이학박사 학위논문

A study on Lefschetz fibration structures of symplectic 4-manifolds

(4차원 사교다양체의 레프셰츠 파이브레이션 구조에 대한 연구)

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A study on Lefschetz fibration structures of symplectic 4-manifolds

A dissertation
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Abstract

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Since it has been known due to S. Donaldson and R. Gompf that a closed 4-manifold admits a symplectic structure if and only if it admits a Lefschetz fibration structure possibly after blowing-ups of the manifold, a study on Lefschetz fibrations is one of the central research themes in symplectic 4-manifolds topology. In this thesis, we study Lefschetz fibration structures on a family of symplectic 4-manifolds. A Lefschetz fibration structure on a smooth 4-manifold is a map from the manifold to a complex curve whose fibers are Riemann surfaces: some of them are possibly singular.

The first part of thesis deals with a relation between Lefschetz fibration structures and diffeomorphism types of a family of knot surgery 4-manifolds. In particular, we investigate the isomorphism classes of Lefschetz fibration structures on knot surgery 4-manifolds with the same Seiberg-Witten invariant using a representation of the corresponding monodromy group. In the second part of thesis, we provide an explicit algorithm for Lefschetz fibration structures on any minimal symplectic filling of the link of quotient surface singularities and we show that they are related by rational blow-downs.

Key words: Knot surgery, Lefschetz fibration, quotient surface singularity, rational blow-down, Stein filling, symplectic filling.
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Contents

Chapter 1

Introduction

A symplectic manifold is an even-dimensional smooth manifold equipped with a closed non-degenerate 2-form ω which is locally $(\mathbb{R}^{2n}, \sum_{i=1}^{n} dp_i \wedge dq_i)$. After a theorem of S. Donaldson [12] that every symplectic 4-manifold admits a Lefschetz pencil structure and a theorem of R. Gompf [30] that a Lefschetz fibration on a smooth 4-manifold whose fibers are nonzero in homology gives a symplectic structure, a study on Lefschetz fibrations has been one of the main research themes in 4-dimensional symplectic topology. In fact, Lefschetz fibrations and Lefschetz pencils have long been studied extensively by algebraic geometers and topologists in the complex category, and these notions have been extended to the symplectic category.

Definition 1.0.1. Suppose that X and Σ are oriented smooth 4- and 2-manifolds respectively. A proper smooth map $f : X \to \Sigma$ is called *a Lefschetz fibration* if it satisfies the following conditions:

- The set of critical points $C = \{p_1, p_2, \cdots, p_n\}$ lies in the interior of X.
- $f : X \setminus C \to \Sigma$ is a fiber bundle.
- For each p_i, there is a local complex coordinate chart on which f is of the form $z_1^2 + z_2^2$

Furthermore, since the isomorphism class of a Lefschetz fibration structure is determined by its monodromy factorization, it gives a combinatorial approach to study the topology of symplectic 4-manifolds. On the other hand, another way to understand topology of symplectic manifolds is to study techniques for constructing symplectic manifolds. To find such techniques, topologists want to know how to apply topological cut-and-paste techniques in the symplectic category. For example, R. Gompf [28] showed that a fiber sum technique can be done in the symplectic category, and M. Symington [54] showed that a rational blow-down surgery under a certain condition can be also performed in the symplectic category.

The search for symplectic cut-and-paste techniques has led us to the study of symplectic 4-manifolds with boundary. In general, a symplectic structure on 4-manifold X induces a compatible contact structure on its boundary, say a 3-manifold Y, and in this case we call X a symplectic filling of Y. By the fundamental theorem of E. Giroux [27], a contact structure on a given 3-manifold can be described by its open book decomposition. Topologically an open book decomposition is a surface bundle over the circle on the complement of a fibered link in the 3-manifold. Hence, if the monodromy of an open book decomposition on a given 3-manifold Y is a product of positive Dehn twists, there exists a symplectic 4-manifold X which admits a Lefschetz fibration structure whose generic fiber is a surface with the non-empty boundary and its boundary $\partial X = Y$ is given by the open book decomposition. Furthermore, such a symplectic 4-manifold X admitting a Lefschetz fibration structure is one of symplectic fillings of Y together with a compatible contact structure induced from the open book decomposition.

Note that, unlikely the closed case, not every symplectic filling admits a Lefschetz fibration structure, but it does if there is an additional complex structure on it. In the case, we call it a Stein filling of a given contact 3-manifold. For example, the link of an isolated complex surface singularity carries a canonical Stein fillable contact structure. Although the existence of Lefschetz fibration structures on Stein fillings are known, it is a somewhat different problem to find an explicit monodromy description for it. For example, in the cyclic quotient surface singularity cases, M. Bhupal and B. Ozbagci [5] provided an algorithm to present each minimal symplectic filling as an explicit Lefschetz fibration, and then using their construction, they showed that every minimal symplectic filling of cyclic quotient surface singularities can be obtained by rational blow-downs from the minimal resolution of its singularity. Note that rational blow-downs can be also described in terms of monodromy substitutions. Explicitly they showed that the mondromy factorization of each minimal symplectic filling is obtained by monodromy substitutions of the monodromy factorization of its minimal resolution.

In this thesis, we study Lefschetz fibration structures on minimal symplectic fillings of the link of non-cyclic quotient surface singularity cases. Similar to the cyclic cases, there is a well-known genus-0 Lefschetz fibration structure on the minimal resolution of non-cyclic quotient surface singularities except for the bad vertex cases. Note that there is a bad vertex on the dual graph of its minimal resolution. For the bad vertex cases, we could also find a genus-1 Lefschetz fibration structure on the minimal resolution and monodromy substitutions to any minimal symplectic fillings. In summary, we prove that there is an algorithm for a genus-0 or genus-1 Lefschetz fibration structure on each minimal symplectic filling of the link of non-cyclic quotient surface singularities. Furthermore, each such a filling can be also obtained by rational blow-downs from the minimal resolution of its singularity. Explicitly, we get the following.

Theorem 1.0.2. *Every minimal symplectic filling of the link of non-cyclic quotient surface singularities admits a genus-*0 *or genus-*1 *Lefschetz fibration structure over the disk. Furthermore, each such a filling can be obtained by rational blow-downs from the minimal resolution of its singularity.*

Another part of this thesis discusses the symplectic version of Fintushel-Stern's conjecture: For a certain family of fibered knots K and K' in S^3, knot surgery 4-manifolds X_K and $X_{K'}$ are isomorphic as Lefschetz fibrations if and only if K and K' are equivalent knots. Note that, using Gompf's symplectic fiber sum, one can perform knot surgery along K in the symplectic category as long as K is a fibered knot in S^3. For the $E(n)_{K_{p,q}}$ case, one can also obtain explicit Lefschetz fibration structures on $E(n)_{K_{p,q}}$ from a genus $(n-1)$ fibration structure of $E(n)$. Regarding this conjecture, J. Park and K-H. Yun ([49], [50]) studied it by using a family of Kanenobu knots

$\{K_{p,q} \mid -p \leq q \leq p, (p,q) \neq (0,0)\}$ which are inequivalent fibered knots with the same Alexander polynomial, and they showed that there are at least four non-isomorphic classes of Lefschetz fibration structures on knot surgery 4-manifolds $E(2)_{K_{p,q}}$. They distinguished the isomorphism classes by investigating the corresponding monodromy groups. By monodromy group we mean the group generated by right-handed Dehn twists in the monodromy factorization. Indeed, by using a representation of the monodromy group, one can construct a bilinear form invariant of Lefschetz fibration structures. A quandle is an algebraic structure equipped with a binary operation motivated from Reidemeister moves. There is a invariant of knots and links in S^3, which is called state-sum invariant, defined by sum of quandle 2-cocycle over all crossings and colorings for a given diagram with a finite quandle. The axioms of quandles guarantee that the sum is well defined, i.e., it is invariant under Reidemeister moves. Using a Alexander quandle, a quandle obtained from a group and a representation, one can construct a bilinear form invariant of knots and links instead of finite quandles. Note that each monodromy factorization of a given Lefschetz fibration over S^2 can be interpreted as a quandle coloring on a certain torus link. Recall that the isomorphism classes of Lefschetz fibrations are determined by its monodromy factorization up to Hurwitz equivalences and global conjugations. Hence, by interpreting Hurwitz moves as Reidemeister moves, one can construct a bilinear form invariant of Lefshcetz fibrations for each representation of the monodromy group. Using the bilinear form invariant from the canonical symplectic representation, we are able to recover the J. Park and K-H. Yun's result for $E(2)_{K_{p,q}}$.

This thesis is organized as follows. We first review several definitions and properties of Lefschetz fibrations in Chapter 2, and the theory of quandles for the bilinear form invariants of Lefschetz fibrations is presented in Chapter 3. In Chapter 4, we investigate the isomorphism classes of Lefschetz fibration structures on knot surgery 4-manifolds using the invariant from the quandle theory. In Chapter 5, we review the basic notions of symplectic fillings of contact structures in terms of open book decompositions. Especially we focus on the correspondence between Stein fillings and Lefschetz fibrations. Finally, we recall the classification of minimal symplectic fillings of the link of quotient surface singularities and we discuss Lefschetz fibration structures on them in Chapter 6. We provide an explicit algorithm to obtain a Lefschetz fibration structure on each minimal symplectic filling. Furthermore, we show that they are related by rational blow-downs in the final chapter.

Chapter 2

Preliminary on Lefschetz fibrations

It is well known that a closed symplectic 4-manifold admits a purely topological description in terms of Lefschetz fibrations and Lefschetz pencils. In this chapter we recall the necessary definitions and sketch these topological descriptions of symplectic 4-manifolds. For more details see [30] and [43].

2.1 Topology of Lefschetz fibrations

Definition 2.1.1. (i) A *Lefschetz fibration* on a 4-manifold X is a map $\pi : X \to \Sigma$ which has finitely many critical points $\{t_1, \ldots, t_n\}$ with distinct critical values $\{\pi(t_1), \ldots, \pi(t_n)\} \subset int(\Sigma)$, and near each critical point we can choose a complex coordinates (z_1, z_2) such that in the coordinates π is of the form $z_1^2 + z_2^2$.

(ii) A *Lefschetz pencil* on a 4-manifold X is a set of nonempty finite points $B \subset X$, called the base locus, together with a map $\pi : X \setminus B \to \mathbb{CP}^1$ such that each point $b \in B$ has a coordinates on which π can be given by the projectivization $\mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1$ and near critical points π behaves as in (i).

For an oriented 4-manifold X, we also require the complex charts near critical points preserve the orientation of X. From now on we always assume

X is oriented. We can turn a Lefschetz pencil into a Lefschetz fibration over S^2 by blowing-up on each points in the base locus. If there is no singular fiber (or no critical value) a Lefschetz fibration is a fiber bundle with a connected base. If a generic fiber is connected, its genus is called the genus of the Lefschetz pencil or fibration.

Definition 2.1.2. Two Lefschetz fibrations $\pi : X \to \Sigma$ and $\pi' : X' \to \Sigma'$ are *isomorphic* if there are diffeomorphism $\Phi : X \to X'$ and $\phi : \Sigma \to \Sigma'$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "\pi"'] & X' \arrow[d, "\pi'"] \\ \Sigma \arrow[r, "\phi"'] & \Sigma' \end{tikzcd}$$

Now we would like to describe a topology of Lefschetz fibrations in terms of their handlebody structures. First recall some basic facts about handles and handlebody structures.

Definition 2.1.3. Let X be a smooth n-dimensional manifold with non-empty boundary. For $0 \leq k \leq n$, an *n-dimensional k-handle* h is $D^k \times D^{n-k}$, attached to X by an embedding,

$$\varphi : \partial D^k \times D^{n-k} \to \partial X.$$

Since there is a canonical way to smoothing the corners, $X \cup_\varphi h$ is also a smooth manifold. Note that the diffeomorphism type of $X \cup_\varphi h$ is determined by isotopy class of φ. By the tubular neighborhood theorem, the isotopy class of φ is determined from the following datas:

(i) A smooth isotopy class of an embedding $\varphi_0 : \partial D^k \to \partial X$.

(ii) An identification of the normal bundle of $\varphi_0(\partial D^k)$ with $\partial D^k \times D^{n-k}$.

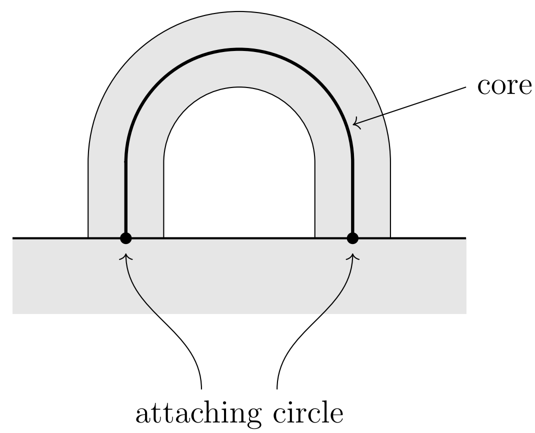


Figure 2.1: A 2-dimensional 1-handle

We call $D^k \times 0$ the *core* of the handle, φ the *attaching map* and $\varphi_0(\partial D^k)$ the *attaching circle* or *attaching sphere*. For the second data, the isomorphism class of a normal bundle can be identified with an element f in the group $\pi_{k-1}(O(n-k))$ and we call f the framing of h.

Example 2.1.4. For the 4-dimensional 2-handle, attaching sphere K is S^1 in M^3. We can specify the framing by choosing a nowhere-vanishing normal vector field to K. The framing of a 2-handle is classified by $\pi_1(O(2))$ which is isomorphic to $\mathbb{Z}$, but there is no canonical identification between the framings and integers until we decide which framing should correspond to 0. Once we pick a 0-framing, we can consider other framings as a curve in D^2 so it can be measured by the winding number the curve around the origin.

Definition 2.1.5. By a *handlebody structure* of X, we mean a decomposition of X into a union of handles. To be precise, a handlebody structure is given as a sequence of submanifolds $X_0 \subset X_1 \subset \cdots \subset X_n = X$ such that $X_i = X_{i-1} \cup i$-handles.

Let $f : X \to \mathbb{R}$ be a Morse function, and $c \in \mathbb{R}$ be a critical value of f of index k with the unique critical point $x = f^{-1}(c)$. Then for sufficiently small $\epsilon > 0$, $X_{c+\epsilon}$ is obtained by attaching a k-handle to $X_{c-\epsilon}$, where

$X_a = \{x \in X \mid f(x) \le a\}$.

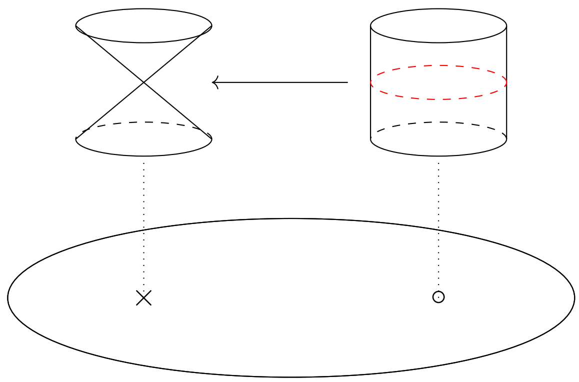


Figure 2.2: Fibers near a Lefschetz singularity

Next we describe the topology of a Lefschetz fibration near its critical points in terms of their handlebody structures. Let U be a neighborhood of a critical point. Then U admit complex coordinate chart (z_1, z_2) such that $\pi(z_1, z_2) = z_1^2 + z_2^2$ on U and the critical point is at $(0, 0)$. Writing $z_j = x_j + iy_j$, we have

$$\pi(x_1 + iy_1, x_2 + iy_2) = (x_1^2 + x_2^2) - (y_1^2 + y_2^2) + 2(x_1y_1 + x_2y_2)i.$$

From this explicit form we see $\pi^{-1}(t > 0)$ is a neighborhood of the circle $C_t = \{x_1^2 + x_2^2 = t, y_1 = y_2 = 0\}$ in a regular fiber F. As t approaches to 0, the circle shrinks to a point. The circle C_t is called a *vanishing cycle*. The collection of C_t with the origin is a disk D_t and is called a *Lefschetz thimble*. From the above observation we see the singular fiber is obtained by shrinking a vanishing cycle from a regular fiber. By taking $g = -\mathrm{Re}(\pi)$ as a Morse function near the origin, the neighborhood νF_0 of singular fiber is obtained by attaching a 4-dimensional 2-handle to the neighborhood νF_t of regular fiber along the vanishing cycle. Suppose that $\partial \nu F_t$ contains a fiber F_s with $0 < s < t$. Then the core of the attaching 2-handle is given by D_s and the attaching circle is the vanishing cycle C_s. Now we describe the framing of this handle attachment by comparing it to the framing coming from a normal vector field in the fiber F_s. At each point of $(\sqrt{s}\cos\theta, \sqrt{s}\sin\theta, 0, 0) \in C_s$, the curve

$$X(t) = (\sqrt{s-t^2}\cos\theta, \sqrt{s-t^2}\cos\theta, -it\sin\theta, it\cos\theta)$$

gives the normal vector $(0, 0, -i\sin\theta, i\cos\theta)$ of C_s in F_s. On the other hand, the framing given by D_s is can be represented by a vector field $(0, 0, 0, i)$. Note that the winding number of $(-i\sin\theta, i\cos\theta)$ is one. It means the framing induced by F_s has one right-handed twist relative to the framing induced by D_S. That is, the 2-handle is attached to the vanishing cycle with framing -1 relative to the framing induced by the surface F_s.

2.2 Monodromy of Lefschetz fibrations

Definition 2.2.1. For a fiber bundle $\pi : E \to B$, consider a loop $l : [0, 1] \to B$ with a base point $b = l(0) = l(1)$ and identify the fiber F with a $\pi^{-1}(b)$ by a diffeomorphism φ. Since the pull back of E, $l^*(E) \to [0, 1]$ is trivial, there is a map $f : [0, 1] \times F \to E$ such that

(i) $\pi(f(t, x)) = l(t)$ for any $(t, x) \in [0, 1] \times F$,

(ii) The map $f_t : F \to F_t(= \pi^{-1}(l(t)))$ defined by $f_t(x) = f(t, x)$ is a diffeomorphism, and

(iii) $f_0 = \varphi$.

Hence for each loop l we get a self diffeomorphism $\varphi \circ f_1^{-1}$ of F. Let $\mathcal{M}_F$ denotes the set of isotopy classes of self-diffeomorphisms of F. Note that the isotopy class $[\varphi \circ f_1^{-1}]$ in $\mathcal{M}_F$ depends on φ and the homotopy type of l. In this case we call $[\varphi \circ f_1^{-1}]$ the *monodromy* associated to l and φ. The induced map from $\pi_1(B) \to \mathcal{M}_F$ is called the *monodromy representation*.

To determine the monodromy around a critical value, we need the following definition.

Definition 2.2.2. Let C be a simple closed curve in an oriented surface F. Then νC, the neighborhood of C in F, is diffeomorphic to $S^1 \times I$ and can be parametrized by (θ, t). Define $\psi : F \to F$ so that $\psi(\theta, t) = (\theta + 2\pi t, t)$ on νC and identity on the outside of νC. We call ψ a right-handed *Dehn twist* of F along C.

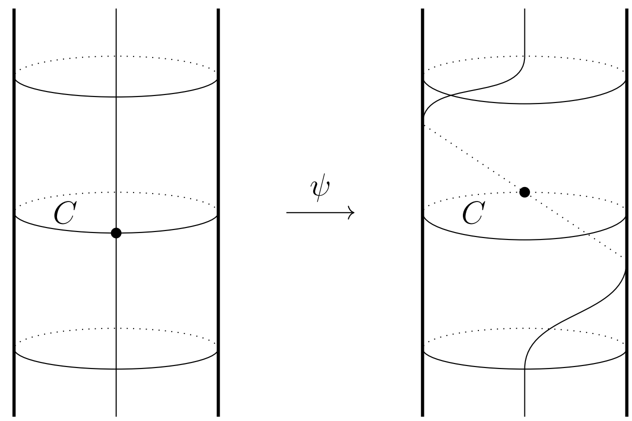


Figure 2.3: Right-handed Dehn twist

For a given Lefschetz fibration $\pi : X \to \Sigma$ with the genric fiber F, let D be a disk in Σ such that there is a unique critical value $p \in D$. A local computation shows the following.

Proposition 2.2.3 ([34]). *The monodromy along ∂D is a right-handed dehn twist of F along a simple closed curve C. Furthermore, C is isotopic to the vanishing cycle of the critical point.*

Now we analyze a Lefschetz fibration over D^2 with n critical points $t_1, \ldots, t_n$. Fix a base point $b_0 \in int(D^2) \setminus \{\text{critical values}\}$, and choose an identification φ of the regular fiber $\pi^{-1}(b_0)$ with F and arcs s_i connecting b_0 and critical value $b_i = \pi(t_i)$. We may assume the index i of critical values b_i are increasing as we travel counterclockwise around b as depicted in Figure 2.4. Let V_i denote a collection of small disjoint open disks with $b_i \in V_i$. Since a Lefschetz fibration is a locally trivial bundle away from the critical points, we have $\pi^{-1}(V_0) \cong F \times D^2$ with $\partial(\pi^{-1}(V_0)) \cong F \times S^1$. Let $\nu(s_i)$ be a regular neighborhood of the arc s_i. By the previous argument we know that $\pi^{-1}(V_0 \cup \nu(s_1) \cup V_1)$ is diffeomorphic to the manifold obtained by attaching

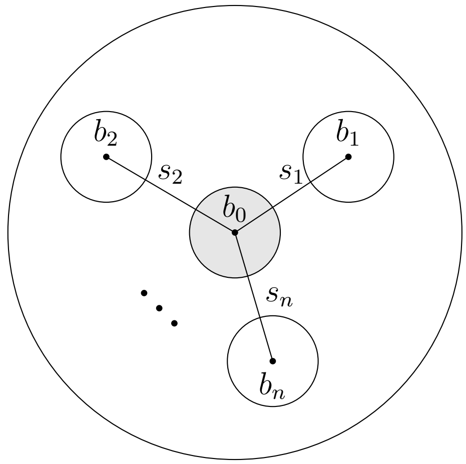


Figure 2.4: A Lefschetz fibration over D^2

a 2-handle H_1 along the vanishing cycle c_1 in $F\times\{\text{point}\}$ to the $F \times D^2$. In addition, $\partial((D^2 \times F) \cup H_1)$ is an F-bundle over S^1 whose monodromy is the right-handed Dehn twist along c_1. Continuing the procedures counterclockwise around b_0, we get

$$X \cong \pi^{-1}(V_0 \cup (\cup_{i=1}^{n} \nu(s_i)) \cup (\cup_{i=1}^{n} V_i).$$

and ∂X is an F-bundle over S^1 whose monodromy is the product of right-handed Dehn twists along vanishing cycles. We will refer to this product $W = t_{c_n} \circ \cdots \circ t_{c_1}$ as the *global monodromy* or *monodromy factoriziation* of the fibration. There are two natural equivalence relations on monodromy factorizations.

(1) Global conjugation : $t_{c_n} \circ \cdots \circ t_{c_1} \sim \phi t_{c_n} \phi^{-1} \circ \cdots \circ \phi t_1 \phi^{-1}$ for any $\phi \in \mathcal{M}_F$.

(2) Hurwitz move: $t_{c_n} \circ \cdots \circ t_{c_1} \sim t_{c_n} \circ \cdots \circ t_{c_i} \circ t_{c_i}^{-1} t_{c_{i+1}} t_{c_i} \circ \cdots \circ t_1$.

A global conjugation corresponds to the change of an identification φ of F, while a Hurwitz move corresponds to the change of the order of arcs $\{s_i\}$ as in Figure 2.5.

It is clear that if the monodromy factorizations of two Lefschetz fibrations

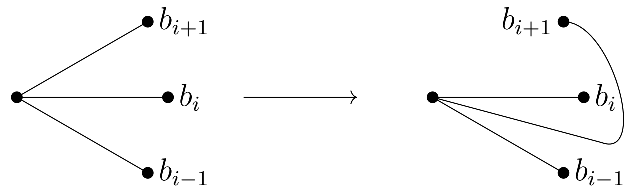


Figure 2.5: The Hurwitz move

W and W' are related by a sequence of the global conjugations and the Hurwitz moves, W and W' are isomorphic Lefschetz fibrations. Furthermore,

Theorem 2.2.4 ([39])**.** *Let $\pi_i : X_i \to D^2, i = 1, 2,$ be two Lefschetz fibrations. Then the two Lefschetz fibrations are isomorphic if and only if their monodromy factorizations are related by a finite sequence of Hurwitz moves and simultaneous conjugation equivalences.*

Example 2.2.5. It is well known that the mapping class group of the tours T^2 is isomorphic to $SL(2;\mathbb{Z})$. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2;\mathbb{Z})$ and $B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \in SL(2;\mathbb{Z})$. Note that each of them can be identified with the right-handed Dehn twist along curve a and b respectively. Since $(B \cdot A)$ has order 6, the Lefschetz fibration X_n over D^2 with global monodromy $(B \cdot A)^{6n}$ extends to a Lefschetz fibration over S^2. In fact, B. Moishezon [40] showed that any Lefschetz fibration over S^2 with torus fiber is isomorphic to X_n for some n.

2.3 Symplectic 4-manifolds and Lefschetz pencils and fibrations

Definition 2.3.1. A 2-form ω on a smooth n-manifold X is a *symplectic form* if ω is closed and nondegenerate. The pair (X, ω) is called a *symplectic manifold.*

Here, the nondegeneracy has its usual meaning in the context of bilinear forms: For any nonzero $v \in T_xX$ there is a vector $w \in T_xX$ such that $\omega(v, w) \neq 0$. An equivalent condition is that the top exterior power ω^n of ω should be nowhere vanishing, i.e., ω^n is a volume form on X. (This indicates why X must have even dimension.) The volume form ω^n determines an orientation on X; we will always use this orientation when considering X as an oriented manifold. The existence of a symplectic structure governs the topology of X. One of the most important result of 4-dimensional symplectic topology is the following S. Donaldson's groundbreaking theorem:

Theorem 2.3.2 ([12])**.** *If (X, ω) is a closed symplectic 4-manifolds and $[\omega] \in H^2(X; \mathbb{R})$ is integral then X admit a Lefschetz fibration structure such that generic fiber is a smooth symplectic submanifold, after blowing-up.*

A converse of the Donaldson's theorem is proved by Gompf.

Theorem 2.3.3 (R. Gompf, [30])**.** *If a smooth, closed 4-manifold X admit a Lefschetz fibration structure such that the homology class of a fiber is nonzero in $H_2(X; \mathbb{R})$ then X admits a symplectic structure with the fibers being symplectic submanifolds.*

The assumption in the above theorem about the homology class of the fiber is not very restrictive. If the fiber genus is greater than 2 or there is at least one singular fiber, then it is fulfilled. By blowing up the base locus, we can turn a Lefschetz pencil into a Lefschetz fibration with sections. Here a section means a map $\sigma : \Sigma \to X$ such that $\pi(\sigma(x)) = x$, so the homology class of the fiber is nonzero. Hence, by Gompf's theorem, there is a symplectic structure on $X \sharp n \overline{\mathbb{CP}}^2$ for which the exceptional spheres are symplectic. By symplectically blowing-down the exceptional spheres, we get the following corollary.

Corollary 2.3.4. *If a 4-manifold X admits a Lefschetz pencil structure, then it admits a symplectic structure.*

Chapter 3

Quandles and monodromy of Lefschetz fibrations

A quandle is a set with binary operation whose definition was motivated by knot theory. In this chapter, we give the definition of quandle, coloring and a bilinear form invariant of Lefschetz fibrations. For more details, see [7].

3.1 Quandles and colorings

Definition 3.1.1. A *quandle*, X is a set with a binary operation $\triangleleft$: $X \times X \to X$ satisfying the followings.

(i) $x \triangleleft x = x$ for any $x \in X$.

(ii) $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$.

(iii) There exist the unique $w \in X$ satisfying $w \triangleleft y = x$.

Example 3.1.2. (Conjugation quandle) A typical example of a quandle is a group $X = G$ with conjugation as the binary operation: $x \triangleleft y = y^{-1}xy$. One can easily check that $(X, \triangleleft)$ satisfying the three axioms of the quandle.

Example 3.1.3. Let M be a right G-module and define $(X, \triangleleft)$ as the following

(i) $X = M \times G$

(ii) For (m_1, g_1) and (m_2, g_2),

$$(m_1, g_1) \lhd (m_2, g_2) = ((m_1 - m_2) \cdot g_2 + m_2, g_2^{-1} g_1 g_2)$$

We call such quandles G-family of Alexander quandles.

Next we define the homology and cohomology for quandles. Let $C_n(X)$ be the free abelian group generated by n-tuples $(x_1, \ldots, x_n)$ of elements of a quandle X. Define a boundary operation $\partial_n : C_n(X) \to C_{n-1}(X)$ by

$$\begin{aligned}(x_1, \ldots, x_n) \to \sum_{i=2}^{n} (-1)^i [&(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \\ &-(x_1 \lhd x_i, \ldots, x_{i-1} \lhd x_i, x_{i+1}, \ldots, x_n)]\end{aligned}$$

for $n \geq 2$ and $\partial_n = 0$ for $n \leq 1$. Then $C_*(X) = (C_n(X), \partial_n)$ is a chain complex. For an abelian group A, define the chain and cochain complexes

$$\begin{aligned} C_*(X; A) &= C_*(X) \otimes A, \partial = \partial \otimes id \\ C^*(X; A) &= Hom(C_*(X), A), \delta(\sigma) = \sigma \circ \partial \text{ for } \sigma \in Hom(\partial, id) \end{aligned}$$

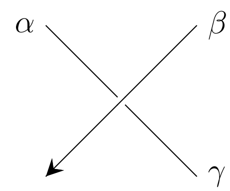
in the usual way.

Example 3.1.4. The cocycle conditions are related to moves on a knot diagram as indicated Figure 3.2. A 2-cocycle φ satisfies the relation:

$$\varphi(x, z) + \varphi(x \lhd z, y \lhd z) = \varphi(x, y) + \varphi(x \lhd y, z).$$

Definition 3.1.5. A *coloring* on an oriented knot digram K is a map $\mathcal{C} : \mathcal{A} \to X$ where X is a quandle and $\mathcal{A}$ is the set of arcs of the diagram, satisfying the relation depicted in Figure 3.1.

Let $Col_X(K)$ denote the set of all X-colorings of K. Note that if K' is a knot digram obtained from K by Reidemeister moves, there is a natural bijection between $Col_X(K)$ and $Col_X(K')$. Figure 3.2 shows the correspondence between Reidemeister type III and colorings of knot diagrams. For the conjugation quandle, each element of $Col_X(K)$ determines a representation



$$\mathcal{C}(\alpha) \lhd \mathcal{C}(\beta) = \mathcal{C}(\gamma)$$

Figure 3.1: Rule at crossings

of $\pi_1(S^3 \setminus K)$ into G since the rule at the crossings corresponds to the relation of the Wirtinger presentation. Together with colorings and quandle 2-cocycle ϕ, we can define the invariants of oriented knots in the case of X is a finite quandle.

Definition 3.1.6. The *partition function*, or *state-sum* of a knot diagram is

$$\sum_{c \in Col_X(K)} \sum_{\tau} (-1)^{\epsilon_\tau} \varphi(\alpha_\tau, \beta_\tau)$$

where τ is all the crossings in the diagram and ϵ_τ is as in Figure 3.5.

There is a one-to-one correspondence between colorings before and after each Reidemeister move. One can check that the state-sum remains unchanged under Reidemeister moves for each coloring. A quandle 2-cocycle condition makes the state-sum unchanged under type I and III moves, while ϵ_τ makes the state-sum unchanged under the type II move. In the following sections, by relating monodromy factorization to the colorings of certain links, we obtain a invariant of Lefschetz fibrations.

3.2 Monodromy of Lefschetz fibrations and colorings on torus link

Now we deal with a Lefschetz fibration whose fiber is a genus g closed surface Σ. In the following, $\mathcal{M}_g$ will denote the mapping class group of Σ. For a given Lefschetz fibration X^4 over S^2 with a monodromy factorization $W =$

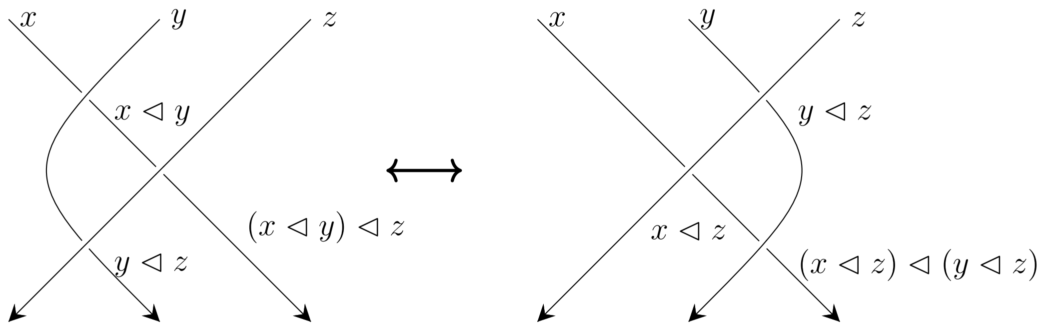


Figure 3.2: Reidemeister move type III and quandle colorings

$w_1 \cdot w_2 \cdot \cdots \cdot w_n = id \in \mathcal{M}_g$, we have a $\mathcal{M}_g$-coloring W of the torus (n, n) link as in Figure 3.3. Here the torus (n, n) links are given by a closure of the full twist of n-braids. Note that Hurwitz moves can be obtained by interchanging ith and $(i+1)$th strands.

3.3 Bilinear form invariants of Lefschetz fibrations

In this section, we construct bilinear form invariants of Lefschetz fibrations using certain quandle 2-cocycle. In [41], T. Nosaka construct a bilinear form invariant with coloring on torus (n, n) link $T_{(n,n)}$. By modifying it slightly we get a generalization of T. Nosaka's bilinear form which is invariant under Hurwitz moves.

For a right $\mathcal{M}_g$ module M and the Alexander quandle $X = M \times \mathcal{M}_g$, we define the lifting of W as follows.

$$Col_{X_W}(T_{n,n}) := \{c \in Col_X(T_{n,n}) \mid pr_2 \circ c = W\}$$

Then, $Col_{X_W}(T_{n,n})$ is given by kernel of a linear map $\Gamma_W : M^n \to M$. So we will consider $Col_{X_W}(T_{n,n})$ as a submodule of M^n later. Furthermore, if W and W' are Hurwitz equivalent, then natural bijection between $Col_{X_W}(T_{n,n})$ and $Col_{X'_W}(T_{n,n})$ is $\mathbb{Z}$-isomorphism. Now we construct bilinear map $Q_{W,\psi,l}$:

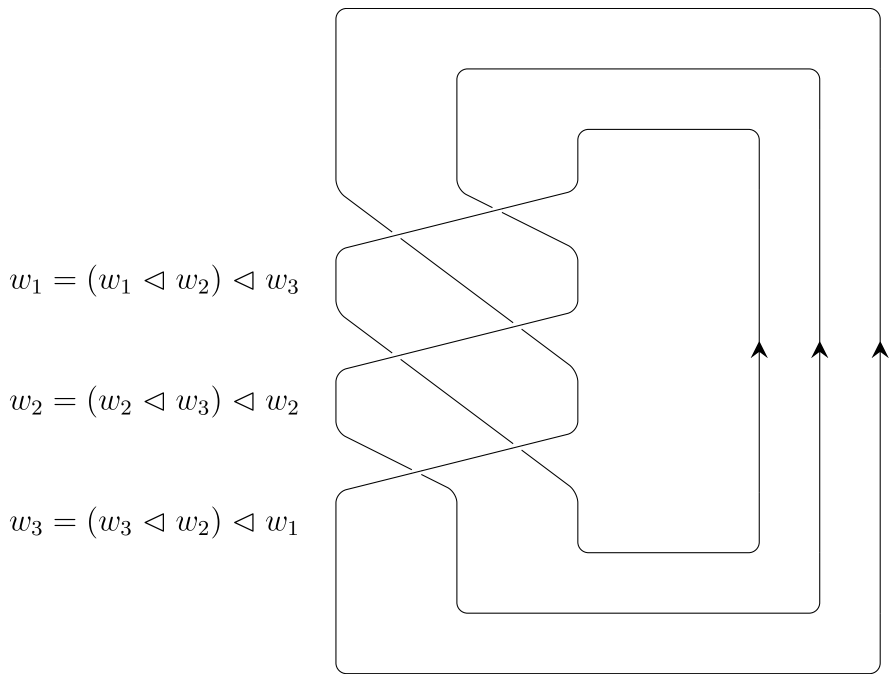


Figure 3.3: $T_{(3,3)}$ as a closure of 3-braids and a coloring W from $w_1 \cdot w_2 \cdot w_3 = id$

$Col_{X_W}(T_{n,n}) \times Col_{X_W}(T_{n,n}) \to \mathbb{Z}$ which is invariant under Reidemeister move so that invariant under Hurwitz moves. Let $\psi : M^2 \to \mathbb{Z}$ be a $\mathcal{M}_g$ invariant bilinear map. Define $\phi_\psi : X^2 \to \mathbb{Z}$ and $D : X^2 \to X$ as the following.

$$\phi_\psi((m_1, w_1), (m_2, w_2)) = \psi(m_1, m_2 \cdot (1 - w_2^{-1}))$$

$$D((m_1, w_1), (m_2, w_2)) = ((m_1 - m_2), w_2)$$

Now we define the bilinear form invariants of Lefschetz fibrations. Note that $\phi_\psi(D((m_1, w_1), (m_2, w_2)), (m_2, w_2))$ satisfies the quandle 2-cocycle condition.

Definition 3.3.1. Let $K = K_1 \cup K_2 \cup \cdots \cup K_n$ be components of $T_{n,n}$. $Q_{W,\psi,l}$ is a bilinear map from $Col_{X_W}(T_{n,n}) \times Col_{X_W}(T_{n,n})$ to $\mathbb{Z}$ of the form

$$Q_{W,\psi,l}(c, c')) = \sum_{\tau} \epsilon_\tau \cdot \phi_\psi(D(c(\alpha_\tau), (\beta_\tau)), c'(\beta_\tau))$$

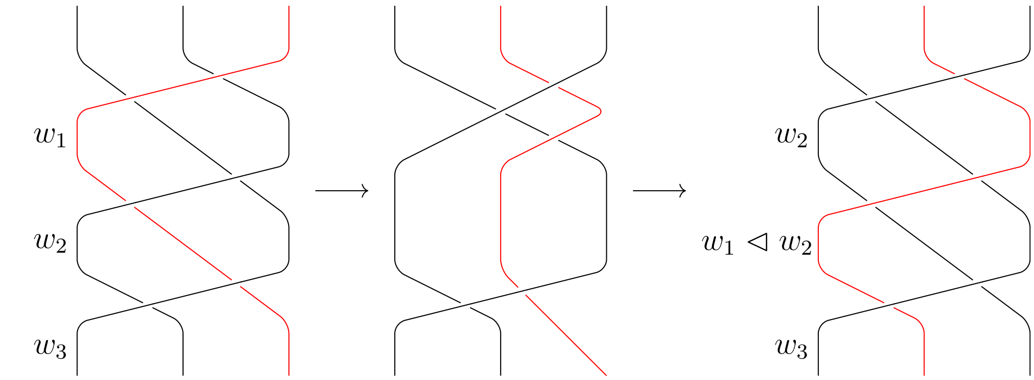


Figure 3.4: Hurwitz move and colorings on $T_{3,3}$

where τ runs over all the crossings such that under arc is from component K_l and ϵ_τ is the sign of τ.

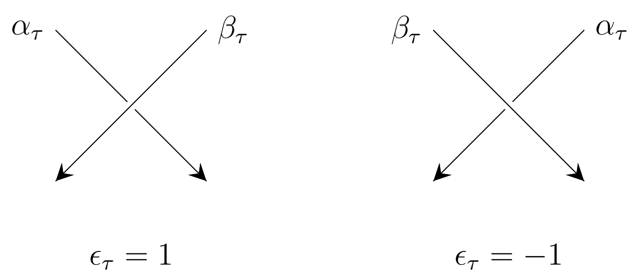


Figure 3.5: ϵ_τ at each crossings

Suppose W' is obtained by W by a single Hurwitz move then natural $\mathbb{Z}$-isomorphism $\mathcal{B}_{W,W'}$ is given by $(m_1, \dots, m_n) \to (m_1, \dots, m_{i-1}, m_{i+1}, m_{i+1} + (m_i - m_{i+1}) \cdot w_{i+1}, m_{i+2}, \dots, m_n)$ and one can check that $\mathcal{B}_{W,W'}$ is also $Q_{W,\psi,l}$ isomorphism. Since ψ is $\mathcal{M}_g$ invariant, it is also invariant under global conjugation. Therefore, $Q_{W,\psi,l}$ itself is indeed an invariants of a Lefschetz fibrations over S^2. In [41], T. Nosaka showed some properties of bilinear form $Q_{W,\psi,l}$.

Proposition 3.3.2. *(I) If ψ is skew-symmetric then the bilinear form*

$Q_{W,\psi,l}$ is symmetric. Furthermore, if ψ is symmetric then $Q_{W,\psi,l}$ is skew-symmetric.

(II) For any l and l', $Q_{W,\psi,l}$ and $Q_{W,\psi,l'}$ are same as the map.

Since $Q_{W,\psi,l}$ does not depend on l, we later only discuss $Q_{W,\psi,1}$ and denote it by $Q_{W,\psi}$. We further review a reduction of $Q_{W,\psi}$ which implies that a large part of $Col_{X_W}(T_{n,n})$ is contained in the kernel of $Q_{W,\psi}$

Proposition 3.3.3. *Consider elements $\mathbf{m} \in M^n$ of the form*

$$(m, \dots, m) \text{ and } (m_1, \dots, m_n)$$

for some $m \in M$ and $m_i \in Ker(1 - w_i) \subset M$. Then $\mathbf{m}$ lies in the $Col_{X_W}(T_{n,n})$, and further lies in the kernel of $Q_{W,\psi}$.

According to the above proposition, we shall regard $Q_{W,\psi}$ as a bilinear form on the quotient module $Col_{X_W}(T_{n,n})/(Diag(M) + (\bigoplus_{1 \leq i \leq n} Ker(1-w_i)))$.

For a given $\mathcal{M}_g$ coloring on $T_{n,n}$, we can trivially get a coloring on $T_{n+1,n+1}$ by $(w_1, \dots, w_n) \to (w_0 = id, w_1, \dots, w_n)$. Although there is canonical isomorphism between $(Col_{X_W}(T_{n,n}), Q_{W,\psi})$ and $(Col_{X_W}(T_{n+1,n+1}), Q_{W,\psi})$, there is advantage that we can extend $(Col_{X_W}(T_{n+1,n+1}), Q_{W,\psi})$ to bigger space. In defining $Q_{W,\psi}$, we regard $c \in Col_{X_W}(T_{n+1,n+1})$ as an element of M^{n+1}. So we can consider natural extension of $Q_{W,\psi}$ to a bilinear form on M^{n+1}. Note that the summands in the definition of $Q_{W,\psi}$ preserve the quantity under Reidemeister type III move. Since Hurwitz move is obtained by Reidemeister type III move (see Figure 3.4), $\mathcal{B}_{W,W'}$ still gives a isomorphism of $(M^{n+1}, Q_{W,\psi})$. While $\bigoplus_{0 \leq i \leq n} Ker(1 - w_i)$ still lies in the kernel of $Q_{W,\psi}$, $Diag(M)$ does not lie in the kernel anymore. Also $Q_{W,\psi}$ is not symmetric or skew-symmetric in general even if ψ is skew-symmetric or symmetric. To avoid an abuse of notation, we will use the notation $(M^{n+1}, B_{W,\psi})$ and $(Col_{X_W}(T_{n+1,n+1}), Q_{W,\psi})$ from now on.

3.3.1 Preliminaries to compute the bilinear forms

In the next chapter, we will investigate the $(M^{n+1}, B_{W,\psi})$ of knot surgery 4-manifold $E(2)_K$ obtained from the standard symplectic representation. Namely, $M = H_1(\Sigma_g;\mathbb{Z}) = \mathbb{Z}^{2g}$ and we set ψ as the symplectic form of the surface Σ_g.

Notation We simply denote $Col_{X_W}(T_{n+1,n+1})/(Diag(M)+(\bigoplus_{0\le i\le n} Ker(1-w_i)))$ as Col_W and $M^{n+1}/(\bigoplus_{0\le i\le n} Ker(1-w_i))$ as $\underline{M}_W$. Since we will use the symplectic form ω for ψ, we write $B_{W,\psi}$ as B_W and $Q_{W,\psi}$ as Q_W.

Proposition 3.3.4. *For a Lefschetz fibration X^4 over S^2 with monodromy factorization $W = w_1 \cdot w_2 \cdot \cdots \cdot w_n = id \in \mathcal{M}_g$, the rank of Col_W is $n_{ns} - 4g + 2b_1(X)$ and the Q_W is a unimodular symmetric bilinear form with $\sigma(Q_W) = \sigma(X) - n_s$. Furthermore, if the intersection form of the X is even, then Q_W is also even. Here, n_{ns} is the number of singular points whose vanishing cycle is a non separating curve in Σ_g and n_s is the number of singular points whose vanishing cycle is a separating curve in Σ_g.*

Proof. First we find a block decomposition of $H_2(X;\mathbb{Z})$: Since $[F]$ is a primitive element of $H_2(X;\mathbb{Z})$, there is an element $[X]$ satisfying $[F]\cdot[X] = 1$ so that $H_2(X;\mathbb{Z}) \cong \langle [F],[X]\rangle \oplus \langle [F],[X]\rangle^{\perp}$. Hence, it suffices to show the following claim.

Claim 3.3.5. *$(\langle [F],[X]\rangle^{\perp}) \subset H_2(X;\mathbb{Z})$ is isomorphic to $Q_W \oplus (-1)^{n_s}$.*

Consider the vanishing cycles as curves in surface, the generator of $M/ker(1-w_i)$ can be represented by a curve m_i which intersect the vanishing cycle c_i at a unique point. For the separating case, we simply take 0 since $ker(1-w_i) = M$. Since $\psi = \omega$ counts a algebraic intersections of given curves, we may choose a orientation on $\{c_i, m_i\}$ satisfying $\omega(c_i, m_i) = [c_i]\cdot[m_i] = 1$. Then $\mathcal{B} = \{(0,\ldots,0,[c_i],0,\ldots,0)\}$ gives the basis of $\underline{M}_W$ and for this basis $B_W = B^{-1} \oplus \langle 0\rangle^{n_s}$ after suitable reindexing where B is as the following. Since the separating vanishing cycles correspond to summand $(-1)^{n_s}$ in the intersection form of X, we may assume $n + \mathrm{s} = 0$.

$$B = \begin{bmatrix} -1 & [c_1]\cdot[c_2] & \cdots & \cdots & [c_1]\cdot[c_{n_{\mathrm{ns}}}] \\ & -1 & [c_2]\cdot[c_3] & \cdots & [c_2]\cdot[c_{n_{\mathrm{ns}}}] \\ & & \ddots & \cdots & \vdots \\ & & & \ddots & \vdots \\ & & & & -1 \end{bmatrix}$$

Since separating vanishing cycles correspond to summand $(-1)^{n_{\mathrm{s}}}$ in the intersection form of X, we may assume $n_{\mathrm{s}} = 0$ from now on. The Mayer-Vietoris's sequence for the $H_2(X;\mathbb{Z})$ tells us that each $\mathbf{r} \in R = \{(r_1, \ldots, r_n) \in \mathbb{Z}^n \mid \sum_{i=1}^{n} r_i \cdot [c_i] = 0 \in H_1(\Sigma_g;\mathbb{Z})\}$ gives a homology class $[\mathbf{r}]$ of X and there intersection is given by B (i.e. $[\mathbf{r_1}]\cdot[\mathbf{r_2}] = \mathbf{r_1}^\intercal \cdot A \cdot \mathbf{r_2} = \mathbf{r_2}^\intercal \cdot A^\intercal \cdot \mathbf{r_1}$). Also we can check that the linear map L_B which is determined by $\mathcal{B}$ and the matrix $B^\intercal$, gives a isomorphism between $ker(\Gamma_W)$ and R. We conclude that claim follows from these two observations. $\square$

Chapter 4

Lefschetz fibration structures on knot surgery 4-manifolds

4.1 Knot surgery 4-manifolds

We say two smooth 4-manifolds X and Y are exotic pair if X and Y are not diffomorphic but homeomorphic. One can use surgery techniques to find exotic structures of given manifolds, because the effects of surgery on homology, homotopy groups or other interesting invariants of the manifold(e.g. Seiberg-Witten invariant, Donaldson invariant) are known.

Knot surgery, introduced by R. Fintushel and R. Stern [21], is one of the most effective one of such surgery techniques.

Definition 4.1.1. Suppose that X is a smooth 4-manifold containing an embedded torus T of square 0. Then, for any knot $K \subset S^3$, one can construct a new smooth 4-manifold, called *a knot surgery 4-manifold*,

$$X_K = X \sharp_{T=T_m} (S^1 \times M_K)$$

by taking a fiber sum along a torus T in X and $T_m = S^1 \times m$ in $S^1 \times M_K$, where M_K is a 3-manifold obtained by doing 0-framed surgery along K and

m is the meridian of K. Equivalently, X_K is defined by

$$X_K = (X \setminus (T^2 \times D^2)) \cup (S^1 \times (S^3 \setminus \nu(K))$$

The two pieces are glued together in such a way that the homology class $[\mathrm{pt} \times \partial D^2]$ is identified with $[\mathrm{pt} \times \lambda]$ where λ is the class of a longitude of K.

Suppose that $\pi_1(X) = 1 = \pi_1(X \setminus T)$. Then, since the class of m normally generates $\pi_1(M_K)$ so the fundamental group of $M_K \times S^1$ is normally generated by the image of $\pi_1(T)$ under gluing map. Hence, by Van Kampen's theorem, X_K is also simply connected. Since the knot complement $S^3 \setminus \nu(K)$ is homology $S^1 \times D^2$, it follows that $(S^3 \setminus \nu(K)) \times S^1$ is homology $T^2 \times D^2$. Thus, the intersection form of X_K is isomorphic to that of X.

Theorem 4.1.2 ([21]). *Suppose that X is a smooth 4-manifold with $b_2^+ > 1$ which contains a homological essential torus T and $\pi_1(X) = 1 = \pi_1(X \setminus T)$. Then X_K is homeomorphic to X and*

$$\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t)$$

where $t = \exp(2[T])$ and Δ_K is the symmetrized Alexander polynomial of K.

Note that there is a infinite family of non isotopic knots with same Alexander polynomial. So it is natural to ask

Does Δ_K determine smooth type of X_K?

Fintushel and Stern initially conjectured [20] for the simply connected elliptic surface $E(2)$ that the classification of all knot surgery 4-manifolds of the form $E(2)_K$ up to diffeomorphism is the same as the classification of all knots in S^3 up to knot equivalence. Although some partial progresses related to the conjecture were obtained by S. Akbulut [2] and M. Akaho [1], the full conjecture is still remained open.

Conjecture 4.1.3. *(Fintushel-Stern conjecture) The smooth classification of knot surgery 4-manifolds is equivalent to the classification of prime knots in S^3 up to mirror image.*

On the other hand, R. Gompf [28] showed that the fiber sum manifold admits a symplectic structure under suitable conditions.

Theorem 4.1.4. *(Symplectic sum) Let (M_i^n, ω_{M_i}) and (N^{n-2}, ω_{N_i}) be closed, symplectic manifolds and let $j_i : N \to M_i$ be symplectic embeddings whose Euler classes satisfy $e(\nu_2) = -e(\nu_1)$. Then for a orientation reversing diffomorphism $\psi : \nu_1 \to \nu_2$, $M_1 \sharp_\psi M_2$ admit a symplectic form ω which is induced by ω_1 and ω_2.*

If K is a fibered knot, there is a fibration map from $S^3 \backslash \nu(K)$ to the meridian of the knot K which naturally extend to the fibration $\pi : S^1 \times M_K \to T^2$. Then by a theorem of W. Thurston [55], $S^1 \times M_K$ is a symplectic 4-manifold with the symplectic torus T_m. Hence, using Gompf's symplectic sum, knot surgery can be done in symplectic category.

Since a symplectic 4-manifold is characterized by a Lefschetz pencil or fibration structure, one can think of symplectic version of Fintushel-Stern's conjecture: For two fibered knot K and K' whether X_K and $X_{K'}$ is isomorphic as a Lefschetz pencil or fibration if and only if K and K' is equivalent as a knot or not. J. Park and K-H. Yun studied this problem [49] [50] with Kanenobu knots $\{K_{p,q}\}$ which are infinite family of non isotopic knots with same Alexander polynomial. They showed that there are at least 4 different isomorphic classes of Lefschetz fibrations along $E(2)_{K_{p,q}}$. In the next section we review the Lefschetz fibration structure of $E(2)_K$ and recover their result using the bilinear form invariant constructed in the chapter 3.

4.2 Nonisomorphic Lefschetz fibration structures on $E(2)_K$

4.2.1 Lefschetz fibration structures of $E(n)_K$

Definition 4.2.1. A complex surface S is an elliptic surface if there is a holomorphic map $\pi : S \to C$ to a complex curve C such that for generic $t \in C$ the fiber $\pi^{-1}(t)$ is a smooth elliptic curve.

Topologically, S is a smooth, closed, oriented 4-manifold with a map to closed surface whose generic fiber is a torus T^2. There are various descriptions of $E(n)$, one of them is the branched cover technique. Let $B_{2,n}$ be the complex curve in $\mathbb{CP}^1 \times \mathbb{CP}^1$ as the union of the 4 horizontal and $2n$ vertical spheres. We define $E(n)$ as the desingularization of the double branched cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ along $B_{2,n}$. As in Figure 4.2.1, there are two natural projection π_1 and π_2. Note that π_2 gives the elliptic fibration structure: the generic fiber of π_2 is a branched double cover of S^2 with 4-branch points hence it is a torus. By locally deforming each singular fiber to Lefchetz type singular fibers we

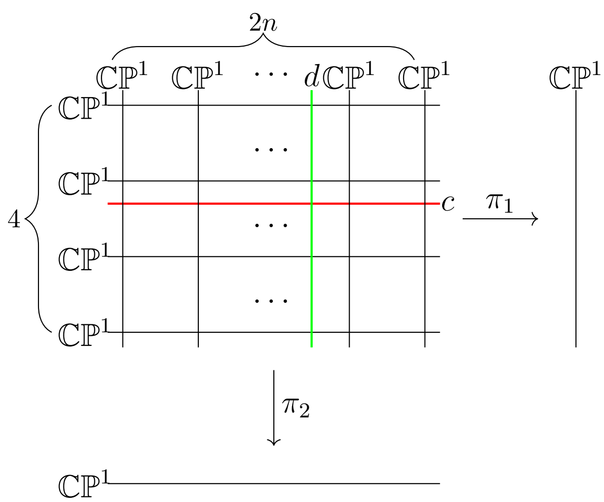


Figure 4.1: Elliptic surfaces $E(n)$

also get a Lefschetz fibration over S^2 with a torus fiber called elliptic fiber. By the $E(n)_K$, we mean the smooth 4-manifolds obtained by knot surgery along this elliptic fiber. On the other hand, π_1 gives another fibration structure whose generic fiber is a surface Σ_{n-1} of genus $(n-1)$. Since the elliptic fiber intersect Σ_{n-1} at exactly two points, $E(n)_K$ has fibration structure h of genus $2g+n-1$ provided that K is a fibered knot of genus g. This fibration has 4 singular fibers, the monodromy η of each singular fiber is the extension of the monodromy ω of singular fiber of π_1: Note that ω is the hyperelliptic involution of Σ_{n-1}. By connect sum two Σ_g to the Σ_{n-1} at p and $\omega(p)$, we can extend ω to a involution η.

Definition 4.2.2. $M(n,g)$ is the complex surface obtained by the desingularization of the branched cover of $\Sigma_g \times S^2$ branched along 2 disjoint copies of $\Sigma_g \times \{pt\}$ and $2n$ disjoint copies of $\{pt\} \times S^2$.

Then singular fibers of horizontal fibration for $M(n,g)$ gives the local holomorphic model for the singular fibers of h hence it can be locally perturbed to the Lefschetz type singular fibers. In conclusion, we have the following theorem.

Theorem 4.2.3 ([22], [57])**.** *Let K be a fibered knot of genus g. Then the knot surgery 4-manifold $E(n)_K$, as a genus $(2g+n-1)$ Lefschetz fibration, has a monodromy factorization*

$$\Phi_K(\eta_{n-1,g}) \cdot \Phi_K(\eta_{n-1,g}) \cdot \eta_{n-1,g} \cdot \eta_{n-1,g}$$

where $\eta^2_{n-1,g}$ is the monodromy factorization of $M(n,g)$ and

$$\Phi_K = \phi_K \oplus id \oplus id : \Sigma_g \sharp \Sigma_{n-1} \sharp \Sigma_g \to \Sigma_g \sharp \Sigma_{n-1} \sharp \Sigma_g$$

is a diffeomorphism obtained by using a monodromy map ϕ_K of K

Remark 4.2.4. There are two ambiguities for the monodromy factorization of $E(2)_K$ given in Theorem 4.2.3: choice of a generic fiber for the monodromy factorization of $M(n,g)$ and the monodromy map of K. If we change the generic fiber F of $M(n,g)$ to $\widetilde{F}$, there exist a diffeomorphism

$$\phi = \phi_1 \oplus \phi_2 \oplus \phi_3 : \Sigma^1_g \sharp \Sigma_{n-1} \sharp \Sigma^1_g \to \Sigma_{2g+n-1}$$

such that $\eta_{n-1,g} = \phi(\eta)$. Then we have

$$\begin{aligned}
\Phi_K(\eta^2) \cdot \eta^2 &= \Phi_K(\phi(\eta^2_{n-1,g})) \cdot \phi(\eta^2_{n-1,g}) \\
&= \phi((\phi^{-1} \cdot \Phi_K \cdot \phi)(\eta^2_{n-1,g}) \cdot \eta^2_{n-1,g}) \\
&\cong (\phi^{-1} \cdot \Phi_K \cdot \phi)(\eta^2_{n-1,g}) \cdot \eta^2_{n-1,g}
\end{aligned}$$

Note that $\phi^{-1} \cdot \Phi_K \cdot \phi$ is also monodromy of K. This means for each equivalence class of the monodromy factorization of $E(n)_K$, we can choose a monodromy factorization with respect to the chosen generic fiber of the monodromy factorization of $M(n, g)$ which depends on the choice of the monodromy of fibered knot K. So if two monodromy factorizations of $E(2)_K$ with respect to the chosen generic fiber F are not Hurwitz equivalent, then they are not isomorphic to each other.

4.2.2 A bilinear form invariant of $E(2)_K$

In this section we compute the bilinear form of $E(2)_{K_{p,q}}$ and investigate the isomorphism classes of fixed Lefschetz fibrations. In [33], T. Kanenobu constructed a inifinite family of non isotopic knots with same Alexander polynomial. He constructed the family from the ribbon fibered knot $4_1\sharp(-4_1^*)$ by using Stalling's twist. We review some propeties of Kanenobu knot $K_{p,q}$ [33].

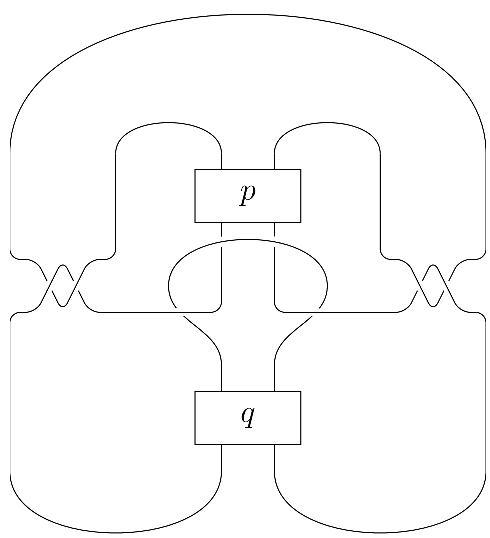


Figure 4.2: Kanenobu knot $K_{p,q}$

Proposition 4.2.5. *In Figure 4.2, p (resp. q) represent a positive p(resp. q) full twists. For a Kanenobu knot $K_{p,q}$*

(i) $K_{p,q}$ *is isotopic to* $K_{r,s}$ *if and only if* $(p,q) = (r,s)$ *or* $(p,q) = (s,r)$

(ii) $K_{p,q}$ *is prime for* $(p,q) \neq (0,0)$

(iii) $\Delta_{K_{p,q}}(t) = (t - 3 + t)^{-2}$

(iv) $K_{p,q}$ *is a fibered ribbon knot with monodromy* $t_d^q \cdot t_{c_2}^p \cdot t_{a_2} \cdot t_{b_2}^{-1} \cdot t_{a_1}^{-1} \cdot t_{b_1}$

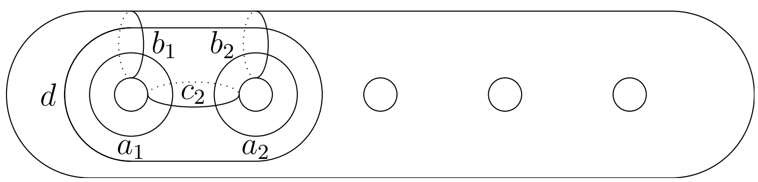


Figure 4.3: Dehn twists in monodromy of $K_{p,q}$

To compute the bilinear form, we need a monodromy factorization as the product of right-handed Dehn twists. The following lemma is useful for the computation.

Lemma 4.2.6 ([57])**.** *$M(2,g)$ has a monodromy factorization $\eta_{1,g}^2$ where $\eta_{1,g} = t_{B_0} \cdot t_{B_1} \cdot t_{B_2} \cdots t_{B_{2g+1}} \cdot t_{b_{g+1}}^2 \cdot t_{b'_{g+1}}^2$ and B_i, b_i, b'_i are simple closed curves on Σ_{2g+1} as in Figure 4.4*

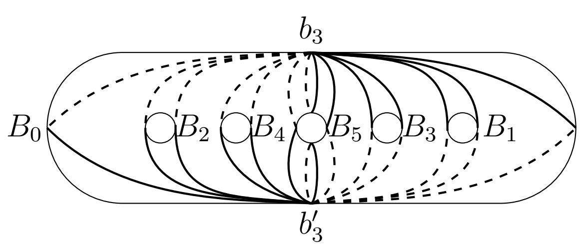


Figure 4.4: Vanishing cycles of $M(2,g)$

Now we are ready to show the following.

Proposition 4.2.7. *For the monodromy factorization* $W_{p,q} = \Phi_{K_{p,q}}(\eta_{1,2}) \cdot \Phi_{K_{p,q}}(\eta_{1,2}) \cdot \eta_{1,2} \cdot \eta_{1,2}$ *of* $E(2)_{K_{p,q}}$, $W_{p,q}$ *and* $W_{r,s}$ *are not isomorphic if* $(p,q) \not\equiv (r,s)$ *mod* 2.

Proof. Since $E(2)_{K_{p,q}}$ is simply connected, there is a subspace S of $\underline{M}_{W_{p,q}}$ such that $\underline{M}_{W_{p,q}} = S \oplus Col_{W_{p,q}}$. By the homological argument as in the Proposition 3.3.4, $\mathbf{B_{p,q}} = B_{W_{p,q}} + B^{\intercal}_{W_{p,q}}$ have a decomposition as a bilinear form (i.e $\mathbf{B_{p,q}} = \mathbf{B_{p,q}}|_S \oplus 2 \cdot Q_{W_{p,q}}$). Note that if $W_{p,q}$ is isomophic to $W_{r,s}$, then $\Gamma_{W_{r,s}}(\mathcal{B}_{W_{p,q},W_{r,s}}(\mathbf{m})) = \Gamma_{W_{p,s}}(\mathbf{m})$ for any $\mathbf{m} \in \underline{M}_{W_{p,q}}$(cf. Remark 4.2.4). Hence for the $\mathbf{m}_1 \in \underline{M}_{W_{p,q}}$ and $\mathbf{m}_2 \in \underline{M}_{W_{r,s}}$ with $\Gamma_{W_{p,q}}(\mathbf{m}_1) = \Gamma_{W_{r,s}}(\mathbf{m}_2) \neq 0$, we have

$$\mathbf{B_{p,q}}(\mathbf{m}_1, \mathbf{m}_1) \equiv \mathbf{B_{r,s}}(\mathbf{m}_2, \mathbf{m}_2) \mod 4$$

since $Q_{W_{p,q}}$ is even by Proposition 3.3.4. By the aid of computer, we can find a nonzero$x = \Gamma_{W_{p,q}}(\mathbf{m}_{p,q})$ with $\mathbf{B_{p,q}}(\mathbf{m}_{p,q}, \mathbf{m}_{p,q}) = -4p^2 - 2p - 2q - 2$. Therefore, the assertion follows. $\square$

Chapter 5

Lefschetz fibration structures on Stein domains

There is a correspondence between Stein fillings and Lefschetz fibrations which allows us to understand geometric structures (complex/symplectic) through topological decompositions. On the boundary we have the Giroux correspondence between contact structures and open book decompositions. In this section, we will explain these notions and their relations to each other. For more details, see [43], [25].

5.1 Contact structures and open book decompositions

E. Giroux [27] proved a central result about the topology of contact 3-manifolds. He showed that there is a one-to-one correspondence between contact structures (up to isotopy) and open book decompositions (up to positive stabilizations and destabilizations) on a closed oriented 3-manifolds. This section is devoted to the introduction of relevant notions.

Definition 5.1.1. Suppose that Y is a 3-manifold. A *contact structure* on Y is a 2-plane field ξ defined by the kernel of a nowhere vanishing 1-form $\alpha \in \Omega^1(M)$ (i.e $\alpha \wedge d\alpha$ is nowhere zero) Such an 1-form α is called contact form.

Example 5.1.2. The standard contact structure ξ_{st} on $\mathbb{R}^3$ is given in the coordinates (x, y, z) as $\ker(dz + xdy + ydx)$. The complex tangents($= TS^3 \cap J(TS^3)$) to $S^3 \subset \mathbb{C}^2$ also form a contact structure.

Example 5.1.3. Let us identify the 3-torus T^3 with $\mathbb{R}^3/\mathbb{Z}^3$. For any positive integer n the 1-form $\sin(2\pi nx)dy + \cos(2\pi nx)dz$ defined on $\mathbb{R}^3$ induces a contact structure ξ_n on T^3.

For an oriented contact 3-manifold Y, we say that ξ is a positive contact structure on Y if the orientation of Y coincides with the orientation given by $\alpha \wedge d\alpha > 0$. Note that the orientation induced on Y by $\alpha \wedge d\alpha$ is independent of the contact form α defining the contact structure ξ. In this thesis, the contact structure ξ is always assumed to be positive. Using α, we orient the normal direction to the contact planes, or equivalently the contact planes are oriented by $d\alpha$. If we choose $-\alpha$ as a contact form then the both orientation of the normal direction and the that of the contact planes will be reversed while the orientation induced by $-\alpha \wedge d(-\alpha)$ on the 3-manifolds is unchanged.

Definition 5.1.4. Two contact 3-manifolds (Y, ξ) and (Y', ξ') are called *contactomorphic* if there is a diffeomorphism $f : Y \to Y'$ such that $f_*(\xi) = \xi'$. If $\xi = \ker\alpha$ and $\xi' = \ker\alpha'$, then this is equivalent to the existence of a nowhere zero function $g : Y \to \mathbb{R}$ such that $f^*\alpha' = g\alpha$. Two contact sturctures ζ and ζ' are said to be *isotopic* if there is a contactomorphism $h : (Y, \zeta) \to (Y, \zeta')$ which is isotopic to the identity.

In fact, two contact structures on a closed manifold are isotopic if and only if they are homotopic through contact structures. There exist contact structures which are contactomorphic but not istopic. Next we introduce the special submanifolds in contact 3-manifolds.

Definition 5.1.5. Suppose that (Y, ξ) is a given contact 3-manifold. A knot $K \subset Y$ is *Legendrian* if the tangent space of K lies in contact plane, i.e., $\alpha(TK) = 0$. The knot K is *transverse* if TK is a transverse to ξ along the knot K. i.e., $\alpha(TK) \neq 0$. The *contact framing* of a Legendrian knot is defined by the normal of ξ along K. Equivalently, we take the framing obtained by pushing K off in the direction of a nonzero vector field transverse to K which lies in the contact planes. We also call the contact framing *Thurston-Bennequin* framing.

If K is null-homologous in Y then it admits a natural 0-framing given by embedded surface $\Sigma \subset Y$ with $\partial\Sigma = K$. Here 'natural' means that the 0-framing does not depend on the choice of surface Σ. In this case the Thurston-Bennequin framing can be thought as an integer $tb(K) \in \mathbb{Z}$, which is called *Thurston-Bennequin invariants*, by measuring the Thurston-Bennequin framing with respect to the Seifert framing (cf. Example 2.1.4).

Example 5.1.6. In $S^3 \subset \mathbb{R}^4$ with its standard contact structure ξ_{st} =ker α, where

$$\alpha = x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2$$

consider the Legendrian unknot

$$K = \{x_1^2 + x_2^2 = 1, y_1 = y_2 = 0\}$$

This circle bounds the 2-disk

$$\Sigma = \{x_1^2 + x_2^2 + y_1^2 = 1, y_1 \geq 0, y_2 = 0\}$$

so the surface framing of K is given by the vector field ∂_{y_1}. In order to compute $tb(K)$, we choose an orientation of K, say the one defined by the unit tangent vector field T.

$$T := x_1 \partial_{x_2} - x_2 \partial_{x_1}$$

Along K, the volume form $\alpha \wedge d\alpha$ on S^3 is

$$2x_1 dy_1 \wedge dx_2 \wedge dy_2 + 2x_2 dy_2 \wedge dx_1 \wedge dy_1$$

The inner product of this volume form with T is $-2dy_1 \wedge dy_2$. So with respect to the chosen orientation of K, $\{T, \partial_{y_2}, \partial_{y_1}\}$ gives the orientation of S^3 along K. The vector field

$$N := x_1 \partial_{y_2} - x_2 \partial_{y_1}$$

along K lies in ξ_{st}, and together with T it spans the contact planes. Now it is easily seen that $tb(K) = -1$.

There are model neighborhoods for Legendrian knots and transverse knots.

Consider the contact structures

$$\zeta_1 = \ker(\cos(2\pi\phi)dx - \sin(2\pi\phi)dy)$$

$$\zeta_2 = \ker(\cos r d\phi + r \sin r d\theta)$$

on $S^1 \times \mathbb{R}^2$ where ϕ is the coordinate for S^1 direction while (x, y) and (r, θ) are coordinates on $\mathbb{R}^2$. Then

Theorem 5.1.7. *If $K \subset (Y, \xi)$ is a Legendrian knot then there are neighborhoods $U_1 \subset Y$ of K and $U_2 \subset S^1 \times \mathbb{R}^2$ of $S^1 \times \{0\}$ such that (U_1, ξ_{U_1}) and (U_2, ξ_{U_2}) are contactomorphic via a contactomorphism mapping K to $S^1 \times \{0\}$. If K is transverse, then some neighborhood of it is contactomorphic to some neighborhood of $S^1 \times \{0\}$ in $(S^1 \times \mathbb{R}^2, \zeta_2)$ again K is mapped to $S^1 \times \{0\}$.*

Next we define the second classical invariant for Legendrian knots, so-called the rotation number. Let K be an oriented Legendrian knot in an oriented contact 3-manifold (Y, ξ) with a positive contact structure. Let Σ be a Seifert surface for K, oriented as the boundary of Σ. Write $c \in H_2(Y, K)$ for the relative homology class represented by Σ. Since every orientable 2-plane bundle over a surface with boundary is trivial, we can find a trivialization of $\xi|_\Sigma$. Let $\gamma : S^1 \to K \subset Y$ be a parametrization of K compatible with its orientation. Given a trivialization $\xi|_\Sigma = \Sigma \times \mathbb{R}^2$ (as oriented bundles), this induces a map $\gamma' : S^1 \to \mathbb{R}^2 \setminus \{0\}$.

Definition 5.1.8. The *rotation number* $r(K, c)$ of the null-homologous oriented Legendrian knot K relative to the class $c \in H_2(Y, K)$ is the degree of the map γ'. In other words, $r(K, c)$ counts the number of rotations of the tangent vector to K relative to the trivialization of $\xi|_\Sigma$ as we go once around K.

The next proposition shows that the definition of rotation number is well defined.

Proposition 5.1.9. *(i) The rotation number $r(K, \Sigma)$ does not depend on the choice of trivialization of $\xi|_\Sigma$.*

(ii) Let Σ, Σ' be Seifert surfaces for K representing the classes $c, c' \in H_2(Y, K))$. Then

$$r(K, \Sigma) - r(K, \Sigma') = \langle e(\xi), c - c' \rangle$$

where $c - c'$ is a class in $H_2(Y)$, and $e(\xi)$ denotes the Euler class.

For a Legendrian knot $K \subset (S^3, \xi_{st})$, $r(K, c)$ is independent of c because $H^2(S^3; \mathbb{Z}) = 0$. Whenever $r(K, c)$ is independent of c, we shall simply write $r(K)$.

Example 5.1.10. Consider the standard Legendrian unknot K in the (S^3, ξ_{st}) as in the previous example. The vector field

$$x_1\partial_{x_2} - x_2\partial_{x_1} + y_2\partial_{y_1} - y_1\partial_{y_2}$$

is a nowhere zero section of ξ_{st} and is also tangent to K. This shows that $rot(K) = 0$.

Next we introduce the notion of open book decompositions. Here Y is always a closed and oriented 3-manifold.

Definition 5.1.11. An open book decomposition of Y is a pair (B, π) where

(i) B is an oriented link in Y called the binding of the open book

(ii) $\pi : Y \setminus B \to S^1$ is a fibration with the fibers Seifert surface of B as the fibers. That is $\partial\overline{\pi^{-1}(\theta)} = B$. The surface $\Sigma = \overline{\pi^{-1}(\theta)}$ is called the page of open book.

Definition 5.1.12. An abstract open book is a pair (Σ, ϕ) where

- Σ is oriented surface with boundary
- $\phi : \Sigma \to \Sigma$ is a diffeomorphism such that ϕ is the identity in a neighborhood of $\partial\Sigma$. ϕ is called the monodromy.

Example 5.1.13. There is an open book decomposition of S^3 given as follows: Take the z-axis to be the binding and consider the half-planes with boundary L as Seifert surface. Explicitly, take $\pi : \mathbb{R}^3 - \{(0,0,z)\} \to S^1 \subset \mathbb{R}^2$ given by

$$\pi(x,y,z) = \frac{1}{\sqrt{x^2+y^2}}(x,y)$$

We can extend this open book decomposition to an open book decomposition of S^3, which is an one point compactification of $\mathbb{R}^3$, with binding the unknot and 2-dimensional disks as pages. The resulting open book decomposition is called the *standard* open book decomposition of S^3. From this viewpoint, we can also observe the well known fact that S^3 is the union of two solid tori, one is the neighborhood of the binding and the other is the union of the pages.

Example 5.1.14. There is a natural open book decomposition structure on the boundary of Lefschetz fibration $\pi : X \to D^2$ if the generic fiber has non-empty boundary : Composing with radial projection $f : D^2 - \{0\} \to S^1$, $f \circ \pi : \partial X - \partial(\pi^{-1}(0)) \to S^1$ is a fibration with fiber F with monodromy equals to the global monodromy of π. Let α be the middle circle $S^1 \times \{\frac{1}{2}\} \subset S^1 \times [0,1]$. Consider the Lefschetz fibration given by t_α, where t_α is the right-handed Dehn twist along α, then it can be built using a single 1-handle and a 2-handle which is 1-handle/2-handle cancelling pair. We denote the corresponding open book decomposition on S^3 by $\mathfrak{ob}_+$. In fact, the binding of the resulting open book decomposition can be identified with the positive Hopf link and the pages are just the obvious Seifert surfaces. By sliding the circles representing $\partial(S^1 \times [0,1])$ over the (-1)-framed 2-handle and cancelling the 1-/2-handle pair, we see the circles $a = S^1 \times 0$ and $b = S^1 \times 1$ linked once in the S^3. Taking t_α^{-1} instead of t_α corresponds to reversing the orientation on the Lefschetz fibration and hence on its boundary S^3. Therefore the resulting open book decomposition has the negative Hopf link as its binding. This open book decomposition will be denoted by $\mathfrak{ob}_-$. An alternative way to give $\mathfrak{ob}_\pm$ is by considering $H_\pm = \{r_1 r_2 = 0\} \subset S^3$ as binding equipped with polar coordinates $(r_1, \theta_1, r_2, \theta_2)$ coming from $S^3 \subset \mathbb{C}^2$ and $\pi_\pm(r_1, \theta_1, r_2, \theta_2) = \theta_1 \pm \theta_2$. The standard open book decomposition becomes $L = \{r_2 = 0\}$ and $\pi(r_1, \theta_1, r_2, \theta_2) = \theta_2$ in these coordinates.

Now we introduce the *Murasugi sum* operation which yields a new (abstract) open book decomposition from two open book decompositions (Σ_i, ϕ_i)

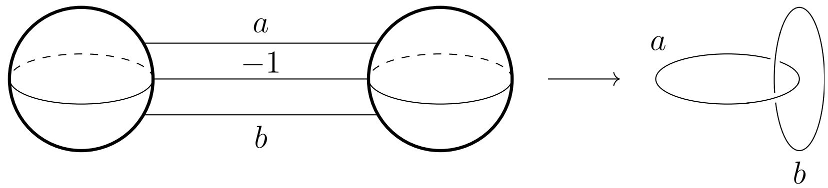


Figure 5.1: The circles a and b after handle cancellation

of Y_i $(i = 1, 2)$. Choose properly embedded arcs α_i and take neighborhoods of the arcs $\nu\alpha_i = \alpha_i \times I$ in pages Σ_i. Let Σ be the surface obtained by gluing two surfaces Σ_i along $\nu\alpha_i$ in a way that α_1 is identified with $I \subset \nu\alpha_2$ and α_2 is identified with $I \subset \nu\alpha_1$. Then there is natural extension of ϕ_i to $\widetilde{\phi}_i \in \mathcal{M}_\Sigma$. In this case we call $(\Sigma, \phi = \widetilde{\phi}_1 \circ \widetilde{\phi}_2)$ is *Murasugi sum* of (Σ_1, ϕ_1) and (Σ_2, ϕ_2). It is well known that (Σ, ϕ) defines a 3-manifold Y diffeomorphic to $Y_1 \sharp Y_2$. For the special case that one of (Σ_i, ϕ_i) is $\mathfrak{ob}_\pm$, we call Murasugi sum plumbing a positive (resp. negative) Hopf band (see Figure 5.2).

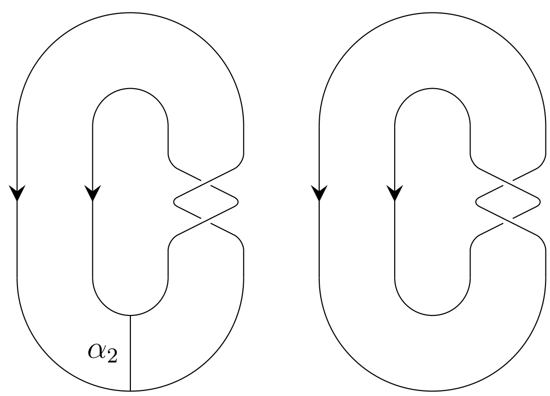


Figure 5.2: Positive and negative Hopf bands

Suppose that an open book decomposition with page Σ is specified by $\phi \in \mathcal{M}_\Sigma$. Attach a 1-handle to the surface Σ to obtain a new surface Σ'. Let α be a simple closed curve in Σ' going over the new 1-handle exactly once. Define a new open book decomposition with $\widetilde{\phi} \circ t_\alpha \in \mathcal{M}_{\Sigma'}$ where t_α stands for the right-handed Dehn twist along α and $\widetilde{\phi} \in \mathcal{M}_{\Sigma'}$ is a natural extension of ϕ. The resulting open book decomposition is called a *positive stabilization* of the one defined by (Σ, ϕ). If we use the left-handed Dehn twist instead then we call the result a *negative stabilization.* Note that the resulting monodromy depends on the chosen curve α.

We can view the stabilization as plumbing Hopf bands (t_α corresponds to the right-handed Dehn twist along middle circle $S^1 \times \{\frac{1}{2}\}$). Since the effect of plumbing a Hopf band on the 3-manifold is topologically taking the connected sum with S^3, by definition we do not change the topology of the underlying 3-manifold.

Definition 5.1.15. A contact structure ξ and an open book decomposition (B, π) on Y are said to be *compatible* if ξ can be represented by a contact 1-form α such that

(i) $d\alpha$ is a positive volume form on each page Σ.

(ii) $\alpha > 0$ on B.

Roughly, an open book decomposition is compatible with a contact structure if we can isotope the contact planes arbitrarily close to the tangent planes of the pages. We close this subsection by stating the fundamental theorem of E. Giroux [27].

Theorem 5.1.16. *(i) For a given open book decomposition of Y there is a compatible contact structure ξ on Y. Contact structures compatible with a fixed open book decomposition are isotopic.*

(ii) For a contact structure ξ on Y there is a compatible open book decomposition of Y. Two open book decompositions compatible with a fixed contact structure admit common positive stabilization.

5.2 Contact 3-manifolds and Symplectic fillings

A symplectic structure of a manifold with boundary usually induces a contact structure on the boundary. The induced contact structure on the boundary provides a model for gluing two symplectic manifolds with boundary.

Definition 5.2.1. We say that (W, ω) is *(strong) filling* of (Y, ξ) if $\partial W \cong Y$ and $\omega|_Y$ is a contact form α of ξ with $d\alpha = \omega|_Y$. Or equivalently there is a *Liouville vector field* near Y which is transverse to Y. By Liouville vector field, we mean the vector field v such that $\mathcal{L}_v \omega = \omega$.

There are two types of strong fillings:

(i) If the orientation of Y given by nowhere vanishing form $\alpha \wedge d\alpha$ agrees with the boundary orientation of ∂W, then we say (W, ω) is a *strong convex filling* of (Y, ξ).

(ii) If two orientations disagree then we say (W, ω) is a *strong concave filling* of (Y, ξ).

For the gluing process one requires a model for collared neighborhood. For a contact 3-manifold $(Y, \xi = \ker\alpha)$, Let $X = Y \times \mathbb{R}$ and $\omega_X = d(e^t\alpha)$. By Cartan's formula,

$$\mathcal{L}_{\partial_t}(d(e^t\alpha)) = \iota_{\partial_t} d(d(e^t\alpha)) + d(\iota_{\partial_t} d(e^t\alpha)) = d(\iota_{\partial_t}(e^t(dt \wedge \alpha + d\alpha))) = d(e^t\alpha)$$

Hence $v = \frac{\partial}{\partial t}$ is a Liouville vector field. The symplectic manifold (X, ω_X) is called the *symplectization* of (Y, ξ). It can be shown that if (W, ω) is a strong symplectic filling (resp. strong concave filling) of (Y, ξ) then there exists a neighborhood N of Y in W, a positive (resp. negative) function $f : Y \to \mathbb{R}$ and a symplectomorphism $\psi : N_X \to N$, where N_X consists of neighborhood N_f of the graph of f in X with points below (resp. above) the graph. This gives a model for a neighborhood of a contact manifold in a strong symplectic filling.

There is also a notion of *weak symplectic filling*, which only requires that $\omega|_\xi$ does not vanish. It is known that there are weakly fillable contact structures which are not strongly fillable: for example, the contact tori (T^3, ξ_n) with $n \geq 2$ (cf. Example 5.1.3) all have this property. However in the special case of rational homology spheres we have

Theorem 5.2.2 ([44])**.** *Suppose that $b_1(Y) = 0$. The symplectic structure ω on a weak symplectic filling W of any contact structure ξ on Y can be* *extended to* $W \cup Y \times [0,1]$ *to a strong filling of* (Y, ξ). *In conclusion, the two notions of symplectic fillings coincide if* Y *is a rational homology sphere.*

In the following we review how to glue a Weinstein 2-handle to a symplectic 4-manifold (W, ω) to the ω convex boundary. Let us take the standard 2-handle H as the closure of the component of

$$\mathbb{R}^4 - \Big(\{x_1^2 + x_2^2 - \frac{1}{2}(y_1^2 + y_2^2) = -1\} \cup \{x_1^2 + x_2^2 - \frac{\epsilon}{6}(y_1^2 + y_2^2)\}\Big)$$

which contains the origin. It inherits the symplectic structure $\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ from the standard symplectic structure of $\mathbb{R}^4$. Consider the vector field

$$v = 2x_1 \frac{\partial}{\partial x_1} - y_1 \frac{\partial}{\partial y_1} + 2x_2 \frac{\partial}{\partial x_2} - y_2 \frac{\partial}{\partial y_2}.$$

Note that

(i) $v = \nabla f$ for $f = x_1^2 + x_2^2 - \frac{1}{2}y_1^2 - \frac{1}{2}y_2^2$ with respect to the standard Riemannian metric and $\mathcal{L}_v \omega_0 = \omega_0$. Hence v is a Liouville vector field which is transverse to ∂H for sufficiently small $\epsilon > 0$.

(ii) The attaching circle $S = \{x_1 = x_2 = 0, y_1^2 + y_2^2 = 2\} \subset \partial H$ is Legendrian with respect to the contact structure $\xi = \ker \alpha$ on ∂H where $\alpha = \iota_v(\omega_0) = 2x_1 dy_1 + y_1 dx_1 + 2x_2 dy_2 + y_2 dx_2$.

(iii) Near S the vector field v points into the H while v points out of H near $B = \{y_1 = y_2 = 0, x_1^2 + x_2^2 = \frac{\epsilon}{2}\}$.

With these observations, we get

Proposition 5.2.3. *Suppose that* (W, ω) *is a symplectic* 4*-manifold with* ω*-convex boundary* ∂W *and* $L \subset \partial W$ *is a Legendrian curve (with respect to the induced contact structure). Then a* 2*-handle* H *can be attached to* X *along* L *in such a way that* ω *extends to* $W \cup H$ *as* ω' *and* $\partial(W \cup H)$ *is* ω'*-convex.*

There are also gluing models for any $2n$-dimensional k-handles with $k \leq n$ (cf. [56]). For 4-dimensional 1-handle cases, starting with the standard 4-disk

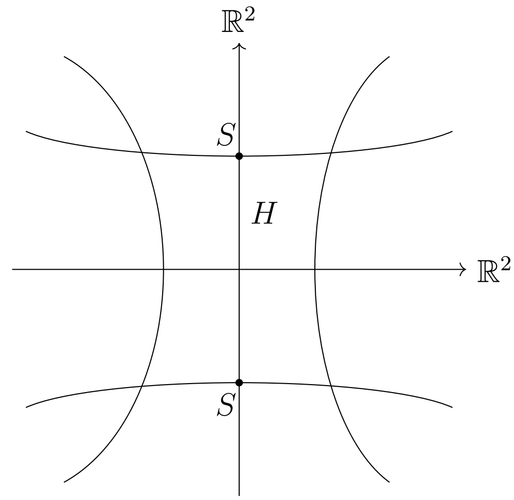


Figure 5.3: Weinstein 2-handle

$(D^4, \omega_{st}|_{D^4})$ and the vector field $v = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}$, we get a symplectic structure on $\sharp_m S^1 \times D^3$ with ω-convex boundary by attaching m 1-handles.

Recall that attaching a 4-dimesional 2-handle consists of two steps: the above gluing scheme tells us the attaching circle should be Legendrian for the Weinstein 2-handle. For a complete picture about the topology of the manifold after attaching the Weinstein 2-handle, we need to specify the framing of the 2-handle H. Recall that the Legendrian knot L admits a canonical contact framing, hence we now compute the framing of the gluing relative to this one. We think of a framing as a vector field in the tangent space $TY|_L$ transverse to tangent vector field of L. Therefore we need to identify two vector fields along L: v_1 is the vector field in ξ which is transverse to the tangent of L (providing the contact framing), while v_2 is the image of the direction we push off the attaching circle to the core of the 2-handle H. Note that the gluing map is contactomorphism, so we can work in the $H \subset \mathbb{R}^4$ for this computation.

(i) For $S = \{x_1 = x_2 = 0, y_1^2 + y_2^2 = 2\}$, fix a parametrization

$$(0, 0, \sqrt{2}\cos t, \sqrt{2}\sin t)$$

(ii) The tangent vector along S is given by

$$(0, 0, -\sin t, \cos t)$$

(iii) Since $\alpha|_S = \sqrt{2}\cos t dx_1 + \sqrt{2}\sin t dx_2$, contact framing can be represented by

$$(\sin t, -\cos t, 0, 0)$$

(iv) The framing comes from the 2-handle H can be given by

$$(1, 0, 0, 0)$$

The above two unit vector fields intersect each other once, implying the difference of the two framings should be ± 1. Taking the orientation into account, we see that the vector field v_1 makes one positive full twist around the origin, therefore the framing we get by pushing the knot slightly in the handle direction is (-1) when compared to the contact framing as a Legendrian knot. In conclusion

Theorem 5.2.4 (A. Weinstein, [56]). *Suppose that (W, ω), ∂W and L are the same as in the above proposition. If we attach a 4-dimensional 2-handle H with framing -1 with respect to its canonical contact framing to ∂W along L then ω extends to $W \cup H$.*

We can also see what happens to the boundary contact 3-manifolds. Note that topologically $\partial(W \cup H)$ is 3-manifold obtained from ∂W by performing (-1) Dehn surgery along L with respect to the contact framing.

Definition 5.2.5. Let L be a Legendrian knot in a contact 3-manifold (M, ξ). Recall that L has a neighborhood N_L that is contactmorphic to a neighborhood of x-axis in

$$(\mathbb{R}^3, \ker(dz - ydx))/\sim$$

where $\sim$ identifies (x, y, z) with $(x+1, y, z)$. With respect to these coordinates on N_L we can remove N_L from M and topologically glue it back with ± 1 twist. We call the resulting manifold $Y_{(L,\pm 1)}$. There is a unique way, up to isotopy, to extend $\xi|_{Y \setminus N_L}$ to a contact structure $\xi_{(L,\pm 1)}$. The contact manifold $(Y_{(L,\pm 1)}, \xi_{(L,\pm 1)})$ is said to be obtained from (Y, ξ) by ± 1- *contact surgery* along L.

Note that Theorem 5.2.4 implies (-1) contact surgery preserves the fillabillity while $(+1)$ surgery preserves the non-fillabillity of the contact structures due to the following lemma.

Lemma 5.2.6. *Let (Y', ξ') be the contact manifold obtained from (Y, ξ) by (-1)-contact surgery along a Legendrian knot K and $(+1)$-contact surgery along a Legendrian push-off K' of K. Then (Y', ξ') is contactomorphic to (Y, ξ).*

On the other hand, one can interpret the contact surgery in terms of open book decompositions.

Theorem 5.2.7 ([16], [26])**.** *Let (Σ, ϕ) be a compatible open book decomposition of (Y, ξ). For a Lengendrian knot L on the page of the open book,*

$$(Y, \xi)_{(L,\pm 1)} = (Y_{(\Sigma, \phi \circ t_L^{\mp 1})}, \xi_{(\Sigma, \phi \circ t_L^{\mp 1})})$$

Here $Y_{(\Sigma, \phi \circ t_L^{\mp 1})}$ is a 3 manifold determind by abstract open book $(\Sigma, \phi \circ t_L^{\mp 1})$ and $\xi_{(\Sigma, \phi \circ t_L^{\mp})}$ is a contact structure compatible with $(\Sigma, \phi \circ t_L^{\mp})$.

5.3 Stein fillings

Complex surfaces are rich source of examples of 4-manifolds. For the complex manifolds with contact boundaries, there is notion of *Stein domain* which is nice near boundary. In this section we briefly review the equivalent definitions of Stein manifolds and understand their topology in terms of handle decompositions and Lefschetz fibration structures.

Definition 5.3.1. *Stein manifolds* are complex manifolds that admit proper biholomorphic embeddings in $\mathbb{C}^N$ for sufficiently large N.

A Stein manifold inherits a symplectic form $\omega = \sum_{i=1}^{N} dx_i \wedge dy_i$ and a Liouville vector field $\sum_{i=1}^{N} \frac{1}{2}(x_i\partial_{x_i} + y_i\partial_{y_i})$ from $\mathbb{C}^N$. Hence by intersecting with large ball in $\mathbb{C}^N$, Stein manifolds give compact complex manifolds with contact boundaries.

A complex manifold (X, J) is Stein if and only if it admits an "exhausting J-convex function". Here exhausting means that $f : X \to \mathbb{R}$ is proper and bonded below and J-convex condition is characterized by the 2-form $\omega_f := -dd^{\mathbb{C}}f$ tames J. Hence the 2-form ω gives a symplectic structure of Stein manifolds. Together with a Liouville vector field ∇f for ω_f, where ∇f is gradient vector field with repect to the Riemannian metric $g_f(u, v) := \omega_f(u, Jv)$, we have a contact structure on regular level set $f^{-1}(c)$.

Definition 5.3.2. A contact 3-manifold (Y, ξ) is called *Stein fillable* if there is a Stein manifold (X, J) with J-convex function $f : X \to \mathbb{R}$, which is bounded below, such that $Y = f^{-1}(c)$ for a non-critical value c and ξ is given by f. We call $W = f^{-1}(-\infty, c]$ a Stein filling of (Y, ξ).

A J-convex function f of a Stein manifold (X, J) becomes *Morse* after small perturbation. By J. Milnor, it was proven that every J-convex Morse function has critical points with index at most n if the real dimension of X is $2n$. So the existence of almost complex structure and the handle decompositon of index $\leq n$ is the necesary condition for a given smooth manifold to carry a Stein structure. Eliashberg's theorem shows that these necesary conditions are also sufficient conditions for admitting the Stein structure.

Theorem 5.3.3 (Y. Eliashberg, [13])**.** *For $n > 2$, a smooth, almost-complex, open $2n$-manifold admits a Stein structure if and only if it is the interior of a (possibly infinite) handlebody without handles of index $> n$. The Stein structure can be chosen to be homotopic to the given almost-complex structure, and the given handle decomposition is induced by an exhausting J-convex Morse function.*

Unlikely in higher dimensions, in dimension 2, there is one more condition to be satisfied. We introduce a way to adapt the handle attachment scheme to the case of Stein surfaces. From now on we assume that $W = f^{-1}(-\infty, c]$, which is called a *Stein domain*, and we consider the contact structure on ∂W provided by f which is also given by the complex tangencies. The main theorem is the following.

Theorem 5.3.4 (Y. Eliashberg, [13])**.** *Suppose that W is a 2-dimensional Stein domain and $L \subset \partial W$ is a Legendrian knot. The Stein structure on W can be extended to $W \cup H$ where H is a Weinstein 2-handle attached along L.*

Similar results hold for the attachment of 1-handles: after attaching a 1-handle to a Stein domain the Stein structure always extends. so we get

Theorem 5.3.5 (R. Gompf, [29])**.** *A smooth, oriented, open 4-manifold X admits a Stein structure if and only if it is the interior of a (possibly infinite) handlebody such that the following hold:*

(i) Each handle has index ≤ 2

(ii) Each 2-handle h_i is attached along a Legendrian curve L_i in the contact structure induced on the boundary of the underlying 0- and 1-handles

(iii) The framing for attaching each h_i is obtained from the canonical framing on L_i by adding a single left (negative) twist.

A smooth, oriented, compact 4-manifold X admits a Stein structure if and only if it has a handle decomposition satisfying these conditions. In either case, any such handle decomposition comes from a J-convex function (with ∂X a level set).

R. Gompf developed the theory of Stein manifolds in the point of view of handle calculus. Here we highlight one result, which will be important in our later considerations: Suppose that the Stein domain (W, J) is given by attaching Weinstein 1-handles and 2-handles to (D^4, J) along the Legendrian link $L = \cup_{i=1}^{n} L_i \subset (\sharp S^1 \times S^2, \xi)$. The first Chern class of $c_1(W, J)$ of the resulting complex structure is represented by a cocycle whose value on each 2-handle with attaching circle L_i is $r(L_i)$ [29], [18].

We now turn our attention to the Lefschetz fibration structures on Stein domains.

Definition 5.3.6. We say a Lefschetz fibration $\pi : X \to D^2$ with bounded fiber F (i.e. generic fiber F is a surface with boundary) is *allowable* if all its vanishing cycles are homologically essential. We call such a Lefschetz fibration a *Positive allowable Lefschetz fibration* or PALF for short. Here the 'positive' means that all singular points correspond to the right-handed Dehn twist in the monodromy.

Recall that there is a natural open book decomposition of ∂X. If there are no singular points, then X is just a $F \times D^2$. There is a Stein structure on $F \times D^2$ since it is obtained from D^4 by attaching only 1-handles. Furthermore, the contact structure from the Stein structure is isotopic to the one induced from open book decomposition (F, id_F). We may assume each vanishing cycle C_i are Legendrian since it is non-separating curves on F (Legendrian Realization Principle). Note that the surface framing of C_i is equal to the contact framing as Legendrian curve. Using Eliashberg's handle attachment (Theorem 5.3.4), we get

Theorem 5.3.7 ([38])**.** *For a PALF $\pi : X \to D^2$ with fiber F and monodromy $\phi = t_{c_1} \cdot t_{c_2} \cdots t_{c_n} \in \mathcal{M}_{\mathcal{F}}$, there is a Stein structure on X and X is a Stein filling of $(\partial X, \xi)$ where ξ is a contact structure on ∂X which is compatible to the open book decomposition (F, ϕ) of ∂X.*

In fact, the converse is also true.

Theorem 5.3.8 ([3], [38])**.** *If W is a Stein domain then it admits a Lefschetz fibration structure. In addition, we can assume that the vanishing cycles in the resulting fibration are homologically essential.*

Chapter 6

Lefschetz fibration structures on symplectic fillings of quotient surface singularities

For the link L of a quotient surface singularity $(X, 0)$, there is a natural contact structure ξ_{st} given by $TL \cap J(TL)$. In this chapter we give an algorithm for a Lefschetz fibration structure of each minimal symplectic fillings of (L, ξ_{st}) and show that each such a filling is obtained by a sequence of rational blow-downs from the minimal resolution of its singularity. We begin by reviewing the elementary facts about quotient surface singularities.

6.1 Quotient surface singularities

Definition 6.1.1. A surface singularity $(X, 0)$ is defined by

$$(X, 0) = (\{f_1 = \cdots = f_m = 0\}, 0) \subset (\mathbb{C}^N, 0)$$

with germs of analytic function f_i satisfying rank of $\left[\dfrac{\partial f_i}{\partial z_j}\right]$ is equal to $N - 2$ and strictly less than $N - 2$ at 0.

Then the local ring of analytic functions $\mathcal{O}_{X,0}$ is given by

$$\mathcal{O}_{X,0} \cong \mathbb{C}^N, 0/\langle f_1, \dots, f_m \rangle$$

Consider the natural action of $GL(2,\mathbb{C})$ on $\mathbb{C}^2$ then it induces an action on $\mathcal{O}_{\mathbb{C}^N,0}$. By the quotient surface singularities we mean the isolated surface singularities given by $(\mathbb{C}^2, 0)/G$ which is determined by $(\mathcal{O}_{\mathbb{C}^2,0})^G$: the algebra of functions invariant under G where G is a finite subgroup of $GL(2,\mathbb{C})$.

Example 6.1.2. Let $\mathbb{Z}_n$ acts on $(\mathbb{C}^2, 0)$ as the following

$$\xi \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \xi \cdot x \\ \xi^{-1} \cdot y \end{pmatrix} \qquad \xi^n = 1$$

then $(\mathcal{O}_{\mathbb{C}^2,0})^{\mathbb{Z}_n} \cong \langle x^n, xy, y^n \rangle \cong \mathbb{C}[u,v,w]/\langle uw = v^n \rangle$ hence $(X,0) = \{(u,v,w) \in \mathbb{C}^3 | uw = v^n\}$.

For an embedding $(X,0) \subset \mathbb{C}^N$, consider an intersection of a small ball B_ϵ centered at origin with X. For sufficiently small $\epsilon > 0$, they intersect transversally, and small neighborhood of the intersection $X \cap B_\epsilon$ is homoeomorphic to cone over $L_X := X \cap \partial B_\epsilon$ which called the link of surface singularities. It is well known that the diffeomorphism type of L_X is independent of the embedding and L_X determines the topology of $(X,0)$.

Example 6.1.3.

(i) $(X,0)$ is smooth if and only if $L_X \cong S^3$

(ii) For $(X,0) = (\{x^2 + y^2 + z^n = 0\}, 0)$, L_X is homeomorphic to the lens space $L(n, n-1)$. In this case $(X,0)$ is also given by $(\mathbb{C}^2,0)/\mathbb{Z}_n$ where $\mathbb{Z}_n$ acts on $(\mathbb{C}^2, 0)$ as the following.

$$\xi \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \xi \cdot x \\ \xi^{n-1} \cdot y \end{pmatrix} \qquad \xi^n = 1$$

(iii) If $(X,0)$ is given by $(\{x^a + y^b + z^c\}, 0)$ with $(a,b) = (b,c) = (c,a) = 1$ then the link L_X is an integeral homology sphere also known as the small Seifert 3-manifold $\Sigma(a,b,c)$.

The link L admits a natural contact structure, so called *Milnor fillable contact structure*, given by the $TL \cap J(TL)$: the J invariant subspace of TL. A *smoothing* of $(X, 0)$ is a proper flat map $\pi : \mathcal{X} \to \Delta$ where $\Delta = \{|t| < \epsilon\} \subset \mathbb{C}$ where $\mathcal{X}$ is a threefold isolated singularity with $\pi^{-1}(0) = (X, 0)$ and $\pi^{-1}(t)$ is smooth. For sufficiently small $|t|$, we call $\pi^{-1}(t)$ a *Milnor fiber* of $(X, 0)$. Topologically Milnor fiber M is a compact 4-manifold with L as the boundary and the diffeomorphism type of M is determined by π. Also there is natural Stein structure on M, providing a Stein filling of (L, ξ_{st}).

From now on, we will work on the $(\mathbb{C}^2, 0)/G$ where G is a finite subgroup of $GL(2, \mathbb{C})$. It is known that every quotient surface singularities is isomorphic to $(\mathbb{C}^2, 0)/G$ with small G, by small we mean G does not contain any reflections, and that for small G_1 and G_2, $(\mathbb{C}^2, 0)/G_i$ are analytically isomorphic if and only if G_1 is conjugate to G_2. Since G is finite, we may assume $G \subset U(2)$. The action of G can be extended to the action on the blowing-up of $\mathbb{C}^2$ at the origin. So we have an action on the exceptional divisor $E \cong \mathbb{CP}^1$, induced by the double covering $G \subset U(2) \to PU(2) \cong SO(3)$. The image of G in $SO(3)$ can be divided into 5 cases: a cyclic subgroup, a dihedral subgroup,the tetrahedral subgroup, the octahedral subgroup, and the icosahedral subgroup.

6.1.1 Minimal resolution of quotient surface singularities

Definition 6.1.4. For a surface singularity $(X, 0)$, a surjective proper analytic map $\pi : \widetilde{X} \to X$ is called a *resolution* if

(i) $\widetilde{X}$ is smooth,

(ii) $\pi : \widetilde{X} \setminus \pi^{-1}(0) \to X \setminus 0$ is an isomorphism.

$E := \pi^{-1}(0)$ is called the exceptional divisor. We say $\widetilde{X}$ is the minimal resolution of X if there is no (-1) curves in the exceptional divisor E. A geometric way to present resolution data is by the *dual resolution graph* $\Gamma_{\widetilde{X}}$. For a quotient surface singularity X, $\Gamma_{\widetilde{X}}$ is constructed as follows.

(i) Each vertex v in $\Gamma_{\widetilde{X}}$ corresponds to the irreducible component E_v of the exceptional divisor E.

(ii) If E_{v_i} and E_{v_j} intersect at the unique point then there is a edge between v_i and v_j in $\Gamma_{\widetilde{X}}$.

(iii) Each vertex v has a weights: the self intersection of E_v.

We call the number edge connected to the vertex v the *valence* of v and the self intersection of E_v the *degree* of v. If the absolute value of the degree is strictly less than valence, we call the vertex v *bad vertex*.

Hirzebruch-Jung continued fractions. We denote by $[c_1, \ldots, c_t] (c_i \geq 1)$ the *Hirzebruch-Jung continued fraction* defined recursively by $[c_t] = c_t$, and

$$[c_i, c_{i+1}, \ldots, c_t] = c_i - \frac{1}{[c_{i+1}, \ldots, c_t]}$$

Because a continued fraction $[c_i, c_{i+1}, \ldots, c_t]$ often describes a chain of smooth rational curves on a surface whose dual graph is given by

$$\overset{-c_1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{-c_2}{\bullet}\!\!-\!\!\!-\ \cdots\ -\!\!\!-\!\!\overset{-c_{t-1}}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{-c_t}{\bullet}$$

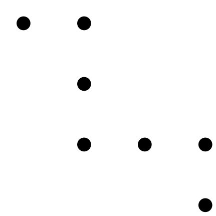
, we use by analogy the term 'blowing up' for the following operations and for their inverses 'blowing down'

$$\begin{aligned}
[c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_t] &\rightarrow [c_1, \ldots, c_{i-1}+1, 1, c_{i+1}+1, \ldots, c_t] \\
[c_1, \ldots, c_{t-1}] &\rightarrow [c_1, \ldots, c_{t-1}+1, 1]
\end{aligned}$$

Let n and q be positive integers with $n > q$. Then the continued fraction expansions of $\frac{n}{q}$ and $\frac{n}{n-q}$ are *dual*. By dual we mean that one can find the expansion of one from the expansion of the other using Riemenschneider's point diagram [51]: place in the ith row $a_i - 1$ dots, the first one under the last one of the $(i-1)$-st row then the jth column contains $b_j - 1$ dots.

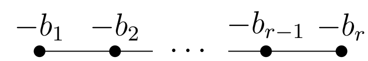
$$\begin{aligned} \frac{n}{q} &= [b_1, b_2, \ldots, b_r] \\ \frac{n}{n-q} &= [a_1, a_2, \ldots, a_e] \end{aligned}$$

Example 6.1.5. $\frac{29}{12} = [3, 2, 4, 2]$ and $\frac{29}{17} = [2, 4, 2, 3]$. Then Riemenschneider's point diagram is given by

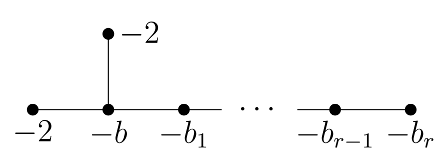


We remark that $[b_1, \ldots, b_r, 1, a_e, \ldots, a_1] = 0$.

Cyclic quotient surface singularities $A_{n,q}, (n, q) = 1, n > q > 0$.
For $\frac{n}{q} = [b_1, b_2, \ldots, b_{r-1}, b_r], b_i \geq 2$, the dual graph of minimal resolution is given by



Dihedral quotient surface singularities $D_{n,q}, (n, q) = 1, n > q > 1$.
For $\frac{n}{q} = [b, b_1, \ldots, b_{r-1}, b_r], b_i \geq 2$, the dual graph of minimal resolution is given by



Tetrahedral quotient surface singularities T_m, $m = 1, 3, 5$ mod 6.
For the list of dual resolution graphs, see Table 6.1.

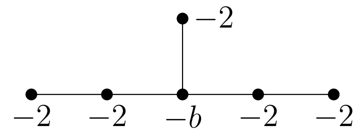
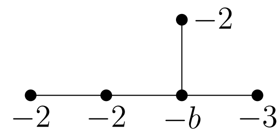
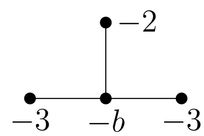
Tetrahedral singularity	Dual resolution graph
$m = 6(b-2)+1$	
$m = 6(b-2)+3$	
$m = 6(b-2)+5$	

Table 6.1: Dual resolution graphs of tetrahedral singularities

Octahedral quotient surface singularities O_m, $(m, 6) = 1$.
For the list of dual resolution graphs, see Table 6.2.

Icosahedral quotient surface singularities I_m, $(m, 30) = 1$.
For the list of dual resolution graphs, see Table 6.3 and Table 6.4.

6.2 Compactifying divisors and minimal symplectic fillings

From a symplectic convex filling X of given contact 3-manifold L, we get a closed symplectic 4-manifold Z by gluing a symplectic concave filling Y of L. For a link L of quotient surface singularity, Y can be given by a regular neighborhood of string of symplectically embedded 2-spheres so called the *compactifying divisor* K of singularity. By *string* we mean a configuration of spheres whose dual graph is an unbranched tree.

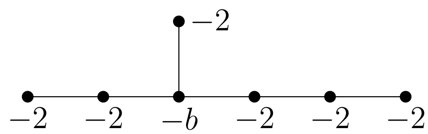
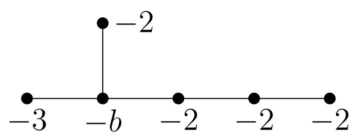
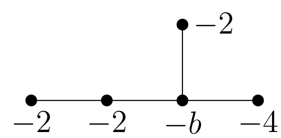
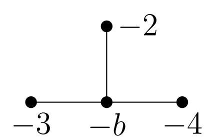
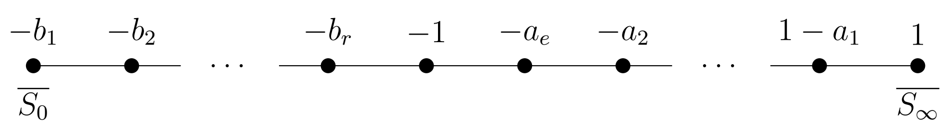
Octahedral singularity	Dual resolution graph
$m = 12(b-2)+1$	
$m = 12(b-2)+5$	
$m = 12(b-2)+7$	
$m = 12(b-2)+11$	

Table 6.2: Dual resolution graphs of octahedral singularities

6.2.1 Compactifying divisors of the quotient surface singularities

Cyclic quotient singularities Let $\mathbb{F}_1$ be a Hirzebruch surface, a ruled surface over $\mathbb{CP}^1$. Consider the sections S_0 and S_∞ with $S_0 \cdot S_0 = -1$ and $S_\infty \cdot S_\infty = 1$. By a sequence of blowing ups, starting from $p = S_0 \cap F$ where F is a fiber, we get the following string of curves with proper transform $\overline{S_0}$ and $\overline{S_\infty}$.



Then for the $A_{n,q}$ singularity with $n/q = [b_1, \ldots, b_r]$, the following string of

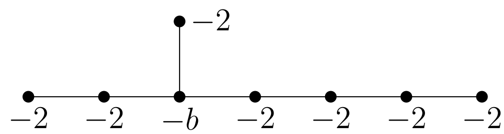
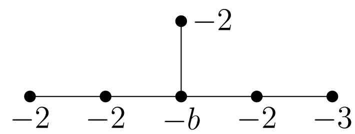
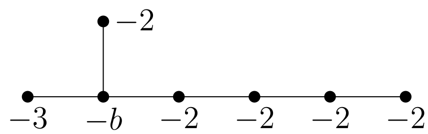
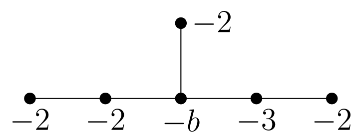
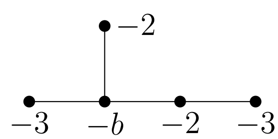
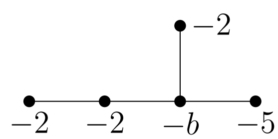
Icosahedral singularity	Dual resolution graph
$m = 30(b-2)+1$	
$m = 30(b-2)+7$	
$m = 30(b-2)+11$	
$m = 30(b-2)+13$	
$m = 30(b-2)+17$	
$m = 30(b-2)+19$	

Table 6.3: Dual resolution graphs of icosahedral singularities

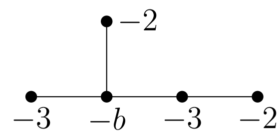
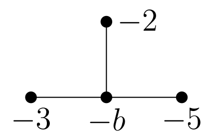
Icosahedral singularity	Dual resolution graph
$m = 30(b-2)+23$	
$m = 30(b-2)+29$	

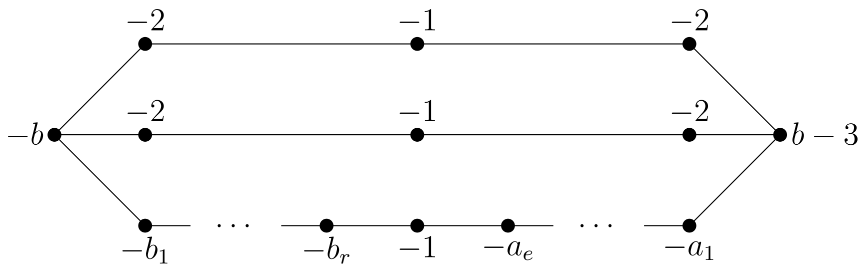
Table 6.4: Dual resolution graphs of icosahedral singularities

curves is the compactifying divisor for the singularity.

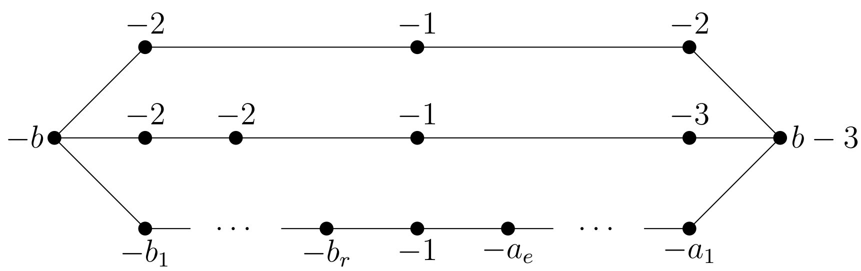
Non-cyclic quotient singularities The minimal resolution of non-cyclic quotient singularities has a central vertex with 3 arms connected to it. For the dihedral case, there are two arms of type $\overset{-2}{\bullet}$. Bhupal and Ono [5] divided tetrahedral, octahedral, icosahedral quotient singularities into two types: If one of the arms is $\overset{-2}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{-2}{\bullet}$, then we call $(3,2)$- type singularities and if one of the arms is $\overset{-3}{\bullet}$, then we call $(3,1)$- type singularities while there is an another arm of type $\overset{-2}{\bullet}$ both cases.

For a non-cyclic singularity whose degree of the central vetax is b, we get the string of curves by a sequence of blowing-ups, starting from $p_i = S_\infty \cap F_i$ $(1 \le i \le 3)$ where F_i are distinct fibers and S_∞ is a section with $S_\infty \cdot S_\infty = b$ in $\mathbb{F}_b$. Then the right part of Figure 6.2.1 gives the compactifying divisors of the non-cyclic singularities.

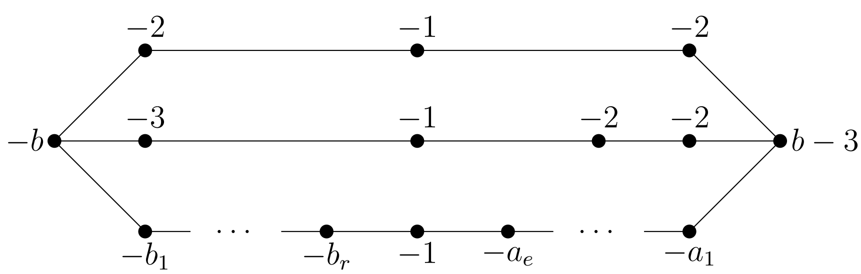
For the symplectic structure of compactifying divisor, we refer the following



Dihedral case



(3,2) type case



(3,1) type case

Figure 6.1: Strings of rational curves

theorem of H. Ohta and K. Ono [45].

Theorem 6.2.1. *Let S be a projective algebraic variety, which is non-singular away from an isolated surface singularity X. Then the outside of the link L of P is a strong concave filling of L.*

6.2.2 Minimal symplectic fillings

Minimal symplectic fillings are classified by P. Lisca [36], [37] for cyclic cases and by M. Bhupal and K. Ono [5] for non-cyclic case. Their strategy for classification is as follows: As we have already mentioned, for each minimal symplectic filling W of X, we have a closed symplectic 4-manifold Z obtained by gluing a regular neighborhood Y of the compactifying divisor K. Hence the classification problem of minimal symplectic fillings reduces to the classification of the pair (Z, K). Explicitly,

Proposition 6.2.2 ([5])**.** *Let W be a minimal symplectic fillings of quotient surface singularities. Then there is a symplectic structure on smooth 4-manifold*

$$Z = W \cup_L Y$$

which is compatible with the symplectic structure of W. Furthermore, Z is a rational symplectic 4-manifold.

Since there is a symplectic sphere with self-intersection 1 in each compactifying divisor for cyclic singularities $(X, 0)$, Z is a rational symplectic 4-manifold by D. Mcduff. We briefly review P. Lisca's classification. For the classification of minimal symplectic fillings for a cyclic case, we need the following definition.

Definition 6.2.3. A e-tuple of nonnegative integers $(n_1 \dots, n_e)$ is called *admissible* if each of the denominators in the continued fraction $[n_1 \dots, n_e]$ is positive.

It is easy to see that an admissible e-tuple of nonnegative integers is either 0 or consists only of positive integers. Let $\mathcal{Z}_e$ be the set of admissible e-tuples of nonnegative integers such that $[n_1 \dots, n_k] = 0$, or equivalently, $\mathcal{Z}_e$ be the set of e-tuple of integers which can be obatained from $[1, 1]$ by a squence of blow-ups. For $\dfrac{n}{n-q} = [a_1, \dots, a_e]$, let

$$\mathcal{Z}_e(\frac{n}{n-q}) = \{(n_1, \dots, n_e) |\ 0 \le n_i \le a_i, \quad i = 1, \dots, e\}$$

Now we construct a smooth 4-manifold $W_{n,q}(\mathbf{n})$ whose boundary is diffeomorphic to the the link of the cyclic singularity of type $A_{n,q}$ which is also known as lens space $L(n, q)$ for each e-tuple $\mathbf{n} \in \mathcal{Z}_e(\dfrac{n}{n-q})$ using a corbodism $w_{n,q}(\mathbf{n})$ between $S^1 \times S^2$ and $L(n, q)$: Consider the chain of e-unknots in S^3 with respective framings $n_1, \dots, n_k$. Let $N(\mathbf{n})$ be the 3-manifolds obtained by Dehn surgery on this framed link. Since $[n_1, \dots, n_e] = 0$, it is clear that $N(\mathbf{n})$ is diffeomorphic to $S^1 \times S^2$. Consider a framed link L in $N(\mathbf{n})$ as in Figure 6.2. Then the cobordism $w_{n,q}(\mathbf{n})$ is obtained by attaching 4-dimensional 2-handles to the $L \subset S^1 \times S^2 \times \{1\} \subset S^1 \times S^2 \times I$. For some diffeomorphis $\varphi : N(\mathbf{n}) \to S^1 \times S^2$,

$$W_{n,q}(\mathbf{n}) := w_{n,q}(\mathbf{n}) \cup_\varphi S^1 \times D^3$$

Since any self-diffeomorphism φ of $S^1 \times S^2$ extends to $S^1 \times D^3$, the diffeomorphism type of $W_{n,q}(\mathbf{n})$ is independent to the choice of φ.

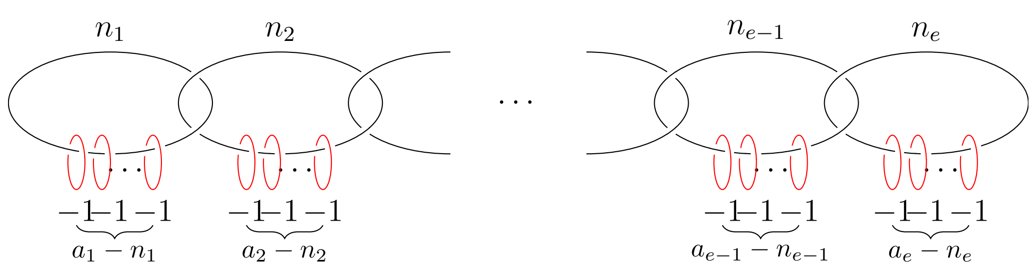


Figure 6.2: The framed link $L \subset N(\mathbf{n})$

According to P. Lisca, any symplectic filling of $(L(n, q), \xi_{st})$ is orientation preserving diffeomorphic to a blow up of $W_{n,q}(\mathbf{n})$ for some $\mathbf{n} \in \mathcal{Z}_e(\dfrac{n}{n-q})$. Explicitly,

Theorem 6.2.4 ([36], [37])**.** *For $n > q \geq 1$ be coprime integers. Then,*

(i) Let (W, ω) be a symplectic filling of the $(L(n,q), \xi_{st})$. Then, there exist $\mathbf{n} \in \mathcal{Z}_e(\frac{n}{n-q})$ such that W is orientation preserving diffeomorphic to a smooth blow up of $W_{n,q}(\mathbf{n})$

(ii) For every $\mathbf{n} \in \mathcal{Z}_e(\frac{n}{n-q})$, the 4-manifold $W_{n,q}(\mathbf{n})$ carries a symplectic form ω such that $(W_{n,q}(\mathbf{n}), \omega)$ is a symplectic filling of $(L(n,q), \xi_{st})$. Moreover, there are no classes in $H_2((W_{n,q}(\mathbf{n}); \mathbb{Z})$ with self-intersection -1.

(iii) For $\mathbf{n} \in \mathcal{Z}_e(\frac{n}{n-q})$ and $\mathbf{n}' \in \mathcal{Z}'_e(\frac{n'}{n'-q'})$, $(W_{n,q}(\mathbf{n}) \sharp r\overline{\mathbb{CP}^2})$ is orientation preserving diffeomorphic to $(W_{n',q'}(\mathbf{n}')\sharp s\overline{\mathbb{CP}^2})$ if and only if:

(a) $(n, r) \equiv (n', s)$ and $(q, \mathbf{n}) \equiv (q', \mathbf{n}')$, or

(b) $(n, r) \equiv (n', s)$ and $(\overline{q}, \overline{\mathbf{n}}) \equiv (q', \mathbf{n}')$.

Here $\overline{q}$ is the only integer satisfying

$$n > \overline{q} \geq 1, \quad q\overline{q} \equiv 1 \ mod \ n.$$

and $\overline{\mathbf{n}}$ is the involution of $\mathbf{n}$. That is

$$\mathbf{n} = [n_1, \ldots n_e] \to \overline{\mathbf{n}} = [n_e, \ldots n_1]$$

For the non-cyclic cases, M. Bhupal and K. Ono proved that Z is rational by showing that there is a sequence of blow-downs and blow-ups transforming the compactifying divisor K in Z into a configuration containing a cuspidal curve with positive self-intersection number; then Z is rational by H. Ohta and K. Ono [44].

We present these blow-downs and blow-ups in Figure 6.3 for dihedral singularities and in Figure 6.4 and Figure 6.5 for tetrahedral, octahedral, and icosahedral singularities. We first blow-up successively (if necessary) the intersection point of the central curve and the third branch until the self-intersection number of the central curve has dropped to -1. Then we blow down or blow up as described in figures so that we get a rational 4-manifold Z_2 with a cuspidal curve C with $C \cdot C > 0$ and a string of 2-spheres $C_1, \ldots, C_k$ (plus some extra 2-spheres intersecting C at the cusp). Let $\widetilde{K} \subset Z_2$ be the proper transform of $K \subset Z$. Since the blow-ups and blow-downs occur only on K and its proper transforms, we have

$$W \cong Z \setminus \nu(K) \cong Z_2 \setminus \nu(\widetilde{K})$$

Since W is minimal, every (-1)-curve in Z_2 should intersect $\widetilde{K}$. Let Z_1 be the rational 4-manifold obtained by contracting all (-1)-curves in $Z_2 \setminus C$, that is, the (-1)-curves not intersecting C. Then, according to Bhupal and Ono , Z_1 can be obtained by a sequence of blowing-ups at p (including infinitely near points over p) in a cuspidal curve C in $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$ and (if necessary) some points on C as shown in Figures. Once we get Z_1, we blow up points on $C'_i, \ldots C'_k$ to get $C_i, \ldots C_k$ so that we have Z_2. Note that the sequence of blow-ups $\pi : Z_2 \to Z_1$ occur only on C'_i. For details, refer M. Bhupal and K. Ono [5].

They also showed that there are only finitely many ways to blow up $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$ for constructing Z_2, and they presented all possible ways of blow-ups, or equivalently, they presented all possibilities for the ways that (-1)-curves in Z_2 intersect $\widetilde{E}_\infty$. Then they concluded that

Proposition 6.2.5 ([5]). *There are only finitely many symplectic deformation types of minimal symplectic fillings of a non-cyclic quotient surface singularity.*

6.2.3 P-resolution

We first define a normal surafce singularity *of class T* which appears in the definition of P-resolution of a quotient surface singularity.

Definition 6.2.6. A normal surface singularity is *of class T* if it is a quotient surface singularity and it admits a $\mathbb{Q}$-Gorenstein one-parameter smoothing. Equivalently, it is a rational double point singularity or a cyclic quotient

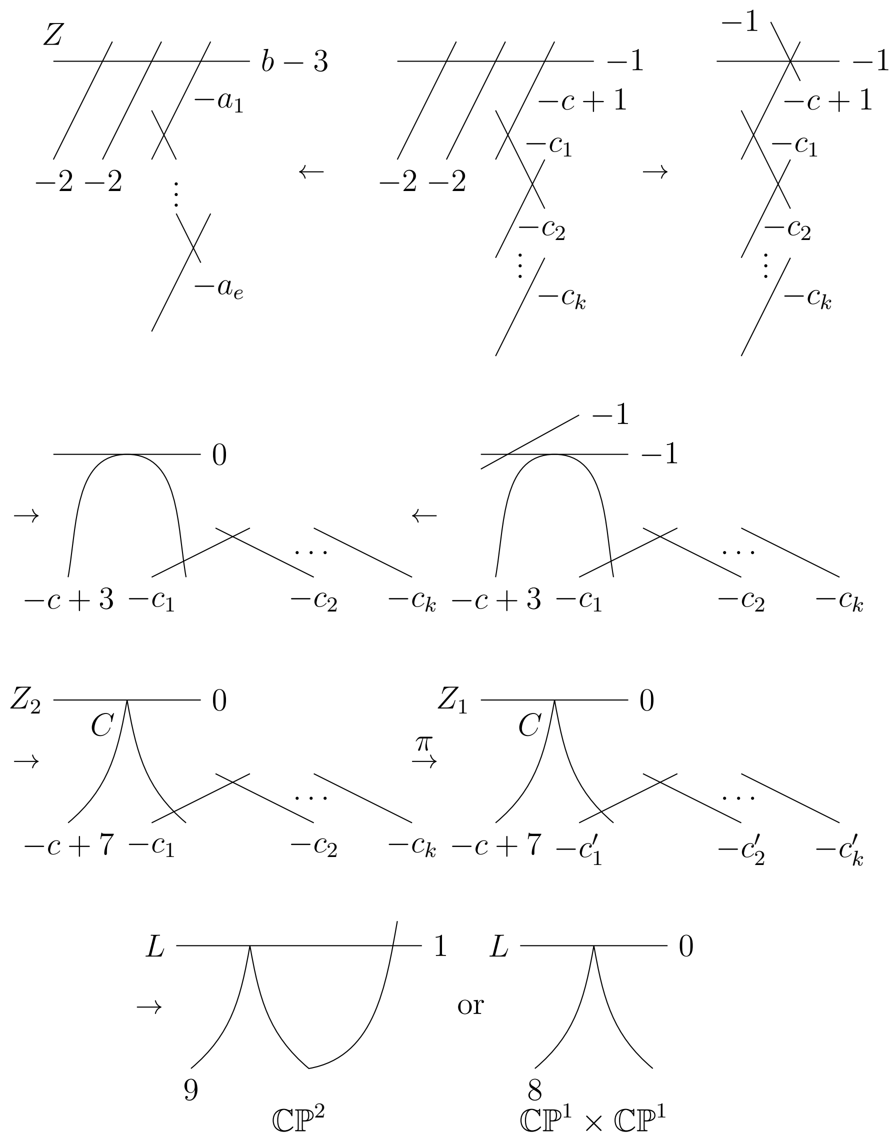


Figure 6.3: Dihedral singularity cases

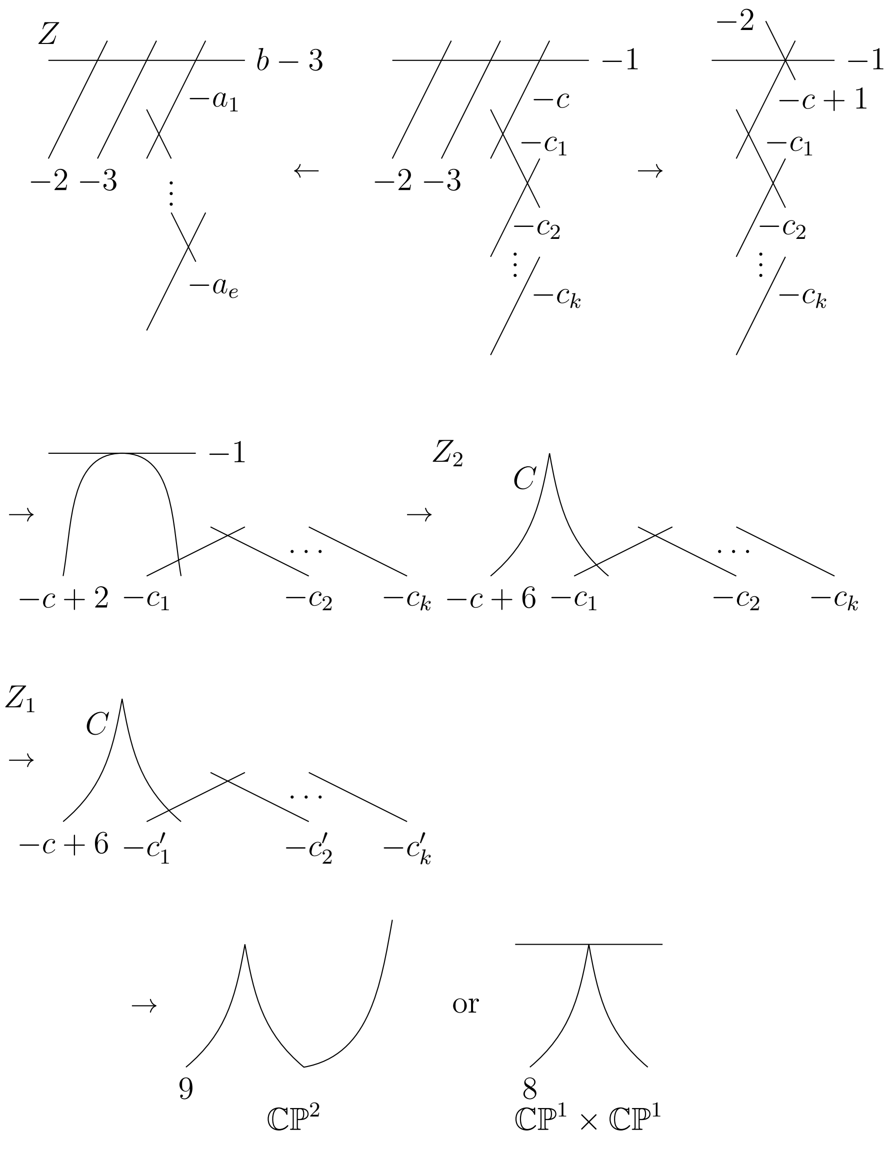


Figure 6.4: $(3, 2)$ type singularity cases

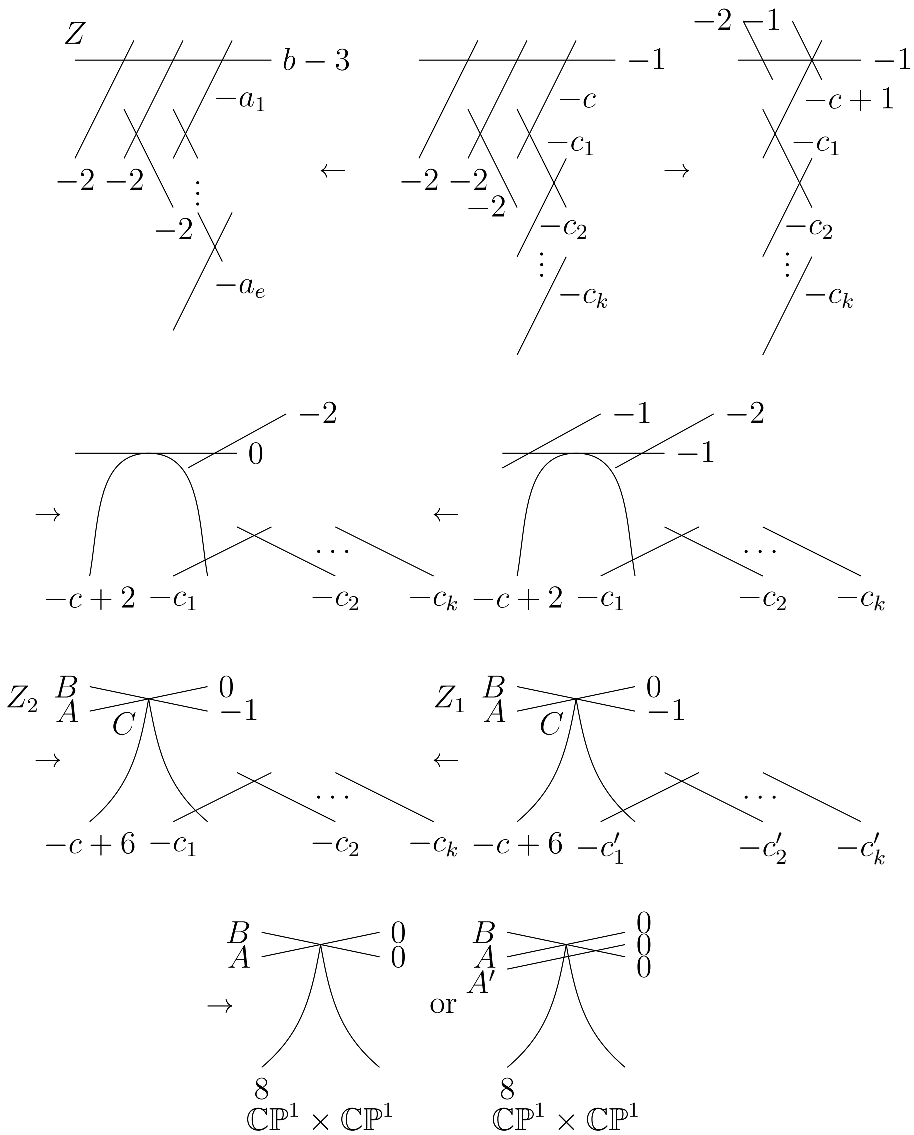


Figure 6.5: $(3, 1)$ type singularity cases

surface singularity of type $\frac{1}{dn^2}(1, dna-1)$ with $d \geq 1$, $n \geq 2$, $1 \leq a < n$, and $(n, a) = 1$.

Note that the one-parameter $\mathbb{Q}$-Gorenstein smoothing of a singularity of class T is interpreted topologically as a rational blowdown surgery defined by R. Fintushel and R.Stern [19] and extended by J. Park [48]. Furthermore, due essentially to J. Wahl [56], a cyclic quotient surface singularity of class T can be recognized from its minimal resolution as follows:

Proposition 6.2.7. *(i) The singularities* $\overset{-4}{\bullet}$ *and* $\overset{-3}{\bullet}\!-\!\overset{-2}{\bullet}\!-\cdots-\!\overset{-2}{\bullet}\!-\!\overset{-3}{\bullet}$ *are of type T.*

(ii) If $\overset{-b_1}{\bullet}\!-\!\overset{-b_2}{\bullet}\!-\cdots-\!\overset{-b_{r-1}}{\bullet}\!-\!\overset{-b_r}{\bullet}$ *is of type T, then also*

$$\overset{-2}{\bullet}-\overset{-b_1}{\bullet}-\cdots-\overset{-b_{r-1}}{\bullet}-\overset{-(b_r+1)}{\bullet}$$

and

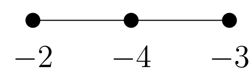
$$\overset{-(b_1+1)}{\bullet}-\overset{-b_2}{\bullet}-\cdots-\overset{-b_r}{\bullet}-\overset{-2}{\bullet}$$

(iii) Every singularity of class T that is not a rational double point can be obtained by starting with one of the singularities described in (i) and iterating the steps described in (ii) above.

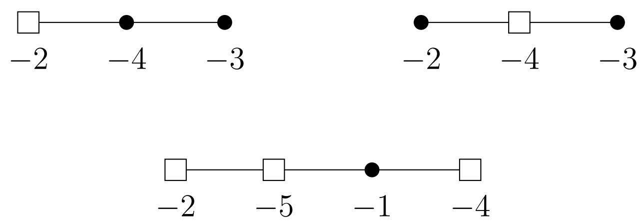
Definition 6.2.8. A *P-resolution* $f : (Y, E) \to (X, 0)$ of a quotient surface singularity $(X, 0)$ is a partial resolution such that Y has at most rational double points or singularities of type T and K_Y is ample relative to f.

We describe a P-resolution $Y \to X$ by indicationg the f-exceptional divisors on the minimal resolution $f : Z \to Y$ of Y. The intersection condition in the definition of a P-resolution can be checked on Z: every (-1) curve on Z must intersect two curves E_1 and E_2, which are exceptional for singularities Y_1 and Y_2 of type T on Y, and the sum of the coefficients k_i of E_i in the canonical cycles K_{Z_i} is less than -1.

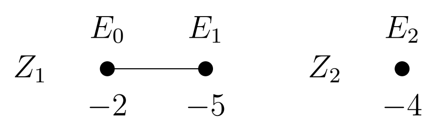
Example 6.2.9. Let X be a cyclic singularity of type $(19, 11)$. Since $19/11 = [2, 4, 3]$, the minimal resolution of X is given by



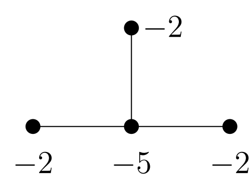
There are three P-resolutions of X as the following.



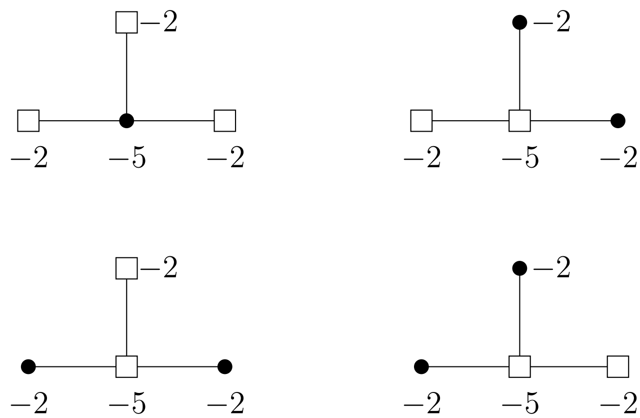
For the last one, we can check the intersection condition by computing K_{Z_i}: $K_{Z_1} = -\frac{1}{3}E_0 - \frac{2}{3}E_1$, $K_{Z_2} = -\frac{1}{2}E_2$ by the adjunction equality.



Example 6.2.10. Let X be a dihedral singularity of type (9,2). Since $9/2 = [5, 2]$, the minimal resolution of X is given by



There are four P-resolutions of X as the following. Note that there are certain symmetries in the list of P-resolutions.



On the other hand, Milnor fibers are invariants of the irreducible components of the reduced versal deformation space of X. According to Kollar-Shepherd-Barron [35], there is a one-to-one correspondence between the irreducible components and P-resolutions of X. Therefore Milnor fibers of irreducible components are in one-to-one correspondence with P-resolutions. J. Stevens [53] determined all P-resolutions of quotient surface singularities.

As mentioned above, minimal symplectic fillings of a cyclic quotient surface singularity of type (n, q) are parametrized by P. Lisca by the set $\mathcal{Z}_e(\frac{n}{n-q})$. On the other hand, J.A. Christophersen [9] , J. Stevens [52], and T. de Jong and D. van Straten [32] parametrized by the same set $\mathcal{Z}_e(\frac{n}{n-q})$ (but with different methods) the reduced irreducible components of the versal deformation space of $(X, 0)$. So P. Lisca raised the following conjecture: The Milnor fiber of the irreducible component of the reduced versal base space of the cyclic quotient surface singularity $(X, 0)$ parametrized by $\mathbf{n} \in \mathcal{Z}_e(\frac{n}{n-q})$ is diffeomorphic to $W_{n,q}(\mathbf{n})$. And then the conjecture was solved affirmatively by Nemethi-Popescu-Pampu [42].

For non-cyclic quotient surface singularities, note that the number of P-resolutions in Stevens and that of minimal symplectic fillings in M. Bhupal and K. Ono are the same. Hence it would be natural to raise the same question as the cyclic case. Since a Milnor fiber is a minimal symplectic fillings, the question is if every minimal symplectic filling can be realized as a Milnor fiber. In [47], H. Park, J. Park, D shin and G. Urzúa gives an explicit one-to-one correspondence between Milnor fibers and minimal symplectic fillings.

6.3 Lefschetz fibration structures on minimal symplectic fillings

Since the Milnor fiber has a natrual Stein structure, our previous arguments say that every minimal symplectic filling of the link of quotient surface singularity is Stein filling. Although the existence of Lefschetz fibration structure(in fact PALF structure) on Stein fillings are known, finding an explicit monodromy description is somewhat a different problem. For the cyclic quotient singularity case, M. Bhupal and B. Ozbagci [6] provided an algorithm to present each minimal symplectic filling as an explicit Lefschetz fibration structure. Furthermore, they showed that each PALF structure on minimal symplectic fillings can be obtained from the minimal resolution by monodromy substitutions which corresponds to rational blow-downs topologically. Generalizing their result, the main goal of this section is to show the following theorem.

Theorem 6.3.1 ([10])**.** *Every minimal symplectic fillings of any quotient surface singularity admits a genus-*0 *or genus-*1 *Lefschetz fibration structure. Furthermore, each such a filling is obtained by rational blow-downs from the minimal resolution of its singularity.*

6.3.1 Monodromy substitutions and rational blow-downs

In [19], R. Fintushel and R. Stern introduced a rational blow-down surgery: Let C_p be a smooth 4-manifold obtained by plumbing disk bundles over 2-sphere according to the following linear diagram:

$$\overset{-(p+2)}{\bullet}\!\!-\!\!-\!\!\overset{-2}{\bullet}\!-\!-\ \cdots\ -\!-\overset{-2}{\bullet}\!\!-\!\!-\!\!\overset{-2}{\bullet}$$

Then the boundary of C_p is a lens space $L(p^2, p-1)$, which bounds a rational ball B_p, i.e., $H_*(B_p; \mathbb{Q}) = H_*(D^4; \mathbb{Q})$. So, if there is a embedding of C_p in a smooth 4-manifold X, one can construct a new smooth 4-manifold X_p by replacing C_p with B_p. This procedure is called a *rational blowdown* surgery and we say that X_p is obtained by rational blowing down X. Furthermore, M. Symington proved that a rational blowdown manifold X_p admits a symplectic structure in some cases. For example, if X is a symplectic 4-manifold containing a configuration C_p such that all 2-spheres in C_p are symplectically embedded and intersect positively, then the rational blowdown manifold X_p also admits a symplectic structure. Later, the Fintushel–Stern's rational blowdown surgery is generalized by J. Park [48] using a configuration $C_{p,q}$ obtained by plumbing disk bundles over 2-sphere according to the dual resolution graph of $L(p^2, pq-1)$ which also bounds a rational ball $B_{p,q}$ as follows:

Definition 6.3.2. Suppose X is a smooth 4-manifold containing a configuration $C_{p,q}$. Then one can construct a new smooth 4-manifold $X_{p,q}$, called a (*generalized*) *rational blowdown* of X, by replacing $C_{p,q}$ with the rational ball $B_{p,q}$. We also call this a *(generalized) rational blowdown* surgery.

Next, we introduce a notion of monodromy substitution which is closely related to a rational blowdown surgery. That is, we briefly explain how to replace a rational blowdown surgery by a monodromy substitution in some cases.

Suppose that a symplectic 4-manifold X with a possibly non-empty boundary admits a Lefschetz fibration structure characterized by a monodromy factorization $\mathcal{W}_X$. Assume that W and W' are distinct products of right-handed Dehn twists which give the same element as a global monodromy in the mapping class group of the fiber. If there is a partial monodromy factorization equal to W in the monodromy factorization $\mathcal{W}_X$ of X, then we can obtain a Lefschetz fibration structure on a new symplectic 4-manifold X' whose monodromy factorization $\mathcal{W}_{X'}$ is obtained by replacing W with W'. Note that the diffeomorphism types and the induced contact structures of ∂X and $\partial X'$ are the same. We call this procedure a *monodromy substitution*. For example, a famous lantern relation gives a rational blowdown surgery involving the lens space $L(4,1)$ [15]: The PALF with a monodromy $abcd$ gives a configuration C_2 while the PALF with a monodromy xyz gives a rational ball B_2. As another example, the *daisy relation*, introduced in [14], gives the monodromy substitution for a configuration C_p and a rational ball B_p. One can also find the monodromy substitution for a (generalized) rational blowdown surgery in [14].

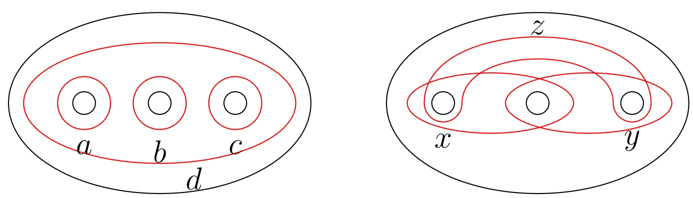


Figure 6.6: Lantern relation

6.3.2 Review for the cyclic singularity cases

Now we briefly review the algorithm of Bhupal-Ozbagci for minimal symplectic fillings of cyclic quotient singularities. Recall that minimal symplectic fillings of cyclic quotient singularities of type (n,q) is parametrized by $\mathcal{Z}_e(\dfrac{n}{n-q})$. For each $\mathbf{n} \in \mathcal{Z}_e(\dfrac{n}{n-q})$, they construct a genus-0 PALF structure on $S^1 \times D^3$ so that the attaching circles of (-1)-framed 2-handles in $W_{n,q}(\mathbf{n})$ lies on the generic fiber.

Depending on blow up sequence from (0), one can construct PALF corresponds to $\mathbf{n} \in \mathcal{Z}_e$ as follows: For each $\mathbf{n} \in \mathcal{Z}_e$, the generic fiber $F_{\mathbf{n}}$ is e holed disk. We may assume the holes in the disk are ordered linearly from left to right as in Figure. If $\mathbf{n} \in \mathcal{Z}_e$ is obtained from $\mathbf{n}' \in \mathcal{Z}_{e-1}$ by blow up at jth term $(1 \le j \le e-2)$, then we consider the generic fiber $F_{\mathbf{n}}$ as the surface obtained from $F_{\mathbf{n}'}$ by splitting $j+1$ hole so that vanishing cycles $\{x_i, i = 1,2,\dots,e-2\}$ for $\mathbf{n}'$ are naturally extend $\{\widetilde{x}_i, i = 1,2,\dots,e-2\}$ to $F_{\mathbf{n}}$. Then monodromy changes from $x_1x_2\cdots x_{e-2}$ to $\widetilde{x}_1\widetilde{x}_2\cdots\widetilde{x}_{e-2}\beta_j$, where β_j

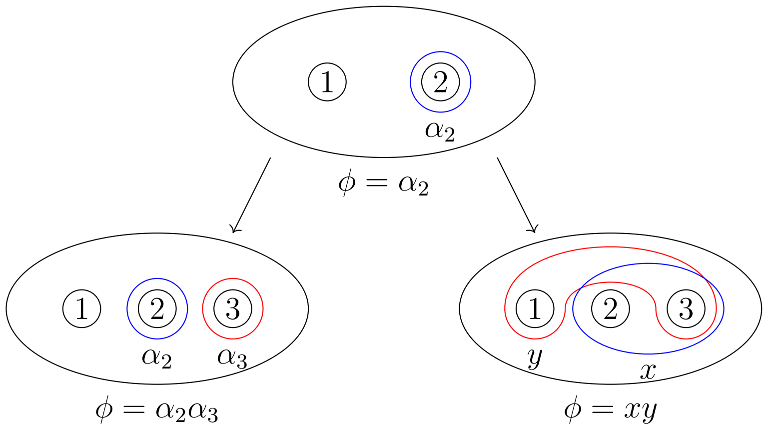


Figure 6.7: PALF on $S^1 \times D^3$ corresponds to $(1,1)$, $(1,2,1)$ and $(2,1,2)$

is curve on F_n such that encircling the holes labelled $1, \dots j, j+2$, skipping the hole labelled $j+1$. For the blow up at $(e-1)$th term, just add eth hole to $F_{\mathbf{n}'}$ at the right of $(e-1)$th hole and add a Dehn twist on curve encircling eth hole. To complete PALF structure on $W_{n,q}(\mathbf{n})$, note that a (-1) framed 2-handle h in $w_{n,q}(\mathbf{n})$ corresponds to a Dehn twist on curve encircling first i holes if the attaching circle of h is meridian of n_i framed unknot.

Bhupal-Ozbagci also showed that the word for each minimal symplectic filling is obtained by a sequence of monodromy substitution from the minimal resolution inductively. Each ith step of the sequence consists of the monodromy substitution of the form

$$w_i \gamma_1^{m_{i,1}} \cdots \gamma_e^{m_{i,e}} = w_{i+1}\gamma_{a_{i+1}}$$

where w_i is word for some $\mathbf{n} \in \mathcal{Z}_e(\dfrac{n}{n-q})$ with a_ith component of $\mathbf{n}$ is 1. One can easily check that PALF with monodromy $w_{i+1}\gamma_{a_{i+1}}$ is a rational ball. Note that they didn't use the explicit relations in [14].

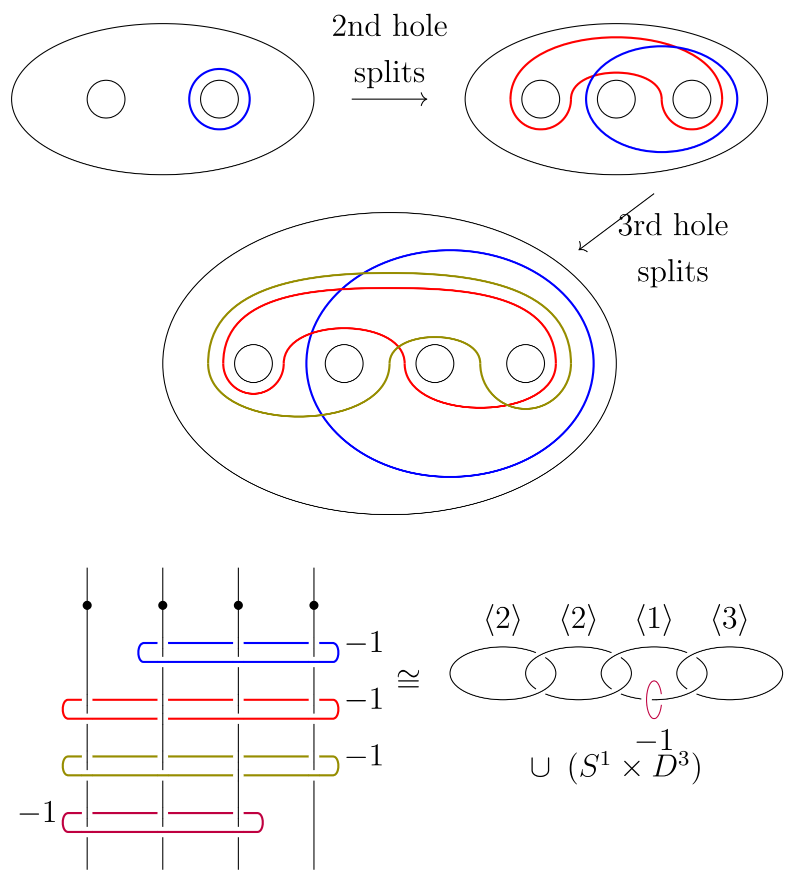


Figure 6.8: a PALF on $W_{9,2}((2,2,1,3))$

6.3.3 Lefschetz fibrations on the minimal resolution of non-cyclic cases

Our strategy for the proof of theorem is the following: For a given P-resolution Y of $(X, 0)$, find a PALF X on the minimal resolution of $(X, 0)$ so that there is a monodromy substitution such that after the substitution, underlying 4-manifold is diffeomorphic to the manifold obtained by rationally blowing down the singularities of type T in the minimal resolution of Y.

Non bad vertex cases. There is a well known genus-0 PALF structure on the minimal resolution of quotient singularity if the dual graph Γ does not have any bad vertex [23]: For each vertex v_i with degree $-b_i$, consider the b_i holed sphere Σ_i. Then fiber surface Σ is obtained by gluing Σ_i along their boundaries according to Γ and vanishing cycles are the boundary parallel curves of each Σ_i. Note that we end up with only one right-handed Dehn twist on the connecting neck.

Bad vertex cases. Now we construct a genus-1 PALF structures on the minimal resolution of non cyclic quotient singularities whose dual graph Γ has a bad vertex: First consider a PALF structure X_L on maximal linear subgraph Γ_L of Γ. We could get a 4-dimensional Kirby diagram of Γ by adding a 2-handle h or two 2-handles $\{h_1, h_2\}$ to X_L depending on the type of arm which is not in Γ_L. After introducing a cancelling 1-handle/2 handle pair, the 2-handles which are not from X can be thought of as a vanishing cycles of new fiber F which is obtained by attaching a 1-handle to the F_L as in Figure 6.10. It remains to check that the induced contact structure on boundary is Milnor fillable contact structure. Recall that a contact 3-manifold (Y, ξ), 2-plane field ξ indueces a $Spin^c$ structure t_ξ. If (X, J) is a Stein filling of (Y, ξ), then t_ξ is restriction of $Spin^c$ structure of X to ∂X induced by its complex structure J. On the other hand there is a theorem of Gay-Stipsciz [24] which characterize the Milnor fillable structures on the links of quotient surface singularities

Theorem 6.3.3. *Suppose that the small seifert fibered 3-manifold* $M =$

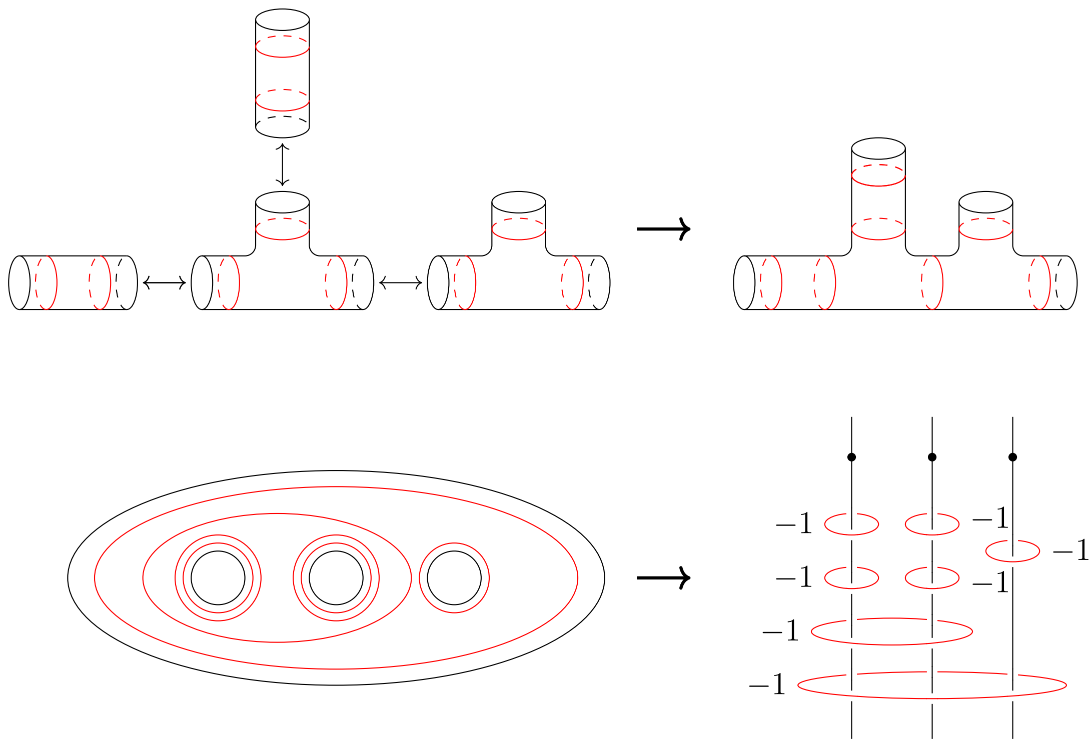
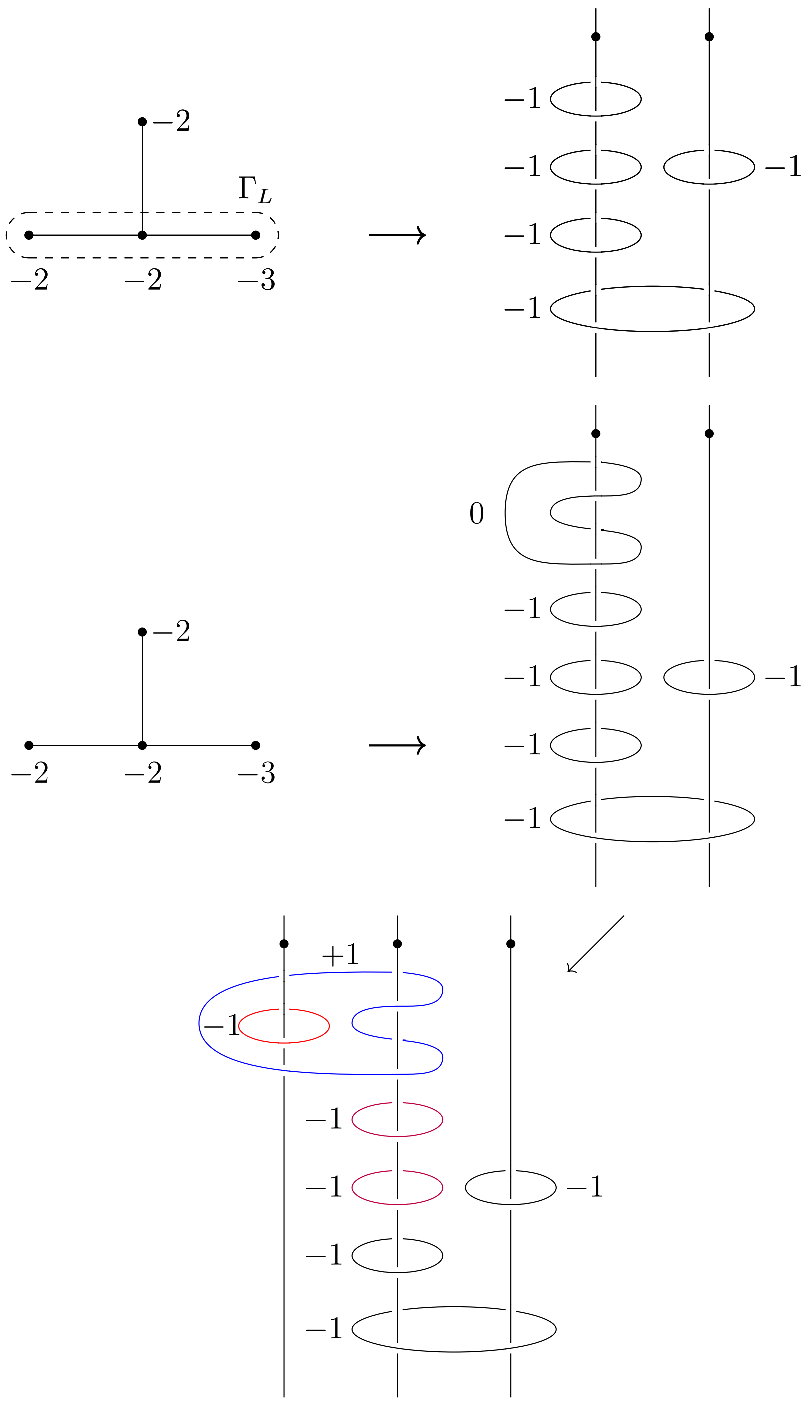


Figure 6.9: genus-0 PALF structure on the minimal resolution of $D_{8,3}$

$M(s_0; r_1, r_2, r_3)$ satifies $s_0 \leq -2$ and M is an L-space. Then two tight contact structure ξ_1, ξ_2 on M are isotopic if and only if $t_{\xi_1} = t_{\xi_2}$.

Since minimal resolution (X, J) is simply connected, the $Spin^c$ structure α of X is determined by Chern class $c_1(\alpha) = c_1(J)$. For the minimal resolution of quotient surface singularities $c_1(J)$ determined by its value on each vertex and it satisfies the Adjunction equality. From the PALF structure of X, we can compute first Chern class in terms of vanishing cycles C_i: $c_1(J)$ is represented by a cocycle whose value on 2-handle corresponding to the C_i is $r(C_i)$ [18], [29]. For the vertices in Γ_L satisfy the Adjunction equlity since the Lefschetz fibration structure for no bad vertex cases induces Milnor fillable contact structure [46]. The homology class of the vertices not in Γ_L can be represented by new vanishing cycles together with some vanishing cycles in Γ_L. For example, in Figure 6.10, -2 sphere which is not in Γ_L is obtained


−2
Γ_L
−2 −2 −3
−1
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−1
−1
0
−1
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−1 −1
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−1

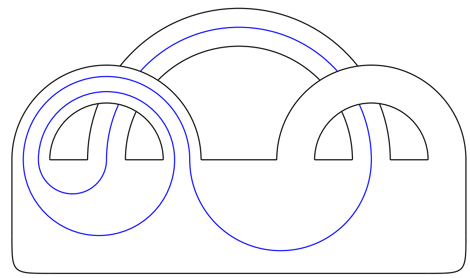


Figure 6.10: genus-1 PALF structure on the minimal resolution of $D_{5,3}$

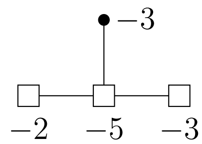
by blue, red and two purple vanishing cycles. By computing the rotation number of these vanishing cycles, we can check that the induced contact structure is indeed Milnor fillable.

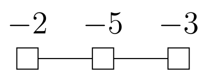
6.3.4 Symplectic fillings as Lefschetz fibrations

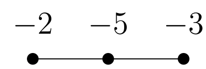
Now we construct PALF structures on minimal symplectic fillings corresponding to P-resolutions. Here, corresponding means that the diffeomorphism type of fillings is obtained by rationally blowing down the singularities of type T from the minimal resolution of Y. Note that every Milnor fiber of a quotient singularity can be obtained topologically by rationally blowing down the corresponding P-resolution Y. Let Γ_Z be the dual graph of the minimal resolution Z of Y.

Case 1 One can observe that most of P-resolutions satisfy the following condition: There is a maximal linear subgraph Γ_L of Γ_Z containing all the singularities of type T. For the P-resolution Y satisfying above condition, one can construct PALF structure on corresponding symplectic filling of X as follows. Note that Γ_L with singularities of type T is also P-resolution Y' of some cyclic quotient singularity X'. Starting from PALF structure $(\Sigma'_X, z_1 z_2 \cdots z_m)$ on the minimal resolution of X', one can get a PALF structure $(\Sigma_X, x_1 \cdots x_n \widetilde{z}_1 \widetilde{z}_2 \cdots \widetilde{z}_m)$ where genus of Σ_X and n depends on the existence of bad vertex and type of arm that does not contained in Γ_L. Here $\widetilde{z}_i$ is a natural extension of z_i to Σ_X. Since there is corresponding monodromy substitution for each P-resolutions of cyclic singularities, one could get the explicit PALF structure corresponding to Y.

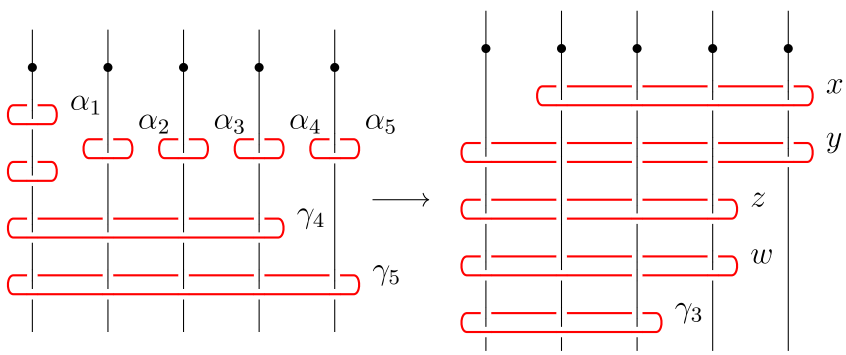
Example 6.3.4. The tetrahedral singularity $T_{6(5-2)+5}$ has the following P-resolution Y.



A PALF structure on P-resolution 

of 

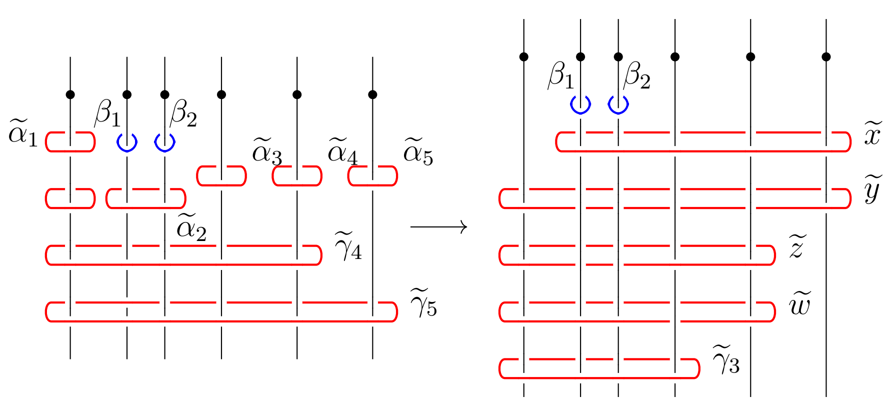
can be obtained by monodromy substitution depicted in figure



and

$$\alpha_1^2\alpha_2\alpha_3\alpha_4\alpha_5\gamma_4\gamma_5 = xyzw\gamma_3.$$

Hence we have desired PALF structure on Y



by using monodromy substitution of the form

$$\beta_2\widetilde{\alpha}_1^2\widetilde{\alpha}_2\widetilde{\alpha}_3\widetilde{\alpha}_4\widetilde{\alpha}_5\widetilde{\gamma}_4\widetilde{\gamma}_5 = \beta_2\widetilde{x}\widetilde{y}\widetilde{z}\widetilde{w}\widetilde{\gamma}_3.$$

which can be interpreted as rational blow down topologically.

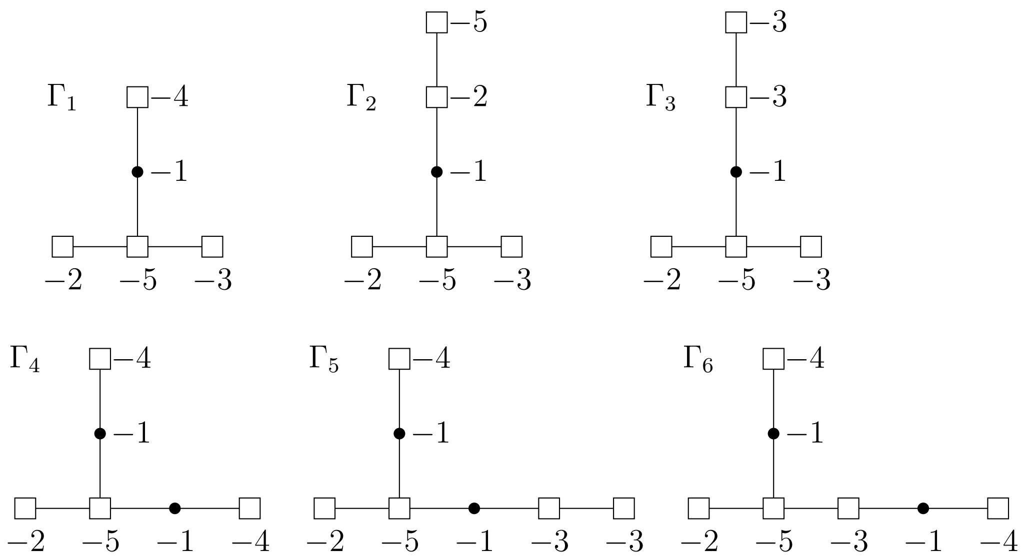


Figure 6.11: Types of subgraph Γ_i

Case 2 One can observe that each P-resolutions which does not satisfying the condition (i.e., for any maximal linear subgraph Γ_L of Y, there is a singularity type of T does not belong to Γ_L) contains one of the subgraphs Γ_i in Figure 6.11 except 2 cases which we will see later.

Let X_i be the 4-manifold which is obtained from the $\{\Gamma_i\}$ by blowing down all (-1) curves. Since the subgraphs contain all the singularities of type T of Y, it suffices to find explicit PALF structures for subgraphs for the PALF structure of Y. The P-resolution of the above types involves two steps; First we rationally blow down all the singularities of type T except the one contains central vertex. This yields the 4-manifold X_i which also can be obtained from Γ_i by blowing down all the (-1) curves. Then we can find the monodromy substitution corresponding to the singularities of type T containing center vertex from the natural PALF structures for X_i.

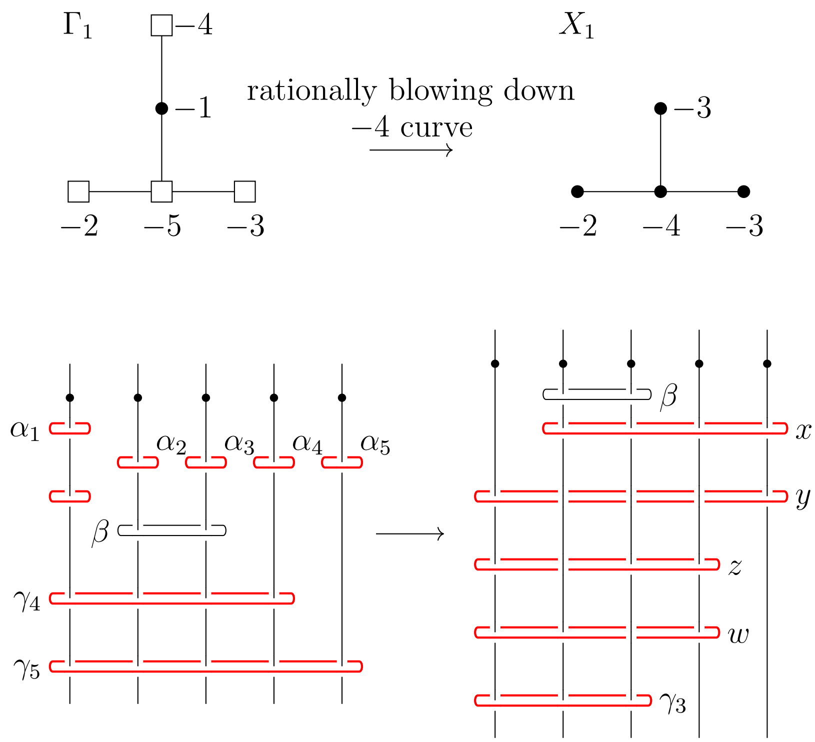
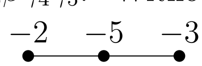
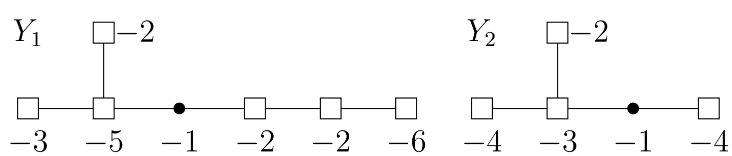


Figure 6.12: PALF structure of Γ_1

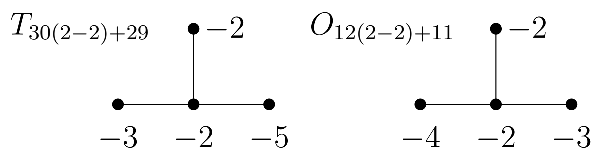
In Figure 6.12 we have PALF structure on X_1 whose monodromy is $\alpha_1^2\alpha_2\alpha_3\alpha_4\alpha_5\beta\gamma_4\gamma_5$. Without β, the monodromy gives the manifold diffeomorphic to 

hence right side of figure gives the desired PALF structure on Γ_1.

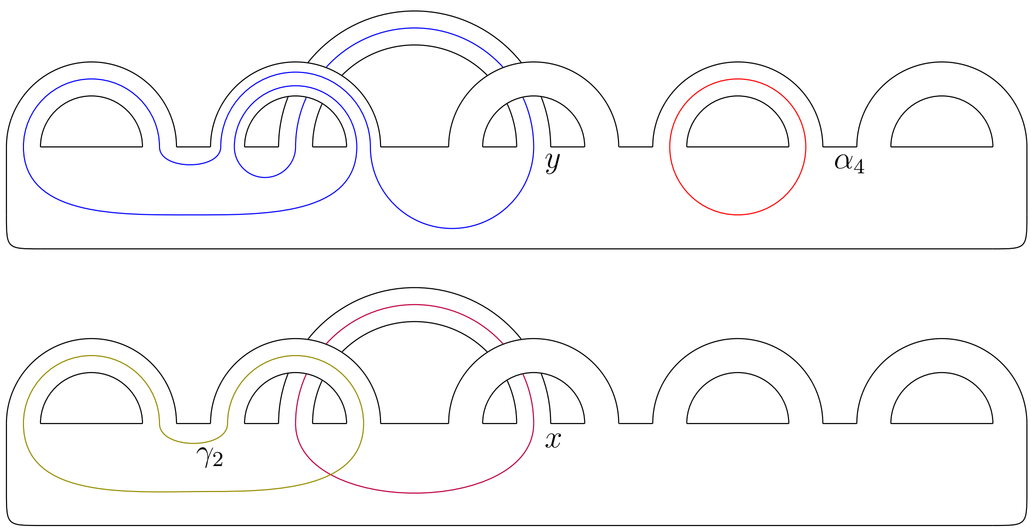
Exceptional cases



Now we rationally blow down the singularity of type T which does not contain central vertex so that the result manifold is diffeomorphic to the minimal resolution.



Since there is a bad vertex in the dual resolution graph, we consider genus-1 PALF structure on it.



The minimal resolution of $T_{30(2-2)+29}$ has monodromy of the following form

$$xy\gamma_2^2\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\gamma_5$$

Here α_i and γ_i are curves encircling ith hole and first i holes respectively. Note that $y = t_{\alpha_2}(t_{\gamma_2}(x))$ where t_{α_2} denote the right-handed Dehn twist over

α_2. Using Hurwitz moves, we can change the mododromy as follows.

$$
\begin{aligned}
& xy\gamma_2^2\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\gamma_5 \\
\sim\ & x\alpha_2\gamma_2 x\alpha_3\alpha_1\alpha_4\alpha_5\gamma_2\gamma_5 \\
\sim\ & t_x(\alpha_2)t_x(\gamma_2)x^2\alpha_3\alpha_1\alpha_4\alpha_5\gamma_2\gamma_5 \\
\sim\ & t_x(\alpha_2)t_x(\gamma_2)t_x^2(\alpha_3)x^2\alpha_3\alpha_1\alpha_4\alpha_5\gamma_2\gamma_5 \\
\sim\ & t_x^2(\alpha_3)(t_x^2\cdot t_{\alpha_3}^{-1}\cdot t_x^{-1})(\alpha_2)(t_x^2\cdot t_{\alpha_3}^{-1}\cdot t_x^{-1})(\gamma_2)x^2\alpha_3\alpha_1\alpha_4\alpha_5\gamma_2\gamma_5
\end{aligned}
$$

Now we take a global conjugate to each monodromy with $f = t_x \cdot t_{\alpha_3}\cdot t_x^{-2}$. Then the global monodromy becomes

$$t_x(\alpha_3)\alpha_2\gamma_2\alpha_3^2\alpha_1\alpha_4\alpha_5 f(\gamma_2)\gamma_5$$

The subword $\alpha_2\gamma_2\alpha_3^2\alpha_1\alpha_4\alpha_5\gamma_5$ corresponds to $\overset{-3}{\bullet}\!\!-\!\!-\!\!\overset{-5}{\bullet}\!\!-\!\!-\!\!\overset{-2}{\bullet}$ which can be rationally blow down to the PALF structure of Y_1. One could get the PALF structure of Y_2 similarly. Summarizing we obtain

Theorem 6.3.5. *There is an explicit algorithm for a genus-*0 *or genus-*1 *PALF structure on any minimal symplectic filling of the link of non-cyclic quotient surface singularities.*

In the construction of PALF structure on each P-resolutions Y of X, we divided it into two families: Those with maximal subgraph Γ_L such that containing all the singularities of type T, and those with no such maximal subgraph. The algorithm of the PALF structure for first family is essentially the algorithm for cyclic cases, which means that the Milnor fiber corresponding to Y is obtained by rational blow-downs along linear plumbing graphs from the minimal resolution of X topologically. On the other hand we find the subword diffeomorphic to neighborhood of a linear string of spheres in a smooth 4-manifold whose boundary is $L(p^2, pq-1)$ for the second family which can be rationally blow-down [14], [15]. Hence we have

Theorem 6.3.6. *Any Milnor fiber of the link of quotient surface singularities can be obtained, up to diffeomorphism, by a sequence of rational blow-downs from the minimal resolution of the singularity.*

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국문초록

S. Donaldson과 R. Gompf에 의해 닫힌 4차원 다양체가 사교 구조를 가질 필요충분조건이 유한번의 블로우 업 이후에 레프셰츠 파이브레이션 구조를 가지는 것이라는 사실이 알려진 이후로 레프셰츠 파이브레이션은 사교 4차원 위상의 핵심 연구 주제 중 하나가 되었다.

이 논문에서 우리는 특정한 4차원 사교 다양체 위에 레프셰츠 파이브레이션 구조에 대해 연구한다. 레프셰츠 파이브레이션 구조는 일부의 파이버가 특이한 미분 가능한 4차원 다양체에서 리만 곡면으로 가는 사상을 말한다. 첫 번째로 우리는 레프셰츠 파이브레이션 구조와 매듭 수술로 얻어진 4차원 다양체의 미분 구조 사이의 연관성에 대해 알아본다. 특히 우리는 모노드로미 군의 표현을 이용하여 같은 사이버그 위튼 불변량을 가지는 매듭 수술 4차원 다양체 위에 레프셰츠 파이브레이션의 동형류에 대해 연구한다. 둘째로 우리는 뭉 곡면 특이점의 사교 채움 위에 레프셰츠 구조에 대한 알고리즘을 만들고 그로부터 사교 채움들 사이에 유리적 블로우 다운 관계가 있음을 보인다.

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