A hybrid univariate dimension reduction method for statistical moments assessment

Yu Zhang
Graduate Student, College of Civil Engineering, Hunan University, Changsha, P. R. China

Jun Xu*
Associate Professor (Corresponding author) College of Civil Engineering, Hunan University, Changsha, P. R. China

ABSTRACT: This paper presents an index α to evaluate the type of response function, which can help judging the application of two existing form of univariate dimensional reduction (UDRM), say the additive and multiplicative UDRMs for statistical moments assessment. It can determine which one is more effective to decompose a multivariate response function. Then, a new hybrid univariate dimensional reduction (HUDRM) is proposed to compute the raw moments of the response function. The results show that the proposed HUDRM can significantly decrease the relative errors of statistical moments in cases where both the additive and multiplicative UDRMs are not able to provide satisfactory results.

1. INTRODUCTION

The key problem of structural reliability analysis is to compute the failure probability. A large number of methods have been proposed to address this problem. The Monte Carlo simulation (MCS), which can provide accurate results, consumes too much computational time. The improved MCS, such as the Importance Sampling*(Melchers 1989, Engelund and Rackwitz 1993),* Directional Sampling*(Ditlevsen, Bjerager et al. 1988)* also requires a large computational time to obtain the accurate results.

The method of moments can be an effective way to deal with this problem *(Zhao and Ono 2001)*, however, the calculation of the first-four moments of a multivariate function, which are the high dimensional integral, is always not an easy task. The univariate dimensional reduction method (UDRM)*(Rahman and Xu 2004)* is widely used in the moments evaluation of a multivariate function, where the key idea is that a multivariate function can be decomposed into the sum of several univariate functions. It is also well known that two different forms exist for UDRM, which are called the additive UDRM and the multiplicative UDRM*(Zhang and Pandey 2013).* These two forms of UDRM are adapted to different types of response functions respectively. In other words, none of these two forms of UDRM is applicable to an arbitrary case. The choice of a specific form of UDRM depends on whether the response function is strongly additive or multiplicative. This paper first proposes an index α to judge the type of the response function. As for some systems, which are not strongly additive or multiplicative, a new hybrid form of the univariate dimensional reduction method is also proposed in this paper for statistical moments assessment.

2. THE EVALUATION INDEX OF RESPONSE FUNCTION

Without loss of generality, a scalar random variable $Y=\eta(\mathbf{X})$ can be defined as a response function of an engineering system, and $\mathbf{X}$ is the *n*-dimensional random vector. Then, the first-four raw moments of the response function can be defined as

$$E\left[Y^i\right]=\int_Y y^i f_Y(y)\,dy \quad i=1,2,3,4 \tag{1}$$

Where solving the moments of the response function could not be an easy task in the *n*-dimensional random-variate space. Recently, the univariate dimensional reduction method (UDRM) is known as an effective way to simplify this problem, which decomposes the multivariate response function into the sum of multiple univariate functions. It is also known that two forms of UDRM exist in practical computations, which are called the additive UDRM (AUDRM) and the multiplicative UDRM (MUDRM). But for a specific response function, we may not be sure which one is the suitable one to evaluate the statistical moments. In other word, in some cases, the AUDRM may work well, however, the errors produced by AUDRM could be large in other cases. In this regard, a criterion needs to develop to discriminate the specific form of UDRM utilized in a given case. For this purpose, the α index is first proposed as follows.

2.1. ADDITIVE UDRM (AUDRM)

For the response function $Y=\eta(\mathbf{X})$, it can be decomposed by additive UDRM*(Rahman and Xu 2004)* as

$$Y \approx Y_A = \sum_{i=1}^{n} \eta\left(x_i, \boldsymbol{\mu}_{-i}\right) - (n-1)\eta(\boldsymbol{\mu}) \qquad (2)$$

where $\boldsymbol{\mu}$ is the mean vector of the random variable $\mathbf{X}$, $\eta\left(x_i, \boldsymbol{\mu}_{-i}\right)$ should be noted as $\eta\left(\mu_1, ..., \mu_{i-1}, x_i, \mu_{i+1}, ..., \mu_n\right)$,and y_A is the value of the function Y_A, which is computed by additive UDRM.

Then the first-four raw moments can be derived as

$$\begin{aligned} M_A^r &\approx E\left[\left(\sum_{i=1}^{n} \eta\left(x_i, \boldsymbol{\mu}_{-i}\right) - (n-1)\eta(\boldsymbol{\mu})\right)^r\right] \\ &= \sum_{i=0}^{r} S_n^i \left[-(n-1)\eta(\boldsymbol{\mu})\right]^{r-i} \quad r = 1,2,3,4 \end{aligned} \qquad (3)$$

in which S_n^i is a recursive formula as follows:

$$S_n^i = \sum_{k=0}^{i} \binom{i}{k} S_{n-1}^k E\left[\eta^{i-k}\left(\mu_1, \mu_2, ..., \mu_{n-1}, x_n\right)\right] \qquad (4)$$

$$i = 0,1,...,\mathrm{r}$$

the initial value is

$$S_1^i = E\left[\eta^i\left(\mu_1, \mu_2, ...\mu_n\right)\right] \quad i = 0,1,...,\mathrm{r} \qquad (5)$$

2.2. MULTIPLICATIVE UDRM (MUDRM)

Actually, the multiplicative UDRM*(Zhang and Pandey 2013)* is derived from the additive UDRM. Consider a logarithmic transformation of response function as follows:

$$g(\mathbf{x}) = \log\left[\eta(\mathbf{x})\right] \quad \eta(\mathbf{x}) > 0 \qquad (6)$$

The approximate expression of the response function can be obtained by the additive UDRM:

$$g(\mathbf{x}) \approx \sum_{i=1}^{n} g\left(x_i, \boldsymbol{\mu}_{-i}\right) - (n-1) g(\boldsymbol{\mu}) \qquad (7)$$

where

$$\begin{aligned} g(\boldsymbol{\mu}) &= \log\left[\eta\left(\mu_1, \mu_2, ...\mu_n\right)\right] \\ g\left(x_i, \boldsymbol{\mu}_{-i}\right) &= \log\left[\eta\left(\mu_1, ...\mu_{i-1}, x_i, \mu_{i+1}, ...\mu_n\right)\right] \end{aligned} \qquad (8)$$

The original response function $\eta(\mathbf{x})$ can be written as

$$\eta(\mathbf{x}) = \exp\left[g(\mathbf{x})\right] \approx Y_M = \eta^{1-n}(\boldsymbol{\mu}) \prod_{i=1}^{n} \eta\left(x_i, \boldsymbol{\mu}_{-i}\right) \qquad (9)$$

and the first-four raw moments can be computed as

$$M_M^r \approx E\left\{\left[\eta^{1-n}(\boldsymbol{\mu}) \prod_{i=1}^{n} \eta\left(x_i, \boldsymbol{\mu}_{-i}\right)\right]^r\right\} \quad r = 1,2,3,4 \qquad (10)$$

2.3. THE DETERMINATION OF INDEX α

Consider using the function values $\left(y_A, y_M\right)$ obtained by AUDRM, MUDRM and α to solve the real function value y, which is defined as

$$y_j = y_{Aj}^{\alpha_j} y_{Mj}^{1-\alpha_j} \quad j = 1,...,\mathrm{n} \qquad (11)$$

where n is the number of the sample points selected, α_j is the solution of Eq.(11) at the sample point j.

In order not to increase the amount of calculation too much, normally, the Gaussian points used in computing the raw moments can be selected. Besides, the sample points should be

selected to make sure both the y_{Aj} and y_{Mj} are positive, the index α_j can be solved by

$$\alpha_j = \frac{\log\left(\frac{y_j}{y_{Mj}}\right)}{\log\left(\frac{y_{Aj}}{y_{Mj}}\right)} \quad j = 1,\ldots \mathrm{n} \tag{12}$$

Define α be the mean of α_j such that

$$\alpha = \frac{1}{n}\sum_{j=1}^{n}\alpha_j \tag{13}$$

It is easy to infer that

$$\begin{aligned} &\alpha < 0 \quad y_A < y_M < y \;\; or \;\; y_A > y_M > y \\ &\alpha > 1 \quad y_M < y_A < y \;\; or \;\; y_M > y_A > y \\ &\alpha \to 0 \quad y_M \to y \\ &\alpha \to 1 \quad y_A \to y \\ &0 < \alpha < 1 \quad y_A < y < y_M \;\; or \;\; y_A > y > y_M \end{aligned} \tag{14}$$

Obviously, when $\alpha < 0$, the function value computed by multiplicative UDRM is closer to the real one; when $\alpha > 1$, the value solved by additive UDRM has a smaller error. We suggest that if $\alpha \to 0$, we need to use the multiplicative UDRM to decompose the response function, and when $\alpha \to 1$, additive UDRM is more effective.

2.4. EXAMPLES

In this section, two examples are used to verity the validity of the proposed index α by comparing the relative errors of the first-four raw moments of the response function, which are obtained by AUDRM and MUDRM, respectively.

2.4.1. Example 1: nonlinear response function

A response function for reinforced concrete beam is given by a nonlinear explicit form*(Zhou and Nowak 1988, Breitung and Faravelli 1994)* :

$$G(\mathbf{X}) = X_1 X_2 X_3 - \frac{X_1^2 X_2^2 X_4}{X_5 X_6} \tag{15}$$

where $G(\mathbf{X})$ is the resistance moment of the reinforced concrete beam, all the random variables are statistically independent and the description and distribution are listed in Table 1.

The selection of sample points for computing the index α is based on the combination of the six random variables' Gaussian points. This example uses 5-points Gaussian quadrature, it has $\left(5\times5\times5\times5\times5\times5=5^6\right)$ combinations to get the sample points. However, it is not necessary to employed so many samples points for calculations. Only a portion of points for the evaluation is effective enough, and the selected sample points should make α_j be a real number. Table 2 shows the sample points for estimating the index α.

The function value y_j and the value y_{Aj}, y_{Mj} obtained by AUDRM and MUDRM , are provided in Table 3. And the index α_j is computed by Eq.(12) are also shown in Table 3. Obviously, α_1 is not a real number, and α_3 and α_7 have much larger values than others, where

$$\alpha = \frac{1}{7}\left(\alpha_1+\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_8+\alpha_9+\alpha_{10}\right) = -0.05$$

Judging from Eq.(14), the function in Example 1 should be decomposed by MUDRM.

Table 4 shows that the first four moments is much more accurate computed by MUDRM than those of AUDRM, which demonstrates the efficacy of using the proposed index α.

2.4.2. Example 2: linear mathematical function

A linear mathematical function is involved in this example*(Kiureghian, Lin et al. 1991)* such that

$$G(X) = X_1 + 2X_2 + 2X_3 + X_4 - 5X_5 - 5X_6 + 350 \tag{16}$$

where all the random variables are lognormally distributed, listed in Table 5.

Table 1: The information of random variables: Example 1

Variable	Description	Distribution	Mean	COV
X_1	Area of reinforcement	Lognormal	$1260\,mm^2$	0.2
X_2	Yield stress of reinforcement	Lognormal	$300\,N/mm^2$	0.2
X_3	Effective stress of reinforcement	Lognormal	$770\,mm$	0.2
X_4	Stress-strain factor of concrete	Lognormal	0.35	0.1
X_5	Compressive strength of concrete	Weibull	$25\,N/mm^2$	0.2
X_6	Width of beam	Normal	$200\,mm$	0.2

Table 2: The sample points for estimating α : Example 1

Samples points \ Variable	X_1	X_2	X_3	X_4	X_5	X_6
1	701.6608	167.0621	428.7927	0.261901	21.45165	85.72
2	944.6275	224.9113	577.2723	0.304215	28.6598	145.776
3	1235.532	294.1742	755.0471	0.348263	33.66946	200
4	1616.022	384.7671	987.5688	0.398689	37.8476	254.224
5	2175.607	518.0018	1329.538	0.463102	41.82167	314.28
6	701.6608	224.9113	755.0471	0.398689	41.82167	314.28
7	2175.607	384.7671	755.0471	0.304215	21.45165	85.72
8	944.6275	294.1742	987.5688	0.463102	41.82167	314.28
9	1616.022	294.1742	577.2723	0.261901	21.45165	85.72
10	1235.532	384.7671	1329.538	0.463102	41.82167	314.28

Table 3: Computation of α : Example 1

Function value	Sample points									
	1	2	3	4	5	6	7	8	9	10
y_{Aj}	-104.5	70.1	267.5	519.0	880.37	91.1	731.7	294.0	764.50	567.5
y_{Mj}	46.5	118.6	267.6	594.3	1415.3	120.5	973.5	273.9	1068.3	623.3
y_j	48.3	119.4	267.6	598.0	1453.6	118.4	516.1	271.7	1088.3	624.1
α_j	0.003-0.01i	-0.01	0.36	-0.05	-0.06	0.06	2.22	-0.11	-0.06	-0.01

The sample points and α_j are given in Table 6, where

$$\alpha=\frac{1}{9}\left(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_9+\alpha_{10}\right)=1.00$$

In this regard, the response function should be decomposed by AUDRM, which is much more effective than MUDRM. Table 7 shows the relative errors by AUDRM is much smaller than those of MDRM, which again proves the index α is reliable.

Table 4: The errors of AUDRM and MUDRM: Example 1

Moments	*1st*	*2nd*	*3rd*	*4th*
AUDRM	279.12	8.6456×10^{4}	2.9166×10^{7}	1.0593×10^{10}
error(%)	0.00006	*-1.07*	*-4.94*	*-12.32*
MUDRM	279.66	8.7717×10^{4}	3.0842×10^{7}	1.2151×10^{10}
error(%)	*0.19*	*0.38*	*0.52*	*0.58*
$MCS\left(10^{7}\right)$	279.12	8.7387×10^{4}	3.0683×10^{7}	1.2081×10^{10}

Table 5: The information of random variables: Example 2

Variable	*Distribution*	*Mean*	*COV*
X_1	*Lognormal*	*120*	*0.1*
X_2	*Lognormal*	*120*	*0.1*
X_3	*Lognormal*	*120*	*0.1*
X_4	*Lognormal*	*120*	*0.1*
X_5	*Lognormal*	*50*	*0.3*
X_6	*Lognormal*	*40*	*0.3*

Table 6: Computation of α : Example 2

Function value	*Sample points*									
	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
y_{Aj}	*702.5*	*686.3*	*635.4*	*528.5*	*305.6*	*26.79*	*939.0*	*71.50*	*837.6*	*222.0*
y_{Mj}	*671.9*	*677.0*	*635.5*	*518.8*	*274.5*	*177.8*	*992.8*	*191.5*	*843.6*	*241.8*
y_j	*702.4*	*686.3*	*635.4*	*528.5*	*305.6*	*26.79*	*939.0*	*128.2*	*837.6*	*222.0*
α_j	*1.00*	*1.00*	*1.00*	*1.00*	*1.00*	*1.00*	*1.00*	*0.41*	*1.00*	*1.00*

Table 7 The error of AUDRM and MUDRM: Example 2

Moments	*1st*	*2nd*	*3rd*	*4th*
AUDRM	6.2000×10^{2}	3.9506×10^{5}	2.5758×10^{8}	1.7133×10^{11}
error(%)	*0.0108*	*0.0210*	*0.0303*	*0.0382*
MUDRM	6.2002×10^{2}	3.9516×10^{5}	2.5806×10^{8}	1.7227×10^{11}
error(%)	*0.0132*	*0.0460*	*0.2172*	*0.5881*
$MCS\left(10^{7}\right)$	6.1993×10^{2}	3.9498×10^{5}	2.5751×10^{8}	1.7126×10^{11}

3. A HYBRID UNIVARIATE DIMENSIONAL REDUCTION METHOD

It can be noted that different response functions adapt to different forms of UDRM. Sometimes, both the additive and multiplicative UDRM can not decompose the response function effectively when $0<\alpha<1$, compared with the form of the Eq.(11), a hybrid form of UDRM is proposed in this section.

3.1. The form of hybrid UDRM (HUDRM)

As seen from above, compared with the definition of α_j, define the *kth* power of the response function $\eta(x)$ as

$$\eta^k(x) = \eta^\beta(x)\eta^{k-\beta}(x) \quad k = 0,1,...,n \tag{17}$$

Where β is a real number, $\eta^\alpha(x)$, $\eta^{i-\alpha}(x)$ are decomposed by AUDRM and MUDRM respectively, which can be written as

$$\eta^\beta(x) \approx \sum_{i=1}^{n} \eta^\beta(x_i, \boldsymbol{\mu}_{-i}) - (n-1)\eta^\beta(\boldsymbol{\mu}) \tag{18}$$

$$\eta^{k-\beta_k}(\mathbf{x}) \approx \eta^{(1-n)(k-\beta_k)}(\boldsymbol{\mu}) \prod_{i=1}^{n} \eta^{k-\beta_k}(x_i, \boldsymbol{\mu}_{-i}) \quad k = 0,1,...,n \tag{19}$$

then $\eta^k(x)$ can be approximated as

$$\begin{aligned}
\eta^k(x) \approx Y_H^k &= \left[\sum_{i=1}^{n} \eta^{\beta_k}(x_i, \boldsymbol{\mu}_{-i}) - (n-1)\eta^{\beta_k}(\boldsymbol{\mu})\right]\left[\eta^{(1-n)(k-\beta_k)}(\boldsymbol{\mu}) \prod_{i=1}^{n} \eta^{k-\beta_k}(x_i, \boldsymbol{\mu}_{-i})\right] \\
&= \frac{\eta^k(x_1, \boldsymbol{\mu}_{-1})\eta^{k-\beta_k}(x_2, \boldsymbol{\mu}_{-2})\eta^{k-\beta_k}(x_3, \boldsymbol{\mu}_{-3}) \cdots \eta^{k-\beta_k}(x_n, \boldsymbol{\mu}_{-n})}{\eta^{(1-n)(k-\beta_k)}(\boldsymbol{\mu})} \\
&+ \frac{\eta^k(x_2, \boldsymbol{\mu}_{-2})\eta^{k-\beta_k}(x_1, \boldsymbol{\mu}_{-1})\eta^{k-\beta_k}(x_3, \boldsymbol{\mu}_{-3}) \cdots \eta^{k-\beta_k}(x_n, \boldsymbol{\mu}_{-n})}{\eta^{(1-n)(k-\beta_k)}(\boldsymbol{\mu})} + \cdots \\
&+ \frac{\eta^k(x_n, \boldsymbol{\mu}_{-1})\eta^{k-\beta_k}(x_1, \boldsymbol{\mu}_{-1})\eta^{k-\beta_k}(x_2, \boldsymbol{\mu}_{-2}) \cdots \eta^{k-\beta_k}(x_{n-1}, \boldsymbol{\mu}_{-(n-1)})}{\eta^{(1-n)(k-\beta_k)}(\boldsymbol{\mu})} \\
&- (n-1)\eta^{\beta_k}(\boldsymbol{\mu})\left[\eta^{(1-n)(k-\beta_k)}(\boldsymbol{\mu}) \prod_{i=1}^{n} \eta^{k-\beta_k}(x_i, \boldsymbol{\mu}_{-i})\right]
\end{aligned} \tag{20}$$

The *kth* raw moment of the response function is given as follows:

$$E\left[Y^k\right] \approx E\left[Y_H^k\right] \tag{21}$$

3.2. The determination of index β

The computation of index β is similar to index α, both based on the sample points. At j sample points, the equation is as follows:

$$\begin{aligned}
y_{Aj}^{\beta_{kj}} &= \left[\sum_{i=1}^{n} \eta_j^{\beta_{kj}}(x_i, \boldsymbol{\mu}_{-i}) - (n-1)\eta^{\beta_{kj}}(\boldsymbol{\mu})\right] \\
y_{Mj}^{k-\beta_{kj}} &= \left[\eta^{(1-n)(k-\beta_{kj})}(\boldsymbol{\mu}) \prod_{i=1}^{n} \eta_j^{k-\beta_{kj}}(x_i, \boldsymbol{\mu}_{-i})\right] \\
y^k &= \eta_j^k(x) = y_{Aj}^{\beta_{kj}} y_{Mj}^{k-\beta_{kj}} \quad j = 1,2,...,n
\end{aligned} \tag{22}$$

It is seen that Eq.（22）above is a nonlinear equation. If every single equation has a real number β_j, n roots will be found based on n sample points, β is the mean of β_j:

$$\beta = \sum_{j=1}^{n} \beta_j \tag{23}$$

3.3. Example: a nonlinear mathematical function

A nonlinear mathematical function is given as

$$G(X) = \frac{X_1^2 X_2}{50000} + X_3 + X_4 + \frac{X_5}{15} - X_6 \tag{24}$$

where the information of each random variable is listed in Table 8. And the index α of this function lies between (0,1) listed in Table 9，which indicates the HUDRM could be more effective for this example.

Table 10 shows the roots of the Eq.(22) based on five sample points. The first-four raw moments computed by Eq.(21), which are based on the hybrid method and the index β_{kj}, are listed in Table 11. The results show that the proposed HUDRM yields very small relative errors of statistical moments than those of AUDRM and MUDRM. The number of deterministic model evaluations for AUDRM and MUDRM are all 30 (5×6), while the number for HUDRM is just 35 $(5\times6+5)$.

Table 8: The information of random variables

Variable	*Distribution*	*Mean*	*COV*
X_1	*Lognormal*	*120*	*0.4*
X_2	*Lognormal*	*120*	*0.4*
X_3	*Lognormal*	*120*	*0.4*
X_4	*Lognormal*	*120*	*0.4*
X_5	*Lognormal*	*50*	*0.3*
X_6	*Lognormal*	*40*	*0.3*

Table 9: The value of index α based on five sample points

Sample points	*1*	*2*	*3*	*4*	*5*	*Mean*
α_j	*0.45*	*0.50*	*0.52*	*0.45*	*0.41*	*0.47*

Table 10: The value of β_{kj} based on five sample points

k \ *j*	*1*	*2*	*3*	*4*	*5*	*mean*
1	*0.786077*	*0.753454*	*0.728498*	*0.63962*	*0.571636*	*0.695857*
2	*0.970853*	*0.996397*	*1.021646*	*0.944006*	*0.882482*	*0.963077*
3	*1.069871*	*1.157979*	*1.243203*	*1.193914*	*1.152744*	*1.163542*
4	*1.129935*	*1.278533*	*1.427715*	*1.415874*	*1.402852*	*1.330982*

Table 11: Comparisons of the first-four raw moments

Moments	*1st*	*2nd*	*3rd*	*4th*
AUDRM	*242.6864*	6.4911×10^4	1.9071×10^7	6.1401×10^9
error(%)	*-0.29*	*-1.29*	*-3.60*	*-8.76*
MUDRM	*243.4280*	6.5976×10^4	2.0265×10^7	7.2180×10^9
error(%)	*0.02*	*0.33*	*2.43*	*7.26*
HUDRM	*243.3862*	6.5628×10^4	1.9778×10^7	6.7384×10^9
error(%)	*-0.0013*	*-0.20*	*-0.03*	*0.13*
$MCS\left(10^7\right)$	*243.3895*	6.5761×10^4	1.9784×10^7	6.7293×10^9

4. CONCLUSIONS

In conclusion, the index α of the response function can be an effective tool for judging the usage of different forms of univariate dimensional reduction method. Further, when the index α lies between (0, 1), a hybrid UDRM is established accordingly. The computational results show that the proposed hybrid UDRM can significantly improve the accuracy for statistical moments assessment without losing efficiency.

5. REFERENCES

Breitung, K. and L. Faravelli (1994). "Log-likelihood maximization and response surface in reliability assessment." *Nonlinear Dynamics*, 5(3), 273-285.

Ditlevsen, O., et al. (1988). "Directional simulation in Gaussian processes." *Probabilistic Engineering Mechanics*, 3(4), 207-217.

Engelund, S. and R. Rackwitz (1993). "A benchmark study on importance sampling techniques in structural reliability." *Structural Safety*, 12(4), 255-276.

Kiureghian, A. D., et al. (1991). "Closure of "Second-Order Reliability Approximations"." *Journal of Engineering Mechanics*, 117, 433-434.

Melchers, R. E. (1989). "Importance sampling in structural system." *Structural Safety*, 6(1), 3-10.

Rahman, S. and H. Xu (2004). "A univariate dimension-reduction method for multi-dimensional integration in stochastic mechanics." *Probabilistic Engineering Mechanics*, 19(4), 393-408.

Zhang, X. and M. D. Pandey (2013). "Structural reliability analysis based on the concepts of entropy, fractional moment and dimensional reduction method." *Structural Safety*, 43(9), 28-40.

Zhao, Y. G. and T. Ono (2001). "Moment methods for structural reliability." *Structural Safety*, 23(1), 47-75.

Zhou, J. and A. S. Nowak (1988). "Integration formulas to evaluate functions of random variables." *Structural Safety*, 5(4), 267-284.