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Higher Order Convergence Rates in Theory of Homogenization of Fully Nonlinear Partial Differential Equations

(완전비선형방정식에 대한 균질화 이론에서의 고차수렴속도)

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Abstract

Homogenization theory is a study of the averaging behavior of a family of partial differential equations that exhibit rapid oscillation in small scales under certain pattern. This thesis consists of three papers concerning higher order convergence rates in periodic homogenization of fully nonlinear partial differential equations. The first paper focuses on interior corrections of uniformly elliptic partial differential equations in non-divergence form, and the second paper studies the effect coming from highly oscillatory initial data for uniformly parabolic Cauchy problems. In the last paper we discover an interesting issue regarding viscous Hamilton-Jacobi equations that initial data has to possess special geometric property determined with respect to the effective Hamiltonian, in order to achieve higher order convergence rates. In all three papers, the heart of analysis lies in developing a regularity theory in non-oscillatory variables in small scales, which allows us to construct higher order correctors through a careful induction scheme. Here the higher order correctors are designed to fix the errors coming from the nonlinear structure of the highly oscillatory partial differential equations, and the higher order convergence rates follows after a suitable barrier argument.

Keywords: homogenization, fully nonlinear equation, periodic setting, convergence rate, corrector, viscosity solution, higher order, regularity
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Contents

Chapter 1

Introduction

When a composite material has a complex structure in the microscopic scale, it affects the effective property in the macroscopic scale in a nontrivial manner. Such a phenomenon is called *homogenization*, which has been of great interest to many scientists in physics, biology, material science and engineering, but also to mathematicians, due to the necessity of rigorous justification of homogenization process, and its potential to open up new fields and ideas in mathematical analysis.

Homogenization theory in mathematics is a study of a family of partial differential equations, abbreviated by PDEs in sequel, which are rapidly oscillating in small scales. The pattern of complex structure constitutes an essential part of the study. The fundamental pattern is the periodic structure, in which case the heterogeneous structure repeats from one cell to another, and one can always obtain an average in a compact set. One may also generalize the oscillating pattern from periodic one, as long as one can obtain average in macroscopic scales. This opens up a room for randomness to be involved in homogenization theory. Nevertheless, there are still many important problems in homogenization theory left open under periodic settings. We refer to the classicial monographs [7, 32] for the general overview of homogenization theory.

A homogenization problem can be formulated as follows. Let $\varepsilon > 0$ be the parameter describing the size of small scales, and F^ε a differential operator that encodes rapidly oscillating structure in ε-scale. We are interested in the behavior of solution u^ε to

$$F^\varepsilon[u^\varepsilon] = 0,$$

and homogenization theory aims to answer the following questions.

- Does u^ε converges? If so, in which sense?
- If u^ε converges in a correct sense, what is the limiting PDE for the limit profile?
- How fast does the convergence take place?

Let us elaborate more on what these questions mean in the mathematical context. The first question asks a correct space X for which

$$u^\varepsilon \to \bar{u} \quad \text{in } X$$

for a certain $\bar{u} \in X$. The second question asks if one can determine an operator $\bar{F}$ such that

$$\bar{F}[\bar{u}] = 0.$$

The operator $\bar{F}$ is often called the *effective operator* corresponding to F^ε, and we require that $\bar{F}$ is homogeneous in small scales. For instance, if $F^\varepsilon[u^\varepsilon] = F(D^2u^\varepsilon, \frac{x}{\varepsilon})$, then we ask $\bar{F}$ to be independent of the rapidly oscillating variable $\frac{x}{\varepsilon}$ and the effective problem becomes $\bar{F}[\bar{u}] = \bar{F}(D^2\bar{u})$.

The last question is about the quantitative error estimate between u^ε and the limit profile $\bar{u}$. More specifically, one seeks a quantity $\delta(\varepsilon) > 0$ that decays as $\varepsilon \to 0$ such that

$$\|u^\varepsilon - \bar{u}\|_X \leq \delta(\varepsilon).$$

The problem becomes very difficult when one attempts to establish a sharp estimate so that the above inequality cannot be improved further.

This thesis is concerned with higher order convergence rates in homogenization of fully nonlinear PDEs, established by the series of collaboration [34] - [36] with my Ph.D. advisor K.-A. Lee. The results in a nutshell provide a rigorous justification of the formal expansion,

$$u^\varepsilon(x) = \bar{u}(x) + \varepsilon w_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 w_2\left(x, \frac{x}{\varepsilon}\right) + \cdots,$$

under a various class of fully nonlinear PDEs having periodically oscillating structure. Here $w_1, w_2, \cdots$, are called the *correctors* that captures the precise oscillating pattern of u^ε at each order of error correction.

Let us briefly summarize the main results. The first result [34] is the higher order convergence rates in the framework of uniformly elliptic, fully nonlinear PDEs in non-divergence form. Due to the nonlinear structure of the PDE, the error subject to the correction at a fixed order becomes an accumulation of the errors left from all the previous orders, making the problem qualitatively different from the framework considered in the classical literature. Here we overcome the difficulty by establishing a *regularity theory in slow variables*, which are non-oscillatory variables in small scales, and obtain higher regularity for the main corrector function and the effective functional. This allows us to construct a sequence of higher order interior correctors in an inductive manner, which eventually leads us to the higher order convergence rates up to the order of the regularity of the given operator, the prescribed domain and the boundary data.

The second result [35] is the higher order convergence rates in the framework of uniformly parabolic, fully nonlinear PDEs with a periodically oscillating initial data. This work was initiated for the purpose of studying the effect of rapid oscillation coming from a lower dimensional data. In order to neglect the curvature influence of a boundary and its interplay with the underlying periodic structure, we considered Cauchy problems, where the initial layer can be considered a *flat surface* with respect to the interior domain. Despite such a simple structure of the lower dimensional object, we found a zone near the initial layer, where the nonlinear structure of the operator becomes highly sensitive and produces coupling effect between *initial layer correctors* and interior correctors. One of the key features in this work is the regularity theory for initial layer correctors in slow variables, which gives exponential decay estimates of the effect coming from the rapid oscillation of the initial data.

The last result [36] is the higher order convergence rates in the framework of viscous Hamilton-Jacobi equations. In this work, we address an interesting issue regarding Hamilton-Jacobi equations that one has to choose an initial data with respect to the effective Hamiltonian in order to achieve the higher order convergence rates. Here we obtain a sufficient class of such initial data, which turns out to be very geometric, and closely related to the level surface of effective Hamiltonian where the gradient vanishes.

This thesis is organized as follows. Next chapter is devoted to the preliminaries for the entire thesis, beginning from the notion of viscosity solution to the associated existence and regularity theory. In Chapter 3, 4 and 5, we present the main results in [34], [35] and respectively [36].

Chapter 2

Preliminaries

Let us define the notion of viscosity solution and list up some important properties and associated regularity theory, which will be required in this thesis. Viscosity solution is a notion of weak solutions to a certain class of PDEs, which naturally have the comparison principle, i.e., if a subsolution is less than or equal to a supersolution on the boundary, then the inequality continues to hold in the interior. Here we shall only present the theory for the class of fully nonlinear, uniformly elliptic, second order PDEs, in order to simplify the exposition. For a more comprehensive overview on the theory of viscosity solution, especially for parabolic PDEs, and viscous Hamilton-Jacobi equations subject to study of this thesis, we refer to [20].

2.1 Existence Theory of Viscosity Solution

Set $n \geq 1$ to the dimension of the underlying space, and Ω a domain, that is, an open connected set, of $\mathbb{R}^n$. Denote by $\mathcal{S}^n$ the space of all real symmetric $n \times n$ matrices. A functional $F : \mathcal{S}^n \times \Omega \to \mathbb{R}$ is said to be uniformly elliptic, if there are $0 < \lambda \leq \Lambda$ such that

$$\lambda|N| \leq F(M + N, x) - F(M, x) \leq \Lambda|N|,$$

for any $M, N \in \mathcal{S}^n$ and any $x \in \Omega$.

Definition 2.1.1 (Viscosity Solution)**.** *Let $F : \mathcal{S}^n \times \Omega \to \mathbb{R}$ be a uniformly elliptic functional, and let $u : \Omega \to \mathbb{R}$ be a function.*

(a) *u is called a viscosity subsolution of $F(D^2u, x) = 0$ in Ω, if $u \in C(\Omega)$, and for any $x_0 \in \Omega$ and any $\phi \in C^2(\Omega)$ such that $u - \phi$ has a local maximum at x_0, one has*

$$F(D^2\phi(x_0), x_0) \geq 0.$$

(b) *u is called a viscosity supersolution of $F(D^2u, x) = 0$ in Ω, if $u \in C(\Omega)$, and for any $x_0 \in \Omega$ and any $\phi \in C^2(\Omega)$ such that $u - \phi$ has a local minimum at x_0, one has*

$$F(D^2\phi(x_0), x_0) \leq 0.$$

(c) *u is called a viscosity solution of $F(D^2u, x) = 0$ in Ω, if u is simultaneously a viscosity subsolution and a viscosity supersolution of $F(D^2u, x) = 0$ in Ω.*

Remark 2.1.2. *Let us remark that one only requires the continuity to define a viscosity solution. Moreover, viscosity solution is designed in such a way that any classical subsolution cannot touch it strictly from below, and similarly any classical supersolution cannot touch it strictly from above.*

Here we shall collect some basic existence and stability results of viscosity solution to a uniformly elliptic, fully nonlinear PDE. For a more thorough review with complete proofs, we ask the reader to consult [20].

Basic existence theory of viscosity solution begins with a comparison principle, which roughly states that if a subsolution stays always below a supersolution on the boundary, then the relation continues to hold in the interior. A precise statement of a comparison principle is given as below.

Theorem 2.1.3 (Comparison Principle; Bounded Domain)**.** *Let Ω be a bounded domain, and $F : \mathcal{S}^n \times \Omega \to \mathbb{R}$ be a uniformly elliptic continuous functional. Suppose that $v, w \in C(\bar{\Omega})$ is a viscosity subsolution and respectively a viscosity supersolution of*

$$F(D^2u, x) = 0 \quad \text{in } \Omega. \tag{2.1}$$

If $v \leq w$ on $\partial\Omega$, then $v \leq w$ in Ω.

Note that the theorem above is formulated for bounded domains. However, we also encounter PDEs on unbounded domains, and the treatment is somewhat different from the case of bounded domains. First we need to restrict the class of viscosity solutions to $BUC(\Omega)$, which consists of all bounded uniformly continuous functions on Ω. Here we shall only consider the case of the entire space, $\Omega = \mathbb{R}^n$.

Theorem 2.1.4 (Comparison Principle; Entire Space). *Let $\mu > 0$ be given and $F : \mathcal{S}^n \times \mathbb{R}^n \to \mathbb{R}$ be a uniformly elliptic continuous functional. Suppose that $v, w \in BUC(\mathbb{R}^n)$ are a viscosity subsolution and respectively a viscosity supersolution of*

$$F(D^2u, x) - \mu u = 0 \quad \text{in } \mathbb{R}^n. \tag{2.2}$$

Then $v \le w$ in $\mathbb{R}^n$.

With the comparison principle, one obtains the existence of viscosity solution by Perron's method. In fact, Perron's method is available whenever a PDE has the comparison principle. We shall again divide the statement into the case of bounded domains and unbounded domains.

Theorem 2.1.5 (Perron's Method; Bounded Domain). *Let Ω be a bounded domain, $F : \mathcal{S}^n \times \bar{\Omega} \to \mathbb{R}$ a uniformly elliptic continuous functional and $\phi \in C(\partial\Omega)$. Then the function $u : \bar{\Omega} \to \mathbb{R}$, defined by the supremum over all viscosity subsolutions $v \in C(\bar{\Omega})$ of* (2.1) *with $v \le \phi$ on $\partial\Omega$, is a viscosity solution of* (2.1) *satisfying $u \le \phi$ on $\partial\Omega$.*

The well-definedness of u above follows easily from the uniform ellipticity and continuity of F together with the boundedness of Ω. Clearly, one can also obtain a viscosity solution by taking the infimum among all viscosity supersolutions which is no less than the specified boundary data on $\partial\Omega$.

Next we state Perron's method for the case of entire space.

Theorem 2.1.6 (Perron's Method; Entire Space). *Let $F : \mathcal{S}^n \times \mathbb{R}^n \to \mathbb{R}$ be a uniformly elliptic continuous functional such that $F(0, \cdot)$ is bounded on $\mathbb{R}^n$, and let $\mu > 0$ be arbitrary. Then the function $u : \mathbb{R}^n \to \mathbb{R}$, defined by*

$$u(x) = \sup\{v(x) : \ v \in BUC(\mathbb{R}^n) \text{ is a viscosity subsolution of } (2.2)\},$$

is the unique viscosity solution of (2.2).

Let us finish this subsection with the stability theorem of viscosity solution.

Theorem 2.1.7 (Stability Under Uniform Convergence)**.** *Let $F_k : \mathcal{S}^n \times \Omega \to \mathbb{R}$ be uniformly elliptic continuous functionals with fixed ellipticity constants for $k = 1, 2, \cdots$. Suppose that u_k is a viscosity solution of $F_k(D^2 u_k, x) = 0$ in Ω for each $k = 1, 2, \cdots$. Assume further that there are a uniformly elliptic continuous functional $F : \mathcal{S}^n \times \Omega \to \mathbb{R}$ and a continuous function $u : \Omega \to \mathbb{R}$ for which $F_k \to F$ locally uniformly in $\mathcal{S}^n \times \Omega$ and $u_k \to u$ locally uniformly in Ω. Then u is also a viscosity solution of $F(D^2 u, x) = 0$ in Ω.*

2.2 Regularity Theory of Viscosity Solution

Here we shall collect some of the basic regularity results for viscosity solutions. Especially we will focus on elliptic PDEs, mainly following the classical monograph [9]. For corresponding results regarding parabolic PDEs, we ask the reader to consult [50] - [52].

Let λ and Λ be positive numbers with $\lambda \leq \Lambda$ and throughout this section, we shall use them to denote ellipticity bounds, unless stated otherwise. Let $F : \mathcal{S}^n \times \Omega \to \mathbb{R}$ be a uniformly elliptic functional with ellipticity bounds λ and Λ. By definition, we have

$$\frac{\lambda}{n} \operatorname{tr}(M^+) - \Lambda \operatorname{tr}(M^-) \leq F(M, x) - F(0, x) \leq \Lambda \operatorname{tr}(M^+) - \frac{\lambda}{n} \operatorname{tr}(M^-),$$

for any $M \in \mathcal{S}^n$ and $x \in \Omega$, where by M^+ and M^- we denoted the unique matrices with nonnegative eigenvalues such that $M = M^+ - M^-$. This motivates us to study the extremal operators, called the Pucci operators. We shall consider a wider class of viscosity solutions associated with these operators.

Definition 2.2.1 (Class S)**.** *Let $f \in C(\Omega)$. We say $u \in S^+(\lambda, \Lambda, f)$ in Ω, if u is a viscosity subsolution of*

$$\Lambda \operatorname{tr}((D^2 u)^+) - \lambda \operatorname{tr}((D^2 u)^-) = f(x) \quad \text{in } \Omega.$$

Similarly, we say $u \in S^-(\lambda, \Lambda, f)$ in Ω, if u is a viscosity supersolution of

$$\lambda \operatorname{tr}((D^2 u)^+) - \Lambda \operatorname{tr}((D^2 u)^-) = f(x) \quad \text{in } \Omega.$$

Finally, we call $u \in S(\lambda, \Lambda, f)$ in Ω if $u \in S^+(\lambda, \Lambda, f) \cap S^-(\lambda, \Lambda, f)$ in Ω.

Let us begin with the Alexander-Bellman-Pucci estimate for the class S^- and S^+, which roughly states that a viscosity supersolution cannot be too negative in the interior, if it is nonnegative on the boundary, and the interior maximum negative value is controlled by the amount of large positive values of the associated source term. This can be easily visualized if one notice that the positive value of a source term contributes to convexity of a solution, since the Hessian at the point has to have a large positive eigenvalue due to the ellipticity of the operator. A precise statement is given as follows.

Theorem 2.2.2 (Alexander-Bellman-Pucci Estimate)**.** *Let $f \in C(B_R)$, and suppose that $u \in S^-(\lambda, \Lambda, f)$ in B_R. If $u \geq 0$ on ∂B_R, then there is $C > 0$ depending only on n, λ and Λ such that*

$$\sup_{B_R} u^- \leq CR \left\| f^+ \right\|_{L^n(B_R \cap \{u = \Gamma_u\})},$$

where Γ_u is the convex envelope of $-u^-$ in B_{2R}, with u extended by zero outside of B_R.

With the Alexander-Bellman-Pucci estimate, one can extend the Krylov-Safanov theory, which mainly states the Harnack inequality, for the class S. The Harnack inequality gives the comparability of values between any pair of interior points, so that if a viscosity solution is large at one point, then it cannot be too small at any other interior point. This yields an algebraic decay estimate on the oscillation of a solution as one goes from a ball to an half ball. Hence, a universal interior Hölder regularity follows easily as an easy corollary.

Theorem 2.2.3 (Krylov-Safanov Theory)**.** *Let $f \in C(\bar{Q}_1)$ and suppose that $u \in S(\lambda, \Lambda, f)$ in Q_1.*

(a) Harnack inequality: If $u \geq 0$ in Q_1, there exists a constant $C > 1$ depending only on n, λ and Λ such that

$$\sup_{Q_{1/2}} u \leq C \left(\inf_{Q_{1/2}} u + \|f\|_{L^n(Q_1)} \right).$$

(b) Interior C^α estimate: $u \in C^\alpha(\bar{Q}_{1/2})$ and

$$\|u\|_{C^\alpha(\bar{Q}_{1/2})} \leq C \left(\|u\|_{L^\infty(Q_1)} + \|f\|_{L^n(Q_1)} \right),$$

where $0 < \alpha < 1$ and $C > 0$ depend only on n, λ and Λ.

We also obtain a uniform regularity estimate up to the boundary.

Theorem 2.2.4 (Boundary Regularity)**.** *Let $f \in C(\bar{\Omega})$ and suppose that Ω satisfies a uniform exterior sphere condition with radius R, that is, for any $x_0 \in \partial\Omega$, there exists a ball $B \subset \Omega^c$ of radius R such that $B \cap \partial\Omega = \{x_0\}$. Also let $\varphi \in C(\partial\Omega)$ with a modulus of continuity ρ. Then for any $u \in S(\lambda, \Lambda, f)$ in Ω, there exists a modulus of continuity ρ^* determined only by n, λ, Λ,* $\mathrm{diam}(\Omega)$*,* $\mathbb{R}$*, $\|\varphi\|_{L^\infty(\Omega)}$ and $\|f\|_{L^\infty(\Omega)}$ such that*

$$|u(x) - u(y)| \leq \rho^*(|x - y|),$$

for any $x, y \in \bar{\Omega}$.

Another application of Krylov-Safanov theory is the Liouville theorem.

Theorem 2.2.5 (Liouville Theorem)**.** *If $u \in S(\lambda, \Lambda, 0)$ in $\mathbb{R}^n$ is globally bounded, then u is a constant.*

Next let us state the higher regularity theory for homogeneous PDEs of the form

$$F(D^2u) = 0.$$

First comes the universal interior $C^{1,\alpha}$ estimates, or the Krylov theory, which only requires uniform ellipticity of the operator. However, it is not always true that a viscosity solution becomes $C^{2,\alpha}$, even if F is a smooth functional. One requires an additional structure condition to achieve $C^{2,\alpha}$ regularity, and a typical sufficient condition is the convexity of the operator. This is the so-called Evans-Krylov theory. Let us also remark that the minimal condition for $C^{2,\alpha}$ regularity still remains open.

Theorem 2.2.6 (Interior Regularity; Homogeneous Case)**.** *Let F be a uniformly elliptic functional on $\mathcal{S}^n$ with ellipticity constants λ and Λ. Suppose that u is a viscosity solution of $F(D^2u) = 0$ in B_1.*

(a) Interior $C^{1,\alpha}$ estimate: $u \in C^{1,\alpha}(\bar{B}_{1/2})$ and

$$\|u\|_{C^{1,\alpha}(\bar{B}_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + |F(0)|\right),$$

where $0 < \alpha < 1$ and $C > 0$ depend only on n, λ and Λ.

(b) *Interior $C^{2,\alpha}$ estimate: Assume further that F is convex on $\mathcal{S}^n$. Then $u \in C^{2,\alpha}(\bar{B}_{1/2})$ and*

$$\|u\|_{C^{2,\alpha}(\bar{B}_{1/2})} \leq C \left(\|u\|_{L^\infty(B_1)} + |F(0)| \right),$$

where $0 < \alpha < 1$ and $C > 0$ depend only on n, λ and Λ.

With the interior higher regularity for homogeneous PDEs, one may expect a similar regularity for heterogeneous PDEs with smooth coefficients. The proof follows the basic idea of the Schauder theory, that is, first approximating the solution by that of a homogeneous PDE obtained by "freezing coefficient", and then iterating the approximation with the updated solution under an appropriate rescaling. Now that we are dealing with fully nonlinear PDEs, we encounter different operators at each iteration step, and the key is to establish a uniform estimate for the class of the operators appear in the entire iteration process.

For an appropriate Hölder class of operators with a linear growth, let us introduce a class $C^\alpha_*(\mathcal{S}^n \times \Omega)$ which consists of all functional $F : \mathcal{S}^n \times \Omega \to \mathbb{R}$ such that

$$\|F\|_{C^\alpha_*(\mathcal{S}^n \times \Omega)} = \sup_{M \in \mathcal{S}^n} \left(\frac{1}{1+|M|} \|F(M, \cdot)\|_{C^\alpha(\Omega)} \right) < \infty.$$

Theorem 2.2.7 (Interior Regularity; Heterogeneous Case)**.** *Let $F \in C(\mathcal{S}^n \times B_1)$ be a uniformly elliptic functional with ellipticity constants λ and Λ, satisfying $F(0, \cdot) = 0$ in B_1, and let $f \in C(B_1)$. Suppose that u is a viscosity solution of $F(D^2u, x) = f(x)$ in B_1.*

(a) *Interior $C^{1,\alpha}$ estimate: Suppose that there exist $0 < \bar{\alpha} \leq 1$ and $\bar{C} > 0$ such that for each $x_0 \in B_{1/2}$, any viscosity solution $v \in C(\bar{B}_{1/2}(x_0))$ of $F(D^2v, x_0) = F(0, x_0)$ in $B_{1/2}(x_0)$ belongs to $C^{1,\bar{\alpha}}(\bar{B}_{1/4}(x_0))$ and*

$$\|v\|_{C^{1,\bar{\alpha}}(\bar{B}_{1/4}(x_0))} \leq \bar{C} \|v\|_{L^\infty(B_{1/2}(x_0))}.$$

Then for any $0 < \alpha < \bar{\alpha}$, a viscosity solution u of $F(D^2u, x) = f(x)$ in B_1 belongs to $C^{1,\alpha}(\bar{B}_{1/4})$ and

$$\|u\|_{C^{1,\alpha}(\bar{B}_{1/4})} \leq C \left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)} \right),$$

where $0<\alpha<\bar\alpha$ and $C>0$ depend only on n, λ, Λ, $\bar\alpha$ and $\bar C$.

(b) *Interior $C^{2,\alpha}$ estimate: Suppose that there exist $0<\bar\alpha\leq 1$ and $\bar C>0$ such that for each $x_0\in B_{1/2}$ and each $M\in\mathcal{S}^n$, any viscosity solution of $F(D^2v+M,x_0)=F(M,x_0)$ in $B_{1/2}(x_0)$ belongs to $C^{2,\bar\alpha}(\bar B_{1/4}(x_0))$ and*

$$\|v\|_{C^{2,\bar\alpha}(\bar B_{1/4}(x_0))}\leq \bar C\,\|v\|_{L^\infty(B_{1/2}(x_0))}.$$

Let $0<\alpha<\bar\alpha$ and suppose further that $F\in C^\alpha_*(\mathcal{S}^n\times B_1)$ and $f\in C^\alpha(B_1)$. Then a viscosity solution u of $F(D^2u,x)=f(x)$ in B_1 belongs to $C^{2,\alpha}(\bar B_{1/4})$ and

$$\|u\|_{C^{2,\alpha}(\bar B_{1/4})}\leq C\left(\|u\|_{L^\infty(B_1)}+\|f\|_{C^\alpha(B_1)}\right),$$

where $0<\alpha<\bar\alpha$ depends only on n, λ, Λ, $\bar\alpha$ and $\bar C$, and $C>0$ depends further but at most on the seminorm $[F]_{C^\alpha_*(\mathcal{S}^n\times B_1)}$.

Remark 2.2.8. *Note that assertion (a) and (b) hold under more general assumptions. Especially, (a) holds even with a source term f having singularities at interior points, only if the blowup rate is of order $r^{-\alpha}$ uniformly around the singularities. More specifically, it is required that $\|f\|_{L^n(B_r(x_0)}\leq cr^{1-\alpha}$ for some $c>0$.*

Due to Theorem 2.2.6, Theorem 2.2.7 (a) holds with a universal exponent α. Moreover, Theorem 2.2.7 (b) also holds with a universal exponent α, provided that F is a convex functional on $\mathcal{S}^n$.

Finally, we have higher regularity for classical solutions in $C^{2,\alpha}$ class, when the operator and the data are smooth.

Theorem 2.2.9 (Higher Regularity)**.** *Suppose that $F\in C^{m,1}_*(\mathcal{S}^n\times\bar\Omega)$, $\partial\Omega\in C^{m+2,1}$ and $g\in C^{m+2,1}(\partial\Omega)$. If $u\in C^{2,\alpha}(\Omega)$ is a solution of $F(D^2u,x)=0$ in Ω with boundary condition $u=g$ on $\partial\Omega$. Then $u\in C^{m+2,\alpha}(\bar\Omega)$.*

Chapter 3

Higher Order Convergence Rates in Theory of Homogenization: Equations in Non-Divergence Form

3.1 Introduction

We establish higher order convergence rates in the theory of periodic homogenization of both linear and fully nonlinear uniformly elliptic equations of non-divergence form. It is known that the equations containing highly oscillating variables $\frac{x}{\varepsilon}$, where the oscillation takes place periodically in the microscopic scale, exhibit a limiting behavior as $\varepsilon \to 0$. More precisely, for the following ε-problems with linear operators,

$$\begin{cases} a_{ij}\left(\frac{x}{\varepsilon}\right) D_{ij} u^\varepsilon = f & \text{in } \Omega, \\ u^\varepsilon = g & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

the solutions u^ε converge to a function u as $\varepsilon \to 0$, which solves a boundary value problem

$$\begin{cases} \bar{a}_{ij} D_{ij} u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

whose operator is homogenous (i.e., the matrix $(\bar{a}_{ij})$ is constant) with respect to the environment. For more details, one may refer to [7] and [32]. A similar behavior also exists when the operator consists of nonlinearity, namely,

$$\begin{cases} F\left(D^2u^\varepsilon, x, \frac{x}{\varepsilon}\right) = 0 & \text{in } \Omega, \\ u^\varepsilon = g & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

As in the linear case, the solutions u^ε exhibit a limiting behavior, and the limit profile u turns out to be a solution of the following PDE,

$$\begin{cases} \bar{F}(D^2u, x) = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

where $\bar{F}$ is no longer oscillatory in the microscopic scale. For more details, see [22].

In this paper, we give a quantitative analysis on the rate of convergence between the solution u^ε and its limit profile u, and we further accelerate the rate by involving appropriate corrector functions for both interior and boundary layer of the physical domain. Finally we end up with a rigorous justification of the following two scale expansion of the solution u^ε:

$$u^\varepsilon(x) = u(x) + \varepsilon(w_1^\varepsilon(x) + z_1^\varepsilon(x)) + \cdots + \varepsilon^m(w_m^\varepsilon(x) + z_m^\varepsilon(x)) + O(\varepsilon^{m-1}), \tag{3.5}$$

where w_k^ε and z_k^ε are the k-th order correctors which fix the error occurring in the interior and on the boundary layer respectively, and m is the positive integer related to the regularity of the operator of the ε-problem. The above expression is explicit if the ε-problem is linear, but rather implicit when a nonlinearity comes in. We make a remark that our result is true also for operators with lower order dependence; essentially most of the challenges lie in proving the case for (3.1) and (3.3) while the desired extensions and generalizations are fairly straightforward to obtain.

3.1.1 Main Result

Our main results are as follows. First we consider the higher order convergence rates for linear equations.

Theorem 3.1.1. *Let $m \geq 2$ be an integer. Set Ω to be a bounded domain in $\mathbb{R}^n$ with $C^{m+2,\alpha}$ boundary and let $f \in C^{m,\alpha}(\bar{\Omega})$ and $g \in C^{m+2,\alpha}(\bar{\Omega})$ for some* *exponent* $0 < \alpha \leq 1$ *and an integer* $m \geq 2$. *Suppose that* (3.1) *satisfies the structure conditions (L1)-(L3) given in Section 3.2. Assume that* $\{u^\varepsilon\}_{\varepsilon>0}$ *is the family of the solutions of* (3.1) *and* u *is the homogenized limit of* $\{u^\varepsilon\}_{\varepsilon>0}$ *which solves* (3.2). *Then there are interior correctors* w_k^ε *and boundary layer correctors* z_k^ε, *respectively defined by* (3.24) *and* (3.25), *for* $k = 1, \ldots, m$ *such that*

$$\|u^\varepsilon - \eta_m^\varepsilon - \theta_m^\varepsilon\|_{L^\infty(\Omega)} \leq C\varepsilon^{m-1} \tag{3.6}$$

for any $\varepsilon \in (0,1)$, *where*

$$\eta_m^\varepsilon = u + \varepsilon w_1^\varepsilon + \varepsilon^2 w_2^\varepsilon + \cdots + \varepsilon^m w_m^\varepsilon, \quad \theta_m^\varepsilon = \varepsilon z_1^\varepsilon + \varepsilon^2 z_2^\varepsilon + \cdots + \varepsilon^m z_m^\varepsilon$$

on $\bar{\Omega}$ *and* C *depends only on* $n, m, \alpha, \sigma, \lambda, \Lambda, \Omega, \|f\|_{C^{m,\alpha}(\bar{\Omega})}$ *and* $\|g\|_{C^{m+2,\alpha}(\bar{\Omega})}$.

The result concerning fully nonlinear equations is stated below.

Theorem 3.1.2. *Let* $m \geq 2$ *be an integer. Set* Ω *to be a bounded domain of* $\mathbb{R}^n$ *with* $\partial\Omega \in C^{m+2,1}$ *and let* $g \in C^{m+2,1}(\bar{\Omega})$. *Suppose that* $F \in C^m(\mathcal{S}^n \times \bar{\Omega} \times \mathbb{R}^n)$ *satisfies the structure conditions (F1)-(F4) given in Section 3.2. Then there are interior correctors* w_k^ε *for* $k = 1, \ldots, [\frac{m}{2}] + 1$ *and the boundary layer corrector* θ_m^ε, *respectively defined by* (3.52) *and* (3.53) *such that for any* $\varepsilon_* \in (0,1)$,

$$\|u^\varepsilon - \eta_m^\varepsilon - \theta_m^\varepsilon\|_{L^\infty(\Omega)} \leq C\varepsilon^{[\frac{m}{2}]}, \quad \forall \varepsilon \in (0, \varepsilon_*], \tag{3.7}$$

where

$$\eta_m^\varepsilon = u + \varepsilon w_1^\varepsilon + \varepsilon^2 w_2^\varepsilon + \cdots + \varepsilon^{[\frac{m}{2}]+1} w_{[\frac{m}{2}]+1}^\varepsilon$$

on $\bar{\Omega}$ *and* $C > 0$ *depends only on* $n, m, \varepsilon_*, \sigma, \lambda, \Lambda, F, g$ *and* Ω.

3.1.2 Historical Background

Classical results in the theory of homogenization could be found in the books [7] and [8], and the references therein. In particular, the notion of higher order correctors are introduced in these books, and one can find a higher order convergence rate for divergent operator on 1-dimensional space. This problem, however, is still open for higher dimensions where boundary oscillation plays a crucial role.

Periodic homogenizations for first and second order nonlinear equations have been studied by many authors, such as Lions, Papanicolaou and Varadhan [39], Evans [21, 22], Caffarelli [10] and Majda and Souganidis [41] and

Evans and Gomes [24], etc. For homogenization with respect to an almost periodic or stationary ergodic environment has been considered by Ishii [30], Lions and Souganidis [40] and Caffarelli, Souganidis and Wang [13], etc.

Rates of convergence in the theory of periodic homogenization were considered by several authors in various circumstances; for example, Capuzzo Dolcetta and Ishii [17] and Camilli and Marchi [15] and Marchi [45], etc. In a stationary ergodic setting, also see Caffarelli and Souganidis [12]. However, as far as we know, there has been no literature concerning higher order convergence rates for homogenization of both linear and nonlinear elliptic equations in nondivergence form.

3.1.3 Heuristic Discussion and Main Difficulties

Let us summarize the main strategies of this paper and make a few remarks on the key features observed in achieving the rates.

The main feature of this work is the construction of *higher order correctors* based on a new regularity theory in *slow variables*. In order to find the next order approximation, we consider the linearized operator near the previous approximation. Since the linearized operator belongs to the same class of the previous one, we are able to proceed our argument in an inductive manner. The relationship between the current approximation and the next one is quite complicated in the nonlinear setting, unlike the linear case; however, such difficulty could be overcome by capturing the stability of correctors with respect to the shape of the limit profile, but not to the physical variable x.

Our induction argument consists of two substeps at each main step. First substep is to improve the previous approximation by constructing a globally periodic corrector and then bending it based on the shape of the limit profile. Then the improved interior approximation creates new errors, of a higher order, away from the given boundary data. The second substep is to fix the new errors by constructing a boundary layer corrector.

Additionally it is noteworthy that at each step of finding the k-th order interior corrector, we encounter a compatibility condition which uniquely determines the $(k-2)$-th order interior corrector. It illustrates the reason why the higher order asymptotic expansion (3.5) starts from ε-order but not from ε^2-order, as seen in many literatures (e.g., [21, 22]). It is closely related to the invariance of the quadratic rescaling of the governing equation.

There are two main differences between the linear and fully nonlinear settings. First the asymptotic expansion (3.5) is made inside of the operator for the fully nonlinear case, which creates an additional error unlike the linear case. Readers may compare the equation (3.48) to (3.23). Fortunately, all the additional errors are controllable and have no influence on determining the order of convergence rates.

Secondly, there is a coupling effect of the fast variable $y = \varepsilon^{-1}x$ and the slow variable x of the interior correctors in the fully nonlinear case, unlike the linear case. Moreover, it causes the difference in the order of convergence rates as seen in Main Theorem I and II. The order is closely related to the regularity of interior correctors, and the coupling effect in the nonlinear case forces the next corrector to have two "degrees" less regularity than the current one (see Lemma 3.4.19).

3.1.4 Outline

This chapter is organized as follows. In the next section, we list up notation, terminology and the standing assumptions throughout this chapter. Section 3.3 is devoted to linear equations. We review the basic homogenization scheme via the viscosity method in Section 3.3.1. Interior and boundary layer correctors of higher order are obtained in Subsection 3.3.2. We present the proof of Theorem 3.1.1 in Section 3.3.3. Section 3.4 is devoted to fully nonlinear equations. The basic homogenization scheme of fully nonlinear equations is shown in Subsection 3.4.1. In Section 3.4.2 we investigate the regularity of the effective operator and the corrector function in the slow variable. In Section 3.4.3 we seek the higher order interior and boundary layer correctors, and finally prove Theorem 3.1.2 in Section 3.4.4.

3.2 Notation and Standing Assumptions

Throughout this chapter, we shall use the following notation. $\mathcal{S}^n$ is the space of all $n \times n$ symmetric matrices. $|M|$ denotes the (L^2, L^2)-norm of M (i.e., $|M| = \sup_{|x|=1} |Mx|$). By $B_r(x)$, we denote the ball of radius $r > 0$ centered at a point x, which belongs either to $\mathbb{R}^n$ or to $\mathcal{S}^n$. By B_r we denote $B_r(0)$. Similarly, $Q_r(x)$ denotes the cube centered with side length $r > 0$ centered at a point $x \in \mathbb{R}^n$. As above, by Q_r we denote $Q_r(0)$.

$S(\lambda, \Lambda, f)$ and $S^*(\lambda, \Lambda, f)$ are the classes of viscosity solutions defined in Definition 2.2.1. $C^{k,\alpha}(\Omega)$ is the space of all k-times continuously differentiable function in Ω whose k-th order derivatives are in $C^\alpha(\Omega)$. Also by $C^{k,\alpha}_{loc}(\Omega)$ we denote the space consisting of all functions belonging to $C^{k,\alpha}(K)$ in any compact $K \subset \Omega$. We shall also use the adimensional norms $\|\cdot\|^*_{C^{k,\alpha}(\Omega)}$ as well as $\|\cdot\|^{(j)}_{C^{k,\alpha}(\Omega)}$, whose definition can be found in [25, Chapter 4].

By c_n, C_n we denote dimensional constants. By c_0, c, C_0, C we denote the positive constants which depends only on the structure constants appearing in the structure conditions (L1)-(L3) or (F1)-(F4) given below. By $C_{f_1,\dots,f_k}$ and $C(f_1,\cdots,f_k)$ we denote positive constants depending on the constants in the structure conditions and further on $f_1,\dots,f_k$ where f_i can be a constant, a function, etc. We will use the summation convention of repeated indices.

Now let us list up the standing assumptions associated with the operators of (3.1) and (3.3). The linear coefficient $A(y) = (a_{ij}(y)) \in C^{m,\alpha}(\mathbb{R}^n; \mathcal{S}^n)$ will satisfy the following conditions.

(L1) (Periodicity) $A(y+k) = A(y)$;

(L2) (Uniform Ellipticity) $\lambda|\xi|^2 \leq a_{ij}(y)\xi_i\xi_j \leq \Lambda|\xi|^2$;

(L3) (Regularity) $\|A\|_{C^{m,\alpha}(\mathbb{R}^n)} \leq \sigma$,

where $y, \xi \in \mathbb{R}^n$ and $k \in \mathbb{Z}^n$ and λ, Λ and σ are positive constants such that $\lambda \leq \Lambda$.

On the other hand, we shall impose the following conditions to the fully nonlinear functional $F \in C^m(\mathcal{S}^n \times \bar{\Omega} \times \mathbb{R}^n)$.

(F1) (Periodicity) $F(M, x, y+k) = F(M, x, y)$;

(F2) (Uniform Ellipticity) $\lambda|N| \leq F(M+N, x, y) - F(M, x, y) \leq \Lambda|N|$;

(F3) (Regularity) $\|F\|_{C^{m,1}(\bar{B}_L \times \bar{\Omega} \times \mathbb{R}^n)} \leq \sigma(1+L)$;

(F4) (Concavity) $F(tM + (1-t)P) \geq tF(M) + (1-t)F(P)$,

where M, N, $P \in \mathcal{S}^n$ with $N \geq 0$, $x \in \bar{\Omega}$, $y \in \mathbb{R}^n$, $k \in \mathbb{Z}^n$, and $t \in [0,1]$ and $L > 0$, and λ, Λ and σ are positive constants such that $\lambda \leq \Lambda$.

3.3 Linear Equations

3.3.1 Basic Homogenization Scheme

Let us fix $\varepsilon > 0$. The coefficient matrix $(a_{ij}(\cdot/\varepsilon))$ of (3.1) is uniformly elliptic in $\bar{\Omega}$ with constants λ and Λ, and belongs to $C^{m,\alpha}(\bar{\Omega})$. According to Theorem 2.2.7 and Theorem 2.2.9, there exists a unique solution $u^\varepsilon \in C^{m+2,\alpha}(\bar{\Omega})$ of (3.1). In [22] it is shown that $\{u^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $C^\alpha(\bar{\Omega})$ and hence has a limit. For the sake of completeness, we prove a weaker result that $\{u^\varepsilon\}_{\varepsilon>0}$ has a uniform modulus of continuity, which still guarantees the existence of limit.

Lemma 3.3.1. *Let $\{u^\varepsilon\}_{\varepsilon>0} \subset C^{m+2,\alpha}(\bar{\Omega})$ be the unique family which solve (3.1) for each $\varepsilon > 0$. Then there is a function $u \in C(\bar{\Omega})$ and a subsequence $\{u^{\varepsilon_k}\}_{k=1}^\infty$ of $\{u^\varepsilon\}_{\varepsilon>0}$ such that $u^{\varepsilon_k} \to u$ uniformly in $\bar{\Omega}$ as $k \to \infty$.*

Proof. We have $u^\varepsilon \in S(\lambda, \Lambda, f)$ in Ω for all $\varepsilon > 0$ by the assumption (L2). By the setting, g has a modulus of continuity $\rho(r) = [g]_{C^\alpha(\bar{\Omega})} r^\alpha$. Since $\partial\Omega \in C^{m+2,\alpha}$, Ω satisfies a uniform sphere condition, say with radius $R > 0$. Thus, Theorem 2.2.4 implies that u^ε has a modulus of continuity ρ^*, which depends only on $n, \lambda, \Lambda, \|f\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}, \mathrm{diam}(\Omega), R$ and ρ.

As the modulus of continuity ρ^* is independent on ε, the family $\{u^\varepsilon\}_{\varepsilon>0}$ is equicontinuous on $\bar{\Omega}$. Moreover, by an a priori estimate we have $\|u^\varepsilon\|_{L^\infty(\Omega)} \leq C(\|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\Omega)})$, where C depends only on n, λ, Λ and $\mathrm{diam}(\Omega)$, for each $\varepsilon > 0$.

Now the conditions for the Arzela-Ascoli theorem are met, which ensures the existence of a subsequence $\{u^{\varepsilon_k}\}_{k=1}^\infty$ of $\{u^\varepsilon\}_{\varepsilon>0}$ which converges uniformly in $\bar{\Omega}$. $\square$

The limit function u will later turn out to be unique and satisfy (3.2) in the classical sense. The next lemma plays a key role in proving this fact. The proof can be also found in [22]; nevertheless we contain the proof for completeness.

Lemma 3.3.2. *For each $M \in \mathcal{S}^n$ there exists a unique $\gamma \in \mathbb{R}$ for which the following equation admits a 1-periodic solution*

$$a_{ij} D_{y_i y_j} w + a_{ij} M_{ij} = \gamma \quad \text{in } \mathbb{R}^n. \tag{3.8}$$

Moreover, the solutions of (3.8) lie in $C^{2,\alpha}(\mathbb{R}^n)$ and are unique up to an additive constant.

To prove this lemma we consider the following penalized problem for $\delta \in (0,1)$

Lemma 3.3.3. *Let $M \in \mathcal{S}^n$. There exists a unique bounded 1-periodic solution w^δ of*

$$a_{ij} D_{y_i y_j} w^\delta + a_{ij} M_{ij} - \delta w^\delta = 0 \quad \text{in } \mathbb{R}^n. \tag{3.9}$$

for each $\delta \in (0,1)$. Moreover, w^δ lies in $C^{2,\alpha}(\mathbb{R}^n)$ with the estimate

$$\sup_{0<\delta<1} \left\| \delta w^\delta \right\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C|M|. \tag{3.10}$$

Proof. In view of Theorem 2.1.4 (a) (with $F(N,y) = a_{ij}(y)N_{ij} + a_{ij}(y)M_{ij}$), we know that (3.9) has a comparison principle. By the hypothesis (L2), all the eigenvalues of (a_{ij}) lie in the interval $[\lambda, \Lambda]$, which implies that

$$\|a_{ij}M_{ij}\|_{L^\infty(\mathbb{R}^n)} \leq n \|A(y)\|_{C^\alpha(\mathbb{R}^n)} |M| \leq n\sigma|M|. \tag{3.11}$$

It then follows that the constant functions $w_-^\delta = -n\sigma|M|/\delta$ and $w_+^\delta = n\sigma|M|/\delta$ are a subsolution and a supersolution respectively to (3.9) for each $\delta \in (0,1)$. Thus, Perron's method (Theorem 2.1.6 with $F(N,y) = a_{ij}(y)N_{ij} + a_{ij}(y)M_{ij}$, $u = w_-^\delta$ and $v = w_+^\delta$) ensures that there is a unique bounded 1-periodic viscosity solution $w^\delta \in C(\mathbb{R}^n)$. It is immediate that

$$\sup_{0<\delta<1} \left\| \delta w^\delta \right\|_{L^\infty(\mathbb{R}^n)} \leq n\sigma|M|. \tag{3.12}$$

Let us apply an interior Schauder estimate in a ball $B_{\sqrt{n}}(y_0)$ for $y_0 \in \mathbb{R}^n$ (see Theorem 2.2.7). Then $w^\delta \in C^{2,\alpha}(B_{\sqrt{n}/2}(y_0))$ and there is c_0 such that

$$\left\| w^\delta \right\|^*_{C^{2,\alpha}(B_{\sqrt{n}/2}(y_0))} \leq c_0 \left(\left\| w^\delta \right\|_{L^\infty(B_{\sqrt{n}}(y_0))} + n\sigma|M| \right) \leq 2n\delta^{-1}c_0\sigma|M|.$$

Since y_0 was chosen in an arbitrary way and $B_{\sqrt{n}/2}(y_0)$ contains a periodic cube, the estimate (3.10) is verified with $C = 2n\delta^{-1}c_0\sigma$. □

We observe that the oscillation of w^δ is bounded independent of δ, although its L^∞ norm is not bounded in a uniform way.

Lemma 3.3.4. *Let $M \in \mathcal{S}^n$ and w^δ be the unique solution to (3.9). Then*

$$\sup_{0<\delta<1} \operatorname*{osc}_{\mathbb{R}^n} w^\delta \leq C|M|. \tag{3.13}$$

Moreover,

$$\sup_{0<\delta<1} \left\| \tilde{w}^\delta \right\|_{C^{2,\alpha}(B_1(y_0))} \leq C|M|, \tag{3.14}$$

where $\tilde{w}^\delta := w^\delta - w^\delta(0)$.

Proof. Define $\hat{w}^\delta(y) := w^\delta(y) - \min_{\mathbb{R}^n} w^\delta \geq 0$ in $\mathbb{R}^n$. Note that $\hat{w}^\delta$ and w^δ achieve its global maximum and minimum, and $\hat{w}^\delta \in C^{2,\alpha}(\mathbb{R}^n)$. Additionally, $\operatorname{osc}_{\mathbb{R}^n} w^\delta = \max_{\mathbb{R}^n} \hat{w}^\delta$. Moreover, plugging $\hat{w}^\delta$ into (3.9) we obtain

$$a_{ij} D_{y_i y_j} \hat{w}^\delta - \delta \hat{w}^\delta = \delta \min_{\mathbb{R}^n} w^\delta - a_{ij} M_{ij} \quad \text{in } \mathbb{R}^n. \tag{3.15}$$

Let us restrict our domain to $B_{\sqrt{n}}(y_0)$ where y_0 is an arbitrary point in $\mathbb{R}^n$. Note that $B_{\sqrt{n}/2}(y_0)$ contains a periodic cube $Q_1(y_0)$. This implies that $\sup_{B_{\sqrt{n}/2}(y_0)} \hat{w}^\delta = \sup_{\mathbb{R}^n} \hat{w}^\delta$ and $\inf_{B_{\sqrt{n}/2}(y_0)} \hat{w}^\delta = \inf_{\mathbb{R}^n} \hat{w}^\delta = 0$. Now we apply the Harnack inequality over $B_{\sqrt{n}}(y_0)$ to (3.15) (see Theorem 2.2.3 (a) with $f = \delta \min_{\mathbb{R}^n} w^\delta - a_{ij} M_{ij}$). Then

$$\sup_{B_{\sqrt{n}/2}(y_0)} \hat{w}^\delta \leq c_0 \left\| \lambda^{-1} (\delta \min_{\mathbb{R}^n} w^\delta - a_{ij} M_{ij}) \right\|_{L^\infty(B_{\sqrt{n}}(y_0))} \leq 2 c_0 \lambda^{-1} n \sigma |M|;$$

here we utilized (3.11) and (3.12). Since the above bound is independent of $\delta \in (0,1)$, and since y_0 is an arbitrary point, we have shown (3.13) with $C = 2c_0 \lambda^{-1} n\sigma$.

Define now $\tilde{w}^\delta(y) := w^\delta(y) - w^\delta(0)$ in $\mathbb{R}^n$. By (3.13), $|\tilde{w}^\delta| \leq \tilde{c}_0 |M|$ in $\mathbb{R}^n$ where $\tilde{c}_0 = 4 c_0 \lambda^{-1} n\sigma$. Moreover, $\tilde{w}^\delta \in C^{2,\alpha}(\mathbb{R}^n)$ and satisfies

$$a_{ij} D_{y_i y_j} \tilde{w}^\delta + a_{ij} M_{ij} - \delta \tilde{w}^\delta = \delta w^\delta(0) \quad \text{in } \mathbb{R}^n.$$

Using a similar argument when proving (3.10), we get

$$\sup_{0<\delta<1} \left\| \tilde{w}^\delta \right\|_{C^{2,\alpha}(B_1(y_0))} \leq \tilde{c}_1 c_0 n \sigma (\lambda^{-1} + 1) |M|,$$

which verifies (3.14) with $C = \tilde{c}_1 c_0 n\sigma(\lambda^{-1} + 1)$. $\square$

Now we are ready to prove Lemma 3.3.2

Proof of Lemma 3.3.2. In view of (3.12), we can take a subsequence $\{\delta_k w^{\delta_k}(0)\}_{k=1}^\infty$ of $\{\delta w^\delta\}_{0<\delta<1}$ and a number $\gamma \in \mathbb{R}$ such that $\delta_k w^{\delta_k}(0) \to \gamma$ as $k \to \infty$. Then (3.13) implies that $\delta_k w^{\delta_k} \to \gamma$ uniformly in $\mathbb{R}^n$ as $k \to \infty$.

On the other hand, by the compact embedding, the uniform estimate (3.14) yields that

$$\left\| \delta_k w^{\delta_k} - \gamma \right\|_{L^\infty(\mathbb{R}^n)} + \left\| \tilde{w}^{\delta_k} - w \right\|_{C^2(\mathbb{R}^n)} \to 0 \quad \text{as} \quad k \to \infty, \tag{3.16}$$

for some 1-periodic $w \in C^{2,\alpha}(\mathbb{R}^n)$. Note that one may need to take a further subsequence of $\{\delta_k\}_{k=1}^\infty$ to ensure the convergence above.

By the stability of viscosity solutions, w solves (3.8) in the viscosity sense. Then the $C^{2,\alpha}(\mathbb{R}^n)$-regularity of w forces itself to be a classical solution.

To this end we prove that the constant γ is unique. Suppose to the contrary that there is another $\gamma' \in \mathbb{R}$ to which a subsequence of $\{\delta w^\delta\}_{0<\delta<1}$ converges uniformly in $\mathbb{R}^n$. Denote w', which belongs to $C^{2,\alpha}(\mathbb{R}^n)$, by the corresponding limit of a subsequence of $\{\tilde{w}^\delta\}_{0<\delta<1}$.

Assume without lose of generality that $\gamma < \gamma'$. As w and w' being bounded, we are able to add a constant t_0 to w in such a way that $w'(y_0)+t_0 < w(y_0)$ at a point $y_0 \in \mathbb{R}^n$. Take t_1 by the infimum value of t such that $w' + t \geq w$ in $\mathbb{R}^n$. Then $w' + t_1$ touches w by above at a point y_1. Since w is a solution of (3.8),

$$\gamma \leq a_{ij}(y_1) D_{y_i y_j}(w' + t_1)(y_1) + a_{ij}(y_1) M_{ij} = \gamma',$$

which is a contradiction. It shows that the constant γ must be unique.

Furthermore, the Liouville theorem (e.g., Theorem 2.2.5) implies that the uniform convergence (3.16) could be made along the full sequence; i.e., the limit function is also unique.

The last assertion of Lemma 3.3.2 is also an easy consequence of the Liouville theorem. $\square$

From now on we denote $w^\delta(\cdot; M)$ by the unique solution of (3.9) for a given $M \in \mathcal{S}^n$. Also $\hat{w}^\delta(\cdot; M) := w^\delta(\cdot; M) - \min_{\mathbb{R}^n} w^\delta(\cdot; M)$ and $\tilde{w}^\delta(\cdot; M) := w^\delta(\cdot; M) - w^\delta(0; M)$. In addition, let us write $w(\cdot; M)$ by the solution of (3.8) for a given $M \in \mathcal{S}^n$ which is normalized by 0; i.e., $w(0; M) = 0$.

By Lemma 3.3.2 we can understand γ as a functional $M \mapsto \gamma(M)$ on $\mathcal{S}^n$. The linear structure of the equation (3.8) allows us to obtain further information about the functional γ which is stated in the next lemma.

Lemma 3.3.5. *Let γ be the functional on $\mathcal{S}^n$ obtained from Lemma 3.3.2.*

(i) There is a constant symmetric matrix $(\bar{a}_{ij})$ such that $\gamma(M) = \bar{a}_{ij} M_{ij}$.

(ii) The matrix $(\bar{a}_{ij})$ *is elliptic with the same ellipticity constants of* (a_{ij})*; i.e.,* $\lambda|\xi|^2 \leq \bar{a}_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2$ *for all* $\xi \in \mathbb{R}^n$*.*

Proof. The assertion (i) is a direct consequence of Lemma 3.3.3, and is left to the reader.

We prove the assertion (ii). Since the proofs are similar, we only show the first inequality. Choose any $\varepsilon > 0$ and assume for a contradiction that there exists $\xi \in \mathbb{R}^n$ for which $\bar{a}_{ij}\xi_i\xi_j < (\lambda - \varepsilon)|\xi|^2$. In view of (3.16), there corresponds $\delta \in (0,1)$ for which $\left\| \delta w^\delta(\cdot;\xi\cdot\xi^t) - \bar{a}_{ij}\xi_i\xi_j \right\|_{L^\infty(\mathbb{R}^n)} < \varepsilon|\xi|^2$. For the moment we abbreviate $w^\delta(\cdot;\xi\cdot\xi^t)$ by w^δ. Then

$$a_{ij}D_{y_iy_j}w^\delta = \delta w^\delta - a_{ij}\xi_i\xi_j \leq \left\| \delta w^\delta - \bar{a}_{ij}\xi_i\xi_j \right\|_{L^\infty(\mathbb{R}^n)} + (\bar{a}_{ij}\xi_i\xi_j - \lambda|\xi|^2) < 0$$

in $\mathbb{R}^n$, which is contradictory to the fact that w^δ achieves a global minimum. □

The constant matrix $(\bar{a}_{ij})$ from Lemma 3.3.5 is called *the effective coefficients* of (a_{ij}) in the following lemma. It is proved in [22], but we present the proof for completeness.

Lemma 3.3.6. *Suppose that* (3.1) *satisfies the structure conditions (L1)-(L2) and let* $\{u^\varepsilon\}_{\varepsilon>0} \subset C^{m+2,\alpha}(\Omega)$ *be the family of solutions to* (3.1)*. Then there exists a unique function* u*, which has a modulus of continuity on* $\bar{\Omega}$*, such that* $u^\varepsilon \to u$ *uniformly in* $\bar{\Omega}$ *as* $\varepsilon \to 0$*. Moreover,* $u \in C^{m+2,\alpha}(\bar{\Omega})$ *and it solves* (3.2)*.*

Proof. We already proved part of the first assertion in Lemma 3.3.1. Since $u^\varepsilon \to u$ uniformly in $\bar{\Omega}$ up to a subsequence and $u^\varepsilon = g$ on $\partial\Omega$ for all $\varepsilon > 0$, we have $u = g$ on $\partial\Omega$. On the other hand, the maximum principle implies that (3.2) has at most one solution. Therefore, the convergence of $u^\varepsilon \to u$ is valid without extracting a subsequence.

We claim that u is a viscosity solution to (3.2). If it is true, then Theorem 2.2.7 and Theorem 2.2.9 imply that $u \in C^{m+2,\alpha}(\bar{\Omega})$.

Thus, we are only left with proving the above claim. Let P be a paraboloid which touches u by above at x_0 in a neighborhood. By replacing P by $P + \eta|x-x_0|^2$ ($\eta > 0$) we may assume that P touches u strictly by above. Assume to the contrary that $\bar{a}_{ij}D_{ij}P - f(x_0) < 0$. By the continuity of f, we can choose $r > 0$ in such a way that $B_r(x_0) \subset \Omega$ and $\bar{a}_{ij}D_{ij}P - f(x) < 0$ for any $x \in B_r(x_0)$.

Define $P^\varepsilon(x) := P(x)+\varepsilon^2 w(\varepsilon^{-1}x; D^2P)$. Note that $P^\varepsilon \in C^{2,\alpha}(\bar{\Omega})$. In view of (3.8) we obtain

$$a_{ij}\left(\frac{x}{\varepsilon}\right) D_{ij}P^\varepsilon(x) - f(x) = \bar{a}_{ij}D_{ij}P - f(x) < 0 \quad \text{in } B_r(x_0).$$

Hence, P^ε is a supersolution of (3.1) so that the strong maximum principle implies $(u^\varepsilon - P^\varepsilon)(x_0) < \max_{\partial B_r(x_0)}(u^\varepsilon - P^\varepsilon)$. Letting $\varepsilon \to 0$ then gives $\max_{\partial B_r(x_0)}(u - P) \geq 0$, which violates the assumption that P touches u strictly by above at x_0. Therefore, $\bar{a}_{ij}D_{ij}P - f(x) \geq 0$ for any $x \in \Omega$. It shows that u is a viscosity subsolution of (3.2).

In a similar manner, we are able to prove that u is a viscosity supersolution of (3.2). This completes the proof. □

3.3.2 Interior and Boundary Layer Correctors

In this subsection, we seek the interior and boundary layer correctors. We make a remark from the previous section before we begin. Recall from the linear algebra, $\{E^{ij} | i, j = 1, \dots, n\}$ is the standard basis of $\mathcal{S}^n$. Any matrix $M \in \mathcal{S}^n$ can be written as $M = M_{ij}E^{ij}$ where $M = (M_{ij})$. Set $M = E^{kl}$ in Lemma 3.3.2 for $k, l \in \{1, \dots, n\}$ and write $\chi^{kl} := w(\cdot; E^{kl}) \in C^{2,\alpha}(\mathbb{R}^n)$. Notice that $\chi^{kl}(0) = 0$. In view of (3.8) and Lemma 3.3.5 (i), χ^{kl} solves

$$a_{ij}D_{ij}\chi^{kl} + a_{kl} = \bar{a}_{kl}. \tag{3.17}$$

Multiplying (3.17) with M_{kl} and summing over the indices $k, l = 1, \dots, n$, we see that $\chi^{kl}M_{kl}$ solves (3.8) with $M = (M_{kl})$. Define

$$w_2(y, x) = \chi^{kl}(y)D_{x_kx_l}u(x) + \psi_2(x) \quad (y \in \mathbb{R}^n, x \in \bar{\Omega}),$$

where u is given by Lemma 3.3.6 and ψ_2 is chosen arbitrarily from $C^{m,\alpha}(\bar{\Omega})$ for the moment. By Lemma 3.3.6, $w_2(\cdot, x) \in C^{2,\alpha}(\mathbb{R}^n)$ for each $x \in \bar{\Omega}$ while $w_2(y, \cdot) \in C^{m,\alpha}(\bar{\Omega})$ for each $y \in \mathbb{R}^n$. Moreover, $w_2(\cdot, x)$ solves

$$a_{ij}D_{y_iy_j}w_2(\cdot, x) + a_{ij}D_{x_ix_j}u(x) = 0 \quad \text{in } \mathbb{R}^n$$

for each $x \in \bar{\Omega}$. We call w_2 the second order (interior) corrector of (3.1). The first order corrector will be defined afterward as a compatibility condition of the third order corrector.

Interior correctors of higher orders are discovered in the similar direction.

Lemma 3.3.7. *There are a family* $\{\bar{a}_{i_1\dots i_k} | 1 \leq i_1, \dots, i_k \leq n, k \geq 2\}$ *of constants and a family* $\{\chi^{i_1\dots i_k} | 1 \leq i_1, \dots, i_k \leq n, k \geq 2\}$ *of 1-periodic functions in* $C^{2,\alpha}(\mathbb{R}^n)$ *which satisfy the following recursive equation*

$$a_{ij} D_{ij} \chi^{i_1\dots i_k} + 2a_{i_k j} D_{y_j} \chi^{i_1\dots i_{k-1}} + a_{i_{k-1} i_k} \chi^{i_1\dots i_{k-2}} = \bar{a}_{i_1\dots i_k} \quad \text{in } \mathbb{R}^n \tag{3.18}$$

for each $1 \leq i_1, \dots, i_k \leq n$. *Here we understand* $\chi \equiv 1$ *and* $\chi^i \equiv 0$ *for each* $i = 1, \dots, n$. *Furthermore, for each* $k \geq 2$, $\chi^{i_1\dots i_k}(0) = 0$ *and*

$$|\bar{a}_{i_1\dots i_k}| + \left\| \chi^{i_1\dots i_k} \right\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_k, \quad \forall 1 \leq i_1, \dots, i_k \leq n. \tag{3.19}$$

Proof. We already know $\{\bar{a}_{ij}\}_{i,j=1,\dots,n}$ and $\{\chi^{ij}\}_{i,j=1,\dots,n}$ from the comment above this lemma; one may notice that (3.18) is exactly the same with (3.17) if $k = 2$. The constant C_2 can be taken by the sum of those from (3.10) and (3.14).

The construction of the families $\{\bar{a}_{i_1\dots i_k}\}$ and $\{\chi^{i_1\dots i_k}\}$ (for $k \geq 3$) can be done by an induction argument, mainly following the lines of the proofs of Lemma 3.3.2, 3.3.3 and 3.3.4. To avoid the redundancy, we leave it to the reader. □

Now let $m \geq 3$. By Lemma 3.3.6 we have $u \in C^{m+2,\alpha}(\bar{\Omega})$. For $1 \leq k \leq m-2$, define $\psi_k \in C^{m-k+2,\alpha}(\bar{\Omega})$ recursively by the unique solution of

$$\begin{cases} \bar{a}_{ij} D_{x_i x_j} \psi_k = -\sum_{l=3}^{k+2} \bar{a}_{i_1\dots i_l} D_{x_{i_1}\dots x_{i_l}} \psi_{k-l+2} & \text{in } \Omega, \\ \psi_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.20}$$

where we understand $\psi_0 \equiv u$. This can be done by an induction argument. Fix k and suppose that $\psi_l \in C^{m-l+2,\alpha}(\bar{\Omega})$ for all $0 \leq l < k$. Then the right hand side of (3.20) belongs to $C^{m-k,\alpha}(\bar{\Omega})$. Now the existence and regularity theories ensure that the boundary value problem (3.20) attains a unique solution $\psi_k \in C^{m-k+2,\alpha}(\bar{\Omega})$. This induction holds because the induction hypothesis is met for $k = 1$.

Furthermore, we have the following.

Lemma 3.3.8. *Let* $m \geq 3$ *and set* ψ_k *as above for* $1 \leq k \leq m-2$. *Then*

$$\|\psi_k\|_{C^{m-k+2,\alpha}(\bar{\Omega})} \leq \tilde{C}_{k,m,\Omega} \left(\|f\|_{C^{m,\alpha}(\bar{\Omega})} + \|g\|_{C^{m+2,\alpha}(\bar{\Omega})} \right), \tag{3.21}$$

for each $k = 0, 1, \dots, m-2$, *where we understand* $\psi_0 = u$.

Proof. Since $u \in C^{m+2,\alpha}(\bar{\Omega})$ solves (3.2), $f \in C^{m,\alpha}(\bar{\Omega})$, $g \in C^{m+2,\alpha}(\bar{\Omega})$ and $\partial\Omega \in C^{m+2,\alpha}$, Theorem 2.2.9 and an a priori estimate yield that

$$\|u\|_{C^{m+2,\alpha}(\bar{\Omega})} \leq C_{m,\Omega}\left(\|f\|_{C^{m,\alpha}(\bar{\Omega})} + \|g\|_{C^{m+2,\alpha}(\bar{\Omega})}\right).$$

The proof is finished by adopting an induction argument. One can also prove that

$$\tilde{C}_{k,m,\Omega} \leq C_{m-k+2,\Omega} \sum_{l=3}^{k+2} C_l \tilde{C}_{k-l+2,m,\Omega}.$$

□

Set for each $1 \leq k \leq m$

$$w_k(y,x) = \sum_{l=1}^{k} \chi^{i_1 \dots i_l}(y) D_{x_{i_1} \dots x_{i_l}} \psi_{k-l}(x) + \psi_k(x) \quad (y \in \mathbb{R}^n, x \in \bar{\Omega}), \tag{3.22}$$

where $\psi_{m-1} \in C^{3,\alpha}(\bar{\Omega})$ and $\psi_m \in C^{2,\alpha}(\bar{\Omega})$ are arbitrary functions which satisfy the inequality (3.21) respectively when $k = m-1$ and m. Recall that we have set $\chi^i \equiv 0$ for all $i = 1, \dots, n$, which implies that $w_1(y,x) = \psi_1(x)$; that is, w_1 is independent of the y-variable.

Lemma 3.3.9. *Let $m \geq 3$ be an integer and w_k be given by (3.22) for each $k = 1, \dots, m$. Then $w_k(\cdot, x) \in C^{2,\alpha}(\mathbb{R}^n)$ for each $x \in \bar{\Omega}$ and $w_k(y, \cdot) \in C^{m-k+2,\alpha}(\bar{\Omega})$ for each $y \in \mathbb{R}^n$ with the estimate*

$$\|w_k(\cdot,x)\|_{C^{2,\alpha}(\mathbb{R}^n)} + \|w_k(y,\cdot)\|_{C^{m-k+2,\alpha}(\bar{\Omega})} \leq \bar{C}_{k,m,\Omega}\left(\|f\|_{C^{m,\alpha}(\bar{\Omega})} + \|g\|_{C^{m+2,\alpha}(\bar{\Omega})}\right),$$

where $\bar{C}_{k,m,\Omega} = \sum_{l=1}^{k} n^l C_l \tilde{C}_{k-l,m,\Omega} + \tilde{C}_{k,m,\Omega}$ for each $k = 1, \dots, m$.

Moreover, for $3 \leq k \leq m$, w_k solves recursively

$$a_{ij} D_{y_i y_j} w_k + 2a_{ij} D_{x_i y_j} w_{k-1} + a_{ij} D_{x_i x_j} w_{k-2} = 0 \quad \text{in } \mathbb{R}^n \times \Omega. \tag{3.23}$$

Proof. The estimate follows from (3.19) and (3.21). The equation (3.23) is immediate from (3.18) and (3.20). □

Define now the k-th order interior corrector w_k^ε of (3.1) for each $1 \leq k \leq m$ and $\varepsilon > 0$ by

$$w_k^\varepsilon(x) := w_k\left(\frac{x}{\varepsilon}, x\right) \quad (x \in \bar{\Omega}). \tag{3.24}$$

By Lemma 3.3.9, $w_k^\varepsilon \in C^{2,\alpha}(\bar{\Omega})$ for each $\varepsilon > 0$. Thus, the following boundary value problem has a unique solution lying in $C^{2,\alpha}(\bar{\Omega})$;

$$\begin{cases} a_{ij}\left(\frac{x}{\varepsilon}\right) D_{ij} z_k^\varepsilon = 0 & \text{in } \Omega, \\ z_k^\varepsilon = -w_k^\varepsilon & \text{on } \partial\Omega. \end{cases} \tag{3.25}$$

We denote the solution by z_k^ε and call it the k-th order boundary layer corrector of (3.1). Lemma 3.3.9 yields a uniform bound of z_k^ε, namely,

$$\sup_{\varepsilon>0} \|z_k^\varepsilon\|_{L^\infty(\Omega)} \leq c_0 \sup_{\varepsilon>0} \|w_k^\varepsilon\|_{L^\infty(\Omega)} \leq c_0 \bar{C}_{k,m,\Omega} \left(\|f\|_{C^{m,\alpha}(\bar{\Omega})} + \|g\|_{C^{m+2,\alpha}(\bar{\Omega})} \right).$$

Note that for any $\varepsilon > 0$, $z_1^\varepsilon \equiv 0$ on $\bar{\Omega}$, since $w_1^\varepsilon \equiv \psi_1$ on $\bar{\Omega}$ where ψ_1 vanishes on $\partial\Omega$.

3.3.3 Proof of Theorem 3.1.1

We are now in position to achieve the higher order convergence rates in the framework of linear equations.

Proof of Theorem 3.1.1. Fix $\varepsilon > 0$. Let w_k^ε and z_k^ε be defined as in the previous section for each $k = 1, \ldots, m$. Define

$$\eta_m^\varepsilon := u + \varepsilon w_1^\varepsilon + \varepsilon^2 w_2^\varepsilon + \cdots + \varepsilon^m w_m^\varepsilon, \quad \theta_m^\varepsilon := \varepsilon z_1^\varepsilon + \varepsilon^2 z_2^\varepsilon + \cdots + \varepsilon^m z_m^\varepsilon$$

on $\bar{\Omega}$. Then both η_m^ε and θ_m^ε belong to $C^{2,\alpha}(\bar{\Omega})$. We utilize (3.8), (3.23) and (3.25). A lengthy but elementary computation gives

$$a_{ij}\left(\frac{x}{\varepsilon}\right) D_{ij}(\eta_m^\varepsilon + \theta_m^\varepsilon) = a_{ij}\left(\frac{x}{\varepsilon}\right) D_{ij}\eta_m^\varepsilon = f + \varepsilon^{m-1}\varphi_m^\varepsilon$$

in Ω, where

$$\begin{aligned} \varphi_m^\varepsilon(x) = \sum_{l=2}^{m-1} & \left[2a_{i_l j}\left(\frac{x}{\varepsilon}\right) D_{y_j}\chi^{i_1\ldots i_{l-1}}\left(\frac{x}{\varepsilon}\right) + a_{i_{l-1}i_l}\left(\frac{x}{\varepsilon}\right)\chi^{i_1\ldots i_{l-2}}\left(\frac{x}{\varepsilon}\right)\right] \\ & \times D_{x_{i_1}\cdots x_{i_l}}\psi_{m-l-1}(x) \\ & + \varepsilon \sum_{l=2}^{m} a_{i_{l-1}i_l}\left(\frac{x}{\varepsilon}\right)\chi^{i_1\ldots i_{l-2}}\left(\frac{x}{\varepsilon}\right) D_{x_{i_1}\ldots x_{i_l}}\psi_{m-l}(x) \qquad (x \in \Omega). \end{aligned}$$

Now we set $\varepsilon \in (0,1)$. According to (3.19) and (3.21), we have

$$\|\varphi_m^\varepsilon\|_{L^\infty(\Omega)} \le L_{m,\Omega} \left(\|f\|_{C^{m,\alpha}(\bar\Omega)} + \|g\|_{C^{m+2,\alpha}(\bar\Omega)} \right)$$

where

$$L_{m,\Omega} = \sigma \left[\sum_{l=3}^{m-1} n^{l-1} \left\{ 2(C_{l-1} + C_{l-2}) \tilde C_{m-l-1,\Omega} + C_{l-2} \tilde C_{m-l,\Omega} \right\} + n^{m-1} C_{m-2} \tilde C_{0,\Omega} \right].$$

Here C_k and $\tilde C_k$ are the constants chosen as in (3.19) and (3.21).

On the other hand, we have $\eta_m^\varepsilon + \theta_m^\varepsilon = g + \sum_{k=1}^m \varepsilon^k (w_k^\varepsilon + z_k^\varepsilon) = g$ on $\partial\Omega$. Thus, $u^\varepsilon - \eta_m^\varepsilon - \theta_m^\varepsilon \in C^{2,\alpha}(\bar\Omega)$ solves the following equation,

$$\begin{cases} a_{ij}\left(\frac{x}{\varepsilon}\right) D_{ij} v = -\varepsilon^{m-1} \varphi_m^\varepsilon & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

An a priori estimate then gives

$$\|u^\varepsilon - \eta_m^\varepsilon - \theta_m^\varepsilon\|_{L^\infty(\Omega)} \le c_0 L_{m,\Omega} \left(\|f\|_{C^{m,\alpha}(\bar\Omega)} + \|g\|_{C^{m+2,\alpha}(\bar\Omega)} \right).$$

□

3.4 Fully Nonlinear Equations

3.4.1 Basic Homogenization Scheme

This subsection is devoted to the homogenization process of (3.3) to (3.4). It generalizes the homogenization result of linear equations (see Section 3.3.1). One may find a general argument in [22] for some lemmas. However, we present all the proofs which are adequate for our situation.

Lemma 3.4.1. *Assume for each $\varepsilon > 0$ that $u^\varepsilon \in C(\bar\Omega)$ is a viscosity solution of (3.3). Then there is a function $u \in C(\bar\Omega)$ and a subsequence $\{u^{\varepsilon_k}\}_{k=1}^\infty$ of $\{u^\varepsilon\}_{\varepsilon>0}$ such that $u^{\varepsilon_k} \to u$ uniformly in $\bar\Omega$ as $k \to \infty$.*

Proof. The proof is identical to that of Lemma 3.3.1. One may notice that the proof of Lemma 3.3.1 does not involve the linear structure of (3.1). □

As we did in Section 3.3.1, we will ascertain the effective equation which u solves in the viscosity sense at the end of this section. Before we start, we point out that the argument throughout this subsection is valid by only assuming that $F \in C^{0,1}(\bar{B}_L \times \bar{\Omega} \times \mathbb{R}^n)$ for each $L > 0$ (i.e., (F3) with $m = 0$).

Lemma 3.4.2. *To each $(M, x) \in \mathcal{S}^n \times \bar{\Omega}$ there corresponds a unique $\gamma \in \mathbb{R}$ for which the following equation*

$$F(D_y^2 w + M, x, y) = \gamma \quad \text{in } \mathbb{R}^n \tag{3.26}$$

attains a 1-periodic solution $w \in C^{2,\alpha}(\mathbb{R}^n)$. Moreover, w is unique up to an additive constant. Moreover, if the solution w satisfies $w(0) = 0$, then

$$\|w\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_{|M|}.$$

As we did in the linear case, we start with an approximating problem.

Lemma 3.4.3. *Let $(M, x) \in \mathcal{S}^n \times \bar{\Omega}$ and $\delta \in (0, 1)$. Then there is a unique bounded 1-periodic function $w^\delta \in C^{2,\alpha}(\mathbb{R}^n)$ which solves*

$$F(D_y^2 w^\delta + M, x, y) - \delta w^\delta = 0 \quad \text{in } \mathbb{R}^n, \tag{3.27}$$

with the uniform estimate

$$\sup_{0<\delta<1} \left\| \delta w^\delta \right\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_{|M|}. \tag{3.28}$$

Proof. Fix $(M, x) \in \mathcal{S}^n \times \bar{\Omega}$. The unique existence of the solution w^δ to (3.27) follows the same argument as in Lemma 3.3.3, so is omitted. Moreover, we have

$$\sup_{0<\delta<1} \left\| \delta w^\delta \right\|_{L^\infty(\mathbb{R}^n)} \leq \sigma(1 + |M|). \tag{3.29}$$

To improve the regularity of w^δ to $C^{2,\alpha}(\mathbb{R}^n)$ we make use of interior $C^{2,\alpha}$ estimate (Theorem 2.2.7) instead of the interior Schauder estimate. We know from the hypothesis (F4) that F is concave with respect to M and from the hypothesis (F3) that for any $y, y_0 \in \mathbb{R}^n$

$$\beta(y, y_0) := \sup_{N \in \mathcal{S}^n} \frac{|F(M + N, x, y) - F(M + N, x, y_0)|}{1 + |N|} \leq \sigma(1 + |M|)|y - y_0|.$$

On the other hand, since w^δ is a solution to (3.27) in $\mathbb{R}^n$, we have $w^\delta \in S(\lambda/n, \Lambda, \delta w^\delta - F(M, x, \cdot))$ in $\mathbb{R}^n$. As we restrict ourselves to the cube Q_2, we obtain from Theorem 2.2.3 (b) that $w^\delta \in C^{\tilde{\alpha}}(\bar{Q}_1)$ and $\| w^\delta \|_{C^{\tilde{\alpha}}(\bar{Q}_1)} \leq c_0(\delta^{-1} + 2)\sigma(1 + |M|)$, for each $\delta > 0$. Since Q_1 is a periodic cube of w^δ, we obtain a uniform Hölder estimate on δw^δ over $\mathbb{R}^n$, namely,

$$\sup_{0<\delta<1} \| \delta w^\delta \|_{C^{\bar{\alpha}}(\mathbb{R}^n)} \leq 3c_0\sigma(1 + |M|).$$

Now Theorem 2.2.7 applies to w^δ so that we get a constant $C_{|M|} > 1$ for which $w^\delta \in C^{2,\alpha}(\bar{B}_{C_{|M|}^{-1}\sqrt{n}}(y_0))$ and

$$\| w^\delta \|^*_{C^{2,\alpha}(\bar{B}_{C_{|M|}^{-1}\sqrt{n}}(y_0))} \leq C_{|M|} \left(\| w^\delta \|_{L^\infty(B_{\sqrt{n}}(y_0))} + 1 \right) \leq \tilde{C}_{|M|}\delta^{-1},$$

where $\|\cdot\|^*_{C^{2,\alpha}(E)}$ is the adimensional $C^{2,\alpha}$ norm on E. Since $y_0 \in \mathbb{R}^n$ was an arbitrary point and $B_{\sqrt{n}}(y_0)$ contains a periodic cube of w^δ, we obtain the estimate (3.28). $\square$

Our next step is to find a uniform bound of the oscillation of w^δ for $\delta \in (0, 1)$.

Lemma 3.4.4. *Let $M \in \mathcal{S}^n$, $x \in \bar{\Omega}$ and w^δ be the unique solution to* (3.27)*. Then*

$$\sup_{0<\delta<1} \operatorname*{osc}_{\mathbb{R}^n} w^\delta \leq C(1 + |M|).$$

Moreover, there holds

$$\sup_{0<\delta<1} \| \tilde{w}^\delta \|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_{|M|}, \tag{3.30}$$

where $\tilde{w}^\delta := w^\delta - w^\delta(0)$ in $\mathbb{R}^n$.

Proof. The proof follows the line of the proof of Lemma 3.3.4. $\square$

It is noteworthy to observe that the derivatives of w^δ are bounded independent of $\delta \in (0, 1)$. To be specific, since $Dw^\delta = D\tilde{w}^\delta$ and $D^2 w^\delta = D^2\tilde{w}^\delta$, we obtain from (3.30) that

$$\sup_{0<\delta<1} \left(\| Dw^\delta \|_{L^\infty(\mathbb{R}^n)} + \| D^2 w^\delta \|_{L^\infty(\mathbb{R}^n)} + [D^2 w^\delta]_{C^\alpha(\mathbb{R}^n)} \right) \leq C_{|M|}. \tag{3.31}$$

We are now in position to prove Lemma 3.4.2.

Proof of Lemma 3.4.2. One may notice that the proof of Lemma 3.3.2 has nothing to do with the linear structure of (3.8). Indeed, (3.29) and (3.30) respectively correspond to (3.12) and (3.14). Hence, by the compact embedding, we are able to extract a subsequence $\{\delta_k w^{\delta_k}, \tilde{w}^{\delta_k}\}_{k=1}^{\infty}$ from $\{\delta w^{\delta}, \tilde{w}^{\delta}\}_{0<\delta<1}$ such that

$$\left\| \delta_k w^{\delta_k} - \gamma \right\|_{L^\infty(\mathbb{R}^n)} + \left\| \tilde{w}^{\delta_k} - w \right\|_{C^2(\mathbb{R}^n)} \to 0 \quad \text{as} \quad k \to \infty, \tag{3.32}$$

for some $\gamma \in \mathbb{R}$ and $w \in C^{2,\alpha}(\mathbb{R}^n)$. In addition, we have that $|\gamma| \leq \sigma(1+|M|)$ and $\|w\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_{|M|}$. The rest of the proof is exactly the same with that of Lemma 3.3.2 and hence is omitted. □

Definition 3.4.5. *Let $(M,x) \in \mathcal{S}^n \times \bar{\Omega}$.*

(i) For each $\delta \in (0,1)$, we denote $w^\delta(\cdot; M, x)$ by the unique bounded 1-periodic solution of (3.27) and $\tilde{w}^\delta(\cdot; M,x) = w^\delta(\cdot; M,x) - w^\delta(0; M,x)$ in $\mathbb{R}^n$. By the uniqueness of the solution, we can understand $w^\delta(y; \cdot, \cdot)$ as the mapping $(M,x) \mapsto w^\delta(y; M,x)$ defined on $\mathcal{S}^n \times \bar{\Omega}$ for each $y \in \mathbb{R}^n$.

(ii) In a similar way, we write $\bar{F}(M,x)$ by the unique number γ of (3.26) and $w(\cdot; M,x)$ by the bounded 1-periodic solution of (3.26) which is normalized by $w(0; M,x) = 0$. Again the uniqueness allows us to understand $\bar{F}$ [resp., $w(y; \cdot, \cdot)$ for each $y \in \mathbb{R}^n$] as the mapping $(M,x) \mapsto \bar{F}(M,x)$ [resp., $w(y; M,x)$] defined on $\mathcal{S}^n \times \bar{\Omega}$.

Note that (3.26) now reads

$$\begin{cases} F(D_y^2 w + M, x, y) = \bar{F}(M,x) & \text{in } \mathbb{R}^n, \\ w \text{ is 1-periodic.} \end{cases} \tag{3.33}$$

The next lemma states that δw^δ and $\tilde{w}^\delta$ are locally Lipschitz continuous in (M,x). One may also find a proof for (3.34) in [2] and [22] regarding a more general situation. The proof for (3.35) can also be found in [42] with a different argument.

Lemma 3.4.6. *For any $L > 0$ and $(M,x), (M',x') \in \bar{B}_L \times \bar{\Omega}$, we have*

$$\left\| \delta w^\delta(\cdot; M', x') - \delta w^\delta(\cdot; M, x) \right\|_{L^\infty(\mathbb{R}^n)} \leq C_L(\|M' - M\| + |x' - x|), \tag{3.34}$$

and

$$\left\| \tilde{w}^\delta(\cdot; M', x') - \tilde{w}^\delta(\cdot; M, x) \right\|_{L^\infty(\mathbb{R}^n)} \le C_L(\|M' - M\| + |x' - x|). \quad (3.35)$$

Proof. For brevity, let us denote by v_1^δ [resp., v_2^δ] the function $w^\delta(\cdot; M', x')$ [resp., $w^\delta(\cdot; M, x)$]. Also by $\tilde{v}_1^\delta$ [resp., $\tilde{v}_2^\delta$] let us denote $\tilde{w}^\delta(\cdot; M', x')$ [resp., $\tilde{w}^\delta(\cdot; M, x)$].

We prove (3.34) first. By the Lipschitz continuity of F, we get

$$F(D_y^2 v_2^\delta + M, x, y) \ge \delta v_2^\delta - \sigma(1 + L)(\|M' - M\| + |x' - x|)$$

which implies that $v_2^\delta - \delta^{-1}\sigma(1 + L)(\|M' - M\| + |x' - x|)$ is a subsolution of (3.27). By the comparison principle (Theorem 2.1.4), we arrive at

$$\delta v_2^\delta - \delta v_1^\delta \le \sigma(1 + L)(\|M' - M\| + |x' - x|) \quad \text{in } \mathbb{R}^n.$$

By a similar argument, we obtain (3.34) with $C_L \ge \sigma(1 + L)$.

Now we move on to the proof of (3.35). The main idea is to use the linearisation of F. Define $a_{ij}^\delta = \int_0^1 F_{p_{ij}}(N_t^\delta, x_t, \cdot)dt$ and $b_k^\delta = \int_0^1 F_{x_k}(N_t^\delta, x_t, \cdot)dt$ where $N_t^\delta := t\{D^2 v_1^\delta + M'\} + (1 - t)\{D^2 v_2^\delta + M\}$ and $x_t := tx + (1 - t)x'$. It is immediate from the structure conditions (F1)-(F3) that a_{ij}^δ and b_k^δ $(i, j, k = 1, \dots, n)$ are 1-periodic and uniformly bounded in $\mathbb{R}^n$ by the Lipschitz constant of F. Furthermore, (a_{ij}^δ) is uniformly elliptic with the same ellipticity constants λ and Λ of F.

Now define $v^\delta := v_1^\delta - v_2^\delta$ and $\tilde{v}^\delta := \tilde{v}_1^\delta - \tilde{v}_2^\delta$. Then $v^\delta, \tilde{v}^\delta \in C^{2,\alpha}(\mathbb{R}^n)$ solve

$$a_{ij}^\delta D_{ij} w + a_{ij}^\delta(M'_{ij} - M_{ij}) + b_k^\delta(x'_k - x_k) = \delta v^\delta \quad \text{in } \mathbb{R}^n. \quad (3.36)$$

As this equation belongs to the same class of (3.9), we arrive the conclusion by the same argument used in Lemma 3.3.4. We left the details to the reader. □

Lemma 3.4.7. *The convergence in* (3.32) *is uniform in* $(M, x) \in \bar{B}_L \times \bar{\Omega}$ *for each* $L > 0$*; i.e.,*

$$\lim_{\delta \to 0} \sup_{(M,x) \in \bar{B}_L \times \bar{\Omega}} \left\| \delta w^\delta(\cdot; M, x) - \bar{F}(M, x) \right\|_{L^\infty(\mathbb{R}^n)} = 0,$$

and

$$\lim_{\delta \to 0} \sup_{(M,x) \in \bar{B}_L \times \bar{\Omega}} \left\| \tilde{w}^\delta(\cdot; M, x) - w(\cdot; M, x) \right\|_{C^2(\mathbb{R}^n)} = 0.$$

Proof. Fix $L > 0$. Put $C_L = \sup\{C_{|M|} : M \in \bar{B}_L\}$ and then take $\tilde{C}_L = \max\{\sigma(1+L), C_L\}$. Then it follows from (3.29) and (3.30) that for any $\delta \in (0,1)$,

$$\sup_{(M,x)\in\bar{B}_L\times\bar{\Omega}} \left\{ \left\| \delta w^\delta(\cdot; M, x) \right\|_{L^\infty(\mathbb{R}^n)}, \left\| \tilde{w}^\delta(\cdot; M, x) \right\|_{C^{2,\alpha}(\mathbb{R}^n)} \right\} \le \tilde{C}_L.$$

The above uniform estimates allow us to extract a subsequence $\{\delta_k w^{\delta_k}\}_{k=1}^\infty$ [resp. $\{\tilde{w}^{\delta_k}\}_{k=1}^\infty$] from $\{\delta w^\delta\}_{0<\delta<1}$ [resp. $\{\tilde{w}^\delta\}_{0<\delta<1}$] such that (3.32) holds regardless of a particular choice of $(M,x) \in \bar{B}_L \times \bar{\Omega}$. The rest of the proof is the same with that in Lemma 3.4.2. □

It is an immediate consequence of Lemma 3.4.6 and 3.4.7 that the effective operator $\bar{F}$ and the corresponding corrector $w(y;\cdot,\cdot)$ are locally Lipschitz continuous (uniform in y). Due to its particular role in the rest of this paper, we present the statement without proof.

Lemma 3.4.8. *$\bar{F}$ and $w(y;\cdot,\cdot)$ are Lipschitz continuous locally in $\mathcal{S}^n$ and globally in $\bar{\Omega}$. Moreover, the Lipschitz continuity of the latter is uniform in $y \in \mathbb{R}^n$.*

There are additional properties of $\bar{F}$. A more general proof is contained in [22]. Here we make a slight adjustment of the proof according to our situation; the main difference is that we have $C^{2,\alpha}$-corrector, which makes the proof simpler.

Lemma 3.4.9. *(i) $\bar{F}$ is uniformly elliptic with the same constants λ and Λ of F.*

(ii) $\bar{F}$ is concave on $\mathcal{S}^n$.

Proof. The proof for the assertion (i) is similar to that of the assertion (ii) of Lemma 3.3.5, so is omitted.

Now we establish the proof of (ii). Let $M, N \in \mathcal{S}^n$ and $x \in \bar{\Omega}$ be given. For simplicity let us write w^M by the solutions of (3.33) with respect to M.

Suppose toward a contradiction that there is some $t \in (0,1)$ and $M, N \in \mathcal{S}^n$ such that

$$\bar{F}(tM + (1-t)N, x) < t\bar{F}(M,x) + (1-t)\bar{F}(N,x).$$

Put $X := tM + (1-t)N \in \mathcal{S}^n$. Adding a constant to w^X if necessary, we may assume that $w^X < tw^M + (1-t)w^N$ in $\mathbb{R}^n$. Then we obtain from the concavity of F that

$$\bar{F}(X,x) < t\bar{F}(M,x) + (1-t)\bar{F}(N,x) \leq F(X + D_y^2(tw^M + (1-t)w^N), x, y)$$

in $\mathbb{R}^n$. However, since $\bar{F}(X + D_y^2 w^X, x, y) = \bar{F}(X,x)$ in $\mathbb{R}^n$, the comparison principle implies that $w^X \geq tw^M + (1-t)w^N$ in $\mathbb{R}^n$, which is a contradiction. □

As we mentioned in the beginning of this section, we determine the equation which u solves in the viscosity sense.

Lemma 3.4.10. *Assume that $F \in C(\mathcal{S}^n \times \bar{\Omega} \times \mathbb{R}^n)$ satisfy the hypotheses (F1)-(F4). Then the function u from Lemma 3.4.1 solves* (3.4)*. Moreover, u is unique and belongs to the class of $C^{2,\alpha}(\bar{\Omega})$.*

Proof. The proof of that u is a viscosity solution of (3.4) is similar to that of Lemma 3.3.6. Instead of using strong maximum principle, one may take advantage of Theorem 2.1.3. The details are left to the reader.

As long as we know that u solves (3.4), the fact that $u \in C^{2,\alpha}(\Omega)$ follows readily from Theorem 2.2.7. The proof is similar to that in Lemma 3.4.3, so the details are omitted; instead of taking advantage of (F1)-(F4), we use Lemma 3.4.9 (i)-(iii). We make a remark here that the exponent α is the same with which we chose in Lemma 3.4.3 because the ellipticity constants of $\bar{F}$ coincide with those of F (Lemma 3.4.9 (i)). □

3.4.2 Regularity Theory in Slow Variables

In the previous subsection, we observed that the Lipschitz regularity of F, in particular in the (M,x)-variable, yields the Lipschitz regularity of $\bar{F}$ and $w(y;\cdot,\cdot)$, where the regularity for the latter is uniform in $y \in \mathbb{R}^n$. Then, it is natural to ask whether higher regularity of F in (M,x)-variable gives higher regularity for $\bar{F}$ and $w(y;\cdot,\cdot)$, and we prove in this subsection that the answer is affirmative. Specifically, we observe that they have the same regularity as F does. This regularity result plays the key role in the rest of this paper, especially in seeking higher order interior correctors. To be precise, we observe the following.

Proposition 3.4.11. *$\bar{F}$ and $w(y;\cdot,\cdot)$ are $C^{m,1}$ locally in $\mathcal{S}^n$ and globally in $\bar{\Omega}$ and for any $L > 0$,*

$$\|\bar{F}\|_{C^{m,1}(\bar{B}_L \times \bar{\Omega})} + \|w(y,\cdot,\cdot)\|_{C^{m,1}(\bar{B}_L \times \bar{\Omega})} \leq C_{L,m} \tag{3.37}$$

Moreover, for any $(M',x'),(M,x) \in \bar{B}_L \times \bar{\Omega}$ there holds

$$\begin{aligned}\sum_{0\leq i+j\leq m-1} \left\| D_M^i D_x^j w(\cdot;M',x') - D_M^i D_x^j w(\cdot;M,x) \right\|_{C^{2,\alpha}(\mathbb{R}^n)} \\ \leq C_{L,m}(\|M'-M\| + |x'-x|).\end{aligned} \tag{3.38}$$

Remark 3.4.12. *Note that the estimate* (3.38) *implies that $D_y^i w(y;\cdot,\cdot) \in C^{m-1,1}(\bar{B}_L \times \bar{\Omega})$ for $i = 1,2$. This will turn out as the coupling effect as we mentioned in Sect. 3.1.*

Before we begin the proof, let us illustrate the heuristics of our argument. In the first place, we only assume that F satisfies the structure condition (F3) with $m = 1$, which means that F is $C^{1,1}$ locally in $\mathcal{S}^n$ and globally in $\bar{\Omega}\times\mathbb{R}^n$, and arrive at the conclusion that $\bar{F}$ and $w(y;\cdot,\cdot)$ are also $C^{1,1}$ locally in $\mathcal{S}^n$ and globally in $\bar{\Omega}$. We also observe that the equation, which involves the partial derivatives of $\bar{F}$ and $w(y;\cdot,\cdot)$ in M and x-variable, satisfies the same structure conditions of F. This implies that under our original assumption (F3) we are able to iterate the argument to get $C^{m,1}$ regularity of $\bar{F}$ and $w(y;\cdot,\cdot)$ which is local in $\mathcal{S}^n$ and global in $\bar{\Omega}$.

As the first step, we prove that if $F \in C^{1,1}$, then the L^∞-norm in (3.34) and (3.35) can be improved by $C^{2,\alpha}$-norm.

Lemma 3.4.13. *For each $L > 0$ and $(M,x),(M',x') \in \bar{B}_L \times \bar{\Omega}$, there hold for all $\delta \in (0,1)$,*

$$\left\| \delta w^\delta(\cdot;M',x') - \delta w^\delta(\cdot;M,x) \right\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_L(\|M'-M\| + |x'-x|) \tag{3.39}$$

and

$$\left\| \tilde{w}^\delta(\cdot;M',x') - \tilde{w}^\delta(\cdot;M,x) \right\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_L(\|M'-M\| + |x'-x|).$$

Proof. The main idea has been already introduced in the proof of Lemma 3.4.6. We only need to obtain a uniform $C^{0,\alpha}(\mathbb{R}^n)$-estimate on the linearized coefficients a_{ij}^δ and b_k^δ; recall all the notations used in Lemma 3.4.6. Here we only present the proof for a_{ij}^δ, since that of b_k^δ follows the same argument.

By the estimate (3.31), we have that for any $t \in [0,1]$, $\left\| N_t^\delta \right\|_{L^\infty(\mathbb{R}^n)} \leq C_L + L$. Hence, we deduce from the condition (F3) that $\left\| a_{ij}^\delta \right\|_{L^\infty(\mathbb{R}^n)} \leq \sigma(C_L + L + 1)$. Again by (3.31), for any $y_1, y_2 \in Q_1$, $\left\| N_t^\delta(y_1) - N_t^\delta(y_2) \right\| \leq$

$C_L|y_1 - y_2|^\alpha$. Thus, the periodicity of a_{ij}^δ yields that $[a_{ij}^\delta]_{C^{0,\alpha}(\mathbb{R}^n)} \leq \tilde{C}_L$, where $\tilde{C}_L = \sigma(C_L + L + 1)(C_L + 1)$. Summing up we get that $\left\| a_{ij}^\delta \right\|_{C^{0,\alpha}(\mathbb{R}^n)} \leq 2\tilde{C}_L$.

It is also easy to see that $\left\| \delta v^\delta \right\|_{C^{0,\alpha}(\mathbb{R}^n)} \leq 6c_0\sigma(1 + L)$. Therefore, we may apply the interior Schauder estimate to (3.36) in a ball $B_{\sqrt{n}}$ containing a periodic cube to get the conclusion, as in Lemma 3.3.3 and 3.3.4. $\square$

As a corollary, we obtain the same Lipschitz continuity of $w(y;\cdot,\cdot)$ in (M,x)-variable which is uniform in the $C^{2,\alpha}(\mathbb{R}^n)$-norm.

Lemma 3.4.14. *For each $L > 0$ and $(M,x),(M',x') \in \bar{B}_L \times \bar{\Omega}$, there holds*

$$\|w(\cdot;M',x') - w(\cdot;M,x)\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_L(\|M' - M\| + |x' - x|).$$

Proof. Apply the uniform convergence (Lemma 3.4.7) to get

$$\|w(\cdot;M',x') - w(\cdot;M,x)\|_{C^{2}(\mathbb{R}^n)} \leq C_L(\|M' - M\| + |x' - x|).$$

Then use the uniform boundedness of $C^{2,\alpha}(\mathbb{R}^n)$-norm of $w(\cdot;M',x') - w(\cdot;M,x)$ (Lemma 3.4.2) and the compactness embedding to improve this inequality to $C^{2,\alpha}(\mathbb{R}^n)$-norm. $\square$

In the subsequent two lemmas, we show that $\bar{F}$ and $w(y;\cdot,\cdot)$ are differentiable and further that the partial derivatives are locally Lipschitz continuous on $\mathcal{S}^n \times \bar{\Omega}$. The former is done by linearizing the equation (3.33). In order to get the latter, however, we need to begin our argument from the linearized equation (3.36).

Lemma 3.4.15. *There exist $\bar{F}_{p_{kl}}$, $\bar{F}_{x_k}$, $D_{p_{kl}}w(y;\cdot,\cdot)$ and $D_{x_k}w(y;\cdot,\cdot)$ for each $y \in \mathbb{R}^n$ on $\mathcal{S}^n \times \bar{\Omega}$. In addition, there hold for any $L > 0$ and $(M,x) \in \bar{B}_L \times \bar{\Omega}$,*

$$\begin{aligned} &|\bar{F}_{p_{kl}}(M,x)| + |\bar{F}_{x_k}(M,x)| \\ &+ \|D_{p_{kl}}w(\cdot;M,x)\|_{C^{2,\alpha}(\mathbb{R}^n)} + \|D_{x_k}w(\cdot;M,x)\|_{C^{2,\alpha}(\mathbb{R}^n)} \\ &\leq C_L. \end{aligned} \tag{3.40}$$

Proof. Here we only provide the proof for the M-partial derivatives of $\bar{F}$ and $w(y;\cdot,\cdot)$. The argument for the x-partial derivatives is similar so we omit it to avoid the redundancy.

Pick any $L > 0$ and $(M, x) \in \bar{B}_L \times \bar{\Omega}$. By v_h we denote $h^{-1}[w(\cdot; M + hE^{kl}, x) - w(\cdot; M, x)]$. As we linearize the equation (3.33) with $M + hE^{kl}$ and M, and divide the both sides by h, we observe that v_h satisfies

$$a_{ij,h} D_{ij} v_h + a_{kl,h} = \gamma_h \tag{3.41}$$

where $a_{ij,h} := \int_0^1 F_{p_{ij}}(N_{t,h}, x, \cdot)dt$, $\gamma_h := h^{-1}[\bar{F}(M + hE^{kl}, x) - \bar{F}(M, x)]$ and $N_{t,h} := tD_y^2 w(\cdot; M + hE^{kl}, x) + (1-t)D_y^2 w(\cdot; M, x) + M + thE^{kl}$.

By following the argument in the proof of Lemma 3.4.13, we observe that for any h with $|h|$ small, $a_{ij,h}$ is also uniformly elliptic with the ellipticity constants λ and Λ, and belongs to $C^{0,\alpha}(\mathbb{R}^n)$ with $\|a_{ij,h}\|_{C^{0,\alpha}(\mathbb{R}^n)} \leq c_L$. Also we know from Lemma 3.4.8 that $|\gamma_h| \leq \tilde{c}_L$.

Therefore, the linearized equation (3.41) belongs to the same class of (3.9). Even though the coefficients of (3.41) vary with respect to the parameter h, the proof of Lemma 3.3.2 is still applicable because we have a uniform convergence of $a_{ij,h}$ as $h \to 0$; indeed, Lemma 3.4.14 implies that $a_{ij,h} \to a_{ij} := F_{p_{ij}}(D_y^2 w(\cdot; M, x) + M, x, \cdot)$ uniformly in $\mathbb{R}^n$ as $h \to 0$. Consequently, there exist a unique constant γ and a bounded 1-periodic function $v \in C^{2,\alpha}(\mathbb{R}^n)$ such that

$$|\gamma_h - \gamma| + \|v_h - v\|_{C^2(\mathbb{R}^n)} \to 0$$

as $h \to 0$ and that v satisfies

$$a_{ij} D_{ij} v + a_{kl} = \gamma \quad \text{in } \mathbb{R}^n. \tag{3.42}$$

By the convergence above, $\gamma = \bar{F}_{p_{kl}}(M, x)$ and $v = D_{p_{kl}} w(\cdot; M, x)$. One should notice that we do not force $v(0)$ to be 0 here; otherwise, we could not say that $v = D_{p_{kl}} w(\cdot; M, x)$. The uniform estimate (3.40) now follows from Lemma 3.4.8 and 3.4.14. □

Lemma 3.4.16. *$\bar{F}_{p_{kl}}$, $\bar{F}_{x_k}$, $D_{p_{kl}} w(y; \cdot, \cdot)$ and $D_{x_k} w(y; \cdot, \cdot)$ are Lipschitz continuous locally in $\mathcal{S}^n$ and globally in $\bar{\Omega}$. Moreover, the Lipschitz continuity of the latter two is uniform $y \in \mathbb{R}^n$.*

Proof. Here we only present the proof for the M-partial derivatives. The proof for the x-partial derivatives is the same, and we leave it to the reader.

Substituting M' [resp., x'] with $M + hE^{kl}$ [resp., x] in the equation (3.36) and dividing by h the both sides, one obtains

$$a^{\delta}_{ij,h} D_{ij} v^{\delta}_{h} + a^{\delta}_{kl,h} - \delta v^{\delta}_{h} = 0 \quad \text{in } \mathbb{R}^n,$$

where $a^{\delta}_{ij,h} := \int_0^1 F_{p_{ij}}(N^{\delta}_{t,h}, x, \cdot)dt$, $v^{\delta}_{h} := h^{-1}[w^{\delta}(\cdot; M + hE^{kl}, x) - w^{\delta}(\cdot; M, x)]$ and $N^{\delta}_{t,h} := tD^2_y w^{\delta}(\cdot; M + hE^{kl}, x) + (1-t)D^2_y w^{\delta}(\cdot; M, x) + M + thE^{kl}$.

By Lemma 3.4.13, we have $\left\| v^{\delta}_{h} \right\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_L$ for any $0 < |h| < 1$ and $\delta > 0$. Then the Arzela-Ascoli theorem yields that for each $\delta > 0$, there is a bounded 1-periodic $v^{\delta} \in C^{2,\alpha}(\mathbb{R}^n)$ such that $v^{\delta}_{h} \to v^{\delta}$ in $C^2(\mathbb{R}^n)$ along a subsequence of h. Moreover, this lemma implies that $a^{\delta}_{ij,h} \to a^{\delta}_{ij} := F_{p_{ij}}(D^2_y w^{\delta}(\cdot; M, x) + M, x, \cdot)$ uniformly in $\mathbb{R}^n$ as $h \to 0$. Since a^{δ}_{ij} is also uniformly elliptic with the same ellipticity constants λ and Λ, the stability of the viscosity solutions (c.f. the proof of Lemma 3.3.2) then ensures that the limit function v^{δ} solves

$$a^{\delta}_{ij} D_{ij} v^{\delta} + a^{\delta}_{kl} - \delta v^{\delta} = 0 \quad \text{in } \mathbb{R}^n. \tag{3.43}$$

Due to the uniqueness of the solution of (3.43) (c.f. Lemma 3.3.3), we now know that $v^{\delta}_{h} \to v^{\delta}$ in $C^2(\mathbb{R}^n)$ as $h \to 0$; i.e., the convergence is valid for the full sequence of h.

From now on we write $a^{\delta}_{ij} = a^{\delta}_{ij}(\cdot; M, x)$ [resp., $v^{\delta} = v^{\delta}(\cdot; M, x)$] to specify the dependency on (M, x). We claim that the equation (3.43) is a δ-penalization of the equation (3.42); i.e., the limit of the normalized function $\tilde{v}^{\delta}(\cdot; M, x) := v^{\delta}(\cdot; M, x) - v^{\delta}(0; M, x)$ solves the equation (3.42). It is enough to prove that $a^{\delta}_{ij}(\cdot; M, x) \to a_{ij}(\cdot; M, x) = F_{p_{ij}}(D^2_y w(\cdot; M, x) + M, x, \cdot)$ uniformly in $\mathbb{R}^n$ as $\delta \to 0$, since then the rest of the proof follows the lines of Lemma 3.4.2. However, by Lemma 3.4.7 and 3.4.13, we have

$$\lim_{(\delta,h)\to(0+,0)} \sup_{(M,x)\in \bar{B}_L \times \bar{\Omega}} \left\| a^{\delta}_{ij,h}(\cdot; M, x) - a_{ij}(\cdot; M, x) \right\|_{L^{\infty}(\mathbb{R}^n)} = 0,$$

which gives the desired convergence.

Next we claim that for each $L > 0$, $a^{\delta}_{ij}(y; \cdot, \cdot)$ is Lipschitz continuous in $\bar{B}_L \times \bar{\Omega}$ uniformly for $y \in \mathbb{R}^n$ and $\delta \in (0, 1)$. If so, then we arrive at our conclusion by applying Lemma 3.4.6, since the equations (3.43) and (3.27) are in the same class.

To see this, choose any $L > 0$ and $(N, z), (N', z') \in \bar{B}_L \times \bar{\Omega}$. According to (3.31), the $C^{2,\alpha}(\mathbb{R}^n)$-norm of both $w^{\delta}(\cdot; N, z)$ and $w^{\delta}(\cdot; N', z')$ is uniformly bounded by C_L. Thus, the structure condition (F3) together with (3.39) yields that

$$\left\| a^\delta_{ij}(\cdot; N, z) - a^\delta_{ij}(\cdot; N', z') \right\|_{L^\infty(\mathbb{R}^n)} \leq \tilde{C}_L(\|N - N'\| + |z - z'|),$$

where $\tilde{C}_L = C_L\sigma(1 + C_L)$, which proves the claim. □

Remark 3.4.17. *Note that the limit of the normalized function* $\tilde{v}^\delta(\cdot; M, x)$ *may not be equal to* $D_{p_{kl}}w(\cdot; M, x)$*, since we cannot assure that* $D_{p_{kl}}w(0; M, x) = 0$*. In fact, those two functions differ by an additive constant. It is the main reason why we do not use the* δ*-penalization argument to derive Lemma 3.4.15, although the proofs are essentially the same.*

We are now in position to present the proof of our main proposition of this subsection.

Proof of Proposition 3.4.11. Observe from Lemma 3.4.16 the first order partial derivatives of $\bar{F}$ and $w(y; \cdot, \cdot)$ satisfies the equations (e.g., (3.42)) which belong to the same class of (3.26), and admit the δ-approximating problems (e.g., (3.43)) which correspond to (3.27). Thus, we can repeat the argument used through Lemma 3.4.13-3.4.16 again to get the Lipschitz continuity of the second order partial derivatives of $\bar{F}$ and $w(y; \cdot, \cdot)$. We iterate this process by m-times to reach the conclusion. We leave the details to the reader. □

3.4.3 Interior and Boundary Layer Correctors

Now we are in position to construct higher order correctors which correct the error occurring in the interior and on the boundary layer of our physical domain Ω. This subsection involves many iterative arguments, so before we make our argument rigorous, we would like to provide the key idea.

First and foremost, we emphasize that the asymptotic expansion of u^ε occurs inside of the operator F, which differs from the linear case. That is, if $\eta^\varepsilon_r := u + \sum_{k=1}^r \varepsilon^k w_k(\varepsilon^{-1}x, x)$ is our expansion, then after a computation we get

$$F\left(D^2\eta^\varepsilon_r, x, \frac{x}{\varepsilon}\right) = F\left(X^0 + \varepsilon Y^r, \frac{x}{\varepsilon}, x\right)$$

where

$$X^k = \begin{cases} D_x^2 u(\cdot) + D_y^2 w_2(\cdot/\varepsilon, \cdot) & \text{if } k = 0 \\ D_x^2 w_k(\cdot/\varepsilon, \cdot) + D_{x,y} w_{k+1}(\cdot/\varepsilon, \cdot) + D_y^2 w_{k+2}(\cdot/\varepsilon, \cdot) & \text{if } 1 \le k \le r-2, \\ D_x^2 w_{r-1}(\cdot/\varepsilon, \cdot) + D_{x,y} w_r(\cdot/\varepsilon, \cdot) & \text{if } k = r-1, \\ D_x^2 w_r(\cdot/\varepsilon, \cdot) & \text{if } k = r, \end{cases} \tag{3.44}$$

and Y^r defined by

$$Y^r = X^1 + \varepsilon X^2 + \cdots + \varepsilon^{r-1} X^r. \tag{3.45}$$

Here we have denoted $D_x D_y + D_y D_x$ by $D_{x,y}$. To further simplify our notation, let us drop the dependency of $(\varepsilon^{-1}x, x)$. Then a Taylor expansion of F with respect to the Hessian gives,

$$\begin{aligned} F(X^0 + \varepsilon Y^r) = {} & F(X^0) + \varepsilon F_{p_{ij}}(X^0) Y^r_{ij} + \cdots + \frac{\varepsilon^r}{r!} F_{p_{i_1 j_1} \cdots p_{i_r j_r}}(X^0) Y^r_{i_1 j_1} \cdots Y^r_{i_r j_r} \\ & + O(\varepsilon^{r+1}), \end{aligned}$$

which would be valid provided that $\|Y^r\|_{L^\infty(\Omega)} \le C$ with a positive constant independent of ε. This in turn requires us to have a uniform control (i.e., independent of ε) on the supremum norm of second order derivatives of w_k in both x and y-variables.

Moreover, one should note that $Y^r = \sum_{k=1}^r \varepsilon^{k-1} X^k$ is a summation of the terms of different ε-order. For this reason we rearrange the terms in the Taylor expansion according to the ε-power as below.

$$\begin{aligned} F(X^0 + \varepsilon Y^r) = {} & F(X^0) + \varepsilon F_{p_{ij}}(X^0) X^1_{ij} + \cdots \\ & + \varepsilon^r \sum_{l=1}^r \frac{1}{l!} \sum_{n_1 + \cdots + n_l = r} F_{p_{i_1 j_1} \cdots p_{i_l j_l}}(X^0) X^{n_1}_{i_1 j_1} \cdots X^{n_l}_{i_l j_l} \\ & + \sum_{l=1}^r \sum_{r+1 \le n_1 + \cdots + n_l \le rl} \frac{\varepsilon^{n_1 + \cdots + n_l}}{l!} F_{p_{i_1 j_1} \cdots p_{i_l j_l}}(X^0) X^{n_1}_{i_1 j_1} \cdots X^{n_l}_{i_l j_l} \\ & + O(\varepsilon^{r+1}). \end{aligned} \tag{3.46}$$

It suggests us to find $w_1, \ldots, w_r$ in such a way that $F(X^0) = 0$, $F_{p_{ij}}(X^0) X^1_{ij} =$

0, and so on.

To satisfy $F(X^0) = 0$, w_2 must be chosen such that $D_y^2 w_2 = D_y^2 w(\cdot; D_x^2 u, x)$. Then $F(X^0) = \bar{F}(D_x^2 u) = 0$ by Lemma 3.4.10. Furthermore, one should obtain, for $k = 1, \ldots, r-2$,

$$\begin{aligned} 0 &= \sum_{l=1}^{k} \frac{1}{l!} \sum_{n_1+\cdots+n_l=k} F_{p_{i_1j_1}\cdots p_{i_lj_l}}(X^0) X_{i_1j_1}^{n_1} \cdots X_{i_lj_l}^{n_l} \\ &= F_{p_{ij}}(X^0) X_{ij}^k + \sum_{l=2}^{k} \frac{1}{l!} \sum_{n_1+\cdots+n_l=k} F_{p_{i_1j_1}\cdots p_{i_lj_l}}(X^0) X_{i_1j_1}^{n_1} \cdots X_{i_lj_l}^{n_l} \\ &= F_{p_{ij}}(X^0) D_{y_iy_j} w_{k+2} + \Phi_{k+2}, \end{aligned} \tag{3.47}$$

which yields the equation for w_k, where

$$\begin{aligned} \Phi_{k+2} = {} & F_{p_{ij}}(X^0) D_{x_ix_j} w_k + 2F_{p_{ij}}(X^0) D_{x_iy_j} w_{k+1} \\ & + \sum_{l=2}^{k} \frac{1}{l!} \sum_{n_1+\cdots+n_l=k} F_{p_{i_1j_1}\cdots p_{i_lj_l}}(X^0) X_{i_1j_1}^{n_1} \cdots X_{i_lj_l}^{n_l}. \end{aligned}$$

Notice that the summation on the right hand side involves X^l for $l \leq k-1$ only; in other words, the term Φ_{k+2} has nothing to do with the functions w_r with $r \geq k+2$. Thus, we are able to obtain w_{k+2} by solving the equation (3.47) as long as Φ_{k+2} satisfies certain inductive hypotheses. On the other hand, since w_{k+2} makes the ε^k-th order term in (3.46) to vanish, there is no opportunity to kill the ε^{r-1} and ε^r-th order terms; recall that the same situation has happened in the linear setting. This in turn suggests that we can have at most

$$F(X^0 + \varepsilon Y^r) = O(\varepsilon^{r-1}),$$

which would lead us to $O(\varepsilon^{r-1})$-rate of convergence (Theorem 3.1.2). Finally we make a remark that as in the linear case, we would come up with the compatibility condition of w_{k+2}, which determines uniquely w_k. Unlike the linear case (Lemma 3.3.7), however, this relationship is more hidden in the induction argument. We will discuss this issue in the proof in more detail.

Now we make our argument rigorous. Throughout this subsection we set $m \geq 2$. First we enhance the regularity of u, since now we have $\bar{F} \in C^{m,1}$.

Lemma 3.4.18. *Assume that F verifies the hypotheses (F1)-(F4). Then*

$u \in C^{m+2,\alpha}(\bar{\Omega})$ and

$$\|u\|_{C^{m+2,\alpha}(\bar{\Omega})} \leq C_{m,g,\Omega}.$$

Proof. By Proposition 3.4.11 we know that F is $C^{1,1}$ locally in $\mathcal{S}^n$ and globally in $\bar{\Omega}$. Since u solves (3.4) where $g \in C^{m+2,1}(\bar{\Omega})$ and $\partial\Omega \in C^{m+2,1}$, the regularity theory (Theorem 2.2.7) implies that $u \in C^{m+2,\alpha}(\bar{\Omega})$ and

$$\|u\|_{C^{m+2,\alpha}(\bar{\Omega})} \leq C_{\bar{F},\Omega}(\|u\|_{L^\infty(\Omega)} + \|g\|_{C^{m+2,1}(\bar{\Omega})}),$$

where $C_{\bar{F},\Omega}$ is a constant depending only on the derivatives of $\bar{F}$ up to m-th order, and on Ω. By (3.37), $C_{\bar{F},\Omega}$ in turn depends only on the constants appearing in the structure conditions (F1)-(F4) and m. By an a priori estimate, on the other hand, we may bound the supremum norm of u by a constant depending only on λ, Λ, Ω and $\|g\|_{L^\infty(\Omega)}$. It completes the proof. □

Next we construct the interior higher order correctors. The regularity theory established in Subsection 3.4.2 now plays an essential role in proving the existence of the correctors and obtaining a uniform control on L^∞-bound of their second order derivatives.

Lemma 3.4.19. *Suppose $m \geq 2$. Then there exist a family of non-trivial 1-periodic functions $\{w_k : \mathbb{R}^n \times \bar{\Omega} \to \mathbb{R}\}_{1 \leq k \leq [\frac{m}{2}]+1}$ for which the following holds.*

(i) $w_k(\cdot, x) \in C^{2,\alpha}(\mathbb{R}^n)$ uniformly for all $x \in \bar{\Omega}$ and $\|w_k(\cdot, x)\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_{m,k,g,\Omega}$.

(ii) $w_k(y, \cdot) \in C^{m-2k+2,1}(\bar{\Omega})$ uniformly for all $y \in \mathbb{R}^n$ and

$$\|w_k(y, \cdot)\|_{C^{m-2k+2,1}(\bar{\Omega})} \leq C_{m,k,g,\Omega}.$$

Moreover, there holds for any $x_1, x_2 \in \bar{\Omega}$ that

$$\sum_{l=0}^{m-2k+1} \left\| D_x^l w_k(\cdot, x_1) - D_x^l w_k(\cdot, x_2) \right\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_{m,k,g,\Omega}|x_1 - x_2|.$$

(iii) Provided that $k \geq 3$, for each $x \in \bar{\Omega}$, $w_k(\cdot, x)$ solves

$$a_{ij}(\cdot, x) D_{y_i y_j} w_k(\cdot, x) + \Phi_k(\cdot, x) = 0 \quad \text{in } \mathbb{R}^n, \tag{3.48}$$

where

$$
\begin{aligned}
\Phi_k &= a_{ij} D_{x_i x_j} w_{k-2} + 2 a_{ij} D_{x_i y_j} w_{k-1} \\
&\quad + \sum_{l=2}^{k-2} \frac{1}{l!} \sum_{n_1+\cdots+n_l=k-2} a_{i_1 j_1 \dots i_l j_l} X_{i_1 j_1}^{n_1} \cdots X_{i_l j_l}^{n_l}, \\
X_{i_r j_r}^{n_r} &= D_{x_{i_r} x_{j_r}} w_{n_r} + 2 D_{x_{i_r} y_{j_r}} w_{n_r+1} + D_{y_{i_r} y_{j_r}} w_{n_r+2}, \quad r = 1, \dots, l, \\
a_{i_1 j_1 \dots i_l j_l} &= F_{p_{i_1 j_1} \dots p_{i_l j_l}} (D_x^2 u + D_y^2 w(\cdot; D_x^2 u, \cdot), \cdot, \cdot), \quad l = 1, \dots, k-2.
\end{aligned}
$$

Proof. We are going to use an induction argument to construct $\{w_k\}_{1 \le k \le [\frac{m}{2}]+1}$ as well as families of functions $\{\psi_k : \bar{\Omega} \to \mathbb{R}\}_{-1 \le k \le [\frac{m}{2}]+1}$ and $\{\phi_k : \mathbb{R}^n \times \bar{\Omega} \to \mathbb{R}\}_{1 \le k \le [\frac{m}{2}]+1}$, which verify the following conditions:

(IP1) $\phi_k(\cdot, x) \in C^{2,\alpha}(\mathbb{R}^n)$ uniformly for all $x \in \bar{\Omega}$ and $\|\phi_k(\cdot, x)\|_{C^{2,\alpha}(\mathbb{R}^n)} \le C_{m,k,g,\Omega}$.

(IP2) $\phi_k(y, \cdot) \in C^{m-2k+4,1}(\bar{\Omega})$ uniformly for $y \in \mathbb{R}^n$ and $\|\phi_k(y, \cdot)\|_{C^{m-2k+4,1}(\bar{\Omega})} \le \tilde{C}_{m,k,g,\Omega}$. Moreover, $\phi_k(0, \cdot) = 0$ in $\bar{\Omega}$ and there holds for any $x_1, x_2 \in \bar{\Omega}$ that

$$
\sum_{l=0}^{m-2k+3} \left\| D_x^l \phi_k(\cdot, x_1) - D_x^l \phi_k(\cdot, x_2) \right\|_{C^{2,\alpha}(\mathbb{R}^n)} \le \tilde{C}_{m,k,g,\Omega} |x_1 - x_2|.
$$

(IP3) $\psi_k \in C^{m-2k+2,1}(\bar{\Omega})$ satisfying $\|\psi_k\|_{C^{m-2k+2,1}(\bar{\Omega})} \le \bar{C}_{m,k,g,\Omega}$.

It will turn out at the end that as we define

$$
w_k(y, x) = \phi_k(y, x) + \chi^{ij}(y, x) D_{x_i x_j} \psi_{k-2}(x) + \psi_k(x), \tag{3.49}
$$

where $\chi^{ij}(y, x) := D_{p_{ij}} w(y; D_x^2 u, x)$, $\{w_k\}_{1 \le k \le [\frac{m}{2}]+1}$ satisfies Lemma 3.4.19.

Let us make a few remarks on the function $\chi^{ij}(y, x)$, which has the particular importance in this proof. First we observe from Proposition 3.4.11 and Lemma 3.4.18 that $\chi^{ij}(\cdot, x) \in C^{2,\alpha}(\mathbb{R}^n)$ for all $x \in \bar{\Omega}$ and $\|\chi^{ij}(\cdot, x)\|_{C^{2,\alpha}(\mathbb{R}^n)} \le C_{m,g,\Omega}^{(1)}$. In addition, $\chi^{ij}(y, \cdot) \in C^{m-1,1}(\bar{\Omega})$ uniformly for $y \in \mathbb{R}^n$ and

$$
\left\| \chi^{ij}(y, \cdot) \right\|_{C^{m-1,1}(\bar{\Omega})} \le C_{m,g,\Omega}^{(2)},
$$

and, in particular for $x_1, x_2 \in \bar{\Omega}$, there holds

$$\sum_{l=0}^{m-2} \left\| D_x^l \chi^{ij}(\cdot, x_1) - D_x^l \chi^{ij}(\cdot, x_2) \right\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_{m,g,\Omega}^{(2)} |x_1 - x_2|.$$

It is noteworthy to see that, in view of the equation (3.42), $\chi^{ij}(\cdot, x)$ solves

$$a_{rs}(\cdot, x) D_{y_r y_s} \chi^{ij}(\cdot, x) + a_{ij}(\cdot, x) = \bar{a}_{ij}(x) \quad \text{in } \mathbb{R}^n,$$

where $\bar{a}_{ij}(x) = \bar{F}_{p_{ij}}(D_x^2 u, x) \in C^{m-1,1}(\bar{\Omega})$ whose $C^{m-1,1}(\bar{\Omega})$-norm is bounded above by $C_{m,g,\Omega}^{(2)}$.

Let us now begin our induction argument. As the first step, we define $\psi_{-1}(x) = \psi_0(x) = \psi_{[\frac{m}{2}]}(x) = \psi_{[\frac{m}{2}]+1}(x) \equiv 0$ on $\bar{\Omega}$ and $\phi_1(y, x) \equiv 0$, $\phi_2(y, x) = w(y; D_x^2 u, x)$ on $\mathbb{R}^n \times \bar{\Omega}$. If $m = 2$ or 3, then $w_1(y, x) = 0$ and $w_2(y, x) = w(y; D_x^2 u, x)$, as we define them according to (3.49). The assertions (i) and (ii) of Lemma 3.4.19 are then immediate from Lemma 3.4.2 and Proposition 3.4.11. Since we have $k \leq 2$ when $m = 2$ or 3, the assertion (iii) can be dismissed. Thus, Lemma 3.4.19 is proved for the case $m = 2$ and 3.

Now we consider the case when $m \geq 4$. One can easily see that ϕ_1 and ϕ_2 [resp., ψ_{-1}, ψ_0, $\psi_{[\frac{m}{2}]}$ and $\psi_{[\frac{m}{2}]+1}$] chosen in the first step still verify (IP1)-(IP2) [resp., (IP3)].

In order to run the induction argument, we choose $3 \leq k \leq [\frac{m}{2}] + 1$ and suppose that we have already found the families $\{\psi_{l-2}\}_{1 \leq l \leq k-1}$, $\{\phi_l\}_{1 \leq l \leq k-1}$ and $\{w_l\}_{1 \leq l \leq k-1}$ which satisfy (IP1)-(IP3) and Lemma 3.4.19 respectively. We then define $\tilde{\Phi}_k : \mathbb{R}^n \times \bar{\Omega} \to \mathbb{R}$ by

$$\begin{aligned} \tilde{\Phi}_k = {} & a_{ij} D_{x_i x_j} (\phi_{k-2} + \chi^{ab} D_{x_a x_b} \psi_{k-4}) + 2 a_{ij} D_{x_i y_j} (\phi_{k-1} + \chi^{ab} D_{x_a x_b} \psi_{k-3}) \\ & + \sum_{l=2}^{k-2} \frac{1}{l!} \sum_{n_1 + \cdots + n_l = k-2} a_{i_1 j_1 \ldots i_l j_l} X_{i_1 j_1}^{n_1} \cdots X_{i_l j_l}^{n_l}. \end{aligned}$$

One may notice that $\tilde{\Phi}_k$ does not involve the functions ψ_{r-2} and ϕ_r for $r \geq k$.

Consider the following problem: For each $x \in \bar{\Omega}$, there exists a unique constant $\Psi_{k-2}(x)$ such that the following PDE,

$$a_{ij}(\cdot, x) D_{y_i y_j} v + \tilde{\Phi}_k(\cdot, x) = \Psi_{k-2}(x) \quad \text{in } \mathbb{R}^n, \tag{3.50}$$

attains a bounded 1-periodic solution v. Note that $a_{ij}(\cdot, x)$ is uniformly elliptic with the ellipticity constants λ and Λ. Moreover, $a_{i_1j_1\ldots i_lj_l}(\cdot,x)$ is 1-periodic and belongs to $C^{m-l,1}(\mathbb{R}^n)$ whose $C^{m-l,1}(\mathbb{R}^n)$-norm is bounded above by a constant $K_{m,l,g,\Omega}$. This fact together with our induction hypotheses, (IP1)-(IP3) and Lemma 3.4.19 (i) and (ii), yields that $\tilde{\Phi}_k(\cdot,x) \in C^{0,\alpha}(\mathbb{R}^n)$ where its $C^{0,\alpha}(\mathbb{R}^n)$-norm is bounded above by a constant $\tilde{K}_{m,k,g,\Omega}$. Therefore, Lemma 3.3.2 yields that the PDE (3.50) is solvable with a $C^{2,\alpha}(\mathbb{R}^n)$-solution, and denote it by $\phi_k(\cdot,x)$. In particular, let us choose $\phi_k(\cdot,x)$ such that $\phi_k(0,x)=0$. Since the domain Ω is bounded, $\phi_k(\cdot,x) \in C^{2,\alpha}(\mathbb{R}^n)$ uniformly for $x \in \bar{\Omega}$ and $\|\phi_k(\cdot,x)\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_{m,k.g,\Omega}$. Therefore, ϕ_k verifies (IP1).

To know the regularity of ϕ_k in x-variable, we utilize Proposition 3.4.11. We know that $a_{i_1j_1\ldots i_mj_m}(y,\cdot) \in C^{m-l,1}(\bar{\Omega})$ and its $C^{m-l,1}(\bar{\Omega})$-norm is bounded above by $L_{m,k,g,\Omega}$. Then again by using our induction hypotheses, we obtain $\tilde{\Phi}_k(y,\cdot) \in C^{m-2k+4,1}(\bar{\Omega})$ whose $C^{m-2k+4,1}(\bar{\Omega})$-norm is bounded above by $\tilde{L}_{m,k,g,\Omega}$. Thus, Proposition 3.4.11 implies that both Ψ_{k-2} and $\phi_k(y,\cdot)$ belong to $C^{m-2k+4,1}(\bar{\Omega})$ with the estimate that

$$\max\{\|\Psi_{k-2}\|_{C^{m-2k+4,1}(\bar{\Omega})}, \|\phi_k(y,\cdot)\|_{C^{m-2k+4,1}(\bar{\Omega})}\} \leq \tilde{C}_{m,k,g,\Omega}.$$

In particular, we obtain for any $x_1, x_2 \in \bar{\Omega}$ that

$$\sum_{i=0}^{m-2k+3} \left\| D_x^i \phi_k(\cdot,x_1) - D_x^i \phi_k(\cdot,x_2) \right\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq \tilde{C}_{m,k,g,\Omega}|x_1-x_2|.$$

Hence, ϕ_k satisfies (IP2) as well.

Moreover, we choose the function $\psi_{k-2} : \bar{\Omega} \to \mathbb{R}$ by the solution of

$$\begin{cases} \bar{a}_{ij} D_{x_ix_j}\psi_{k-2} = -\Psi_{k-2} & \text{in } \Omega, \\ \psi_{k-2} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.51}$$

Recall from Lemma 3.4.9 that $\bar{a}_{ij}$ is uniformly elliptic in $\bar{\Omega}$ with the ellipticity constants λ and Λ. Also Proposition 3.4.11 implies that $\bar{a}_{ij} \in C^{m-1,1}(\bar{\Omega})$ whose $C^{m-1,1}(\bar{\Omega})$-norm is bounded above by $C_{m,g,\Omega}$. Since $\Psi_{k-2} \in C^{m-2k+4,1}(\bar{\Omega})$, there exists a unique solution $\psi_{k-2} \in C^{m-2k+6,1}(\bar{\Omega})$ of (3.51) and

$$\|\psi_{k-2}\|_{C^{m-2k+6,1}(\bar{\Omega})} \leq C_{\|\bar{a}_{ij}\|_{C^{m-1,1}(\bar{\Omega})},\Omega}(\|\psi\|_{L^\infty(\Omega)} + \|\Psi\|_{C^{m-2,1}(\bar{\Omega})}) \leq \bar{C}_{m,k-2,g,\Omega}.$$

Thus, ψ_{k-2} satisfies the induction hypothesis (IP3).

Define $v_k : \mathbb{R}^n \times \bar{\Omega} \to \mathbb{R}$ by

$$v_k(y,x) := \phi_k(y,x) + \chi^{ij}(y,x) D_{x_i x_j} \psi_{k-2}(x).$$

It then follows from the observations above that $v_k(\cdot,x) \in C^{2,\alpha}(\mathbb{R}^n)$ with the estimate $\|v_k(\cdot,x)\|_{C^{2,\alpha}(\mathbb{R}^n)} \le A_{m,k,g,\Omega}$ and that $v_k(y,\cdot) \in C^{m-2k+2,1}(\bar{\Omega})$ with the estimate

$$\|v_k(y,\cdot)\|_{C^{m-2k+2,1}(\bar{\Omega})} \le \tilde{A}_{m,k,g,\Omega}.$$

Furthermore, we have for any pair of $x_1, x_2 \in \bar{\Omega}$ that

$$\sum_{i=0}^{m-2k+1} \left\| D_x^i v_k(\cdot,x_1) - D_x^i v_k(\cdot,x_2) \right\|_{C^{2,\alpha}(\mathbb{R}^n)} \le \tilde{A}_{m,k,g,\Omega} |x_1 - x_2|.$$

One may also check that $A_{m,k,g,\Omega} = C_{m,k,g,\Omega} + C^{(1)}_{m,g,\Omega} \bar{C}_{m,k-2,g,\Omega}$ and $\tilde{A}_{m,k,g,\Omega} = \tilde{C}_{m,k,g,\Omega} + C^{(2)}_{m,g,\Omega} \bar{C}_{m,k-2,g,\Omega}$. Moreover, we combine (3.50) and (3.51) and obtain that

$$\begin{aligned} & a_{ij}(\cdot,x) D_{y_i y_j} v_k(\cdot,x) + \Phi_k(\cdot,x) \\ &= a_{ij}(\cdot,x) D_{y_i y_j} \phi_k(\cdot,x) + \tilde{\Phi}_k(\cdot,x) + \\ &\quad + [a_{rs}(\cdot,x) D_{y_r y_s} \chi^{ij}(\cdot,x) + a_{ij}(\cdot,x)] D_{x_i x_j} \psi_{k-2}(x) \\ &= \Psi_{k-2}(x) + \bar{A}_{ij} D_{x_i x_j} \psi_{k-2}(x) \\ &= 0 \quad \text{in } \mathbb{R}^n. \end{aligned}$$

Hence, v_k satisfies Lemma 3.4.19.

We have obtained so far ψ_{k-2}, ϕ_k and v_k which satisfy (IP1)-(IP3) and Lemma 3.4.19 respectively. Now we apply the same argument above using

$$\begin{aligned} \hat{\hat{\Phi}}_{k+1} &= a_{ij} D_{x_i x_j} (\phi_{k-1} + \chi^{ab} D_{x_a x_b} \psi_{k-3}) + 2 a_{ij} D_{x_i y_j} (\phi_k + \chi^{ab} D_{x_a x_b} \psi_{k-2}) \\ &\quad + \sum_{l=2}^{k-1} \frac{1}{l!} \sum_{n_1 + \cdots + n_l = k-1} a_{i_1 j_1 \ldots i_l j_l} \hat{X}^{n_1}_{i_1 j_1} \cdots \hat{X}^{n_l}_{i_l j_l}, \end{aligned}$$

where $\hat{X}^l_{i_r j_r} = X^l_{i_r j_r}$ for $1 \le l \le k-3$ and $\hat{X}^{k-2}_{i_r j_r} = D_{x_{i_r} x_{j_r}} w_{k-2} + 2 D_{x_{i_r} y_{j_r}} w_{k-1} + D_{y_{i_r} y_{j_r}} v_k$. Then we obtain ψ_{k-1}, ϕ_{k+1} and v_{k+1} which satisfy (IP1)-(IP3) and

Lemma 3.4.19 respectively. Applying the same argument once again using

$$\begin{aligned}\hat{\tilde{\Phi}}_{k+2} &= a_{ij}D_{x_ix_j}(\phi_k + \chi^{ab}D_{x_ax_b}\psi_{k-2}) + 2a_{ij}D_{x_iy_j}(\phi_{k+1} + \chi^{ab}D_{x_ax_b}\psi_{k-1}) \\ &\quad + \sum_{l=2}^{k}\frac{1}{l!}\sum_{n_1+\cdots+n_l=k} a_{i_1j_1\ldots i_lj_l}\hat{X}^{n_1}_{i_1j_1}\cdots\hat{X}^{n_l}_{i_lj_l},\end{aligned}$$

where $\hat{X}^l_{i_rj_r} = X^l_{i_rj_r}$ for $1 \le l \le k-3$, $\hat{X}^{k-2}_{i_rj_r} = D_{x_{i_r}x_{j_r}}w_{k-2} + 2D_{x_{i_r}y_{j_r}}w_{k-1} + D_{y_{i_r}y_{j_r}}v_k$ and $\hat{X}^{k-1}_{i_rj_r} = D_{x_{i_r}x_{j_r}}w_{k-1} + 2D_{x_{i_r}y_{j_r}}v_k + D_{y_{i_r}y_{j_r}}v_{k+1}$, we get ψ_k, ϕ_{k+2} and v_{k+2} satisfying (IP1)-(IP3) and Lemma 3.4.19 respectively.

Now let us define w_k as in (3.49); i.e., $w_k(y,x) = v_k(y,x) + \psi_k(x)$. Then w_k satisfies Lemma 3.4.19; in particular, the estimates are satisfied with the constant $\max\{A_{m,k,g,\Omega} + \tilde{A}_{m,k,g,\Omega}\} + \bar{C}_{m,k,g,\Omega}$. In addition, one can check that

$$\begin{aligned}\hat{\tilde{\Phi}}_{k+1} &= a_{ij}D_{x_ix_j}(\phi_{k-1} + \chi^{ab}D_{x_ax_b}\psi_{k-3}) + 2a_{ij}D_{x_iy_j}(\phi_k + \chi^{ab}D_{x_ax_b}\psi_{k-2}) \\ &\quad + \sum_{l=2}^{k-1}\frac{1}{l!}\sum_{n_1+\cdots+n_l=k-1} a_{i_1j_1\ldots i_lj_l}X^{n_1}_{i_1j_1}\cdots X^{n_l}_{i_lj_l} \\ &=: \tilde{\Phi}_{k+1},\end{aligned}$$

which implies that the functions ψ_{k-1} and ϕ_{k+1} are not changed by replacing v_k by w_k in the induction argument. Therefore, our induction argument runs through $k = 3, \cdots, [\frac{m}{2}]+1$, by which we obtain the families $\{\psi_{k-2}\}_{1\le k\le[\frac{m}{2}]+1}$, $\{\phi_k\}_{1\le k\le[\frac{m}{2}]+1}$ and $\{w_k\}_{1\le k\le[\frac{m}{2}]+1}$, where $w_{[\frac{m}{2}]} = v_{[\frac{m}{2}]}$ and $w_{[\frac{m}{2}]+1} = v_{[\frac{m}{2}]+1}$. Recall that we have chosen $\psi_{[\frac{m}{2}]} = \psi_{[\frac{m}{2}]+1} \equiv 0$. Thus, we have constructed all the desired families $\{\psi_k\}_{-1\le k\le[\frac{m}{2}]+1}$, $\{\phi_k\}_{1\le k\le[\frac{m}{2}]+1}$ and $\{w_k\}_{1\le k\le[\frac{m}{2}]+1}$ which satisfy (IP1)-(IP3) and Lemma 3.4.19 respectively. It completes our proof. □

Remark 3.4.20. *As we note in the remark below Proposition 3.4.11, we see how the coupling effect contribute to the regularity of $x \mapsto w_k(y,x)$. If the x and y-variables were decoupled, we would have obtained $w_k(\cdot,x) \in C^{m-k+2,1}(\bar{\Omega})$.*

To this end we define the k-th order interior corrector w^ε_k of (3.6) for each $1 \le k \le [\frac{m}{2}]+1$ and $\varepsilon > 0$ by

$$w^\varepsilon_k(x) = w_k\left(\frac{x}{\varepsilon}, x\right) \quad (x \in \bar{\Omega}), \tag{3.52}$$

where w_k's are given in accordance with Lemma 3.4.19, and define $\eta_m^\varepsilon : \bar{\Omega} \to \mathbb{R}$ by

$$\eta_m^\varepsilon = u + \varepsilon w_1^\varepsilon + \cdots + \varepsilon^{[\frac{m}{2}]+1} w^\varepsilon_{[\frac{m}{2}]+1}.$$

Now we are in position to introduce the boundary layer corrector. The underlying idea of seeking the boundary layer corrector is the same as in the linear case; we correct the boundary oscillation occurred by the interior correctors by solving the corresponding boundary value problem (c.f. (3.25)). Due to the nonlinearity of the problem (3.3), however, we cannot find the boundary layer corrector in an order-wise manner. Instead, we consider a boundary value problem which involves the entire boundary oscillation caused by the interior correctors; i.e., we solve for each $\varepsilon > 0$ the following PDE,

$$\begin{cases} F\left(D^2\eta_m^\varepsilon + D^2\theta_m^\varepsilon, x, \varepsilon^{-1}x\right) = F\left(D^2\eta_m^\varepsilon, x, \varepsilon^{-1}x\right) & \text{in } \Omega, \\ \theta_m^\varepsilon = -\eta_m^\varepsilon + g & \text{on } \partial\Omega. \end{cases} \tag{3.53}$$

One may notice from Lemma 3.4.19 that $\eta_m^\varepsilon \in C^2(\bar{\Omega})$ that the right hand side of (3.53) is a uniformly continuous function on $\bar{\Omega}$ for each $\varepsilon > 0$. Thus, Perron's method (e.g., Theorem 2.1.5) ensures the unique existence of a viscosity solution $\theta_m^\varepsilon \in C(\bar{\Omega})$ of (3.53).

3.4.4 Proof of Theorem 3.1.2

We shall now prove our main result concerning the higher order convergence rates for fully nonlinear equations.

Proof of Theorem 3.1.2. Suppose that $m \geq 4$. The first part of the proof verifies the discussion we made in the beginning of the previous subsection. Fix $\varepsilon_* \in (0,1)$ and pick any $\varepsilon > 0$. We will skip the calculation if it has already been done in the previous subsection.

In what follows let us denote by r_m the positive integer $[\frac{m}{2}] + 1$. We choose the family $\{w_k\}_{1 \leq k \leq r_m}$ from Lemma 3.4.19. Next we define the family $\{X^k\}_{1 \leq k \leq r_m}$ as in (3.44) and then the function Y^{r_m} as in (3.45). By Lemma 3.4.19 (i)-(ii), we have a uniform bound on the matrix norm of X^k, which is independent of ε, namely,

$$\left\| X^k(\cdot/\varepsilon, \cdot) \right\|_{L^\infty(\Omega)} \leq C_{m,k,g,\Omega}. \tag{3.54}$$

It is then immediately follows that

$$\sup_{0<\varepsilon\leq\varepsilon_*} \|Y^{r_m}(\cdot/\varepsilon,\cdot)\|_{L^\infty(\bar\Omega)} \leq (1-\varepsilon_*)L_*\frac{1-\varepsilon^{r_m}}{1-\varepsilon} < L_* \tag{3.55}$$

where $L_* = (1-\varepsilon_*)^{-1}\max\{1, C_{m,1,g,\Omega},\dots,C_{m,r_m,g,\Omega}\}$.

In the rest of this proof, we set $\varepsilon \in (0,\varepsilon_*]$ to be fixed. We choose any $x\in\Omega$ and adopt the Taylor expansion of $F(D^2\eta_m^\varepsilon, x, x/\varepsilon)$ with respect to the M-variable up to (r_m-1)-th order. For brevity, we omit the dependency on $(\varepsilon^{-1}x, x)$. Then, by the choice of our interior correctors w_k^ε, we end up with

$$\begin{aligned}
F(D^2\eta_m^\varepsilon) &= F(X^0+\varepsilon Y^{r_m})\\
&= F(X^0) + \sum_{k=1}^{r_m-1}\frac{\varepsilon^k}{k!}F_{p_{i_1j_1}\cdots p_{i_kj_k}}(X^0)Y^{r_m}_{i_1j_1}\cdots Y^{r_m}_{i_kj_k} + R_m^\varepsilon\\
&= F(X^0) + \sum_{k=1}^{r_m-1}\varepsilon^k\sum_{l=1}^{k}\frac{1}{l!}\sum_{n_1+\cdots+n_l=k}F_{p_{i_1j_1}\cdots p_{i_lj_l}}(X^0)X^{n_1}_{i_1j_1}\cdots X^{n_l}_{i_lj_l}\\
&\quad + \tilde R_m^\varepsilon\\
&= \tilde R_m^\varepsilon,
\end{aligned} \tag{3.56}$$

where

$$\begin{aligned}
R_m^\varepsilon &= \frac{\varepsilon_0^{r_m}}{r_m!}F_{p_{i_1j_1}\cdots p_{i_{r_m}i_{r_m}}}(X^0)Y^{r_m}_{i_1j_1}\cdots Y^{r_m}_{i_{r_m}j_{r_m}} \quad \text{for some } \varepsilon_0\in[0,\varepsilon],\\
\tilde R_m^\varepsilon &= R_m^\varepsilon + \sum_{k=1}^{r_m-2}\sum_{r_m-1\leq n_1+\cdots+n_k\leq r_m k}\frac{\varepsilon^{n_1+\cdots+n_k}}{k!}F_{p_{i_1j_1}\cdots p_{i_kj_k}}(X^0)X^{n_1}_{i_1j_1}\cdots X^{n_k}_{i_kj_k}.
\end{aligned}$$

One should note that $F_{p_{i_1j_1}\cdots p_{i_kj_k}}(X^0)$ are exactly the coefficients $a_{i_1j_1\dots i_kj_k}$ appearing in (3.48). Now due to (3.54) and (3.55), we have

$$|R_m^\varepsilon| \leq \tilde C_{m,g,\Omega}L_*^{r_m}\varepsilon^{r_m},$$

and thus,

$$|\tilde R_m^\varepsilon| \leq |R_m^\varepsilon| + \hat C_{m,g,\Omega}L_*^{(r_m-2)r_m}\varepsilon^{r_m-1} \leq C_0\varepsilon^{r_m-1}.$$

The second part of this proof is devoted to the establishment of the estimate (3.7). The essence is to construct barriers and argue by the comparison principle. Choose $R > 0$ in such a way that $\bar{\Omega} \subset B_R(0)$. Consider the functions $\eta_m^{\varepsilon,\pm} : \bar{\Omega} \to \mathbb{R}$ defined by

$$\eta_m^{\varepsilon,\pm} = \eta_m^\varepsilon + \theta_m^\varepsilon \pm (2\lambda)^{-1} C_0 \varepsilon^{r_m - 1}(R^2 - |x|^2) \quad (x \in \bar{\Omega}).$$

By the uniform ellipticity of F (structure condition (F2)) and the choice of the boundary layer corrector (3.53), there holds

$$F(D^2\eta_m^{\varepsilon,+}) \leq F(D^2\eta_m^\varepsilon + D^2\theta_m^\varepsilon) - C_0\varepsilon^{r_m-1} = F(D^2\eta_m^\varepsilon) - C_0\varepsilon^{r_m-1} \leq 0$$

in the viscosity sense, and $\eta_m^{\varepsilon,+}|_{\partial\Omega} \geq \eta_m^\varepsilon + \theta_m^\varepsilon = g$. Thus, $\eta_m^{\varepsilon,+}$ is a viscosity supersolution of (3.3). In a similar manner, one can verify that $\eta_m^{\varepsilon,-}$ is a viscosity subsolution of (3.3). Thus, the comparison principle yields $\eta_m^{\varepsilon,-} \leq u^\varepsilon \leq \eta_m^{\varepsilon,+}$ in $\bar{\Omega}$. It then follows that

$$\|u^\varepsilon - \eta_m^\varepsilon - \theta_m^\varepsilon\|_{L^\infty(\Omega)} \leq (2\lambda)^{-1} C_0 \varepsilon^{r_m-1},$$

which proves (3.7).

The proof for the case $m = 2$ or 3 shares the same idea presented above, but is simpler. In this case, $\eta_m^\varepsilon(x) = u(x) + \varepsilon^2 w_2(\varepsilon^{-1}x, x)$, and thus, we do not need the expansion (3.56); instead we can directly argue as in the second part. The rest of the proof is exactly the same, so is omitted. $\square$

Chapter 4

Higher Order Convergence Rates in Theory of Homogenization: Oscillatory Initial Data

4.1 Introduction

We are interested in higher order convergence rates in periodic homogenization of fully nonlinear uniformly parabolic Cauchy problems, accompanied with rapidly oscillating initial data. We conduct our analysis based on the theory of viscosity solutions. Readers may consult [20], [50], [51] and [52] for standard existence and regularity theory of viscosity solutions.

4.1.1 Main Result

The governing problem under our consideration is formulated as

$$\begin{cases} u_t^\varepsilon = \dfrac{1}{\varepsilon^2} F\left(\varepsilon^2 D^2 u^\varepsilon, x, t, \dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^2}\right) & \text{in } \mathbb{R}^n \times (0,T), \\ u^\varepsilon(x,0) = g\left(x, \dfrac{x}{\varepsilon}\right) & \text{on } \mathbb{R}^n, \end{cases} \tag{4.1}$$

Our main result is stated as follows.

Theorem 4.1.1. *Assume that F and g verify (4.6) - (4.8) and (4.10) - (4.11) respectively. Let u^ε be the bounded viscosity solution to (4.1) for $\varepsilon > 0$. Then for each integer $d \geq 0$, there exist sequences $\{\tilde{v}_{d,k}\}_{k=0}^\infty$, $\{\tilde{w}_{d,k}\}_{k=0}^\infty$ of spatially periodic functions on $\mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,\infty)$ and a sequence $\{\tilde{w}^\#_{d,k}\}_{k=0}^\infty$ of periodic functions on $\mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times \mathbb{R}$ such that for any integer $m \geq 2$, any $\varepsilon \leq \frac{1}{2}$, any $x \in \mathbb{R}^n$ and $0 \leq t \leq T$,*

$$\left| u^\varepsilon(x,t) - \sum_{d=0}^{\lfloor \frac{m}{2} \rfloor - 1} \sum_{k=0}^{m-2d} \varepsilon^{k+2d} \left(\tilde{v}_{d,k}\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right) + \tilde{w}_{d,k}\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right)\right)\right| \leq C_m \varepsilon^{m-1}, \tag{4.2}$$

and in particular for $c_m \varepsilon^2 |\log \varepsilon| \leq t \leq T$,

$$\left| u^\varepsilon(x,t) - \sum_{d=0}^{\lfloor \frac{m}{2} \rfloor - 1} \sum_{k=0}^{m-2d} \varepsilon^{k+2d} \tilde{w}^\#_{d,k}\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right)\right| \leq C_m \varepsilon^{m-1}, \tag{4.3}$$

where c_m and C_m depend only on n, λ, Λ, α, m, T and K.

Let us make a few remarks regarding Theorem 4.1.1 as follows.

Remark 4.1.2. *In Section 4.4.3, we observe that $\tilde{v}_{d,k}$ is of the form*

$$\tilde{v}_{d,k}(x,t,y,s) = v_{d,k}(x,t,y,s) - \bar{v}_{d,k}(x,t),$$

and satisfies an exponential decay estimate in $s \to \infty$. The functions $v_{d,k}$ and $\bar{v}_{d,k}$ will be called the initial layer corrector and respectively the effective initial data of order $k+2d$. The exponential decay estimate amounts to (4.3), which is the higher order convergence rate (4.3) away from the initial time zone.

Remark 4.1.3. *Moreover, $\tilde{w}_{d,k}$ is of the form*

$$\tilde{w}_{d,k}(x,t,y,s) = w_{d,k}(x,t,y,s) + \bar{u}_{d,k}(x,t),$$

where $w_{d,k}$ and $\bar{u}_{d,k}$ will be called the interior corrector and respectively the effective limit profile of order $k+2d$. Furthermore, $w_{d,k}$ will be paired with a space-time-periodic function $w^\#_{d,k}$ such that

$$\tilde{w}^\#_{d,k}(x,t,y,s) = w^\#_{d,k}(x,t,y,s) + \bar{u}_{d,k}(x,t).$$

Here $w_{d,k}^{\#}$ *will play the role of the interior corrector in purely periodic homogenization problems. In addition,* $\tilde{w}_{d,k}$ *and* $\tilde{w}_{d,k}^{\#}$ *for* $k = 0, 1$ *will turn out to be constant in the fast variables* (y, s)*, so the interior error estimate* (4.3) *is of the form* $u^\varepsilon - \bar{u}_{0,0} - \bar{u}_{0,1} - \varepsilon^2 \tilde{w}_{0,2} - \cdots$*. This indicates that there is no rapid oscillation in the interior up to order* ε*.*

Remark 4.1.4. *In Theorem 4.1.1 and to the rest of this paper, we assume that* F *is concave in its matrix variable. Such an assumption is made to have* $C^{2,\alpha}$ *correctors in fast variables* $(y, s) = (\varepsilon^{-1}x, \varepsilon^{-2}t)$*, and smooth limit profiles in slow variables* (x, t)*. These are essential, at least in our approach, to establish higher order convergence rates, since it requires accurate error correction at each order of* ε*.*

On the other hand, the interior equation for the effective problem of (4.1) *is a linear equation. However, this does not make the problem easier in the sense that we have strong nonlinear coupling effect near the initial time layer when we construct higher order correctors. The particular scaling is used to derive smooth initial layer correctors in slow variables. We shall discuss more on this issue later.*

4.1.2 Historical Background

Periodic homogenization of (4.1) (or (4.5), to be more exact) is rigorously justified in [2] and [42]; see also the references therein, and [29] for first order fully nonlinear equations as well as [3] for iterated homogenization. There is a wide range of literature on the rate of convergence regarding the homogenization problems of type (4.1) or (4.5), provided that the initial data is non-oscillatory; that is, g is independent on its second argument. Recent development can be found, for instance, in [31], [44] and [15] using continuous dependence estimates, [28] based on a different approach, and [37], [40] in stationary ergodic settings; see also the references therein for classical results in this regard.

Higher order convergence rate in the theory of homogenization has been studied in various settings. We refer to [12], [33] for divergence type elliptic equations, [46] for perforated domains with mixed boundary conditions, [18] for Maxwell equations, [5] for wave equations, [27] for some numerical results, and also the references therein. Recently, the authors proved in [34] higher order convergence rate for non-divergence type elliptic equations, based on the theory of viscosity solutions.

As far as we know, however, there has not yet been any result on (higher order) convergence rate in homogenization of (4.1), which is the main concern of this paper. Here we also achieve higher order convergence rate (in Proposition 4.5.2) for uniformly parabolic equations with non-oscillatory initial data, that is,

$$\begin{cases} u_t^\varepsilon = F\left(D^2u^\varepsilon, x, t, \dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^2}\right) & \text{in } \mathbb{R}^n \times (0,T), \\ u^\varepsilon(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{4.4}$$

Moreover, we achieve a quantitative error estimate (in Proposition 4.5.3) in the following homogenization problem away from the initial time zone,

$$\begin{cases} u_t^\varepsilon = F\left(D^2u^\varepsilon, x, t, \dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^2}\right) & \text{in } \mathbb{R}^n \times (0,T), \\ u^\varepsilon(x,0) = g\left(x, \dfrac{x}{\varepsilon}\right) & \text{on } \mathbb{R}^n. \end{cases} \tag{4.5}$$

The estimate depends on a particular structure on F, which will be specified later, and in some cases, we obtain the sharp estimate, which is a convergence rate of order ε.

4.1.3 Heuristic Discussion and Main Difficulties

The main difficulties in achieving higher order convergence rate are due to the nonlinear structure of (4.1). If our operator were linear, that is, $F(P, x, t, y, s) = \mathrm{tr}(A(x, t, y, s)P)$, the construction of the higher order correctors would be independent to each iterative step. However, since we deal with fully nonlinear operators, the effect coming from the rapid oscillation of the correctors are accumulated in the Hessian variable as we iterate the approximation process.

A notable observation here is that the coupling effect due to the nonlinear structure of the governing operator changes the nature of the interior correctors, if one desires to establish a higher order convergence rate. Let us remark that the effect coming from the rapid oscillation of the initial layer corrector does not completely vanish in the interior and even remains to be of order 1 near the initial time layer, although it eventually becomes very small as one stays away from the initial time zone. Now that the interior equation is fully nonlinear, such an effect is recorded in the Hessian variable and does not go away, resulting a strong correlation between the constructions of initial layer correctors and interior correctors. Let us remark that one does not encounter such a coupling effect in the context of linear equations, since the linearity lets us to construct initial layer corrector and interior correctors independently.

In order to resolve this issue, we find a new type of cell problems for higher order interior correctors, which are only spatially periodic. Still we observe that each interior corrector is paired with a space-time-periodic corrector, i.e., the standard one, and the difference between these two turns out to decay exponentially fast in time. This allows us to iterate the approximation scheme and eventually leads us to a higher order error correction in the interior as well.

Another interesting observation is the regularity theory in slow variables. This is especially new for the initial layer corrector and the effective initial data, where we achieve exponential decay estimates for the difference between them; let us note that the case of interior corrector is rather a duplicate of our previous work [34], which concerns non-divergence type elliptic equations. The exponential decay estimate regarding initial layer corrector is deduced from the Harnack inequality for viscosity solutions, and can be considered classical if one thinks of linear elliptic equations on spatially periodic domain. One may also find some variations in this regard in several other places. For example, see [31], [15] for continuous dependence estimates, and [1] for elliptic boundary correctors. The novelty here is that we establish the exponential decay estimate for the derivatives of any order, which is certainly not precedented.

Let us make the final remark on homogenization of (4.5). The key difference in the homogenization process between (4.1) and (4.5) is that the initial layer corrector of the latter problem may not be differentiable in the slow variables in general, while the former produces smooth initial layer correctors. The main reason for such a distinction is that the operator of (4.1) oscillates in accordance with the oscillation of the initial data, which stabilizes the influence of the fully nonlinearity of the operator near the initial time layer to a controllable level. This ensures the base-case initial layer corrector (and the base-case effective initial data) to be smooth enough in the slow variables to induce higher order ones. However, the operator of (4.5) makes too much impact on the oscillation of the solution near the initial layer and, as a result, defects the regularity of the base-case initial layer corrector in a substantial way.

We observe that the higher order convergence rate in the framework of

(4.5) is a highly sophisticated matter that requires a thorough analysis on the limiting behavior of the sequence $\{\varepsilon^2 F(\frac{1}{\varepsilon^2}P, x, t, y, s)\}_{\varepsilon>0}$ as $\varepsilon \to 0$. Instead, we prove the convergence rate for certain uniformly parabolic operators F and initial data g. We observe that the convergence rate is determined by that of the sequence $\{\varepsilon^2 F(\frac{1}{\varepsilon^2}P, x, t, y, s)\}_{\varepsilon>0}$ to its limit operator, if any; and if the speed of the latter convergence is fast enough, we obtain the optimal rate of convergence for u^ε to its limit profile, away from the initial time layer by the order of $\varepsilon^2|\log\varepsilon|$.

4.1.4 Outline

This paper is organized as follows. In Section 4.2, we introduce the notations and the standing assumptions that will be used throughout this paper, unless stated otherwise. In Section 4.3, we establish the regularity theory in the slow variables and, in Section 4.4, we construct higher order initial layer correctors and interior correctors. Especially, our main result, Theorem 4.1.1, is proved in Section 4.4.3. Section 4.5 is devoted to proving some additional results, namely the higher order convergence rate in homogenization of (4.4), and the convergence rate in homogenization of (4.5).

4.2 Notation and Standing Assumptions

Let $n \geq 1$ be the spatial dimension and $T > 0$ be the terminal time. We will call x (resp., t, y, and s) the slow spatial (resp., slow temporal, fast spatial, and fast temporal) variable.

By $\mathcal{S}^n$ we denote the space of all real symmetric matrices of order n, endowed with (L^2, L^2)-norm; that is, $|P| = (\sum_{i,j=1}^n p_{ij}^2)^{1/2}$ for any $P \in \mathcal{S}^n$. By $E_{ij} = (e_{KL1}^{ij})$ we will denote the (i,j)-th standard basis matrix for $\mathcal{S}^n$ that is $e_{KL1}^{ij} = 2^{-1}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ with δ being the Kronecker delta. By $\mathrm{tr}(P)$ we denote the trace of P.

Let F be a smooth functional on $\mathcal{S}^n$. By $\frac{\partial F}{\partial p_{ij}}(P)$ we denote the derivative of F in direction E_{ij} at P. By $D_p^k F$ we denote the k-th order derivative of F on $\mathcal{S}^n$ such that

$$D_p^k F(P) = \left(\frac{\partial^k F}{\partial p_{i_1 j_1} \cdots \partial p_{i_l j_l} \cdots \partial p_{i_k j_k}}(P) \right) = \left(\frac{\partial^k F}{\partial p_{i_1 j_1} \cdots \partial p_{j_l i_l} \cdots \partial p_{i_k j_k}}(P) \right).$$

For notational convenience, we also understand $D_p^k F$ in the sense of Fréchet derivatives. That is, for each $P \in \mathcal{S}^n$, we consider $D_p^k F(P)$ as the (symmetric) multilinear map from $(\mathcal{S}^n)^k$ to $\mathbb{R}$ such that

$$D_p^k F(P)(Q_1, \cdots, Q_k) = \frac{\partial^k F}{\partial p_{i_1 j_1} \cdots \partial p_{i_k j_k}}(P) q_{i_1 j_1}^1 \cdots q_{i_k j_k}^k,$$

for any $Q_l = (q_{ij}^l) \in \mathcal{S}^n$ with $1 \leq l \leq k$; here and thereafter we use the summation convention for repeated indices. In particular, we have

$$D_p F(P)(Q) = \operatorname{tr}(D_p F(P) Q).$$

By $C^{k,\alpha}(X)$ we shall denote the usual Hölder space on X. Especially, when $X = \mathbb{R}^n \times [0, \infty)$, we shall define $E^{k,\alpha}(\mathbb{R}^n \times [0, \infty); \beta)$, with $\beta > 0$, by the subspace of $C^{k,\alpha}(\mathbb{R}^n \times [0, \infty))$ consisting of functions $f : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ satisfying

$$\|f\|_{E^{k,\alpha}(\mathbb{R}^n \times [0,\infty);\beta)} = \|f\|_{C^{k,\alpha}(\mathbb{R}^n \times [0,\infty))} + \sup_{s>0} \left(e^{\beta s} \|f(\cdot, s)\|_{C^{k,\alpha}(\mathbb{R}^n)} \right) \leq C,$$

for some finite $C \geq 0$.

Given a function or a mapping f on $X \times Y$, with X a space of slow variables and Y a space of fast variables (y, s), f is said to be spatially periodic, if

$$f(\cdot, y + k, s) = f(\cdot, y, s), \quad k \in \mathbb{Z}^n,$$

while f is said to be periodic, if

$$f(\cdot, y + k, s + l) = f(\cdot, y, s), \quad k \in \mathbb{Z}^n, l \in \mathbb{Z}.$$

Here X may consist of x, (x, t) or (P, x, t).

We will use the parabolic terminologies, such as $|(x, t)| = (|x|^2 + |t|)^{1/2}$. For more details, we refer to [50]. See [20] for the classical existence theory, the comparison principle and the stability theory of viscosity solutions. Also see [50], [51] and [52] for the basic regularity theory for viscosity solutions, such as the Harnack inequality, and interior and boundary regularity.

Now let us make the standing assumptions throughout this paper. Assume that $F : \mathcal{S}^n \times \mathbb{R}^n \times [0, T] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is uniformly elliptic, periodic and concave. By uniform ellipticity and concavity, we indicate that there are

$0 < \lambda \leq \Lambda$ such that

$$\lambda|Q| \leq F(P+Q,x,t,y,s) - F(P,x,t,y,s) \leq \Lambda|Q|, \quad Q \in \mathcal{S}^n, Q \geq 0, \tag{4.6}$$

and

$$\frac{1}{2}F(P,x,t,y,s) + \frac{1}{2}F(Q,x,t,y,s) \leq F\left(\frac{P+Q}{2},x,t,y,s\right), \quad Q \in \mathcal{S}^n.$$

By periodicity we mean that

$$F(P,x,t,y+k,s+l) = F(P,x,t,y,s), \quad k \in \mathbb{Z}^n, l \in \mathbb{Z}. \tag{4.7}$$

Suppose further that $F \in C^\infty(\mathcal{S}^n \times \mathbb{R}^n \times [0,T]; C^\alpha(\mathbb{R}^n \times \mathbb{R}))$ for some $0 < \alpha < 1$, i.ede is some $K > 0$ for which

$$\sum_{|\kappa|+|\mu|+2\nu=l} \left\| D_p^\kappa D_x^\mu \partial_t^\nu F(P,x,t,\cdot,\cdot) \right\|_{C^\alpha(\mathbb{R}^n\times\mathbb{R})} \leq K|P|^{(1-|\kappa|)_+}, \quad l \geq 0. \tag{4.8}$$

Let us remark that (4.8) implies the zero source term condition,

$$F(0,x,t,y,s) = 0. \tag{4.9}$$

On the other hand, let $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a periodic function, by which we indicate

$$g(x,y+k) = g(x,y), \quad k \in \mathbb{Z}^n. \tag{4.10}$$

Also suppose that $g \in C^\infty(\mathbb{R}^n; C^{2,\alpha}(\mathbb{R}^n))$ and

$$\sum_{|\mu|=l} \|D_x^\mu g(x,\cdot)\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq K, \quad l \geq 0. \tag{4.11}$$

4.3 Regularity Theory in Slow Variables

Let us establish the regularity theory in slow variables, (x,t), regarding viscosity solutions to uniformly parabolic problems.

4.3.1 Spatially Periodic Cauchy Problem

Let $f : \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ be function satisfying

$$f(x,t,y+k,s) = f(x,t,y,s), \quad k \in \mathbb{Z}^n, \tag{4.12}$$

and assume that $f \in C^\infty(\mathbb{R}^n \times [0,T]; E^\alpha(\mathbb{R}^n \times [0,\infty);\beta))$, for some $\beta > 0$, such that

$$\sup_{|\mu|+2\nu=l} \|D_x^\mu \partial_t^\nu f(x,t,\cdot,\cdot)\|_{E^\alpha(\mathbb{R}^n\times[0,\infty);\beta)} \leq K, \quad l \geq 0, \tag{4.13}$$

with $0 < \alpha < 1$ and $K > 0$ being the same constants used in (4.8).

For each $(x,t) \in \mathbb{R}^n \times [0,T]$, let us consider the following spatially periodic and uniformly parabolic Cauchy problem,

$$\begin{cases} v_s = F(D_y^2 v, x,t,y,s) + f(x,t,y,s) & \text{in } \mathbb{R}^n \times (0,\infty), \\ v(x,t,y,0) = g(x,y) & \text{on } \mathbb{R}^n. \end{cases} \tag{4.14}$$

By the standard existence theory [20], we know that there exists a unique viscosity solution $v(x,t,\cdot,\cdot) \in BUC(\mathbb{R}^n \times [0,\infty))$ to (4.14). Due to the periodicity of F, f and g, we deduce that v satisfies

$$v(x,t,y+k,s) = v(x,t,y,s), \quad k \in \mathbb{Z}^n, \tag{4.15}$$

for any $(x,t,y,s) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,\infty)$.

We shall begin with an easy observation that the spatial oscillation of $v(x,t,y,s)$ in y decays exponentially fast as $s \to \infty$. The exponential rate will turn out to be independent of (x,t).

Lemma 4.3.1. *For each $(x,t) \in \mathbb{R}^n \times [0,T]$, there exists a unique $\gamma \in \mathbb{R}$ such that*

$$e^{\beta_0 s}|v(x,t,y,s) - \gamma| \leq C, \tag{4.16}$$

for any $(x,t,y,s) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,\infty)$, where $0 < \beta_0 < \beta$ depends only on n, λ, Λ and β, and $C > 0$ depend only on n, λ, Λ, β, β_0 and K.

Proof. Since (x,t) will be fixed throughout the proof, let us write $v = v(y,s)$, $F = F(M,y,s)$, $f = f(y,s)$ and $g = g(y)$ for notational convenience. By $S(s)$, $I(s)$ and $O(s)$ let us denote $\sup_{\mathbb{R}^n} v(\cdot,s)$, $\inf_{\mathbb{R}^n} v(\cdot,s)$ and respectively $\operatorname{osc}_{\mathbb{R}^n} v(\cdot,s)$. Also write $Y = (0,1)^n$ and $2Y = (0,2)^n$. By the spatial periodicity (4.15) of v, we have $S(s) = \sup_{2Y} v(\cdot, s) = \sup_Y v(\cdot, s)$, and similar identities for $I(s)$ and $O(s)$ as well.

Fix $s_0 \geq 0$. Then

$$\partial_s \left(S(s_0) + \frac{K}{\beta}(e^{-\beta s_0} - e^{-\beta s}) \right) = K e^{-\beta s} \geq f(y, s),$$

for any $y \in \mathbb{R}^n$ and $s \geq s_0$, due to (4.13). Since we have (4.9), we deduce that $S(s_0) + \frac{K}{\beta}(e^{-\beta s_0} - e^{-\beta s})$ is a supersolution to (4.14) in $\mathbb{R}^n \times [s_0, \infty)$. Similarly, one can observe that $I(s_0) - \frac{K}{\beta}(e^{-\beta s_0} - e^{-\beta s})$ is a subsolution to (4.14) in $\mathbb{R}^n \times [s_0, \infty)$. Thus, by the comparison principle [20] for viscosity solutions, we deduce that

$$I(s_0) - \frac{K}{\beta}(e^{-\beta s_0} - e^{-\beta s}) \leq v(y, s) \leq S(s_0) + \frac{K}{\beta}(e^{-\beta s_0} - e^{-\beta s}), \qquad (4.17)$$

for any $y \in \mathbb{R}^n$ and $s \geq s_0$.

Now for each nonnegative integer k, let us define

$$v_k(y, s) = v(y, s + k) - I(k) + \frac{K}{\beta} e^{-\beta k}, \quad y \in \mathbb{R}^n, s \geq 0.$$

From (4.17) with s and s_0 replaced by $s + k$ and k respectively, we deduce that

$$v_k(y, s) \geq 0, \quad y \in \mathbb{R}^n, s \geq 0.$$

On the other hand, we see that v_k is a (spatially periodic) viscosity solution to

$$\partial_s v_k = F(D_y^2 v_k, y, s + k) + f(y, s + k) \quad \text{in } 2Y \times (0, 1).$$

Therefore, we may apply the Harnack inequality in $\bar{Y} \times [\frac{1}{2}, 1]$ and deduce from the spatial periodicity of v_k that

$$S\left(k + \frac{1}{2}\right) - I(k) + \frac{K}{\beta} e^{-\beta k} \leq c_1 \left(I(k+1) - I(k) + \frac{K}{\beta} e^{-\beta k} \right),$$

where c_1 depends only on n, λ and Λ. Utilizing (4.17) with $s_0 = k + \frac{1}{2}$ and $s = k + 1$, we obtain that $S(k+1) \leq S(k + \frac{1}{2}) + \frac{K}{\beta} e^{-\beta k}$. Combining these two inequalities, we arrive at

$$S(k+1) - I(k) \leq c_1 \left(I(k+1) - I(k) + \frac{K}{\beta} e^{-\beta k} \right). \tag{4.18}$$

Now we define

$$w_k(y,s) = S(k) + \frac{K}{\beta} e^{-\beta k} - v(y, s+k), \quad y \in \mathbb{R}^n, s \geq 0.$$

Then by (4.17) and (4.14), w_k is a spatially periodic nonnegative viscosity solution to

$$\partial_s w_k = -F(-D_y^2 w_k, y, s+k) - f(y, s+k) \quad \text{in } 2Y \times (0,1).$$

Notice that the operator $-F(-M, y, s)$ satisfies the same ellipticity condition (4.6). Hence, we may invoke a similar argument as above and prove that

$$S(k) - I(k+1) \leq c_1 \left(S(k) - S(k+1) + \frac{K}{\beta} e^{-\beta k} \right). \tag{4.19}$$

Notice that the constant c_1 here is the same as that in (4.18).

By (4.18) and (4.19), we have

$$O(k+1) \leq \frac{c_1 - 1}{c_1 + 1} O(k) + \frac{2c_1 K}{(c_1+1)\nu} e^{-\beta k}. \tag{4.20}$$

Iterating (4.20) with respect to k and using $O(0) = \operatorname{osc}_{\mathbb{R}^n} g \leq 2K$, we arrive at

$$e^{\beta_0 s} O(s) \leq c_2 K \quad \text{with } 0 < \beta_0 < \min\left(\beta, \log \frac{c_1+1}{c_1-1} \right), \tag{4.21}$$

where $c_2 > 0$ is another constant depending only on n, λ, Λ, β and β_0.

The estimate (4.21) implies that $O(s) \to 0$ as $s \to \infty$. On the other hand, we know from (4.17) that both $S(s)$ and $I(s)$ converge as $s \to \infty$. Combining these two observations, we deduce that $S(s)$ and $I(s)$ converge to the same limit, which we shall denote by γ. Then (4.16) follows immediately from (4.21). □

Remark 4.3.2. *The proof of Lemma 4.3.1 does not involve the periodicity of F in s.*

By Lemma 4.3.1, we are able to define $\bar{v} : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ by

$$\bar{v}(x,t) = \lim_{s\to\infty} v(x,t,0,s). \tag{4.22}$$

The limit value in the right hand side of (4.22) is precisely the unique constant γ in the statement of Lemma 4.3.1. With $\bar{v}$ at hand, (4.16) reads

$$e^{\beta_0 s}|v(x,t,y,s) - \bar{v}(x,t)| \leq C, \tag{4.23}$$

for any $(x,t,y,s) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,\infty)$.

One may notice that the proof of Lemma 4.3.1 does not involve the assumptions on the concavity of F in P, the C^α regularity of F and f in (y,s) and the $C^{2,\alpha}$ regularity of g in y. Assuming these conditions additionally, we are allowed to use the interior and boundary $C^{2,\bar{\alpha}}$ estimates (the so-called Schauder theory) for viscosity solutions (with some $0 < \bar{\alpha} \leq \alpha$). As a result, we improve the estimate (4.23) in terms of $C^{2,\bar{\alpha}}$ norm.

Lemma 4.3.3. *There exists $0 < \bar{\alpha} < \alpha$, depending only on n, λ, Λ and α, such that $\bar{v} \in L^\infty(\mathbb{R}^n \times [0,T])$ and $v \in L^\infty(\mathbb{R}^n \times [0,T]; C^{2,\bar{\alpha}}(\mathbb{R}^n \times [0,\infty)))$ with*

$$|\bar{v}(x,t)| + \|v(x,t,\cdot,\cdot) - \bar{v}(x,t)\|_{E^{2,\bar{\alpha}}(\mathbb{R}^n\times[0,\infty);\beta_0)} \leq C, \tag{4.24}$$

for any $(x,t,s) \in \mathbb{R}^n \times [0,T]$, where $C > 0$ depends only on n, λ, Λ, β, β_0 and K.

Proof. Let us fix $(x,t) \in \mathbb{R}^n \times [0,T]$ and simply write $F(P,y,s)$, $f(y,s)$, $g(y)$, $v(y,s)$, and γ for $F(P,x,t,y,s)$, $f(x,t,y,s)$, $g(x,y)$, $v(x,t,y,s)$ and, respectively, $\bar{v}(x,t)$. Let us denote by Y and $2Y$ the cubes $(0,1)^n$ and $(0,2)^n$.

In view of (4.14), the function $\tilde{v}(y,s) = v(y,s) - \gamma$ is a viscosity solution to

$$\begin{cases} \tilde{v}_s = F(D_y^2\tilde{v}, y, s) + f(y,s) & \text{in } 2Y \times (0,\infty), \\ \tilde{v}(y,0) = g(y) - \gamma & \text{on } 2Y. \end{cases} \tag{4.25}$$

Since F is uniformly elliptic and concave in P, and since F and f are C^α while g is $C^{2,\alpha}$ in (y,s), we may apply the boundary $C^{2,\bar{\alpha}}$ estimate [51] to (4.25) for some $0 < \bar{\alpha} \leq \alpha$, depending only on n, λ, Λ and α. This yields that $\tilde{v} \in C^{2,\bar{\alpha}}(\bar{Y} \times [0,s_0])$ with

$$\|\tilde{v}\|_{C^{2,\bar{\alpha}}(\bar{Y}\times[0,s_0])} \leq c_1 \left(\|\tilde{v}\|_{L^\infty(2Y\times[0,1))} + \|f\|_{C^\alpha(2Y\times[0,1))} + \|g\|_{C^{2,\alpha}(2Y)} \right),$$

where $0 < s_0 \leq \frac{1}{2}$ and $c_1 > 0$ depend only on n, λ, Λ, α and K. Utilizing (4.16), (4.13) and (4.11) (with $m = 0$), we derive that

$$\|\tilde{v}\|_{C^{2,\bar{\alpha}}(Y\times[0,s_0])} \leq c_2, \tag{4.26}$$

where $c_2 > 0$ is determined only by n, λ, Λ, α, β, β_0 and K.

Now let us fix a nonnegative integer k and define

$$\tilde{v}_k(y,s) = \tilde{v}(y,s+k) \quad (y \in \mathbb{R}^n, s \geq 0).$$

Then from (4.25), we know that $\tilde{v}_k$ solves

$$\partial_s \tilde{v}_k = F(D_y^2\tilde{v}_k, y, s+k) + f(y,s+k) \quad \text{in } 2Y \times (0,2).$$

Hence, it follows from the interior $C^{2,\bar{\alpha}}$ estimate (with $\bar{\alpha}$ being the same as that in (4.26)) that $\tilde{v}_k \in C^{2,\bar{\alpha}}(\bar{Y} \times [s_0, s_0+1])$ with

$$\|\tilde{v}_k\|_{C^{2,\bar{\alpha}}(\bar{Y}\times[s_0,s_0+1])} \leq c_3 \left(\|\tilde{v}_k\|_{L^\infty(2Y\times(0,2))} + \|f\|_{C^\alpha(2Y\times(0,2))} \right),$$

where $c_3 > 0$ depends only on n, λ, Λ, α and K. Utilizing (4.16) and (4.13) (with $m = 0$), we deduce that

$$\|\tilde{v}_k\|_{C^{2,\bar{\alpha}}(\bar{Y}\times[s_0,s_0+1])} \leq c_4 e^{-\beta_0 k}, \tag{4.27}$$

where $c_4 > 0$ is determined only by n, λ, Λ, α, β, β_0 and K.

Iterating (4.27) with respect to k and utilizing (4.26) for the initial case of this iteration argument, we arrive at (4.24). $\square$

Let us remark that Lemma 4.3.3 yields the compactness (in (y,s)) of $\{v(x_i,t_i,y,s)\}_{i=1}^\infty$ and $\{\tilde{v}(x_i,t_i,y,s)\}_{i=1}^\infty$ when $(x_i,t_i) \to (x,t)$. By the stability theory [20] of viscosity solutions, we obtain that v and $\tilde{v}$ are continuous in (x,t), stated as below. Let us also point out that the following lemma is a version of continuous dependence estimates, and we refer to [31], [15] and other literature for more discussions in this regard.

Lemma 4.3.4. *Let $\bar{\alpha}$ be the Hölder exponent chosen in Lemma 4.3.3. Then $\bar{v} \in C(\mathbb{R}^n \times [0,T])$ and $v \in C(\mathbb{R}^n \times [0,T]; C^{2,\hat{\alpha}}_{loc}(\mathbb{R}^n \times [0,\infty)))$ for any $0 < \hat{\alpha} < \bar{\alpha}$.*

Proof. As in the proof of Lemma 4.3.3, we will fix $(x,t) \in \mathbb{R}^n \times [0,T]$ and continue with using the simplified notation for F, f, g, v, γ and $\tilde{v}$. Let us take any sequence $(x_i, t_i) \to (x, t)$ as $i \to \infty$. For notational convenience, let us write $F_i(P, y, s) = F(P, x_i, t_i, y, s)$, $f_i(y, s) = f(x_i, t_i, y, s)$, $g_i(y) = g(x_i, y)$, $v_i(y, s) = v(x_i, t_i, y, s)$, $\gamma_i = \bar{v}(x_i, t_i)$ and $\tilde{v}_i(y, s) = v_i(y, s) - \gamma_i$. By C we denote a positive constant that depends only on n, λ, Λ, α, β, β_0 and K, and will let it vary from one line to another.

We prove $v_i \to v$ first. By (4.24) we have

$$\|v_i\|_{C^{2,\bar{\alpha}}(\mathbb{R}^n \times [0,\infty))} \leq C,$$

for any $i = 1, 2, \cdots$. Hence, we know from the Arzela-Ascoli theorem that for any subsequence $\{w_j\}_{j=1}^{\infty}$ of $\{v_i\}_{i=1}^{\infty}$, there exist a further subsequence $\{w_{j_k}\}_{k=1}^{\infty}$ and a certain function $w \in C^{2,\bar{\alpha}}(\mathbb{R}^n \times [0, \infty))$ such that $w_{j_k} \to w$ in $C^{2,\hat{\alpha}}_{loc}(\mathbb{R}^n \times [0, \infty))$ as $k \to \infty$, for any $0 < \hat{\alpha} < \bar{\alpha}$. One may notice that w_{j_k} solves

$$\begin{cases} \partial_s w_{j_k} = F_{j_k}(D_y^2 w_{j_k}, y, s) + f_{j_k}(y, s) & \text{in } \mathbb{R}^n \times (0, \infty), \\ w_{j_k}(y, 0) = g_{j_k}(y) & \text{in } \mathbb{R}^n, \end{cases}$$

in the viscosity sense. Due to the regularity assumptions (4.8), (4.11) and (4.13) on F, g and respectively f, we know that $F_i \to F$ uniformly on $\mathcal{S}^n \times \mathbb{R}^n \times [0, \infty)$, $g_i \to g$ uniformly on $\mathbb{R}^n$ and $f_i \to f$ uniformly on $\mathbb{R}^n \times [0, \infty)$, as $i \to \infty$. Hence, letting $k \to \infty$, we observe from the stability theory [20] that the limit function w also solves

$$\begin{cases} w_s = F(D_y^2 w, y, s) + f(y, s) & \text{in } \mathbb{R}^n \times (0, \infty), \\ w(y, 0) = g(y) & \text{on } \mathbb{R}^n, \end{cases}$$

in the viscosity sense. However, the above equation is identical with the equation (4.14). Since v is the unique solution to (4.14), we deduce that $w = v$ on $\mathbb{R}^n \times [0, \infty)$.

What we have proved so far is that for any subsequence of $\{v_i\}_{i=1}^{\infty}$, there exists a further subsequence which converges to v. Thus, $v_i \to v$ as $i \to \infty$ in $C^{2,\hat{\alpha}}_{loc}(\mathbb{R}^n \times [0, \infty))$ for any $0 < \hat{\alpha} < \bar{\alpha}$.

Now we are left with showing that $\gamma_i \to \gamma$. Due to (4.24), we have

$$e^{\beta_0 s} \|\tilde{v}_i(\cdot, s)\|_{C^{2,\bar{\alpha}}(\mathbb{R}^n)} \leq C, \tag{4.28}$$

for any $s \geq 0$, uniformly for all $i = 1, 2, \cdots$. Since $\tilde{v}_i(y, s) = v_i(y, s) - \gamma_i$ and

$\tilde{v}(y,s) = v(y,s) - \gamma$, we deduce from (4.28) and (4.24) that

$$|\gamma_i - \gamma| \leq 2Ce^{-\beta_0 s} + |v_i(0,s) - v(0,s)|.$$

Given any $\delta > 0$, we fix a sufficiently large s_0 such that $4Ce^{-\beta_0 s_0} \leq \delta$, and correspondingly choose i_0 such that $2|v_i(0,s_0) - v(0,s_0)| \leq \delta$ for all $i \geq i_0$. Then we have $|\gamma_i - \gamma| \leq \delta$ for all $i \geq i_0$, proving that $\gamma_i \to \gamma$ as $i \to \infty$. Thus, the proof is finished. □

By Lemma 4.3.4, we are ready to prove the differentiability of v and $\bar{v}$ in the slow variables (x,t), and an exponential decay estimate for the derivatives of $v - \bar{v}$. Here we use Lemma 4.3.4 to obtain compactness (in (y,s)) of the difference quotients (in (x,t)) of v. Arguing similarly as in the proof of Lemma 4.3.4, we deduce that the difference quotients converge to a single limit, proving the differentiability of v.

Lemma 4.3.5. *Let $\bar{\alpha}$ be the Hölder exponent chosen in Lemma 4.3.3. Then there exist $D_{x_k}\bar{v}(x,t)$ and $D_{x_k}v(x,t,\cdot,\cdot) \in C^{2,\bar{\alpha}}(\mathbb{R}^n \times [0,\infty))$, for any $1 \leq k \leq n$, such that*

$$|D_{x_k}\bar{v}(x,t)| + \|D_{x_k}(v(x,t,\cdot,\cdot) - \bar{v}(x,t))\|_{E^{2,\bar{\alpha}}(\mathbb{R}^n \times [0,\infty);\beta_1)} \leq C,$$

for any $(x,t) \in \mathbb{R}^n \times [0,T]$, where $0 < \beta_1 < \beta_0$ depends only on n, λ, Λ and β_0, and $C > 0$ depends only on n, λ, Λ, α, β, β_0, β_1 and K. Moreover, we have $\bar{v} \in C^1(\mathbb{R}^n \times [0,T])$ and $v \in C^1(\mathbb{R}^n \times [0,T]; C^{2,\hat{\alpha}}_{loc}(\mathbb{R}^n \times [0,\infty)))$ for any $0 < \hat{\alpha} < \bar{\alpha}$.

Remark 4.3.6. *According to the parabolic terminology, C^1 regularity in (x,t) only involves derivatives in x. For more details, see Section 4.2.*

Proof of Lemma 4.3.5. Throughout this proof, let us write by C a positive constant depending only on n, λ, Λ, α, β and K, and allow it to vary from one line to another. Fix $(x,t) \in \mathbb{R}^n \times [0,T]$ and $1 \leq k \leq n$. We shall omit the dependence on t for notational convenience. Let us define

$$
\begin{aligned}
A_\sigma(y,s) &= \int_0^1 D_pF(\rho D_y^2 v(x+\sigma e_k,y,s)+(1-\rho)D_y^2v(x,y,s),x,y,s)d\rho,\\
\Psi_\sigma(y,s) &= \frac{F(D_y^2v(x+\sigma e_k,y,s),x+\sigma e_k,y,s)-F(D_y^2v(x+\sigma e_k,y,s),x,y,s)}{\sigma}\\
&\quad+\frac{f(x+\sigma e_k,y,s)-f(x,y,s)}{\sigma},\\
G_\sigma(y) &= \frac{g(x+\sigma e_k,y)-g(x,y)}{\sigma},
\end{aligned}
$$

for $(y,s)\in\mathbb{R}^n\times[0,\infty)$, and nonzero $\sigma\in\mathbb{R}$.

Clearly, A_σ, Ψ_σ and G_σ are periodic in y. The ellipticity of A_σ follows immediately from (4.6). Indeed, A_σ satisfies

$$
\lambda|Q|\leq \operatorname{tr}(A_\sigma(y,s)Q)\leq\Lambda|Q|\quad (Q\in\mathcal{S}^n, Q\geq 0), \tag{4.29}
$$

for any $(y,s)\in\mathbb{R}^n\times[0,\infty)$. It should be remarked that the lower and the upper ellipticity bounds of A_σ are not only independent of σ but also the same as those of F.

By (4.8) and (4.24), we know that $A_\sigma\in C^{\bar\alpha}(\mathbb{R}^n\times[0,\infty))$ and

$$
\|A_\sigma\|_{C^{\bar\alpha}(\mathbb{R}^n\times[0,\infty))}\leq C. \tag{4.30}
$$

Let us remark here that we need Lipschitz regularity of D_pF in P in order to have (4.30).

Similarly, we may deduce from (4.8), (4.13) and (4.24) that $\Psi_\sigma\in C^{\bar\alpha}(\mathbb{R}^n\times[0,\infty))$ satisfies

$$
\|\Psi_\sigma\|_{E^{\bar\alpha}(\mathbb{R}^n\times[0,\infty);\beta_0)}\leq C. \tag{4.31}
$$

On the other hand, it follows directly from (4.11) that $G\in C^{2,\alpha}(\mathbb{R}^n)$ and

$$
\|G_\sigma\|_{C^{2,\alpha}(\mathbb{R}^n)}\leq K. \tag{4.32}
$$

Now we define

$$
V_\sigma(y,s)=\frac{v(x+\sigma e_k,y,s)-v(x,y,s)}{\sigma}\quad\text{and}\quad\Gamma_\sigma=\frac{\bar v(x+\sigma e_k)-\bar v(x)}{\sigma},
$$

for $(y,s)\in\mathbb{R}^n\times[0,\infty)$ and nonzero $\sigma\in\mathbb{R}$. Linearizing the equation (4.14), we see that V_σ is a viscosity solution to

$$\begin{cases} \partial_s V_\sigma = \mathrm{tr}(A_\sigma(y,s) D_y^2 V_\sigma) + \Psi_\sigma(y,s) & \text{in } \mathbb{R}^n \times (0,\infty), \\ V_\sigma(y,0) = G_\sigma(y) & \text{on } \mathbb{R}^n. \end{cases} \tag{4.33}$$

Owing to (4.30) - (4.32), we observe that the equation (4.33) belongs to the same class of (4.14). Hence, Lemma 4.3.3 is applicable to the problem (4.33). In particular, the exponent β in the statement of Lemma 4.3.3 is replaced here by β_0. Thus, we obtain some $0 < \beta_1 < \beta_0$, depending only on n, λ, Λ and β_0, such that

$$|\Gamma_\sigma| + \|V_\sigma\|_{E^{2,\bar{\alpha}}(\mathbb{R}^n \times [0,\infty);\beta_1)} \leq C. \tag{4.34}$$

Now we invoke the compactness argument used in the proof of Lemma 4.3.4. Choose any sequence $\sigma_i \to 0$ as $i \to \infty$. Then by (4.34), there exist a subsequence $\{\tau_j\}_{j=1}^\infty$ of $\{\sigma_i\}_{i=1}^\infty$ and a function $V \in C^{2,\bar{\alpha}}(\mathbb{R}^n \times [0,\infty))$ such that $V_{\tau_j} \to V$ in $C^{2,\hat{\alpha}}_{loc}(\mathbb{R}^n \times [0,\infty))$ as $j \to \infty$, for any $0 < \hat{\alpha} < \bar{\alpha}$.

On the other hand, from the regularity assumptions on F and f ((4.8) and (4.13) respectively) and the continuity of $D_y^2 v(x,t,y,s)$ in (x,t) (Lemma 4.3.4), we deduce that $A_\sigma \to A$ and $\Psi_\sigma \to \Psi$ locally uniformly in $\mathbb{R}^n \times [0,\infty)$ as $\sigma \to 0$, where

$$\begin{aligned} A(y,s) &= D_p F(D_y^2 v(x,y,s), x, y, s), \\ \Psi(y,s) &= D_{x_k} F(D_y^2 v(x,y,s), x, y, s) + D_{x_k} f(x,y,s). \end{aligned}$$

It also follows from the regularity assumption (4.11) on g that $G_\sigma \to G$ uniformly in $\mathbb{R}^n$ with

$$G(y) = D_{x_k} g(x,y).$$

Hence, it follows from the stability of viscosity solutions (see [20] for the details) that the limit function V of V_{τ_j} is a viscosity solution to

$$\begin{cases} V_s = \mathrm{tr}(A(y,s) D_y^2 V) + \Psi(y,s) & \text{in } \mathbb{R}^n \times (0,\infty), \\ V(y,0) = G(y) & \text{on } \mathbb{R}^n. \end{cases} \tag{4.35}$$

However, A, G and Ψ also satisfy (4.29), (4.32) and respectively (4.31). Thus, (4.35) belongs to the same class of (4.14), which implies that V is the unique (spatially periodic) viscosity solution to (4.35). This shows that $V_\sigma \to V$ in

$C^{2,\hat{\alpha}}_{loc}(\mathbb{R}^n \times [0,\infty))$ as $\sigma \to 0$, for any $0 < \hat{\alpha} < \bar{\alpha}$. In other words,

$$V(y,s) = D_{x_k} v(x,y,s).$$

Equipped with the uniform estimate (4.34) and the observation that $V_\sigma \to V$, we may also prove that $\Gamma_\sigma \to \Gamma$ as $\sigma \to 0$, for some $\Gamma \in \mathbb{R}$. Since this part repeats the argument used in the end of the proof of Lemma 4.3.4, we skip the details. Let us remark that

$$\Gamma = D_{x_k} \bar{v}(x).$$

The second assertion of Lemma 4.3.5 can be justified by following the proof of Lemma 4.3.4 regarding (4.35). To avoid the redundancy of the argument, we omit the details. $\square$

From the proof of Lemma 4.3.5, we observe that the regularity of v and $\bar{v}$ in (x,t) can be improved in a systematic way. Induction on the order of the derivatives (in (x,t)) of v and $\bar{v}$ leads us to the following proposition.

Proposition 4.3.7. *Under the assumptions* (4.6) - (4.11) *and* (4.12) - (4.13), $v \in C^\infty(\mathbb{R}^n \times [0,T]; C^{2,\bar{\alpha}}(\mathbb{R}^n \times [0,\infty)))$ *and* $\bar{v} \in C^\infty(\mathbb{R}^n \times [0,T])$ *with*

$$\sum_{|\mu|+2\nu=m} \left[|D_x^\mu \partial_t^\nu \bar{v}(x,t)| + \| D_x^\mu \partial_t^\nu (v(x,t,\cdot,\cdot) - \bar{v}(x,t)) \|_{E^{2,\bar{\alpha}}(\mathbb{R}^n \times [0,\infty);\beta_m)} \right] \leq C_m, \tag{4.36}$$

for all $(x,t) \in \mathbb{R}^n \times [0,T]$ *and any* $m \geq 0$, *where* $0 < \beta_m < \beta$ *depends only on* n, λ, Λ, m *and* β, *and* $C_m > 0$ *depends only on* n, λ, Λ, α, β, m *and* K.

Remark 4.3.8. *As pointed out in Remark 4.3.2, the proof of this proposition does not use the periodicity of F in s. Moreover,* $0 < \beta_m < \cdots < \beta_0 < \beta$ *for any* $m \geq 1$ *and* C_m *depends on the choice of* $\beta_0, \cdots, \beta_m$.

Proof of Proposition 4.3.7. The proof of this proposition repeats the argument of Lemma 4.3.5. One may notice that although the statement of this lemma only involves the derivatives in x, the proof works equally well for the derivatives in t. Here we will only provide the sketch of the proof, and leave out the details to avoid redundancy.

Let V_k and $\bar{V}_k$ be the k-th order derivative (in (x,t)) of v and respectively $\bar{v}$. Let (P_k) be the equation which V_k solves, and suppose (as the induction hypothesis) that the coefficient A_k, the source term Ψ_k and the initial data G_k of (P_k) belong to the same class of (4.14). We know that this hypothesis is satisfied when $k = 1$, since in that case the equation (P_k) is precisely (4.35). By the induction hypothesis, Lemma 4.3.4 is applicable, which gives us higher regularity of V_k in the fast variables.

Now let $\{V_{k,\sigma}\}_{\sigma\neq 0}$ be the sequence of difference quotients of V_k (in (x,t)). To avoid confusion, let us denote by $(P_{k,\sigma})$ the equation for $V_{k,\sigma}$. Let us also denote by $A_{k,\sigma}$, $\Psi_{k,\sigma}$ and $G_{k,\sigma}$ the coefficient, the source term and respectively the initial data of $(P_{k,\sigma})$.

Following the proof of Lemma 4.3.5, we may observe that $(P_{k,\sigma})$ is obtained by linearizing (P_k). Utilizing the structure conditions of F, f and g, one may deduce that $(P_{k,\sigma})$ belongs to the same class of (P_k) with the structure conditions for $(P_{k,\sigma})$ being independent of σ. Moreover, one may observe from the regularity assumptions on F and f that $A_{k,\sigma}$ and $\Psi_{k,\sigma}$ converge to some A_{k+1} and Ψ_{k+1}, respectively, as $\sigma \to 0$ locally uniformly in the underlying domain of (y,s). Here one needs to use the continuity of $D_y^2 V_k$ in (x,t) that will be given in the induction hypotheses. On the other hand, $G_{k,\sigma}$ will converge to some G_{k+1} uniformly in y, due to the regularity assumption on G.

Hence, the stability theory of viscosity solutions will ensure that any limit of $V_{k,\sigma}$ is a viscosity solution to the problem (P_{k+1}) having A_{k+1}, Ψ_{k+1} and G_{k+1} as the coefficient, the source term and, respectively, the initial data. Then the uniqueness of (viscosity) solutions to (P_{k+1}) will lead us to the observation that $V_{k,\sigma}$ converges to a single limit function, say V_{k+1}. In other words, V_k is differentiable (in (x,t)) and the corresponding derivative is V_{k+1}. Utilizing this fact, one may also prove that $\bar{V}_k$ is differentiable with the derivative being $\bar{V}_{k+1}$.

We observe that Lemma 4.3.4 provides us the desired estimate for V_k and $\bar{V}_k$, while Lemma 4.3.5 yields that for V_{k+1} and $\bar{V}_{k+1}$. The rest of the proof can now be finished by an induction argument. □

4.3.2 Cell Problem

Due to the uniform ellipticity and the periodicity of F, we know from the classical work [22] that there is a functional $\bar{F} : \mathcal{S}^n \times \mathbb{R}^n \times [0,T] \to \mathbb{R}$ such that for each $(P,x,t) \in \mathcal{S}^n \times \mathbb{R}^n \times [0,T]$, the following equation,

$$w_s = F(D_y^2 w + P, x, t, y, s) - \bar{F}(P, x, t) \quad \text{in } \mathbb{R}^n \times \mathbb{R}, \tag{4.37}$$

has a periodic viscosity solution $w \in BUC(\mathbb{R}^n \times \mathbb{R})$. We also know that $\bar{F}$ is uniformly elliptic with the same ellipticity constants of F, and it is concave in the Hessian variable P.

Moreover, periodic viscosity solutions to (4.37) are unique up to an additive constant, if any. This also allows us to define another functional $w : \mathcal{S}^n \times \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that $w(P,x,t,\cdot,\cdot)$ is the unique viscosity solution to (4.37) which also satisfies

$$w(P,x,t,0,0) = 0.$$

We shall now study the regularity of $\bar{F}$ and w in (P,x,t), which follows closely to the authors' previous work [34]. We begin by improving the regularity of w in the fast variables (y,s), based on the interior $C^{2,\alpha}$ estimates [51] for viscosity solutions to concave equations. We leave the proof to the reader, as it is straightforward from the classical regularity result, and the property of the cell problem.

Lemma 4.3.9. *There exists $0 < \bar{\alpha} \leq \alpha$ depending only on n, λ, Λ and α such that $w(P,x,t,\cdot,\cdot) \in C^{2,\bar{\alpha}}(\mathbb{R}^n \times \mathbb{R})$ with*

$$\|w(P,x,t,\cdot,\cdot)\|_{C^{2,\bar{\alpha}}(\mathbb{R}^n \times \mathbb{R})} \leq C|P|,$$

for each $(P,x,t) \in \mathcal{S}^n \times \mathbb{R}^n \times [0,T]$, where $C > 0$ depends only on n, λ, Λ, α and K. Moreover, $w \in C(\mathcal{S}^n \times \mathbb{R}^n \times [0,T]; C^{2,\hat{\alpha}}(\mathbb{R}^n \times \mathbb{R}))$ for any $0 < \hat{\alpha} < \bar{\alpha}$.

With the above lemma at hand, we can proceed with the proof of (continuous) differentiability of $\bar{F}$ and w in (P,x,t). The proof is also similar to that of Lemma 4.3.5.

Lemma 4.3.10. *Let $\bar{\alpha}$ be the Hölder exponent chosen in Lemma 4.3.9. Then there exist $D_p^\kappa D_x^\mu \bar{F}(P,x,t)$ and $D_p^\kappa D_x^\mu w(P,x,t,\cdot,\cdot) \in C^{2,\bar{\alpha}}(\mathbb{R}^n \times \mathbb{R})$, for any pair (κ,μ) of multi-indices satisfying $|\kappa| + |\mu| = 1$, such that*

$$\left|D_p^\kappa D_x^\mu \bar{F}(P,x,t)\right| + \left\| D_p^\kappa D_x^\mu w(P,x,t,\cdot,\cdot) \right\|_{C^{2,\bar{\alpha}}(\mathbb{R}^n \times \mathbb{R})} \leq C|P|^{1-|\kappa|},$$

for any $(P,x,t) \in \mathcal{S}^n \times \mathbb{R}^n \times [0,T]$, where $C > 0$ depends only on n, λ, Λ, α and K. Moreover, we have $\bar{F} \in C^1(\mathcal{S}^n \times \mathbb{R}^n \times [0,T])$ and $w \in C^1(\mathcal{S}^n \times \mathbb{R}^n \times [0,T]; C^{2,\hat{\alpha}}(\mathbb{R}^n \times \mathbb{R}))$ for any $0 < \hat{\alpha} < \bar{\alpha}$.

Remark 4.3.11. *As pointed out in Remark 4.3.6, C^1 regularity in (P,x,t) does not involve that in t, according to the parabolic terminology.*

Proof of Lemma 4.3.10. In this proof, we use C to denote a positive constant that depends only on n, λ, Λ, α and K, and allow it to vary from one line to another. We shall prove this lemma for the derivatives in P only, since the same argument applies to the proof for the derivatives in x. Fix $(P,x,t) \in \mathcal{S}^n \times \mathbb{R}^n \times [0,T]$ and $1 \leq i,j \leq n$. Recall from Section 4.2 that by E_{ij} we denote the (i,j)-th standard basis matrix in $\mathcal{S}^n$. For notational convenience, we shall skip the dependence of F, w and $\bar{F}$ on (x,t). Define

$$\begin{aligned}
&A_\sigma(y,s)\\
&= \int_0^1 D_pF(\rho D_y^2 w(P+\sigma E_{ij},y,s) + (1-\rho)D_y^2 w(P,y,s) + \rho\sigma E_{ij},y,s)d\rho,\\
&W_\sigma(y,s) = \frac{w(P+\sigma E_{ij},y,s) - w(P,y,s)}{\sigma},\\
&\Gamma_\sigma = \frac{\bar{F}(P+\sigma E_{ij}) - \bar{F}(P)}{\sigma},
\end{aligned}$$

for $(y,s) \in \mathbb{R}^n \times \mathbb{R}$. By linearization, we deduce that W_σ is a (viscosity) solution to

$$\partial_s W_\sigma = \operatorname{tr}(A_\sigma(y,s)(D_y^2 W_\sigma + E_{ij})) - \Gamma_\sigma \quad \text{in } \mathbb{R}^n \times \mathbb{R}. \tag{4.38}$$

Clearly, A_σ is periodic on $\mathbb{R}^n \times \mathbb{R}$. More importantly, A_σ is uniformly elliptic in the sense of (4.29) and Hölder continuous with the uniform estimate (4.30). It should be stressed that the lower and upper ellipticity bounds for of A_σ are given by λ and, respectively, Λ and are independent of σ. Hence, (4.38) belongs to the same class of (4.37). As a result, Lemma 4.3.9 is applicable to (4.38). This yields that $W_\sigma \in C^{2,\bar{\alpha}}(\mathbb{R}^n \times \mathbb{R})$ and

$$|\Gamma_\sigma| + \|W_\sigma\|_{C^{2,\bar{\alpha}}(\mathbb{R}^n\times\mathbb{R})} \leq C|E_{ij}| \leq C. \tag{4.39}$$

Notice that Lemma 4.3.9 ensures $w \in C(\mathcal{S}^n; C^{2,\hat{\alpha}}(\mathbb{R}^n \times \mathbb{R}))$ for any $0 < \hat{\alpha} < \bar{\alpha}$. This combined with uniform ellipticity (4.6) of F yields that we have $A_\sigma \to A$ in $C^{\hat{\alpha}}(\mathbb{R}^n \times \mathbb{R})$ as $\sigma \to 0$ for any $0 < \hat{\alpha} < \bar{\alpha}$, where

$$A(y,s) = D_pF(D_y^2 w(P,y,s) + P, y, s).$$

On the other hand, the uniform estimate (4.39) and the periodicity of W_σ implies that any subsequence of $\{(\Gamma_\sigma, W_\sigma)\}_{\sigma\neq 0}$ contains a further subsequence that converges in $\mathbb{R} \times C^{2,\hat{\alpha}}(\mathbb{R}^n \times \mathbb{R})$, for any $0 < \hat{\alpha} < \bar{\alpha}$. However, the stability [20] of viscosity solutions ensures that a (uniform) limit (Γ, W) of $\{(\Gamma_\sigma, W_\sigma)\}_{\sigma \neq 0}$, if any, should satisfy

$$W_s = \operatorname{tr}(A(y,s)(D_y^2 W + E_{ij})) - \Gamma \quad \text{in } \mathbb{R}^n \times \mathbb{R},$$

in the viscosity sense. Since A is periodic and uniformly elliptic (in the sense of (4.29)) and W is also periodic, the classical argument [22] ensures the uniqueness of Γ. Moreover, since $W_\sigma(0,0) = 0$ for all nonzero σ, the limit W should also be unique. Therefore, $\Gamma_\sigma \to \Gamma$ and $W_\sigma \to W$ as $\sigma \to 0$, where the latter holds in $C^{2,\hat{\alpha}}(\mathbb{R}^n \times \mathbb{R})$ for any $0 < \hat{\alpha} < \bar{\alpha}$.

By the definition of Γ_σ and W_σ, we conclude that $\bar{F}$ and w are differentiable at P in direction E_{ij} with

$$\Gamma = D_{p_{ij}} \bar{F}(P) \quad \text{and} \quad W(y,s) = D_{p_{ij}} w(P,y,s).$$

The rest of the proof then follows from Lemma 4.3.9, and hence we omit the details. □

The following proposition is obtained by induction on the order of derivatives of $\bar{F}$ and w in the slow variables (P,x,t).

Proposition 4.3.12. *Assume that F verifies* (4.6) - (4.8). *Then $\bar{F} \in C^\infty(\mathcal{S}^n \times \mathbb{R}^n \times [0,T])$ and $w \in C^\infty(\mathcal{S}^n \times \mathbb{R}^n \times [0,T]; C^{2,\bar{\alpha}}(\mathbb{R}^n \times \mathbb{R}))$ and*

$$\begin{aligned}&\sum_{|\kappa|+|\mu|+2\nu=m} \left[\left| D_p^\kappa D_x^\mu \partial_t^\nu \bar{F}(P,x,t) \right| + \left\| D_p^\kappa D_x^\mu \partial_t^\nu w(P,x,t,\cdot,\cdot) \right\|_{C^{2,\bar{\alpha}}(\mathbb{R}^n \times \mathbb{R})} \right] \\ &\leq C_m |P|^{(1-|\kappa|)_+},\end{aligned} \tag{4.40}$$

for all $(P,x,t) \in \mathcal{S}^n \times \mathbb{R}^n \times [0,T]$ and for each integer $m \geq 0$, where $0 < \bar{\alpha} \leq \alpha$ depends only on n, λ, Λ and α, and $C_m > 0$ depends only on n, λ, Λ, α, m and K.

Proof. One may notice that the higher regularity of $\bar{F}$ and w in the slow variables (P,x,t) can be obtained by inductively applying Lemma 4.3.10 on the number of derivatives. Since the whole argument resembles that of the proof of Proposition 4.3.7, we omit the details. □

4.4 Higher Order Convergence Rate

This section is devoted to achieving the higher order convergence rates of the homogenization process of (4.1). We expect that away from the initial time zone, by which we indicate the strip $0 \leq t \leq \varepsilon^2$, the solution, u^ε, of (4.1) becomes less affected by the rapidly oscillatory behavior of the initial data, and that it behaves more as a solution to certain Cauchy problem with a non-oscillatory initial data. Thus, it is reasonable to split u^ε into the non-oscillatory part and the oscillatory part near the initial time layer.

For this reason, we construct two types of the higher order correctors associated with the homogenization problem (4.1), namely the initial layer corrector and the interior corrector. The former type captures the oscillatory behavior of u^ε near the initial time layer, while the latter describes its behavior in the interior. The construction of these correctors of higher orders will be based on the regularity theory in the slow variables established in Section 4.3.

Throughout this section, the constants $K > 0$ and $0 < \alpha < 1$ will be used to denote those in (4.8) and (4.11). Also we shall denote by $\beta_{k,l}$ a positive, generic constant that depends only on n, λ, Λ, k and l, and by $C_{k,l}$ a positive, generic constant that depends only on n, λ, Λ, α, K, k and l.

Let $S(m, \bar{\alpha}; d, k)$ be the class of all spatially periodic functions or mappings, f, on $\mathbb{R}^n \times [0, T] \times \mathbb{R}^n \times [0, \infty)$ such that $f \in C^\infty(\mathbb{R}^n \times [0, T]; C^{k,\bar{\alpha}}(\mathbb{R}^n \times [0, \infty))$ and for each integer $l \geq 0$, there is a positive constant $C_{m,\bar{\alpha},d,k}$, depending at most on n, λ, Λ, α, K, m, $\bar{\alpha}$, d, k and l, such that

$$\sum_{|\mu|+2\nu=l} \|D_x^\mu \partial_t^\nu f(x, t, \cdot, \cdot)\|_{C^{m,\alpha}(\mathbb{R}^n \times [0,\infty))} \leq C_{m,\bar{\alpha},d,k,l},$$

for all $(x, t) \in \mathbb{R}^n \times [0, T]$. By $E(m, \bar{\alpha}; d, k)$ be the subclass of $S(m, \bar{\alpha}; d, k)$ consisting of all functions or mappings f such that for each integer $l \geq 0$, there are positive constants $\beta_{d,k,l}$, depending only on n, λ, Λ, d, k and l, and $C_{m,\bar{\alpha},d,k}$, depending on the same parameters as above, such that

$$\sum_{|\mu|+2\nu=l} \|D_x^\mu \partial_t^\nu f(x, t, \cdot, \cdot)\|_{E^{m,\alpha}(\mathbb{R}^n \times [0,\infty); \beta_{d,k,l})} \leq C_{m,\bar{\alpha},d,k,l},$$

for all $(x, t) \in \mathbb{R}^n \times [0, T]$. In particular, we shall write by $S(d, k)$ the space of all functions or mappings, f, on $\mathbb{R}^n \times [0, T]$, whose obvious extension to

$\mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,\infty)$ belongs to the class $S(0,0;d,k)$.

4.4.1 Initial Layer Corrector

In this subsection, we aim at proving the following proposition.

Proposition 4.4.1. *Assume F and g verify* (4.6) - (4.8) *and* (4.10) - (4.11). *Then there exist a sequence $\{v_k \in S(2,\bar{\alpha};k)\}_{k=0}^{\infty}$ of spatially periodic functions on $\mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,\infty)$ and a sequence $\{\bar{v}_k \in S(k)\}_{k=0}^{\infty}$ of functions on $\mathbb{R}^n \times [0,T]$ such that $v_k - \bar{v}_k \in E(2,\bar{\alpha};k)$ for each $k \geq 0$. Moreover, for any integer $m \geq 0$, set*

$$\tilde{v}_m^\varepsilon(x,t) = \sum_{k=0}^{m} \varepsilon^k \left(v_k\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right) - \bar{v}_k(x,t)\right), \quad \bar{g}_m^\varepsilon(x) = \sum_{k=0}^{m} \varepsilon^k \bar{v}_k(x,0). \tag{4.41}$$

Then one has

$$\begin{cases} \partial_t \tilde{v}_m^\varepsilon = \dfrac{1}{\varepsilon^2} F\left(\varepsilon^2 D^2 \tilde{v}_m^\varepsilon, x, t, \dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^2}\right) + \psi_m^\varepsilon\left(x,t,\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon^2}\right) & \text{in } \mathbb{R}^n \times (0,T), \\ \tilde{v}_m^\varepsilon(x,0) + \bar{g}_m^\varepsilon(x) = g\left(x, \dfrac{x}{\varepsilon}\right) & \text{on } \mathbb{R}^n, \end{cases} \tag{4.42}$$

where ψ_m^ε satisfies

$$|\psi_m^\varepsilon(x,t,y,s)| \leq C_m \varepsilon^{m-1} e^{-\beta_m s}, \tag{4.43}$$

for any $0 < \varepsilon \leq \frac{1}{2}$.

Remark 4.4.2. *We shall see later that this proposition holds even when F is periodic only in the fast spatial variable y. Moreover, we shall call $v_k(x,t,y,s)$ the k-th order initial layer corrector and the function $\bar{g}_k : \mathbb{R}^n \to \mathbb{R}$, defined by*

$$\bar{g}_k(x) = \bar{v}_k(x,0), \tag{4.44}$$

the k-th order effective initial data.

Let us begin with heuristic arguments by the formal expansion. The computation presented here uses the Taylor expansion of F in its matrix variable P. We should mention that such an approach has already been shown in the authors' previous work [34].

Differentiating $\tilde{v}_m^\varepsilon$ with respect to t, we obtain

$$\begin{aligned}\varepsilon^2 \partial_t \tilde{v}_m^\varepsilon(x,t) &= \sum_{k=0}^m \varepsilon^k \partial_s v_k \left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) \\ &\quad + \sum_{k=0}^m \varepsilon^{k+2} \partial_t \left(v_k \left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) - \bar{v}_k(x,t)\right).\end{aligned} \tag{4.45}$$

In order to proceed with the derivatives of $\tilde{v}_m^\varepsilon$ in variable x, let us introduce a $\mathcal{S}^n$-valued mapping $V_k = V_k(x,t,y,s)$ on $\mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,\infty)$ defined by

$$V_k = \begin{cases} D_y^2 v_0, & k = 0, \\ D_y^2 v_1 + D_{xy} v_0, & k = 1, \\ D_y^2 v_k + D_{xy} v_{k-1} + D_x^2 (v_{k-2} - \bar{v}_{k-2}), & k \geq 2, \end{cases} \tag{4.46}$$

and corresponding define

$$\tilde{V}_k = \begin{cases} V_k, & 0 \leq k \leq m, \\ D_{xy} v_m + D_x^2 (v_{m-1} - \bar{v}_{m-1}), & k = m+1, \\ D_x^2 (v_m - \bar{v}_m), & k = m+2. \end{cases} \tag{4.47}$$

With $\tilde{V}_k$, one can write

$$\varepsilon^2 D^2 \tilde{v}_m^\varepsilon(x,t) = \sum_{k=0}^m \varepsilon^k \tilde{V}_k \left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) = V_0 \left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) + \varepsilon V_m^\varepsilon \left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right), \tag{4.48}$$

where in the second identity we wrote V_m^ε for the sum of $\varepsilon^{k-1} \tilde{V}_k$ over $1 \leq k \leq m+2$.

For notational convenience, let us write

$$A_k(x,t,y,s) = D_p^k F(V_0, x, t, y, s), \quad k \geq 1, \tag{4.49}$$

for $(x,t,y,s) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,\infty)$, and especially

$$A = A_1,$$

which is a $\mathcal{S}^n$-valued mapping, uniformly elliptic in the sense that $\mathrm{tr}(A(x,t,y,s)N)$ satisfies the ellipticity condition (4.6). What we shall do is the Taylor expansion of F in the matrix variable with base point V_0 and perturbation $\varepsilon V_m^\varepsilon$. In the following computation, we shall omit the variables $(x, t, \varepsilon^{-1}x, \varepsilon^{-2}t)$ in the exposition, since they do not play any important role.

$$\begin{aligned}
&F\left(\varepsilon^2 D^2 \tilde{v}_m^\varepsilon\right)\\
&= F(V_0 + \varepsilon V_m^\varepsilon)\\
&= F(V_0) + \varepsilon \operatorname{tr}(AV_m^\varepsilon) + \sum_{k=2}^m \frac{\varepsilon^k}{k!} A_k(V_m^\varepsilon, \cdots, V_m^\varepsilon) + R_m^\varepsilon\\
&= F(V_0) + \varepsilon \operatorname{tr}(AV_1) + \sum_{k=2}^m \varepsilon^k \left(\operatorname{tr}(AV_k) + \sum_{l=2}^k \frac{1}{l!} \sum_{\substack{i_1+\cdots+i_l=k\\ i_1,\cdots,i_l\geq 1}} A_l(V_{i_1}, \cdots, V_{i_l}) \right)\\
&\quad + E_m^\varepsilon,
\end{aligned} \tag{4.50}$$

where R_m^ε is the remainder term from the Taylor expansion, i.e.,

$$R_m^\varepsilon = F(V_0 + \varepsilon V_m^\varepsilon) - F(V_0) - \varepsilon \operatorname{tr}(AV_m^\varepsilon) - \sum_{k=2}^m \frac{\varepsilon^k}{k!} A_k(V_m^\varepsilon, \cdots, V_m^\varepsilon), \tag{4.51}$$

and E_m^ε is the term that contains further errors,

$$E_m^\varepsilon = R_m^\varepsilon + \sum_{k=2}^{m+2} \sum_{\substack{m+1\leq i_1+\cdots+i_k\leq k(m+2)\\ 1\leq i_1,\cdots,i_k\leq m+2}} \frac{\varepsilon^{i_1+\cdots+i_k}}{k!} A_k(\tilde{V}_{i_1}, \cdots, \tilde{V}_{i_k}), \tag{4.52}$$

Hence, plugging $\tilde{v}_m^\varepsilon$ into (4.1) and equating the power of ε, and noting (4.45) and (4.50), we obtain a sequence of equations that v_k should solve. The next lemma gives a rigorous justification of the above heuristic arguments.

Lemma 4.4.3. *One can recursively construct sequences $\{v_k \in S(2, \bar{\alpha}; k)\}_{k=0}^\infty$ and $\{\bar{v}_k \in S(k)\}_{k=0}^\infty$, with $v_k - \bar{v}_k \in E(2, \bar{\alpha}; k)$, as follows.*

(i) $v_0(x, t, \cdot, \cdot)$ is the spatially periodic solution of

$$\begin{cases} \partial_s v_0 = F(D_y^2 v_0, x, t, y, s) & \text{in } \mathbb{R}^n \times (0, \infty),\\ v_0(x, t, y, 0) = g(x, y) & \text{on } \mathbb{R}^n, \end{cases} \tag{4.53}$$

(ii) For each $2 \leq k \leq m$, $v_k(x,t,\cdot,\cdot)$ is the spatially periodic solution of

$$\begin{cases} \partial_s v_k = \mathrm{tr}(A(x,t,y,s)D_y^2 v_k) + \Phi_k(x,t,y,s) & \text{in } \mathbb{R}^n \times (0,\infty), \\ v_k(x,t,y,0) = 0 & \text{on } \mathbb{R}^n, \end{cases} \tag{4.54}$$

where

$$\Phi_k = \begin{cases} 0, & k = 1, \\ \mathrm{tr}(A(D_{xy}v_{k-1} + D_x^2(v_{k-2} - \bar{v}_{k-2})) - \partial_t(v_{k-2} - \bar{v}_{k-2}) & \\ + \displaystyle\sum_{l=2}^{k} \frac{1}{l!} \sum_{\substack{i_1+\cdots+i_l=k \\ i_1,\cdots,i_l \geq 1}} A_l(V_{i_1}, \cdots, V_{i_l}), & k \geq 2. \end{cases} \tag{4.55}$$

(iii) For each $k \geq 0$,

$$\bar{v}_k(x,t) = \lim_{s\to\infty} v_k(x,t,0,s).$$

Proof. It is clear from Proposition 4.3.7 that $v_0 \in S(2,\bar{\alpha};0)$ and $\bar{v}_0 \in S(0)$ with $v_0 - \bar{v}_0 \in E(2,\bar{\alpha};0)$. Henceforth, we shall suppose $m \geq 1$, and assume further, as the induction hypothesis, that we have already found $v_k \in S(2,\bar{\alpha};k)$ and $\bar{v}_k \in S(k)$ satisfying $v_k - \bar{v}_k \in E(2,\bar{\alpha};k)$, for $0 \leq k \leq m-1$.

Recall the mappings V_k and A_k from (4.46) and (4.49). Since $v_k - \bar{v}_k \in E(2,\bar{\alpha};k)$, we have $V_k \in E(0,\bar{\alpha};k)$ for each $0 \leq k \leq m-1$. This along with the structure condition (4.8) of F that $A_k \in E(0,\bar{\alpha};k)$ for each $k \geq 0$ as well.

Now let Φ_m be as in (4.55). One may notice that Φ_m only involves functions v_k and $\bar{v}_k$, for $0 \leq k \leq m-1$, which are assumed to be known already. Hence, combining the induction hypothesis that $v_k - \bar{v}_k \in E(2,\bar{\alpha};k)$, and the observation that $V_k, A_k \in E(0,\bar{\alpha};k)$, deduce that $\Phi_m \in E(0,\bar{\alpha};m)$. Thus, one can apply Proposition 4.3.7 again to the viscosity solution $v_m(x,t,\cdot,\cdot)$ of (4.54), and verify that $v_m \in S(2,\bar{\alpha};m)$, $\bar{v}_m \in S(2,\bar{\alpha};m)$ and $v_m - \bar{v}_m \in E(2,\bar{\alpha};m)$. The proof is then completed by the induction principle. □

Remark 4.4.4. *Let us remark that the proof above does not involve the periodicity of F in the fast temporal variable s. This is why Proposition 4.4.1 holds even if we only assume the spatial periodicity of F (that is periodicity in y), as mentioned in Remark 4.4.2.*

We are now ready to prove Proposition 4.4.1

Proof of Proposition 4.4.1. Let $\{v_k\}_{k=0}^{\infty}$ and $\{\bar{v}_k\}_{k=0}^{\infty}$ be the sequence taken from Lemma 4.4.3, and let $\tilde{v}_m^\varepsilon$ and $\bar{g}_m^\varepsilon$ be as in (4.41). Then it follows from (4.45), (4.50), (4.53) and (4.54) that the functions $\tilde{v}_m^\varepsilon$ and $\bar{g}_m^\varepsilon$ defined by (4.41) satisfy (4.42) with

$$\psi_m^\varepsilon(x,t,y,s) = \sum_{k=m-1}^{m} \varepsilon^k \partial_t (v_k(x,t,y,s) - \bar{v}(x,t)) - \varepsilon^{-2} E_m^\varepsilon(x,t,y,s),$$

where E_m^ε is given by (4.52). The rest of the proof is devoted to the proof of (4.43).

From the fact that $v_k - \bar{v}_k \in E(2,\bar{\alpha};k)$ for any $k \geq 0$, we know that

$$\sum_{k=m-1}^{m} \varepsilon^k \left|\partial_t (v_k(x,t,y,s) - \bar{v}_k(x,t))\right| \leq C_m \varepsilon^{m-1} e^{-\beta_m s}, \tag{4.56}$$

for any $0 < \varepsilon \leq \frac{1}{2}$. On the other hand, from the observation that $V_k, A_k \in E(0,\bar{\alpha};k)$, the remainder term R_m^ε in (4.51) can be estimated as

$$|R_m^\varepsilon(x,t,y,s)| \leq \frac{\varepsilon^{m+1}}{(m+1)!} \left|B_{m+1}(V_m^\varepsilon, \cdots, V_m^\varepsilon)\right|(x,t,y,s) \leq C_m \varepsilon^{m+1} e^{-\beta_m s}, \tag{4.57}$$

for any $0 < \varepsilon \leq \frac{1}{2}$. Noting that the summation indices $i_1, \cdots, i_k$ in the definition of E_m^ε are subject to the restriction $i_1 + \cdots + i_k \geq m+1$, we deduce from (4.57) that

$$|E_m^\varepsilon(x,t,y,s)| \leq C_m \varepsilon^{m+1} e^{-\beta_m s}, \tag{4.58}$$

for all $0 < \varepsilon \leq \frac{1}{2}$. Thus, (4.43) follows from (4.56) and (4.58). This completes the proof. □

4.4.2 Interior Corrector

In this subsection, we shall construct the higher order interior correctors. Here we shall consider a more general class of homogenization problems compared to (4.1). This will be essential in achieving the higher order convergence rate away from the initial time layer, and we shall discuss more in this direction in Section 4.4.3.

Proposition 4.4.5. *Assume that F verifies (4.6) - (4.8). Let $\{V_k \in E(0,\bar{\alpha};k)\}_{k=0}^{\infty}$ be a sequence of $\mathcal{S}^n$-valued, spatially periodic mappings on $\mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,\infty)$ and let $\{\bar{g}_k \in S(k)\}_{k=0}^{\infty}$ be a sequence of functions on $\mathbb{R}^n$.*

Then there exist a sequence $\{w_k \in S(2,\bar{\alpha};k)\}_{k=0}^{\infty}$ of spatially periodic functions on $\mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,\infty)$ and a sequence $\{w_k^{\#} \in S(2,\bar{\alpha};k)\}_{k=0}^{\infty}$ of periodic functions on $\mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times \mathbb{R}$, satisfying $w_k - w_k^{\#} \in E(2,\bar{\alpha};k)$ with $w_0 = w_1 = w_0^{\#} = w_1^{\#} = 0$, and a sequence $\{\bar{u}_k \in S(k)\}_{k=0}^{\infty}$ of functions on $\mathbb{R}^n \times [0,T]$, satisfying $\bar{u}_k(\cdot,0) = \bar{g}_k$ for any integer $k \geq 0$, such that the following is true. Define

$$\tilde{w}_m^{\varepsilon}(x,t) = \sum_{k=0}^{m} \varepsilon^k \left(w_k\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right) + \bar{u}_k(x,t)\right),$$
$$\tilde{w}_m^{\#,\varepsilon}(x,t) = \sum_{k=0}^{m} \varepsilon^k \left(w_k^{\#}\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right) + \bar{u}_k(x,t)\right).$$

Then one has

$$\begin{aligned}\partial_t \tilde{w}_m^{\varepsilon} = \frac{1}{\varepsilon^2} F\left(\varepsilon^2 D^2 \tilde{w}_m^{\varepsilon} + \sum_{k=0}^{m} \varepsilon^k V_k, x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)&\\ - \frac{1}{\varepsilon^2} F\left(\sum_{k=0}^{m} \varepsilon^k V_k, x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) + \psi_m^{\varepsilon}\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right) &\quad \text{in } \mathbb{R}^n \times (0,T),\end{aligned} \tag{4.59}$$

and

$$\partial_t \tilde{w}_m^{\#,\varepsilon} = \frac{1}{\varepsilon^2} F\left(\varepsilon^2 D^2 \tilde{w}_m^{\#,\varepsilon}, x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) + \psi_m^{\#,\varepsilon}\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right) \quad \text{in } \mathbb{R}^n \times (0,T),$$

with some ψ_m^{ε} spatially periodic and some $\psi_m^{\#,\varepsilon}$ periodic such that

$$\left|\psi_m^{\#,\varepsilon}(x,t,y,s)\right| + e^{\beta_m s}\left|(\psi_m^{\varepsilon} - \psi_m^{\#,\varepsilon})(x,t,y,s)\right| \leq C_m \varepsilon^{m-1}, \tag{4.60}$$

for any $0 < \varepsilon \leq \frac{1}{2}$.

Remark 4.4.6. *The function $w_k^{\#}$ is the time-periodic version of w_k, i.e., the former is also periodic in the fast time variable s as well as y. In what* *follows, we shall call*

$$\tilde{w}_k(x,t,y,s) = w_k(x,t,y,s) + \bar{u}_k(x,t),$$

the k-th order interior corrector, and $\bar{u}_k$ *the k-th order effective limit profile.*

To illustrate the idea why we consider two functions w_k and $w_k^{\#}$, let us consider a Cauchy problem with a non-oscillatory initial data, but with an operator of the type considered in (4.59), say

$$\begin{cases} \partial_t u^\varepsilon = \dfrac{1}{\varepsilon^2} F\left(\varepsilon^2 D^2 u^\varepsilon + V_0, x, t, \dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^2}\right) & \\ \qquad\quad - \dfrac{1}{\varepsilon^2} F\left(V_0, x, t, \dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^2}\right) & \text{in } \mathbb{R}^n \times (0,T), \\ u^\varepsilon(x,0) = \bar{g}(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{4.61}$$

As $\bar{g}$ being non-oscillatory in ε-scales, we expect $D^2 u^\varepsilon$ to be of order 1 in the interior. Now that

$$\frac{1}{\varepsilon^2}\left(F\left(\varepsilon^2 P + V_0, x, t, y, s\right) - F(V_0, x, t, y, s)\right) \to \operatorname{tr}(A(x,t,y,s)P),$$

with A given by (4.49), it is reasonable to guess that the effective problem of (4.61) is the same with the one corresponding to

$$\begin{cases} \partial_t \tilde{u}^\varepsilon = \operatorname{tr}\left(A\left(x, t, \dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^2}\right) D^2 \tilde{u}^\varepsilon\right) & \text{in } \mathbb{R}^n \times (0,T), \\ \tilde{u}^\varepsilon(x,0) = \bar{g}(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{4.62}$$

However, $A(x,t,y,s) = D_p F(V_0, x, t, y, s)$ and $V_0(x,t,y,s)$ decays exponentially fast in s, so the effective problem of (4.62) will also coincide with that of

$$\begin{cases} \partial_t \hat{u}^\varepsilon = \operatorname{tr}\left(A^{\#}\left(x, t, \dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^2}\right) D^2 \hat{u}^\varepsilon\right) & \text{in } \mathbb{R}^n \times (0,T) \\ \hat{u}^\varepsilon(x,0) = \bar{g}_0(x) & \text{on } \mathbb{R}^n. \end{cases}$$

where $A^{\#} : \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times \mathbb{R} \to \mathcal{S}^n$ is defined by

$$A^{\#}(x,t,y,s) = D_p F(0, x, t, y, s).$$

It is noteworthy that $A^{\#}$ is periodic in both y and s and uniformly elliptic. Hence, to $A^{\#}$ corresponds a unique effective coefficient $\bar{A} : \mathbb{R}^n \times [0,T] \to \mathcal{S}^n$. Moreover, due to [22], there exists a unique $\mathcal{S}^n$-valued mapping $\chi^{\#} = (\chi_{ij}^{\#})$ on $\mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times \mathbb{R}$ such that for each $(x,t) \in \mathbb{R}^n \times [0,T]$, $\chi_{ij}^{\#}(x,t,\cdot,\cdot)$ is the unique periodic solution to the following cell problem,

$$\begin{cases} \partial_s \chi_{ij}^{\#} = \operatorname{tr}(A^{\#}(x,t,y,s)(D_y^2 \chi_{ij}^{\#} + E_{ij})) - \operatorname{tr}(\bar{A}(x,t)E_{ij}) & \text{in } \mathbb{R}^n \times \mathbb{R}, \\ \chi_{ij}^{\#}(x,t,0,0) = 0. \end{cases} \tag{4.63}$$

In particular, $\bar{A}$ is uniformly elliptic with the same ellipticity bounds as those of $A^{\#}$. Moreover by Proposition 4.3.12, we know that $\bar{A} \in S(0)$, $\chi^{\#} \in S(2,\alpha;0)$, where α is the Hölder exponent in the regularity assumption (4.8) of F.

The following lemma ensures the existence of the matrix corrector mapping that exactly captures the oscillatory behavior of the coefficient A.

Lemma 4.4.7. *There exists a unique mapping $\chi = (\chi_{ij}) : \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,\infty) \to \mathcal{S}^n$ such that $\chi_{ij}(x,t,\cdot,\cdot)$ is the spatially periodic solution to*

$$\begin{cases} \partial_s \chi_{ij} = \operatorname{tr}(A(x,t,y,s)(D_y^2 \chi_{ij} + E_{ij})) - \operatorname{tr}(\bar{A}(x,t)E_{ij}) & \text{in } \mathbb{R}^n \times (0,\infty), \\ \chi_{ij}(x,t,y,0) = \chi_{ij}^{\#}(x,t,y,0) & \text{on } \mathbb{R}^n, \end{cases} \tag{4.64}$$

for each $(x,t) \in \mathbb{R}^n \times [0,T]$. Moreover, $\chi \in S(2,\bar{\alpha};0)$ and $\chi - \chi^{\#} \in E(2,\bar{\alpha};0)$.

Proof. Fix $(x,t) \in \mathbb{R}^n \times [0,T]$, $1 \le i,j \le n$ and consider the following spatially periodic Cauchy problem,

$$\begin{cases} \partial_s \varphi_{ij} = \operatorname{tr}(A(x,t,y,s)D_y^2 v_{ij}) + b_{ij}(x,t,y,s) & \text{in } \mathbb{R}^n \times (0,\infty), \\ \varphi_{ij}(x,t,y,0) = 0 & \text{on } \mathbb{R}^n, \end{cases}$$

with

$$b_{ij}(x,t,y,s) = \operatorname{tr}((A(x,t,y,s) - A^{\#}(x,t,y,s))(D_y^2 \chi_{ij}^{\#}(x,t,y,s) + E_{ij})).$$

Since $V_k \in E(0,\bar{\alpha};k)$ and $\chi^{\#} \in S(k)$, we know that b_{ij} decays exponentially fast as $s \to \infty$. Thus, Lemma 4.3.1 implies that the function,

$$\bar{\varphi}_{ij}(x,t) = \lim_{s\to\infty} \varphi_{ij}(x,t,0,s),$$

is well defined. Now it follows from the regularity assumption (4.8) on F together with Proposition 4.3.7 that this lemma is satisfied by

$$\chi_{ij}(x,t,y,s) = \chi_{ij}^{\#}(x,t,y,s) + (\varphi_{ij}(x,t,y,s) - \bar{\varphi}_{ij}(x,t)).$$

We omit the details. $\square$

In what follows, let us write

$$W_k = \begin{cases} 0, & k = 0,1, \\ D_y^2 w_k + D_{xy} w_{k-1} + D_x^2(w_{k-2} + \bar{u}_{k-2}), & k \geq 2. \end{cases} \tag{4.65}$$

Note that we set $W_0 = W_1 = 0$, which is coherent the assertion in Proposition 4.4.5 that $w_0 = w_1 = 0$. Next set $W_k^{\#}$ by the time-periodic version of W_k, that is,

$$W_k^{\#} = \begin{cases} 0, & k = 0,1, \\ D_y^2 w_k^{\#} + D_{xy} w_{k-1}^{\#} + D_x^2(w_{k-2}^{\#} + \bar{u}_{k-2}), & k \geq 2. \end{cases} \tag{4.66}$$

Also let A_k be as in (4.49), and set $A_k^{\#} \in S(0,\alpha;k)$ to its time-periodic version,

$$A_k^{\#}(x,t,y,s) = D_p^k F(0,x,t,y,s), \quad k \geq 1.$$

It follows from $V_0 \in S(0,\bar{\alpha};0)$ that $A_k - A_k^{\#} \in E(0,\bar{\alpha};k)$ for any $k \geq 0$.

We are now ready to construct the higher order interior correctors as follows.

Lemma 4.4.8. *One can recursively construct a sequence $\{w_k \in S(2,\bar{\alpha};k)\}_{k=0}^{\infty}$ of spatially periodic functions on $\mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,\infty)$, a sequence $\{w_k^{\#} \in S(2,\bar{\alpha};k)\}_{k=0}^{\infty}$ of periodic functions on $\mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times \mathbb{R}$ with $w_k - w_k^{\#} \in E(2,\bar{\alpha};k)$, a sequence $\{\bar{u}_k \in S(k)\}_{k=0}^{\infty}$ of functions on $\mathbb{R}^n \times [0,T]$ satisfying the following.*

(i) $w_k^{\#}(x,t,\cdot,\cdot)$ is the periodic solution to

$$\begin{cases} \partial_s w_k^{\#} = \mathrm{tr}(A^{\#}(x,t,y,s) D_y^2 w_k^{\#}) + \Phi_k^{\#}(x,t,y,s) & \text{in } \mathbb{R}^n \times \mathbb{R}, \\ w_k^{\#}(x,t,0,0) = 0, \end{cases} \tag{4.67}$$

for each $(x,t) \in \mathbb{R}^n \times [0,T]$, *where*

$$\Phi_k^{\#} = \begin{cases} 0, & k = 0,1, \\ \operatorname{tr}(A^{\#}(D_{xy} w_{k-1}^{\#} + D_x^2(w_{k-2}^{\#} + \bar{u}_{k-2}))) - \partial_t(w_{k-2}^{\#} + \bar{u}_{k-2}) & \\ \displaystyle + \sum_{l=2}^{k} \frac{1}{l!} \sum_{\substack{i_1+\cdots+i_l=k \\ i_1,\cdots,i_l \geq 1}} A_l^{\#}(W_{i_1}^{\#}, \cdots, W_{i_l}^{\#}). & k \geq 2. \end{cases}$$

(ii) $w_k(x,t,\cdot,\cdot)$ *is the spatially periodic solution to*

$$\begin{cases} \partial_s w_k = \operatorname{tr}(A(x,t,y,s) D_y^2 w_k) + \Phi_k(x,t,y,s) & \text{in } \mathbb{R}^n \times (0,\infty), \\ w_k(x,t,y,0) = w_k^{\#}(x,t,y,0) & \text{on } \mathbb{R}^n, \end{cases} \tag{4.68}$$

for each $(x,t) \in \mathbb{R}^n \times [0,T]$, *where*

$$\Phi_k = \begin{cases} 0, & k = 0,1, \\ \operatorname{tr}(A(D_{xy} w_{k-1} + D_x^2(w_{k-2} + \bar{u}_{k-2}))) & \\ \quad - \partial_t(w_{k-2} + \bar{u}_{k-2}) & \\ \displaystyle + \sum_{l=2}^{k} \frac{1}{l!} \sum_{\substack{i_1+\cdots+i_l=k \\ i_1,\cdots,i_l \geq 1}} (A_l(V_{i_1} + W_{i_1}, \cdots, V_{i_l} + W_{i_l}) & k \geq 2. \\ \quad - A_l(V_{i_1}, \cdots, V_{i_l})), & \end{cases}$$

(iii) $\bar{u}_k$ *is the unique solution of*

$$\begin{cases} \partial_t \bar{u}_k = \operatorname{tr}(\bar{A}(x,t) D_x^2 \bar{u}_k) + \bar{\Phi}_k(x,t) & \text{in } \mathbb{R}^n \times (0,T), \\ \bar{u}_k(x,0) = \bar{g}_k(x) & \text{on } \mathbb{R}^n, \end{cases} \tag{4.69}$$

where $\bar{\Phi}_k(x,t)$ *is the unique number for which there exists a unique periodic solution to*

$$\begin{cases} \partial_s \phi_k^{\#} = \operatorname{tr}(A^{\#}(x,t,y,s) D_y^2 \phi_k^{\#}) + \Phi_k^{\#}(x,t,y,s) - \bar{\Phi}_k(x,t) & \text{in } \mathbb{R}^n \times \mathbb{R}, \\ \phi_k^{\#}(x,t,0,0) = 0, & \end{cases} \tag{4.70}$$

for each $(x,t) \in \mathbb{R}^n \times [0,T]$.

Proof. Since $\Phi_k^\# = 0$ for $k = 0, 1$, one should have $w_k^\# = 0$ for $k = 0, 1$ as well, since $w_k^\#$ is the unique periodic solution to (4.67). This also implies that $w_k = 0$ for $k = 0, 1$. Hence, we only need to construct $w_k^\#$ and w_k for $k \geq 2$.

Let us remark that the construction of $w_k^\#$ and $\bar{u}_{k-2}$, for $k \geq 2$, is independent of w_k. Moreover, the construction is very similar with the elliptic case, which can be found in the previous work [34, Lemma 3.3.2] by the authors. Especially, $w_k^\#$ is given by

$$w_k^\#(x,t,y,s) = \phi_k^\#(x,t,y,s) + \operatorname{tr}\left(\chi^\#(x,t,y,s) D_x^2 \bar{u}_{k-2}(x,t)\right),$$

with $\chi^\#$ and $\phi_k^\#$ given as the unique periodic solutions to (4.63) and respectively (4.70); here one can also deduce that $\phi_k^\# \in S(2, \bar{\alpha}; k)$. We shall leave this part to the reader, and proceed directly with the construction of w_k only.

Fix any $m \geq 2$, and suppose that we have already found $w_k^\#$, $\phi_k^\#$ and $\bar{u}_{k-2}$, for $k \leq m$, and w_k, for $k \leq m-1$, that satisfy the assertions of this lemma. Note that Φ_m only involves these functions. Since $w_k \in S(2, \bar{\alpha}; k)$ and $w_k^\# \in S(2, \bar{\alpha}; k)$ together satisfy $w_k - w_k^\# \in E(2, \bar{\alpha}; k)$ as the induction hypotheses for $0 \leq k \leq m-1$, one can derive along with the assumption $V_k \in S(0, \bar{\alpha}; k)$ that $\Phi_m \in S(0, \bar{\alpha}; m)$ and $\Phi_m - \Phi_m^\# \in S(0, \bar{\alpha}; m)$. Hence, one can argue analogously as with the proof of Lemma 4.4.7 and deduce that for each $(x,t) \in \mathbb{R}^n \times [0,T]$, there exists a unique, spatially periodic solution $\phi_m(x,t,\cdot,\cdot)$ to

$$\begin{cases} \partial_s \phi_m = \operatorname{tr}(A(x,t,y,s) D_y^2 \phi_m) + \Phi_m(x,t,y,s) - \bar{\Phi}_m(x,t) & \text{in } \mathbb{R}^n \times (0,\infty), \\ \phi_m(x,t,y,0) = \phi_m^\#(x,t,y,0) & \text{on } \mathbb{R}^n, \end{cases} \tag{4.71}$$

and $\phi_m \in S(2, \bar{\alpha}; m)$ with $\phi_m - \phi_m^\# \in E(2, \bar{\alpha}; m)$; the last inclusion follows from $\Phi_m - \Phi_m^\# \in E(0, \bar{\alpha}; m)$ and $A - A^\# \in E(0, \bar{\alpha}; 0)$. Finally, we define $w_m : \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ by

$$w_m(x,t,y,s) = \phi_m(x,t,y,s) + \operatorname{tr}(\chi(x,t,y,s) D_x^2 \bar{u}_{m-2}(x,t)).$$

Then it follows from (4.71) and (4.64) that

$$\begin{aligned} \partial_s w_m &= \partial_s \phi_m + \operatorname{tr}((\partial_s \chi) D_x^2 \bar{u}_{m-2}) \\ &= \operatorname{tr}(A D_y^2 \phi_m) + \Phi_m - \bar{\Phi}_m + \operatorname{tr}((A D_y^2 \chi - \bar{A}) D_x^2 \bar{u}_{m-2}) \\ &= \operatorname{tr}(A D_y^2 w_m) + \Phi_m, \end{aligned}$$

in (y,s), with each (x,t) fixed, which verifies the interior equation in (4.68). The initial condition is straightforward. On the other hand, $w_m \in S(2,\bar{\alpha};m)$ and $w_m - w_m^\# \in E(2,\bar{\alpha};m)$, since we have $\phi_m - \phi_m^\# \in E(2,\bar{\alpha};m)$, $\chi - \chi^\# \in E(2,\bar{\alpha};0)$ and $\bar{u}_{m-2} \in S(m-2)$. We omit the details. □

Equipped with Lemma 4.4.8, we are ready to prove Proposition 4.4.5.

Proof of Proposition 4.4.5. Here we shall only address the notable difference in the computation involving the Taylor expansion, when proving Proposition 4.4.5 (iv), and leave the rest of the argument to the reader, since the main argument follows closely to the proof of Proposition 4.4.1. Let us define

$$X_k = V_k + W_k,$$

with W_k given as in (4.65). Since $W_0 = W_1 = 0$, we have $X_0 = V_0$ and $X_1 = V_1$. Thus, one can proceed as in the computation in (4.50) and deduce that

$$\begin{aligned}
&F(V_0 + \varepsilon X_m^\varepsilon) - F(V_0 + \varepsilon V_m^\varepsilon) + \psi_m^\varepsilon \\
&= \sum_{k=2}^m \varepsilon^k \operatorname{tr}(A(X_k - V_k)) \\
&\quad + \sum_{k=2}^m \varepsilon^k \sum_{l=2}^k \frac{1}{l!} \sum_{\substack{i_1+\cdots+i_l=k \\ i_1,\cdots,i_l\geq 1}} (A_l(X_{i_1},\cdots,X_{i_l}) - A_l(V_{i_1},\cdots,V_{i_l})) \\
&= \sum_{k=2}^m \varepsilon^k \operatorname{tr}(AW_k) \\
&\quad + \sum_{k=2}^m \varepsilon^k \sum_{l=2}^k \frac{1}{l!} \sum_{\substack{i_1+\cdots+i_l=k \\ i_1,\cdots,i_l\geq 1}} (A_l(V_{i_1}+W_{i_1},\cdots,V_{i_l}+W_{i_l}) - A_l(V_{i_1},\cdots,V_{i_l})),
\end{aligned}$$

where ψ_m^ε is the error term of the form (4.52), $V_m^\varepsilon = \sum_{k=1}^m \varepsilon^{k-1} V_k$ and $X_m^\varepsilon = \sum_{k=1}^m \varepsilon^{k-1} X_k + \sum_{k=m+1}^{m+2} \varepsilon^{k-1}\tilde{W}_k$, with

$$\tilde{W}_k = \begin{cases} W_k, & 0 \leq k \leq m, \\ D_{xy}w_m + D_x^2(w_{m-1} + \bar{u}_{m-1}), & k = m+1, \\ D_x^2(w_m + \bar{u}_m), & k = m+2. \end{cases} \tag{4.72}$$

Hence, it follows from the recursive equations (4.68) of w_k that

$$F(V_0 + \varepsilon X_m^\varepsilon) - F(V_0 + \varepsilon V_m^\varepsilon) - E_m^\varepsilon = \sum_{k=2}^{m} \varepsilon^k (\partial_s w_k + \partial_t (w_{k-2} + \bar{u}_{k-2})).$$

This shows that $\tilde{w}_m^\varepsilon$ solves (4.59) with the remainder term ψ_m^ε. We skip the rest of the proof. □

4.4.3 Nonlinear Coupling Effect and the Bootstrap Argument

The main goal of this subsection is to establish our main result, Theorem 4.1.1. Throughout this subsection, let us assume that F and g verify (4.6) - (4.8) and (4.10) - (4.11).

We shall begin with the analyze the effect arising from the rapid oscillation on the initial data of (4.1), and approximate the associated solution u^ε with the higher order initial layer correctors. Here the approximation is up to a viscosity solution to a new homogenization problem, but this time with a non-oscillatory initial condition.

Lemma 4.4.9. *Under the assumption of Theorem 4.1.1, one can construct $\{v_k\}_{k=0}^\infty$, $\{\bar{v}_k\}_{k=0}^\infty$, $\tilde{v}_m^\varepsilon$ and $\bar{g}_m^\varepsilon$ be as in Proposition 4.4.1. Let $\tilde{u}_m^\varepsilon$ be the bounded viscosity solution of*

$$\begin{cases} \partial_t \tilde{u}_m^\varepsilon = \dfrac{1}{\varepsilon^2} F\left(\varepsilon^2 D^2 \tilde{u}_m^\varepsilon + \displaystyle\sum_{k=0}^{m} \varepsilon^k V_k, x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) & \\ \qquad\qquad - \dfrac{1}{\varepsilon^2} F\left(\displaystyle\sum_{k=0}^{m} \varepsilon^k V_k, x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) & \text{in } \mathbb{R}^n \times (0,T), \\ \tilde{u}_m^\varepsilon(x,0) = \bar{g}_m^\varepsilon(x) & \text{on } \mathbb{R}^n, \end{cases} \tag{4.73}$$

with V_k given as in (4.47). Then one has, for any $0 < \varepsilon \le \frac{1}{2}$,

$$|u^\varepsilon(x,t) - \tilde{v}_m^\varepsilon(x,t) - \tilde{u}_m^\varepsilon(x,t)| \le C_m \varepsilon^{m-1}, \tag{4.74}$$

for all $x \in \mathbb{R}^n$ and $0 \le t \le T$. In particular,

$$|u^\varepsilon(x,t) - \tilde{u}_m^\varepsilon(x,t)| \le C_m \varepsilon^{m-1}, \tag{4.75}$$

for all $x \in \mathbb{R}^n$ *and* $c_m \varepsilon^2 |\log \varepsilon| \leq t \leq T$.

Proof. We claim that $\tilde{u}_m^\varepsilon + \tilde{v}_m^\varepsilon$ solves

$$\begin{cases} \partial_t \left(\tilde{u}_m^\varepsilon + \tilde{v}_m^\varepsilon\right) = \dfrac{1}{\varepsilon^2} F\left(\varepsilon^2 D^2 \left(\tilde{u}_m^\varepsilon + \tilde{v}_m^\varepsilon\right), x, t, \dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^2}\right) & \\ \qquad\qquad\qquad\qquad + r_m^\varepsilon \left(x, t, \dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^2}\right) & \text{in } \mathbb{R}^n \times (0,T), \\ \left(\tilde{u}_m^\varepsilon + \tilde{v}_m^\varepsilon\right)(x,0) = g\left(x, \dfrac{x}{\varepsilon}\right) & \text{on } \mathbb{R}^n, \end{cases} \tag{4.76}$$

with the remainder term r_m^ε satisfying

$$|r_m^\varepsilon(x,t,y,s)| \leq C_m \varepsilon^{m-1} e^{-\beta_m s}. \tag{4.77}$$

Then since both $\tilde{u}_m^\varepsilon + \tilde{v}_m^\varepsilon$ and u^ε are bounded uniformly continuous in $\mathbb{R}^n \times [0,T]$, one can deduce from the standard comparison principle [20] and the exponential decay estimate (4.77) of r_m^ε that

$$|u^\varepsilon(x,t) - \tilde{v}_m^\varepsilon(x,t) - \tilde{u}_m^\varepsilon(x,t)| \leq \sup_{\xi \in \mathbb{R}^n} \int_0^t \left| r_m^\varepsilon(\xi, \tau, \varepsilon^{-1}\xi, \varepsilon^{-2}\tau) \right| d\tau \leq C_m \varepsilon^{m-1},$$

for any $x \in \mathbb{R}^n$ and $0 \leq t \leq T$, as desired. The error estimate (4.75) away from the initial time layer follows immediately from (4.74) and the exponential decay estimate that $v_k - v_k^\# \in E(2, \bar{\alpha}; k)$.

From the initial conditions of (4.73) and (4.42), one can easily verify the initial condition of (4.76). Hence, it only remains to check the interior equation and the exponential decay estimate of the remainder term. However, from the computation (4.48) of $D^2\tilde{v}_m^\varepsilon$, one can proceed as

$$
\begin{aligned}
&F\left(\varepsilon^2 D^2\left(\tilde{u}_m^\varepsilon + \tilde{v}_m^\varepsilon\right), x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)\\
&= F\left(\varepsilon^2 D^2 \tilde{u}_m^\varepsilon + \sum_{k=0}^{m} \varepsilon^k V_k + \sum_{k=m+1}^{m+2} \varepsilon^k \tilde{V}_k, x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)\\
&= F\left(\varepsilon^2 D^2 \tilde{u}_m^\varepsilon + \sum_{k=0}^{m} \varepsilon^k V_k, x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) + h_m^{1,\varepsilon}\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)\\
&= \varepsilon^2 \partial_t \tilde{u}_m^\varepsilon + F\left(\sum_{k=0}^{m} \varepsilon^k V_k, x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) + h_m^{1,\varepsilon}\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)\\
&= \varepsilon^2 \partial_t \tilde{u}_m^\varepsilon + F\left(\sum_{k=0}^{m} \varepsilon^k V_k + \sum_{k=m+1}^{m+2} \varepsilon^k \tilde{V}_k, x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)\\
&\quad + \left(h_m^{1,\varepsilon} + h_m^{2,\varepsilon}\right)\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)\\
&= \varepsilon^2 \partial_t\left(\tilde{u}_m^\varepsilon + \tilde{v}_m^\varepsilon\right) + \left(-\varepsilon^2 \psi_m^\varepsilon + h_m^{1,\varepsilon} + h_m^{2,\varepsilon}\right)\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right),
\end{aligned}
$$

where $\tilde{V}_k$ is given by (4.47), ψ_m^ε is given as in Proposition 4.4.1 and by $h_m^{1,\varepsilon}$ and $h_m^{2,\varepsilon}$ we simply denoted the terms so that we have the equalities above. Let us remark that $h_m^{1,\varepsilon}$ and $h_m^{2,\varepsilon}$ are well-defined, since $D^2\tilde{u}_m^\varepsilon$ and $\partial_t \tilde{u}_m^\varepsilon$ exist in the classical sense. This is because the operator governing the interior equation (4.73) for $\tilde{u}_m^\varepsilon$ is uniformly elliptic, smooth and concave; here the smoothness comes from the fact that $V_k \in E(0, \bar{\alpha}; k)$. Hence, the standard regularity theory [51] ensures the smoothness of $\tilde{u}_m^\varepsilon$, although it may not possess a uniform regularity for the time derivative and the spatial Hessian. In addition, it follows from the ellipticity condition (4.6) of F that for each $i = 1, 2$, one has

$$
\left|h_m^{i,\varepsilon}(x, t, y, s)\right| \le C_0 \sum_{k=m+1}^{m+2} \varepsilon^k \left|\tilde{V}_k(x, t, y, s)\right| \le C_m \varepsilon^{m+1} e^{-\beta_m s}, \tag{4.78}
$$

for any $0 < \varepsilon \le \frac{1}{2}$, where the second estimate follows from the exponential decay estimate of $v_k - \bar{v}_k$.

This shows that $\tilde{u}_m^\varepsilon + \tilde{v}_m^\varepsilon$ satisfies the interior equation of (4.76) with

$$r_m^\varepsilon = \psi_m^\varepsilon - \varepsilon^{-2}\left(h_m^{1,\varepsilon} + h_m^{2,\varepsilon}\right).$$

The decay estimate (4.77) of r_m^ε can be deduced from (4.43) and (4.78), which finishes the proof. □

One may compare (4.73) with (4.61), and realize that we are in a position to invoke Proposition 4.4.5 to construct the higher order interior correctors for the new homogenization problem. This eventually leads us to a higher order approximation of $\tilde{u}_m^\varepsilon$ by the interior correctors, again up to some function with order ε^2. The function turns out to be a viscosity solution to a new homogenization problem essentially belongs to the same class of (4.1), which allows us to run a bootstrap argument.

Lemma 4.4.10. *Under the conclusion of Lemma 4.4.9, let $\{V_k\}_{k=0}^\infty$ and $\{\bar{g}_k\}_{k=0}^\infty$ be as in* (4.46) *and respectively* (4.44)*. Then one can construct $\{w_k\}_{k=0}^\infty$, $\{w_k^\#\}_{k=0}^\infty$, $\{\bar{u}_k\}_{k=0}^\infty$, $\tilde{w}_m^\varepsilon$ and $\tilde{w}_m^{\#,\varepsilon}$ as in Proposition 4.4.5. Let $u_{1,m}^\varepsilon$ be the bounded viscosity solution to*

$$\begin{cases} \partial_t u_{1,m}^\varepsilon = \dfrac{1}{\varepsilon^4} F\left(\varepsilon^4 D^2 u_{1,m}^\varepsilon + \displaystyle\sum_{k=0}^m \varepsilon^k (V_k + W_k), x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) & \\ \qquad\qquad - \dfrac{1}{\varepsilon^4} F\left(\displaystyle\sum_{k=0}^m \varepsilon^k (V_k + W_k), x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) & \text{in } \mathbb{R}^n \times (0,T), \\ u_{1,m}^\varepsilon(x,0) = -\displaystyle\sum_{k=0}^{m-2} \varepsilon^k w_{k+2}\left(x, 0, \frac{x}{\varepsilon}, 0\right) & \text{on } \mathbb{R}^n, \end{cases} \tag{4.79}$$

where W_k is given as in (4.65)*. Then for any $0 < \varepsilon \le \frac{1}{2}$,*

$$\left|\tilde{u}_m^\varepsilon(x,t) - \tilde{w}_m^\varepsilon(x,t) - \varepsilon^2 u_{1,m}^\varepsilon(x,t)\right| \le C_m \varepsilon^{m-1}, \tag{4.80}$$

for all $x \in \mathbb{R}^n$ and $0 \le t \le T$. Moreover, one has

$$\left|\tilde{u}_m^\varepsilon(x,t) - \tilde{w}_m^{\#,\varepsilon}(x,t) - \varepsilon^2 u_{1,m}^\varepsilon(x,t)\right| \le C_m \varepsilon^{m-1} \tag{4.81}$$

for all $x \in \mathbb{R}^n$ and $c_m \varepsilon^2 |\log \varepsilon| \le t \le T$.

Proof. We assert that $\tilde{w}_m^\varepsilon + \varepsilon^2 u_{1,m}^\varepsilon$ is a viscosity solution to

$$\begin{cases} \partial_t \left(\tilde{w}_m^\varepsilon + \varepsilon^2 u_{1,m}^\varepsilon\right) \\ = \dfrac{1}{\varepsilon^2} F\left(\varepsilon^2 D^2 \left(\tilde{w}_m^\varepsilon + \varepsilon^2 u_{1,m}^\varepsilon\right) + \displaystyle\sum_{k=0}^m \varepsilon^k V_k, x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) & \text{in } \mathbb{R}^n \times (0,T), \\ \quad - \dfrac{1}{\varepsilon^2} F\left(\displaystyle\sum_{k=0}^m \varepsilon^k V_k, x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) + r_m^\varepsilon\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) \\ \left(\tilde{w}_m^\varepsilon + \varepsilon^2 u_{1,m}^\varepsilon\right)(x,0) = \bar{g}_m^\varepsilon(x) & \text{on } \mathbb{R}^n, \end{cases} \tag{4.82}$$

with some remainder term r_m^ε satisfying

$$|r_m^\varepsilon(x,t,y,s)| \leq C_m \varepsilon^{m-1}. \tag{4.83}$$

Then one can deduce the desired estimate (4.80) by means of the comparison principle, as in the proof of Lemma 4.4.9. Moreover the error estimate (4.81) away from the initial time layer follows from (4.80) and the exponential decay estimate of $w_k - w_k^\#$.

Note that $V_k \in E(0, \bar{\alpha}; k)$ and that $\bar{v}_k \in S(k)$, which implies $\bar{g}_k = \bar{v}_k(\cdot, 0) \in S(k)$, for any $k \geq 0$, so the sequences $\{V_k\}_{k=0}^\infty$ and $\{\bar{g}_k\}_{k=0}^\infty$ satisfy the assumption of Proposition 4.4.5. Thus, we obtain the sequence $\{w_k\}_{k=0}^\infty$ of higher order interior correctors, and the sequence $\{\bar{u}_k\}_{k=0}^\infty$ of higher order effective limits. From Proposition 4.4.5 and the definitions, (4.44) and (4.41), of $\bar{g}_k$ and $\bar{g}_m^\varepsilon$, we observe that (4.79) that

$$\tilde{w}_m^\varepsilon(x,0) + \varepsilon^2 u_{1,m}^\varepsilon(x,0) = \sum_{k=0}^m \varepsilon^k \bar{u}_k(x,0) = \sum_{k=0}^m \varepsilon^k \bar{g}_k(x) = \bar{g}_m^\varepsilon(x).$$

This verifies the initial condition of (4.82).

On the other hand, since we have

$$\begin{aligned} \varepsilon^2 D^2 \tilde{w}_m^\varepsilon(x,t) &= \sum_{k=0}^{m+2} \varepsilon^k \tilde{W}_k\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) \\ &= \sum_{k=0}^{m} \varepsilon^k W_k\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) + \sum_{k=m+1}^{m+2} \varepsilon^k \tilde{W}_k\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right), \end{aligned}$$

with $\tilde{W}_k$ given as in (4.72), it follows from the interior equations of (4.59) that

$$
\begin{aligned}
&F\left(\varepsilon^2 D^2\left(\tilde{w}_m^\varepsilon+\varepsilon^2 u_{1,m}^\varepsilon\right)+\sum_{k=0}^m \varepsilon^k V_k, x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)\\
&=F\left(\varepsilon^4 D^2 u_{1,m}^\varepsilon+\sum_{k=0}^m \varepsilon^k (V_k+W_k)+\sum_{k=m+1}^{m+2}\varepsilon^k \tilde{W}_k, x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)\\
&=F\left(\varepsilon^4 D^2 u_{1,m}^\varepsilon+\sum_{k=0}^m \varepsilon^k (V_k+W_k), x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)+h_m^{1,\varepsilon}\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)\\
&=\varepsilon^4\partial_t u_{1,m}^\varepsilon+F\left(\sum_{k=0}^m \varepsilon^k (V_k+W_k), x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)+h_m^{1,\varepsilon}\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)\\
&=\varepsilon^4\partial_t u_{1,m}^\varepsilon+F\left(\sum_{k=0}^m \varepsilon^k (V_k+W_k)+\sum_{k=m+1}^{m+2}\varepsilon^k \tilde{W}_k, x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)\\
&\quad+\left(h_m^{1,\varepsilon}+h_m^{2,\varepsilon}\right)\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)\\
&=\varepsilon^2\partial_t\left(\tilde{w}_m^\varepsilon+\varepsilon^2 u_{1,m}^\varepsilon\right)+\left(-\varepsilon^2\psi_m^\varepsilon+h_m^{1,\varepsilon}+h_m^{2,\varepsilon}\right)\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right),
\end{aligned}
$$

where ψ_m^ε is given by (4.59), and by $h_m^{1,\varepsilon}$ and $h_m^{2,\varepsilon}$ we simply denoted the terms so that we have the equalities above. Arguing similarly as with the proof of Lemma 4.4.9, one can justify the well-definedness of $h_m^{i,\varepsilon}$, for $i=1,2$, and deduce that

$$
\left|h_m^{i,\varepsilon}(x,t,y,s)\right| \leq C_0 \sum_{k=m+1}^{m+2} \varepsilon^k \left|\tilde{W}_k(x,t,y,s)\right| \leq C_m \varepsilon^{m+1}, \tag{4.84}
$$

for any $0<\varepsilon\leq\frac{1}{2}$, where the second estimate follows from the observation that $w_k \in S(2,\bar{\alpha};k)$.

Hence, $\tilde{w}_m^\varepsilon+\varepsilon^2 u_{1,m}^\varepsilon$ satisfies the interior equation of (4.82) with

$$
r_m^\varepsilon = \psi_m^\varepsilon - \varepsilon^{-2}\left(h_m^{1,\varepsilon}+h_m^{2,\varepsilon}\right),
$$

and the estimate (4.83) of r_m^ε follows from (4.60) and (4.84). This finishes the proof. □

As a corollary to the above lemmas, we achieve the following.

Corollary 4.4.11. *One has, for any* $0 < \varepsilon \leq \frac{1}{2}$,

$$\left|u^\varepsilon(x,t) - \tilde{v}^\varepsilon_m(x,t) - \tilde{w}^\varepsilon_m(x,t) - \varepsilon^2 u^\varepsilon_{1,m}(x,t)\right| \leq C_m \varepsilon^{m-1}, \tag{4.85}$$

for all $x \in \mathbb{R}^n$ *and* $0 \leq t \leq T$. *In addition,*

$$\left|u^\varepsilon(x,t) - \tilde{w}^{\#,\varepsilon}_m(x,t) - \varepsilon^2 u^\varepsilon_{1,m}(x,t)\right| \leq C_m \varepsilon^{m-1}, \tag{4.86}$$

for all $x \in \mathbb{R}^n$ *and* $c_m \varepsilon^2 |\log \varepsilon| \leq t \leq T$.

It is worthwhile to repeat that $u^\varepsilon_{1,m}$ is a solution to a homogenization problem essentially of the same type with (4.1). Hence, we can iterate the above arguments, provided that we can construct the higher order initial layer correctors and interior correctors in a more general setting. Here we shall only present the argument and skip the proof, since the main idea and the computations are already shown in the proofs of Proposition 4.4.1 and Proposition 4.4.5.

First comes the construction of higher order initial layer correctors.

Proposition 4.4.12. *Assume that* F *verifies* (4.6) - (4.8). *Fix integers* $d \geq 0$ *and* $m \geq 2d$. *Let* $\{X_{d,k} \in S(0,\bar{\alpha}; d,k)\}_{k=0}^\infty$ *be a sequence of* $\mathcal{S}^n$*-valued, spatially periodic mappings on* $\mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,\infty)$, *and* $\{X^\#_{d,k} \in S(0,\bar{\alpha}; d,k)\}_{k=0}^\infty$ *be a sequence of* $\mathcal{S}^n$*-valued periodic mappings on* $\mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times \mathbb{R}$ *such that* $X_{d,k} - X^\#_{d,k} \in E(0,\bar{\alpha}; d,k)$. *Also let* $\{g_{d,k} \in S(2,\bar{\alpha}; d,k)\}_{k=0}^\infty$ *be a sequence of periodic functions on* $\mathbb{R}^n \times \mathbb{R}^n$.

Then there exist a sequence $\{v_{d,k} \in S(2,\bar{\alpha}; d,k)\}_{k=0}^\infty$ *of spatially periodic functions on* $\mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,\infty)$ *and a sequence* $\{\bar{v}_{d,k} \in S(d,k)\}_{k=0}^\infty$ *of functions on* $\mathbb{R}^n \times [0,T]$ *such that* $v_{d,k} - \bar{v}_{d,k} \in E(2,\bar{\alpha}; d,k)$ *and the following is true. Set*

$$\tilde{v}^\varepsilon_{d,m}(x,t) = \sum_{k=0}^{m-2d} \varepsilon^k \left(v_{d,k}\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right) - \bar{v}_{d,k}(x,t)\right),$$

$$\bar{g}^\varepsilon_{d,m}(x) = \sum_{k=0}^{m-2d} \varepsilon^k \bar{v}_{d,k}(x,0).$$

Then $\tilde{v}^{\varepsilon}_{d,m}$ and $\bar{g}^{\varepsilon}_{d,m}$ satisfy

$$\begin{cases} \partial_t \tilde{v}^{\varepsilon}_{d,m} & \\ \quad = \dfrac{1}{\varepsilon^{2d+2}} F\left(\varepsilon^{2d+2} D^2 \tilde{v}^{\varepsilon}_{d,m} + \displaystyle\sum_{k=0}^{m} \varepsilon^k X_{d,k}, x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) & \text{in } \mathbb{R}^n \times (0,T), \\ \quad - \dfrac{1}{\varepsilon^{2d+2}} F\left(\displaystyle\sum_{k=0}^{m} \varepsilon^k X_{d,k}, x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) + \psi^{\varepsilon}_{d,m}\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) & \\ \tilde{v}^{\varepsilon}_{d,m}(x,0) + \bar{g}^{\varepsilon}_{d,m}(x) = \displaystyle\sum_{k=0}^{m-2d} \varepsilon^k g_{d,k}\left(x, \frac{x}{\varepsilon}\right) & \text{on } \mathbb{R}^n, \end{cases}$$

with some spatially periodic function $\psi^{\varepsilon}_{d,m}$ verifying

$$\left|\psi^{\varepsilon}_{d,m}(x,t,y,s)\right| \leq C_{d,m} \varepsilon^{m-2d-1} e^{-\beta_{d,m} s},$$

for any $0 < \varepsilon \leq \frac{1}{2}$.

Remark 4.4.13. *One may notice that Proposition 4.4.1 is simply the special case with $d = 0$, and $X_{0,k} = X^{\#}_{d,k} = 0$ for any $k \geq 0$, $g_{0,0} = g$ and $g_{0,k} = 0$ for any $k \geq 1$. Moreover, the new homogenization problem* (4.79) *falls under the case $d = 1$, $X_{1,k} = V_k + W_k$, $X^{\#}_{1,k} = W^{\#}_k$ and $g_{1,k}(x,y) = \bar{u}_{k+2}(x,0) - w_{k+2}(x,0,y,0)$.*

Next follows the construction of the higher order interior correctors.

Proposition 4.4.14. *Assume that F verifies* (4.6) - (4.8). *Fix integers $d \geq 0$ and $m \geq 2d$. Let $\{Y_{d,k} \in S(0,\bar{\alpha};d,k)\}_{k=0}^{\infty}$ be a sequence of $\mathcal{S}^n$-valued, spatially periodic mappings on $\mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,\infty)$, and $\{Y^{\#}_{d,k} \in S(0,\bar{\alpha};d,k)\}_{k=0}^{\infty}$ be a sequence of $\mathcal{S}^n$-valued periodic mappings on $\mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times \mathbb{R}$ such that $Y_{d,k} - Y^{\#}_{d,k} \in E(2,\bar{\alpha};d,k)$. Also let $\{\bar{g}_{d,k} \in S(d,k)\}_{k=0}^{\infty}$ be a sequence of functions on $\mathbb{R}^n$.*

Then there are a sequence $\{w_{d,k} \in S(2,\bar{\alpha};d,k)\}_{k=0}^{\infty}$ of spatially periodic functions on $\mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,\infty)$, a sequence $\{w^{\#}_{d,k} \in S(2,\bar{\alpha};d,k)\}_{k=0}^{\infty}$ of periodic functions on $\mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times \mathbb{R}$, satisfying $w_{d,k} - w^{\#}_{d,k} \in E(2,\bar{\alpha};d,k)$ and $w_{d,0} = w_{d,1} = w^{\#}_{d,0} = w^{\#}_{d,1} = 0$, and a sequence $\{\bar{u}_{d,k} \in S(d,k)\}_{k=0}^{\infty}$ of functions on $\mathbb{R}^n \times [0,T]$, satisfying $\bar{u}_{d,k}(\cdot,0) = \bar{g}_{d,k}$, such that the following is *true. Define*

$$\tilde{w}^{\varepsilon}_{d,m}(x,t) = \sum_{k=0}^{m-2d} \varepsilon^k \left(w_{d,k}\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right) + \bar{u}_{d,k}(x,t)\right),$$
$$\tilde{w}^{\#,\varepsilon}_{d,m}(x,t) = \sum_{k=0}^{m-2d} \varepsilon^k \left(w^{\#}_{d,k}\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right) + \bar{u}_{d,k}(x,t)\right).$$

Then one has

$$\begin{aligned}\partial_t \tilde{w}^{\varepsilon}_{d,m} &\\ = \frac{1}{\varepsilon^{2d+2}} F &\left(\varepsilon^{2d+2} D^2 \tilde{w}^{\varepsilon}_m + \sum_{k=0}^{m} \varepsilon^k Y_{d,k}, x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) \\ &- \frac{1}{\varepsilon^{2d+2}} F\left(\sum_{k=0}^{m} \varepsilon^k Y_{d,k}, x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) + \psi^{\varepsilon}_{d,m}\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right) \quad \textit{in } \mathbb{R}^n \times (0,T),\end{aligned}$$

and

$$\begin{aligned}\partial_t \tilde{w}^{\#,\varepsilon}_{d,m} &\\ = \frac{1}{\varepsilon^{2d+2}} F &\left(\varepsilon^{2d+2} D^2 \tilde{w}^{\#,\varepsilon}_m + \sum_{k=0}^{m} \varepsilon^k Y^{\#}_{d,k}, x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) \\ &- \frac{1}{\varepsilon^{2d+2}} F\left(\sum_{k=0}^{m} \varepsilon^k Y^{\#}_{d,k}, x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) + \psi^{\varepsilon,\#}_{d,m}\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right) \quad \textit{in } \mathbb{R}^n \times (0,T),\end{aligned}$$

with some spatially periodic function $\psi^{\varepsilon}_{d,m}$ *and some periodic function* $\psi^{\#,\varepsilon}_{d,m}$ *satisfying*

$$\left|\psi^{\#,\varepsilon}_{d,m}(x,t,y,s)\right| + e^{\beta_{d,m} s}\left|(\psi^{\varepsilon}_{d,m} - \psi^{\#,\varepsilon}_{d,m})(x,t,y,s)\right| \le C_{d,m}\varepsilon^{m-2d-1},$$

for any $0 < \varepsilon \le \frac{1}{2}$.

Remark 4.4.15. *Proposition 4.4.5 is the special case with* $d = 0$, $Y_{0,k} = V_k$ *and* $Y^{\#}_{d,k} = 0$.

Finally, we are ready to prove our main result.

Proof of Theorem 4.1.1. Since the case $2 \le m \le 3$ is already proved in

Corollary 4.4.11, we shall consider the case $m \geq 4$ only.

Let us construct $\{X_{d,k}\}_{k=0}^{\infty}$, $\{X_{d,k}\}_{k=0}^{\infty}$, $\{g_{d,k}\}_{k=0}^{\infty}$, $\{Y_{d,k}\}_{k=0}^{\infty}$, $\{Y_{d,k}^{\#}\}_{k=0}^{\infty}$ and $\{\bar{g}_{d,k}\}_{k=0}^{\infty}$, for $d \geq 1$, as follows. For the initial case $d = 1$, take

$$X_{1,k} = V_k + W_k, \quad X_{1,k}^{\#} = W_k^{\#},$$

where V_k, W_k and $W_k^{\#}$ are as in (4.46), (4.65) and respectively (4.66). Also define

$$g_{1,k} = -w_{k+2}(\cdot, 0, \cdot, 0),$$

with w_k as in Lemma 4.4.10. One can see from above that $\{X_{1,k}\}_{k=0}^{\infty}$, $\{X_{1,k}^{\#}\}_{k=0}^{\infty}$ and $\{g_{1,k}\}_{k=0}^{\infty}$ satisfy the assumption of Proposition 4.4.12 with $d = 1$.

Now let $d \geq 1$ be any, and suppose that $\{X_{d,k}\}_{k=0}^{\infty}$, $\{X_{d,k}^{\#}\}_{k=0}^{\infty}$ and $\{g_{d,k}\}_{k=0}^{\infty}$ are already given as in Proposition 4.4.12. Then we obtain $\{v_{d,k}\}_{k=0}^{\infty}$ and $\{\bar{v}_{d,k}\}_{k=0}^{\infty}$ from which one can define, as in (4.46),

$$V_{d,k} = \begin{cases} D_y^2 v_{d,0}, & k = 0, \\ D_y^2 v_{d,1} + D_{xy} v_{d,0}, & k = 1, \\ D_y^2 v_{d,k} + D_{xy} v_{d,k-1} + D_x^2 (v_{d,k-2} - \bar{v}_{d,k-2}), & k \geq 2, \end{cases}$$

and

$$\bar{g}_{d,k} = \bar{v}_{d,k}(\cdot, 0).$$

Then we set, for $k \geq 0$,

$$Y_{d,k} = \begin{cases} X_{d,k}, & 0 \leq k \leq 2d - 1, \\ X_{d,k} + V_{d,k-2d}, & k \geq 2d, \end{cases}$$

and

$$Y_{d,k}^{\#} = X_{d,k}^{\#}.$$

Then from the assumptions that $X_{d,k}, X_{d,k}^{\#} \in S(0, \bar{\alpha}; d, k)$ with $X_{d,k} - X_{d,k}^{\#} \in E(0, \bar{\alpha}; d, k)$ and the observation that $V_{d,k} \in E(0, \bar{\alpha}; d, k)$ it follows that $\{Y_{d,k}\}_{k=0}^{\infty}$, $\{Y_{d,k}^{\#}\}_{k=0}^{\infty}$ and $\{\bar{g}_{d,k}\}_{k=0}^{\infty}$ defined as above satisfy the conditions of Proposition 4.4.14. Thus, one obtains $\{w_{d,k}\}_{k=0}^{\infty}$, $\{w_{d,k}^{\#}\}_{k=0}^{\infty}$ and $\{\bar{u}_{d,k}\}_{k=0}^{\infty}$ as in the proposition.

With such a choice of higher order interior correctors and effective limit profiles, we set

$$W_{d,k} = \begin{cases} 0, & k = 0, 1, \\ D_y^2 w_{d,k} + D_{xy} w_{d,k-1} + D_x^2 (w_{d,k-2} + \bar{u}_{d,k-2}), & k \geq 2, \end{cases}$$

and its time-periodic version by

$$W^{\#}_{d,k} = \begin{cases} 0, & k = 0, 1, \\ D_y^2 w^{\#}_{d,k} + D_{xy} w^{\#}_{d,k-1} + D_x^2 (w^{\#}_{d,k-2} + \bar{u}_{d,k-2}), & k \geq 2. \end{cases}$$

Now define, for $k \geq 0$,

$$X_{d+1,k} = \begin{cases} Y_{d,k}, & 0 \leq k \leq 2d+1, \\ Y_{d,k} + W_{d,k-2d}, & k \geq 2d+2. \end{cases}$$

and respectively the time-periodic version by

$$X^{\#}_{d+1,k} = \begin{cases} Y^{\#}_{d,k} = X^{\#}_{d,k}, & 0 \leq k \leq 2d+1, \\ Y^{\#}_{d,k} + W^{\#}_{d,k-2d} = X^{\#}_{d,k} + W^{\#}_{d,k-2d}, & k \geq 2d+2, \end{cases}$$

as well as the new oscillatory initial data by

$$g_{d+1,k} = -w_{d,k+2}(\cdot, 0, \cdot, 0).$$

By means of $w_{d,k} \in S(2, \bar{\alpha}; d, k)$, $w^{\#}_{d,k} \in S(2, \bar{\alpha}; d, k)$ with $w_{d,k} - w^{\#}_{d,k} \in E(2, \bar{\alpha}; d, k)$, $\bar{u}_{d,k} \in S(d, k)$ and the assumptions on $Y_{d,k}$ and $Y^{\#}_{d,k}$, one can verify that $\{X_{d+1,k}\}_{k=0}^{\infty}$, $\{X^{\#}_{d+1,k}\}_{k=0}^{\infty}$ and $\{g_{d+1,k}\}_{k=0}^{\infty}$ also satisfy the assumptions of Proposition 4.4.12, which allows us to run an induction argument.

To this end, given $m \geq 4$ and $1 \leq d \leq \lfloor \frac{m}{2} \rfloor - 1$, we obtain $\tilde{v}^{\varepsilon}_{d,m}$, $\bar{g}^{\varepsilon}_{d,m}$ satisfying Proposition 4.4.12 (iii), and $\tilde{w}^{\varepsilon}_{d,m}$, $\tilde{w}^{\#,\varepsilon}_{d,m}$ satisfying Proposition 4.4.14 (iv). Following the arguments from Lemma 4.4.9 to Lemma 4.4.10, one can prove, as in the conclusion of Corollary 4.4.11, that

$$\left| u^{\varepsilon}_{d,m}(x,t) - \tilde{v}^{\varepsilon}_{d,m}(x,t) - \tilde{w}^{\varepsilon}_{d,m}(x,t) - \varepsilon^2 u^{\varepsilon}_{d+1,m}(x,t) \right| \leq C_{d,m} \varepsilon^{m-1}, \quad (4.87)$$

for all $x \in \mathbb{R}^n$ and $0 \leq t \leq T$, where we wrote by $u^{\varepsilon}_{r,m}$ for $r \in \{d, d+1\}$ by the bounded viscosity solution to

$$\begin{cases} \partial_t u^\varepsilon_{r,m} = \dfrac{1}{\varepsilon^{2r+2}} F\left(\varepsilon^{2r+2} D^2 u^\varepsilon_{r,m} + \displaystyle\sum_{k=0}^{m-2r} \varepsilon^k X_{r,k}, x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) & \\ \qquad\qquad - \dfrac{1}{\varepsilon^{2r+2}} F\left(\displaystyle\sum_{k=0}^{m-2r} \varepsilon^k X_{r,k}, x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) & \text{in } \mathbb{R}^n \times (0,T), \\ u^\varepsilon_{r,m}(x,0) = \displaystyle\sum_{k=0}^{m-2r} \varepsilon^k g_{r,k}\left(x, \frac{x}{\varepsilon}\right) & \text{on } \mathbb{R}^n. \end{cases} \tag{4.88}$$

It also follows immediately from the exponential decay estimates of $\tilde{v}^\varepsilon_{d,m}$ and $\tilde{w}^\varepsilon_{d,m} - \tilde{w}^\#_{d,m}$, which can be deduced by $v_{d,k} - \bar{v}_{d,k} \in E(2, \bar{\alpha}; d, k)$ and respectively $w_{d,k} - w^\#_{d,k} \in E(2, \bar{\alpha}; d, k)$ for all $k \geq 0$, that

$$\left| u^\varepsilon_{d,m}(x,t) - \tilde{w}^{\#,\varepsilon}_{d,m}(x,t) - \varepsilon^2 u^\varepsilon_{d+1,m}(x,t) \right| \leq C_{d,m} \varepsilon^{m-1}, \tag{4.89}$$

for all $x \in \mathbb{R}^n$ and $c_{d,m}\varepsilon^2 |\log \varepsilon| \leq t \leq T$.

Finally, we add up (4.87) side by side for all $1 \leq d \leq \lfloor \frac{m}{2} \rfloor - 1$ and combine it with (4.85). This yields

$$\left| u^\varepsilon(x,t) - \sum_{d=0}^{\lfloor \frac{m}{2} \rfloor - 1} \varepsilon^{2d} \left(\tilde{v}^\varepsilon_{d,m}(x,t) + \tilde{w}^\varepsilon_{d,m}(x,t) \right) - \varepsilon^{2\lfloor \frac{m}{2} \rfloor} u^\varepsilon_{\lfloor \frac{m}{2} \rfloor}(x,t) \right| \leq C_m \varepsilon^{m-1}.$$

Since the governing operator of the initial value problem (4.88) for $u^\varepsilon_{\lfloor \frac{m}{2} \rfloor}$ satisfies the zero source term condition in the sense of (4.9), and the associated initial data is bounded by some constant C_m, $u^\varepsilon_{\lfloor \frac{m}{2} \rfloor}$ as the bounded viscosity solution should also be bounded by the same constant C_m. Thus, we arrive at the global higher order convergence rate (4.2), as desired. The error estimate (4.3) away from the initial time layer can be derived similarly by adding (4.86) with (4.89) side by side for all $1 \leq d \leq \lfloor \frac{m}{2} \rfloor - 1$. This completes the proof. $\square$

4.5 Further Observations

This section is devoted to making some further observations on the (higher order) convergence rates for uniformly parabolic Cauchy problems. In Section 4.5.1, we obtain the higher order convergence rate for (4.4). In Section 4.5.2, we achieve the optimal convergence rate for (4.5) under some additional structure condition on the operator F and the initial data g.

4.5.1 Non-Oscillatory Initial Data and Higher Order Convergence Rate

Based on the construction of the higher order correctors, we are able to achieve the higher order convergence rate of the homogenization process of the problem (4.4). The iteration argument is basically the same with the proof of Theorem 4.1.1. The key difference here is that we begin with the higher order error correction in the interior, not near the initial time layer. This seems to be reasonable, since the initial data of (4.4) is not rapidly oscillatory.

The construction of the higher order interior correctors for (4.4) is essentially the same with Proposition 4.4.5, and has already been studied in the authors' previous work [34] in the framework of elliptic equations.

Proposition 4.5.1. *Assume that F satisfies* (4.6) - (4.8)*. Let $\{\bar{g}_k \in S(k)\}_{k=0}^{\infty}$ be a sequence of functions on $\mathbb{R}^n$. There exist a sequence $\{w_k \in S(2,\bar{\alpha};k)\}_{k=0}^{\infty}$ of periodic functions on $\mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times \mathbb{R}$ and a sequence $\{\bar{u}_k \in S(k)\}_{k=0}^{\infty}$ of functions on $\mathbb{R}^n \times [0,T]$ such that $\bar{u}_k(x,0) = \bar{g}_k(x)$, $w_0 = w_1 = 0$, and the following hold. Set*

$$\tilde{w}_m^{\varepsilon}(x,t) = \sum_{k=0}^{m} \varepsilon^k \left(w_k \left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) + \bar{u}_k(x,t) \right).$$

Then one has

$$\partial_t \tilde{w}_m^{\varepsilon} = F\left(D^2 \tilde{w}_m^{\varepsilon}, x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) + \psi_m^{\varepsilon}\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \quad \text{in } \mathbb{R}^n \times (0,T),$$

with some periodic ψ_m^{ε} satisfying

$$|\psi_m^{\varepsilon}(x,t,y,s)| \leq C_m \varepsilon^{m-1},$$

for any $0 < \varepsilon \leq \frac{1}{2}$.

Proof. The main difference of the proof here from that of Proposition 4.4.5 is that the function $\bar{u}_0$ in Lemma 4.4.8 is chosen by the solution to

$$\begin{cases} \partial_t \bar{u}_0 = \bar{F}(D^2 \bar{u}_0, x, t) & \text{in } \mathbb{R}^n \times (0,T), \\ \bar{u}_0(x,0) = \bar{g}_0(x) & \text{on } \mathbb{R}^n, \end{cases}$$

instead of a linear equation (5.81) for $k = 0$. It should be stressed that the matrix corrector X and the effective coefficient $\bar{A}$ are chosen to be the same as those in Section 4.4.2. We omit the rest of the proof to avoid redundant arguments. □

Equipped with Proposition 4.5.1 together with Proposition 4.4.1 and Proposition 4.4.5, we are ready to state and prove the higher order convergence rate regarding the homogenization problem of (4.4).

Proposition 4.5.2. *Assume that F satisfies* (4.6) - (4.8)*, and Let $g \in C^\infty(\mathbb{R}^n)$ be a function whose derivatives are bounded by K uniformly for all orders. Under these assumptions, let u^ε be the bounded viscosity solution to* (4.4) *for $\varepsilon > 0$. Then for each integer $d \geq 0$, there exist sequences $\{\tilde{v}_{d,k}\}_{k=0}^\infty$, $\{\tilde{w}_{d,k}\}_{k=0}^\infty$ of spatially periodic functions on $\mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,\infty)$ and a sequence $\{\tilde{w}^\#_{d,k}\}_{k=0}^\infty$ of periodic functions on $\mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times \mathbb{R}$ such that one has, for any $m \geq 2$, any $\varepsilon \leq \frac{1}{2}$, any $x \in \mathbb{R}^n$ and any $0 \leq t \leq T$,*

$$\left| u^\varepsilon(x,t) - \sum_{d=0}^{\lfloor \frac{m}{2} \rfloor - 1} \sum_{k=0}^{m-2d} \varepsilon^{k+2d} \left(\tilde{v}_{d,k}\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right) + \tilde{w}_{d,k}\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right)\right)\right|$$
$$\leq C_m \varepsilon^{m-1},$$

and in particular, for $c_m \varepsilon^2 |\log \varepsilon| \leq t \leq T$,

$$\left| u^\varepsilon(x,t) - \sum_{d=0}^{\lfloor \frac{m}{2} \rfloor - 1} \sum_{k=0}^{m-2d} \varepsilon^{k+2d} \tilde{w}^\#_{d,k}\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right)\right| \leq C_m \varepsilon^{m-1},$$

where c_m and C_m depend only on n, λ, Λ, α, m, T and K.

Proof. Let us fix $m \geq 2$. Due to Proposition 4.5.1, we derive that for any

$0 < \varepsilon \leq \frac{1}{2}$,

$$\left|u^\varepsilon(x,t) - \tilde{w}_m^\varepsilon(x,t) - \varepsilon^2 u_{0,m}^\varepsilon\right| \leq C_m \varepsilon^{m-1},$$

for any $x \in \mathbb{R}^n$ and any $0 \leq t \leq T$, where $u_{0,m}^\varepsilon$ is the bounded viscosity solution to to (4.88) for $d = 0$, $X_{0,k} = X_{0,k}^{\#} = W_k$, with W_k given as in (4.65). Thus, $u_{0,m}^\varepsilon$ falls under the setting of Proposition 4.4.12, and hence we may proceed as in the proof of Theorem 4.1.1 and achieve the desired estimates. This finishes the proof. □

4.5.2 General Fully Nonlinear Problem and Convergence Rate

Let us begin with a short overview the homogenization process of the problem (4.5), which can be found in [2] and [42]. First we make an additional assumption on F that there is $F_* : \mathcal{S}^n \times \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times \mathbb{R}$ for which

$$\varepsilon^2 F\left(\frac{1}{\varepsilon^2} P, x, t, y, s\right) \to F_*(P, x, t, y, s) \quad \text{as } \varepsilon \to 0, \tag{4.90}$$

locally uniformly for all $(P, x, t, y, s) \in (\mathcal{S}^n \setminus \{0\}) \times \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times \mathbb{R}$. Here F_* is called the recession operator (corresponding to F). It is clear from its definition that F_* also satisfies the conditions (4.6) - (4.7).

Following Lemma 4.3.1 and the comments above it, we obtain a (unique) function $v : \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ such that $v(x,t,\cdot,\cdot)$ is the spatially periodic solution to

$$\begin{cases} v_s = F_*(D_y^2 v, x, t, y, s) & \text{in } \mathbb{R}^n \times (0,\infty), \\ v(x,t,y,0) = g(y,x) & \text{on } \mathbb{R}^n. \end{cases} \tag{4.91}$$

for each $(x,t) \in \mathbb{R}^n \times [0,T]$. Also it induces $\bar{v} : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ given by

$$\bar{v}(x,t) = \lim_{s \to \infty} v(x,t,0,s). \tag{4.92}$$

On the other hand, let $\bar{F}$ and w be defined as in the beginning of Section 4.3.2. Under these circumstances, the ε-problem (4.5) is homogenized to the following effective problem

$$\begin{cases} \bar{u}_t = \bar{F}(D^2\bar{u}, x, t) & \text{in } \mathbb{R}^n \times (0,T), \\ \bar{u}(x,0) = \bar{g}(x) & \text{on } \mathbb{R}^n, \end{cases} \tag{4.93}$$

according to [2] and [42], in the sense that the viscosity solution u^ε of (4.5) converges to the viscosity solution $\bar{u}$ of (4.93) locally uniformly in $\mathbb{R}^n \times (0,T)$.

The following proposition gives the optimal rate of $u^\varepsilon \to \bar{u}$ under some additional assumptions.

Proposition 4.5.3. *Assume F and g verify* (4.6) - (4.8) *and* (4.10) - (4.11). *Suppose that F_* satisfies, with some $0 \leq \delta < 1$,*

$$|F(P,x,t,y,s) - F_*(P,x,t,y,s)| \leq K|P|^\delta, \tag{4.94}$$

and that v and $\bar{v}$ satisfy the conclusion of Proposition 4.3.7. Under these circumstances, let u^ε and $\bar{u}$ be the viscosity solutions to (4.5) *and, respectively,* (4.93). *Then there are positive constants c and C, depending only on n, λ, Λ, α, δ and K, such that for any $0 < \varepsilon \leq \frac{1}{2}$,*

$$|u^\varepsilon(x,t) - \bar{u}(x,t)| \leq C\varepsilon^{\min(1,2-2\delta)}, \tag{4.95}$$

for all $x \in \mathbb{R}^n$ and $c\varepsilon^2|\log\varepsilon| \leq t \leq T$.

Remark 4.5.4. *The inequality* (4.94) *implies that*

$$\left|\varepsilon^2 F\left(\frac{1}{\varepsilon^2}P, x, t, y, s\right) - F_*(P,x,t,y,s)\right| \leq K\varepsilon^{2-2\delta}|P|^\delta, \tag{4.96}$$

for any $(P,x,t,y,s) \in \mathcal{S}^n \times \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times \mathbb{R}$. In comparison of (4.96) *with* (4.95)*, we realize that the rate of $u^\varepsilon \to \bar{u}$ depends sensitively on the rate of* (4.90).

Remark 4.5.5. *The second additional assumption that v and $\bar{v}$ satisfy the assertion of Proposition 4.3.7 has been made because this assumption fails to hold for general F_*. The main reason is that nonlinear F_* is Lipschitz continuous (in the matrix variable P) at best, which prevents us from having Proposition 4.3.7. We shall provide some concrete example later in this regard.*

Proof of Proposition 4.5.3. Throughout this proof, we will write by c and C positive constants depending at most on n, λ, Λ, α, δ and K, and let them vary from one line to another.

Define $\tilde{v}^\varepsilon : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ by

$$\tilde{v}^\varepsilon(x,t) = v\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) - \bar{v}(x,t).$$

Since v and $\bar{v}$ are assumed to satisfy (4.36) for all $m \geq 0$, we observe that $\tilde{v}^\varepsilon$ is a (classical solution) to

$$\begin{cases} \tilde{v}^\varepsilon_t = F\left(D^2\tilde{v}^\varepsilon, x, t, \dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^2}\right) + \psi^\varepsilon\left(x, t, \dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^2}\right) & \text{in } \mathbb{R}^n \times (0,T), \\ \tilde{v}^\varepsilon(x,0) = g\left(x, \dfrac{x}{\varepsilon}\right) - \bar{v}(x,0) & \text{on } \mathbb{R}^n, \end{cases}$$

with $\psi^\varepsilon : \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ being defined by

$$\psi^\varepsilon(x,t,y,s) = F_*\left(\varepsilon^{-2}V_0, x, t, y, s\right) - F\left(\varepsilon^{-2}V_0 + \varepsilon^{-1}V_1 + V_2, x, t, y, s\right),$$

where

$$V_0 = D_y^2 v, \quad V_1 = D_{xy} v, \quad V_2 = D_x^2(v - \bar{v}).$$

One may notice that (4.36) implies that all V_0, V_1 and V_2 satisfy the exponential decay estimate. Thus, utilizing (4.94), we observe that

$$\left|F_*\left(\varepsilon^{-2}V_0, x, t, y, s\right) - F\left(\varepsilon^{-2}V_0, x, t, y, s\right)\right| \leq C\varepsilon^{-2\delta} e^{-\delta\beta s}, \tag{4.97}$$

for any $(x,y,s) \in \mathbb{R}^n \times \mathbb{R}^n \times [0,\infty)$ and any $0 < \varepsilon \leq \frac{1}{2}$. On the other hand, we have from (4.6) that

$$\left|F\left(\varepsilon^{-2}V_0, x, t, y, s\right) - F\left(\varepsilon^{-2}V_0 + \varepsilon^{-1}V_1 + V_2, x, t, y, s\right)\right| \leq C\varepsilon^{-1} e^{-\beta s}, \tag{4.98}$$

for any $(x,t,y,s) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,\infty)$ and any $0 < \varepsilon \leq \frac{1}{2}$. Combining (4.97) with (4.98), we arrive at

$$|\psi^\varepsilon(x,t,y,s)| \leq C\varepsilon^{-2\delta} e^{-\delta\beta s},$$

for any $(x,t,y,s) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,\infty)$ and any $0 < \varepsilon \leq \frac{1}{2}$.

Thus, arguing analogously as in the proof of Lemma 4.4.9, with the bounded viscosity solution $\tilde{u}^\varepsilon$ of

$$\begin{cases} \tilde{u}^\varepsilon_t = \dfrac{1}{\varepsilon^2} F\left(D^2\tilde{u}^\varepsilon + \varepsilon^{-2}V_0 x, t, \dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^2}\right) & \\ \qquad - \dfrac{1}{\varepsilon^2} F\left(\varepsilon^{-2}V_0, x, t, \dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^2}\right) & \text{in } \mathbb{R}^n \times (0,T), \\ \tilde{u}^\varepsilon(x,0) = \bar{g}(x) & \text{on } \mathbb{R}^n. \end{cases}$$

one has, for any $0 < \varepsilon \leq \frac{1}{2}$,

$$|u^\varepsilon(x,t) - \tilde{u}^\varepsilon(x,t)| \leq C\varepsilon^{2-2\delta}, \tag{4.99}$$

for all $x \in \mathbb{R}^n$ and all $c\varepsilon^2|\log\varepsilon| \leq t \leq T$.

On the other hand, we know that Proposition 4.3.12 is true under the assumptions (4.6) - (4.8) on F. Hence, it follows from the estimate (4.40) and the assumption (4.36), which holds also for $\bar{g}$, that the solution $\bar{u}$ to (4.93) satisfies $\bar{u} \in C^\infty(\mathbb{R}^n \times [0,T])$ and

$$\sum_{|\mu|+2\nu=l} |D^\mu_x \partial_t \bar{u}(x,t)| \leq C_l, \tag{4.100}$$

for any $l \geq 0$. Now let $w(x,t,\cdot,\cdot)$ be the unique periodic solution of

$$\begin{cases} w_s = F\left(D^2_y w + D^2_x\bar{u}(x,t), x, t, y, s\right) - \bar{F}(D^2_x\bar{u}(x,t), x, t) & \text{in } \mathbb{R}^n \times \mathbb{R} \\ w(x,t,0,0) = 0, \end{cases}$$

for each $(x,t) \in \mathbb{R}^n \times [0,T]$. Due to Proposition 4.3.12 and (4.100), we have $w \in C^\infty(\mathbb{R}^n \times [0,T]; C^{2,\hat{\alpha}}(\mathbb{R}^n \times \mathbb{R}))$, for any $0 < \hat{\alpha} < \bar{\alpha}$, and

$$\sum_{|\mu|+2\nu=l} \|D^\mu_x \partial^\nu_t w(x,t,\cdot,\cdot)\|_{C^{2,\bar{\alpha}}(\mathbb{R}^n\times\mathbb{R})} \leq C_l,$$

for any $l \geq 0$.

Therefore, arguing as above, we observe that the function $\tilde{w}^\varepsilon : \mathbb{R}^n \times [0,T] \to \mathbb{R}$, defined by

$$\tilde{w}^\varepsilon(x,t) = \bar{u}(x,t) + \varepsilon^2 w\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right),$$

solves

$$\begin{cases} \tilde{w}^{\varepsilon}_t = F\left(D^2\tilde{w}^{\varepsilon}, x, t, \dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^2}\right) + \psi^{\varepsilon}\left(x, t, \dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^2}\right) & \text{in } \mathbb{R}^n \times (0,T), \\ \tilde{w}^{\varepsilon}(x,0) = \bar{g}(x) & \text{on } \mathbb{R}^n, \end{cases}$$

for some $\psi^{\varepsilon} : \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ satisfying

$$|\psi^{\varepsilon}(x,t,y,s)| \leq C\varepsilon,$$

for any $(x,t,y,s) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times \mathbb{R}$ and any $0 < \varepsilon \leq \frac{1}{2}$.

Now we may proceed as in the proof of Lemma 4.4.10 and deduce that

$$\left| u^{\varepsilon}(x,t) - \tilde{w}^{\varepsilon}\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) - \varepsilon^2 u^{\varepsilon}_1(x,t) \right| \leq C\varepsilon, \tag{4.101}$$

for all $x \in \mathbb{R}^n$ and all $0 \leq t \leq T$, provided $0 < \varepsilon \leq \frac{1}{2}$, where u^{ε}_1 is the bounded viscosity solution of (4.79); we would like to focus on the fact that the governing operator of (4.79) has zero source term in the sense of (4.9), and the initial data of (4.79) is bounded. Thus, u^{ε}_1 is bounded globally, especially independent of ε. Finally, the error estimate (4.95) can be deduced by combining (4.99) and (4.101). □

Let us finish this subsection with an example that reveals that the assumptions of Proposition 4.5.3 are satisfied for certain F and g.

Example 4.5.6. *Let F_* be independent of (x,t,y,s) and satisfy $F_*(P) < -F_*(-P)$ for any nonzero matrix $P \in \mathcal{S}^n$. For instance, one may take F_* by Pucci's minimal operator for the lower ellipticity bound $\lambda' > \lambda$ and the upper ellipticity bound $\Lambda' < \Lambda$. On the other hand, let g be given by $g(x,y) = \psi(x)\phi(y)$ on $\mathbb{R}^n \times \mathbb{R}^n$, with ϕ being a smooth periodic function and ψ being a smooth bounded function.*

Let us write by $F_-(P)$ and $F_+(P)$ the functionals $F_*(P)$ and, respectively, $-F_*(-P)$, and consider the spatially periodic Cauchy problem,

$$\begin{cases} \partial_s v_{\pm} = F_{\pm}(D^2_y v_{\pm}) & \text{in } \mathbb{R}^n \times (0,\infty), \\ v_{\pm}(y,0) = \phi(y) & \text{on } \mathbb{R}^n. \end{cases}$$

According to Lemma 4.3.1, there are unique real numbers γ_+ and γ_- such that $\gamma_{\pm} = \lim_{s\to\infty} v_{\pm}(0,s)$. Notice that $v_{\pm} \in C^{2,\alpha}$ for some $0 < \alpha < 1$ depending *only on n, λ and Λ, owing to the convexity of F_+ and the concavity of F_-.*

Let us observe that $\gamma_+ > \gamma_-$. First it follows from the comparison principle that $v_+ > v_-$ in $\mathbb{R}^n \times (0, \infty)$, which implies $\gamma_+ \geq \gamma_-$. Moreover, since $F_+(P) > F_-(P)$ for any nonzero $P \in \mathcal{S}^n$, the function $w = v_+ - v_-$ solves

$$\partial_s(v_+ - v_-) \geq \mathrm{tr}(A(y,s)D_y^2(v_+ - v_-)) \quad \textit{in } \mathbb{R}^n \times (0, \infty),$$

where A is the linearized coefficient associated with F_+. This implies that the function $W(s) = \min_{\mathbb{R}^n}(v_+(\cdot, s) - v_-(\cdot, s))$ is non-decreasing for $s > 0$, whence we have $\gamma_+ > \gamma_-$.

Now let v be the solution to (4.91). Then the uniqueness of v implies that $v(x,y,s) = \psi(x)v_-(y,s)$ if $\psi(x) \geq 0$ and $v(x,y,s) = \psi(x)v_+(y,s)$ if $\psi(x) \leq 0$. This also implies that the function $\bar{v}$ defined by (4.92) satisfies $\bar{v}(x) = \gamma_+\psi(x)$ if $\psi(x) \geq 0$ and $\bar{v}(x) = \gamma_-\psi(x)$ if $\psi(x) \leq 0$.

This implies that if ψ changes sign at some point, then v and $\bar{v}$ are not even differentiable at that point. On the other hand, we have v and $\bar{v}$ satisfying the conclusion of Proposition 4.3.7, provided that ψ is either uniformly positive or uniformly negative.

Chapter 5

Higher Order Convergence Rates in Theory of Homogenization: Viscous Hamilton-Jaboci Equations

5.1 Introduction

This paper concerns the higher order convergence rates of the homogenization of viscous Hamilton-Jacobi equations. The model problem is of the form,

$$\begin{cases} u_t^\varepsilon - \varepsilon \operatorname{tr}\left(A\left(\frac{x}{\varepsilon}\right) D^2 u^\varepsilon\right) + H\left(Du^\varepsilon, \frac{x}{\varepsilon}\right) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u^\varepsilon = g & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{5.1}$$

Here the diffusion matrix A is periodic and uniformly elliptic, and the Hamiltonian H is periodic in the spatial variable while it is convex and grows quadratically in the gradient variable. The initial data g will be chosen to have smooth solutions for the effective Hamilton-Jacobi equation. At the end of this paper, we shall extend the result to the fully nonlinear, viscous Hamilton-Jacobi equation in the form of

$$\begin{cases} u_t^\varepsilon + H\left(\varepsilon D^2 u^\varepsilon, Du^\varepsilon, \frac{x}{\varepsilon}\right) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u^\varepsilon = g & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{5.2}$$

This paper is in the sequel of the authors' previous works [34] and [35], where the higher order convergence rates were achieved in the periodic homogenization of fully nonlinear, uniformly elliptic and parabolic, second order PDEs. We found it interesting in the previous works that even if we begin with a nonlinear PDE at the first order approximation, we no longer encounter such a nonlinear structure in the second and the higher order approximations. Instead, we always obtain a linear PDE with an external source term, which can be interpreted as the nonlinear effect coming from the error that is left undetected in the previous step of the approximation.

The previous papers were concerned with uniformly elliptic (or parabolic) PDEs that are nonlinear in the second order derivatives, where the nonlinear perturbation is still made in the same order of the linear structure. A key difference in the current paper is that we impose a nonlinear structure (in the gradient term) that has quadratic growth at the infinity, so that this nonlinearity cannot be attained by order 1 perturbations of a linear structure. We believe that the quadratic growth condition can be generalized to superlinear growth condition, only if the solution of the corresponding effective problem is smooth enough.

Another interesting fact we found in studying Hamilton-Jacobi equations is that the geometric shape of the initial data turns out to play an important role in achieving higher order convergence rates. In particular, what we observe in this paper is that the geometric shape of the initial data has to be selected according to the nonlinear structure of the effective Hamiltonian, which to the best of our knowledge has not yet been addressed in any existing literature. The main reason for this requirement is to ensure the solution of the effective problem to be sufficiently smooth such that one can proceed with the approximation as much as one desires.

In this paper, we establish higher order convergence rates when the initial data is convex, while the Hamiltonian is convex. However, a natural question is if one can generalize one of these structure conditions, which seems to be an interesting yet challenging problem. We shall come back to this in the forthcoming paper.

The periodic homogenization of (viscous) Hamilton-Jacobi equations is by now considered to be standard, and one may consult the classical materials [21] and [39] for a rigorous justification. For the notion of viscosity solutions and the standard theory in this framework we refer to [9] and [20].

For the recent development in the rate of convergence in periodic homogenization of (viscous) Hamilton-Jacobi equations, we refer to [14], [17],

[43], [47] and the references therein. Nevertheless, this is the first work on the higher order convergence rates in the regime of (viscous) Hamilton-Jacobi equations. For the higher order convergence rates for other type of equations, we refer to [34], [35] and the references therein.

The paper is organized as follows. In Section 5.2, we introduce basic notation used throughout this paper, and list up the standing assumptions regarding the main problem (5.1). From Section 5.3 to Section 5.5, we are concerned with the homogenization problem of (5.1). In Section 5.3, we summarize some standard results on the cell problem and the effective Hamiltonian. In Section 5.4, we establish the regularity theory of interior correctors in the slow variable. Based on this regularity theory, we construct the higher order interior correctors in Section 5.5 and prove Theorem 5.5.6, which is the first main result. Finally in Section 5.6, we generalize this result to the homogenization of (5.2), and prove Theorem 5.6.6, which is the second main result.

5.2 Notation and Standing Assumptions

Throughout the paper, we set $n \geq 1$ to be the spatial dimension. The parameters λ, Λ, α, α', β, β', K, L, and $\bar{\mu}$ will be fixed positive constants, unless stated otherwise. By $\mathbb{Z}^n$ we denote the space of n-tuple of integers. By $\mathcal{S}^n$ we denote the space of all symmetric $n \times n$ matrices.

Definition 5.2.1. *Given $k, l \geq 0$ integers, $0 < \mu \leq 1$ real number, X and Y metric spaces, we define $C^l(X; C^{k,\mu}(Y))$ by the space of functions $f = f(x,y)$ on $X \times Y$ satisfying the following.*

(i) $f(\cdot, y) \in C^l(X)$ for all $y \in Y$.

(ii) $\{D_x^m f(x,\cdot)\}_{x\in X}$ is uniformly bounded in $C^{k,\mu}(Y)$ for any $0 \leq m \leq l$.

(iii) Given any sequence $x_k \to x$ in X, one has $D_x^m f(x_k,\cdot) \to D_x^m f(x,\cdot)$ in $C^{k,\mu}(Y)$ for any $0 \leq m \leq l$.

From Section 5.3 to Section 5.5, we study the higher order convergence rates in homogenization of (5.1). Throughout these sections, we assume that the diffusion matrix A satisfies the following, for any $y \in \mathbb{R}^n$.

(i) A is periodic:

$$A(y + k) = A(y). \tag{5.3}$$

(ii) A is uniformly elliptic:

$$\lambda I \leq A(y) \leq \Lambda I. \tag{5.4}$$

(iii) $A \in C^{0,1}(\mathbb{R}^n)$ and

$$\|A\|_{C^{0,1}(\mathbb{R}^n)} \leq K. \tag{5.5}$$

On the other hand, we shall assume that the Hamiltonian H verifies the following, for any $(p, y) \in \mathbb{R}^n \times \mathbb{R}^n$.

(i) H is periodic in y:

$$H(p, y + k) = H(p, y), \tag{5.6}$$

for any $k \in \mathbb{Z}^n$.

(ii) H has quadratic growth in p:

$$\alpha|p|^2 - \alpha' \leq H(p, y) \leq \beta|p|^2 + \beta'. \tag{5.7}$$

(iii) H is convex in p:

$$H(tp + (1-t)q, y) \leq tH(p, y) + (1-t)H(q, y), \tag{5.8}$$

for any $0 \leq t \leq 1$ and any $q \in \mathbb{R}^n$.

(iv) $H \in C^\infty(\mathbb{R}^n; C^{0,1}(\mathbb{R}^n))$ and

$$\left\| D_p^k H(p, \cdot) \right\|_{C^{0,1}(\mathbb{R}^n)} \leq K\left(1 + |p|^{(2-k)_+}\right), \tag{5.9}$$

for any nonnegative integer k.

The assumptions on the initial data g will be given in the beginning of Section 5.5, since we need to derive the effective Hamiltonian beforehand. On the other hand, the structure conditions for (5.2) will be given in the beginning of Section 5.6.

5.3 Preliminaries

Let us begin with the well-known cell problem for our model equation (5.1), stated as below. This lemma is by now considered to be standard (for instance, see [21] and [22]), since the diffusion coefficient A is uniformly elliptic and the Hamiltonian H is convex. Nevertheless, we shall present a proof for the reader's convenience.

Lemma 5.3.1. *For each $p \in \mathbb{R}^n$, there exists a unique real number, γ, for which the following PDE,*

$$-\operatorname{tr}(A(y)D^2w) + H(Dw + p, y) = \gamma \quad \text{in } \mathbb{R}^n, \tag{5.10}$$

has a periodic viscosity solution $w \in C^{2,\mu}(\mathbb{R}^n)$ for any $0 < \mu < 1$. Moreover, we have

$$\alpha|p|^2 - \alpha' \leq \gamma \leq \beta|p|^2 + \beta'. \tag{5.11}$$

Furthermore, a periodic solution w of (5.10) is unique up to an additive constant, and satisfies

$$\|w - w(0)\|_{L^\infty(\mathbb{R}^n)} + \|Dw\|_{C^{1,\mu}(\mathbb{R}^n)} \leq C_{|p|}, \tag{5.12}$$

where $C > 0$ depends only on n, λ, Λ, α, α', β, β', K, μ and $|p|$.

We shall divide the proof into two steps. The first step concerns the approximating problem and a uniform Lipschitz estimate.

Lemma 5.3.2. *For each $p \in \mathbb{R}^n$ and $\delta > 0$, there exists a unique periodic viscosity solution $w^\delta \in C^{0,1}(\mathbb{R}^n)$ to*

$$-\operatorname{tr}(A(y)D^2w^\delta) + H(Dw^\delta + p, y) + \delta w^\delta = 0 \quad \text{in } \mathbb{R}^n, \tag{5.13}$$

which satisfies

$$-\beta|p|^2 - \beta' \leq \delta \left\| w^\delta \right\|_{L^\infty(\mathbb{R}^n)} \leq -\alpha|p|^2 + \alpha', \tag{5.14}$$

and a uniform Lipschitz estimate

$$\left\| Dw^\delta \right\|_{L^\infty(\mathbb{R}^n)} \leq C(1 + |p|), \tag{5.15}$$

where $C > 0$ depends only on λ, Λ, α, α' and K.

Proof. Due to (5.7), we know that $-\delta(\alpha|p|^2 - \alpha')$ and $-\delta(\beta|p|^2 + \beta')$ are a supersolution and, respectively, a subsolution of (5.13). Thus, the comparison principle yields a unique viscosity solution, w^δ, of (5.13), satisfying (5.14). The uniqueness of w^δ implies its periodicity, that is, $w^\delta(y + k) = w^\delta(y)$ for all $y \in \mathbb{R}^n$ and all $k \in \mathbb{Z}^n$.

Hence, we are only left with proving the uniform Lipschitz estimate (5.15). We shall show it first under the claim that $w^\delta \in C^3(\mathbb{R}^n)$, and then justify this claim at the end of the proof.

Under the assumption that $w^\delta \in C^3(\mathbb{R}^n)$, let us differentiate (5.13) in y and take the inner product with Dw, which yields a uniformly elliptic PDE for the function $v = |Dw^\delta|^2$,

$$-2\,\mathrm{tr}(A(y)[D^2 w^\delta(y)]^2) - \mathrm{tr}(A(y)D^2 v) + B(y)\cdot Dv + 2E(y)\cdot Dw^\delta(y) + 2\delta v = 0, \tag{5.16}$$

in $\mathbb{R}^n$, where $B(y) = D_p H(Dw^\delta(y) + p, y)$ and $E(y) = D_y H(Dw^\delta(y) + p, y)$.

Since v is periodic and continuous, v achieves a global maximum at some point $y_0 \in \mathbb{R}^n$. Denote

$$M = D^2 w^\delta(y_0) \quad \text{and} \quad q = Dw^\delta(y_0).$$

Then it follows from (5.16) that

$$-\,\mathrm{tr}(A(y_0)M^2) + E(y_0)\cdot q \leq 0. \tag{5.17}$$

However, due to the ellipticity condition (5.4), we have

$$-\,\mathrm{tr}(A(y_0)M^2) \geq \lambda |M|^2. \tag{5.18}$$

On the other hand, it follows from the regularity assumption (5.9) that

$$|E(y_0)| \leq K(1 + |M| + |p+q|^2). \tag{5.19}$$

Inserting (5.18) and (5.19) into (5.17), we obtain

$$|M|^2 \leq \frac{K}{\lambda}(1 + |M| + |p+q|^2)|q|. \tag{5.20}$$

By means of Young's inequality $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon}b^2$, one may continue with the estimation in (5.20) as

$$|M|^2 \leq C_1(1 + |q| + |q|^2 + |p|^2)|q|,$$

where C_1 depends only on λ and K.

Let us return to (5.13). Due to (5.4), (5.7) and (5.14), the PDE (5.13) evaluated at y_0 becomes

$$\alpha|q+p|^2-\alpha' \leq H(q+p,y_0) = \mathrm{tr}(A(y_0)M)-\delta w(y_0) \leq \Lambda|M|+\beta|p|+\beta'. \quad (5.21)$$

After some manipulation using Young's inequality and (5.21), we obtain that

$$|q|^2 \leq C_2\left(1+|p|^2\right) + C_1\sqrt{(1+|q|+|q|^2)|q|}, \quad (5.22)$$

where $C_2 > 0$ depends only on n, λ, Λ, α, α', β and β'.

Let $C_3 > 0$ be such that

$$(1+t+t^2)t \leq \frac{1}{4C_1}t^4 \quad \text{for all } t \geq C_3,$$

in which case C_3 depends only on C_1, whence on λ and K only. Then we conclude from (5.22) that

$$|q|^2 \leq \max\left\{2C_2(1+|p|^2), C_3^2\right\} \leq C_4(1+|p|^2),$$

where $C_4 = \max\{2C_2, C_3^2\}$, proving the uniform Lipschitz estimate (5.15).

Thus, we are left with proving that any viscosity solution w^δ of (5.13) belongs to the class $C^3(\mathbb{R}^n)$. As a matter of fact, it is sufficient to show that any such viscosity solution belongs to the class $C^{2,\mu}(\mathbb{R}^n)$, for some $0 < \mu < 1$, since improving the regularity from $C^{2,\mu}$ to C^3 follows immediately from a bootstrap argument; for instance, see Theorem 2.2.9.

By means of the weak Bernstein method [6], we know at least that w^δ is locally Lipschitz in $\mathbb{R}^n$. Thus, for each ball B_R, w can be viewed as the viscosity solution of

$$\mathrm{tr}(A(y)D^2w^\delta) = f(y),$$

with $f = -H(Dw^\delta + p, \cdot) + \delta w^\delta \in L^\infty(B_R)$. Hence, it follows from the $C^{1,\mu}$ estimate (Theorem 2.2.7 (a)) that $w \in C^{1,\mu}_{loc}(B_{R/2})$ for some $0 < \mu < 1$. Thus, $f \in C^\mu(B_{R/2})$, and under assumption (ii) on H in this lemma, we can apply the $C^{2,\nu}$ estimate (Theorem 2.2.7 (b)) and obtain $w^\delta \in C^{2,\nu}(B_{R/4})$ for some $0 < \nu < \min\{\mu, \bar{\mu}\}$. Since this holds for any $R > 0$, we conclude that $w^\delta \in C^{2,\nu}_{loc}(\mathbb{R}^n)$, from which one can improve the regularity (Theorem 2.2.9) so that $w \in C^3(\mathbb{R}^n)$. This finishes the proof. □

With the uniform Lipschitz estimate for the approximating solution w^δ, we can finish the proof of Lemma 5.3.1.

Proof of Lemma 5.3.1. Throughout the proof, $C_{|p|}$ will denote a positive, generic constant that depends at most on n, λ, Λ, α, α', β, β', K, μ and $|p|$, unless stated otherwise. Moreover, we shall fix $0 < \mu < 1$.

Let $p \in \mathbb{R}^n$ be given. We know *a priori* that periodic viscosity solutions of (5.10), if any, are unique up to an additive constant. Suppose that w' is another periodic viscosity solution of (5.10). Then $v = w - w'$ satisfies the following linearized equation,

$$-\operatorname{tr}(A(y)D^2 v) + B(y) \cdot Dv = 0 \quad \text{in } \mathbb{R}^n,$$

where $B(y) = \int_0^1 D_p H(tD_y w + (1-t)D_y w' + p, y)dt$. Now that v is bounded, we deduce from the Liouville theorem that v is a constant function on $\mathbb{R}^n$.

Henceforth, we prove the existence of a unique real number, γ, such that the cell problem (5.10) admits a periodic viscosity solution. Let $w^\delta \in C^{0,1}(\mathbb{R}^n)$ be the unique periodic viscosity solution of (5.13), satisfying (5.14) and the uniform Lipschitz estimate (5.15).

By periodicity and (5.15), we have $\operatorname{osc}_{\mathbb{R}^n} w^\delta \leq C(1+|p|)$. This also yields that $w^\delta - w^\delta(0) \in C^{0,1}(\mathbb{R}^n)$ and

$$\left\| w^\delta - w^\delta(0) \right\|_{L^\infty(\mathbb{R}^n)} + \left\| Dw^\delta \right\|_{L^\infty(\mathbb{R}^n)} \leq C(1+|p|).$$

Due to (5.7), (5.14) and (5.15), we know that

$$\left\| H(Dw^\delta + p, \cdot) + \delta w^\delta \right\|_{L^\infty(\mathbb{R}^n)} \leq C(1+|p|)^2.$$

Considering the second and the third terms on the left hand side of (5.13) as an external force, we may apply the interior $C^{1,\mu}$ estimate (Theorem 2.2.7 (a)) in a ball such that the concentric ball with half the radius contains the periodic cell, and then use the periodicity of w^δ to derive that $w^\delta - w^\delta(0) \in C^{1,\mu}(\mathbb{R}^n)$ and

$$\left\| w^\delta - w^\delta(0) \right\|_{L^\infty(\mathbb{R}^n)} + \left[Dw^\delta \right]_{C^\mu(\mathbb{R}^n)} \leq C_{|p|}.$$

Now the $C^{1,\mu}$ regularity of w^δ yields that

$$\left\| H(Dw^\delta + p, \cdot) + \delta w^\delta \right\|_{C^\mu(\mathbb{R}^n)} \leq C_{|p|}.$$

Hence, it follows from the interior $C^{2,\mu}$ estimates (Theorem 2.2.7 (b)) (again we choose a large ball containing the periodic cell, as we did when applying the interior $C^{1,\mu}$ estimates) and the periodicity of w^δ that $w^\delta - w^\delta(0) \in C^{2,\mu}(\mathbb{R}^n)$ and

$$\left\| w^\delta - w^\delta(0) \right\|_{L^\infty(\mathbb{R}^n)} + \left\| Dw^\delta \right\|_{C^{1,\mu}(\mathbb{R}^n)} \leq C_{|p|}. \tag{5.23}$$

Due to the compactness of both of the sequences $\{w^\delta - w^\delta(0)\}_{\delta>0}$ and $\{-\delta w^\delta\}_{\delta>0}$ in $C^{2,\mu}(\mathbb{R}^n)$, we know that $w^\delta - w^\delta(0) \to w$ and $-\delta w^\delta \to \gamma$ in $C^{2,\mu'}(\mathbb{R}^n)$, for any $0 < \mu' < \mu$, for some $w \in C^{2,\mu}(\mathbb{R}^n)$ and some $\gamma \in \mathbb{R}$, along a subsequence. Now that viscosity solutions are stable under the uniform convergence, we know that w is a viscosity solution of (5.10) with the limit γ on the right hand side. This proves the existence part of Lemma 5.3.1.

To investigate the uniqueness of γ, we suppose towards a contradiction that there is another real number γ', corresponding to the same p, such that (5.10) has a periodic viscosity solution, say w'. Without losing any generality, let us assume $\gamma > \gamma'$. Then it is easy to see that w' is a strict subsolution of (5.10). However, due to the periodicity of $w' - w$, $w' - w$ attains a local maximum at some point, whence we arrive at a contradiction. Thus, γ must be unique.

The inequality (5.11) follows immediately from the inequality (5.14) and the fact that $-\delta w^\delta \to \gamma$ uniformly in $\mathbb{R}^n$. To see that the estimate (5.12) holds, we first observe from the convergence of $w^\delta - w^\delta(0) \to w$ in $C^{2,\mu'}(\mathbb{R}^n)$, for any $0 < \mu' < \mu$, and the estimate (5.23) that $w \in C^{2,\mu}(\mathbb{R}^n)$ and satisfies (5.12). Note that we used $w(0) = 0$, which follows from the construction of w. Now if w' is another periodic viscosity solution of (5.24), then due to the uniqueness that we have shown in the beginning of this proof, we have $w' - w'(0) = w$. Therefore, w' satisfies (5.12), which completes the proof of this lemma. □

Due to the uniqueness of γ in Lemma 5.3.1, we may define a functional $\bar{H} : \mathbb{R}^n \to \mathbb{R}$ in such a way that for each $p \in \mathbb{R}^n$, $\bar{H}(p)$ is the unique real number for which the following PDE,

$$-\operatorname{tr}(A(y)D^2 w) + H(Dw + p, y) = \bar{H}(p) \quad \text{in } \mathbb{R}^n, \tag{5.24}$$

has a periodic solution in $C^{2,\mu}(\mathbb{R}^n)$ (for any $0 < \mu < 1$). Moreover, the second part of Lemma 5.3.1 yields a functional $w : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that for each $p \in \mathbb{R}^n$, $w(p, \cdot) \in C^{2,\mu}(\mathbb{R}^n)$ (for any $0 < \mu < 1$) is the unique periodic viscosity solution of (5.24) that is normalized so as to satisfy

$$w(p,0) = 0. \tag{5.25}$$

Let us list up some basic properties of $\bar{H}$ that were already found in [22]. We provide the proof for the sake of completeness.

Lemma 5.3.3. *$\bar{H}$ satisfies the following properties.*

(i) $\bar{H}$ has the same quadratic growth as that of H:

$$\alpha|p|^2 - \alpha' \leq \bar{H}(p) \leq \beta|p|^2 + \beta', \tag{5.26}$$

for any $p \in \mathbb{R}^n$.

(ii) $\bar{H}$ is also convex:

$$\bar{H}(tp + (1-t)q) \leq t\bar{H}(p) + (1-t)\bar{H}(q), \tag{5.27}$$

for any $0 \leq t \leq 1$, and any $p, q \in \mathbb{R}^n$.

(iii) $\bar{H} \in C^{0,1}_{loc}(\mathbb{R}^n)$ and

$$|\bar{H}(p) - \bar{H}(q)| \leq C(1 + |p| + |q|)|p - q|, \tag{5.28}$$

where $C > 0$ depends only on n, λ, Λ, α, α', β, β' and K.

Proof. Notice that (5.26) follows immediately from (5.11) and (5.24). Thus, we shall only prove (ii) and (iii).

For the notational convenience, let us write $w_p(y) = w(p, y)$. To prove (5.27), we assume to the contrary that there are some $p, q \in \mathbb{R}^n$ and $0 < t < 1$ such that

$$t\bar{H}(p) + (1-t)\bar{H}(q) < \bar{H}(tp + (1-t)q). \tag{5.29}$$

For the notational convenience, let us write $r = tp + (1-t)q$ and $\tilde{w}_r = tw_p + (1-t)w_q$. Then due to (5.29) and (5.8), one can easily deduce that $\tilde{w}_r$ is a periodic viscosity solution of

$$-\operatorname{tr}(A(y)D^2\tilde{w}_r) + H(D\tilde{w}_r + r, y) < \bar{H}(r) \quad \text{in } \mathbb{R}^n.$$

In other words, $\tilde{w}_r$ is a strict viscosity subsolution of the PDE for w_r which is precisely the cell problem (5.10) with $p = r$. Therefore, it follows from the comparison principle that $\tilde{w}_r - w_r$ cannot attain any local maximum. However, as $\tilde{w}_r - w_r$ being a non-constant continuous periodic function, it surely attains local maximum at some point, whence we arrive at a contradiction. Therefore, we must have (5.27) for any $0 \leq t \leq 1$ and any $p, q \in \mathbb{R}^n$.

Finally let us prove (5.28). To do so, we go back to the penalized problem (5.13). Analogous with the notation w_p, let us denote by w_p^δ the unique viscosity solution of (5.13) corresponding to p. Due to the uniform gradient estimate (5.15) and the regularity assumption (5.9), we have

$$|H(Dw_p^\delta + p, y) - H(Dw_p^\delta + q, y)| \leq C(1 + |p| + |q|)|p - q|,$$

where $C > 0$ depends only on n, λ, Λ, α, α', β, β' and K. Therefore, we have

$$-\operatorname{tr}(A(y)D^2 w_p^\delta) + H(Dw_p^\delta + q, y) + \delta w_p^\delta \leq C(1 + |p| + |q|)|p - q| \quad \text{in } \mathbb{R}^n,$$

in the viscosity sense. In other words, $w_p^\delta - \delta^{-1}C(1 + |p| + |q|)|p - q|$ is a viscosity subsolution of (5.13) with p replaced by q. Hence, it follows from the comparison principle that

$$\delta w_p^\delta - \delta w_q^\delta \leq C(1 + |p| + |q|)|p - q|,$$

on $\mathbb{R}^n$. Passing to the limit $\delta \to 0$ in the last inequality, we arrive at

$$\bar{H}(p) - \bar{H}(q) \leq C(1 + |p| + |q|)|p - q|$$

Similarly, one may also obtain that

$$\bar{H}(q) - \bar{H}(p) \leq C(1 + |p| + |q|)|p - q|,$$

proving (5.28). This completes the proof of Lemma 5.3.3. □

5.4 Regularity in Slow Variables

In this section, we shall investigate the regularity of $\bar{H}$ and w in the slow variable p. Such a regularity has been established in the authors' previous works [34] and [35], for fully nonlinear elliptic and, respectively, parabolic PDEs. Let us first observe the continuity of w in p variable.

Lemma 5.4.1. *$w \in C(\mathbb{R}^n; C^{2,\mu}(\mathbb{R}^n))$, for any $0 < \mu < 1$, and given $L > 0$ and $p \in B_L$, one has*

$$\|w(p,\cdot)\|_{L^\infty(\mathbb{R}^n)} + \|D_y w(p,\cdot)\|_{C^{1,\mu}(\mathbb{R}^n)} \leq C_L \tag{5.30}$$

where $C > 0$ depends only on n, λ, Λ, α, α', β, β', K, μ and L.

Proof. Let us fix $0 < \mu < 1$. The estimate (5.30) follows immediately from (5.12) and the choice of w that $w(p,0) = 0$. Thus, we prove that w is continuous in p variable with respect to the $C^{2,\mu}$ norm in y variable.

Let $\{p_k\}_{k=1}^\infty$ be a sequence of vectors in $\mathbb{R}^n$ converging to some $p_0 \in \mathbb{R}^n$ as $k \to \infty$. Let us write, for the notational convenience, $w_k(y) = w(p_k, y)$ and $\gamma_k = \bar{H}(p_k)$ for $k = 0, 1, 2, \cdots$. We already know from (5.28) that $\gamma_k \to \gamma_0$ as $k \to \infty$. Hence, it suffices to prove that $w_k \to w_0$ in $C^{2,\mu'}(\mathbb{R}^n)$ as $k \to \infty$, for any $0 < \mu' < \mu$.

Due to (5.30), we know that $\{w_k\}_{k=1}^\infty$ is uniformly bounded in $C^{2,\mu}(\mathbb{R}^n)$, for any $0 < \mu < 1$. Now that w_k is periodic for all $k = 1, 2, \cdot, s$, the Arzela-Ascoli theorem yields that for any subsequence $\{v_k\}_{k=1}^\infty \subset \{w_k\}_{k=1}^\infty$ there are a further subsequence $\{v_{k_i}\}_{i=1}^\infty$ and a periodic function $v \in C^{2,\mu}(\mathbb{R}^n)$ such that $v_{k_i} \to v$ in $C^{2,\mu}(\mathbb{R}^n)$, for any $0 < \mu < 1$, as $i \to \infty$. Now that $p_{k_i} \to p_0$ and $\gamma_{k_i} \to \gamma_0$ as $i \to \infty$, we deduce from the stability of viscosity solutions that v and γ_0 satisfies

$$-\operatorname{tr}(A(y)D^2 v) + H(Dv + p_0, y) = \gamma_0 \quad \text{in } \mathbb{R}^n.$$

Since $v(0) = 0$, the second part of Lemma 5.3.1 implies that $v = w_0$. This shows that any subsequence of $\{w_k\}_{k=1}^\infty$ contains a further subsequence that converges to w_0 in $C^{2,\mu'}(\mathbb{R}^n)$, for any $0 < \mu' < \mu$. Moreover, we know from $v \in C^{2,\mu}(\mathbb{R}^n)$ that $w_0 \in C^{2,\mu}(\mathbb{R}^n)$ as well. Therefore, $w_k \to w_0$ in $C^{2,\mu'}(\mathbb{R}^n)$, for any $0 < \mu' < \mu$ as $k \to \infty$, which completes the proof in view of Definition 5.2.1. □

Next we prove that $\bar{H}$ and w are continuously differentiable in p.

Lemma 5.4.2. *$\bar{H} \in C^1(\mathbb{R}^n)$ and*

$$|D_p \bar{H}(p)| \leq C(1 + |p|),$$

where $C > 0$ depends only on n, λ, Λ, α, α', β, β' and K. Moreover, $w \in C^1(\mathbb{R}^n; C^{2,\mu}(\mathbb{R}^n))$, for any $0 < \mu < 1$, such that for any $L > 0$ and any

$p \in B_L$,

$$\|D_p w(p,\cdot)\|_{C^{2,\mu}(\mathbb{R}^n)} \leq C_L,$$

where $C_L > 0$ depends only on n, λ, Λ, α, α', β, β', K, μ and L.

Proof. Let us fix $0 < \mu < 1$. Throughout this proof, we shall write by C_L a positive constant depending at most on n, λ, Λ, α, α', β, β', K, μ and L. We will also let it differ from one line to another, unless stated otherwise.

Fix $L > 0$, $p \in B_L$, $0 < \mu < 1$ and $1 \leq k \leq n$. Write $w_h(y) = w(p+he_k, y)$ and $\gamma_h = \bar{H}(p+he_k)$ for any $h \in \mathbb{R}$ with $|h| \leq 1$. Also write $W_h(y) = h^{-1}(w_h(y) - w_0(y))$, and $\Gamma_h = h^{-1}(\gamma_h - \gamma_0)$. Then W_h turns out to be a periodic viscosity solution to

$$-\operatorname{tr}(A(y)D^2 W_h) + B_h(y)\cdot(DW_h + e_k) = \Gamma_h \quad \text{in } \mathbb{R}^n, \tag{5.31}$$

where

$$B_h(y) = \int_0^1 D_p H(tD_y w_h + (1-t)D_y w_0 + p + the, y)dt.$$

It follows from (5.30) and (5.9) that $B_h \in C^\mu(\mathbb{R}^n)$ and

$$\|B_h\|_{C^\mu(\mathbb{R}^n)} \leq C_L, \tag{5.32}$$

for any $h \in \mathbb{R}$ with $0 < |h| \leq L - |p|$; recall from $p \in B_L$ that $L - |p| > 0$. Moreover, we know from (5.28) that

$$|\Gamma_h| \leq C_0(1 + |p|), \tag{5.33}$$

for any $h \in \mathbb{R}$ with $0 < |h| \leq L - |p|$, where $C_0 > 0$ depends only on n, α, α', β, β' and K.

One may notice that (5.31) belongs to the same class of (5.37), whence it follows from Lemma 5.4.3 below that $W_h \in C^{2,\mu}(\mathbb{R}^n)$ and

$$\|W_h\|_{C^{2,\mu}(\mathbb{R}^n)} \leq C_L, \tag{5.34}$$

for any $h \in \mathbb{R}$ with $0 < |h| \leq L - |p|$. On the other hand, from the fact that Lemma 5.4.1 implies $Dw_h \to Dw_0$ in $C^{1,\mu}(\mathbb{R}^n)$, we know that $B_h \to B_0$ in

$C^\mu(\mathbb{R}^n)$, where B_0 is defined by

$$B_0(y) = D_p H(D_y w_0 + p, y).$$

As with the estimate (5.32), we also know that

$$\|B_0\|_{C^\mu(\mathbb{R}^n)} \leq C_L.$$

According to the Arzela-Ascoli theorem, there is some $W_0 \in C^{2,\mu}(\mathbb{R}^n)$ such that $W_h \to W_0$ in $C^{2,\mu'}(\mathbb{R}^n)$ for any $0 < \mu' < \mu$, along a subsequence. Moreover, we may choose $\Gamma_0 \in \mathbb{R}$ such that $\Gamma_h \to \Gamma_0$ along a further subsequence. Then by the stability of viscosity solutions, W_0 becomes a periodic solution to

$$-\operatorname{tr}(A(y)D^2W_0) + B_0(y) \cdot (D_y W_0 + e_k) = \Gamma_0 \quad \text{in } \mathbb{R}^n. \tag{5.35}$$

Let us remark that the convergence $\Gamma_h \to \Gamma_0$ along a subsequence implies

$$|\Gamma_0| \leq C_0(1 + |p|), \tag{5.36}$$

where C_0 is the same constant chosen in (5.33).

Now that (5.35) belongs to the same class of (5.37), it follows from Lemma 5.4.3 below that Γ_0 is unique. From the uniqueness of the limit Γ_0, we infer that $\Gamma_h \to \Gamma_0$ without extracting any subsequence. By definition, $\Gamma_0 = D_{p_k}\bar{H}(p)$. The estimate on $D_p\bar{H}(p)$ in Lemma 5.4.2 now follows from (5.36).

Moreover, since any limit W_0 of $\{W_h\}_{0<|h|\leq 1}$ satisfies $W_0(0) = 0$, we also have from the last part of Lemma 5.4.3 below that W_0 is unique, and belongs to $C^{2,\mu}(\mathbb{R}^n)$, with the estimate

$$\|W_0\|_{C^{2,\mu}(\mathbb{R}^n)} \leq C_L.$$

Owing to the uniqueness of the limit W_0, we conclude that $W_h \to W_0$ in $C^{2,\mu'}(\mathbb{R}^n)$ along the full sequence, which implies that $W_0 = D_{p_k} w(p, \cdot)$.

The continuity of $D_{p_k}\bar{H}$ and $D_{p_k} w$ in variable p can be proved similarly as in the proof of Lemma 5.4.1. To avoid repeating arguments, we omit the details and leave this part to the reader. □

Lemma 5.4.3. *Let $B \in C^\mu(\mathbb{R}^n)$ be a periodic, vector-valued mapping. Then for each $p \in \mathbb{R}^n$, there exists a unique real number, γ, for which the following*

PDE,

$$-\operatorname{tr}(A(y)D^2 v) + B(y)\cdot(Dv+p) = \gamma \quad \text{in } \mathbb{R}^n, \tag{5.37}$$

admits a periodic viscosity solution $v \in C^{2,\mu}(\mathbb{R}^n)$. Moreover, γ satisfies

$$|\gamma| \le |p| \, \|B\|_{L^\infty(\mathbb{R}^n)}\,.$$

Furthermore, a periodic viscosity solution v of (5.37) *is unique up to an additive constant, and satisfies*

$$\|v - v(0)\|_{C^{2,\mu}(\mathbb{R}^n)} \le C|p|,$$

where $C > 0$ depends only on n, λ, Λ, μ and $\|B\|_{C^\mu(\mathbb{R}^n)}$.

Proof. The proof is essentially the same with that of Lemma 5.3.1, and hence it is omitted. □

In what follows, let us write $\bar{B}(p) = D_p\bar{H}(p)$, $v(p,y) = D_p w(p,y)$ and

$$B(p,y) = D_p H(D_y w(p,y) + p, y). \tag{5.38}$$

In view of the proof of Lemma 5.4.2, we may understand $\bar{B}(p)$ as the unique real vector in $\mathbb{R}^n$ for which the following (decoupled) system,

$$-\operatorname{tr}(A(y)D^2 v) + B(p,y)\cdot(D_y v + I) = \bar{B}(p), \tag{5.39}$$

has a periodic viscosity solution, where I is the identity matrix in $\mathcal{S}^n$. Moreover, $v(p,\cdot)$ can be considered as the unique periodic viscosity solution of (5.39) such that

$$v(p,0) = 0. \tag{5.40}$$

It is remarkable that after linearization in (5.10), we end up with a cell problem whose gradient part has a linear growth, as shown in (5.39). Moreover, one may expect that the linear structure of the "new" cell problem (5.39) will be preserved throughout the linearization we do in the future to obtain higher regularity of $\bar{H}$ and w in p. This is the brief idea behind the proof of the following proposition. One may find a similar proposition for uniformly elliptic, fully nonlinear PDEs in the authors' previous work [34] and [35].

Lemma 5.4.4. *$\bar{H} \in C^\infty(\mathbb{R}^n)$ and $w \in C^\infty(\mathbb{R}^n; C^{2,\mu}(\mathbb{R}^n))$, for any $0 < \mu < 1$, such that for any $k = 0, 1, 2, \cdots$, any $L > 0$ and any $p \in B_L$,*

$$\left|D_p^k \bar{H}(p)\right| + \left\| D_p^k w(p, \cdot) \right\|_{C^{2,\mu}(\mathbb{R}^n)} \leq C_{k,L}, \tag{5.41}$$

where $C_{k,L} > 0$ depends only on n, λ, Λ, α, α', β, β', K, μ, k and L.

Proof. We follow the proof of Lemma 5.4.2. Due to Lemma 5.4.2 and the regularity assumption (5.9), we already know that $B \in C^1(\mathbb{R}^n; C^{1,\mu}(\mathbb{R}^n))$, for any $0 < \mu < 1$, with B defined in (5.38). Thus, in order to run the same argument in the proof of Lemma 5.4.2, we need the Lipschitz regularity of $\bar{B} = D_p \bar{H}$ in p. However, this can be shown as in the proof of (5.28) of Lemma 5.3.3. This is because we can also understand the constant vector $\bar{B}(p)$ as the limit of $\{-\delta v^\delta\}_{\delta>0}$, with v^δ being the unique periodic viscosity solution of

$$-\operatorname{tr}(A(y)D^2 v^\delta) + B(p, y) \cdot (D_y v^\delta + I) + \delta v^\delta = 0 \quad \text{in } \mathbb{R}^n.$$

Once we know that $\bar{B}$ is Lipschitz in p, it follows from Lemma 5.4.2 and the elliptic regularity theory that the difference quotient $V_h = h^{-1}(v_h - v_0)$ is uniformly bounded in $C^{2,\mu}(\mathbb{R}^n)$, as it being a periodic viscosity solution of

$$-\operatorname{tr}(A(y)D^2 V_h) + B_h(y) \cdot D_y V_h + B_h(y) \cdot (D_y v_0(y) + I) = \bar{B}_h \quad \text{in } \mathbb{R}^n,$$

with $v_h = v(p + he_k, \cdot)$, $B_h = B(p + he_k, \cdot)$, $B_h = h^{-1}(B_h - B_0)$ and $\bar{B}_h = h^{-1}(\bar{B}_h - \bar{B}_0)$. Hence, we deduce from the stability of viscosity solutions that any pair $(V_0, \bar{B}_0)$ of $\{V_h\}_{0<|h|\leq 1}$ and, respectively, $\{\bar{B}_h\}_{0<|h|\leq 1}$ must satisfy

$$-\operatorname{tr}(A(y)D^2 V_0) + B_0(y) \cdot D_y V_0 + B_0(y) \cdot (D_y v_0(y) + I) = \bar{B}_0 \quad \text{in } \mathbb{R}^n. \tag{5.42}$$

Since (5.42) belongs to the same class of (5.37), we know from Lemma 5.4.3 that V_0 and $\bar{B}_0$ are unique. Thus, we derive the differentiability of $\bar{B}$ and v in p. Arguing as in the proof of Lemma 5.4.1, we may also observe that $D_p \bar{B}$ and $D_p v$ are continuous in p.

One may now iterate this argument to obtain higher regularity of $\bar{B}$ and v in p, which automatically implies that of $\bar{H}$ and w. We leave out the details to the reader. $\square$

5.5 Interior Corrector and Higher Order Convergence Rate

In this section, we construct the higher order interior correctors for the homogenization problem (5.1), based on the regularity result achieved in Section 5.4.

We begin with the effective Hamilton-Jacobi equation for (5.1), which is given by

$$\begin{cases} \partial_t \bar{u}_0 + \bar{H}(D\bar{u}_0) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ \bar{u}_0 = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{5.43}$$

The characteristic curve, which starts from $x_0 \in \mathbb{R}^n$, is given by

$$\xi(t;x_0) = x_0 + D_p\bar{H}(D_x g(x_0))t. \tag{5.44}$$

Note that this is indeed a line with direction $D_p\bar{H}(D_x g(x_0))$. Moreover, the gradient of $\bar{u}$ is constant along this curve. To be specific, we have

$$D_x\bar{u}(\xi(t;x_0),t) = D_x g(x_0). \tag{5.45}$$

It is noteworthy that the initial data, g, does not play any role when deriving the effective Hamiltonian $\bar{H}$, as shown in Section 5.4. This allows us to choose the initial data g *a posteriori* so as to make sure that

$$\{(\xi(t;x),t) : t > 0\} \bigcap \{(\xi(t;x'),t) : t > 0\} = \emptyset, \tag{5.46}$$

if and only if $x \neq x'$, as well as that

$$\bigcup_{x\in\mathbb{R}^n} \{(\xi(t;x),t) : t > 0\} = \mathbb{R}^n \times (0,\infty). \tag{5.47}$$

One may easily observe that there are infinitely many initial data g that satisfy the conditions (5.46) and (5.47), once $\bar{H}$ is determined. A trivial example is an affine function, which can be generalized to any smooth, convex and globally Lipschitz function.

Once we have the initial data g, we can observe from the characteristic equations for (5.43) that $\bar{u}_0 \in C^\infty(\mathbb{R}^n \times [0,\infty))$ (see Lemma 5.5.1). Setting

$$\bar{B}(x,t) = D_p\bar{H}(D_x\bar{u}_0(x,t)), \tag{5.48}$$

we obtain $\bar{B} \in C^\infty(\mathbb{R}^n \times [0,\infty))$, according to Lemma 5.4.4. In order to construct the higher order correctors, we need to have smooth solutions for first order linear PDEs with $\bar{B}$ as the drift term. For this reason, we require that

$$\bar{B}(x,t) \neq 0,$$

for any $(x,t) \in \mathbb{R}^n \times (0,\infty)$. In view of (5.45) and (5.47), the image of $\bar{B}$ on $\mathbb{R}^n \times (0,\infty)$ coincides with that of $D_p\bar{H}(D_xg)$ on $\mathbb{R}^n$. Hence, we ask D_xg not to be the critical points of $\bar{H}$.

Let us list up the conditions for g to be imposed in the rest of this paper:

(i) g is convex:

$$g(tx + (1-t)x') \leq tg(x) + (1-t)g(x'), \tag{5.49}$$

for any $0 \leq t \leq 1$ and any $x, x' \in \mathbb{R}^n$.

(ii) $g \in C^\infty(\mathbb{R}^n) \cap \mathrm{Lip}(\mathbb{R}^n)$, and there is $L > 0$ such that

$$\left\| D_x^k g \right\|_{L^\infty(\mathbb{R}^n)} \leq L, \tag{5.50}$$

for any $k = 1, 2, \cdots$. Moroever, g is normalized so as to satisfy

$$g(0) = 0. \tag{5.51}$$

(iii) D_xg is not a critical point of $\bar{H}$:

$$D_p\bar{H}(D_xg(x)) \neq 0, \tag{5.52}$$

for any $x \in \mathbb{R}^n$.

Under these assumptions, we obtain a unique smooth solution of (5.43) that is semi-concave in the sense of (5.53), as stated below.

Lemma 5.5.1. *Let $\bar{H}$ satisfy* (5.26) – (5.28) *and* (5.41)*, and g satisfy* (5.49) – (5.51)*. Then there exists a unique solution $\bar{u}_0 \in C^\infty(\mathbb{R}^n \times [0,\infty))$ of* (5.43) *satisfying*

$$\bar{u}_0(x+z,t) - 2\bar{u}_0(x,t) + \bar{u}_0(x-z,t) \leq C\left(1 + \frac{1}{t}\right)|z|^2, \tag{5.53}$$

for some constant $C \geq 0$ and all $x, z \in \mathbb{R}^n$ and $t > 0$. Moreover, one has, for any $i, j = 0, 1, 2, \cdots$ and any $T > 0$,

$$\left| D_x^i \partial_t^j \bar{u}_0(x,t) \right| \leq C_{i,j,T}, \tag{5.54}$$

uniformly for all $(x,t) \in \mathbb{R}^n \times [0,T]$, where $C_{i,j,T}$ is a positive constant depending at most on n, α, α', β, β', K, L, i, j and T.

In order to prove this lemma, observe first a basic regularity result for first order PDEs.

Lemma 5.5.2. *Let $T > 0$ and $B \in C^1(\mathbb{R}^n \times (0,T); \mathbb{R}^n)$ and $f \in C^1(\mathbb{R}^n \times (0,T))$ be such that for some $L > 0$, one has, for any $i, j \geq 0$ with $i + j \leq 1$,*

$$|D_x^i \partial_t^j B(x,t)| + |D_x^i \partial_t^j f(x,t)| \leq L, \tag{5.55}$$

for all $(x,t) \in \mathbb{R}^n \times (0,T)$. Suppose that $v \in C^1(\mathbb{R}^n \times [0,T))$ is a solution of

$$\begin{cases} v_t + B(x,t) \cdot Dv + f(x,t) = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ v = 0 & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Then v satisfies, for each $T > 0$ and $i, j \geq 0$ with $i + j \leq 1$,

$$|D_x^i \partial_t^j v(x,t)| \leq C_{L,T}, \tag{5.56}$$

for all $(x,t) \in \mathbb{R}^n \times [0,T]$, where $C_{L,T}$ depends only on n, L and T.

Proof. Let us denote by $(\xi(t), t)$ the characteristic curve of v starting from $x_0 \in \mathbb{R}^n$. Then we know from [23, Section 3] that $p(t) = D_x v(\xi(t), t)$, $q(t) = \partial_t v(\xi(t), t)$ and $z(t) = v(\xi(t), t)$ satisfy

$$\begin{cases} \dot{p}(t) & = -D_x B(\xi(t), t) \cdot p(t) - D_x f(\xi(t), t), \\ \dot{q}(t) & = -\partial_t B(\xi(t), t) \cdot p(t) - \partial_t f(\xi(t), t), \\ \dot{z}(t) & = -f(\xi(t), t). \end{cases}$$

Therefore, it follows from (5.55) that

$$|p(t)| + |q(t)| \leq C_L e^{Lt} + Lt,$$

and

$$|z(t)| \leq Lt,$$

for any $0 < t < T$, from which the estimate (5.56) follows immediately. $\square$

Let us present the proof of Lemma 5.5.1.

Proof of Lemma 5.5.1. Due to (5.26), (5.27), (5.41) and (5.50), we know that there exists a unique solution $\bar{u}_0$ of (5.43) satisfying (5.53); see [23, Theorem 7 in Section 3]. Moreover, it follows from [23, Theorem 8 in Section 3] that $\bar{u}_0$ is given by the Hopf-Lax formula, i.e.,

$$\bar{u}_0(x,t) = \min_{y \in \mathbb{R}^n} \left\{ t\bar{L}\left(\frac{x-y}{t}\right) + g(y) \right\},$$

where $\bar{L}$ is the Legendre transform of $\bar{H}$. According to [23, Lemma 2 in Section 3], $\bar{u}_0$ is Lipschitz continuous on $\mathbb{R}^n \times [0,\infty)$, with

$$|D_x^i \partial_t^j \bar{u}_0(x,t)| \leq C,$$

for any $(x,t) \in \mathbb{R}^n \times [0,\infty)$ and $i, j \geq 0$ with $i + j = 1$, where $C > 0$ depends only on n, α, α', β, β' and L (with $L > 0$ being the constant in (5.50)).

Under the convexity assumptions (5.27) and (5.49) on $\bar{H}$ and g, we are able to prove the smoothness of $\bar{u}_0$ on $\mathbb{R}^n \times [0,\infty)$. To see this, we only need to verify that the characteristic curves (5.44) corresponding to the problem (5.43) exist for all time $t > 0$. Since $\bar{H}$ and g are convex and twice continuously differentiable, we know that $D_p^2\bar{H}(p)$ and $D_x^2 g(x)$ are nonnegative definite for any $p \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Hence, the composition $D_p\bar{H}(D_x g)$ is monotone in the sense that

$$(D_p\bar{H}(D_x g(x_1)) - D_p\bar{H}(D_x g(x_2))) \cdot (x_1 - x_2) \geq 0, \tag{5.57}$$

for any $x_1, x_2 \in \mathbb{R}^n$. Now if the characteristic curves $\{(\xi(t;x_1),t) : t > 0\}$ and $\{(\xi(t;x_2),t) : t > 0\}$ coincide with each other at some $t = t_0 > 0$, then we must have

$$(D_p\bar{H}(D_x g(x_1)) - D_p\bar{H}(D_x g(x_2)))t_0 = -(x_1 - x_2),$$

which violates (5.57). Thus, we verify that $\{(\xi(t;x_1),t) : t > 0\} \cap \{(\xi(t;x_2),t) : t > 0\} = \emptyset$ for distinct pair of points $x_1, x_2 \in \mathbb{R}^n$. Thus, the characteristic curve exist globally, which implies $\bar{u}_0 \in C^\infty(\mathbb{R}^n \times [0,\infty))$.

Now that we know the smoothness of $\bar{u}_0$, we can derive the estimate (5.54) by applying Lemma 5.5.2 inductively on each derivative of $\bar{u}_0$. To be more specific, given integers $k, l \geq 0$, the function $v(x,t) = D_x^k \partial \partial_t^l \bar{u}_0(x,t) - D_x^k g(x)$ satisfies

$$\begin{cases} v_t + \bar{B}(x,t) \cdot Dv + f(x,t) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ v = 0 & \text{on } \mathbb{R}^n \times \{t=0\}, \end{cases}$$

where $\bar{B}(x,t) = D_p \bar{H}(D_x \bar{u}_0(x,t))$ and $f(x,t)$ consists of lower order terms. Hence, assuming that (5.54) holds for any $i,j \geq 0$ with $0 \leq i \leq k$, $0 \leq j \leq l$ and $i+j < k+l$, one may verify that $\bar{B}$ and f satisfy (5.55), under the regularity assumptions (5.41) and (5.50) of $\bar{H}$ and g, the normalization condition (5.51) of g together with the estimate (5.54) for lower order derivatives of $\bar{u}_0$. This finishes the proof. □

Recall from (5.48) the function $\bar{B}$ associated with the limit profile $\bar{u}$ and the effective Hamiltonian $\bar{H}$. Due to (5.54) and (5.41), we know that $\bar{B} \in C^\infty(\mathbb{R}^n \times [0,\infty))$ and, for each $i,j = 0,1,2,\cdots$, and any $T > 0$,

$$\left| D_x^i \partial_t^j \bar{B}(x,t) \right| \leq C_{i,j,T}, \tag{5.58}$$

uniformly for all $(x,t) \in \mathbb{R}^n \times [0,T]$, where $C_{i,j,T}$ is another positive constant determined by the same parameters listed above.

In what follows, we shall seek a sequence of the interior correctors for the homogenization problem (5.1). The first order interior corrector w_1 will be in the form of

$$w_1(x,t,y) = \phi_1(x,t,y) + \bar{u}_1(x,t), \tag{5.59}$$

where ϕ_1 denotes

$$\phi_1(x,t,y) = w(D_x \bar{u}_0(x,t), y), \tag{5.60}$$

with $w = w(p,y)$ being the periodic (viscosity) solution of (5.24) normalized so as to satisfy (5.25). Here $\bar{u}_1$ is an effective data that is not determined yet. Let us remark that one may choose $\bar{u}_1$ by any regular data, if one stops seeking interior correctors at this step. However, if one would like to go further and construct the second order corrector w_2, one needs to select $\bar{u}_1$ specifically by the solution of an effective limit equation, which arises from the solvability condition of w_2.

We will continuously observe such a relationship between the consecutive correctors. In fact, in the proof of Lemma 5.5.3 below, it will turn out that the k-th order interior corrector w_k, for $k \geq 2$, is in the form of

$$w_k(x,t,y) = \phi_k(x,t,y) + \chi(x,t,y) \cdot D_x \bar{u}_{k-1}(x,t) + \bar{u}_k(x,t), \tag{5.61}$$

where $\phi_k(x,t,\cdot)$ will be the periodic viscosity solution of a certain cell problem normalized so as to satisfy $\phi_k(x,t,0) = 0$, and $\chi : \mathbb{R}^n \times [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ will be defined by

$$\chi(x,t,y) = v(D_x\bar{u}_0(x,t), y),$$

with $v = v(p,y)$ being the periodic solution of (5.39) normalized so as to satisfy (5.40). Here $\bar{u}_{k-1}$ will be determined specifically such that the cell problem for ϕ_k is solvable, while $\bar{u}_k$ will be "free" to choose before one tries to construct the $(k+1)$-th corrector w_{k+1}.

It is noteworthy that, owing to Lemma 5.4.4, we have $\chi \in C^\infty(\mathbb{R}^n \times [0,\infty); C^{2,\mu}(\mathbb{R}^n))$ and, for any $i,j = 0,1,2,\cdots$ and any $T > 0$,

$$\left\| D_x^i \partial_t^j \chi(x,t,\cdot) \right\|_{C^{2,\mu}(\mathbb{R}^n)} \le C_{i,j,R,T}, \tag{5.62}$$

uniformly for all $(x,t) \in \mathbb{R}^n \times [0,T]$. In addition, we know from (5.39) and (5.40) that for each $(x,t) \in \mathbb{R}^n \times (0,\infty)$, $\chi(x,t,\cdot)$ is the unique periodic viscosity solution of

$$-\operatorname{tr}(A(y)D_y^2\chi) + B(x,t,y)\cdot(D_y\chi + I) = \bar{B}(x,t) \quad \text{in } \mathbb{R}^n, \tag{5.63}$$

which also satisfies

$$\chi(x,t,0) = 0. \tag{5.64}$$

For the rest of this section, we will justify the existence of the higher order interior correctors in a rigorous way. The corresponding work has been done by the authors in [34] and [35] in the framework of fully nonlinear, uniformly elliptic or parabolic, second order PDEs in non-divergence form.

To simplify the notation, let us write

$$w_0(x,t,y) = \bar{u}_0(x,t), \tag{5.65}$$

and by W_k, for $k = 0,1,2,\cdots$, the vector-valued mapping,

$$W_k(x,t,y) = D_y w_{k+1}(x,t,y) + D_x w_k(x,t,y). \tag{5.66}$$

Note from (5.59), (5.60) and (5.65) that

$$W_0(x,t,y) = D_y\phi_1(x,t,y) + D_x\bar{u}_0(x,t). \tag{5.67}$$

We shall also write by B_k, for $k = 1, 2, \cdots$, the mapping,

$$B_k(x, t, y) = D_p^k H(W_0(x, t, y)), \tag{5.68}$$

where $D_p^k H$ is understood in the sense of Fréchet derivatives, and to make the notation coherent to the notation of $\bar{B}$, we will write

$$B(x, t, y) = B_1(x, t, y).$$

Let us also remark that, due to (5.9), (5.54) and (5.41), we have $B_k \in C^\infty(\mathbb{R}^n \times [0, \infty); C^\mu(\mathbb{R}^n))$, for any $0 < \mu < 1$. In particular, we obtain, for any $i, j = 0, 1, 2, \cdots$, any $k = 1, 2, \cdots$ and any $T > 0$,

$$\left\| D_x^i \partial_t^j B_k(x, t, \cdot) \right\|_{C^\mu(\mathbb{R}^n)} \leq C_{i,j,k,T}, \tag{5.69}$$

uniformly for all $(x, t) \in \mathbb{R}^n \times [0, T]$, where $C_{i,j,k,T} > 0$ depends only on n, λ, Λ, α, α', β, β', K, L, μ, i, j, k and T.

Lemma 5.5.3. *Suppose that A, H and g satisfy* (5.3) – (5.5), (5.6) – (5.9) *and, respectively,* (5.49) – (5.52). *Then there exists a sequence $\{w_k\}_{k=1}^\infty$ satisfying the following.*

(i) $w_k \in C^\infty(\mathbb{R}^n \times [0, \infty); C^{2,\mu}(\mathbb{R}^n))$, *for any $0 < \mu < 1$, and*

$$\left\| D_x^i \partial_t^j w_k(x, t, \cdot) \right\|_{C^{2,\mu}(\mathbb{R}^n)} \leq C_{i,j,k,T}, \tag{5.70}$$

for each $i, j = 0, 1, 2, \cdots$, any $T > 0$, and uniformly for all $(x, t) \in \mathbb{R}^n \times [0, T]$, where $C_{i,j,k,T} > 0$ depends only on n, λ, Λ, α, α', β, β', K, L, μ, i, j, k and T.

(ii) w_k *satisfies*

$$w_k(x, 0, 0) = 0. \tag{5.71}$$

(iii) For each $(x, t) \in \mathbb{R}^n \times (0, \infty)$, $w_k(x, t, \cdot)$ is a periodic solution of

$$\partial_t w_0(x, t, y) - \mathrm{tr}(A(y) D_y^2 w_1) + H(D_y w_1 + D_x w_0(x, t, y), y) = 0 \quad \textit{in } \mathbb{R}^n, \tag{5.72}$$

for $k = 1$, and

$$\begin{aligned}&\partial_t w_{k-1}(x,t,y) - \operatorname{tr}(A(y)D_y^2 w_k)\\&\quad + B(x,t,y)\cdot(D_y w_k + D_x w_{k-1}(x,t,y)) + \Phi_{k-1}(x,t,y) = 0 \quad \text{in } \mathbb{R}^n,\end{aligned} \tag{5.73}$$

for $k \geq 2$, where

$$\begin{aligned}\Phi_{k-1}(x,t,y) &= -2\operatorname{tr}(A(y)(D_xD_y w_{k-1}(x,t,y) + D_x^2 w_{k-2}(x,t,y)))\\&\quad + \sum_{l=2}^{k-1}\frac{1}{l!}\sum_{\substack{i_1+\cdots+i_l=k-1\\ i_1,\cdots,i_1\geq 1}} B_l(x,t,y)(W_{i_1}(x,t,y),\cdots,W_{i_l}(x,t,y)),\end{aligned} \tag{5.74}$$

with the last summation term understood as zero when $k = 2$.

Remark 5.5.4. *The summation term in the definition* (5.74) *of Φ_k amounts to the nonlinear effect of the Hamiltonian H in p. In view of* (5.68)*, one may easily observe that the whole summation term becomes zero when H is linear in p. The choice of Φ_k is specifically designed to achieve* (5.92)*, which will eventually leads us to the higher order convergence rate for the homogenization problem* (5.1)*. We will also see later in* (5.110) *and* (5.112) *that the choice of Φ_k changes according to the type of nonlinearity that needs to be taken care of.*

Proof of Lemma 5.5.3. Throughout this proof, we shall fix $0 < \mu < 1$, and denote by $C_{*,\cdots,*}$ a positive constant depending only on the subscripts as well as the parameters n, λ, Λ, α, α', β, β', K, L and μ. We will also allow it to vary from one line to another, for notational convenience.

Define ϕ_1 by (5.60). Since $\bar{u}_0 \in C^\infty(\mathbb{R}^n \times [0,\infty))$, we know from (5.41) that $\phi_1 \in C^\infty(\mathbb{R}^n \times [0,\infty); C^{2,\mu}(\mathbb{R}^n))$. Moreover, it follows from (5.54) that for each $i, j = 0, 1, 2, \cdots$, and any $T > 0$,

$$\left\| D_x^i \partial_t^j \phi_1(x,t,\cdot)\right\|_{C^{2,\mu}(\mathbb{R}^n)} \leq C_{i,j,T},$$

uniformly for all $(x,t) \in \mathbb{R}^n \times [0,T]$. In view of the definition of w_0 in (5.65), $\phi_1(x,t,\cdot)$ is a periodic viscosity solution of (5.72), for each $(x,t) \in \mathbb{R}^n \times (0,\infty)$, as $\bar{u}_0$ being the solution of (5.43).

Let us now fix $k \geq 1$ and suppose that we have already found $\{w_l\}_{l=0}^{k-1}$ that satisfies the assertions (i) and (ii) of this lemma. Moreover, assume that we have already obtained $\bar{u}_{k-1} \in C^{\infty}(\mathbb{R}^n \times [0,\infty))$ such that

$$\left| D_x^i \partial_t^j \bar{u}_{k-1}(x,t) \right| \leq C_{i,j,k,T}, \tag{5.75}$$

for any $i, j = 0, 1, 2, \cdots$, any $T > 0$ and any $(x,t) \in \mathbb{R}^n \times [0,T]$. Additionally, suppose that we have also found $\phi_k \in C^{\infty}(\mathbb{R}^n \times [0,\infty); C^{2,\mu}(\mathbb{R}^n))$ such that for each $(x,t) \in \mathbb{R}^n \times [0,\infty)$, $\phi_k(x,t,\cdot)$ is a periodic function normalized by

$$\phi_k(x,t,0) = 0, \tag{5.76}$$

and that we have, for any $i, j = 0, 1, 2, \cdots$ and any $T > 0$,

$$\left\| D_x^i \partial_t^j \phi_k(x,t,\cdot) \right\|_{C^{2,\mu}(\mathbb{R}^n)} \leq C_{i,j,k,T}, \tag{5.77}$$

uniformly for all $(x,t) \in \mathbb{R}^n \times [0,T]$.

Define $\tilde{w}_k$ by

$$\tilde{w}_1(x,t,y) = \phi_1(x,t,y),$$

if $k = 1$, and by

$$\tilde{w}_k(x,t,y) = \phi_k(x,t,y) + \chi(x,t,y) \cdot D_x \bar{u}_{k-1}(x,t),$$

if $k \geq 2$. We deduce from (5.75), (5.77) and (5.62) that $\tilde{w}_k \in C^{\infty}(\mathbb{R}^n \times [0,\infty); C^{2,\mu}(\mathbb{R}^n))$ and satisfies

$$\left\| D_x^i \partial_t^j \tilde{w}_k(x,t,\cdot) \right\|_{C^{2,\mu}(\mathbb{R}^n)} \leq C_{i,j,k,T}, \tag{5.78}$$

for any $i, j = 0, 1, 2, \cdots$, any $T > 0$ and any $(x,t) \in \mathbb{R}^n \times [0,T]$.

In view of the estimate (5.78), we observe that $\tilde{w}_k$ satisfies the assertion (i) of Lemma 5.5.3. Moreover, it follows from the hypothesis (5.76), and the fact (5.64) that $\tilde{w}_k$ verifies the assertion (ii) of this lemma as well. Henceforth, we shall assume, as the last hypothesis for this induction argument, that $\tilde{w}_k$ satisfies the assertion (iii) of this lemma.

In order to find $\bar{u}_k$, we first define

$$
\begin{aligned}
f_k(x,t,y) = {} & \partial_t \tilde{w}_k(x,t,y) + B(x,t,y)\cdot D_x\tilde{w}_k(x,t,y) \\
& - 2\operatorname{tr}(A(y)(D_xD_y\tilde{w}_k(x,t,y) + D_x^2 w_{k-1}(x,t,y))) \\
& + \sum_{l=2}^{k}\frac{1}{l!}\sum_{\substack{i_1+\cdots+i_l=k\\ i_1,\cdots,i_1\geq 1}} B_l(x,t,y)(W_{i_1}(x,t,y),\cdots,W_{i_l}(x,t,y)).
\end{aligned}
$$

Using (5.5), (5.69), (5.75), (5.77), (5.62) and (5.78) together with the induction hypothesis (5.70), we deduce that $f_k \in C^\infty(\mathbb{R}^n\times[0,\infty);C^\mu(\mathbb{R}^n))$ and

$$
\left\| D_x^i\partial_t^j f_k(x,t,\cdot)\right\|_{C^\mu(\mathbb{R}^n)} \leq C_{i,j,k,T}, \tag{5.79}
$$

for any $i,j=0,1,2,\cdots$, any $T>0$ and any $(x,t)\in\mathbb{R}^n\times[0,T]$.

Now that f_k is periodic in y, we may consider the following cell problem: there exists a unique function $\bar{f}_k:\mathbb{R}^n\times(0,\infty)\to\mathbb{R}$ such that for each $(x,t)\in\mathbb{R}^n\times[0,\infty)$, the PDE,

$$
-\operatorname{tr}(A(y)D_y^2\phi_{k+1}) + B(x,t,y)\cdot D_y\phi_{k+1} + f_k(x,t,y) = \bar{f}_k(x,t) \quad \text{in } \mathbb{R}^n, \tag{5.80}
$$

has a periodic viscosity solution. Following the argument in the proof of Lemma 5.4.3, we see that the cell problem (5.80) is solvable. Moreover, if we normalize ϕ_{k+1} so as to satisfy

$$
\phi_{k+1}(x,t,0) = 0,
$$

such a periodic viscosity solution ϕ_{k+1} is unique. Furthermore, applying the regularity theory in the slow variable established in Lemma 5.4.4, we deduce from (5.69) and (5.79) that $\bar{f}_k\in C^\infty(\mathbb{R}^n\times(0,\infty))$ and $\phi_{k+1}\in C^\infty(\mathbb{R}^n\times[0,\infty);C^{2,\mu}(\mathbb{R}^n))$. In particular, we have, for any $i,j=0,1,2,\cdots$ and any $T>0$,

$$
\left|D_x^i\partial_t^j\bar{f}_k(x,t)\right| + \left\| D_x^i\partial_t^j\phi_{k+1}(x,t,\cdot)\right\|_{C^{2,\mu}(\mathbb{R}^n)} \leq C_{i,j,k,T},
$$

uniformly for all $(x,t)\in\mathbb{R}^n\times[0,T]$.

With $\bar{f}_k$ at hand, we consider the first order linear PDE,

$$
\begin{cases}
\partial_t\bar{u}_k + \bar{B}(x,t)\cdot D_x\bar{u}_k + \bar{f}_k(x,t) = 0 & \text{in } \mathbb{R}^n\times(0,\infty),\\
\bar{u}_k = 0 & \text{on } \mathbb{R}^n\times\{t=0\},
\end{cases} \tag{5.81}
$$

where $\bar{B}$ is defined by (5.48). Recall from (5.52) that $\bar{B}$ vanishes nowhere in $\mathbb{R}^n \times (0, \infty)$. Thus, it follows from the classical existence and regularity theory for the first order linear PDE (see e.g. Lemma 5.5.2) that there exists a unique solution $\bar{u}_k \in C^\infty(\mathbb{R}^n \times [0, \infty))$ of (5.81) such that

$$\left| D_x^i \partial_t^j \bar{u}_k(x,t) \right| \leq C_{i,j,k,T}, \tag{5.82}$$

for any $i, j = 0, 1, 2, \cdots$, any $T > 0$ and any $(x,t) \in \mathbb{R}^n \times [0, T]$.

Define w_k by

$$w_k(x,t,y) = \tilde{w}_k(x,t,y) + \bar{u}_k(x,t), \tag{5.83}$$

which coincides with the expression (5.59) and (5.61) for any $k \geq 1$. Using (5.78) and (5.82), we see that w_k, defined by (5.83), verifies the assertions (i) and (ii) of Lemma 5.5.3. Besides, let us notice that

$$f_k(x,t,y) = \partial_t \tilde{w}_k(x,t,y) + B(x,t,y) \cdot D_x \tilde{w}_k(x,t,y) + \Phi_k(x,t,y), \tag{5.84}$$

where Φ_k is defined by (5.74), since we have $D_y \tilde{w}_k(x,t,y) = D_y w_k(x,t,y)$.

To this end, let us set $\tilde{w}_{k+1}$ by

$$\tilde{w}_{k+1}(x,t,y) = \phi_{k+1}(x,t,y) + \chi(x,t,y) \cdot D_x \bar{u}_k(x,t).$$

Then we observe from (5.80), (5.81), (5.63) and (5.84) that

$$\begin{aligned}
&\partial_t w_k(x,t,y) - \operatorname{tr}(A(y) D_y^2 \tilde{w}_{k+1}(x,t,y)) \\
&+ B(x,t,y) \cdot (D_y \tilde{w}_{k+1}(x,t,y) + D_x w_k(x,t,y)) + \Phi_k(x,t,y) \\
&= \partial_t \bar{u}_k(x,t) - \operatorname{tr}(A(y) D_y^2 \phi_{k+1}(x,t,y)) + B(x,t,y) \cdot \phi_{k+1} + f_k(x,t,y) \\
&\quad + (-\operatorname{tr}(A(y) D_y^2 \chi(x,t,y)) + B(x,t,y) \cdot (D_y \chi(x,t,y) + I)) \cdot D_x \bar{u}_k(x,t) \\
&= \partial_t \bar{u}_k(x,t) + \bar{f}_k(x,t) + \bar{B}(x,t) \cdot D_x \bar{u}_k(x,t) \\
&= 0.
\end{aligned}$$

Hence, we have proved that $\tilde{w}_{k+1}$ satisfies the assertion (iii) of Lemma 5.5.3.

Recall that we have started with $\{w_l\}_{k=0}^{k-1}$, $\bar{u}_{k-1}$, ϕ_k and $\tilde{w}_k$, and obtained w_k, $\bar{u}_k$, ϕ_{k+1} and $\tilde{w}_{k+1}$ that satisfy all the induction hypotheses (5.75), (5.76) and assertion (i) - (iii) of this lemma. Moreover, we have established the initial case for the induction hypotheses in the beginning of this proof. Thus, the proof is complete. □

We shall call w_k, chosen from Lemma 5.5.3, the k-th order interior corrector for the homogenization problem (5.1), due to the following lemma. Although the computation involved in the proof below is similar to what can be found in [34, Section 3.3] and [35, Section 4.1], we present it in detail for the sake of completeness.

Lemma 5.5.5. *Let $\{w_k\}_{k=1}^{\infty}$ be chosen as in Lemma 5.5.3. Then for each integer $m \geq 1$ and each $0 < \varepsilon \leq \frac{1}{2}$, the function η_m^ε, defined by*

$$\eta_m^\varepsilon(x,t) = \bar{u}_0(x,t) + \sum_{k=1}^{m} \varepsilon^k w_k\left(x,t,\frac{x}{\varepsilon}\right), \tag{5.85}$$

is a viscosity solution of

$$\begin{cases} \partial_t \eta_m^\varepsilon - \varepsilon \operatorname{tr}\left(A\left(\dfrac{x}{\varepsilon}\right) D^2 \eta_m^\varepsilon\right) + H\left(D\eta_m^\varepsilon, \dfrac{x}{\varepsilon}\right) = \psi_m^\varepsilon\left(x,t,\dfrac{x}{\varepsilon}\right) & \text{in } \mathbb{R}^n \times (0,\infty), \\ \eta_m^\varepsilon = g & \text{on } \mathbb{R}^n \times \{t=0\}, \end{cases} \tag{5.86}$$

where $\psi_m^\varepsilon \in C(\mathbb{R}^n \times [0,\infty); L^\infty(\mathbb{R}^n))$ satisfies, for any $T > 0$,

$$\|\psi_m^\varepsilon(x,t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq C_{m,T}\varepsilon^m, \tag{5.87}$$

uniformly for all $(x,t) \in \mathbb{R}^n \times [0,T]$, where $C_{m,T} > 0$ is a constant depending only on n, λ, Λ, α, α', β, β', K, L, μ, m and T.

Proof. Aligned with the notation (5.66) of W_k, let us denote by X_k, the matrix-valued mapping,

$$X_k(x,t,y) = D_y^2 w_{k+1}(x,t,y) + (D_xD_y + D_yD_x)w_k(x,t,y) + D_x^2 w_{k-1}(x,t,y), \tag{5.88}$$

for $k = 1, 2, \cdots$, with w_{-1} being understood as the identically zero function. One may notice from (5.60), (5.59) and (5.65) that

$$X_0(x,t,y) = D_y^2 \phi_1(x,t,y). \tag{5.89}$$

Fix $m \geq 1$ and $0 < \varepsilon \leq \frac{1}{2}$. For the moment, we shall replace w_{m+1} and w_{m+2} by the identically zero functions, only to simplify the exposition. With this replacement, we have $W_m = D_x w_m$, $X_m = (D_xD_y + D_yD_x)w_m + D_x^2 w_{m-1}$ and $X_{m+1} = D_x^2 w_m$.

In view of (5.66) and (5.88), we have

$$D\eta_m^\varepsilon(x,t) = \sum_{k=0}^{m} \varepsilon^k W_k\left(x,t,\frac{x}{\varepsilon}\right),$$

and

$$\varepsilon D^2\eta_m^\varepsilon(x,t) = \sum_{k=0}^{m+1} \varepsilon^k X_k\left(x,t,\frac{x}{\varepsilon}\right).$$

Let us define Ψ_k by

$$\Psi_0(x,t,y) = -\operatorname{tr}(A(y)X_0(x,t,y)) + H(W_0(x,t,y),y),$$

if $k=0$, and by

$$\begin{aligned}\Psi_k(x,t,y) = &-\operatorname{tr}(A(y)X_k(x,t,y))\\ &+\sum_{l=1}^{k}\frac{1}{l!}\sum_{\substack{i_1+\cdots+i_l=k\\ i_1,\cdots,i_l\geq 1}} B_l(x,t,y)(W_{i_1}(x,t,y),\cdots,W_{i_l}(x,t,y)),\end{aligned}$$

if $1\leq k\leq m-1$. Using Ψ_k, one may rephrase the PDEs (5.72) and (5.73)

$$\partial_t w_k(x,t,y) + \Psi_k(x,t,y) = 0 \quad \text{in } \mathbb{R}^n, \tag{5.90}$$

for $0\leq k\leq m-1$.

Denoting by $T_{m-1}(p_0,p)$ the $(m-1)$-th order Taylor polynomial of H in p at p_0, namely,

$$T_{m-1}(p_0,p)(y) = \sum_{k=0}^{m-1}\frac{1}{k!}D_p^k H(p_0,y)(p,\cdots,p),$$

we have

$$
\begin{aligned}
&T_{m-1}\left(W_0(x,t,y), \sum_{k=1}^{m} \varepsilon^k W_k(x,t,y)\right)(y) - \sum_{k=0}^{m-1} \varepsilon^k \operatorname{tr}(A(y)X_k(x,t,y)) \\
&= \sum_{k=0}^{m-1} \varepsilon^k \Psi_k(x,t,y) \\
&\quad + \sum_{k=2}^{m} \sum_{\substack{m \le i_1+\cdots+i_k \le km \\ 1 \le i_1,\cdots,i_k \le m}} \frac{\varepsilon^{i_1+\cdots+i_k}}{k!} B_k(x,t,y)(W_{i_1}(x,t,y),\cdots,W_{i_k}(x,t,y)).
\end{aligned} \tag{5.91}
$$

Hence, we apply the Taylor expansion of H in p at W_0 up to $(m-1)$-th order and derive that

$$
\begin{aligned}
&-\varepsilon \operatorname{tr}\left(A\left(\frac{x}{\varepsilon}\right) D^2 \eta_m^\varepsilon(x,t)\right) + H\left(D\eta_m^\varepsilon(x,t), \frac{x}{\varepsilon}\right) \\
&= \sum_{k=0}^{m-1} \varepsilon^k \Psi_k\left(x,t,\frac{x}{\varepsilon}\right) + E_m^\varepsilon\left(x,t,\frac{x}{\varepsilon}\right),
\end{aligned} \tag{5.92}
$$

where E_m^ε is defined so as to satisfy

$$
\begin{aligned}
&E_m^\varepsilon(x,t,y) - R_{m-1}\left(W_0(x,t,y), \sum_{k=1}^{m} \varepsilon^k W_k(x,t,y)\right)(y) \\
&\quad + \sum_{k=m}^{m+1} \varepsilon^k \operatorname{tr}(A(y)X_k(x,t,y)) \\
&= \sum_{k=2}^{m} \sum_{\substack{m \le i_1+\cdots+i_k \le km \\ 1 \le i_1,\cdots,i_k \le m}} \frac{\varepsilon^{i_1+\cdots+i_k}}{k!} B_k(x,t,y)(W_{i_1}(x,t,y),\cdots,W_{i_k}(x,t,y)),
\end{aligned} \tag{5.93}
$$

with $R_{m-1}(p_0,p)$ being the $(m-1)$-th order remainder term of H in p at p_0.

Now using (5.90), we observe that η_m^ε solves (5.86) with

$$
\psi_m^\varepsilon(x,t,y) = \varepsilon^m \partial_t w_m(x,t,y) + E_m^\varepsilon(x,t,y). \tag{5.94}
$$

Note that the initial condition of (5.86) is satisfied, due to that of (5.43) and the assertion (ii) of Lemma 5.5.3. Hence, we are only left with proving the estimate (5.87) for ψ_m^ε.

It is clear that (5.70) implies

$$\|\partial_t w_m(x,t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq C_{m,T},$$

for any $T > 0$ and any $(x,t) \in \mathbb{R} \times [0,T]$, where $C_{m,T} > 0$ is a constant depending only on n, λ, Λ, α, α', β, β', K, L, μ, m and T. On the other hand, using (5.9), (5.41) and (5.70), and noting that $\varepsilon^{i_1+\cdots+i_k} \leq \varepsilon^m$ for any $1 \leq i_1, \cdots, i_k \leq m$ satisfying $m \leq i_1 + \cdots + i_k \leq km$, we deduce from (5.93) that

$$\|E_m^\varepsilon(x,t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq C_{m,T}\varepsilon^m, \tag{5.95}$$

for any $T > 0$ and any $(x,t) \in \bar{B}_R \times [0,T]$, with $C_{m,T} > 0$ being yet another constant depending only on the same parameters listed above. This finishes the proof. □

With the aid of Lemma 5.5.5, we prove the first main result of this paper.

Theorem 5.5.6. *Suppose that the diffusion coefficient A, the Hamiltonian H and the initial data g satisfy* (5.3) – (5.5), (5.6) – (5.9), *and respectively* (5.49) – (5.52). *Under these conditions, let $\{u^\varepsilon\}_{\varepsilon>0}$ be the sequence of the viscosity solutions of* (5.1). *Then with the viscosity solution $\bar{u}_0$ of* (5.43) *and the sequence $\{w_k\}_{k=1}^\infty$ of k-th order interior correctors chosen in Lemma 5.5.3, we have, for each integer $m \geq 1$, any $0 < \varepsilon \leq \frac{1}{2}$ and any $T > 0$,*

$$\left| u^\varepsilon(x,t) - \bar{u}_0(x,t) - \sum_{k=1}^m \varepsilon^k w_k\left(x,t,\frac{x}{\varepsilon}\right) \right| \leq C_{m,T}\varepsilon^m,$$

uniformly for all $(x,t) \in \mathbb{R}^n \times [0,T]$, where $C_{m,T} > 0$ depends only on n, λ, Λ, α, α', β, β', K, L, μ, m and T.

Proof. The proof follows from Lemma 5.5.5 and the comparison principle for viscosity solutions. Let η_m^ε be as in (5.85). Due to (5.87), we see that $\eta_m^\varepsilon + C_{m,T}\varepsilon^m t$ and $\eta_m^\varepsilon - C_{m,T}\varepsilon^m t$ are a viscosity supersolution and, respectively, a viscosity subsolution of (5.1). Thus, the comparison principle yields that

$$|u^\varepsilon(x,t) - \eta_m^\varepsilon(x,t)| \leq TC_{m,T}\varepsilon^m,$$

uniformly for all $(x,t) \in \mathbb{R}^n \times [0,T]$, which finishes the proof. □

5.6 Generalization to Fully Nonlinear Hamiltonian

In this section, we generalize Theorem 5.5.6 to the fully nonlinear, viscous Hamilton-Jacobi equation, (5.2), whose gradient term is convex and grows quadratically at the infinity. Henceforth, we shall assume that the nonlinear functional H satisfies the following conditions, for any $(M,p,y) \in \mathcal{S}^n \times \mathbb{R}^n \times \mathbb{R}^n$.

(i) H is periodic in y:

$$H(M,p,y+k) = H(M,p,y), \tag{5.96}$$

for any $k \in \mathbb{Z}^n$.

(ii) H is uniformly elliptic in M:

$$\lambda |N| \leq H(M,p,y) - H(M+N,p,y) \leq \Lambda |N|, \tag{5.97}$$

for any $N \in \mathcal{S}^n$ with $N \geq 0$.

(iii) H has interior $C^{2,\bar{\mu}}$ estimates: Let $(M,p,y_0) \in \mathcal{S}^n \times \mathbb{R}^n \times \mathbb{R}^n$ and $a \in \mathbb{R}$ with $t \in \mathbb{R}$ such that $H(M+tI,p,y_0) = H(tI,p,y_0) = a$ and $|t| \leq \lambda^{-1}|H(0,p,y_0) - a|$. Then for any $v_0 \in C(\partial B_1(y_0))$, there exists a viscosity solution $v \in C(\bar{B}_1(y_0)) \cap C^2(B_1(y_0)) \cap C^{2,\bar{\mu}}(\bar{B}_{1/2}(y_0))$ of

$$\begin{cases} H(D^2 v + M + tI, p, y_0) = a & \text{in } B_1(y_0), \\ v = v_0 & \text{on } \partial B_1(y_0), \end{cases}$$

such that

$$\|v\|_{C^{2,\bar{\mu}}(\bar{B}_{1/2}(y_0))} \leq K \|v_0\|_{L^\infty(\partial B_1(y_0))}.$$

(iv) H is convex in p:

$$H(M, tp + (1-t)q, y) \leq tH(M,p,y) + (1-t)H(M,q,y),$$

for any $0 \leq t \leq 1$ and any $q \in \mathbb{R}^n$.

(v) H has quadratic growth in p:

$$\alpha|p|^2 - \alpha' \le H(0,p,y) \le \beta|p|^2 + \beta'. \tag{5.98}$$

(vi) $H \in C^\infty(\mathcal{S}^n \times \mathbb{R}^n; C^{0,1}(\mathbb{R}^n))$ and

$$\left\| D_M^k D_p^l H(M,p,\cdot) \right\|_{C^{0,1}(\mathbb{R}^n)} \le K\left(1 + |M|^{(1-k)_+} + |p|^{(2-l)_+}\right), \tag{5.99}$$

for any pair (k,l) of nonnegative integers.

We shall impose the conditions (5.49) – (5.52) to the initial data g, as we did in the preceding section, once the effective Hamiltonian $\bar{H}$ is determined. The effective Hamiltonian $\bar{H}$ is derived by solving the cell problem (5.100), stated as follows.

Lemma 5.6.1. *For each $p \in \mathbb{R}^n$, there exists a unique real number γ, for which the following PDE,*

$$H(D^2 w, Dw + p, y) = \gamma \quad \text{in } \mathbb{R}^n, \tag{5.100}$$

has a periodic solution $w \in C^{2,\mu}(\mathbb{R}^n)$, for some $0 < \mu < \bar{\mu}$ depending only on n, λ, Λ and $\bar{\mu}$. Moreover, γ satisfies (5.11) *and, furthermore, a periodic solution w of* (5.100) *is unique up to an additive constant and it satisfies* (5.12).

Proof. As in the proof of Lemma 5.3.1, we shall consider the approximated problem, with $\delta > 0$,

$$H(D^2 w^\delta, Dw^\delta + p, y) + \delta w^\delta = 0 \quad \text{in } \mathbb{R}^n. \tag{5.101}$$

Due to the uniform ellipticity (5.97) and the presence of the term δw^δ, (5.101) admits the comparison principle. Thus, there exists a unique viscosity solution w^δ of (5.101) satisfying (5.14). On the other hand, the uniqueness along with the periodicity (5.96) of H implies that w^δ is also periodic in y.

The uniform Lipschitz estimate (5.15) of w^δ can be deduced similarly as in the proof of Lemma 5.3.2. The only difference is that when performinig the classical Bernstein technique, the coefficients in the PDE (5.16) are given by $A(y) = D_M H(D^2 w^\delta(y), Dw^\delta(y) + p, y)$, $B(y) = D_p H(D^2 w^\delta(y), Dw^\delta(y) + p, y)$ and $E(y) = D_y H(D^2 w^\delta(y), Dw(y) + p, y)$. The rest of the argument follows only with a minor modification, and we leave out the detail to the reader.

Once we have the uniform Lipschitz estimate (5.15), we may consider

$$\begin{aligned} G_\delta(M, y) &= -H(M, Dw^\delta(y) + p, y) + H(0, Dw^\delta(y) + p, y), \\ g_\delta(y) &= -H(0, Dw^\delta(y) + p, y) + \delta w^\delta(y), \end{aligned}$$

and rephrase (5.101) as

$$G_\delta(D^2 w^\delta, y) = g_\delta(y) \quad \text{in } B_1(y_0), \tag{5.102}$$

where $y_0 \in \mathbb{R}^n$ is arbitrary. By (5.97), G_δ is uniformly elliptic in M over $\delta > 0$, while by (5.15) and (5.98), g_δ is uniformly bounded over $\delta > 0$. Hence, applying the interior $C^{1,\eta}$ estimate [9, Theorem 8.3] to (5.102), we obtain that $w^\delta - w^\delta(0) \in C^{1,\eta}(\bar{B}_{1/2}(y_0))$ for some $0 < \eta < 1$, depending only on n, λ and Λ, and

$$\begin{aligned} \left\| w^\delta - w^\delta(0) \right\|_{C^{1,\eta}(\bar{B}_{1/2}(y_0))} &\leq C_1 \left(\left\| w^\delta - w^\delta(0) \right\|_{L^\infty(B_1(y_0))} + \|g_\delta\|_{L^\infty(B_1(y_0))} \right) \\ &\leq \tilde{C}_1 \left(\operatorname*{osc}_{\mathbb{R}^n} w^\delta + 1 \right), \end{aligned}$$

where $C_1, \tilde{C}_1 > 0$ depend at most on n, λ, Λ, α, α', β', β', K and $|p|$. Here the second inequality can be deduced from (5.14), (5.15) and (5.98). Since y_0 is an arbitrary of $\mathbb{R}^n$, we observe from the last inequality and (5.15) that $w^\delta - w^\delta(0) \in C^{1,\eta}(\mathbb{R}^n)$ and

$$\left\| w^\delta - w^\delta(0) \right\|_{C^{1,\eta}(\mathbb{R}^n)} \leq C_2, \tag{5.103}$$

where $C_2 > 0$ depends at most on n, λ, Λ, α, α', β', β', $\bar{\mu}$, K and $|p|$.

Now that $w^\delta - w^\delta(0) \in C^{1,\eta}(\mathbb{R}^n)$, we may fix any $y_0 \in \mathbb{R}^n$ and consider

$$\begin{aligned} F_\delta(M, y) &= -H(M + tI, Dw^\delta(y + y_0) + p, y + y_0) - \delta w^\delta(y_0), \\ f_\delta(y) &= \delta w^\delta(y + y_0) - \delta w^\delta(y_0), \end{aligned}$$

where $t \in \mathbb{R}$ is chosen such that $H(tI, q, y_0) = a$ and $|t| \leq \lambda^{-1}|H(0, q, y_0) - a|$ with $q = Dw^\delta(y_0) + p$ and $a = -\delta w^\delta(y_0)$. Due to (5.15), we know that $|t| \leq C_{|p|}$. With F_δ and f_δ, we can rewrite (5.101) as

$$F_\delta(D^2 v^\delta, y) = f_\delta(y) \quad \text{in } B_1 \text{ with } v^\delta(y) = w^\delta(y+y_0) - w^\delta(y_0) - \frac{t}{2}|y|^2. \tag{5.104}$$

Note that $F_\delta(0,0) = f_\delta(0) = 0$. Moreover, due to the structure condition (iii) of H, F_δ satisfies F_δ also has the interior $C^{2,\bar{\mu}}$ estimate for fixed coefficient (cf. [9, Theorem 8.1]). On the other hand, it follows from (5.103) that F_δ and f_δ is C^ν at the origin (cf. [9, Eq. (8.3)]) with $\nu = \min\{\eta, \bar{\mu}\}$. Hence, the interior $C^{2,\nu}$ estimate [9, Theorem 8.1 and Remark 3] is applicable for (5.104), from which we deduce that $v^\delta \in C^{2,\nu}(\bar{B}_{1/2})$ with

$$\begin{aligned}\left\| v^\delta \right\|_{C^{2,\nu}(\bar{B}_{1/2})} &\leq C_3 \left(\left\| v^\delta \right\|_{L^\infty(B_1)} + \|f_\delta\|_{C^\nu(B_1)} \right) \\ &\leq \tilde{C}_3 \left(\operatorname*{osc}_{B_1(y_0)} w^\delta + |t| + \delta [w^\delta]_{C^\nu(B_1(y_0))} \right),\end{aligned}$$

where $C_3, \tilde{C}_3 > 0$ depends at most on n, λ, Λ, α, α', β', β', $\bar{\mu}$, ν, K and $|p|$. Since $|t| \leq C_{|p|}$ and $y_0 \in \mathbb{R}^n$ is arbitrary, we conclude from (5.15), (5.103) and the last inequality that $w^\delta - w^\delta(0) \in C^{2,\nu}(\mathbb{R}^n)$ and

$$\left\| w^\delta - w^\delta(0) \right\|_{C^{2,\nu}(\bar{B}_{1/2}(y_0))} \leq C_4, \tag{5.105}$$

where $C_4 > 0$ depends at most on n, λ, Λ, α, α', β', β', $\bar{\mu}$, ν, K and $|p|$.

With (5.105), we have, in particular, $w^\delta - w^\delta(0) \in C^{1,1}(\mathbb{R}^n)$, whence one can repeat the argument above now with any $0 < \nu = \mu < \bar{\mu}$, so that we have (5.23). The rest of the proof concerning the uniqueness of the limit of $\{\delta w^\delta\}_{\delta > 0}$ follows the same argument in that of Lemma 5.3.1, and we omit the details. □

As in Section 5.4, we shall denote by $\bar{H}$ the effective Hamiltonian of H. That is, $\bar{H} : \mathbb{R}^n \to \mathbb{R}$ is a function defined in such a way that for each $p \in \mathbb{R}^n$, $\bar{H}(p)$ is the unique real number for which the following PDE,

$$H(D^2 w, Dw + p, y) = \bar{H}(p) \quad \text{in } \mathbb{R}^n, \tag{5.106}$$

has a periodic viscosity solution in $C^{2,\mu}(\mathbb{R}^n)$. Moreover, we shall also denote by $w : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by the functional such that for each $p \in \mathbb{R}^n$, $w(p, \cdot) \in C^{2,\mu}(\mathbb{R}^n)$ is the unique periodic solution of (5.106) that is normalized so as to satisfy (5.25).

Following the same arguments in their proofs, one may prove that $\bar{H}$ and w satisfy Lemma 5.3.3 and Lemma 5.4.1, except for that $w \in C(\mathbb{R}^n; C^{2,\mu}(\mathbb{R}^n))$ for some fixed $0 < \mu < \bar{\mu}$, rather than any $0 < \mu < 1$. This is because the proofs of those lemmas do not rely on the linear structure of the diffusion coefficient, but more on its uniform ellipticity. A more important observation is the generalization of Lemma 5.4.4, which amounts to the regularity of $\bar{H}$ and w in the slow variables.

Lemma 5.6.2. *$\bar{H} \in C^\infty(\mathbb{R}^n)$ and $w \in C^\infty(\mathbb{R}^n; C^{2,\mu}(\mathbb{R}^n))$, for any $0 < \mu < \bar{\mu}$, such that* (5.41) *holds, for any $k = 0, 1, 2, \cdots$, any $L > 0$ and any $p \in B_L$.*

Proof. Let us fix $0 < \mu < \bar{\mu}$. It suffices to prove that $\bar{H}$ and w verify Lemma 5.4.2. Moreover, to see this fact, it is enough to show that the linearization argument in the proof of Lemma 5.4.2 also works out when the Hamiltonian H depends nonlinearly on the Hessian variable M.

Let w_h, γ_h, W_h and Γ_h be as in the proof of Lemma 5.4.2. Then by linearizing the cell problem (5.106) (in both of the Hessian and the gradient variables), we observe that W_h solves

$$- \operatorname{tr}(A_h(y) D^2 W_h) + B_h(y) \cdot (DW_h + e_k) = \Gamma_h \quad \text{in } \mathbb{R}^n, \tag{5.107}$$

where

$$A_h(y) = \int_0^1 -D_M H(tD_y^2 w_h + (1-t) D_y^2 w_0, D_y w_h + p, y) dt,$$

and

$$B_h(y) = \int_0^1 D_p H(D_y^2 w_0, tD_y w_h + (1-t) D_y w_0 + p + the, y) dt.$$

In comparison of (5.107) with (5.31), one may see that the only major difference here is that the diffusion coefficient, A_h, here is not fixed but depends on the parameter h.

Nevertheless, A_h is uniformly elliptic not only in y but also in h, due to the assumption (5.97). This implies that Lemma 5.4.3 is still applicable, and thus $W_h \in C^{2,\mu}(\mathbb{R}^n)$ and satisfies (5.34) uniformly for h.

Moreover, since w satisfies (5.30), it follows from the regularity assumption (5.99) of H that $A_h \in C^\mu(\mathbb{R}^n)$ and

$$\|A_h\|_{C^\mu(\mathbb{R}^n)} \leq C,$$

where $C > 0$ depends only on n, λ and Λ. For the same reason, we deduce that $B_h \in C^\mu(\mathbb{R}^n)$ and satisfies (5.32). Furthermore, since $w \in C(\mathbb{R}^n; C^{2,\mu}(\mathbb{R}^n))$, we have $A_h \to A_0$ and $B_h \to B_0$ in $C^{\mu'}(\mathbb{R}^n)$, for any $0 < \mu' < \mu$, with

$$A_0(y) = -D_M H(D_y^2 w_0, D_y w_0 + p, y),$$

and

$$B_0(y) = D_p H(D_y^2 w_0, D_y w_0 + p, y).$$

The rest of the proof follows similarly with that of of Lemma 5.4.2. In particular, we obtain unique $W_0 \in C^{2,\mu}(\mathbb{R}^n)$ and $\Gamma_0 \in \mathbb{R}$ such that W_0 is the periodic solution

$$-\operatorname{tr}(A_0(y)D^2 W_0) + B_0(y) \cdot (DW_0 + e_k) = \Gamma_0 \quad \text{in } \mathbb{R}^n,$$

satisfying $W_0(0) = 0$. We leave out the details to the reader. □

Now we are in position to construct the higher order interior correctors of the homogenization problem (5.2). We shall now let g satisfy the structure conditions (5.49) – (5.52), with $\bar{H}$ being the effective Hamiltonian chosen to satisfy the cell problem (5.106). Next we shall denote by $\bar{u}_0$ the solution of (5.43), with the updated data $\bar{H}$ and g, and write by $\bar{B}$ the function defined by (5.48). Once again, we have from Lemma 5.5.1 and Lemma 5.6.2 that $\bar{u}_0 \in C^\infty(\mathbb{R}^n \times [0,\infty))$ and $\bar{B} \in C^\infty(\mathbb{R}^n \times [0,\infty))$ with the estimates (5.54) and (5.58).

Let w_0, $\{W_k\}_{k=0}^\infty$ and $\{X_k\}_{k=0}^\infty$ denote those defined in (5.65), (5.66) and, respectively, (5.88), where the sequence $\{w_k\}_{k=1}^\infty$ of higher order interior correctors will be given as below.

Now that the Hamiltonian H is nonlinear in M, we need to apply the Taylor expansion not only in the variable p but also in the variable M, in order to obtain the PDEs (or, more precisely, the cell problems) for the higher order interior correctors. In this direction, we consider the coefficient $B_{k,l}$ defined by

$$B_{k,l}(x,t,y) = D_M^k D_p^l H(X_0(x,t,y), W_0(x,t,y), y),$$

for $k, l = 0, 1, 2, \cdots$. In particular, we shall write

$$A(x,t,y) = -B_{1,0}(x,t,y) = -D_M H(X_0(x,t,y), W_0(x,t,y), y),$$

and

$$B(x,t,y) = B_{0,1}(x,t,y) = D_pH(X_0(x,t,y), W_0(x,t,y), y).$$

Note that A is uniformly elliptic with the same ellipticity bounds as those of H.

Lemma 5.6.3. *Suppose that H and g satisfy* (5.96) – (5.99) *and, respectively,* (5.49) – (5.52)*. Then there exists a sequence $\{w_k\}_{k=1}^{\infty}$ satisfying the following.*

(i) $w_k \in C^\infty(\mathbb{R}^n \times [0,\infty); C^{2,\mu}(\mathbb{R}^n))$*, for any* $0 < \mu < \bar{\mu}$*, and satisfies the estimate* (5.70)*, for any* $i, j = 0, 1, 2, \cdots$*, any* $T > 0$ *and any* $(x,t) \in \mathbb{R}^n \times [0,T]$.

(ii) w_k *is normalized so as to satisfy* (5.71).

(iii) For each $(x,t) \in \mathbb{R}^n \times (0,\infty)$*,* $w_k(x,t,\cdot)$ *is a periodic solution of*

$$\partial_t w_0(x,t,y) + H(D_y^2 w_1, D_y w_1 + D_x w_0(x,t,y), y) = 0 \quad \text{in } \mathbb{R}^n, \tag{5.108}$$

for $k = 1$*, and*

$$\begin{aligned}&\partial_t w_{k-1}(x,t,y) - \operatorname{tr}(A(x,t,y) D_y^2 w_k) \\ &\quad + B(x,t,y) \cdot (D_y w_k + D_x w_{k-1}(x,t,y)) + \Phi_{k-1}(x,t,y) = 0 \quad \text{in } \mathbb{R}^n,\end{aligned} \tag{5.109}$$

for $k \geq 2$*, where*

$$\begin{aligned}&\Phi_{k-1}(x,t,y) \\ &= -2\operatorname{tr}(A(x,t,y)(D_x D_y w_{k-1}(x,t,y) + D_x^2 w_{k-2}(x,t,y))) \\ &\quad + \sum_{l=2}^{k-1} \frac{1}{l!} \sum_{\substack{i_1+\cdots+i_l=k-1 \\ i_1,\cdots,i_1\geq 1}} \sum_{r=0}^{l} B_{r,l-r}(x,t,y)(X_{i_1}(x,t,y), \cdots, X_{i_r}(x,t,y), \\ &\qquad\qquad W_{i_{r+1}}(x,t,y), \cdots, W_{i_l}(x,t,y))\end{aligned} \tag{5.110}$$

with the last summation term understood as zero when $k = 2$.

Remark 5.6.4. *As mentioned in Remark 5.5.4,* Φ_k *now takes care of the nonlinear effect produced by* H *in both* M *and* p *variables. Moreover, the* *summation term in the definition* (5.110) *of* Φ_k *is specifically constructed to have* (5.112)*, by which we will eventually derive the higher order convergence rates for the homogenization problem* (5.2)*.*

Proof of Lemma 5.6.3. The proof follows essentially the same induction argument presented in that of Lemma 5.5.3. To avoid any repeating argument, we shall only point out the major difference from the proof of Lemma 5.5.3, and ask the reader to fill in the details.

Here we define ϕ_1 by (5.60) with w being the (normalized) periodic solution of (5.106) (instead of (5.24)), and accordingly set w_1 by (5.59) with some $\bar{u}_1$ to be determined. Then we observe that W_0 and X_0 verify the expressions (5.67) and, respectively, (5.89). Moreover, we verify that $B_{l,k-l}$ satisfy the estimate (5.69), for any $l = 0, 1, \cdots, k$ and any $k = 1, 2, \cdots$.

The function f_k, which takes cares of all the nonlinear effect caused in the k-th step of approximation, is now replaced by

$$\begin{aligned} f_k(x,t,y) = {} & \partial_t \tilde{w}_k(x,t,y) + B(x,t,y) \cdot D_x \tilde{w}_k(x,t,y) \\ & - 2\,\mathrm{tr}(A(x,t,y)(D_x D_y \tilde{w}_k(x,t,y) + D_x^2 w_{k-1}(x,t,y))) \\ & + \sum_{l=2}^{k-1} \frac{1}{l!} \sum_{\substack{i_1+\cdots+i_l=k-1 \\ i_1,\cdots,i_1 \geq 1}} \sum_{r=0}^{l} B_{r,l-r}(x,t,y)(X_{i_1}(x,t,y), \cdots, X_{i_r}(x,t,y), \\ & \qquad W_{i_{r+1}}(x,t,y), \cdots, W_{i_l}(x,t,y)), \end{aligned}$$

Due to the periodicity of f_k in y, we consider the following cell problem: there exists a unique $\bar{f}_k : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}^n$ such that for each $(x,t) \in \mathbb{R}^n \times (0,\infty)$, the PDE,

$$-\mathrm{tr}(A(x,t,y) D_y^2 \phi_{k+1}) + B(x,t,y) \cdot D_y \phi_{k+1} + f_k(x,t,y) = \bar{f}_k(x,t) \quad \text{in } \mathbb{R}^n,$$

has a periodic viscosity solution. The rest of the proof can be derived by following that of Lemma 5.5.3, whence we omit the details. □

The next lemma is the corresponding version of Lemma 5.5.5 for fully nonlinear Hamiltonian H.

Lemma 5.6.5. *Let* $\{w_k\}_{k=1}^\infty$ *be chosen as in Lemma 5.6.3. Then for each integer* $m \geq 1$ *and each* $0 < \varepsilon \leq \frac{1}{2}$*, the function* η_m^ε*, defined by* (5.85)*, is a* *viscosity solution of*

$$\begin{cases} \partial_t \eta_m^\varepsilon + H\left(\varepsilon D^2 \eta_m^\varepsilon, D\eta_m^\varepsilon, \dfrac{x}{\varepsilon}\right) = \psi_m^\varepsilon\left(x, t, \dfrac{x}{\varepsilon}\right) & \text{in } \mathbb{R}^n \times (0,\infty), \\ \eta_m^\varepsilon = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases} \tag{5.111}$$

where $\psi_m^\varepsilon \in C(\mathbb{R}^n \times [0,\infty); L^\infty(\mathbb{R}^n))$ *satisfies* (5.87), *for any* $T > 0$ *and all* $(x,t) \in \mathbb{R}^n \times [0,T]$.

Proof. As in the proof of Lemma 5.6.3, we shall mention the key points that need to be modified from the proof of Lemma 5.5.5, in order to take care of the nonlinear effect in the Hessian variable of H. Let us begin by fixing $m \geq 1$ and $0 < \varepsilon \leq \frac{1}{2}$, and replacing w_{m+1} and w_{m+2} by the identically zero functions, again for the notational convenience.

We shall define Ψ_k by

$$\Psi_0(x,t,y) = H(X_0(x,t,y), W_0(x,t,y), y),$$

if $k = 0$, and by

$$\Psi_k(x,t,y) = \sum_{l=1}^{k-1} \frac{1}{l!} \sum_{\substack{i_1+\cdots+i_l=k-1 \\ i_1,\cdots,i_1 \geq 1}} \sum_{r=0}^{l} B_{r,l-r}(x,t,y)(X_{i_1}(x,t,y), \cdots, X_{i_r}(x,t,y),$$
$$W_{i_{r+1}}(x,t,y), \cdots, W_{i_l}(x,t,y))$$

if $1 \leq k \leq m-1$. Then it follows from the PDEs (5.108) and (5.109) that (5.90) holds for $0 \leq k \leq m-1$.

Applying the Taylor expansion of H in (M,p) at (X_0, W_0) up to $(m-1)$-th order, and after some calculations similar to those in (5.91), we obtain that

$$H\left(\varepsilon D^2 \eta_m^\varepsilon(x,t), D\eta_m^\varepsilon(x,t), \frac{x}{\varepsilon}\right) = \sum_{k=0}^{m-1} \varepsilon^k \Psi_k\left(x,t,\frac{x}{\varepsilon}\right) + E_m^\varepsilon\left(x,t,\frac{x}{\varepsilon}\right), \tag{5.112}$$

where E_m^ε is defined so as to satisfy

$$\begin{aligned}
&E_m^\varepsilon(x,t,y)\\
&-R_{m-1}\left((X_0(x,t,y),W_0(x,t,y)),\left(\sum_{k=1}^{m+1}\varepsilon^k X_k(x,t,y),\sum_{k=1}^{m}\varepsilon^k W_k(x,t,y)\right)\right)(y)\\
&=\sum_{k=2}^{m}\sum_{\substack{m\le i_1+\cdots+i_k\le km\\ 1\le i_1,\cdots,i_k\le m}}\frac{\varepsilon^{i_1+\cdots+i_k}}{k!}\sum_{l=0}^{k}B_{l,k-l}(x,t,y)(X_{i_1}(x,t,y)\cdots,X_{i_l}(x,t,y),\\
&\qquad\qquad\qquad\qquad W_{i_{l+1}}(x,t,y),\cdots,W_{i_k}(x,t,y)),
\end{aligned}$$

where $R_{m-1}((M_0,p_0),(M,p))$ denotes the $(m-1)$-th order remainder term of H in (M,p) at (M_0,p_0).

We deduce from (5.112) that η_m^ε solves (5.111) with ψ_m^ε defined by (5.94). The rest of the proof follows similarly to that of Lemma 5.5.5. In particular, we have (5.95), since $B_{l,k-l}$ and w_k satisfy the estimate (5.69) and, respectively, (5.70). We leave out the details to the reader. □

Finally, we generalize Theorem 5.5.6 to the regime of fully nonlinear, viscous Hamilton-Jacobi equation, as stated below.

Theorem 5.6.6. *Suppose that the Hamiltonian H and the initial data g satisfy* (5.96) – (5.99) *and, respectively,* (5.49) – (5.52). *Under these conditions, let $\{u^\varepsilon\}_{\varepsilon>0}$ be the sequence of the viscosity solutions of* (5.2). *Then with the viscosity solution $\bar u_0$ of* (5.43) *and the sequence $\{w_k\}_{k=1}^\infty$ of k-th order interior correctors chosen in Lemma 5.6.3, we have, for each integer $m\ge 1$, any $0<\varepsilon\le\frac12$ and any $T>0$,*

$$\left|u^\varepsilon(x,t)-\bar u_0(x,t)-\sum_{k=1}^m\varepsilon^k w_k\left(x,t,\frac{x}{\varepsilon}\right)\right|\le C_{m,T}\varepsilon^m,$$

uniformly for all $(x,t)\in\mathbb{R}^n\times[0,T]$, where $C_{m,T}>0$ depends only on n, λ, Λ, α, α', β, β', K, L, μ, $\bar\mu$, m and T.

Proof. The proof follows the same comparison argument as that in the proof of Theorem 5.5.6. Let η_m^ε be as in Lemma 5.6.5. According to Lemma 5.6.5, $\eta_m^\varepsilon+C_{m,T}\varepsilon^m t$ and $\eta_m^\varepsilon-C_{m,T}\varepsilon^m t$ are a viscosity supersolution and, respectively, a viscosity subsolution of (5.2), for some constant $C_{m,T}>0$ depending only on n, λ, Λ, α, α', β, β', K, L, μ, $\bar{\mu}$, m and T. Therefore, the comparison principle yields that

$$|u^\varepsilon(x,t) - \eta_m^\varepsilon(x,t)| \leq TC_{m,T}\varepsilon^m,$$

uniformly for all $(x,t) \in \mathbb{R}^n \times [0,T]$. This completes the proof. $\square$

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국문초록

균질화 이론은 미세영역에서 특정한 패턴으로 빠르게 진동하는 일련의 편미분방정식의 평균화 현상을 연구한다. 본 학위논문은 완전비선형방정식의 주기적 균질화에서의 고차수렴속도에 대한 세 편의 연구논문으로 구성되어 있다. 첫 번째 논문은 비축중성 고른 타원형 방정식의 내부 수정에 주안점을 두고 있고, 두 번째 논문은 고른 포물선형 코쉬 문제에서 빠르게 진동하는 초기조건이 끼치는 영향에 대하여 연구한다. 마지막 논문에서 우리는 점성적 해밀턴-야코비 방정식에 관한 흥미로운 현상을 발견하는데, 고차수렴속도를 얻으려면 균질화된 해밀턴 작용소에 따라 결정되는 특수한 기하학적 성질이 만족되도록 초기조건을 결정해야한다는 사실이다. 세 편의 연구논문을 관통하는 핵심적인 해석기법은 미세영역에서 진동하지 않는 변수에 대한 정칙성 이론을 개발하는 것이다. 이러한 정칙성 결과는 고차 수정자를 귀납적으로 정 의할 수 있는 이론적 토대로 작용한다. 여기서 고차 수정자는 매우 진동적인 편미분방정식의 비선형적 구조로부터 기인하는 오차를 수정할 수 있도록 설계되어 있고, 고차수렴속도는 고차 수정자를 사용한 적절한 장벽 논리로부터 얻게 된다.

주요어: 균질화, 완전비선형방정식, 주기적 상황, 수렴속도, 수정자, 점성해, 고차, 정칙성
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감사의 글

먼저 석박통합과정을 지도해주신 이기암 교수님께 감사의 마음을 전합니다. 교수님께서 처음 주신 연구주제는 제가 연구해나가는 데에 있어서 소중한 밑거름이 되었습니다. 연구에 자신감을 잃을 때에도 따뜻하게 격려해주시고 늘 발전적인 방향으로 지도하여주신 덕분에 본 박사과정을 수월하게 마칠 수 있었습니다. 교수님을 본 받아 성실하고 열정적인 연구자가 되도록 끊임없이 노력하겠습니다.

바쁘신 가운데에도 박사학위논문 심사를 맡아주신 변순식 교수님, 이상혁 교수님, 박형빈 교수님 그리고 김성훈 교수님께도 감사드립니다. 교수님들의 세심한 조언과 새로운 안목은 박사논문을 완성하는데 큰 도움이 되었습니다.

박사과정 동안 연구실을 함께 사용한 동료들에게도 감사를 표합니다. 특히 같은 분단을 사용한 성하 형, 민현, 태훈 그리고 탁원은 다소 지루할 수도 있었던 연구생활에 활력소가 되었고, 제가 막혀 있는 부분을 자기 문제인 것처럼 고민해준 고마운 친구들입니다.

여러 차례 스웨덴 왕립공과대학에 초대하여 방문 연구할 기회를 주신 Henrik Shahgholian 교수님께도 감사를 표합니다. 교수님의 세심한 지도로 여러 좋은 연구 성과를 낼 수 있었습니다. 스웨덴에 머무는 동안 여러모로 도와준 Andreas Minne 박사와 Martin Strömqvist 박사에게도 고마운 마음을 전합니다.

끝으로 모난 저를 사랑으로 길러주신 부모님, 하루가 다르게 귀여워지는 아들 유진이, 그리고 이 세상에서 가장 사랑하는 아내 수연이에게 깊은 감사의 마음을 전합니다. 이들의 한결같은 응원과 사랑 없이는 외로운 박사과정을 결코 마칠 수 없었을 것입니다.