Comparative Investigations of Modified Bivariate Dimension Methods for Statistical Moments Assessment of response

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ABSTRACT: Evaluating the statistical moments of performance functions which aims at keeping the tradeoff of accuracy and efficiency is still a challenge in structure reliability analysis. This paper proposes several modified bivariate dimension reduction methods (BDRM), based on two-dimensional and one-dimensional Gauss-Hermite quadrature. Compared to the original BDRM, evaluating central moments by these modified BDRMs requires less computational effort. One numerical example is investigated, which demonstrate that two of the modified BDRMs can achieve good balance between accuracy and efficiency for statistical moment assessment.

1. INTRODUCTION

The statistical central moments estimation of structural performance functions with uncertain parameters is one of the main topics for analyzing random structures *(Fan et al. 2016)* and plays an important role in the moment methods based reliability assessment *(Zhao and Ono 2001)*. This problem usually presents as the first few central moments with basic random variables involved.

Generally, the interested response function is written as $Z = G(X)$, where $\mathbf{X} = [x_1, x_2, \ldots, x_n]^T$ is the vector of basic random variables related to structural properties and loading conditions. In practice, such a response function is always a highly nonlinear implicit function with multiple random variables, thus the analytical solutions of the central moments are unavailable. This difficulty has drawn increasing attention on numerical solutions of central moments of response. Usually, these solutions are divided into two categories. The first category is the expansion based method such as the Taylor expansion method *(Ibrahim 1987)*. Yet calculating derivatives is required in these methods, which is difficult to solve when highly nonlinear functions are involved. The second category is the point estimate method (PEM), which involves the calculation of a set of finite points and weights of the interested functions *(Zhao and Ono 2000)*. It can be seen that the difficulty of calculation is reduced, because there is no need to solve the derivatives in PEM. In the early PEM, the number of points increases with dimension increasing *(Rosenblueth 1975)*. Yet it is inadequate for solving nonlinear problems especially for the estimation of higher-order statistical moments because of the low accuracy of the early PEM. Furthermore, the tensor product method *(Issacson and Keller 1994)* and the sparse grid method *(Gerstner and Griebel 1998)* can be employed to evaluate the central moments. However a huge amount of computation effort is still needed *(Xu et al. 2012)*. The dimension reduction method (DRM), *(Rahman and Xu 2004)* also a type of PEM, transforms a multi-dimensional problem into low dimensions, and then sums up a series of low-dimensional integrations. The univariate dimension reduction method (UDRM) *(Rahman and Xu 2004)* is highly efficient for central moment estimation, and thus it is widely used in practical applications. However, it is insufficient in evaluating moments of multi-dimensional and highly-interacted performance functions, since only one-dimensional integrations are involved in UDRM *(Xu and Rahman 2004)*. Alternatively, the bivariate dimension reduction method (BDRM) helps to improve the accuracy and theoretical adaptation *(Xu and Rahman 2004)*. The BDRM including several evaluations of two-dimensional and one-dimensional integrations, only requires the Gaussian points and related weights consistent with standard normal random variables since the Rosenblatt or Nataf transformation can be employed to transform the arbitrarily distributed random variables to be standard normal ones. Generally, tensor product rule for the two dimensional numerical integration will be widely applied, resulting in large computation effort. Therefore, numerical methods contributed to improving efficiency without losing accuracy for evaluating the central moments of response is of necessity to be found.

Aiming at balancing the computation effort and accuracy, this paper develops modified BDRMs in order to evaluate first-four central moments of response. This paper is organized as follows. In Section 2, the BDRM applied in unbiased moment estimation of response is first introduced. Then the advantages and disadvantages of the original BDRM are discussed, and the proposed modified BDRMs for moment estimation of response are presented. In Section 3, one numerical example is presented to check the accuracy as well as the efficiency. Finally, conclusions are given in Section 4.

2. PROPOSED MODIFIED BIVARIATE REDUCTION METHODS

2.1. Problem formulation

The first four central moments of a performance function $Z = G(\mathbf{X})$ usually relate to the moment methods based reliability analysis, which can be expressed as follows:

$$\mu_Z = \int_{\Omega_{\mathbf{X}}} z \cdot p_z(z)\,dz = \int_{\Omega_{\mathbf{X}}} G(\mathbf{x}) \cdot p_{\mathbf{X}}(\mathbf{x})\,d\mathbf{x} \tag{1}$$

$$\begin{aligned} M_{kZ} &= \int_{\Omega_{\mathbf{X}}} (z-\mu_Z)^k \cdot p_z(z)\,dz \\ &= \int_{\Omega_{\mathbf{X}}} \left(G(\mathbf{x})-\mu_Z\right)^k \cdot p_{\mathbf{X}}(\mathbf{x})\,d\mathbf{x} \end{aligned} \tag{2}$$

where $\mathbf{X} = [x_1, x_2, \ldots, x_n]^T$ are input basic random variables, $\Omega_{\mathbf{X}}$ is the distribution space of $\mathbf{X}$; $p_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function of random vector $\mathbf{X}$; μ_Z is the mean of *Z*; M_{kZ} is the k-order central moment of *Z*.

As a matter of fact, the integrations in Eq. (1) and (2) can be transformed to be

$$\begin{aligned} \mu_Z &= \int_{\Omega_{\mathbf{U}}} G\left[T^{-1}(\mathbf{U})\right] \cdot p_{\mathbf{U}}(\mathbf{u})\,d\mathbf{x} \\ &= \int_{\Omega_{\mathbf{U}}} H(\mathbf{U}) \cdot p_{\mathbf{U}}(\mathbf{u})\,d\mathbf{x} \end{aligned} \tag{3}$$

$$\begin{aligned} M_{kZ} &= \int_{\Omega_{\mathbf{U}}} \left(G\left[T^{-1}(\mathbf{U})\right]-\mu_Z\right)^k \cdot p_{\mathbf{U}}(\mathbf{u})\,d\mathbf{x} \\ &= \int_{\Omega_{\mathbf{U}}} \left(H(\mathbf{U})-\mu_Z\right)^k \cdot p_{\mathbf{U}}(\mathbf{u})\,d\mathbf{x} \end{aligned} \tag{4}$$

where $H(\mathbf{U}) = G\left[T^{-1}(\mathbf{U})\right]$; $T^{-1}(\bullet)$ denotes the inverse of the Rosenblatt transformation or Nataf transformation which transforms the non-normal random vector $\mathbf{X}$ into the independent standard normal vector $\mathbf{U} = [u_1, u_2, \ldots, u_n]^T$.

The bivariate dimension method BDRM for decomposing the response function gives

$$\begin{aligned} \mu_Z &= \sum_{k_1<k_2} \int_{U_{k_1}} \int_{U_{k_2}} H\left(u_{k_1}, u_{k_2}, \mathbf{0}\right) \prod_{j=2}^{2} p_{U_{k_j}}(u_{k_j})\,du_{k_j} \\ &\quad -(n-2)\sum_{k_1=1}^{n} \int_{U_{k_1}} H\left(u_{k_1}, \mathbf{0}\right) \cdot p_{U_{k_1}}(u_{k_1})\,du_{k_1} \\ &\quad + \frac{(n-1)(n-2)}{2} H(\mathbf{0}) \end{aligned} \tag{5}$$

$$M_{kZ} = \sum_{k_1<k_2} \int_{U_{k_1}} \int_{U_{k_2}} \left[H\left(u_{k_1}, u_{k_2}, \mathbf{0}\right) - \mu_Z \right]^k \prod_{j=2}^{2} p_{U_{k_j}}(u_{k_j}) du_{k_j}$$
$$-(n-2) \sum_{k_1=1}^{n} \int_{U_{k_1}} \left[H\left(u_{k_1}, \mathbf{0}\right) - \mu_Z \right]^k \cdot p_{U_{k_1}}(u_{k_1}) du_{k_1}$$
$$+ \frac{(n-1)(n-2)}{2} \left[H(\mathbf{0}) - \mu_Z \right]^k \quad (6)$$

where $\mathbf{0}$ is the zero vector. Usually, to evaluate the s-dimensional numerical integration $(s \geq 2)$ the tensor product method *(Xu et al. 2012)* is adopted. In this regard, the number of function evaluation in BDRM is

$$N = 1 + n \cdot (d-1) + \frac{n(n-1)}{2} \cdot (d-1)^2 \quad (7)$$

where n denotes the degree of the input random variables and $2 \leq n \leq \infty$; d is the number of Gaussian quadrature points.

In the original BDRM, the five-point Gaussian-Hermite rule is always used to evaluate the integration in Eq. (5) and (6). However, the computational effort of the original BDRM will be quite large. For example, if n=10, the total number of required evaluation function will be 761, where the number of points for calculating the two dimensional integrations is large as 720. Although the high accuracy of central moments can be yielded in the original BDRM for practical problems, massive computational effort may prohibit its application. Therefore, to achieve higher efficiency without losing accuracy, some modified BDRMs will be developed as follows.

2.2. Sparse grid based modified BDRM

It is noted that the sparse grid method *(Smolyak 1963)* can assure the same level of accuracy while utilizing fewer Gaussian points to evaluate the two-dimensional integration.

Let $U_1^{i_j} = \left\{ u_{j,m} \right\}$, $m = 1, 2, \ldots, d$ and $\Omega_1^{i_j} = \left\{ \omega_{j,m}^{i_j} \right\}$, $m = 1, 2, \ldots, d$ *(Xiu 2009)* denote the one-dimensional quadrature points and weights given by Gaussian-Hermite rule, where the subscript i_j is the quadrature's level of accuracy in the j-th dimension and the corresponding algebraic accuracy is $2i_j - 1$ *(Heiss and Winschel 2008)*.

Then, through applying the Smolyak algorithm, the sparse grid in two dimension, say $U_{(k_1,k_2)}^r$ with r -level $(r \geq 0)$ accuracy, can be defined as *(Xiong et al. 2010)*

$$U_{(k_1,k_2)}^r = \bigcup_{r+1 \leq |\mathbf{i}| \leq q} U_1^{i_{k_1}} \otimes U_1^{i_{k_2}} \quad (8)$$

where $q = r + 2$, and $|\mathbf{i}| = i_{k_1} + i_{k_2}$.

The corresponding weight for the l-th point $\mathbf{u}_l \in U_{(k_1,k_2)}^r$ reads

$$\omega_{2,l} = (-1)^{q-|\mathbf{i}|} \binom{d-1}{q-|\mathbf{i}|} \left(\omega_{j_{i_{k_1}}}^{i_{k_1}} \cdot \omega_{j_{i_{k_2}}}^{i_{k_2}} \right) \quad (9)$$

Thereby, the tensor product can reduce points from the full grid, improving the required algebraic accuracy. Namely, the integration accuracy of the sparse grid method is nearly identical with that of tensor product method. It has been proven that the sparse gird method can achieve $2r+1$-order of algebraic accuracy.

The formation of sparse grid in two-dimension with $r = 2$ is shown in Figure 1 *(Xiong et al. 2010)*. It is clear that compared to the tensor product method, much fewer integration points are utilized in the sparse grid method. In other words, the sparse grid method is able to fit in the bivariate dimensional reduction method to evaluate the two dimensional integration parts, and thus formulate a new modified BDRM. This method is named as the sparse grid based modified BDRM, and is referred to BDRM-SG for short.

2.3. Cubature based modified BDRM

Similarly, the cubature formulas with fixed algebraic accuracy may satisfy both the efficiency and accuracy for evaluating the involved two dimensional integrations in BDRM. Commonly, the fifth-order algebraic accuracy is

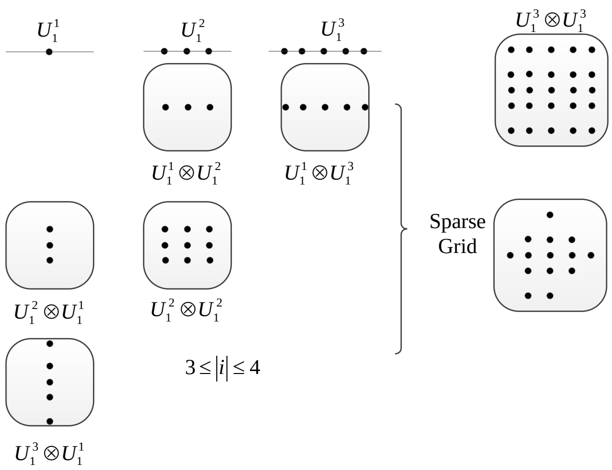

Figure 1:the distribution of the sparse grid

employed. In this subsection, three typical cubature formulas reappear and are incorporated into BDRM to raise the computational efficiency.These cubature formulas are called as Cubature I – III.

2.3.1. Cubature I

Such two dimensional cubature formula uses 8 integration points *(Stroud 1971)*, and is defined as

$$I[f] = A\left[f\left(\sqrt{2}\eta,\sqrt{2}\eta\right)+f\left(-\sqrt{2}\eta,-\sqrt{2}\eta\right)\right] + B\left[\begin{array}{l} f\left(\sqrt{2}\lambda,\sqrt{2}\xi\right)+f\left(\sqrt{2}\xi,\sqrt{2}\lambda\right) \\ +f\left(-\sqrt{2}\lambda,-\sqrt{2}\xi\right)+f\left(-\sqrt{2}\xi,-\sqrt{2}\lambda\right)\end{array}\right] + C\left[f\left(\sqrt{2}\mu,\sqrt{2}\mu\right)+f\left(-\sqrt{2}\mu,-\sqrt{2}\mu\right)\right] \quad (10)$$

where the parameters are listed in Table 1. This formula reduces the Gaussian points so that the efficiency of evaluating the two dimensional numerical integration will be pretty improved. Hence, this formula is denoted as BDRM-C-1.

2.3.2. Cubature II

This formula considers only 9 integration points *(Stroud and Secrest 1963)*, which is expressed as

$$I[f] = \frac{1}{2}f(0,0)+\frac{1}{16}\left[f(2,0)+f(0,2)+f(-2,0)+f(0,-2)\right] + \frac{1}{16}\left[f\left(\sqrt{2},\sqrt{2}\right)+f\left(\sqrt{2},-\sqrt{2}\right)+f\left(-\sqrt{2},\sqrt{2}\right)+f\left(-\sqrt{2},-\sqrt{2}\right)\right] \quad (11)$$

Table 1: Parameters in Cubature I

Parameter	*Value*
η	0.446103183094540
λ	1.366025403784440
ξ	-0.366025403784439
μ	1.981678829458710
A	0.328774019778636
B	0.083333333333333
C	0.004559313554697

When combining this formula and BDRM, the consequent modified BDRM is defined as BDRM-C-2.

2.3.3. Cubature III

Similarly, merely 9 integration points are needed in this two-dimensional cubature formula *(McNamee and Stenger 1967)*. The formula is written as

$$I[f] = \frac{4}{9}f(0,0)+\frac{1}{9}\left[f\left(\sqrt{3},0\right)+f\left(0,\sqrt{3}\right)+f\left(-\sqrt{3},0\right)+f\left(0,-\sqrt{3}\right)\right] + \frac{1}{36}\left[f\left(\sqrt{3},\sqrt{3}\right)+f\left(\sqrt{3},-\sqrt{3}\right)+f\left(-\sqrt{3},\sqrt{3}\right)+f\left(-\sqrt{3},-\sqrt{3}\right)\right] \quad (12)$$

Also, the resulting modified BDRM is called as BDRM-C-3.

2.3.4. Advantages

Obviously, the amount of required Gaussian integration points which are involved in the cubature based modified BDRMs are much fewer than that of the original BDRM, contributing much higher efficiency. Moreover, the computation effort will be further alleviated if recurring points exist in the modified BDRMs. For instance, let $n=3$, μ_Z can be calculated as

$$\mu_Z = \sum_{l=1}^{d_1}\omega_{2,l}H\left(u_{1,l},u_{2,l},0\right)+\sum_{l=1}^{d_1}\omega_{2,l}H\left(u_{1,l},0,u_{3,l}\right)+\sum_{l=1}^{d_1}\omega_{2,l}H\left(0,u_{2,l},u_{3,l}\right) -(n-2)\cdot\left[\sum_{s=1}^{d}\omega_{1,l}H\left(u_{1,s},0,0\right)+\sum_{s=1}^{d}\omega_{1,l}H\left(0,u_{2,s},0\right)+\sum_{s=1}^{d}\omega_{1,l}H\left(0,0,u_{3,s}\right)\right] + \frac{(n-1)(n-2)}{2}H(0,0,0) \quad (13)$$

Furthermore, if applying the BDRM-C-2 we can find that $H(0,0,0)$ is repeatedly evaluated in each summation. In this regard, the efficiency may be further improved through performing reoccurred deterministic analysis once.

2.4. High-order unscented transformation based modified BDRM

The high-order unscented transformation (HUT) can evaluate the first-four moments of input random variables with accuracy and efficiency. The basic idea of HUT is to capture a series of points (named sigma points) and corresponding weights which fit the high-order moments of input variables, and then evaluate the moments of output variables by weights and nonlinear transformation of the sigma points *(Julier and Uhlmann 1997).*

Consider the standard independent normal vector $\mathbf{\Theta} \sim N(\mathbf{0},\mathbf{1})$, three types of sigma points and relative weights which match the first-four moments are utilized in HUT, and is defined as *(Zhang et al. 2014)*

Type I:

$$\boldsymbol{\theta}_0 = \mathbf{0},\quad \boldsymbol{\omega}_0 = \frac{-2n^2 + (4-2n)\beta^2 + (4\beta+4)n}{(n+\beta)^2(4-n)} \tag{14}$$

Type II:

$$\begin{cases} \boldsymbol{\theta}_{j_1} = +\sqrt{\dfrac{(4-n)(n+\beta)}{(\beta+2-n)}}\mathbf{e}_{j_1} \\ \boldsymbol{\theta}_{j_1+n} = -\sqrt{\dfrac{(4-n)(n+\beta)}{(\beta+2-n)}}\mathbf{e}_{j_1},\ j_1 = 1,2,\ldots,n \\ \boldsymbol{\omega}_1 = \dfrac{(\beta+2-n)^2}{2(n+\beta)^2(4-n)} \end{cases} \tag{15}$$

where $\mathbf{e}_{j_1} = [0,\ldots,0,1,0,\ldots,0]^T$, $i = 1,2,\ldots,n$.

Type III:

$$\begin{cases} \boldsymbol{\theta}_{j_2} = +\sqrt{(n+\beta)}\mathbf{s}_{j_2}^{+} \\ \boldsymbol{\theta}_{j_2+0.5n(n-1)} = -\sqrt{(n+\beta)}\mathbf{s}_{j_2}^{+} \\ \boldsymbol{\theta}_{j_2+n(n-1)} = +\sqrt{(n+\beta)}\mathbf{s}_{j_2}^{-},\ j_2 = 1,2,\ldots,0.5n(n-1) \\ \boldsymbol{\theta}_{j_2+1.5n(n-1)} = -\sqrt{(n+\beta)}\mathbf{s}_{j_2}^{-} \\ \boldsymbol{\omega}_2 = \dfrac{1}{(n+\beta)^2} \end{cases} \tag{16}$$

where β is a free parameter;

$$\begin{aligned} \mathbf{s}_{j_2}^{+} &= \left\{ \sqrt{\frac{1}{2}}(\mathbf{e}_k + \mathbf{e}_l) : k<l, k,l = 1,2,\ldots,n \right\}, \\ \mathbf{s}_{j_2}^{-} &= \left\{ \sqrt{\frac{1}{2}}(\mathbf{e}_k - \mathbf{e}_l) : k<l, k,l = 1,2,\ldots,n \right\} \end{aligned} \tag{17}$$

Then, we can see that when $n=2$ (means two-dimension) the HUT contains 9 points which are different from those of the original BDRM, resulting in improving the computation efforts. When evaluating the two-dimensional HUT, the free parameter β is recommended to be 7.2 according to massive computational experiences. In this regard, the sigma points and their weights for the two-dimensional integration are as follows *(Xu and Dang 2019)*

Type I:

$$\vartheta_0 = (0,0), \omega_0 = \frac{180}{529} \tag{18}$$

Type II:

$$\begin{cases} \vartheta_1 = \left(\dfrac{\sqrt{23}}{3}, 0\right), \vartheta_2 = \left(0, \dfrac{\sqrt{23}}{3}\right), \\ \vartheta_1 = \left(-\dfrac{\sqrt{23}}{3}, 0\right), \vartheta_4 = \left(0, -\dfrac{\sqrt{23}}{3}\right) \\ \omega_1 = \omega_2 = \omega_3 = \omega_4 = \dfrac{81}{529} \end{cases} \tag{19}$$

Type III:

$$\begin{cases} \vartheta_5 = \left(\frac{\sqrt{115}}{5}, \frac{\sqrt{115}}{5}\right), \vartheta_6 = -\left(\frac{\sqrt{115}}{5}, \frac{\sqrt{115}}{5}\right), \\ \vartheta_7 = \left(\frac{\sqrt{115}}{5}, -\frac{\sqrt{115}}{5}\right), \vartheta_8 = \left(-\frac{\sqrt{115}}{5}, \frac{\sqrt{115}}{5}\right) \\ \omega_5 = \omega_6 = \omega_7 = \omega_8 = \frac{25}{2116} \end{cases} \tag{20}$$

Since the two-dimensional HUT well assures the accuracy and efficiency of evaluating the first-four moments, this method could be introduced to the modified BDRM, which is called as high-order unscented transformation based modified BDRM, and is denoted simply as BDRM-HUT.

Similarly, employing the BDRM-HUT to calculate the two-dimensional integration also analyzes repeatedly on the original point $(0,0)$. Therefore, the computation effort will be further enhanced by performing deterministic analysis on each reoccurred point only one time.

3. NUMERICAL EXAMPLE

This section presents one numerical example in order to compare the accuracy and efficiency of the proposed modified BDRMs. The results provided from the proposed modified BDRMs are compared with those of Monte Carlo Simulations (MCS) and the original BDRM. Besides, to evaluate the one-dimensional integration related to the modified BDRMs, the three-point Gaussian quadrature is applied. Moreover, concerning the involved two-dimensional integrations in the modified BDRMs, the fifth-order algebraic accuracy is set for BDRM-SG, BDRM-C-1, BDRM-C-2 and BDRM-C-3.

3.1. Example 1

This example considers a nonlinear undamped single-degree-of-freedom oscillator problem illustrated in Figure 2.

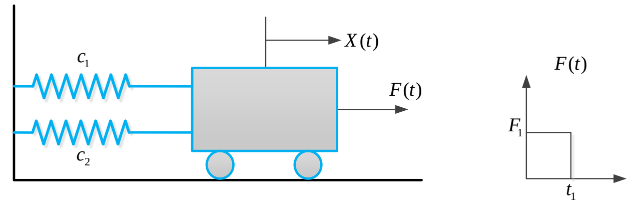

Figure 2:a nonlinear undamped single-degree-of-freedom oscillator

The performance function can be written as

$$Z = G(X) = 3r - \left| 2\frac{F_1}{m\omega_0^{\ 2}} \sin\left(\frac{\omega_0^{\ 2} t_1}{2}\right) \right| \tag{21}$$

where $\omega_0 = \sqrt{(c_1 + c_2)/m}$ and r is the displacement at which one of the spring yields. The descriptions and statistical information of six involved random variables are listed in Table 2.

Table 2: Parameters of variables in Example 1.

Variable	*Distribution*	*Mean*	*C.O.V*
m	*Normal*	*1*	*0.15*
c_1	*Normal*	*1*	*0.2*
c_2	*Normal*	*0.1*	*0.2*
r	*Normal*	*0.5*	*0.2*
F_1	*Normal*	*1*	*0.2*
t_1	*Normal*	*1*	*0.1*

Note: C.O.V means Coefficient of variation.

The first-four central moments calculated by different methods are presented in Table 3. Moreover, the results computed by MCS (10^7 runs) are provided for comparisons. It is seen that the original BDRM provides the most exact results whereas the computational effort is the largest. In comparison, all the proposed modified BDRMs concern on both the accuracy and efficiency. However, for BDRM-SG, the relative error of skewness (γ_Z) is 9.6548%, which is considerably deviated from the exact result. It can be seen that other modified BDRMs need the same number of deterministic calculations, which is 133, and nearly 1/2 of that of the original BDRM.

Table 3: Comparisons of results in Example 1

Methods	*N*	μ_Z *(R.E.)*	σ_Z *(R.E.)*	γ_Z *(R.E.)*	κ_Z *(R.E.)*
BDRM-SG	*193*	*0.5367 (0.0118%)*	*0.3922 (0.0086%)*	*-0.1346 (9.6548%)*	*2.9034 (6.9227%)*
BDRM-C-1	*133*	*0.5368 (0.0032%)*	*0.3926 (0.0884%)*	*-0.1587 (6.4961%)*	*3.1179 (0.0461%)*
BDRM-C-2	*133*	*0.5367 (0.0033%)*	*0.3923 (0.0012%)*	*-0.1428 (4.1459%)*	*2.9729 (4.6937%)*
BDRM-C-3	*133*	*0.5367 (0.0050%)*	*0.3923 (0.0033%)*	*-0.1451 (2.6153%)*	*3.0259 (2.9932%)*
BDRM-HUT	*133*	*0.5368 (0.0050%)*	*0.3923 (0.0035%)*	*-0.1557 (4.5377%)*	*3.1423 (0.7383%)*
Original BDRM	*265*	*0.5368 (0.0040%)*	*0.3923 (0.0014%)*	*-0.1487 (0.1992%)*	*3.0481 (2.2828%)*
MCS	10^7	*0.5367*	*0.3923*	*-0.1490*	*3.1193*

Note: N means the number of required deterministic evaluation.
R.E. means relative error

It is noted that they yield quite accurate results of statistical moments. The relative error of γ_Z by BDRM-C-1 is 6.4961%, while the smallest relative errors for higher-order moments are given by BDRM-C-3, for example, the largest relative errors of γ_Z and κ_Z are less than 3%. It is found that the relative errors produced by BDRM-C-2 as well as BDRM-HUT are comparably larger than that of BDRM-C-3 in this case, however, these small relative errors can be also accepted in most of engineering cases. Therefore, the BDRM-C-2, BDRM-C-3 and BDRM-HUT can provide quite accurate results with low computation efforts for central moment assessment in this example.

4. CONCLUSIONS

The statistical moment estimation of response is an important topic in structural reliability analysis, and assuring the balance between the accuracy and efficiency is of great concern. In this regard, this paper gives several modified bivariate dimension reduction methods (BDRMs) to evaluate the central moments of response. Since the evaluation of two-dimensional integrations in BDRM greatly affects the tradeoff between accuracy and computation effort, the efficient methods for two-dimensional numerical integration are introduced and incorporated into BDRM to form some new modified BDRMs. Some of these new methods, compared with the original BDRM, largely improve the computation effort without losing accuracy. One numerical example is presented to verify the evaluation efforts. The following conclusions can be achieved:

(1). Although utilizing the original BDRM will get very accurate central moments of response, the computation effort is always quite large which badly influence its practical applications.

(2). The BDRM-SG requires much fewer integration points to evaluate the central moments. However, relatively lower accuracy of higher-order moments will be also obtained.

(3). The BDRM-C-1 as well as the BDRM-C-3, which are highly efficient, could not always assure the accuracy of the interested central moments.

(4). The BDRM-C-2 and BDRM-HUT can achieve both high accuracy and high efficiency when evaluating the required central moments. Therefore, these two methods are highly recommended to use for computing the statistical central moments.

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