A random assignment of indivisible goods in single-peaked economies with a common peak

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Abstract

Bogomolnaia and Moulin (2001) show that there is no rule satisfying *equal treatment of equals, sd-efficiency* and *sd-strategyproofness* for a random assignment problem of indivisible goods. Furthermore, Kasajima (2013) shows that the incompatibility result holds when agents have single-peaked preferences. In this paper, we restrict the domain by requiring that all agents have a common peak and investigate the existence of rules satisfying the three axioms. We show that the three axioms are still incompatible. As it turns out, the three axioms are still incompatible even though all agents have the same preferences except the three least-preferable objects.

Keywords: Probabilistic assignment, indivisible goods, single-peaked preferences with a common peak, impossibility result.

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1. Introduction

We consider the problem of allocating indivisible goods or objects among a group of agents when each agent is supposed to recieve exactly one of them. Each agent has a complete, transitive, and strict binary relation over objects.

Not surprisingly, the indivisibility of objects casues serious difficulties in achieving fairness. For instance, suppose there are two desirable objects to be allocated to two agents, and they prefer the same object. It is clear that each of the two possible allocations will violate any reasonable notion of fairness. Therefore, we assign a probability vector to get each object for each agent called random assignment.

There are many ways to allocate a random assignment to each agent. A natural approach to the problem is to generalize the simple lottery mechanism: order the agents uniformly at random and let them successively choose an available object according to this (random) order; thus the first agent picks his favorite object, the second agent picks his favorite object among the remaining objects, etc. This is the random priority rule (Abdulkadiroğlu and Sönmez, 1998), and has a number of attractive features: it is ex post efficient and truthful (or strategyproof: revealing true preferences is a dominant strategy for all the agents); it is fair in the weak sense of equal treatment of equals (agents with identical preferences are treated in an identical manner); however, it is not efficient when agents are endowed with utility functions consistent with their preferences, that is, it is not ex ante efficient.

A second solution to this problem is probabilistic serial (PS) rule (Bogomolnaia and Moulin, 2001). This solution computes a random assignment that is sd-efficient and envy-free. sd-efficiency is stronger than ex post efficiency but weaker than ex ante efficiency; given the ordinal nature of the input to the rule (only preference rankings are used, not complete utility functions), this is perhaps the most meaningful notion of efficiency in an economy where ordinal preferences are reported. Finally, this PS rule proposed by Bogomolnaia and Moulin is not truthful.

When each agent has single-peaked preferences, i.e., there is an order on the object set such that each agent has a single most preferred object and becomes monotonically worse off when moving away from her most preferred object, Kasajima(2013) shows that *equal treatment of equals*, *sd-efficiency* and *sd-strategy-proofness* are incompatible.

To make it compatible, we restrict this domain even more in strong sense. Every agent has the same object for the most preferred object in single-peaked preference.

There are many real life examples for this economy where every agent has the same most preferred object. For example, every korean high school student wants to go to Seoul Nat'l University the most in common, and there is a single-peaked preference over universities. Unfortunately, even on this more restricted domain than the one that Kasajima (2013) considered, they are still incompatible. Moreover, if every agent has the same preference over objects except the least-preferable three objects, a result is the same.

2. Model

There is a finite set of **agents** $N = \{1, 2, 3, \ldots, |N|\}$ and finite set of distinct indivisible goods, or **objects** $O = \{o_1, o_2, o_3 \ldots, o_n\}$. and assume that $|N| = |O|$. Each agent is supposed to receive exactly one object. Each agent $i \in N$ has a complete, transitive, and strict binary relation R_i over objects. We refer to R_i as agent i's **preference relation** and each agent $i \in N$ is characterized by her preference relation R_i. Let $\mathcal{R}$ be a domain of preference, and $R = (R_i)_{i \in N}$ be a preference profile. We consider the following restriction on preferences. Let Φ^O be the set of strict orders on O. Without loss of generality, let O be ordered in such a way that

$$o_1 \prec o_2 \prec \cdots \prec o_n$$

For each $i \in N$, $R_i \in \mathcal{R}^N$ is **single-peaked** on O (with respect to $\prec$) if and only if following condition holds:

- there is $t \in \{2, \ldots, n-1\}$ such that
  $o_t \; R_i \; o_{t-1} \; R_i \cdots R_i \; o_1$ and $o_t \; R_i \; o_{t+1} \; R_i \cdots R_i \; o_n$
- $o_n \; R_i \; o_{n-1} \; R_i \cdots R_i \; o_1$,
- $o_1 \; R_i \; o_2 \; R_i \cdots R_i \; o_n$

We call the most preferred object $o_t \in O$ simply a peak. An economy where every agent has single-peaked preference is called a **single-peaked economy**. Let $\mathcal{R}^N$ be a set of all single-peaked economies.

With a restriction of preferences, a possibility of obtaining positive results should increase. But Kasajima(2013) shows a hope that the restriction of single-peaked preferences would help to get a positive results comes to nothing. So we consider even more stronger restriction on preferences that every agent has the same most preferred object in the single-peaked economies. So the peak $o_t \in O$ is the same for all agents. When the peak $o_t \in O$ is either the first or the last object in ordered line over objects, the preference relation is the same for every agent. i.e. for all $i \in N$,

- $o_1 \; R_i \; o_2 \; R_i \cdots R_i \; o_n$, or
- $o_n \; R_i \; o_{n-1} \; R_i \cdots R_i \; o_1$

In these two cases, every agent is considered as the same person since they have the same preference relations. These economies can be regarded as very extreme cases since there is only one agent with the size of n. So we consider the domain except these two extreme cases. In other words, with a common peak $o_t \in O$, we only consider the case with $t \in \{2, \ldots, n-1\}$ throughout the paper. When all agents in single-peaked economies have a common peak, we call these economies **single-peaked economies with a common peak**.

A **deterministic allocation** is a one-to-one correspondence between the set of agents, N, and the set of objects, O. Often, it is convenient to think of a deterministic assignment as a 0 - 1 matrix, with rows indexed by agents and columns indexed by objects: a 0 - 1 matrix represents a deterministic assignment if and only if it contains exactly one 1 in each row and each column. (Such matrices are called permutation matrices.) We let $\mathcal{D}$ be the set of all deterministic assignments.

A **random allocation** is a probability distribution over deterministic allocations; the corresponding convex combination of permutation matrices is a stochastic matrix, whose (i, j)th entry represents the probability with which agent i receives object j. More formally, a random allocation is a matrix $M = [M_{ik}]_{i \in N, k \in O}$ such that

- for each $i \in N$ and each $k \in O$, $M_{ik} \in \{0, 1\}$
- for each $i \in N$, $\sum_{k \in O} M_{ik} = 1$ and
- for each $k \in O$, $\sum_{i \in N} M_{ik} = 1$

Let $\mathcal{M}$ be the set of all random allocations. For each $i \in N$, her **random assignment** in $M \in \mathcal{M}$ is a vector $M_i = [M_{ik}]_{k \in O}$, i.e. the ith row of M. A **rule** is a function that associates with each problem a matrix in $\mathcal{M}$. A generic rule is denoted by φ.

We consider three requirements on rules. Let φ be an arbitrary rule. A rule φ satisfies **equal treatment of equals** if agents with identical preferences get identical allocations ($\varphi_i(R) = \varphi_j(R)$ if $R_i = R_j$).

The next definition requires specifying how an agent compares two assignments. Upon enumerating O from the best to the worst according to $R_i : \tilde{o_1}\ R_i\ \tilde{o_2}\ R_i \ldots R_i\ \tilde{o_n}$, An assignment $M_i = [M_{ik}]_{k \in O}$ **stochastically dominates** another assignment $N_i = [N_{ik}]_{k \in O}$ at R_i, which we write $M_i\ R_i^{sd}\ N_i$, if

$$\sum_{k=1}^{t} M_{i\tilde{o_k}} \geq \sum_{k=1}^{t} N_{i\tilde{o_k}} \text{ for } t = 1, \ldots, n$$

A random allocation $M \in \mathcal{M}$ stocahstically dominates another random allocation $N \in \mathcal{M}$ if $M_i\ R_i^{sd}\ N_i$ for all $i \in N$.

A rule φ satisfies **stochastic dominance efficiency** (simply, **sd-efficiency**) if a random allocation selected by the rule φ is not stochastically dominated by any other random allocation $N \in \mathcal{M}$.

A rule φ is **stochastic dominance strategyproof** (simply, **sd-strategyproof**) if for any preference profile R, agent $i \in N$ and her preference R_i, we have $\varphi_i(R)\ R_i^{sd}\ \varphi_i(R_i', R_{-i})$. In words, a rule is sd-strategyproof if a random assignment under truthtelling stochastically dominates the one under any misreported preferences.

Now we introduce two pervasively discussed rules. The first one is a **random priority** (RP) rule. In random priority, there are all possibile orderings of agents with equal probability. For each realization of the ordering, the first agent in the ordering takes her the most preferred object, and the second agent in the ordering takes her the most preferred object among the remaining objects, and so on. Random priority rule is sd-strategyproof and satisfies equal treatment of equals. But there is an efficiency loss, that is, random priority rule is not sd-efficient.

The second one is **probabilistic serial** (PS) rule introduced by Bogomolnaia and Moulin (2001) eliminates the ineffiency present in RP. In PS, each indivisible object is an infinitely divisible cake of probability share. Time runs from 0 to 1. Given reported preferences, each agent "eats" her favorite object that have not been completely eaten away at every point in time with the speed of 1. When all objects are eaten away, resulting profile of object share eaten by each agent induces a random allocation. The probabilistic serial rule satiesfies equal treatment of equals and is sd-efficient if all agents reveal their preferences truthfully. However, the PS rule does not satisfy sd-strategyproofness. In other words, an agent may receive a more desirable random assignment by misreporting her preferences.

3. Results

For three agents, *equal treatment of equals*, *sd-efficiency*, and *sd-strategy-proofness* are compatible (Bogomolnaia and Moulin, 2001). However, for four or more agents, they are not. And they are still incompatible in restricted preference domain, that is, single-peaked economies (Kasajima, 2013). This paper restrict the preference domain in even more stronger sense. We consider single-peaked economies with a common peak.

For easy understanding of this restricted domain, if $N = \{1, 2, 3, 4\}, O = \{a, b, c, d\}$, assume a common peak is b. Then available preference relations for each agent are only for 3 cases

- $b\ R_i\ a\ R_i\ c\ R_i\ d$ or
- $b\ R_i\ c\ R_i\ a\ R_i\ d$ or
- $b\ R_i\ c\ R_i\ d\ R_i\ a$

With this restriction, agents' preference over objects is different only for 3 objects $\{a, c, d\}$. If we extend to n objects, these least three preferable objects in different preferences order are significant factors for results.

In this way, suppose that every agent has a common peak $o_t \in O$ such that $t \in \{2, \ldots, n-1\}$. Among many possible preference profiles with a common peak o_t, we divide this domain into following three :

- $o_t\ R_i\ \cdots\ R_i\ o_1\ R_i\ o_{n-1}\ R_i\ o_n \quad (R')$
- $o_t\ R_i\ \cdots\ R_i\ o_{n-1}\ R_i\ o_1\ R_i\ o_n \quad (R'')$
- $o_t\ R_i\ \cdots\ R_i\ o_{n-1}\ R_i\ o_n\ R_i\ o_1 \quad (R''')$

and each preference profile is denoted by R', R'' and R'''

Theorem Domain : single-peaked economies with a common peak. Let $n \geq 4$. No rule satisfies the following three requirements : equal treatment of equals, sd-efficiency and sd-strategyproofness.

Proof.

Here, we only consider the cases that n is even. In Appendix, we provide a proof for the cases that n is odd. Suppose, by a way of contradiction, that there is a rule φ that satisfies the three requirements.

Profile 1: For all $i \in N$, preference relation is R''. Then, by equal treatment of equals, for each $i \in N$ and each $k \in O$, $\varphi_{ik}(R) = \frac{1}{n}$

Profile 1-1: For each $i \in N \setminus \{1\}$, preference relation is R''. And agent 1 has preference relation R'. Then by sd-strategyproofness

$$\varphi_1(R''_1, R_{-1}) R'^{sd}_1 \varphi_1(R) \quad and \quad \varphi_1(R) R''^{sd}_1 \varphi_1(R''_1, R_{-1}).$$

Thus, $\varphi_{1o_2}(R) = \varphi_{1o_2}(R''_1, R_{-1})$, $\ldots$, $\varphi_{1o_{n-2}}(R) = \varphi_{1o_{n-2}}(R''_1, R_{-1})$ and $\varphi_{1o_n}(R) = \varphi_{1o_n}(R''_1, R_{-1})$. Invoking our conclusion for Profile 1, $\varphi_{1o_2}(R) = \cdots = \varphi_{1o_{n-2}}(R) = \varphi_{1o_n}(R) = \frac{1}{n}$.

Next, we claim that $\varphi_{1o_{n-1}}(R) = 0$. Suppose by way of contradiction that $\varphi_{1o_{n-1}}(R) > 0$. Since $\varphi_{1o_1}(R) < \frac{2}{n}$, there is $i \in N \setminus \{1\}$ such that $\varphi_{io_1}(R) > 0$. let $\delta = \min\{\varphi_{1o_{n-1}}(R), \varphi_{io_1}(R)\}$. Let $M \in \mathcal{M}$ be such that $M_{1o_{n-1}} = \varphi_{1o_{n-1}}(R) - \delta$, $M_{1o_1} = \varphi_{1o_1}(R) + \delta$, $M_{io_{n-1}} = \varphi_{io_{n-1}} + \delta$, $M_{io_1} = \varphi_{io_1}(R) - \delta$, and other entries are the same. Then, it is easy to see that M stochastically Pareto dominates $\varphi(R)$, in violation of sd-efficiency. Thus $\varphi_{1o_{n-1}}(R) = 0$
Then by equal treatment of equals

$$\varphi(R) = \begin{pmatrix} \frac{2}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{1}{n} \\ \frac{n-2}{n(n-1)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{n-2}{n(n-1)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n} \end{pmatrix}$$

Profile 1-2: For each $i \in N \setminus \{1, 2\}$, preference relation is R''. And agent 1,2 has preference relation R'. Then by sd-strategyproofness

$$\varphi_2(R) R'^{sd}_2 \varphi_2(\text{Profile 1-1}) \quad and \quad \varphi_2(\text{Profile 1-1}) R''^{sd}_2 \varphi_2(R).$$

Thus, $\varphi_{2o_2}(R) = \cdots = \varphi_{2o_{n-2}}(R) = \varphi_{2o_n}(R) = \frac{1}{n}$ Next we claim that $\varphi_{2o_{n-1}}(R) = 0$. Suppose by way of contradiction that $\varphi_{2o_{n-1}}(R) > 0$. Then $\varphi_{2o_1}(R) < \frac{2}{n}$. By equal treatment of equals, $\varphi_{1o_1}(R) < \frac{2}{n}$. Then, there is $i \in \{3, 4, \ldots, n\}$ such that $\varphi_{io_1}(R) > 0$. As the argument in the previous profile, it is easy to construct $M \in \mathcal{M}$ such that M stochastically dominates $\varphi(R)$ at R, in violation of *sd-efficiency*. Then, by equal treatment of equals,

$$\varphi(R) = \begin{pmatrix} \frac{2}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{1}{n} \\ \frac{2}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{1}{n} \\ \frac{n-4}{n(n-2)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-2} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{n-4}{n(n-2)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-2} & \frac{1}{n} \\ \frac{n-4}{n(n-2)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-2} & \frac{1}{n} \end{pmatrix}$$

Repeat this process until exact half of agents has a preference profile R'.

Profile 1-k: Agents $i \in \{1, 2, \dots, \frac{n}{2}\}$ has R' and agents $i \in \{\frac{n}{2}+1, \dots, n\}$ has R''. By *sd-strategyproofness* $\varphi_{io_2}(R) = \dots = \varphi_{io_{n-2}}(R) = \varphi_{io_n}(R) = \frac{1}{n}$ for $i \in N$ as it was discussed above. By *sd-efficiency* $\varphi_{io_{n-1}} = 0$ for $i \in \{1, 2, \dots, \frac{n}{2}\}$. Suppose by way of contradiction that $\varphi_{io_{n-1}}(R) > 0$, let $\delta = min\{\varphi_{io_{n-1}}(R), \varphi_{no_1}(R)\}$. Let $M \in \mathcal{M}$ be such that $M_{io_{n-1}} = \varphi_{io_{n-1}}(R) - \delta$, $M_{io_1} = \varphi_{io_1}(R) + \delta$, $M_{no_{n-1}}(R) = \varphi_{no_{n-1}}(R) + \delta$, $M_{no_1} = \varphi_{no_1}(R) - \delta$. Then, M stochastically dominates $\varphi(R)$ at R. Thus, by equal treatment of equals,

$$\varphi(R) = \begin{pmatrix} \frac{2}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{2}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \end{pmatrix}$$

Profile 2-1: For each $i \in N \setminus \{n\}$, preference relation is R''. And agent n has preference relation R'''. Then by sd-strategyproofness

$$\varphi_n(R) R_n'''^{sd} \varphi_n(R_n'', R_{-n}) \quad and \quad \varphi_n(R_n'', R_{-n}) R_n''^{sd} \varphi_n(R)$$

$\varphi_{io_2}(R) = \varphi_{io_3}(R) = \dots = \varphi_{io_{n-1}}(R) = \frac{1}{n}$ for all agent $i \in N$. By *sd-efficiency*, $\varphi_{na}(R) = 0$ as the argument in the previous profile. Then by *equal treatment of equals*

$$\varphi(R) = \begin{pmatrix} \frac{1}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & \frac{n-2}{n(n-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & \frac{n-2}{n(n-1)} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & \frac{2}{n} \end{pmatrix}$$

Profile 2-2: Agent 1 has preference relation R', agent $i \in \{2, 3, \dots, n-1\}$ has R'' and agent n has R'''. Invoking our conclusion for Profile 1-1 and Profile 2-1, by *sd-strategyproofness*,

$$\varphi(R) R_1'^{sd} \varphi(\text{Profile 2-1}) \quad and \quad \varphi(\text{Profile 2-1}) R_1''^{sd} \varphi(R)$$

$$\varphi(R) R_n'''^{sd} \varphi(\text{Profile 1-1}) \quad and \quad \varphi(\text{Profile 1-1}) R_n''^{sd} \varphi(R)$$

Thus, $\varphi_{io_2}(R) = \varphi_{io_3}(R) = \dots = \varphi_{io_{n-2}}(R) = \frac{1}{n}$ for all agent $i \in N$. Also, $\varphi_{1o_n}(R) = \frac{n-2}{n(n-1)}$ and $\varphi_{no_{n-1}}(R) = \frac{1}{n-1}$. By *sd-efficiency*, $\varphi_{1o_{n-1}}(R) = 0$ and

$\varphi_{no_1}(R) = 0$. By *equal treatment of equals*,

$$\varphi(R) = \begin{pmatrix} \frac{2n-1}{n(n-1)} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{n-2}{n(n-1)} \\ \frac{n^2-3n+1}{n(n-1)(n-2)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{n^2-4n+5}{n(n-1)(n-2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{n^2-3n+1}{n(n-1)(n-2)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{n^2-4n+5}{n(n-1)(n-2)} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{2n-3}{n(n-1)} \end{pmatrix}$$

Profile 2-3: Agent 1,2 has preference relation R', agent $i \in \{3, 4, \ldots, n-1\}$ has R'' and agent n has R'''. Invoking our conclusion from Profile 2-2 and Profile 1-2,

$$\varphi_2(R) R_2'^{sd} \varphi_2(\text{Profile 2-2}) \quad \textit{and} \quad \varphi_2(\text{Profile 2-2}) R_2''^{sd} \varphi_2(R)$$

$$\varphi_n(R) R_n'''^{sd} \varphi_n(\text{Profile 1-2}) \quad \textit{and} \quad \varphi_n(\text{Profile 1-2}) R_n''^{sd} \varphi_n(R)$$

by *sd-strategyproofness*, $\varphi_{io_2}(R) = \varphi_{io_3}(R) = \cdots = \varphi_{io_{n-2}}(R) = \frac{1}{n}$ for all agent $i \in N$. Also, $\varphi_{no_{n-1}}(R) = \frac{1}{n-2}$ and $\varphi_{2o_n}(R) = \frac{n^2-4n+5}{n(n-1)(n-2)}$. By *sd-efficiency*, $\varphi_{2o_{n-1}}(R) = 0$ and $\varphi_{no_1}(R) = 0$. Then, by equal treatment of equals,

$$\varphi(R) = \begin{pmatrix} \frac{2n^2-5n+1}{n(n-1)(n-2)} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{n^2-4n+5}{n(n-1)(n-2)} \\ \frac{2n^2-5n+1}{n(n-1)(n-2)} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{n^2-4n+5}{n(n-1)(n-2)} \\ \frac{n^3-7n^2+48n-34}{n(n-1)(n-2)(n-3)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-2} & \frac{(n-2)(n^2-5n+8)}{n(n-1)(n-2)(n-3)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{n^3-7n^2+48n-34}{n(n-1)(n-2)(n-3)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-2} & \frac{(n-2)(n^2-5n+8)}{n(n-1)(n-2)(n-3)} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-2} & \frac{2n-6}{n(n-2)} \end{pmatrix}$$

Profile 2-4: Agent $1, 2, 3$ has preference relation R', agent $i \in \{4, 5, \ldots, n-1\}$ has R'' and agent n has R'''. Invoking our conclusion from Profile 2-3 and Profile 1-3

$$\varphi_3(R) R_3'^{sd} \varphi_3(\text{Profile 2-3}) \quad \textit{and} \quad \varphi_3(\text{Profile 2-3}) R_3''^{sd} \varphi_3(R)$$

$$\varphi_n(R) R_n'''^{sd} \varphi_n(\text{Profile 1-3}) \quad \textit{and} \quad \varphi_n(\text{Profile 1-3}) R_n''^{sd} \varphi_n(R)$$

By *sd-strategyproofness*, $\varphi_{3o_n}(R) = \frac{(n-2)(n^2-5n+8)}{n(n-1)(n-2)(n-3)}$ and $\varphi_{no_{n-1}}(R) = \frac{1}{n-3}$. By *sd-efficiency* $\varphi_{no_1}(R) = 0$ and $\varphi_{3o_{n-1}}(R) = 0$. Then by *equal treatment of equals*,

$$\varphi(R)=\begin{pmatrix} \frac{(n-2)(2n^2-7n+1)}{n(n-1)(n-2)(n-3)} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{(n-2)(n^2-5n+8)}{n(n-1)(n-2)(n-3)} \\ \frac{(n-2)(2n^2-7n+1)}{n(n-1)(n-2)(n-3)} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{(n-2)(n^2-5n+8)}{n(n-1)(n-2)(n-3)} \\ \frac{(n-2)(2n^2-7n+1)}{n(n-1)(n-2)(n-3)} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{(n-2)(n^2-5n+8)}{n(n-1)(n-2)(n-3)} \\ \frac{(n-2)(n^3-10n^2+24n-3)}{n(n-1)(n-2)(n-3)(n-4)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-3} & \frac{(n-2)(n-3)(n^2-6n+11)}{n(n-1)(n-2)(n-3)(n-4)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{(n-2)(n^3-10n^2+24n-3)}{n(n-1)(n-2)(n-3)(n-4)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-3} & \frac{(n-2)(n-3)(n^2-6n+11)}{n(n-1)(n-2)(n-3)(n-4)} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-3} & \frac{2n-9}{n(n-3)} \end{pmatrix}$$

From these results, we can get a sequence for $\varphi_{(n-1)o_n}(R)$ and $\varphi_{no_n}(R)$ satisfying *sd-strategyproofness*. Let k be the number of agent who have preference profile R'. Then $\varphi_{(n-1)o_n}(R)=\frac{(n-2)(n-3)...(n-k)(n^2-(k+3)n+(3k+2))}{n(n-1)(n-2)...(n-(k+1))}$ and $\varphi_{no_n}(R)=\frac{2n-3k}{n(n-k)}$.

Profile 2-k: Agent $i\in\{1,2,\ldots,\frac{n}{2}-1\}$ has R' and agent $i\in\{\frac{n}{2},\ldots,n-1\}$ has R'' and agent n has R'''. So the number of agents who have preference profile of R',R'',R''' is $(\frac{n}{2}-1)$, $\frac{n}{2}$, 1 each. Since majority have preference R'', by *sd-efficiency* $\varphi_{no_1}(R)=0$ and $\varphi_{io_{n-1}}(R)=0$ for $i\in\{1,2,\ldots,\frac{n}{2}-1\}$. By *equal treatment of equals*,

$$\varphi(R)=\begin{pmatrix} \frac{2(n^2+n+1)}{(n-1)n(n+2)} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{n^2+n-8}{(n-1)n(n+2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{2(n^2+n+1)}{(n-1)n(n+2)} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{n^2+n-8}{(n-1)n(n+2)} \\ \frac{2(2n^2-n+2)}{n^2(n-1)(n+2)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n+2} & \frac{(n-2)(n-3)...(\frac{n}{2}+1)(n^2-n-2)}{2n(n-1)...(\frac{n}{2}+1)\frac{n}{2}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{2(2n^2-n+2)}{n^2(n-1)(n+2)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n+2} & \frac{(n-2)(n-3)...(\frac{n}{2}+1)(n^2-n-2)}{2n(n-1)...(\frac{n}{2}+1)\frac{n}{2}} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n+2} & \frac{n+6}{n(n+2)} \end{pmatrix}$$

Profile 3: Every agent $i\in N$ has preference relation R'. By equal treatment of equals, $\varphi(R)=\frac{1}{n}$

Profile 3-1: Agent $i\in N\backslash\{n\}$ has R' and agent n has R''. By *sd-strategyproofness*, $\varphi_{no_2}(R)=\cdots=\varphi_{no_{n-2}}(R)=\varphi_{no_n}(R)=\frac{1}{n}$. By *sd-efficiency*, $\varphi_{no_1}(R)=0$. Then by *equal treatment of equals*,

$$\varphi(R)=\begin{pmatrix} \frac{1}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-2}{n(n-1)} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-2}{n(n-1)} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \end{pmatrix}$$

Profile 3-2: Agent $i \in N \setminus \{n\}$ has R' and agent n has R'''. Invoking our conclusion from Profile 3-1, agent n change preference relation from $o_1 \succ o_n$ to $o_n \succ o_1$ and other relations are the same. But agent n has nothing of object o_1. Thus allocation will be the same to conclusion for Profile 3-1.

$$\varphi(R) = \begin{pmatrix} \frac{1}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-2}{n(n-1)} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-2}{n(n-1)} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \end{pmatrix}$$

Profile 3-3: Agent $i \in N \setminus \{n-1, n\}$ has R', agent $n-1$ has R'' and agent n has R'''. Invoking our conclusion from Profile 3-2

$$\varphi_{n-1}(R) R''^{sd}_{n-1} \varphi_{n-1}(\text{Profile 3-2}) \quad \textit{and} \quad \varphi_{n-1}(\text{Profile 3-2}) R'^{sd}_{n-1} \varphi_{n-1}(R)$$

By *sd-strategyproofness*, $\varphi_{n-1 o_n}(R) = \frac{1}{n}$. By *sd-efficiency*, $\varphi_{n-1 o_1}(R) = 0$. Then, by *equal treatment of equals*,

$$\varphi(R) = \begin{pmatrix} \frac{1}{n-2} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-4}{n(n-2)} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{n-2} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-4}{n(n-2)} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \end{pmatrix}$$

Repeat this process until agent $i \in \{\frac{n}{2}+2, \frac{n}{2}+3, \ldots, n-1\}$ has a preference profile R''

Profile 3-k: Agent $i \in \{1, 2, \frac{n}{2}+1\}$ has R', agent $i \in \{\frac{n}{2}+2, \ldots, n-1\}$ has R'' and agent n has R'''. By *sd-strategyproofness* $\varphi_{io_n}(R) = \frac{1}{n}$ for all $i \in N$. By *sd-efficiency* $\varphi_{io_1}(R) = 0$ for $i \in \{\frac{n}{2}+2, \ldots, n\}$. Then by *equal treatment of equals*,

$$\varphi(R) = \begin{pmatrix} \frac{2}{n+1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-(\frac{n}{2}-1)}{n(n-\frac{n-1}{2})} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{2}{n+1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-(\frac{n}{2}-1)}{n(n-\frac{n-1}{2})} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \end{pmatrix}$$

Profile 4: Agent $i \in \{1, 2, \ldots, \frac{n}{2}\}$ has R', agent $i \in \{\frac{n}{2}+1, \ldots, n-1\}$ has R'' and agent n has R'''. Invoking our conclusion from Profile 1-k, Profile 2-k and 3-k, by sd-strategy-proofness

$$\varphi_{\frac{n}{2}}(R) R'^{sd}_{\frac{n}{2}} \varphi(\text{Profile 2-k}) \quad \textit{and} \quad \varphi(\text{Profile 2-k}) R''^{sd}_{\frac{n}{2}} \varphi_{\frac{n}{2}}(R)$$

$$\varphi_{\frac{n}{2}+1}(R) R''^{sd}_{\frac{n}{2}+1} \varphi(\text{Profile 3-k}) \quad and \quad \varphi(\text{Profile 3-k}) R'^{sd}_{\frac{n}{2}+1} \varphi_{\frac{n}{2}+1}(R)$$

$$\varphi_n(R) R'''^{sd}_n \varphi_n(\text{Profile 1-k}) \quad and \quad \varphi_n(\text{Profile 1-k}) R''^{sd}_n \varphi_n(R)$$

$\varphi_{no_{n-1}}(R) = \frac{2}{n}$. $\varphi_{\frac{n}{2}o_n}(R) = \frac{(n-2)(n-3)...(\frac{n}{2}+1)(n^2-n-2)}{2n(n-1)...(\frac{n}{2}+1)\frac{n}{2}} = \frac{n^2-n-2}{n^2(n-1)}$ and $\varphi_{(\frac{n}{2}+1)o_n}(R) = \frac{1}{n}$. By *sd-efficiency*, $\varphi_{no_1}(R) = 0$. So assignment of object o_n for all agents will be

$$\underbrace{\frac{n^2-n-2}{n^2(n-1)}}_{\text{assignment for } i \in \{1,...,\frac{n}{2}\}} \cdot \frac{n}{2} + \underbrace{\frac{1}{n}}_{i \in \{\frac{n}{2}+1,...,n-1\}} \cdot (\frac{n}{2} - 1) + \frac{1}{n} < 1$$

which means $\varphi(R) \notin \mathcal{M}$ ■

The result of Kasajima (2013) strengthens the results by Bogomolnaia and Moulin (2001) since the impossibility holds on a smaller domain of preferences. And our result strengthens the results of both Kasajima (2013) and Bogomolnaia and Moulin (2001) by restricting the domain in even more stronger sense than the one Kasajima considered.

Remark 1 In two extreme economies where a common peak is either the first or the last object, every agent is considered as the same agent. The rule $\varphi_{ik}(R) = \frac{1}{n}$ for all $i \in N, k \in O$ satisfies the three requirements. In these two economies, the impossibility result does not hold.

Remark 2 Even if we impose a restriction, not only a common peak but also the same preference from the peak to $(n-3)$th object, the impossibility result still holds.

Remark 3 Whenever there are at least three possible preference profile, the impossibility result holds. For example, assume there is an economy where $n-2$ agents have R', 1 agent has R'' and 1 has R'''. Even though a majority of agents have the same preference R', because of two different preference profiles, the impossibility result holds.

4. Conclusion

We showed that *sd-efficiency*, *sd-strategyproofness* and *equal treatment of equals* are incompatible in a single-peaked economies with a common peak. The result contributes to the literature on the assignment problem to find preference domain to make these axioms compatible. Bogomolnaia and Moulin (2002) consider that all agents has same preference over objects while receiving no object(opting out) choice is only difference for agents. In this case considered, three axioms are compatible. It may be interesting to find the exact domain for these axioms to be compatible for future research.

Appendix : A formal proof for odd n

Consider the case which is extended to n (odd) object cases. A common peak is o_t for $t \in \{2, 3, \dots n-1\}$, then possible preference relations are as follows:

- $o_t\ R_i\ \dots\ R_i\ o_1\ R_i\ o_{n-1}\ R_i\ o_n\ (R')$
- $o_k\ R_i\ \dots\ R_i\ o_{n-1}\ R_i\ o_1\ R_i\ o_n\ (R'')$
- $o_k\ R_i\ \dots\ R_i\ o_{n-1}\ R_i\ o_n\ R_i\ o_1\ (R''')$

Suppose, by a way of contradiction, that there is a rule φ that satisfies the three requirements.

Profile 1: All agents have R'. Then by *equal treatment of equals*, $\varphi_{ik}(R) = \frac{1}{n}$ for all $i \in N$ and $k \in O$.

Profile 1-1: Agents $i \in N \backslash \{n\}$ have R' and agent n has R''. By *sd-strategyproofness*,

$$\varphi_n(R) R_n'^{sd} \varphi_n(\text{Profile 1}) \quad \textit{and} \quad \varphi_n(\text{Profile 1}) R_n'^{sd} \varphi_n(R)$$

Then $\varphi_{no_n}(R) = \frac{1}{n}$. By *sd-efficiency*, $\varphi_{no_1}(R) = 0$. Then by *equal treatment of equals*,

$$\varphi(R) = \begin{pmatrix} \frac{1}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-2}{n(n-1)} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-2}{n(n-1)} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \end{pmatrix}$$

Profile 1-2: Agents $i \in N \setminus \{n\}$ have R', and agent n has R'''. Agent n has just changed her preference from $o_1 R_n o_n$ to $o_n R_n o_1$. But she already have 0 of object o_1. Thus a random assignment for agent n will be the same as the previous profile.

$$\varphi(R) = \begin{pmatrix} \frac{1}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-2}{n(n-1)} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-2}{n(n-1)} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \end{pmatrix}$$

Profile 1-3: Agents $i \in N \setminus \{n-1, n\}$ have R'. and agent n-1,n have R''. Then by *sd-strategyproofness*,

$$\varphi_{n-1}(R) R_{n-1}''^{sd} \varphi_{n-1}(\text{Profile 1-1}) \quad \textit{and} \quad \varphi_{n-1}(\text{Profile 1-1}) R_{n-1}'^{sd} \varphi_{n-1}(R)$$

Then $\varphi_{(n-1)o_n}(R) = \frac{1}{n}$. By *sd-efficiency*, $\varphi_{(n-1)o_1}(R) = 0$. By *equal treatment of* *equals*,

$$\varphi(R)=\begin{pmatrix}\frac{1}{n-2} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-4}{n(n-2)} & \frac{1}{n}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ \frac{1}{n-2} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-4}{n(n-2)} & \frac{1}{n}\\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n}\\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n}\end{pmatrix}$$

In Profile 1, Since majority agents have R', to satisfy *sd-efficiency*, $\varphi_{io_1}(R)=0$ for $i\in N\setminus\{j\in N|\text{agent } j \text{ who has preference } R'\}$. And we repeat this process until we get Profile 1-k.

Profile 1-k: Agents $i\in\{1,2,\frac{n+1}{2}\}$ have R' and agent $i\in\{\frac{n+1}{2}+1,\dots,n\}$ have R'''. By *sd-strategyproofness*, $\varphi_{io_n}(R)=\frac{1}{n}$ for all $i\in N$. By *sd-efficiency*, $\varphi_{io_1}(R)=0$ for $i\in\{\frac{n+1}{2}+1,\dots,n\}$. Then by *equal treatment of equals*,

$$\varphi(R)=\begin{pmatrix}\frac{2}{n+1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n(n+1)} & \frac{1}{n}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ \frac{2}{n+1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n(n+1)} & \frac{1}{n}\\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n}\end{pmatrix}$$

Profile 2: All agents $i\in N$ have R'''. Then by *equal treatment of equals*, $\varphi_{ik}(R)=\frac{1}{n}$ for all $i\in N$ and $k\in O$.

Profile 2-1: Agents $i\in N\setminus\{1\}$ have R''', and agent 1 has R''. By *sd-strategyproofness*, $\varphi_{1o_{n-1}}(R)=\frac{1}{n}$. By *sd-efficiency*, $\varphi_{1o_n}(R)=0$. By *equal treatment of equals*,

$$\varphi(R)=\begin{pmatrix}\frac{2}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & 0\\ \frac{n-2}{n(n-1)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & \frac{1}{n-1}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ \frac{n-2}{n(n-1)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & \frac{1}{n-1}\end{pmatrix}$$

Profile 2-2: Agents $i\in N\setminus\{1\}$ have R''' and agent 1 has R''. By *sd-strategyproofness*, $\varphi_{1o_n}(R)=0$. By *sd-efficiency*, $\varphi_{1o_{n-1}}(R)=0$. Then by *equal treatment of equals*,

$$\varphi(R)=\begin{pmatrix}\frac{3}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & 0\\ \frac{n-3}{n(n-1)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-1}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ \frac{n-3}{n(n-1)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-1}\end{pmatrix}$$

Profile 2-3: Agents $i\in N\setminus\{1,2\}$ have R''' and agent 1,2 have R''. By *sd-strategyproofness*, $\varphi_{2o_{n-1}}(R)=\frac{1}{n}$. By *sd-efficiency*, $\varphi_{2o_n}(R)=0$. Then by *equal* *treatment of equals*,

$$\varphi(R) = \begin{pmatrix} \frac{2}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & 0 \\ \frac{2}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & 0 \\ \frac{n-4}{n(n-2)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & \frac{1}{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{n-4}{n(n-2)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & \frac{1}{n-2} \end{pmatrix}$$

Profile 2-4: Agents $i \in N \setminus \{1,2\}$ have R''', agent 2 has R'' and agent 1 has R'. By *sd-strategyproofness*, $\varphi_{2o_{n-1}}(R) = \frac{1}{n-1}$, $\varphi_{1o_n}(R) = 0$. By *sd-efficiency*, $\varphi_{2o_n}(R) = 0$. Then by *equal treatment of equals*,

$$\varphi(R) = \begin{pmatrix} \frac{3}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & 0 \\ \frac{2}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & 0 \\ \frac{n-5}{n(n-2)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{n-5}{n(n-2)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} \end{pmatrix}$$

Since n is an arbitrary large number, there exists s (smaller than n) such that $\frac{3}{n} \cdot s > 1$ and $\frac{3}{n} \cdot (s-1) + \frac{2}{n} < 1$. Let s be the smallest integer satisfying these two requirements.

Profile 2-x: Agent $i \in N \setminus \{1,2,\ldots,s\}$ has R''' and agent $i \in \{1,2,\ldots,s-1\}$ has R' and agent s has R''. Then by *equal treatment of equals*,

$$\varphi(R) = \begin{pmatrix} \frac{3}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{3}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & 0 \\ \frac{2n-3s+3}{n(n-s+1)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-(s-1)} & 0 \\ \frac{n^2-4ns+2n+3s^2-3s}{n(n-s)(n-s+1)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-(s-1)} & \frac{1}{n-s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{n^2-4ns+2n+3s^2-3s}{n(n-s)(n-s+1)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-(s-1)} & \frac{1}{n-s} \end{pmatrix}$$

Profile 2-x$'$: Agent $i \in N \setminus \{1,2,\ldots,s\}$ has R''' and agent $i \in \{1,2,\ldots,s\}$ has R'. By *sd-strategyproofness*,

$$\varphi_s(R) R_s^{'sd} \varphi_s(\text{Profile 2-s}) \quad \textit{and} \quad \varphi_s(\text{Profile 2-x}) R_s^{''sd} \varphi_s(R)$$

Then $\varphi_{so_n}(R) = 0$. Since $\frac{3}{n} \cdot s > 1$, $\varphi_{so_{n-1}}(R) \neq 0$. Thus $\varphi_{io_1}(R) = 0$ for

$i \in \{s+1, s+2, \dots, n\}$ to satisfy *sd-efficiency*. By *equal treatment of equals*,

$$\varphi(R) = \begin{pmatrix} \frac{1}{s} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{3s-n}{n\cdot s} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{s} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{3s-n}{n\cdot s} & 0 \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2n-3s}{n(n-s)} & \frac{1}{n-s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2n-3s}{n(n-s)} & \frac{1}{n-s} \end{pmatrix}$$

From this point, agents who have preference profile R^3 have zero probability of o_1

Profile 2-k: Agent $i \in \{1, 2, \dots, \frac{n-1}{2}\}$ has R' and agent $i \in \{\frac{n+1}{2}, \dots, n\}$ has R'''. As the argument in the previous profile, $\varphi_{io_1}(R) = 0$ for $i \in \{\frac{n+1}{2}, \dots, n\}$ by *sd-efficiency*. By *equal treatment of equals*,

$$\varphi(R) = \begin{pmatrix} \frac{2}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-3}{n(n-1)} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{2}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-3}{n(n-1)} & 0 \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n+3}{n(n+1)} & \frac{2}{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n+3}{n(n+1)} & \frac{2}{n+1} \end{pmatrix}$$

Profile 3: Agent $i \in \{1, 2, \dots, \frac{n-1}{2}\}$ has R' and agent $\frac{n+1}{2}$ has R'' and agent $i \in \{\frac{n+3}{2}, \dots, n\}$ has R'''. By *sd-strategyproofness*,

$$\varphi_{\frac{n+1}{2}}(R) R''^{sd}_{\frac{n+1}{2}} \varphi_{\frac{n+1}{2}}(\text{Profile 1-k}) \quad \textit{and} \quad \varphi_{\frac{n+1}{2}}(\text{Profile 1-k}) R'^{sd}_{\frac{n+1}{2}} \varphi_{\frac{n+1}{2}}(R)$$

$$\varphi_{\frac{n+1}{2}}(R) R''^{sd}_{\frac{n+1}{2}} \varphi_{\frac{n+1}{2}}(\text{Profile 2-k}) \quad \textit{and} \quad \varphi_{\frac{n+1}{2}}(\text{Profile 2-k}) R'''^{sd}_{\frac{n+1}{2}} \varphi_{\frac{n+1}{2}}(R)$$

Then, $\varphi_{\frac{n+1}{2}o_{n-1}}(R) = \frac{n+3}{n(n+1)}$ and $\varphi_{\frac{n+1}{2}o_n}(R) = \frac{1}{n}$. And by *sd-efficiency*, $\varphi_{io_1}(R) = 0$ for $i \in \{\frac{n+1}{2}+1, \dots, n\}$ and $\varphi_{io_{n-1}}(R) = 0$ for $i \in \{1, 2, \dots, \frac{n-1}{2}\}$.

$$\varphi(R) = \begin{pmatrix} \frac{2n^2+2}{n(n-1)(n+1)} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{n^2-5}{n(n-1)(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{2n^2+2}{n(n-1)(n+1)} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{n^2-5}{n(n-1)(n+1)} \\ \frac{n-1}{n(n-1)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n+3}{n(n+1)} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2(n^2-3)}{n(n-1)(n+1)} & \frac{n^2+3}{n(n-1)(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2(n^2-3)}{n(n-1)(n+1)} & \frac{n^2+3}{n(n-1)(n+1)} \end{pmatrix}$$

Profile 3-1: Agent $i \in \{1, 2, \dots, \frac{n-1}{2}\}$ has R' and agents $\frac{n+1}{2}, \frac{n+3}{2}$ has R'' and agent $i \in \{\frac{n+1}{2}+2, \dots, n\}$ has R'''. By *sd-strategyproofness*

$$\varphi_{\frac{n+3}{2}}(R) R''^{sd}_{\frac{n+3}{2}} \varphi_{\frac{n+3}{2}}(\text{Profile 3}) \quad \textit{and} \quad \varphi_{\frac{n+3}{2}}(\text{Profile 3}) R'''^{sd}_{\frac{n+3}{2}} \varphi_{\frac{n+3}{2}}(R)$$

$$\varphi_{\frac{n+1}{2}}(R) R''^{sd}_{\frac{n+1}{2}} \varphi_{\frac{n+1}{2}}(\text{Profile 1-k}) \quad \textit{and} \quad \varphi_{\frac{n+1}{2}}(\text{Profile 1-k}) R'^{sd}_{\frac{n+1}{2}} \varphi_{\frac{n+1}{2}}(R)$$

Then $\varphi_{\frac{n+3}{2}o_{n-1}}(R) = \frac{2(n^2-3)}{n(n-1)(n+1)}$, and $\varphi_{\frac{n+1}{2}o_n}(R) = \frac{1}{n}$. Also, by *sd-efficiency*, $\varphi_{io_{n-1}}(R) = 0$ for $i \in \{1, 2, \ldots, \frac{n-1}{2}\}$ and $\varphi_{io_1}(R) = 0$ for $i \in \{\frac{n+1}{2} + 2, \ldots, n\}$. By *equal treatment of equals*,

$$\varphi(R) = \begin{pmatrix} \frac{(2((n-1)n(n+1)-8))}{n(n-1)^2(n+1)} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{n^3-3n^2-n+19}{n(n-1)^2(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{(2((n-1)n(n+1)-8))}{n(n-1)^2(n+1)} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{n^3-3n^2-n+19}{n(n-1)^2(n+1)} \\ \frac{4}{n(n-1)(n+1)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2(n^2-3)}{n(n-1)(n+1)} & \frac{1}{n} \\ \frac{4}{n(n-1)(n+1)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2(n^2-3)}{n(n-1)(n+1)} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{!}{n} & \frac{2(n^2-n-4)}{n(n-1)(n+1)} & \frac{n^2+2n+5}{n(n-1)(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2(n^2-n-4)}{n(n-1)(n+1)} & \frac{n^2+2n+5}{n(n-1)(n+1)} \end{pmatrix}$$

Profile 3-2: Agent $i \in \{1, 2, \ldots, \frac{n-1}{2}\}$ has R' and agents $\{\frac{n+1}{2}, \frac{n+3}{2}, \frac{n+5}{2}\}$ has R'' and agent $i \in \{\frac{n+1}{2} + 3, \ldots, n\}$ has R'''. By the argument above,

$$\varphi(R) = \begin{pmatrix} \frac{2(n^3-7n-18)}{(n-1)^2 n(n+1)} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{n^3-3n^2+11n+39}{2(n-1)n(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{2(n^3-7n-18)}{(n-1)^2 n(n+1)} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{n^3-3n^2+11n+39}{2(n-1)n(n+1)} \\ \frac{2(n+3)}{(n-1)n(n+1)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2(n^2-n-4)}{n(n-1)(n+1)} & \frac{1}{n} \\ \frac{2(n+3)}{(n-1)n(n+1)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2(n^2-n-4)}{n(n-1)(n+1)} & \frac{1}{n} \\ \frac{2(n+3)}{(n-1)n(n+1)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2(n^2-n-4)}{n(n-1)(n+1)} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2\{(n-5)(n-1)n+24\}}{n(n-1)(n+1)(n-5)} & \frac{n^3-3n^2-13n-33}{(n-5)(n-1)n(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2\{(n-5)(n-1)n+24\}}{n(n-1)(n+1)(n-5)} & \frac{n^3-3n^2-13n-33}{(n-5)(n-1)n(n+1)} \end{pmatrix}$$

From this conclusion, we can get a sequence for $\varphi_{no_{n-1}}(R)$. Let k be a number of agents who have preference profile R''.
Then $\varphi_{no_{n-1}}(R) = \frac{2\{(n-(2k-1))(n-(2k-3))...(n-5)(n-1)n-(-1)^k 2^{(k-1)k!}\}}{n(n-1)(n+1)(n-5)(n-7)...(n-(2k-1))}$

Profile 3-k: Agent $i \in \{1, 2, \ldots, \frac{n-1}{2}\}$ has R' and agents $\{\frac{n+1}{2}, \ldots, n-2\}$ has R'' and agents $\{n-1, n\}$ has R'''.

$$
\varphi(R)=\begin{pmatrix}
 & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
 & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \\
 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2\{(n-(n-6))(n-(n-8))...(n-5)(n-1)n-2^{\frac{n-7}{2}\cdot\frac{n-5}{2}!}\}}{n(n-1)(n+1)(n-5)(n-7)...(n-(n-6))} & \frac{1}{n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2\{(n-(n-6))(n-(n-8))...(n-5)(n-1)n-2^{\frac{n-7}{2}\cdot\frac{n-5}{2}!}\}}{n(n-1)(n+1)(n-5)(n-7)...(n-(n-6))} & \frac{1}{n} \\
 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2\{(n-(n-4))(n-(n-6))...(n-5)(n-1)n-2^{\frac{n-5}{2}\cdots\frac{n-3}{2}!}\}}{n(n-1)(n+1)(n-5)(n-7)...(n-(n-4))} & \\
 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2\{(n-(n-4))(n-(n-6))...(n-5)(n-1)n-2^{\frac{n-5}{2}\cdots\frac{n-3}{2}!}\}}{n(n-1)(n+1)(n-5)(n-7)...(n-(n-4))} &
\end{pmatrix}
$$

Profile 4: All agents $i \in N$ have R''. By *equal treatment of equals*, $\varphi_{ik}(R) = \frac{1}{n}$ for all $i \in N$ and $k \in O$.

Profile 4-1: Agents $i \in N\backslash\{1\}$ have R'' and agent 1 has R'. By *sd-strategyproofness*, $\varphi_{1o_n}(R) = \frac{1}{n}$. By *sd-efficiency*, $\varphi_{1o_{n-1}}(R) = 0$. Then by *equal treatment of equals*,

$$
\varphi(R)=\begin{pmatrix}
\frac{2}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{1}{n} \\
\frac{n-2}{n(n-1)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\frac{n-2}{n(n-1)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n}
\end{pmatrix}
$$

Profile 4-k: Agents $i \in \{1, \ldots, \frac{n-1}{2}\}$ have R' and agent $i \in \{\frac{n+1}{2}, \ldots, n\}$ have R''. Then by *equal treatment of equals*,

$$
\varphi(R)=\begin{pmatrix}
\frac{2}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{1}{n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\frac{2}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{1}{n} \\
\frac{2}{n(n+1)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n+1} & \frac{1}{n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\frac{2}{n(n+1)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n+1} & \frac{1}{n}
\end{pmatrix}
$$

Profile 5: Agents $i \in \{1, \ldots, \frac{n-1}{2}\}$ have R', agents $i \in \{\frac{n+1}{2}, \ldots, n-1\}$ have R'' and agent n has R'''. By *sd-strategyproofness*,

$$
\varphi_{\frac{n+1}{2}}(R)R''^{sd}_{\frac{n+1}{2}}\varphi_{\frac{n+1}{2}}(\text{Profile 1-k}) \quad \textit{and} \quad \varphi_{\frac{n+1}{2}}(\text{Profile 1-k})R'^{sd}_{\frac{n+1}{2}}\varphi_{\frac{n+1}{2}}(R)
$$

$$
\varphi_n(R)R'''^{sd}_n\varphi_n(\text{Profile 4-k}) \quad \textit{and} \quad \varphi_n(\text{Profile 4-k})R''^{sd}_n\varphi_n(R)
$$

$$
\varphi_{n-1}(R)R'''^{sd}_{n-1}\varphi_{n-1}(\text{Profile 3-k}) \quad \textit{and} \quad \varphi_{n-1}(\text{Profile 3-k})R''^{sd}_{n-1}\varphi_{n-1}(R)
$$

To satisfy *sd-efficiency*, $\varphi_{io_{n-1}}(R) = 0$ for $i \in \{1, 2, \ldots, \frac{n-1}{2}\}$. Then,

$$\underbrace{\frac{2\{(n-(n-4))(n-(n-6))\ldots(n-5)(n-1)n - 2^{\frac{n-5}{2}\cdots\frac{n-3}{2}!}\}}{n(n-1)(n+1)(n-5)(n-7)\ldots(n-(n-4))}}_{\text{agent n's assignment of } o_{n-1}}$$

$$+ \underbrace{\frac{2\{(n-(n-4))(n-(n-6))\ldots(n-5)(n-1)n - 2^{\frac{n-5}{2}\cdots\frac{n-3}{2}!}\}}{n(n-1)(n+1)(n-5)(n-7)\ldots(n-(n-4))}}_{\text{assignment of } o_{n-1} \text{ for } i \in \{\frac{n+1}{2}, \ldots, n-1\}} \cdot \frac{n-1}{2} + 0 \neq 1$$

leads to a contradiction to $\varphi(R) \in \mathcal{M}$ ■

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