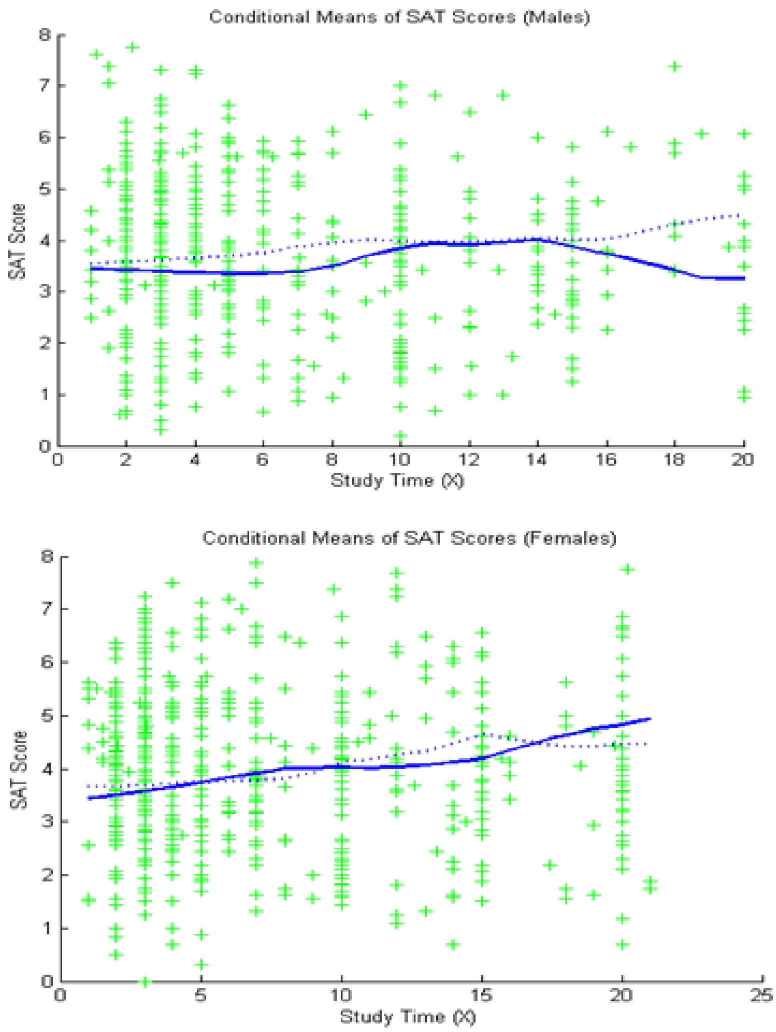

Figure 6: Conditional Mean of KSAT Scores (Control: Study Time)

Note: The figure shows the nonparametric estimates of conditional mean of Korean SAT scores as function of weekly self-study time. The support is constrained from 0.1-quantile to 0.9-quantile of study time. The kernel function used is $K(u) = 3/2(1-(2u)^2)I\{|\ u\ |\leq 0.5\}$. Bandwidth h is computed to be 7.6809 for males, and be 10 for females from cross-validation. The dotted line is for single-sex schools and the other line is for coed schools.

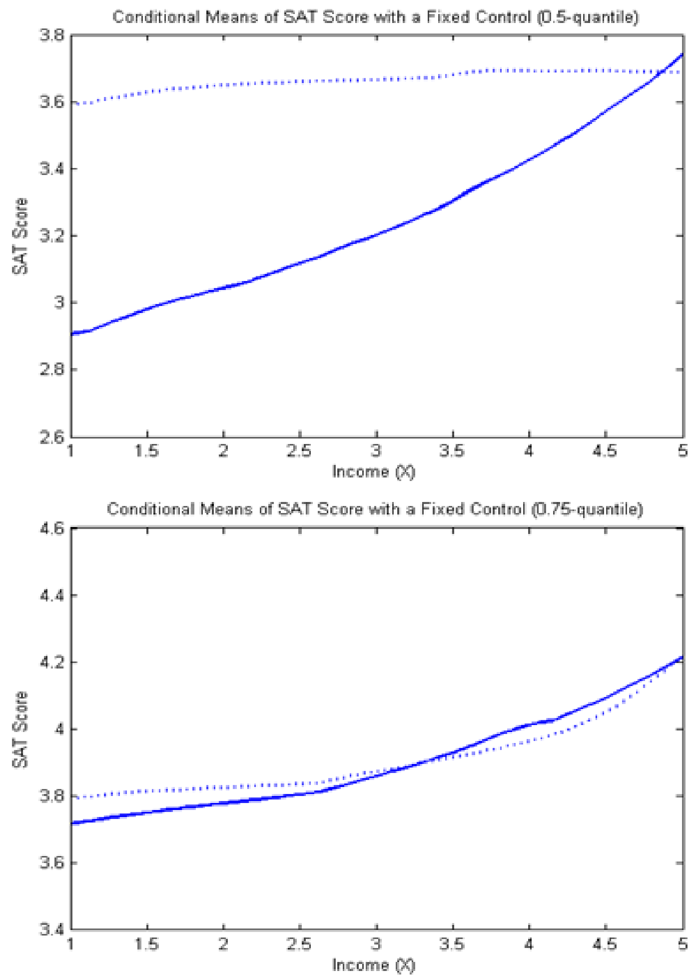

Figure 7: Conditional Mean of KSAT Scores with Fixed Study Time (Males)

Note: The figure shows the nonparametric estimates of conditional mean of Korean SAT scores across income, with fixed study time. The support is constrained from 0.1-quantile to 0.9-quantile of income. The product kernel is used with $K(u) = 3/2(1 - (2u)^2)I\{| u |\leq 0.5\}$. Rule-of-thumb bandwidth h is computed to be 5.7436 for males. Two fixed values of study time are considered: 5 hours (0.5-quantile) and 12 hours (0.75-quantile). The dotted line is for single-sex schools and the other line is for coed schools.

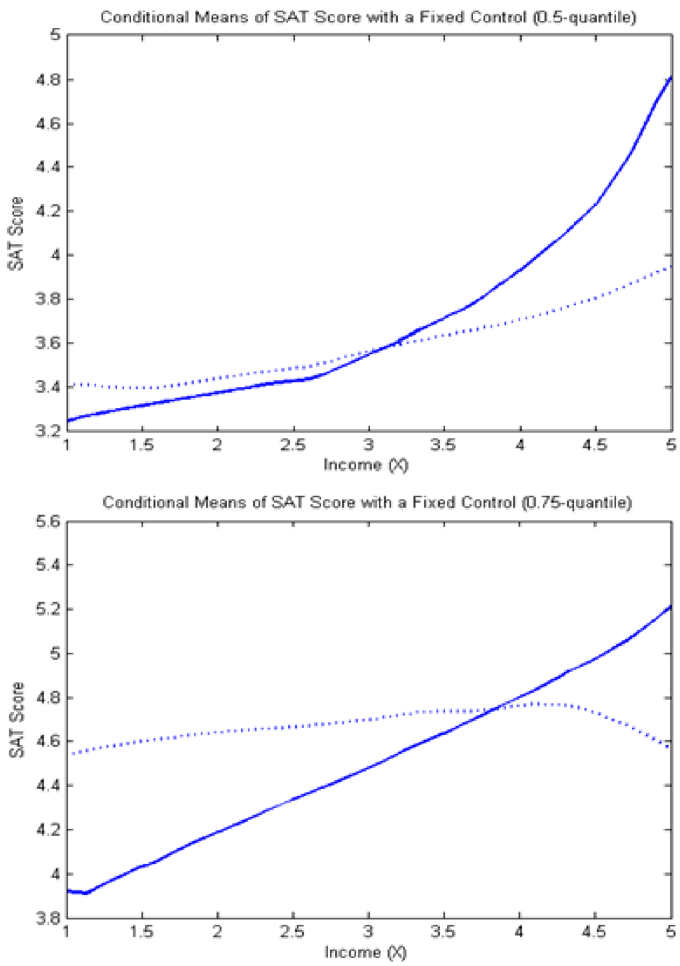

Figure 8: Conditional Mean of KSAT Scores with Fixed Study Time (Females)

Note: The figure shows the nonparametric estimates of conditional mean of Korean SAT scores across income, with fixed study time. The support is constrained from 0.1-quantile to 0.9-quantile of income. The product kernel is used with $K(u) = 3/2(1-(2u)^2)I\{\mid u \mid \leq 0.5\}$. Rule-of-thumb bandwidth h is computed to be 4.7188 for females. Two fixed values of study time are considered: 5 hours (0.5-quantile) and 14 hours (0.75-quantile). The dotted line is for single-sex schools and the other line is for coed schools.

8 Appendix C: Test Statistics and Asymptotics

Suppose $\{(Y_i, X_i, Z_i)'\}_{i=1}^n$ that are *i.i.d.* copies from a random vector. We consider the case where $(Y, X, Z)' \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Define $m(x, z) \equiv E[Y_i \mid X_i = x, Z_i = z]$. This paper is concentrating on the problem of testing functional inequalities while fixing one of the control variables($\mathcal{X}, \mathcal{Z} \subset \mathbb{R}$ are the domain of interest.):

$$\begin{cases} H_0 : m(x, z) \leq 0 & \text{for all } x \in \mathcal{X}, \text{for fixed } z \in \mathcal{Z} \\ H_1 : m(x, z) > 0 & \text{for some } x \in \mathcal{X}, \text{for fixed } z \in \mathcal{Z} \end{cases} \tag{1}$$

Lee, Song and Whang (2012: TFI, hereafter) treated the problem of testing functional inequalities even in cases where $(Y', X')' \in \mathbb{R}^J \times \mathbb{R}^d$. They employed a Poissonization technique in order to test inequality constraints of nonparametric functions. Their focus was not to test the plausibility of certain parametric assumptions but to test whether some functional inequalities hold uniformly across a control variable.

This paper is closely related to TFI in most theoretical parts, however, it extends TFI in that it handles the issue of testing functional inequalities to see whether it holds uniformly over a control variable while fixing another control variable. This issue can be highly relevant to a number of applied settings.

One of the Poissonization techniques invented by Beirlant and Mason(1995) was shown to achieve a central limit theorem for L_p-distance of kernel density estimators. Giné, Mason and Zaitsev(2003: GMZ, hereafter) found the asymptotic distribution of the L_1-distance indexed by kernel functions using a Poissonization technique under weak regularity condition. In this paper, the contribution of Berlant and Mason(1995), GMZ and TFI is used to test the hypothesis (1) throughout the theoretical section.

Let $\Lambda : \mathbb{R} \to \mathbb{R}$ be s.t. $\Lambda(v) = max\{v, 0\}, v \in \mathbb{R}$. To study the test of functional inequalities, we focus on the following L_1-type functionals:

$$\rho \to \Gamma(\rho) \equiv \int_{\mathcal{X}} \Lambda(\rho(x, z))w(x, z)dx$$

Let f denote the density function of X given Z and define $g(x, z) \equiv m(x, z)f(x, z)$. As suggested in TFI, the testing problem can be done rather by testing if $\Gamma(g) = 0$ or $\Gamma(g) > 0$. We need the following expression to build up a test statistic:

$$\hat{g_n}(x, z) \equiv \frac{1}{nh^2} \sum_{i=1}^{n} Y_i K\big(\frac{x - X_i}{h}\big) K\big(\frac{z - Z_i}{h}\big)$$

where $K : \mathbb{R} \to \mathbb{R}$ is a kernel function and h is a bandwidth parameter such that $h \to 0$ as $n \to \infty$. Applying a method of Poissonization, this paper finds that the tests done while fixing one of the control variables are asymptotically distribution free under the least favorable case of the null hypothesis, resulting in asymptotic normal approximation. The difference from TFI's result is the fact that the rate of convergence in our main theorem is $\sqrt{nh^2}$ compared to $\sqrt{n}$ in TFI.

The one-sided L_1 approach in testing functional inequalities after fixing one of the control variables is suggested in this section. Even though it only treats L_1-type functionals, it is in the similar vein of TFI which studied one-sided L_p-type functionals, by adopting a Poissonization technique to test inequality constraints. The following presents useful lemmas and their proof to be used for stating and proving the main theorem.

8.1 Assumptions

Given $z \in \mathcal{Z}$, let $S(z) \equiv \{x \in \mathcal{X} : w(x,z) > 0\}$. Given $\epsilon > 0$, let $S^{\epsilon}(z)$ be an ϵ-enlargement of $S(z)$, i.e., $S^{\epsilon}(z) \equiv \{x + a : x \in S(z), a \in [-\epsilon, \epsilon]\}$. Let $r_p(x,z) \equiv E[\mid Y_i \mid^p \mid X_i = x, Z_i = z] f(x,z)$ for $1 \leq p < \infty$.

Assumption 1 *(i) For some $\epsilon > 0$, given $z \in \mathcal{Z}$, $r_2(x,z)$ is bounded away from zero and $r_{2p+2}(x,z)$ is bounded, both uniformly in $x \in S^{\epsilon}(z)$.*
(ii) Given $z \in \mathcal{Z}$, $w(\cdot,z)$ is nonnegative on $\mathcal{X}$ and $0 < \int_{\mathcal{X}} w^s(x,z)dx < \infty$, where $s \in \{1,2\}$.
(iii) For $\epsilon > 0$ in (i), $S^{\epsilon}(z) \subset \mathcal{X}$.

Note that Assumption 1(i) and 1(iii) require $S(z)$ be a bounded set for given $z \in \mathcal{Z}$. Assumption 1(iii) is suggested in order to avoid the boundary problem of kernel estimator.

We assume the same kernel function K for both x and z.

Assumption 2 *The kernel function $K : \mathbb{R} \to \mathbb{R}$, satisfying that*
(i) $K(u) = 0$ for all $u \in \mathbb{R} \setminus [-1/2, 1/2]$.
(ii) K is of bounded variation.
(iii) $\parallel K \parallel_{\infty} \equiv sup_{u \in \mathbb{R}} \mid K(u) \mid < \infty$ and $\int K(u)du = 1$.

8.2 Relevant Lemmas and Their Proof

Lemma 1 *Given $z \in \mathcal{Z}$, let $J(\cdot,z) : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable, bounded function and $H : \mathbb{R} \to \mathbb{R}$ be a bounded function with compact support S. Then for almost every $y \in \mathbb{R}$, for fixed z and z' such that $z' - z \in [-h/2, h/2]$,*

$$\int_{\mathbb{R}} J(x,z)H_h(y-x)H_h(z-z')dx \to J(y,z)H(z)\int_S H(x)dx \ as \ h \to 0$$

where $H_h(x) \equiv H(x/h)/h$.
Furthermore, suppose that for fixed z, $\bar{J} \equiv \int \mid J(x,z) \mid dx > 0$. Then for all $0 < \epsilon < \bar{J}$, there exist $M > 0, v > 0$, and a Borel set B of finite Lebesgue measure $m(B)$ s.t. $B \subset [-M+v, M-v]$, $\alpha \equiv \int_{\mathbb{R}\setminus[-M,M]} \mid J(x,z) \mid dx > 0, \int_B \mid J(x,z) \mid dx > \bar{J} - \epsilon, J$ is continuous on B, and

$$sup_{y \in B} \mid \int_{\mathbb{R}} J(x,z)H_h(y-x)H_h(z-z')dx - J(y,z)H(z)\int_S H(x)dx \mid \to 0 \ as \ h \to 0.$$

Proof. Consider a simple function $H_m(x) \equiv \sum_{i=1}^m a_i I\{x \in A_i \cap S\}$ for some numbers $a_i \in \mathbb{R}$ and measurable sets $A_i \subset \mathbb{R}$ s.t. $\mid H_m(x) - H(x) \mid \to 0$ as $m \to \infty$. WLOG, let $A_i \cap S$ be a rectangle. Let $H_{h,m}(x) \equiv H_m(x/h)/h$. Note that

$$\int_{\mathbb{R}} J(x,z) H_{h,m}(y-x) H_h(z-z')dx$$

$$= \sum_{i=1}^m \frac{a_i}{h} \int_{\mathbb{R}} J(x,z) I\{(y-x)/h \in A_i \cap S\} H_h(z-z')dx$$

$$= \sum_{i=1}^m \frac{a_i}{h} \int_{y-h(A_i \cap S)} J(x,z) H_h(z-z')dx$$

where $y - h(A_i \cap S) = \{y - hu : u \in A_i \cap S\}$. Hence,

$$\mid \int_{\mathbb{R}} J(x,z) H_{h,m}(y-x) H_h(z-z')dx - J(y,z)H(z) \int_S H_m(x)dx \mid$$

$$= \mid \sum_{i=1}^m a_i \Big\{ \frac{1}{h} \int_{y-h(A_i \cap S)} J(x,z) H_h(z-z')dx - J(y,z) H_h(z-z') m(A_i \cap S) \Big\} \mid$$

$$\leq \mid H_h(z-z') \mid \mid \sum_{i=1}^m \frac{a_i}{h} \int_{y-h(A_i \cap S)} \{J(x,z) - J(y,z)\} dx \mid$$

$$\leq \mid H_h(z-z') \mid \sum_{i=1}^m \frac{a_i}{h} \int_{y-hB_i} \mid J(x,z) - J(y,z) \mid dx$$

where B_i is a compact ball in $\mathbb{R}$ centered at zero containing $A_i \cap S$. For almost every $y \in \mathbb{R}$ (w.r.t. Lebesgue measure), the last sum converges to zero as $h \to 0$ by the Lebesgue Differentiation Theorem. By sending $m \to \infty$ and considering $H_h(z-z') \to H(z)$ as $h \to 0$, we obtain the result of the first statement.

The second statement can be proved following the proof of Lemma 6.1 of GMZ. Since J is Lebesgue integrable, the integral $\int_{\mathbb{R}\setminus[-M,M]} \mid J(u,z) \mid du$ is continuous in M and converges to zero as $M \to \infty$.
For given $z \in \mathcal{Z}$, we can find $M > 0$ and $v > 0$ s.t.

$$\int_{\mathbb{R}\setminus[-M,M]} \mid J(u,z) \mid du = \frac{\epsilon}{8}$$

and

$$\int_{\mathbb{R}\setminus[-M+v,M-v]} \mid J(u,z) \mid du = \frac{\epsilon}{4}.$$

The construction of the desired set $B \subset [-M+v, M-v]$ can be done using the arguments in the proof of Lemma 6.1 of GMZ. ■

Lemma 2 (Sweeting(1977)) *Let $\mathbb{Z} \in \mathbb{R}$ be a mean zero normal random vector with variance 1 and $\{W_i\}_{i=1}^n$ is a set of i.i.d. random vectors in $\mathbb{R}$ s.t. $EW_i = 0, EW_iW_i' = 1, E \parallel W_i \parallel^r < \infty, r \geq 3$. Then for any Borel measurable function $\varphi(\cdot, z), \varphi : \mathbb{R} \to \mathbb{R}$ for given z, satisfying $sup_{x\in\mathbb{R}} \frac{|\varphi(x,z)-\varphi(0,z)|}{1+||x||^r min(||x||,1)} < \infty$, we have*

$$
\begin{aligned}
&| E\left[\varphi\left(\tfrac{1}{\sqrt{n}}\textstyle\sum_{i=1}^n W_i\right)\right] - E[\varphi(\mathbb{Z})] | \\
&\leq C_1 \left(sup_{x\in\mathbb{R}} \tfrac{|\varphi(x,z)-\varphi(0,z)|}{1+\|x\|^r min(\|x\|,1)} \right) \left\{ \tfrac{1}{\sqrt{n}} E \parallel W_i \parallel^3 + \tfrac{1}{n^{(r-2)/2}} E \parallel W_i \parallel^r \right\} \\
&\quad + C_2 E\left[\omega_\varphi\left(\mathbb{Z}; \tfrac{C_3}{\sqrt{n}} E \parallel W_i \parallel^3 \right)\right]
\end{aligned}
$$

where C_1, C_2, C_3 are positive constant that depend only on r and for given z, $\omega_\varphi(x; \epsilon) \equiv sup\{| \varphi(x,z) - \varphi(y,z) |: y \in \mathbb{R}, \parallel x - y \parallel \leq \epsilon\}$.

Lemma 2 is almost the same as Lemma 2 in TFI. Therefore, we omit its proof in this paper.

The following expressions are to be used in the following lemmas:

$$\rho_n^2(x,z) \equiv h^{-2} E\big[Y_i^2 K^2\big(\frac{x - X_i}{h}\big) K^2\big(\frac{z - Z_i}{h}\big)\big]$$

$$k_{n,r}(x,z) \equiv h^{-2} E\big[\mid Y_i K\big(\frac{x - X_i}{h}\big) K\big(\frac{z - Z_i}{h}\big) \mid^r \big], r \geq 1$$

$$a_n \equiv \int_{\mathcal{X}} \rho_n(x,z) w(x,z) dx \cdot Emax\{\mathbb{Z}_1, 0\}$$

Lemma 3 *Suppose that Assumption 1(i), (iii) and 2 hold and $h \to 0$ as $n \to \infty$. Then for $\epsilon > 0$, in Assumption 1(iii), there exist positive integer n_0 and constants $C_1, C_2 > 0$ s.t. for all $n \geq n_0$, all $r \in [1, 4]$, given $z \in \mathcal{Z}$,*

$$0 \leq C_1 \leq inf_{x\in S^{\epsilon/2}(z)} h\rho_n^2(x,z)$$

and

$$sup_{x\in S^{\epsilon/2}(z)} hk_{n,r}(x,z) \leq C_2 < \infty.$$

Proof. Since $h \to 0$ as $n \to \infty$, we apply change of variables to find that from large n on, and for fixed $z \in \mathcal{Z}$,

$$inf_{x\in S^{\epsilon/2}(z)} h\rho_n^2(x,z) = inf_{x\in S^{\epsilon/2}(z)} \frac{1}{h} E\big[Y_i^2 K^2\big(\frac{x - X_i}{h}\big) K^2\big(\frac{z - Z_i}{h}\big)\big]$$

$$\geq inf_{x\in S^{\epsilon}(z)} E[Y_i^2 \mid X=x, Z=z] f(x,z) K^2\big(\frac{z-z'}{h}\big) \int_{[-1/2,1/2]} K^2(u)du \;\; where\; z' \in [z-h/2, z+h/2]$$

$$\geq inf_{x\in \mathcal{X}} E[Y_i^2 \mid X=x, Z=z] f(x,z) K^2\big(\frac{z-z'}{h}\big) \int_{[-1/2,1/2]} K^2(u)du \; > \; C_1$$

for some $C_1 > 0$ by Assumption 1(i) and 2. Similarly, from some large n on,

$$sup_{x\in S^{\epsilon/2}(z)} hk_{n,r}(x,z) \leq sup_{x\in\mathcal{X}} E[\mid Y_i \mid^r \mid X=x, Z=z] f(x,z) \mid K\big(\frac{z-z'}{h}\big) \mid^r \int \mid K(u) \mid^r du \; < \; \infty.$$

■

Define $\hat{g_N}(x,z) \equiv \frac{1}{nh^2}\sum_{i=1}^{N} Y_i K\big(\frac{x-X_i}{h}\big) K\big(\frac{z-Z_i}{h}\big),\ x\in\mathcal{X}$ given $z\in\mathcal{Z}$, where N is a Poisson variable that has mean n and independent of $\{(Y_i, X_i, Z_i)\}_{i=1}^{\infty}$. Let $v_n(x,z) \equiv \hat{g_n}(x,z) - E\hat{g_n}(x,z)$ and $v_N(x,z) \equiv \hat{g_N}(x,z) - E\hat{g_N}(x,z)$. We define

$$\xi_n(x,z) \equiv \frac{\sqrt{nh^2}v_N(x,z)}{\rho_n(x,z)}$$

and

$$V_n(x,z) \equiv \frac{\sum_{i\geq N_1}\big\{Y_i K\big(\frac{x-X_i}{h}\big)K\big(\frac{z-Z_i}{h}\big) - E\big[Y_i K\big(\frac{x-X_i}{h}\big)K\big(\frac{z-Z_i}{h}\big)\big]\big\}}{\sqrt{E\big[Y_i^2 K^2\big(\frac{x-X_i}{h}\big)K^2\big(\frac{z-Z_i}{h}\big)\big]}}$$

where N_1 denotes a Poisson random variable with mean 1 that is independent of $\{(Y_i, X_i, Z_i)\}_{i=1}^{\infty}$. Then $Var(V_n(x,z)) = 1$.

Let $V_n^{(i)}(x,z), i=1,\ldots,n$ be *i.i.d.* copies of $V_n(x,z)$ so that $\xi_n(x,z) \overset{d}{=} \frac{1}{\sqrt{n}}\sum_{i=1}^{n} V_n^{(i)}(x,z)$

Lemma 4 *Suppose that Assumption 1(i), (iii) and 2 hold and $h\to 0$ as $n\to\infty$, and $limsup_{n\to\infty}(nh)^{1-r/2} < C$, for some constant $C > 0$ and for $r\in[2,4]$. Then for $\epsilon > 0$ in Assumption 1(iii),*

$$sup_{x\in S^{\epsilon/2}(z)} E[\mid V_n(x,z) \mid^r] \leq C_1 h^{(1-r/2)}$$

and

$$sup_{x\in S^{\epsilon/2}(z)} E[\mid \xi_n(x,z) \mid^r] \leq C_2$$

where C_1 and C_2 are constants that depend only on r.

Proof. For all $x\in S^{\epsilon/2}(z), E[V_n^2(x,z)] = 1$. Recall the definition of $k_{n,r}(x,z)$,

$k_{n,r}(x,z) \equiv h^{-2}E[| \; Y_i K\big(\frac{x-X_i}{h}\big)K\big(\frac{z-Z_i}{h}\big) \; |^r], \;\; r \geq 1.$
Then for some $C_1 > 0$,

$$sup_{x\in S^{\epsilon/2}(z)}E \mid V_n(x,z) \mid^r \leq sup_{x\in S^{\epsilon/2}(z)} \frac{h^2 k_{n,r}(x,z)}{h^r \rho_n^r(x,z)} \leq h^{1-r/2}C_1 \tag{2}$$

by Lemma 3, completing the proof of the first statement. As for the second statement, using $\xi_n(x,z) \overset{\mathrm{d}}{=} \frac{1}{\sqrt{n}}\sum_{i=1}^n V_n^{(i)}(x,z)$ and apply Rosenthal's inequality: If η_i are independent centered random vectors, then for every $p \geq 2, n \in \mathbb{N}$,

$$E \mid \sum_{i=1}^n \eta_i \mid^p \leq \big(\frac{15p}{logp}\big)^p max\big[\big(\sum_{i=1}^n E\eta_i^2\big)^{p/2}, \sum_{i=1}^n E \mid \eta_i \mid^p \big].$$

Then,

$$sup_{x\in S^{\epsilon/2}(z)}E[| \; \xi_n(x,z) \; |^r] = sup_{x\in S^{\epsilon/2}(z)}E[| \sum_{i=1}^n \frac{1}{\sqrt{n}} V_n^{(i)}(x,z) \; |^r]$$

$$\leq C_3 sup_{x\in S^{\epsilon/2}(z)} max[(n \cdot \frac{1}{n}EV_n^2(x,z))^{r/2}, nE \mid \frac{1}{\sqrt{n}}V_n(x,z) \mid^r]$$

$$= C_3 sup_{x\in S^{\epsilon/2}(z)} max[(EV_n^2(x,z))^{r/2}, n^{1-r/2}E \mid V_n(x,z) \mid^r]$$

$$\leq C_4 max\{1, C_5 n^{1-r/2}h^{1-r/2}\}.$$

The last inequality is from (2). By the given condition that $limsup_{n\to\infty}(nh)^{1-r/2} < C$, we get the desired result. ■

The following lemma is adopted from Lemma 5.3. of GMZ and Lemma A6 of TFI.

Lemma 5 *Suppose that Assumption 1 and 2 hold and* $h \to 0$ *and* $n^{-1/2}h^{-2} \to 0$ *as* $n \to \infty$. *Then for any Borel set* $A \subset \mathbb{R}$,

$$\int_A [\sqrt{n}Emax\{v_N(x,z),0\} - h^{-1}\rho_n(x,z)Emax\{\mathbb{Z}_1,0\}]w(x,z)dx \to 0$$

$$\int_A [\sqrt{n}Emax\{v_n(x,z),0\} - h^{-1}\rho_n(x,z)Emax\{\mathbb{Z}_1,0\}]w(x,z)dx \to 0$$

Proof. Recall the definition of $\xi_n(x,z)$ and write

$$\sqrt{n}Emax\{v_N(x,z),0\} - h^{-1}\rho_n(x,z)Emax\{\mathbb{Z}_1,0\}$$

$$= h^{-1}\rho_n(x,z)[Emax\{\xi_n(x,z),0\} - Emax\{\mathbb{Z}_1,0\}]$$

In view of Lemma 3 and Assumption 1(ii), we find that it suffices for the first statement of the lemma to show that

$$sup_{x\in S} \mid Emax\{\xi_n(x,z),0\} - Emax\{\mathbb{Z}_1,0\} \mid = o(h^{\frac{3}{2}}) \tag{3}$$

By Lemma 4, $sup_{x\in S}E \mid V_n(x,z) \mid^3 \leq Ch^{-1/2}$ for some $C>0$. Using Lemma 2, applying $r=3$, $W_i = V_n^{(i)}(x,z)$, and $\varphi(\cdot) = max\{\cdot,0\}$ for given $z \in \mathcal{Z}$, we deduce that

$$sup_{x\in S} \mid Emax\{\xi_n(x,z),0\} - Emax\{\mathbb{Z}_1,0\} \mid \tag{4}$$

$$\leq C_1 n^{-1/2} sup_{x\in S} E \mid V_n(x,z) \mid^3 + C_2 E[\omega_{max}(\mathbb{Z}_1; \frac{C_3}{\sqrt{n}} E \mid V_n(x,z) \mid^3)]$$

for some constants $C_s > 0, s = 1,2,3$. We can see that the first term is $o(h^{3/2})$ and for the last expectation, observe that for all $x \in S$,

$$E[\omega_{max}(\mathbb{Z}_1; \frac{C_3}{\sqrt{n}} E \mid V_n(x,z) \mid^3)] = sup\{\mid max\{\mathbb{Z}_1,0\} - max\{y,0\} \mid : y \in \mathbb{R}, \mid \mathbb{Z}_1 - y \mid \leq \frac{C_3}{\sqrt{n}} E \mid V_n(x,z) \mid^3\}$$

$$\leq 2 \mid \mathbb{Z}_1 - y \mid = O(n^{-1/2}h^{-1/2})$$

We can see that the RHS of the last inequality is $O(n^{-1/2}h^{-1/2}) = o(h^{3/2})$, this completes the proof of (3).

We consider the second statement. Let $\overline{V_n}^{(i)}(x,z), i = 1,\ldots,n$ be *i.i.d.* copies of

$$\frac{Y_i K\left(\frac{x-X_i}{h}\right) K\left(\frac{z-Z_i}{h}\right) - E\left[Y_i K\left(\frac{x-X_i}{h}\right) K\left(\frac{z-Z_i}{h}\right)\right]}{\sqrt{E\left[Y_i^2 K^2\left(\frac{x-X_i}{h}\right) K^2\left(\frac{z-Z_i}{h}\right)\right] - \left(E\left[Y_i K\left(\frac{x-X_i}{h}\right) K\left(\frac{z-Z_i}{h}\right)\right]\right)^2}}$$

so that $var(\overline{V_n}^{(i)}(x,z)) = 1$. Observe that for some constant $C>0$,

$$sup_{x\in S} E \mid \overline{V_n}^{(i)}(x,z) \mid^3 \leq Ch^{-1} sup_{x\in S} \frac{k_{n,3}(x,z)}{(\rho_n^2(x,z) - b_n^2(x,z))^{\frac{3}{2}}}$$

$$= Ch^{-1/2} sup_{x\in S} \frac{hk_{n,3}(x,z)}{(h\rho_n^2(x,z) - hb_n^2(x,z))^{\frac{3}{2}}}$$

where $b_n \equiv h^{-1}E[Y_i K\left(\frac{x-X_i}{h}\right) K\left(\frac{z-Z_i}{h}\right)]$.
Therefore, by Lemma 3 and for some $C'>0$, the last term is bounded by $C'h^{-1/2}$.
Define $\overline{\xi_n}(x,z) \equiv \frac{\sqrt{nh^2}v_n(x,z)}{\tilde{\rho_n}(x,z)}$ where $\tilde{\rho_n}^2(x,z) \equiv nh^2 var(v_n(x,z))$.
Then, $\overline{\xi_n}(x,z) \stackrel{d}{=} \frac{1}{\sqrt{n}} \sum_{i=1}^n \overline{V_n}^{(i)}(x,z)$.

Using Lemma 2 and following the argument in (4) analogously, we deduce that

$$sup_{x\in S} \mid Emax\{\overline{\xi_n}(x,z),0\} - Emax\{\mathbb{Z}_1,0\} \mid = o(h^{\frac{3}{2}}).$$

This leads us to conclude that

$$\int_A \{\sqrt{n}Emax\{v_n(x,z),0\} - h^{-1}\tilde{\rho_n}(x,z)Emax\{\mathbb{Z}_1,0\}\}dx = o(1).$$

Now, observe that

$$h^{-1} \mid \tilde{\rho_n}(x,z) - \rho_n(x,z) \mid = h^{-1} \mid (\rho_n^2(x,z) - b_n^2(x,z))^{1/2} - (\rho_n^2(x,z))^{1/2} \mid$$

$$\leq hb_n^2(x,z)[h\rho_n^2(x,z) + hb_n^2(x,z)]^{-1/2}h^{1/2}$$

$$= O(h^{1/2}) = o(1) \ \ uniformly\ over\ x \in S.$$

where the inequality above follows from Lemma 3. This completes the proof. ■

Now, let $(Z_{1n}(x,z), Z_{2n}(u,z)) \in \mathbb{R}^2$ be a jointly normal centered random vector whose covariance matrix is the same as that of $(\xi_n(x,z), \xi_n(u,z))$ for all $x, u \in \mathbb{R}$, given $z \in \mathcal{Z}$. We define

$$\sigma_n(A) \equiv nh^2 \int_A \int_A cov(max\{v_N(x,z),0\}, max\{v_N(x',z),0\})w(x,z)w(x',z)dxdx'$$

$$\sigma(A) \equiv \int_A \int_{[-1,1]} g(u)\rho^2(x,z)w^2(x,z)dudx = q_p \int_A \rho^2(x,z)w^2(x,z)dx$$

where $g(u) \equiv cov(max\{\sqrt{1-t^2(u)}\mathbb{Z}_1 + t(u)\mathbb{Z}_2, 0\}, max\{\mathbb{Z}_2, 0\}), t(u) \equiv \frac{\int K(x)K(x+u)dx}{\int K^2(u)du}$, and $q_p \equiv \int_{[-1,1]} g(u)du$.

$$\tau_n(A) \equiv \int_A \int_{[-1,1]} g_n(x,u,z)\lambda_n(x,x+uh,z)hdudx$$

where

$$\lambda_n(x,u,z) \equiv \rho_n(x,z)\rho_n(u,z)w(x,z)w(u,z)I_A(x)I_A(u), \ \ and$$

$$g_n(x,v,z) \equiv cov(max\{Z_{1n}(x,z),0\}, max\{Z_{2n}(x+vh,z),0\})$$

The following lemma is related to Lemma 6.5 of GMZ.

Lemma 6 *Suppose that Assumption 1 and 2 hold and let $h \to 0$ as $n \to \infty$ s.t. $limsup_{n\to\infty}(nh)^{1-r/2}$ for any $r \in [2,4]$ for some $C > 0$.*
(i) Suppose that $A \subset S(z)$ is any Borel set. Then

$$\sigma_n(A) = \tau_n(A) + o(1).$$

(ii) Suppose further that A has a finite Lebesgue measure, $\exists \rho(x,z)$ s.t.

$$sup_{x\in A} \mid \rho_n(x,z) - \rho(x,z) \mid \to 0 \text{ as } n \to \infty$$

and $\rho^2(\cdot,z)$ and $w^2(\cdot,z)$ are continuous and bounded on A given $z \in \mathcal{Z}$.
Then, as $n \to \infty, \tau_n(A) = \sigma(A) + o(1)$, and hence from (i),

$$\sigma_n(A) \to \sigma(A).$$

Proof. By change of variables, we write $\sigma_n(A) = \tilde{\tau_n}(A)$, where

$$\tilde{\tau_n}(A) \equiv \int_A \int_{[-1,1]} cov(max\{\xi_n(x,z),0\}, max\{\xi_n(x+uh,z),0\})\lambda_n(x,x+uh,z)hdudx.$$

Fix $\epsilon_1 \in (0,1]$, and let $c(\epsilon_1) = (1+\epsilon_1)^2 - 1$. Let η_1 and η_2 be two independent random vectors that are independent of $(\{Y_i, X_i, Z_i\}_{i=1}^{\infty}, N)$, each having a two-point distribution that gives two points, $\{\sqrt{c(\epsilon_1)}\}$ and $\{-\sqrt{c(\epsilon_1)}\}$, the equal mass of 1/2, so that $E\eta_1 = E\eta_2 = 0$. $var(\eta_1) = var(\eta_2) = c(\epsilon_1)$. Furthermore, observe that for any $r \geq 1$,

$$E \mid \eta_1 \mid^r = \frac{1}{2} \mid c(\epsilon_1) \mid^{r/2} + \frac{1}{2} \mid c(\epsilon_1) \mid^{r/2} \leq C\epsilon_1^{r/2} \tag{5}$$

for some constant $C > 0$ depending only on r. Define

$$\xi_{n,1}^{\eta}(x,z) \equiv \frac{\xi_n(x,z) + \eta_1}{1+\epsilon_1} \text{ and } \xi_{n,2}^{\eta}(x+uh,z) \equiv \frac{\xi_n(x+uh,z) + \eta_2}{1+\epsilon_1}.$$

Note that $var(\xi_{n,1}^{\eta}(x,z)) = var(\xi_{n,2}^{\eta}(x+uh,z)) = 1$. Let $(Z_{1n}^{\eta}(x,z), Z_{2n}^{\eta}(x+uh,z))$ be a jointly normal centered random vector whose covariance matrix is the same as that of $(\xi_{n,1}^{\eta}(x,z), \xi_{n,2}^{\eta}(x+uh,z))$ for all $(x,u) \in \mathbb{R} \times [-1,1]$. Define

$$\tilde{\tau_n}^{\eta}(A) \equiv \int_A \int_{[-1,1]} cov(max\{\xi_{n,1}^{\eta}(x,z),0\}, max\{\xi_{n,2}^{\eta}(x+uh,z),0\}\lambda_n(x,x+uh,z)hdudx$$

$$\tau_n^{\eta}(A) \equiv \int_A \int_{[-1,1]} cov(max\{Z_{1n}^{\eta}(x,z),0\}, max\{Z_{2n}^{\eta}(x+uh,z),0\}\lambda_n(x,x+uh,z)hdudx$$

Then, observe that

$$| \tilde{\tau_n}(A) - \tilde{\tau_n}^{\eta}(A) | \leq \int_A \int_{[-1,1]} | \Delta^{\eta}_{n,1}(x,u,z) | \lambda_n(x, x+uh, z) h du dx$$

$$+ \int_A \int_{[-1,1]} | \Delta^{\eta}_{n,2}(x,u,z) | \lambda_n(x, x+uh, z) h du dx$$

where $\Delta^{\eta}_{n,1}(x,u,z) \equiv E max\{\xi_n(x,z),0\} E max\{\xi_n(x+uh,z),0\} - E max\{\xi^{\eta}_{n,1}(x,z),0\} E max\{\xi^{\eta}_{n,2}(x+uh,z),0\}$ and
$\Delta^{\eta}_{n,2}(x,u,z) \equiv E max\{\xi_n(x,z),0\} max\{\xi_n(x+uh,z),0\} - E max\{\xi^{\eta}_{n,1}(x,z),0\} max\{\xi^{\eta}_{n,2}(x+uh,z),0\}$.

We already know that for any $a, b \in \mathbb{R}, a_+ = max(a,0), b_+ = max(b,0), | a_+ - b_+ | \leq 2 | a_+ - b_+ | \leq 2 | a - b |$.

We bound $| \Delta^{\eta}_{n,2}(x,u,z) |$ by $2E[| \xi_n(x,z) - \xi^{\eta}_{n,1}(x,z) || \xi_n(x+uh,z) |] + 2E[| \xi_n(x+uh,z) - \xi^{\eta}_{n,2}(x+uh,z) || \xi^{\eta}_{n,1}(x,z) |] \equiv 2[A_{1n}(x,u,z) + A_{2n}(x,u,z)]$, say.

As for $A_{1n}(x,u,z)$,

$$A_{1n}(x,u,z) \leq (E | \xi_n(x,z) - \xi^{\eta}_{n,1}(x,z) |^2)^{1/2} (E[| \xi_n(x+uh,z) |^2])^{1/2}.$$

Note that $E[| \xi_n(x,z) - \xi^{\eta}_{n,1}(x,z) |^2] \leq [E | \xi_n(x,z) - \xi^{\eta}_{n,1}(x,z) |^4]^{1/2}$.

Now,

$$E[| \xi_n(x,z) - \xi^{\eta}_{n,1}(x,z) |^4] = (1+\epsilon_1)^{-4} E[| \epsilon_1 \xi_n(x,z) - \eta_1 |^4]$$

$$\leq 2^3 (1+\epsilon_1)^{-4} \{\epsilon_1^4 E[| \xi_n(x,z) |^4] + E | \eta_1 |^4\}.$$

Applying Lemma 4 and (5) to the last bound, we conclude that

$$sup_{x \in S} E[| \xi_n(x,z) - \xi^{\eta}_{n,1}(x,z) |^4] \leq \frac{C_1(\epsilon_1^4 + \epsilon_1^2)}{(1+\epsilon_1)^4} \leq C_2 \epsilon_1^2$$

for some $C_1, C_2 > 0$. From some large n on,

$$sup_{u \in [-1,1]} sup_{x \in S} E[| \xi_n(x+uh,z) |^2] \leq sup_{x \in S^{\epsilon/2}(z)} E[| \xi_n(x,z) |^2] \leq C_4$$

for $\epsilon > 0$ in Assumption 1(iii). Therefore, for some constant $C > 0, sup_{u \in [-1,1]} sup_{x \in S} A_{1n}(x,u) \leq C\sqrt{\epsilon_1}$. Using similar arguments for $A_{2n}(x,u)$, we deduce that for some constant $C' > 0, sup_{u \in [-1,1]} sup_{x \in S} | \Delta^{\eta}_{n,2}(x,u,z) | \leq C' \sqrt{\epsilon_1}$.

Let us turn to $\Delta^{\eta}_{n,1}(x,u,z)$. Bound $| \Delta^{\eta}_{n,1}(x,u,z) |$ by $2E[| \xi_n(x,z) - \xi^{\eta}_{n,1}(x,z) |] E[| \xi_n(x+uh,z) |] + 2E[| \xi_n(x+uh,z) - \xi^{\eta}_{n,2}(x+uh,z) |] E[| \xi^{\eta}_{n,1}(x,z) |]$.

Using similar arguments for $\Delta^{\eta}_{n,2}(x,u,z)$, we find that for some constant $C'' > 0, sup_{u \in [-1,1]} sup_{x \in S} | \Delta^{\eta}_{n,1}(x,u,z) | \leq C'' \sqrt{\epsilon_1}$.

By Lemma 3 and Assumption 1(ii), there exist $n_0 > 0$ and $C_1, C_2 > 0$ s.t. for all $n \geq n_0$,

$$\int_A \int_{[-1,1]} \lambda_n(x, x+uh, z) h du dx \leq C_1 \int_A \int_{[-1,1]} w(x,z) w(x+uh, z) du dx$$
$$\leq C_2 \sqrt{\int_A w^2(x,z) dx} \sqrt{\int_A \int_{[-1,1]} w^2(x+uh, z) du dx} < \infty.$$

Hence,

$$| \tilde{\tau_n}(A) - \tilde{\tau_n}^{\eta}(A) | \leq C_5 \sqrt{\epsilon_1} \int_A \int_{[-1,1]} \lambda_n(x, x+uh, z) h du dx \leq C_6 \sqrt{\epsilon_1}$$

for some constants $C_5, C_6 > 0$.

Since the choice of $\epsilon_1 > 0$ is arbitrary, it remains for the proof of Lemma 6(i) to prove $| \tilde{\tau_n}^{\eta}(A) - \tau_n(A) | = o(1)$, as $n \to \infty$ and then $\epsilon_1 \to 0$. For any $x \in S$,

$$(\xi_{n,1}^{\eta}(x,z), \xi_{n,2}^{\eta}(x+uh, z))' \stackrel{d}{=} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} W_n^{(i)}(x, u, z)$$

where $W_n^{(i)}(x,u,z)$ are *i.i.d.* copies of $W_n(x,u,z) \equiv (q_n(x,z), q_n(x+uh,z))'$ with

$$q_n(x,z) \equiv \frac{\sum_{i \leq N_1} Y_i K\left(\frac{x-X_i}{h}\right) K\left(\frac{z-Z_i}{h}\right) - E\left[Y_i K\left(\frac{x-X_i}{h}\right) K\left(\frac{z-Z_i}{h}\right)\right] + \eta_1}{h \rho_n(x,z)(1+\epsilon_1)}$$

Using the same argument as in the proof of Lemma 4, we find that,

$$sup_{x \in S} E[| q_n(x,z) |^3] \leq C h^{-1/2}, \;\; for \;\; some \;\; C > 0. \tag{6}$$

Let Σ_{1n} be the 2×2 covariance matrix of $(\xi_{n,1}^{\eta}(x,z), \xi_{n,2}^{\eta}(x+uh,z))'$. Define

$$m\tilde{a}x_n(v) \equiv max\{[\Sigma_{1n}^{1/2} v]_1, 0\} max\{[\Sigma_{1n}^{1/2} v]_2, 0\}, \;\; v \in \mathbb{R}^2$$

where $[a]_j$ of a vector $a \in \mathbb{R}^2$ indicates its j-th entry. There exists some $C > 0$ s.t. for all n,

$$sup_{v \in \mathbb{R}^2} \frac{| m\tilde{a}x_n(v) - m\tilde{a}x_n(0) |}{1 + \| v \|^4 min\{\| v \|, 1\}} \leq C$$

and

$$\int sup_{u \in \mathbb{R}^2 : \|z-u\| \leq \delta} | m\tilde{a}x_n(z) - m\tilde{a}x_n(u) | d\Phi(z) \leq C\delta \;\; for \;\; all \;\; \delta \in (0,1].$$

Hence, for $\tilde{W_n}^{(i)}(x,u,z) \equiv \Sigma_{1n}^{-1/2} W_n^{(i)}(x,u,z)$, and by (6),

$$sup_{x\in S}E \parallel \tilde{W_n}^{(i)}(x,u,z) \parallel^3$$

$$\leq C_1(1-(E[\xi_{n,1}^{\eta}(x,z)\xi_{n,2}^{\eta}(x+uh,z)])^2)^{-\frac{3}{2}}2sup_{x\in S}E \mid q_n(x,z) \mid^3$$

$$\leq C_1(1-(1+\epsilon_1)^{-4})^{-\frac{3}{2}}2sup_{x\in S}E \mid q_n(x,z) \mid^3$$

$$\leq C_2(1-(1+\epsilon_1)^{-4})^{-\frac{3}{2}}h^{-\frac{1}{2}} \ \ for\ \ some\ \ C_1, C_2>0.$$

So that $n^{-1/2}sup_{x\in S}E \parallel \tilde{W_n}^{(i)}(x,u,z) \parallel^3 = O(n^{-1/2}h^{-1/2})$.

By Lemma 2 and following the arguments in (4) analogously,

$$sup_{x\in S} \mid Em\tilde{a}x_n(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\tilde{W_n}^{(i)}(x,u,z)) - Em\tilde{a}x_n(\tilde{Z_n}^{\eta}(x,u,z)) \mid = O(n^{-1/2}h^{-1/2}) = o(1)$$

where $\tilde{Z_n}^{\eta}(x,u,z) \equiv \Sigma_{1n}^{-1/2}(Z_{1n}^{\eta}(x,z), Z_{2n}^{\eta}(x+uh,z))'$.

By (5) and Lemma 4,

$$cov(max\{Z_{1n}^{\eta}(x,z),0\}, max\{Z_{2n}^{\eta}(x+uh,z),0\}) \leq \sqrt{E \mid Z_{1n}^{\eta}(x,z) \mid^2}\sqrt{E \mid Z_{2n}^{\eta}(x+uh,z) \mid^2} < C$$

for some $C>0$ that does not depend on ϵ_1.

Apply the Dominated Convergence Theorem(DCT) to obtain that $\mid \tau_n^{\eta}(A) - \tilde{\tau_n}^{\eta}(A) \mid = o(1)$ as $n\to\infty$ for each $\epsilon_1>0$.

Finally, for all $x\in A$ and all $u\in[-1,1]$,

$$\begin{aligned} &cov(max\{Z_{1n}^{\eta}(x,z),0\}, max\{Z_{2n}^{\eta}(x+uh,z),0\}) \\ &= cov(max\{Z_{1n}(x,z),0\}, max\{Z_{2n}(x+uh,z),0\}) + o(1) \end{aligned}$$

where the $o(1)$ term is one that converges to zero as $n\to\infty$ and then $\epsilon_1\to 0$. Therefore, by the DCT,

$$\mid \tau_n^{\eta}(A) - \tau_n(A) \mid = o(1). \ ((i)\ is\ proved!)$$

To prove (ii), define

$$t_n(x,u,z) \equiv E[\xi_n(x,z)\xi_n(x+uh,z)]$$

$$e_n(x,u,z) \equiv \frac{1}{h^2}E\big[Y_i^2K\big(\frac{x-X_i}{h}\big)K\big(\frac{x-X_i}{h}+u\big)K^2\big(\frac{z-Z_i}{h}\big)\big]$$

$$e(x,u,z) \equiv \rho^2(x,z)\frac{\int K(x)K(x+u)dx}{\int K^2(u)du}$$

By Assumption 1(i), and Lemma 3, for almost every $x \in A$ and for each $u \in [-1,1]$,

$$t_n(x,u,z) = \frac{1}{\rho_n(x,z)\rho_n(x+uh,z)}\frac{1}{h^2}E\big[Y_i^2 K\big(\frac{x-X_i}{h}\big)K\big(\frac{x-X_i}{h}+u\big)K^2\big(\frac{z-Z_i}{h}\big)\big] \quad (7)$$

$$= \frac{e_n(x,u,z)}{\rho_n(x,z)\rho_n(x+uh,z)} = \frac{e_n(x,u,z)}{\rho(x,z)\rho(x+uh,z)} + o(1) = t(u) + o(1)$$

where $t(u) \equiv \frac{e(x,u,z)}{\rho^2(x,z)} = \frac{\int K(x)K(x+u)dx}{\int K^2(u)du}$ by definition.

With the statement in Lemma 6(ii),

$$\tau_n(A) = \int_A \int_{[-1,1]} g_n(x,u,z)\lambda(x,x+uh,z)hdudx + o(1)$$

where $\lambda(x,u,z) \equiv \rho(x,z)\rho(u,z)w(x,z)w(u,z)I_A(x)I_A(u)$.

By (7), for almost every $x \in A$ and for each $u \in [-1,1], g_n(x,u,z) \to g(u)$ as $n \to \infty$, where $g(u) \equiv cov(max\{\sqrt{1-t^2(u)}\mathbb{Z}_1 + t(u)\mathbb{Z}_2, 0\}, max\{\mathbb{Z}_2, 0\})$.

Further, since $\rho(\cdot,z)\rho(\cdot,z)$ and $w(\cdot,z)w(\cdot,z)$ are continuous on A and A has a finite Lebesgue measure, we follow the proof of Lemma 6.4 of GMZ to find that $g_n(x,u,z)\lambda(x,x+uh,z)$ converges in measure to $g(u)\lambda(x,x,z)$ on $A \times [-1,1]$, as $n \to \infty$. Using the bounded convergence theorem, we achieve the proof of the lemma.

■

Lemma 7 *Suppose that Assumption 1 and 2 hold. Moreover, assume that as $n \to \infty, h \to 0, n^{-1/2}h^{-2} \to 0$. Then given $z \in \mathcal{Z}$, there exists a constant $C > 0$ s.t. for any Borel set $A \subset \mathbb{R}$,*

$$limsup_{n\to\infty} E[\mid \sqrt{nh^2}\int_A [max\{v_n(x,z),0\} - Emax\{v_n(x,z),0\}]w(x,z)dx \mid]$$

$$\leq C\int_A w(x,z)dx + C\sqrt{\int_A w^2(x,z)dx}.$$

Proof. It suffices to show there exists $C > 0$ s.t. for any Borel set $A \subset \mathbb{R}$,

Step1) $E[\mid \sqrt{nh^2}\int_A [max\{v_n(x,z),0\} - max\{v_N(x,z),0\}]w(x,z)dx \mid] \leq C\int_A w(x,z)dx$

Step2) $E[\mid \sqrt{nh^2}\int_A [max\{v_N(x,z),0\} - Emax\{v_N(x,z),0\}]w(x,z)dx \mid] \leq C\sqrt{\int_A w^2(x,z)dx}$

Step3) $\sqrt{nh^2} \mid \int_A (Emax\{v_N(x,z),0\} - Emax\{v_n(x,z),0\})w(x,z)dx \mid \to 0$ *as* $n \to \infty$.

Proof of Step1) Define

$$u_n^2(x,z) \equiv E\big[Y_i^2 K^2\big(\frac{x-X_i}{h}\big)K^2\big(\frac{z-Z_i}{h}\big)\big] - \Big(E\big[Y_i K\big(\frac{x-X_i}{h}\big)K\big(\frac{z-Z_i}{h}\big)\big]\Big)^2$$

$$\overline{V_n}(x,z) \equiv \frac{1}{u_n(x,z)}\big\{Y_i K\big(\frac{x-X_i}{h}\big)K\big(\frac{z-Z_i}{h}\big) - E\big[Y_i K\big(\frac{x-X_i}{h}\big)K\big(\frac{z-Z_i}{h}\big)\big]\big\}$$

We write if $N=n, \Sigma_{i=N+1}^n = 0$ and if $N>n, \Sigma_{i=N+1}^n = -\Sigma_{i=n+1}^N$. Using this, we write

$$v_n(x,z) = \frac{1}{nh^2}\sum_{i=1}^N \overline{V_n}(x,z)u_n(x,z) + \frac{1}{nh^2}\sum_{i=N+1}^n \overline{V_n}(x,z)u_n(x,z).$$

Now, observe that

$$\frac{1}{\sqrt{nh^2}}\sum_{i=1}^N \overline{V_n}(x,z)u_n(x,z) = \frac{1}{\sqrt{nh^2}}\sum_{i=1}^N \big\{Y_i K\big(\frac{x-X_i}{h}\big)K\big(\frac{z-Z_i}{h}\big) - E\big[Y_i K\big(\frac{x-X_i}{h}\big)K\big(\frac{z-Z_i}{h}\big)\big]\big\}$$

$$= \sqrt{nh^2}\{\hat{g_N}(x,z) - \frac{1}{h^2}E\big[Y_i K\big(\frac{x-X_i}{h}\big)K\big(\frac{z-Z_i}{h}\big)\big]\} + \sqrt{nh^2}\big(\frac{n-N}{N}\big)\frac{1}{h^2}E\big[Y_i K\big(\frac{x-X_i}{h}\big)K\big(\frac{z-Z_i}{h}\big)\big]$$

$$= \sqrt{nh^2}v_N(x,z) + \sqrt{nh^2}\big(\frac{n-N}{N}\big)\frac{1}{h^2}E\big[Y_i K\big(\frac{x-X_i}{h}\big)K\big(\frac{z-Z_i}{h}\big)\big]$$

Let $\eta_n(x,z) \equiv \sqrt{n}\big(\frac{n-N}{N}\big)\frac{1}{h^2}E\big[Y_i K\big(\frac{x-X_i}{h}\big)K\big(\frac{z-Z_i}{h}\big)\big]$ and $s_n(x,z) \equiv \frac{1}{\sqrt{nh^2}}\sum_{i=N+1}^n \overline{V_n}(x,z)u_n(x,z)$. We can write

$$\sqrt{nh^2}v_n(x,z) = \sqrt{nh^2}v_N(x,z) + (\sqrt{h^2}\eta_n(x,z) + s_n(x,z)).$$

We can see that for some $C>0$,

$$\mid \sqrt{nh^2}\int_A (max\{v_n(x,z),0\} - max\{v_N(x,z),0\})w(x,z)dx \mid$$

$$= \mid \int_A max\{\sqrt{nh^2}v_n(x,z),0\} - max\{\sqrt{nh^2}v_N(x,z),0\}w(x,z)dx \mid$$

$$\leq C\int_A \mid \sqrt{h^2}\eta_n(x,z) \mid w(x,z)dx + C\int_A \mid s_n(x,z) \mid w(x,z)dx$$

$$= A_{1n} + A_{2n}, \ \ say.$$

To deal with A_{1n}, A_{2n}, show the following:
Claim1) $sup_{x\in S}E[\eta_n^2(x,z)h^2] = O(1)$ and Claim2) $sup_{x\in S}E \mid s_n(x,z) \mid^2 = o(1)$.

Proof of Claim1) By the fact that $E \mid n^{-1/2}(n-N) \mid^2 = O(1)$,

$$E \mid \sqrt{n}\big(\frac{n-N}{N}\big) \mid^2 sup_{x\in S} \mid \frac{1}{h^2}E\big[Y_i K\big(\frac{x-X_i}{h}\big)K\big(\frac{z-Z_i}{h}\big)\big]Ch \mid^2 = O(1)$$

for some $C > 0$.

Proof of Claim2) Note that

$$\mid \sqrt{nh^2}\frac{s_n(x,z)}{u_n(x,z)} \mid = \mid \sum_{i=n+1}^{N} \overline{V_n}(x,z) \mid .$$

Certainly, $var(\overline{V_n}(x,z)) = 1$. Hence by Lemma 1(i) of Horváth(1991), for some $C > 0$,

$$E\big(\sqrt{nh^2}\frac{s_n(x,z)}{u_n(x,z)}\big)^2 \leq E \mid N-n \mid E \mid \mathbb{Z}_1 \mid^2 + C\{E \mid N-n \mid^{1/2} E \mid \overline{V_n}(x,z) \mid^3 + E \mid \overline{V_n}(x,z) \mid^4\}.$$

As seen in Lemma 5, $sup_{x\in S}E \mid \overline{V_n}(x,z) \mid^3 \leq Ch^{-1/2}$ for some $C > 0$. Similarly,

$$sup_{x\in S} \mid \overline{V_n}(x,z) \mid^4 \leq \frac{h^2 k_{n,4}(x,z)}{h^4(\rho_n^2(x,z) - b_n^2(x,z))^2} \leq Ch^{-1}$$

for some $C > 0$. Furthermore, $E \mid N-n \mid = O(n^{1/2})$ and $E \mid N-n \mid^{1/2} = O(n^{1/4})$. (e.g. (2.21), (2.22) of Horváth(1991)). Therefore, there exists $C > 0$ s.t.

$$sup_{x\in S}E\big(\sqrt{nh^2}\frac{s_n(x,z)}{u_n(x,z)}\big)^2 \leq C\{n^{1/2} + n^{1/4}h^{-1/2} + h^{-1}\}$$

$$sup_{x\in S}E(\sqrt{nh^2}s_n(x,z))^2 \leq C'\{n^{1/2}h^2 + n^{1/4}h^{3/2} + h\}$$

$$sup_{x\in S}E(s_n(x,z))^2 \leq C'\{h + n^{-1/4}h^{1/2} + \frac{1}{\sqrt{n}}\}.$$

Therefore, $sup_{x\in S}E(s_n(x,z))^2 = o(1)$.
Now, using Claim1 and 2, we prove Step1. Let $\mu(A) \equiv \int_A w(x,z)dx$.

$$E \mid A_{1n} \mid \leq C\int_A E[\mid \sqrt{h^2}\eta_n(x,z) \mid]w(x,z)dx$$

$$\leq C\mu(A)sup_{x\in S}E \mid \sqrt{h^2}\eta_n(x,z) \mid$$

$$\leq C\mu(A)\big(sup_{x\in S}E[h^2\eta_n^2(x,z)]\big)^{1/2}.$$

Therefore, by Claim1, $E \mid A_{1n} \mid\leq C'\mu(A)$ for some $C'>0$. As for A_{2n}, similarly, we obtain that

$$E[A_{2n}] \leq C\int_A \mid s_n(x,z) \mid w(x,z)dx$$

$$\leq C\mu(A)\big(sup_{x\in S}E \mid s_n(x,z) \mid^2 \big)^{1/2}.$$

By Claim2, $E[A_{2n}] = o(1)$.

Proof of Step2) We can follow the proof of Lemma 6(i) to show

$$E[\sqrt{nh^2}\int_A(\mid v_N(x,z) \mid -E \mid v_N(x,z) \mid)w(x,z)dx]^2 = \kappa_n(A) + o(1)$$

where $\kappa_n(A) \equiv \int_A\int_{[-1,1]} r_n(x,u,z)\lambda_n(x,x+uh,z)hdudx$,
$\lambda_n(x,x',z) \equiv \rho_n(x,z)\rho_n(x',z)w(x,z)w(x',z)I_{A\cap S}(x)I_{A\cap S}(x')$,
$r_n(x,u,z) \equiv cov(\mid Z_{n,A}(x,z) \mid, \mid Z_{n,B}(x+uh,z) \mid)$ with $(Z_{n,A}(x,z), Z_{n,B}(x+uh,z))' \in \mathbb{R}^2$ denoting a centered normal random vector whose covariance matrix is equal to that of $(\xi_n(x,z),\xi_n(x+uh,z))'$. By Cauchy-Schwarz inequality and Lemma 4,

$$sup_{x\in S}r_n(x,u,z) \leq sup_{x\in S}\sqrt{E \mid Z_{n,A}(x,z) \mid^2 E \mid Z_{n,B}(x+uh,z) \mid^2} < \infty.$$

Further,

$$\int_A\int_{[-1,1]}\lambda_n(x,x+uh,z)hdudx \leq C\sqrt{\int_A w^2(x,z)dx}\sqrt{\int_{A+uh} w^2(x,z)dx}.$$

By Assumption 1(ii), $\int_{S^\epsilon(z)} w^2(x,z)dx < \infty$ for some $\epsilon>0$, we find that as $h\to 0$, the last term converges to $C\int_A w^2(x,z)dx$. So we obtained the desired result for Step2.

Proof of Step3) The convergence follows from the proof of Lemma 5.

■

Let $C\subset\mathbb{R}$ be a bounded Borel set s.t $\alpha\equiv P\{x\in\mathbb{R}\setminus C\}>0$.
For any Borel set $A\subset C$, let

$$\zeta_n(A) \equiv \int_A max\{v_n(x,z),0\}w(x,z)dx$$

$$\zeta_N(A) \equiv \int_A max\{v_N(x,z), 0\} w(x,z) dx$$

Define

$$S_n(A) \equiv \frac{\sqrt{nh^2}\{\zeta_N(A) - E\zeta_n(A)\}}{\sqrt{\sigma_n(A)}}$$

where $U_n \equiv \frac{1}{\sqrt{n}}\{\sum_{i=1}^N I\{X_i \in C\} - nP\{x \in C\}\}$ and
$V_n \equiv \frac{1}{\sqrt{n}}\{\sum_{i=1}^N I\{X_i \in \mathbb{R} \setminus C\} - nP\{x \in \mathbb{R} \setminus C\}\}$

Lemma 8 *Suppose that Assumption 1 and 2 hold. Also, assume that as* $n \to \infty, h \to 0$ *and* $n^{-1/2}h^{-2} \to 0$. *Let* $A \subset C$ *be s.t.* $\sigma(A) > 0, \alpha \equiv P\{x \in \mathbb{R} \setminus C\} > 0$. $\rho(\cdot, z), w(\cdot, z)$ *are continuous and bounded on* A *for given* $z \in \mathcal{Z}$. *Furthermore,* $sup_{x \in A} \mid \rho_n(x,z) - \rho(x,z) \mid \to 0$ *as* $n \to \infty$ *is satisfied. Then,*

$$(S_n(A), U_n) \xrightarrow{d} (\mathbb{Z}_1, \sqrt{1-\alpha}\mathbb{Z}_2).$$

Proof. First, we show that $cov(S_n(A), U_n) \to 0$. Write

$$cov(S_n(A), U_n) = \frac{\sqrt{nh^2}}{\sqrt{\sigma_n(A)}} \int_A cov(max\{v_N(x,z), 0\}, U_n) w(x,z) dx.$$

It suffices to show that

$$cov(\sqrt{nh^2}\{\zeta_N(A) - E\zeta_N(A)\}, U_n) = o(1) \tag{8}$$

since $\sigma_n(A) \to \sigma(A) > 0$ by Lemma 6. For any $x \in S$,

$$(\frac{\sqrt{nh^2} v_N(x,z)}{\rho_n(x,z)}, \frac{U_n}{\sqrt{P\{x \in C\}}}) \stackrel{d}{=} (\frac{1}{\sqrt{n}} \sum_{i=1}^n Q_n^{(i)}(X,z), \frac{1}{\sqrt{n}} \sum_{i=1}^n U^{(i)})$$

where $(Q_n^{(i)}, U^{(i)})'$ are *i.i.d.* compies of $(Q_n(x,z), U)$ with

$$Q_n(x,z) \equiv \frac{1}{h\rho_n(x,z)} \{ \sum_{i \le N_1} Y_i K(\frac{x - X_i}{h}) K(\frac{z - Z_i}{h}) - E[Y_i K(\frac{x - X_i}{h}) K(\frac{z - Z_i}{h})] \}$$

and

$$U \equiv \frac{\sum_{i \le N} I\{X_i \in C\} - P\{x \in C\}}{\sqrt{P\{x \in C\}}}$$

Uniformly over $x \in S, d_n(x,z) \equiv E[Q_n(x,z)U] = O(h^{1/2}) = o(1)$, by Lemma 3. Let $(\mathbb{Z}_{1n}, \mathbb{Z}_{2n})'$ be a centered normal random vector with the same covariance matrix as that of $(Q_n(x,z), U)'$. Let the 2×2 covariance matrix be $\Sigma_{n,2}$.

Recall the definition of $\xi_n(x,z)$, and write

$$cov(max\{\xi_n(x,z), 0\}, \frac{1}{\sqrt{n}} \sum_{i=1}^{n} U^{(i)}) - cov(max\{\mathbb{Z}_{1n}, 0\}, \mathbb{Z}_{2n})$$

$$= E[max\{\xi_n(x,z), 0\} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} U^{(i)}] - E[max\{\mathbb{Z}_{1n}, 0\}\mathbb{Z}_{2n}] \equiv A_{1n}(x,z), \ \ say.$$

Define $\overline{max_n}(v) \equiv max\{[\Sigma_{n,2}^{1/2} v]_1, 0\}[\Sigma_{n,2}^{1/2} v]_2, v \in \mathbb{R}^2$. There exists some $c > 0$ s.t. for all $n \geq 1$,

$$sup_{v \in \mathbb{R}^2} \frac{| \overline{max_n}(v) - \overline{max_n}(0) |}{1 + \| v \|^2 \ min\{\| v \|, 1\}} \leq c$$

and

$$\int sup_{u \in \mathbb{R}^2 : \|z-u\| \leq \delta} \ | \ \overline{max_n}(z) - \overline{max_n}(u) \ | \ d\Phi(z) \leq c\delta, \ \ for \ \ all \ \ \delta \in (0,1].$$

Letting $W_n^{(i)}(x,z) \equiv \Sigma_{n,2}^{-1/2}(Q_n^{(i)}(x,z), U^{(i)})'$, observe that from large n on, for some $c > 0$,

$$E \ \| \ W_n^{(i)}(x,z) \ \|^3 = E \ \| \ \Sigma_{n,2}^{-1/2}(Q_n^{(i)}(x,z), U^{(i)})' \ \|^3$$

$$= E[\{tr(\Sigma_{n,2}^{-1/2}(Q_n^{(i)}(x,z), U^{(i)})'(Q_n^{(i)}(x,z), U^{(i)})\Sigma_{n,2}^{-1/2})\}^{3/2}]$$

$$\leq C(1 - d_n^2(x,z))^{-3/2} E[| \ Q_n(x,z) \ |^3 + | \ U \ |^3] \leq ch^{-1/2}$$

Hence, by Lemma 2,

$$sup_{x \in S} \ | \ A_{1n}(x,z) \ | = sup_{x \in S} \ | \ E\overline{max_n}\big(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} W_n^{(i)}(x,z)\big) - E\overline{max_n}(\tilde{Z}_n) \ |$$

$$= O(n^{-1/2}h^{-1/2}) = o(1) \ \ where \ \ \tilde{Z}_n \equiv \Sigma_{n,2}^{-1/2}(\mathbb{Z}_{1n}, \mathbb{Z}_{2n})'.$$

This completes the proof of (8), hence, $cov(S_n(A), U_n) \to 0$.

Now define $\Delta_n(x,z) \equiv \sqrt{nh^2}[max\{v_N(x,z), 0\} - Emax\{v_N(x,z), 0\}]w(x,z)$. Let $\{R_{n,i} : i = 1, \ldots, L_n\}$ be the collection of rectangles in $\mathbb{R}$ s.t. all the rectangles $R_{n,i}$ are of the form $R_{n,i} \equiv (a,b]$, where $h \leq b - a \leq 2h$. Let $B_{n,i} \equiv R_{n,i} \cap C$ and $\mathcal{I}_n \equiv \{i : \mathbb{R} \times B_{n,i} \neq \phi\}$. Then $B_{n,i}$ has Lebesgue measure $m(B_{n,i})$ bounded by $C_1 h$ and the cardinality of the set $\mathcal{I}_n$ is bounded by $C_2 h^{-1}$ for some positive constants C_1, C_2. Define

$$\alpha_{i,n} \equiv \frac{1}{\sqrt{\sigma_n(A)}} \int_{B_{i,n}\cap A} \Delta_n(x,z)dx$$

$$u_{i,n} \equiv \frac{1}{\sqrt{n}} \big\{ \sum_{j=1}^{N} I\{X_j \in B_{n,i}\} - nP\{X_j \in B_{n,i}\} \big\}.$$

Then, we can write $S_n(A) = \sum_{i \in \mathcal{I}_n} \alpha_{i,n}$ and $U_n = \sum_{i \in \mathcal{I}_n} u_{i,n}$.
Certainly, $var(S_n(A)) = 1$ and $var(u_n) = 1 - \alpha$. Take $\mu_1, \mu_2 \in \mathbb{R}$ and let

$$y_{i,n} \equiv \mu_1 \alpha_{i,n} + \mu_2 u_{i,n}.$$

Since $cov(S_n(A), U_n) \to 0$,

$$var(\sum_{i \in \mathcal{I}_n} y_{i,n}) \to \mu_1^2 + \mu_2^2(1-\alpha) \ \ as\ n \to \infty.$$

Since $\sigma_n^r(A) = \sigma^r(A) + o(1), r > 0$ by Lemma 6 and $m(B_{n,i}) \le ch$ for a constant $c > 0$, we take $r \in (2, 4]$ and bound

$$\sigma_n^{r/2}(A) \sum_{i \in \mathcal{I}_n} E \mid y_{i,n} \mid^r \le C sup_{x \in A} E \mid \Delta_n(x,z) \mid^r \sum_{i \in \mathcal{I}_n} \big(\int_A \int_A \int_A I_{B_{n,i}}(u,v,s) dudvds \big)^{r/3}$$

where $I_B(u,v,s) \equiv I\{u \in B\} I\{v \in B\} I\{s \in B\}$. Using Jensen's inequality, we have

$$sup_{x \in A} E \mid \Delta_n(x,z) \mid^r \le C_1(\sqrt{nh^2})^r sup_{x \in A} E \mid v_N(x,z) \mid^r \ w^r(x,z)$$

$$\le C_2(\sqrt{nh^2})^r sup_{x \in A \cap S} E \mid v_N(x,z) \mid^r$$

for some $C_1, C_2 > 0$. As for the last term, we apply Rosenthal's inequality(see e.g. Lemma 2.3 of GMZ): for some constant $C > 0$,

$$(\sqrt{nh^2})^r sup_{x \in A \cap S} E \mid v_N(x,z) \mid^r \le ch^r sup_{x \in S} \big(\frac{1}{h^2} E\big[Y_i^2 K^2 \big(\frac{x - X_i}{h} \big) K^2 \big(\frac{z - Z_i}{h} \big) \big] \big)^{r/2}$$

$$+ch^r sup_{x \in S} \big(\frac{n}{\sqrt{n^r} h^{2r}} E\big[\mid Y_i K \big(\frac{x - X_i}{h} \big) K \big(\frac{z - Z_i}{h} \big) \mid^r \big] \big)$$

By Lemma 3, the first term is $O(h^{r/2})$ and the last term is $O(n^{1-r/2} h^{-r+1})$. Hence, we find that

$$\sum_{i \in \mathcal{I}_n} E \mid y_{i,n} \mid^r = cardinality\ of\ \mathcal{I}_n \times O(m(B_{n,i})^r h^{r/2} \{1 + n^{1-r/2} h^{-3r/2+1}\})$$

$$= O(h^{-1}h^r h^{r/2}\{1 + n^{1-r/2}h^{-3r/2+1}\}) = O(h^{3r/2-1} + n^{1-r/2}) = o(1) \ \ for\ any\ r \in (2,4].$$

Therefore, as $n \to \infty, \sum_{i\in\mathcal{I}_n} E \mid y_{i,n} \mid^r \to 0$ for any $r \in (2,4]$.

The sequence $\{y_{i,n}\}_{i=1}^n$ is a one-dependent triangular array. By Corollary 2 of Shergin(1979), and the Cramer-Wold device[e.g. Billinsley(1968)], the desired result follows. ∎

Lemma 9 *Suppose that the conditions of Lemma 8 are satisfied, and let $A \subset \mathbb{R}$ be a Borel set in Lemma 8. Then*

$$\frac{\sqrt{nh^2}\{\zeta_n(A) - E\zeta_n(A)\}}{\sqrt{\sigma_n(A)}} \xrightarrow{d} N(0,1),\ \ as\ n \to \infty$$

Proof.

The conditional distribution of $S_n(A)$ given $N = n$ is equal to that of

$$\frac{\sqrt{nh^2}}{\sqrt{\sigma_n(A)}}\int_A [max\{v_n(x,z),0\} - Emax\{v_N(x,z),0\}]w(x,z)dx$$

Using Lemma 8, and the de-Poissonization argument of Veirlant and Mason(1995), this conditional distribution converges to N(0,1). Now by Lemma 5, it follows that

$$\sqrt{nh^2}\int_A [Emax\{v_N(x,z),0\} - Emax\{v_n(x,z),0\}]w(x,z)dx \to 0\ \ as\ n \to \infty.$$

∎

Before we state the theorem and its proof, note the following expressions:

$$\rho_n^2(x,z) \equiv h^{-2}E\big[Y_i^2K^2\big(\frac{x-X_i}{h}\big)K^2\big(\frac{z-Z_i}{h}\big)\big]$$

$$\sigma \equiv q_p\int_{\mathcal{X}} \rho^2(x,z)w^2(x,z)dx$$

$$\hat{g_n}(x,z) \equiv \frac{1}{nh^2}\sum_{i=1}^n Y_iK\big(\frac{x-X_i}{h}\big)K\big(\frac{z-Z_i}{h}\big)$$

$$\Gamma(\hat{g_n}) \equiv \int_{\mathcal{X}} max\{\hat{g_n}(x,z),0\}w(x,z)dx$$

$$a_n \equiv \int_{\mathcal{X}} \rho_n(x,z)w(x,z)dx \cdot Emax\{\mathbb{Z}_1,0\}$$

$$v_n(x,z) \equiv \hat{g_n}(x,z) - E\hat{g_n}(x,z)$$

$$\zeta_n(A) \equiv \int_A max\{v_n(x,z), 0\} w(x,z) dx.$$

8.3 Theorem and Its Proof

Theorem 10 *Suppose that Assumptions 1-2 hold and that* $h \to 0$ *and* $n^{-1/2}h^{-2} \to 0$ *as* $n \to \infty$*. Further, assume that given* $z \in \mathcal{Z}, E[Y_i \mid X_i = x, Z_i = z] = 0$ *for almost all* $x \in \mathcal{X}$*. Then,*

$$T_n \equiv \frac{1}{\sqrt{\sigma_n}}\{\sqrt{nh^2}\Gamma(\hat{g_n}) - a_n\} \xrightarrow{d} N(0,1)$$

where $\sigma_n \equiv q_p \int_{\mathcal{X}} \rho_n^2(x,z) w^2(x,z) dx, q_p \equiv \int_{[-1,1]} cov(max\{\sqrt{1-t^2(u)}\mathbb{Z}_1 + t(u)\mathbb{Z}_2, 0\}, max\{\mathbb{Z}_2, 0\}) du.$

Proof. For given $z \in \mathcal{Z}$, fix $\epsilon > 0$ as in Assumption 1 (iii), and take $n_0 > 0$ s.t. for all $n \geq n_0$,

$$\{x - uh : x \in S(z), u \in [-1/2, 1/2]\} \subset S^{\epsilon}(z) \in \mathcal{X}.$$

Since we concentrate on the least favorable case of H_0, $\hat{g_n}(x,z) = v_n(x,z)$ for almost all $x \in \mathcal{X}$ and for all $n \geq n_0$. This results from the fact that

$$E\hat{g_n}(x,z) = E\frac{1}{nh^2}\sum_{i=1}^{n} Y_i K\left(\frac{x - X_i}{h}\right) K\left(\frac{z - Z_i}{h}\right)$$

$$= \int_{[-1/2,1/2]} E[Y_i \mid X_i = x - uh, Z_i = z] f(x - uh, z) \frac{1}{h} K\left(\frac{z - Z_i}{h}\right) K(u) du = 0$$

for almost all $x \in S(z)$ and for all $n \geq n_0$. We will only consider the case $n \geq n_0$.

Fix $0 < \epsilon_l \to 0$ as $l \to \infty$ and take a compact set $\mathcal{W}_l \subset S(z)$ s.t. w is bounded and continuous on $\mathcal{W}_l$ and for $s \in \{1, 2\}$,

$$\int_{\mathcal{X} \setminus \mathcal{W}_l} w^s(x,z) dx \to 0 \ as \ l \to \infty.$$

We can find such $\mathcal{W}_l$ following the arguments in the proof of Lemma 6.1 of GMZ because w^s is integrable by Assumption 1 (ii). Take $M_l, v_l > 0$ s.t. for $C_l \equiv [-M_l + v_l, M_l - v_l], P\{X_i \in \mathbb{R} \setminus C_l\} > 0$, and for some Borel $A_l \subset C_l \cap \mathcal{W}_l, \exists \rho(\cdot, z)$ that is bounded and continuous on A_l, s.t.

$$sup_{x \in A_l} \mid \rho_n(x,z) - \rho(x,z) \mid \to 0, \ as \ n \to \infty$$

and

$$\int_{\mathcal{W}_l \backslash A_l} \rho(x,z) w^s(x,z) dx \to 0,\ as\ l \to \infty,\ for\ s \in \{1,2\}. \tag{9}$$

The existence of M_l, v_l, ϵ_l and the set A_l are ensured by Lemma 1. By Assumption 1 (i), we find that the second convergence in (9) above implies that $\int_{\mathcal{W}_l \backslash A_l} w^s(x,z)dx \to 0$ as $l \to \infty$, for $s \in \{1,2\}$.

We can write

$$\frac{\sqrt{nh^2}\Gamma(\hat{g_n}) - a_n}{\sqrt{\sigma_n}} = \frac{\int_{\mathcal{X}} \sqrt{nh^2} max\{v_n(x,z), 0\} w(x,z) dx - \int_{\mathcal{X}} \rho_n(x,z) w(x,z) dx \cdot E max\{\mathbb{Z}_1, 0\}}{\sqrt{\sigma_n}}$$

(With Lemma 5, substitute the second term of numerator with $\int \sqrt{nh^2} E max\{v_n(x,z), 0\} w(x,z) dx$.)

$$= \frac{\sqrt{nh^2}}{\sqrt{\sigma_n}} \{\zeta_n(\mathcal{X} \backslash \mathcal{W}_l) - E\zeta_n(\mathcal{X} \backslash \mathcal{W}_l)\} + \frac{\sqrt{nh^2}}{\sqrt{\sigma_n}} \{\zeta_n(\mathcal{W}_l \backslash A_l) - E\zeta_n(\mathcal{W}_l \backslash A_l)\} + \frac{\sqrt{nh^2}}{\sqrt{\sigma_n}} \{\zeta_n(A_l) - E\zeta_n(A_l)\}. \tag{10}$$

Since $\mathcal{X} \backslash A_l = (\mathcal{X} \backslash \mathcal{W}_l) \cup (\mathcal{W}_l \backslash A_l)$, by Lemma 7 and $\int_{\mathcal{X} \backslash \mathcal{W}_l} w^s(x,z)dx \to 0$, $\int_{\mathcal{W}_l \backslash A_l} w^s(x,z)dx \to 0$ as $l \to \infty$, we can get

$$\sqrt{nh^2}\{\zeta_n(\mathcal{X} \setminus A_l) - E\zeta_n(\mathcal{X} \setminus A_l)\} \xrightarrow{p} 0,\ as\ n \to \infty, l \to \infty. \tag{11}$$

Furthermore, we write $\mid \sigma_n - \sigma_n(A_l) \mid$ as

$$\int_{\mathcal{X}} q_p (1 - I_{A_l}(x)) \rho_n^2(x,z) w^2(x,z) dx$$

$$\leq sup_{x \in S(z)} \mid q_p \rho_n^2(x,z) \mid \int_{\mathcal{X}} (1 - I_{A_l}(x)) w^2(x,z) dx$$

$$= sup_{x \in S(z)} \mid q_p \rho_n^2(x,z) \mid \int_{\mathcal{X} \backslash A_l} w^2(x,z) dx.$$

where $q_p \equiv \int_{[-1,1]} cov(max\{\sqrt{1-t^2(u)}\mathbb{Z}_1 + t(u)\mathbb{Z}_2, 0\}, max\{\mathbb{Z}_2, 0\}) du$.

Observe that as $l \to \infty$, $\left(\int_{\mathcal{X} \backslash A_l} w^2(x,z) dx\right)^2 \to 0$.

With Lemma 6(ii), it follows that

$$lim_{l \to \infty} limsup_{n \to \infty} \mid \sigma_n - \sigma_n(A_l) \mid = 0. \tag{12}$$

Furthermore, since $\sigma_n(A_l) \to \sigma(A_l)$ as $n \to \infty$ for each l by Lemma 6, and $\sigma(A_l) \to \sigma > 0$ as $l \to \infty$, by Assumption 1, it follows that for all $\epsilon_1 > 0$,

$$0 < \sigma - \epsilon_1 \leq liminf_{n\to\infty}\sigma_n \leq limsup_{n\to\infty}\sigma_n \leq \sigma + \epsilon_1 < \infty. \tag{13}$$

Combining this with the expression (11) above, we find that as $n \to \infty, l \to \infty$,

$$\frac{\sqrt{nh^2}}{\sqrt{\sigma_n}}\{\zeta_n(\mathcal{X} \setminus A_l) - E\zeta_n(\mathcal{X} \setminus A_l)\} = o_p(1).$$

As for the last term in (10), by (13) and Lemma 9 as $n \to \infty, l \to \infty$,

$$\sqrt{nh^2} \mid \zeta_n(A_l) - E\zeta_n(A_l) \mid = O_p(1).$$

Therefore, by (12),

$$\frac{\sqrt{nh^2}}{\sqrt{\sigma_n}}\{\zeta_n(A_l) - E\zeta_n(A_l)\} = \frac{\sqrt{nh^2}}{\sqrt{\sigma_n(A_l)}}\{\zeta_n(A_l) - E\zeta_n(A_l)\} + o_p(1).$$

For all $l \geq 1$, the last term converges in distribution to N(0,1) by Lemma 9. Since $\sigma_n(A_l) \to \sigma$ as $n \to \infty, l \to \infty$, we can conclude that

$$\sqrt{nh^2}\Gamma(\hat{g_n}) - a_n \xrightarrow{d} N(0, \sigma).$$

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요약(국문초록)

우리나라에서 1974년부터 2009년까지 시행된 고교평준화 제도가 적용되는 지역의 일반 고등학교에 한하여 학생들은 무작위적으로 자신이 속한 학군 내에서 학교를 배정받았다. 이 논문은 이러한 특수한 제도적 배경을 바탕으로 고등학교의 성별구성이 대입수학능력검정시험(이하 수능)으로 대표되는 학생들의 학업 성적에 미치는 영향을 분석하고자 하였다. 분석대상은 한국직업능력개발원의 KEEP 패널 자료 중 2005년에 고등학교에 입학한 학생들로서, 학생들의 사회경제적인 배경을 나타내는 변수들이 단성학교와 남녀공학 간에 큰 차이가 없음을 확인하였다. 이 논문에서는 남녀공학으로 배정받는 것을 하나의 처리(treatment)로 가정하고 그 효과를 모수적이고 준모수적인 방법으로 추정하였다. 또한 분포적인 영향을 분석하기 위하여 비모수적 가설검정을 실시하였다. 귀무가설은 비양수적인 조건부 평균처리효과(average treatment effect)와 1차 확률적 지배(first-order stochastic dominance) 관계 존재 여부에 관한 것이다. 가설검정 결과 모든 가계소득 또는 공부시간 수준에서 남학생들은 유의하게 단성학교에서 더 높은 수능점수를 얻는 것으로 보이며 여학생들은 남녀공학인지 여부에 따라 수능성적이 유의한 차이를 나타내지 않았다. 그러나 공부시간을 고정시키고 모든 소득수준에 대하여 가설검정을 실시한 결과, 공부시간이 많은 남학생들의 경우 남녀공학 배정이 성적에 부정적인 영향을 미친다는 유의한 증거를 찾을 수 없었다.

주요어: 단성학교, 남녀공학, 처리효과, 비모수 가설검정, 확률적 지배
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