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Fully Decentralized Design of Distributed Observer for Linear Time Invariant Systems

선형 시불변 시스템에 대한 분산 관측기의 비집중 설계

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ABSTRACT

Fully Decentralized Design of Distributed Observer for Linear Time Invariant Systems

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Unlike the classical observer designed for a single sensor, the distributed observer is used when the plant is monitored by a network of agents (sensors), and it consists of multiple local observers, each of which is implemented in its corresponding agent. Each local observer generates the estimate for the full state of the plant by using the associated agent's local measurements and by communicating with the neighboring agents. The essential benefit of distributed estimation is that through the inter-agent cooperation, each agent can reconstruct the full state with a very limited sensor capacity, which leads to improved estimation performance over the networked agent systems.

However, most of the distributed estimation schemes in the literature lack i) flexibility in performance over various types of communication topologies, ii) feasibility in implementing the scheme for networked agent systems, and iii) applicability to various forms of plants.

In this dissertation, we propose a novel design scheme of the distributed observer for continuous linear time-invariant systems. The main idea is to let each agent recover the self-reconstructible portion by itself using its own measurement and accept only insufficient information by the projected diffusive coupling of the neighbors' estimates via the specially designed communication protocol called agent-wise decomposed diffusive coupling.

In addition, the performance of the proposed distributed observer design technique is analyzed and improved from the three points of view mentioned above.

i) We provide an equivalent condition to the existence of the proposed observer for the general linear time invarinat plant over the general communication network, which is not necessarily strongly connected.

ii) We devise an adaptive distributed observer that can be constructed in a 'fully decentralized manner'; each agent can implement the proposed algorithm by itself without any global information such as the shape of entire network or the parameters of other agents. In consequence, this proposed adaptive observer has *plug and play features*; the estimation is conducted robustly to the joining and leaving of members.

iii) Going further from the distributed estimation for autonomous plants, we consider the plant with input, where the input is measured in a distributed fashion as well as the plant state is. The unknown input observer from geometrical control theory is employed to estimate the plant state despite the locally unknown input. As a result, it is shown that the estimates approximately converge to the plant state assuming that the plant input is bounded.

Lastly, we apply our approach to the distributed multi-robot localization problem and present simulation results to confirm the effectiveness of the proposed scheme.
Keywords: distributed estimation, distributed observer, multi-agent systems, sensor network, fully decentralized design, geometrical decomposition, adaptive observer

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To my family and friends

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Symbols and Acronyms

$\mathbb{R}$	the field of real numbers
$\mathbb{C}$	the field of complex numbers
$\mathbb{C}^-$	the open left half complex plane
$\mathbb{C}^{0+}$	the right closed half complex plane
$\mathbb{R}^n$	the Euclidean space of dimension n
$\mathbb{R}^{m\times n}$	the space of $m \times n$ matrices with real entries
$\mathrm{Re}(s)$	the real part of a complex number s
$\mathrm{Im}(s)$	the imaginary part of a complex number s
I_n	the $n \times n$ identity matrix
$0_{m\times n}$	the $m \times n$ matrix with all entries to zero
1_m	the m dimensional column vector comprising all ones
$\mathrm{col}(x_i)_{i\in\mathcal{N}}$	the concatenation of the vectors x_i for all i contained in the index set $\mathcal{N}$
$\mathrm{diag}(A_i)_{i\in\mathcal{N}}$	the block diagonal matrix whose diagonal blocks consist of the matrices A_i for all i contained in the index set $\mathcal{N}$
$\lambda_{\max}(A)$	the maximum eigenvalue of the matrix $A \in \mathbb{R}^{n\times n}$ with real eigenvalues
$\lambda_{\min}(A)$	the minimum eigenvalue of the matrix $A \in \mathbb{R}^{n\times n}$ with real eigenvalues
$\sigma_{\max}(A)$	the maximum singular of the matrix $A \in \mathbb{R}^{n\times n}$

$\sigma_{\min}(A)$	the minimum singular of the matrix $A \in \mathbb{R}^{n\times n}$
$\ker A$	the kernel of the matrix A is denoted as $\ker A := \{x \mid Ax = 0\}$
$\operatorname{im} A$	the image of the matrix A is denoted as $\operatorname{im} A := \{y \mid y = Ax \text{ for some } x\}$
$\mathcal{V}^{\perp}$	the orthogonal complement of the subspace $\mathcal{V}$, i.e., $\mathcal{V}^{\perp} := \{x \in \mathbb{R}^n \mid x^T v = 0 \text{ for all } v \in \mathcal{V}\}$
$\mathsf{a}(A)$	the abscissa of matrix A, i.e., the maximum among the real part of the eigenvalues of A
$\kappa(A)$	the condition number of matrix A, i.e., $\kappa(A) := \|A\|\|A^{-1}\|$
$\lvert\mathcal{N}\rvert$	the cardinality of the set $\mathcal{N}$
$\lvert x\rvert$	the Euclidean norm of the vector $x \in \mathbb{R}^n$
$\|A\|$	the induced 2-norm of the matrix $A \in \mathbb{R}^{m\times n}$
$\Diamond$	end of theorems, lemmas, propositions, assumptions, remarks, and so on
$\Box$	end of proof

- A square matrix A is said to be Hurwitz (matrix) if every eigenvalue λ of A has strictly negative real parts, i.e., $\mathrm{Re}(\lambda) < 0$.
- For any state variable $x(t)$, its initial condition will be denoted by $x(0)$.
- Assume that a set of matrices $\{C_i \in \mathbb{R}^{p_i\times n} \mid i = 1, 2, \cdots, N\}$ is given. For a subset $\bar{\mathcal{N}} := \{i_1, i_2, \cdots, i_{\bar{N}}\} \subseteq \{1, 2, \cdots, N\}$, we define a concatenated matrix $\mathrm{col}(C_i)_{i\in\bar{\mathcal{N}}} := [C_{i_1}^T, C_{i_2}^T, \cdots, C_{i_{\bar{N}}}^T]^T$. With a slight abuse of notation, some matrices $\{C_{j_1}\, C_{j_2}\, \cdots\, C_{j_{N_e}}\}$ may be empty matrices, i.e., $C_{j_k} \in \mathbb{R}^{0\times n}$ for $k \in \mathcal{N}_e := \{1, 2, \cdots, N_e\} \subseteq \{1, 2, \cdots, N\}$. In this case, $\mathrm{col}(C_i)_{i\in\bar{\mathcal{N}}}$ actually means that $\mathrm{col}(C_i)_{i\in\bar{\mathcal{N}}\setminus\mathcal{N}_e}$. For a set of matrices $\{U_i \in \mathbb{R}^{n_i\times\nu_i} \mid i = 1, 2, \cdots, N\}$ and a subset $\bar{\mathcal{N}} \subseteq \{1, 2, \cdots, N\}$, we also use the notation $\mathrm{diag}(U_i)_{i\in\bar{\mathcal{N}}}$ for the matrix formed by arranging the matrices $\{U_i\}_{i\in\bar{\mathcal{N}}}$ in a block diagonal fashion. Special care should be taken when some matrices

$\{U_{j_1}, U_{j_2}, \cdots, U_{j_{N_e}}\}$ are empty matrices. For example, with $U_1 = U_3 = 1_n$ and $U_2 \in \mathbb{R}^{n\times 0}$ which is an empty matrix, it is defined as follows

$$\mathrm{diag}(U_i)_{i\in\{1,2,3\}} := \begin{bmatrix} 1_n & 0_{n\times 1} \\ 0_{n\times 1} & 0_{n\times 1} \\ 0_{n\times 1} & 1_n \end{bmatrix}.$$

Note that zero rows are appended in the block corresponding to the empty matrix. For empty matrices $U \in \mathbb{R}^{n\times 0}$ and $V \in \mathbb{R}^{0\times m}$, we have $UV = 0_{n\times m}$, and all the usual algebraic properties, such as matrix addition, scalar multiplication, and matrix multiplication, equally hold for the empty matrices [NH93].

Acronyms

LTI	Linear time invariant
iSCC	Independent strongly connected component
SISO	Single-input single-output
MIMO	Multi-input multi-output
SCBD	Special coordinate basis decomposition

Chapter 1

Introduction

1.1 Research Background

As the scale of the control plants grew larger and larger, designing one *centralized* controller caused many problems such as increased computational and communicational load, and difficulties in maintenance. Such shortcomings resulted in the emergence of *decentralized control and estimation* in the 1970s, where some important foundations were laid by [CM76], [ŠV76], and [Šil11]. In decentralized control, the large-scale plant is assumed to be partitioned into multiple (weakly interconnected) subsystems and the local controller/estimator is designed for each subsystem with the local input/output channel to perform a control task. Here, the purpose of the decentralized estimation is to generate the estimate of the assigned substate using the local measurements so as to control the subsystem.

Recent advances in computational and communication technologies have expanded the subject of the control study to *the networked agent system*[1], where the term agent indicates the system capable of the independent decision making (or computation) based on its own interaction such as sensing the environment or communicating with other agents[2], and the term networked means each agent

[1]The multi-agent system is composed of multiple interacting agents, in which the interaction is not necessarily limited to the communication. For example, the coupled pendulums can be seen as a multi-agent system with the physical interaction, but it is not a networked agent system unless each pendulum is equipped with a communication device. Consequently, the networked agent system is a subset of the multi-agent system.

[2]In this context, the agent is not necessarily limited to being self-actuated.

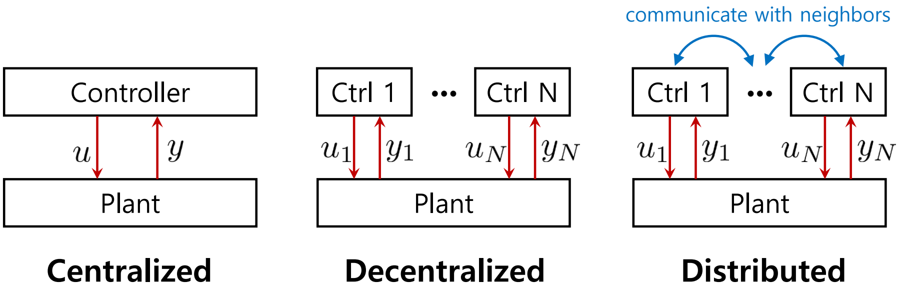


Figure 1.1: The concept of centralized, decentralized, and distributed control systems.

is with the the communication capability. The examples of the networked agents include the spatially deployed vehicles such as the platooning heavy-duty trucks, the group of aerial vehicles flying in a formation, and the sensor networks from networked cameras in the city to the underwater harbor surveillance system.

The inter-agent communication let each agent obtain more information compared to the decentralized estimation, where each agent can estimate only a assigned substate using its own measurement. As a result, a lot of control problems are considered for the networked agent system related to, for example, target tracking [PPRC11], environmental monitoring [PM16], security and surveillance [MS16], and power network monitoring [PCB12], to name a few. All these applications have in common that it requires "every agent" reconstruct the "full state" of the dynamic plant using their measurements and the communications, which provides ample reason to consider the problem of distributed state estimation (or short distributed estimation). The concepts of these paradigms are illustrated in Fig. 1.1.

The distributed estimation problem for linear time invariant plants is defined as follows. Suppose that the plant $\dot{x} = Ax$ is monitored by a network of N agents (sensors) such that the i-th agent measures the partial state $y_i = C_i x$, and communicates according to a given network topology $\mathcal{G}$. The *distributed estimation* problem is to design an algorithm which is embedded in each agent to resolve the full state x under the following constraints: i) the i-th agent accesses the local measurement y_i only, and ii) it communicates with its neighboring agents through the given network $\mathcal{G}$. The key feature of distributed estimation lies on its localized structure; that is each agent exchanges its local estimate through the local communication.

The distributed estimation problem has been studied mainly in two directions: Kalman filter based approach and observer based approach. The goal of the Kalman filter based approach is to find a covariance-minimizing optimal estimator under noisy measurements in a distributed fashion, which is called the *distributed Kalman filter*. See, e.g., [OS05, OS07, OS09, KT08, KJ11, BCM$^+$15, MB12] for discrete-time and [KSW16] for continuous-time settings. Although the optimal performance is guaranteed, the Kalman filter based approach requires an amount of resources such as computing multiple iterations between the innovation updates and exchanging covariance matrices with the neighbors[3].

On the other hand, in the observer based approach, the focus lies on the stabilization of the estimation error using relatively few resources. From this point of view, the observer based distributed estimation problem can be regarded as a dual problem of the distributed control problem. However, the major challenge of designing a distributed observer stems from the heterogeneity of the measurement matrices C_i's. While most works of the distributed control consider a homogeneous and stabilizable input matrix, the measurement matrix C_i of the agent in the distributed estimation problem are neither homogeneous nor detectable with A. Instead, the entire output matrix C, the collection of C_i's, is detectable with A. Therefore, the distributed observer design is not a mere dual problem of the distributed control but requires a new approach to overcome the heterogeneity.

A *distributed state observer* (for short, *distributed observer*) is discussed in many papers. In [KJ14], the authors propose a scalar gain estimator and characterize the class of plants by the notion of scalar tracking capacity. Its simple design is an advantage but also limits the class of systems that can be accommodated by their algorithm. The state-augmented type observer[4] is discussed in [PM17, WM18, WMFL17]. In [PM17], the distributed estimation problem is

[3] Denoting the dimension of the plant state by n, the distributed Kalman filter requires the communication load of dimension $n(n+2)$ per each transmission, whereas only the communication load of dimension n is sufficient for the distributed observer proposed in [KSC16].

[4] The dimension of the observer in each agent should be larger than or equal to the dimension of the plant while at least one agent should have more dimension than that of the plant.

casted as a decentralized stabilization problem under mild assumptions, and an equivalent condition for the existence of the proposed scheme is also derived. The authors in [WM18] propose a simpler design method than [PM17], and the convergence rate of the estimation error can be freely assigned. The time-varying network is also considered in [WMFL17].

Unlike the state-augmented observers, the distributed observers to be introduced from now on have the same dimension as the plant's, and have the structure consists of the Luenberger-type observer part using the local measurement and the part of coupled estimates. In [ZLL$^+$14], the estimation error is stabilized by designing the collection of output injection gains for the collected pair $(A, \mathrm{col}(C_i))$ and by assigning the strong gain for the diffusive coupling of individual state estimates simultaneously. In [MS18], the authors transform the collected pair $(A, \mathrm{col}(C_i))$ of the whole system into a block lower triangular form and assign each agent a transformed substate sequentially along a pre-defined ordering of agents to achieve error stabilization.

The distributed observer using the *agent-wise decomposed diffusive coupling*, which is the diffusive coupling of the state estimates multiplied by the agent-wise decomposition based weighting matrix, is first proposed in our conference paper [KSC16], and this structure is applied in combinations with the LMI-based approach [HTWS18a] and reduced order observer form [HTWS18b]. The underlying idea of the agent-wise decomposed diffusive coupling can be explained intuitively. From the i-th agent's view, the estimation error is decomposed into its own observable/unobservable part of the pair (A, C_i), and its observable part is recovered by the Luenberger observer part using its own measurement y_i. On the other hand, by the specifically designed weighting matrix, the unobservable part is compensated by the diffusive coupled estimates projected onto the unobservable subspace, expecting that other agent's observable parts stabilize it. Thanks to the approach based on the agent's point of view, it achieves the most localized design so far.

While all the literature reviews above show that the distributed estimation has evolved far, it still lacks practicality to apply to general networked agent systems. First, more general plants and more general communication topologies should be addressed in the problem of the distributed estimation. Note that the networked agent system is a gathering of multiple agents, and its communication topology is affected by various factors, such as the relative positions of agents, the relative attitudes among agents, the communication devices' characteristics, and even the environment, which implies that the form of the communication network is very diverse. Therefore, the distributed observer should be designed to accommodate the general plants and the larger class of networks beyond the undirected or the strongly connected networks.

Second, the construction of the distributed observer should be done in a decentralized fashion as its operation does. In decentralized estimation, the construction is done centrally in the sense that there is an architect who knows the entire plant and assigns a partitioned substate to a local estimator and design. For the networked agent system, on the other hand, the agents are equal in the sense that none of them has the power to manipulate all other agents or knows the entire system, so it is unnatural to assume the existence of an architect in general. Unfortunately, all distributed observers introduced so far are centrally constructed. It is well known that finding a structure which supports decentralized construction is challenging, but a lot of advantages follows such as flexibility, or robustness[XBL05]. Furthermore, the distributed construction allows the plug-and-play operation, that is, each agent can automatically reconfigure whenever a new agent is added or removed, which is the ultimate goal of every algorithm runs in distributed environments.

Lastly, it is necessary to consider the distributed estimation problem for the plant with the input, in which the input is measured in a distributed manner as is the plant state. A real-life plant, needless to say, has an input. If we suppose that every agent knows the plant input, then considering the plant input has no meaning because its effect can be canceled in all agents' error dynamics. However,the setting that all agents know the plant input is nonsense for the distributed estimation problem. Assuming the plant input measured in a distributed way is more practical, but more challenging at the same time because, unlike the case of autonomous plants, each agent's estimation suffers from its unknown part of the input. Above all, this issue is meaningful in that it has not yet been addressed in the literature despite its high practicality.

In this dissertation, we address the distributed estimation problem from three aspects listed above. In particular, we propose a distributed observer design scheme and explore in relation to the capability of estimation under the general network and plant, the flexibility to be constructed in a decentralized fashion, and the applicability to the plant with input measured in a distributed manner.

1.2 Contributions and Outline of Dissertation

The following overview reveals the outline of this thesis and briefly summarizes its contributions.

Chapter 2. Graph Theory for Distributed Estimation

In this chapter, we review basic definitions and facts from graph theory and provide new technical tools to handle the Laplacian matrix of general directed graphs to comprehend distributed estimation over the general network. Parts of this chapter are based on [KLS]. The contribution of this chapter is:

- We provide a new interpretation of the Laplacian for a general weighted directed graph by using the concept of nonsingular M-matrix and the independent strongly connected component.
- We provide a way to treat the Laplacian matrix for general directed graphs like one for undirected graphs by analyzing the weighted symmetrized Laplacian matrix.

Chapter 3. Geometrical Subspaces and the Special Coordinate Basis Decomposition

In this chapter, we review the geometrical concepts for linear time invariant systems mostly related to the weakly unobservable subspace, which describes the part of the state that can be vanished in output due to the effect of the input. Furthermore, we introduce the special coordinate basis decomposition, a powerful tool for analyzing the structure of linear time invariant systems, which is developed in the 1980s but gains lots of attention recently. The contents in this chapter will be served as a starting point of the distributed estimation for the plant with input.

Chapter 4. Distributed State Observer for Linear Systems

In this chapter, we address the distributed estimation problem, of which the goal is enabling every agent, or sensor node, to reconstruct the plant state by using its own measurements and communicating with the nearby communications through the given topology. Note that, from the individual agent's viewpoint, the plant state is decomposed into the part that can be reconstructible by the individual agent itself using its own measurements and the rest, which can be recovered through the communication. Motivated by this observation, we propose a novel structure of the distributed observer, which consists of the Luenberger observer for recovering the self-reconstructible part, and the communication protocol, called the agent-wise decomposed diffusive coupling, for recovering the rest of the plant state cooperatively. Most of this chapter is based on [KSC16, KLS]. The contribution of this chapter is listed as follows:

- We first propose a novel structure of the distributed observer based on the agent-wise decomposed diffusive coupling, which is a combination of the agent-wise detectability decomposition and the diffusive coupling.
- We provide a simple design method that guarantees the assigned convergence rate and the implementation procedure for individual agent can be implemented by choosing a sufficiently large scalar gain.
- We provide an equivalent condition to the existence of the proposed observer over a general weighted communication topology without any assumption on the plant system matrix and the output matrices.

Chapter 5. Adaptive Distributed State Observer for Completely Decentralized Construction

In this chapter, we improve the proposed distributed observer to the one that can be constructed in a completely decentralized manner so that each agent can implement its own local observer by itself. As direct advantages, it becomes practically useful and robust against the change of environments such as topological changes of communication or the change of the number of agents caused by their leaving/joining. Moreover, the structure allows the plug and play operation; that is, during the operation, a new agent can design itself without global information and join the network, and any agent may leave the network without hampering the operation as long as the operation conditions are met. This chapter is based on [KLS] and its contribution is:

- We design a specific form of adaptive law for the distributed observer with agent-wise diffusive coupling is introduced to achieve the completely decentralized construction.

- We show that the proposed adaptive distributed observer works under the same condition given in the previous chapter only by using the knowledge of the plant system matrix and the local measurement matrix.

- To the best of our knowledge, this is the first decentralized constructible distributed observer, or the first fully distributed observer, in the literature.

Chapter 6. Distributed State Observer for Linear Systems with Input

In this chapter, we aim at enhancing the practicability of the proposed distributed observer design schemes by considering the plant with inputs, where each agent, or sensor node, can measure only a part of the plant input as well as a part of the plant state. Unlike the classical observer design problem, this problem is not trivial because none of agents knows the entire plant input. This setting is, however, more practical than considering just homogeneous plants because it accommodates more practical applications such as monitoring the large-scale systems with multiple input/output channels or the localization in the multi-agent systems with inputs. Note that, from the individual agent's perspective, this setting naturally leads to the sub-problem of identifying the partial state reconstructible by the agent itself using its own measurements despite the unknown input. To address this sub-problem, the results of the geometric control theory is adopted such as the concept of the weakly unobservable subspace and the special coordinate basis decomposition. With a new structural decomposition, the form of distributed observer is proposed based on the agent-wise decomposed diffusive coupling and it is shown that the practical estimation is achieved. The contribution of this chapter is listed as follows:

- We adopt the concept of the geometric subspace from Chapter 3 to show that the part of the plant state except the unstable weakly unobservable subspace is the maximum amount of the reconstructible part. Furthermore, we construct a partial unknown input observer that estimates self-reconstructible part based on the results of special coordinate basis decomposition.

- We develop a new form of the distributed observer that consist of the partial unknown input observer and the agent-wise decomposed diffusive coupling.

- We provide a sufficient condition to yield the practical convergence of the proposed distributed observer's estimation error.

Chapter 7. Application to Distributed Multi-robot Localization

As an application of the distributed observer, we consider the distributed localization problem for a group of robots. We show that the proposed distributed observer provides the practical estimate of all the robots' position by employing the proposed observer in Chapter 6 even when the acceleration of the robots is nonzero and each robot measures its own acceleration only. This demonstrates the advantage of the proposed scheme over the other approaches.

Chapter 2

Graph Theory for Distributed Estimation

Graph theory is a useful tool for understanding and modeling networked agent systems. Generally one agent in the networked system is associated with one node and the information flow from one agent to another agent can be represented by an edge in graph theory. Moreover, the concept of connectivity of graph theory is one of the most basic properties to understand the distributed estimation over networked agent systems.

The definition, type, and basic concepts of a graph have been studied a lot since long ago [Bon76], but algebraic graph theory, the field of defining a matrix that reflects the characteristics of the graph and associating the properties of the graph with the properties of the matrix, has gained much attention in recent years (see, for example, [GR01] and the reference therein) because the most of distributed control methods such as diffusive coupling or local averaging has direct connection with some graph based matrices including the adjacency matrix, the Laplacian matrix, and so on (refer to, for example, [Wie10], [OS05], and [ME10]).

In this chapter, we review the basic definitions and results from graph theory and provide new technical tools to handle the Laplacian matrix of general directed graphs to tackle distributed estimation problem over the general network topology. The results will be used throughout the dissertation.

2.1 Basic Definitions of Graph Theory

In this part, we introduce the weighted directed graph and investigate the other concepts derived from them.

Definition 2.1.1. (Weighted directed graph) A weighted directed graph $\mathcal{G}$ is a tuple $\mathcal{G} := (\mathcal{N}, \mathcal{E}, \mathcal{A})$ of finite nonempty node set $\mathcal{N} := \{1, 2, \cdots, N\}$, edge set $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$, and weighted adjacency matrix $\mathcal{A} := [\alpha_{ij}] \in \mathbb{R}^{N \times N}$, satisfying the following properties:

1. the graph contains no self-loops, i.e., $(i, i) \notin \mathcal{E}$ and $\alpha_{ii} = 0$ for all $i \in \mathcal{N}$, and

2. the element α_{ji} is positive if the edge (i, j) is contained in $\mathcal{E}$ and $\alpha_{ji} = 0$ otherwise.

$\Diamond$

Using the previous definition, the communication topology of a group of systems(agents) is modeled as a graph under the following conventions. The individual systems of the group are associated with the nodes and the information flow from the i-th system to the j-th system is modeled as an edge $(i, j) \in \mathcal{E}$. Typically, a graph can be depicted in diagrammatic form, where a node is represented as a dot or a circle and the information flow from the i-th node to the j-th node is represented as an arrow pointing from the i-th node to the j-th node.

Some special classes of graphs defined in Definition 2.1.1 are of interest. A graph $\mathcal{G}$ is unweighted if the elements of the adjacency matrix $\mathcal{A}$ are constrained to have values in $\{0, 1\}$. In this case, the adjacency matrix can be computed directly from the edge set $\mathcal{E}$, and thus, $\mathcal{G}$ can be simply written as $\{\mathcal{N}, \mathcal{E}\}$. As another important class of graphs, a graph is said to be undirected if it has the property that $\alpha_{ij} = \alpha_{ji}$ for all $i, j \in \mathcal{N}$. Note that this definition implies that the communication between agents occurs in two-way, sending and receiving and, consequently, the adjacency matrix becomes symmetric, i.e., $\mathcal{A} = \mathcal{A}^T$.

A notion of the induced subgraph is also important. Given a graph $\mathcal{G} = \{\mathcal{N}, \mathcal{E}, \mathcal{A}\}$ and a subset of nodes $\bar{\mathcal{N}} \subseteq \mathcal{N}$, the subgraph $\bar{\mathcal{G}}$ induced by $\bar{\mathcal{N}}$ is the graph obtained from $\mathcal{G}$ by removing the nodes of $\mathcal{N} \setminus \bar{\mathcal{N}}$ together with all edges starting from or ending at that node set, i.e., $\bar{\mathcal{G}} = \{\bar{\mathcal{N}}, \bar{\mathcal{E}}\}$, where $\bar{\mathcal{E}} := \{(i,j) \in \mathcal{E} \,|\, i,j \in \bar{\mathcal{N}}\}$.

2.2 Connected Graph and Laplacian Matrix

In this part, we introduce the concept of connectivity in graphs and the Laplacian matrix, and shows the properties of the Laplacian matrix of the connected graphs.

Definition 2.2.1. (Neighbor) For a given graph $\mathcal{G} = \{\mathcal{N}, \mathcal{E}, \mathcal{A}\}$, the j-th node is called neighbor of the i-th node if $(j,i) \in \mathcal{E}$. The neighbors of the i-th node is the set of all nodes that are neighbor of the i-th node, and denoted by $\mathcal{N}_i$, i.e., $\mathcal{N}_i := \{j \in \mathcal{N} \,|\, (j, i \in \mathcal{E})\}$. $\Diamond$

Note that, according to the convention given in the previous part, the j-th agent is a neighbor of the i-th agent if i-th agent can receive information from the j-th agent. The sequence of neighboring pair of nodes leads to the concept of the path.

Definition 2.2.2. (Path) A path from the i-th node to the j-th node of a graph $\mathcal{G}$ is a sequence of $l > 1$ distinct nodes $\{i_1, i_2, \cdots, i_l\}$ such that $(i_k, i_{k+1}) \in \mathcal{E}$ for $k = 1, 2, \cdots, l$. Here, length of the path is $l - 1$. $\Diamond$

If every node in the graph can receive information from all the other agents, then we say that the graph is strongly connected.

Definition 2.2.3. (Strongly connected graph) A graph $\mathcal{G} = \{\mathcal{N}, \mathcal{E}, \mathcal{A}\}$ is said to be strongly connected if for any ordered pair of distinct nodes (i,j), there exists a path from the i-th node to the j-th node. When $\mathcal{G}$ is undirected, we just drop "strongly" and say connected. $\Diamond$

Now we define the Laplacian matrix of a graph.

Definition 2.2.4. (Laplacian matrix) For a graph $\mathcal{G} = \{\mathcal{N}, \mathcal{E}, \mathcal{A}\}$ with $|\mathcal{N}| = N$, the Lapalacian $\mathcal{L}(\mathcal{G}) = [l_{ij}]$ is the N by N real matrix defined as

$$l_{ij} = \begin{cases} \sum_{j=1, j\neq i}^{N} \alpha_{ij}, & i = j, \\ -\alpha_{ij}, & i \neq j. \end{cases}$$

$\diamondsuit$

By its construction, $\mathcal{L}(\mathcal{G})$ contains a zero eigenvalue with a corresponding eigenvector 1_N (i.e., $\mathcal{L}(\mathcal{G})1_N = 0$) and all the other eigenvalues lie in the closed right-half complex plane, which can be shown by using Gersgorin disk theorem.

For an undirected graph $\hat{\mathcal{G}}$, its Laplacian matrix $\mathcal{L}(\hat{\mathcal{G}})$ is symmetric, and thus, the eigenvalues of $\mathcal{L}(\mathcal{G})$ can be sorted as $0 = \lambda_1(\mathcal{L}(\hat{\mathcal{G}})) \leq \lambda_2(\mathcal{L}(\hat{\mathcal{G}})) \leq \cdots \leq \lambda_N(\mathcal{L}(\hat{\mathcal{G}}))$. In this case, the connectivity of $\mathcal{G}$ can be equivalently described by the condition of its Laplacian matrix as follows.

Lemma 2.2.1. (Connectedness of undirected graph) For an undirected graph $\hat{\mathcal{G}}$, its Laplacian matrix $\mathcal{L}(\hat{\mathcal{G}})$ is symmetric, and the zero eigenvalue of $\mathcal{L}(\hat{\mathcal{G}})$ is simple, or equivalently, $\lambda_2(\mathcal{L}(\hat{\mathcal{G}})) > 0$, if and only if the corresponding graph $\hat{\mathcal{G}}$ is connected. $\diamondsuit$

For directed graphs, the strongly connected condition can be characterized by the following lemma adopted from Lemma 2.1 and Lemma 2.2 of [MRC16].

Lemma 2.2.2. (Laplacian matrix for strongly connected graph) Let $\mathcal{G}$ be a strongly connected directed graph with N nodes and $\mathcal{L}(\mathcal{G}) \in \mathbb{R}^{N\times N}$ be the associated Laplacian matrix. Then the following statements hold.

1. There exists a vector $\theta = [\theta_1, \cdots, \theta_N]^T$ with positive entries, i.e., $\theta_i > 0$ for all $i = 1, \cdots, N$ such that $\sum_{i=1}^{N} \theta_i = 1$ and $\theta^T \mathcal{L}(\mathcal{G}) = 0$ hold.

2. The matrix $\hat{\mathcal{L}}(\mathcal{G}) := \Theta\mathcal{L}(\mathcal{G}) + \mathcal{L}(\mathcal{G})^T\Theta$, where $\Theta := \mathrm{diag}(\theta_1, \cdots, \theta_N)$, is a symmetric Laplacian matrix associated with a connected undirected graph $\hat{\mathcal{G}}$ obtained from $\mathcal{G}$ by ignoring the directions of the edges.

$\diamondsuit$

From the technical aspect, this lemma provides a way to treat the Laplacian matrix of the directed graph like the one for the undirected graph.

2.3 Partitioning Directed Graph using Independent Strongly Connected Component

Let us introduce the concept of the independent strongly connected component (iSCC), which is basically a source or isolated strongly connected component.

Definition 2.3.1. (iSCC) An *independent strongly connected component (iSCC)* of a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ is an induced subgraph $\bar{\mathcal{G}} = (\bar{\mathcal{N}}, \bar{\mathcal{E}})$ which is maximal, subject to being strongly connected, and satisfies $(i, j) \notin \mathcal{E}$ for any $i \in \mathcal{N} \setminus \bar{\mathcal{N}}$ and $j \in \bar{\mathcal{N}}$. $\diamondsuit$

Using the notion of iSCC, any weighted directed graph can be partitioned into the set of iSCCs and the rest. The algebraic property required to describe the rest is *M-matrix.*

Definition 2.3.2. (Nonsingular M matrix) A matrix $\mathcal{M} \in \mathbb{R}^{N \times N}$ is said to be a *nonsingular M-matrix* if $\mathcal{M}$ has non-positive off-diagonal entries and all eigenvalues of $\mathcal{M}$ have positive real parts. $\diamondsuit$

The following describes the algebraic property of nonsingular M-matrix, which is borrowed from [Ple77, Theorem 1].

Lemma 2.3.1. Let $\mathcal{M} \in \mathbb{R}^{N \times N}$ be a nonsingular M-matrix. Then there exists a diagonal matrix $G := \text{diag}(g_1, \cdots, g_N)$ with $\sum_{i=1}^{N} g_i = 1$ and $g_i > 0$ for all $i = 1, \cdots, N$ such that the symmetric matrix

$$\hat{\mathcal{M}} := G\mathcal{M} + \mathcal{M}^T G \tag{2.3.1}$$

is positive definite. $\diamondsuit$

Finally, based on the partitioning, the Laplacian matrix of an arbitrary directed graph can be represented by the special upper triangular form given in the following lemma.

Lemma 2.3.2. [Wie10, Appendix A. 2. 3] Consider a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ with c distinct iSCCs $\bar{\mathcal{G}}_k = (\bar{\mathcal{N}}_k, \bar{\mathcal{E}}_k)$ for $k = 1, \cdots, c$. Let us denote

$\bar{\mathcal{N}}_0 := \mathcal{N} \setminus \cup_{k=1}^{c} \bar{\mathcal{N}}_k$ and $\bar{N}_0 := |\bar{\mathcal{N}}_0|$. Without loss of generality, it is assumed that the index of each agent is re-ordered as

$$\begin{aligned}
\bar{\mathcal{N}}_0 &= \{1, \cdots, \bar{N}_0\}, \\
\bar{\mathcal{N}}_1 &= \{\bar{N}_0 + 1, \cdots, \bar{N}_0 + |\bar{\mathcal{N}}_1|\}, \cdots, \\
\bar{\mathcal{N}}_c &= \{N - |\bar{\mathcal{N}}_c| + 1, \cdots, N\}.
\end{aligned}$$

Then $\mathcal{L}(\mathcal{G})$ is represented in the following upper block diagonal form of

$$\mathcal{L}(\mathcal{G}) = \begin{bmatrix} \mathcal{M}_0 & \mathcal{M}_{\mathsf{r}1} & \cdots & \mathcal{M}_{\mathsf{r}c} \\ 0 & \mathcal{L}(\bar{\mathcal{G}}_1) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \mathcal{L}(\bar{\mathcal{G}}_c) \end{bmatrix}, \tag{2.3.2}$$

where $\mathcal{L}(\bar{\mathcal{G}}_k)$ is the Laplacian of iSCC $\bar{\mathcal{G}}_k$ for $k = 1, 2, \cdots, c$, the matrix $\mathcal{M}_0$ is a nonsingular M-matrix, and $\mathcal{M}_{\mathsf{r}1}, \mathcal{M}_{\mathsf{r}2}, \cdots, \mathcal{M}_{\mathsf{r}c}$ are some matrices with appropriate dimension. ♢

Notice that Lemma 2.3.2 can be applied to any weighted directed graph. Roughly speaking, it says that the every graph's Laplacian matrix has diagonal blocks consist of iSCCs' Laplacian matrices and the nonsingular M-matrix. This lemma will play a key role in the subsequent analysis.

Chapter 3

Geometrical Subspaces and the Special Coordinate Basis Decomposition

The geometric control theory for linear systems has attracted much attention over the past few decades. Most of the concepts in linear systems can be addressed and studied nicely within the geometric framework (see, for example, [Won12] and [TSH01]). In fact, the geometric approach is useful to characterize the system properties and provides the intuitive and concise answer to a given problem in general. As a typical example, the system properties such as controllability, observability, system invertibility can be understood from the geometric perspective and additionally, the problem of disturbance decoupling also can be tackled by using the geometric approach.

In the networked agent system, an agent (or a sensor) obtains information through two routes; the first one is to sense the plant and the second one is to communicate with the neighbors. The most basic question one can ask is this: what is the largest amount of information that can be extracted from the measurement? When the plant is autonomous, the answer is the detectable part, or to be specific, the quotient of state space modulo undetectable subspace. When the plant has input, on the other hand, the answer is not simple. If the input is only partially known, the answer is even harder to find.

In this chapter, we make preparation to answer the question. First, we review the geometrical concepts for linear time invariant systems mostly related to the weakly unobservable subspace and its variations, which describes the part of the state that can be vanished in output due to the effect of the input. Furthermore, we introduce the special coordinate basis decomposition, a powerful tool for analyzing the structure of linear time invariant systems having inputs and outputs, which is developed in the 1980s but gains lots of attention recently. The contents in this chapter will be served as a starting point of the distributed estimation for the plant with input.

3.1 Basic Definitions and Properties of Geometrical Subspaces

The purpose of this part is to introduce the basic concepts of some useful geometric subspaces for LTI systems and recall well-known results from geometric control theory.

Let us consider the following continuous time LTI system

$$\Sigma: \quad \begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx + Du, \end{cases} \tag{3.1.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the output, and the matrices A ,B, C, and D are of appropriated dimension. The following are the definition and properties of the weakly unobservable subspace adopted from [TSH01].

Definition 3.1.1. Consider the LTI system Σ of (3.1.1). An initial state of Σ, $x_0 \in \mathbb{R}^n$, is called *weakly unobservable* if there exists an input signal $u(t)$ such that the corresponding system output $y(t) = 0$ for all $t \geq 0$. The subspace formed by the set of all weakly unobservable points of Σ is called the *weakly unobservable subspace* of Σ and is denoted by $\mathcal{V}^*(\Sigma)$. $\Diamond$

The following theorem of [TSH01] shows that the weakly unobservable subspace can be described in an alternative way.

Theorem 3.1.1. The weakly unobservable subspace of Σ, $\mathcal{V}^*(\Sigma)$, is equal to the largest subspace $\mathcal{V}$ that satisfies either one of the following conditions:

1.

$$\begin{bmatrix} A \\ C \end{bmatrix} \mathcal{V} \subseteq (\mathcal{V} \times 0) + \operatorname{im} \begin{bmatrix} B \\ D \end{bmatrix}, \tag{3.1.2}$$

2. there exists an $F \in \mathbb{R}^{m \times n}$ such that $(A+BF)\mathcal{V} \subseteq \mathcal{V}$ and $(C+DF)\mathcal{V} = 0$.

$\Diamond$

Based on the characterization of the weakly unobservable subspace given in Theorem 3.1.1, we can further define the stable and the unstable weakly unobservable subspace Σ.

Definition 3.1.2. Consider a system Σ of (3.1.1). Then we define *the stable weakly unobservable subspace* $\mathcal{V}^-(\Sigma)$ and *the unstable weakly unobservable subspace* $\mathcal{V}^{0+}(\Sigma)$ to be the largest subspace $\mathcal{V}$ that satisfies

$$(A+BF)\mathcal{V} \subseteq \mathcal{V}, \quad (C+DF)\mathcal{V} = 0, \tag{3.1.3}$$

and the eigenvalues of the restriction $(A+BF)|_{\mathcal{V}}$ are contained in the open left half complex plane $\mathbb{C}^-$ and the right closed half complex plane $\mathbb{C}^{0+}$ for some $F \in \mathbb{R}^{m \times n}$, respectively. $\Diamond$

The following definition characterizes a subspace for which any state starting from within it goes to zero in finite time while its corresponding output signal remains zero.

Definition 3.1.3. Consider a system Σ of (3.1.1). An initial state of Σ, $x_0 \in \mathbb{R}^n$, is called controllable weakly unobservable if there exists an input signal $u(t)$ and $t_1 > 0$ such that the corresponding system output $y(t) = 0$ for $t \in [0, t_1]$ and the resulting state vanishes at $t = t_1$, i.e., $x(t_1) = 0$. The subspace formed by the set of all controllable weakly unobservable points of Σ is called the controllable weakly unobservable subspace of Σ and is denoted by $\mathcal{R}^*(\Sigma)$. $\Diamond$

Obviously it follows from the Definitions 3.1.1 and 3.1.3 that $\mathcal{R}^*(\Sigma)$ is contained in $\mathcal{V}^*(\Sigma)$. Since the controllability is an equivalent condition to the arbitrary pole assignment, it is possible to assign any pole of the dynamics of the controllable weakly unobservable subspace using the input while the output is zero.

In this part, we introduce only a part of the geometric subspaces regarding to the reconstruction of the partial state when the plant has input that is unknown. The geometric approach takes the most conservative stance for the unknown information, so the unknown input is regarded as hostile to the state reconstruction. In fact, the unobservable subspace consists of the state that does not appear in the output under the zero input. On the other hand, the weakly unobservable subspace consists of the state that does not appear in the output under the input that is hostilely designed to prevent that state appear in the output. So it is obvious that the weakly unobservable subspace contains the unobservable subspace. In addition, the controllable weakly unobservable subspace specifies the part of the weakly unobservable subspace that is directly influenced by the input.

3.2 Special Coordinate Basis Decomposition

It is well known that the general LTI SISO system with the relative degree ρ can be transformed into the following specific form

$$\begin{aligned} y &= x_{\mathsf{a},1} \\ \dot{x}_{\mathsf{a},1} &= x_{\mathsf{a},2} \\ \vdots\ &=\ \vdots \\ \dot{x}_{\mathsf{a},(\rho-1)} &= x_{\mathsf{a},\rho} \\ \dot{x}_{\mathsf{a},\rho} &= E_{\mathsf{a}0}x_0 + E_{\mathsf{aa}}x_{\mathsf{a}} + u \\ \dot{x}_0 &= A_{00}x_0 + L_0 y, \end{aligned} \tag{3.2.1}$$

where $x_{\mathsf{a}} := [x_{\mathsf{a},1}\, x_{\mathsf{a},2} \cdots x_{\mathsf{a},\rho}]^T$. This form is obtained by repeatedly differentiating the output until it reaches input and by setting the rest part as x_0. In fact, this process can be seen as identifying a chain of integrators that starts from the input and ends with the output, which is associated with the variable x_{a}. Similar process can be done for general LTI MIMO systems, but there might exist three different types of chain of integrators:

1. Integral chains that start from an input and end with an output: the part of the state that describes this type of chains is denoted as $\tilde{x}_{\mathsf{a}}$ in the following lemma.

2. Integral chains that do not start from an input but end with an output: the part of the state that describes this type of chains is denoted as $\tilde{x}_{\mathsf{b}}$ in the following lemma.

3. Integral chains that start from an input but do not end with an output: the part of the state that describes this type of chains is denoted as $\tilde{x}_{\mathsf{e}}$ in the following lemma.

In general, the collection of these partial state does not cover the whole state space of the given system. Consequently, the remaining part of the state is not directly influenced by the input, and does not directly appear in the output. This remaining part is denoted as $[\tilde{x}_{\mathsf{c}}^T \; \tilde{x}_{\mathsf{d}}^T]^T$ in the following lemma.

The structural decomposition by identifying these integral chains is known as *the special coordinate basis decomposition* (SCBD) given in [SS87]. The following is adopted from [CLS04].

Lemma 3.2.1. (SCBD for the strictly proper system) There is a coordinate change of the form

$$x = P\tilde{x} = \begin{bmatrix} P_{\mathsf{a}} & P_{\mathsf{b}} & P_{\mathsf{c}} & P_{\mathsf{d}} & P_{\mathsf{e}} \end{bmatrix} \begin{bmatrix} \tilde{x}_{\mathsf{a}} \\ \tilde{x}_{\mathsf{b}} \\ \tilde{x}_{\mathsf{c}} \\ \tilde{x}_{\mathsf{d}} \\ \tilde{x}_{\mathsf{e}} \end{bmatrix}, \tag{3.2.2}$$

$$y = R\tilde{y} = R \begin{bmatrix} \tilde{y}_{\mathsf{a}} \\ \tilde{y}_{\mathsf{b}} \end{bmatrix}, \quad u = Q\tilde{u} = Q \begin{bmatrix} \tilde{u}_{\mathsf{a}} \\ \tilde{u}_{\mathsf{e}} \end{bmatrix} \tag{3.2.3}$$

which transforms the system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx \end{aligned}$$

into

$$\begin{aligned} \dot{\tilde{x}} &= P^{-1}AP\tilde{x} + P^{-1}BQ\tilde{u} \\ \tilde{y} &= R^{-1}CP\tilde{x} \end{aligned} \tag{3.2.4}$$

where

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} \tilde{A}_{\mathsf{aa}} & \tilde{B}_{\mathsf{a}}\tilde{A}_{\mathsf{ab}} & \tilde{B}_{\mathsf{a}}\tilde{A}_{\mathsf{ac}} & \tilde{B}_{\mathsf{a}}\tilde{A}_{\mathsf{ad}} & \tilde{B}_{\mathsf{a}}\tilde{A}_{\mathsf{ae}} \\ \tilde{A}_{\mathsf{ba}}\tilde{C}_{\mathsf{a}} & \tilde{A}_{\mathsf{bb}} & 0 & 0 & 0 \\ \tilde{A}_{\mathsf{ca}}\tilde{C}_{\mathsf{a}} & \tilde{A}_{\mathsf{cb}}\tilde{C}_{\mathsf{b}} & \tilde{A}_{\mathsf{cc}} & 0 & 0 \\ \tilde{A}_{\mathsf{da}}\tilde{C}_{\mathsf{a}} & \tilde{A}_{\mathsf{db}}\tilde{C}_{\mathsf{b}} & 0 & \tilde{A}_{\mathsf{dd}} & 0 \\ \tilde{A}_{\mathsf{ea}}\tilde{C}_{\mathsf{a}} & \tilde{A}_{\mathsf{eb}}\tilde{C}_{\mathsf{b}} & \tilde{B}_{\mathsf{e}}\tilde{A}_{\mathsf{ec}} & \tilde{B}_{\mathsf{e}}\tilde{A}_{\mathsf{ed}} & \tilde{A}_{\mathsf{ee}} \end{bmatrix}, \\ P^{-1}BQ &= \begin{bmatrix} \tilde{B}_{\mathsf{a}} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \tilde{B}_{\mathsf{e}} \end{bmatrix}, \\ R^{-1}CP &= \begin{bmatrix} \tilde{C}_{\mathsf{a}} & 0 & 0 & 0 & 0 \\ 0 & \tilde{C}_{\mathsf{b}} & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{3.2.5}$$

Here, the transformed system satisfies following properties:

1. the variables $\tilde{x}_{\mathsf{a}}$, $\tilde{y}_{\mathsf{a}}$, $\tilde{u}_{\mathsf{a}}$ consist of ρ parts

$$\tilde{x}_{\mathsf{a}} = \begin{bmatrix} \tilde{x}_{\mathsf{a},1} \\ \vdots \\ \tilde{x}_{\mathsf{a},\rho} \end{bmatrix}, \quad \tilde{y}_{\mathsf{a}} = \begin{bmatrix} \tilde{y}_{\mathsf{a},1} \\ \vdots \\ \tilde{y}_{\mathsf{a},\rho} \end{bmatrix}, \quad \tilde{u}_{\mathsf{a}} = \begin{bmatrix} \tilde{u}_{\mathsf{a},1} \\ \vdots \\ \tilde{u}_{\mathsf{a},\rho} \end{bmatrix}, \tag{3.2.6}$$

   where $\tilde{u}_{\mathsf{a},k} \in \mathbb{R}$, $\tilde{y}_{\mathsf{a},k} \in \mathbb{R}$, $\tilde{x}_{\mathsf{a},k} \in \mathbb{R}^{q_k}$ for $1 \le k \le \rho$, and the matrices $\tilde{A}_{\mathsf{aa}}$, $\tilde{B}_{\mathsf{a}}$, and $\tilde{C}_{\mathsf{a}}$ is of the form

$$\begin{aligned} \tilde{A}_{\mathsf{aa}} &= \mathrm{diag}(\mathsf{A}_{q_1}, \cdots, \mathsf{A}_{q_\rho}) + \tilde{B}_{\mathsf{a}}E_{\mathsf{a}} + F_{\mathsf{a}}\tilde{C}_{\mathsf{a}} \\ \tilde{B}_{\mathsf{a}} &= \mathrm{diag}(\mathsf{B}_{q_1}, \cdots, \mathsf{B}_{q_\rho}) \\ \tilde{C}_{\mathsf{a}} &= \mathrm{diag}(\mathsf{C}_{q_1}, \cdots, \mathsf{C}_{q_\rho}) \end{aligned} \tag{3.2.7}$$

for some matrices E_{a} and F_{a} with

$$\mathsf{A}_{q_k} := \begin{bmatrix} 0 & I_{q_k-1} \\ 0 & 0 \end{bmatrix}, \quad \mathsf{B}_{q_k} := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{q_k \times 1} \tag{3.2.8}$$

$$\mathsf{C}_{q_k} := \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{1 \times q_k}, \tag{3.2.9}$$

2. the pair $(\tilde{A}_{\mathsf{bb}}, \tilde{C}_{\mathsf{b}})$ is observable,
3. the eigenvalues of $\tilde{A}_{\mathsf{cc}}$ lie in $\mathbb{C}^-$, the eignevalues of $\tilde{A}_{\mathsf{dd}}$ lie in $\mathbb{C}^{0+}$, and
4. the pair $(\tilde{A}_{\mathsf{ee}}, \tilde{B}_{\mathsf{e}})$ is controllable.

$\Diamond$

Here we briefly summarize the procedure to obtain SCBD for MIMO system.

Step i) Partition output $y = Cx \in \mathbb{R}^m$ into scalar outputs so that $y_i = C_i x$ with row matrices C_i for $i = 1, 2, \cdots, m$.

Step ii) Repeatedly differentiate from y_1 to y_m under the following rule. Assume that the iteration number is k, which starts from 0.

- If the derivative $\tilde{y}_i^{(k)}$ reaches input, it is the end of this chain of integrators and it belongs to the subspace associated with $\tilde{x}_{\mathsf{a}}$. In this case, the differentiation associated with the output y_i ends.
- If not, then refine $\tilde{y}_i^{(k)}$ to avoid entanglement with the coordinate of the previously defined integral chains. There are following two sub-cases.
  - If the the derivative $\tilde{y}_i^{(k)}$[1] can be represented by a linear combination of the derivatives $y_i, y_i^{(1)}, \cdots, y_i^{(k-1)}$ and the derivatives are contained in previously computed integral chains, then this chain of integrators ends up without an input and it belongs

[1] Note that $\tilde{y}_i^{(k)}$ is not just derivative of $\tilde{y}_i^{(k-1)}$ but is defined by the specific linear combination of previously defined derivative of outputs.

to the subspace associated with $\tilde{x}_{\mathsf{b}}$. In this case, the differentiation associated with the output y_i ends.

- Otherwise, it implies that there are more integrators in the chain associated with y_i, which must be further identified. In this case, $\tilde{y}_i^{(k+1)}$ is defined as the addition of the derivative of $y_i^{(k)}$ and a linear combination of the previously computed derivatives so that the integral chain structure remains valid.

Step iii) Refine (determine) $\tilde{y}_{\mathsf{a}}$, $\tilde{x}_{\mathsf{a}}$, $\tilde{u}_{\mathsf{a}}$, $\tilde{y}_{\mathsf{b}}$ and $\tilde{x}_{\mathsf{b}}$ so that the dynamics of $\tilde{x}_{\mathsf{a}}$ becomes the first item i.e., (3.2.6) and (3.2.7), and the dynamics of $\tilde{x}_{\mathsf{b}}$ becomes the second line of the $P^{-1}AP$, i.e.,

$$\dot{\tilde{x}}_{\mathsf{b}} = \tilde{A}_{\mathsf{bb}}\tilde{x}_{\mathsf{b}} + \tilde{A}_{\mathsf{ba}}\tilde{y}_{\mathsf{a}}, \qquad \tilde{y}_{\mathsf{b}} = \tilde{C}_{\mathsf{b}}\tilde{x}_{\mathsf{b}}.$$

Here, since $\tilde{y}_i^{(k+1)}$ is defined as the addition of the derivative of $y_i^{(k)}$ and a linear combination of the previously computed derivatives for each i, the partial state $\tilde{x}_{\mathsf{b}}$ consists of $\tilde{x}_{\mathsf{b},j}$ for $j = 1, 2, \cdots, m_b$ where each $\tilde{x}_{\mathsf{b},j} \in \mathbb{R}^{l_j}$, denoted as $[\tilde{x}_{\mathsf{b},j,1}, \tilde{x}_{\mathsf{b},j,2}, \cdots, \tilde{x}_{\mathsf{b},j,l_j}]$ has dynamics given by

$$\begin{aligned}
\tilde{y}_{\mathsf{b},j} &= \tilde{x}_{\mathsf{b},j,1} \\
\dot{\tilde{x}}_{\mathsf{b},j,1} &= \tilde{x}_{\mathsf{b},j,2} + [*]\tilde{y}_{\mathsf{a}} + [**]\tilde{y}_{\mathsf{b}} \\
\vdots \ &= \ \vdots \\
\dot{\tilde{x}}_{\mathsf{b},j,(l_j-1)} &= \tilde{x}_{\mathsf{b},j,l_j} + [*]\tilde{y}_{\mathsf{a}} + [**]\tilde{y}_{\mathsf{b}} \\
\dot{\tilde{x}}_{\mathsf{b},j,l_j} &= [*]\tilde{y}_{\mathsf{a}} + [**]\tilde{y}_{\mathsf{b}},
\end{aligned}$$

where $[*]$ and $[**]$ denote the matrices of less importance. Note that the upper triangular entries of this dynamics are one. The observability of the pair $(\tilde{A}_{\mathsf{bb}}, \tilde{C}_{\mathsf{b}})$ can be checked from the fact that setting $\tilde{y}_{\mathsf{b}} \equiv 0$ implies $\tilde{x}_{\mathsf{b}} \equiv 0$.

Step iv) Let $\tilde{x}_0$ be the rest of the state unrelated to $\tilde{x}_{\mathsf{a}}$ and $\tilde{x}_{\mathsf{b}}$. Then its dynamics becomes

$$\dot{\tilde{x}}_0 = \bar{A}_{00}\tilde{x}_0 + \bar{A}_{0\mathsf{b}}\tilde{x}_\mathsf{b} + \bar{A}_{0\mathsf{a}}\tilde{x}_\mathsf{a} + \bar{B}_{0\mathsf{a}}\tilde{u}_\mathsf{a} + \bar{B}_{0\mathsf{c}}\tilde{u}_\mathsf{c}. \tag{3.2.10}$$

For $i = 1, 2, \cdots, m_d$, let us denote $\tilde{x}_{\mathsf{a},i} \in \mathbb{R}^{q_i} = [\tilde{x}_{\mathsf{a},i,1}, \tilde{x}_{\mathsf{a},i,2}, \cdots, \tilde{x}_{\mathsf{a},i,q_i}]^T$. Then from the structure (3.2.7), it holds that

$$\dot{\tilde{x}}_{\mathsf{a},i,q_i} = [*]\tilde{x}_0 + [**]\tilde{x}_\mathsf{b} + [***]\tilde{x}_\mathsf{a} + \tilde{u}_{\mathsf{a},i}.$$

Therefore, $\tilde{u}_\mathsf{a}$ can be eliminated from (3.2.10) and, with abuse of notation[2], (3.2.10) can be rewritten as

$$\dot{\tilde{x}}_0 = \bar{A}_{00}\tilde{x}_0 + \bar{A}_{0\mathsf{b}}\tilde{x}_\mathsf{b} + \bar{A}_{0\mathsf{a}}\tilde{x}_\mathsf{a} + \bar{B}_{0\mathsf{a}} \begin{bmatrix} \dot{\tilde{x}}_{\mathsf{a},1,q_1} \\ \dot{\tilde{x}}_{\mathsf{a},2,q_2} \\ \vdots \\ \dot{\tilde{x}}_{\mathsf{a},m_d,q_{m_d}} \end{bmatrix} + \bar{B}_{0\mathsf{c}}\tilde{u}_\mathsf{c}.$$

With abuse of notation, by redefine new $\tilde{x}_0$ as

$$\tilde{x}_0 - \bar{B}_{0\mathsf{a}} \begin{bmatrix} \tilde{x}_{\mathsf{a},1,q_1} \\ \tilde{x}_{\mathsf{a},2,q_2} \\ \vdots \\ \tilde{x}_{\mathsf{a},m_d,q_{m_d}} \end{bmatrix},$$

(3.2.10) can be rewritten as

$$\dot{\tilde{x}}_0 = \bar{A}_{00}\tilde{x}_0 + \bar{A}_{0\mathsf{b}}\tilde{x}_\mathsf{b} + \bar{A}_{0\mathsf{a}}\tilde{x}_\mathsf{a} + \bar{B}_{0\mathsf{c}}\tilde{u}_\mathsf{c}.$$

In a similar way, by iterative redefinition of $\tilde{x}_0$, it is possible to represent $\dot{\tilde{x}}_0$ with $\tilde{y}_\mathsf{a}$ and $\tilde{y}_\mathsf{b}$ instead of $\tilde{x}_\mathsf{a}$ and $\tilde{x}_\mathsf{b}$ as follows:

$$\dot{\tilde{x}}_0 = \bar{A}_{00}\tilde{x}_0 + \bar{A}_{0\mathsf{b}}\tilde{y}_\mathsf{b} + \bar{A}_{0\mathsf{a}}\tilde{y}_\mathsf{a} + \bar{B}_{0\mathsf{c}}\tilde{u}_\mathsf{c}. \tag{3.2.11}$$

[2]In this step, the variable $\tilde{x}_0$ is re-defined many times. However, to avoid the notational complexity, only one symbol is used

Step v) By applying the controllability decomposition to (3.2.11), $\tilde{x}_0$ is decomposed into $[\tilde{x}_w^T \, \tilde{x}_{\mathsf{e}}^T]^T$ such that

$$\dot{\tilde{x}}_w = \bar{A}_{ww}\tilde{x}_w + \bar{A}_{w\mathsf{a}}\tilde{y}_{\mathsf{a}} + \bar{A}_{w\mathsf{b}}\tilde{y}_{\mathsf{b}}$$
$$\dot{\tilde{x}}_{\mathsf{e}} = \bar{A}_{\mathsf{e}w}\tilde{x}_w + \tilde{A}_{\mathsf{ee}}\tilde{x}_{\mathsf{e}} + \tilde{A}_{\mathsf{ea}}\tilde{y}_{\mathsf{a}} + \tilde{A}_{\mathsf{eb}}\tilde{y}_{\mathsf{b}} + \tilde{B}_{\mathsf{e}}\tilde{u}_{\mathsf{e}},$$

where the pair $(\tilde{A}_{\mathsf{ee}}, \tilde{B}_{\mathsf{e}})$ is controllable.

Step vi) Finally, by applying the real Jordan decomposition, $\tilde{x}_w$ is decomposed into $[\tilde{x}_{\mathsf{c}}^T \, \tilde{x}_{\mathsf{d}}^T]^T$ such that

$$\dot{\tilde{x}}_{\mathsf{c}} = \tilde{A}_{\mathsf{cc}}\tilde{x}_{\mathsf{c}} + \tilde{A}_{\mathsf{ca}}\tilde{y}_{\mathsf{a}} + \tilde{A}_{\mathsf{cb}}\tilde{y}_{\mathsf{b}}$$
$$\dot{\tilde{x}}_{\mathsf{d}} = \tilde{A}_{\mathsf{dd}}\tilde{x}_{\mathsf{d}} + \tilde{A}_{\mathsf{da}}\tilde{y}_{\mathsf{a}} + \tilde{A}_{\mathsf{db}}\tilde{y}_{\mathsf{b}}.$$

where the eigenvalues of $\tilde{A}_{\mathsf{cc}}$ lie on $\mathbb{C}^-$ and the eigenvalues of $\tilde{A}_{\mathsf{dd}}$ lie on $\mathbb{C}^{0+}$.

This results can be directly extended to the nonstrictly proper systems, i.e., $D \neq 0$ in (3.1.1). There are nonsingular transformations G and H such that

$$GDH = \begin{bmatrix} I_{\mathrm{rank\,D}} & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore, without loss of generality, (3.1.1) can be rewritten as

$$\begin{aligned} \dot{x} &= Ax + \begin{bmatrix} B_0 & B_1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \\ \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} &= \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} x + \begin{bmatrix} I_{\mathrm{rank\,D}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \end{aligned} \tag{3.2.12}$$

where B_0, B_1, C_0, C_1 are matrices with appropriate dimension. Here we have

$$\dot{x} = Ax + B_0(y_0 - C_0x) + B_1u_1 = (A - B_0C_0)x + B_1u_1 + B_0y_0$$

Now apply (3.2.1) to the strictly proper system

$$\begin{aligned} \dot{x} &= (A - B_0C_0)x + B_1u_1 \\ y_1 &= C_1x. \end{aligned}$$

Then there exist nonsingular transformation P, Q, and R such that

$$P^{-1}(A - B_0C_0)P = \bar{A}, \quad P^{-1}B_1Q = \bar{B}_1, \quad R^{-1}C_1P = \bar{C}_1 \tag{3.2.13}$$

where $\bar{A}$, $\bar{B}_1$, and $\bar{C}_1$ denote the matrices of the right hand side of (3.2.5). Consequently, the coordinate change

$$x = P\tilde{x}, \quad y = \begin{bmatrix} y_0 \\ R\tilde{y}_1 \end{bmatrix}, \quad u = \begin{bmatrix} u_0 \\ Q\tilde{u}_1 \end{bmatrix} \tag{3.2.14}$$

with (3.2.13) transforms (3.2.12) into special form. This can be summarized by the following lemma.

Lemma 3.2.2. (SCBD for the nonstrictly proper system) There exist nonsingular transformations P, Q, R which transform the nonstrictly proper system (3.1.1) into

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u \tag{3.2.15}$$

$$y = \tilde{C}\tilde{x} + \tilde{D}\tilde{u} \tag{3.2.16}$$

where

$$\tilde{A} = P^{-1}AP = \begin{bmatrix} \tilde{A}_{\mathsf{aa}} & \tilde{B}_{\mathsf{a}}\tilde{A}_{\mathsf{ab}} & \tilde{B}_{\mathsf{a}}\tilde{A}_{\mathsf{ac}} & \tilde{B}_{\mathsf{a}}\tilde{A}_{\mathsf{ad}} & \tilde{B}_{\mathsf{a}}\tilde{A}_{\mathsf{ae}} \\ \tilde{A}_{\mathsf{ba}}\tilde{C}_{\mathsf{a}} & \tilde{A}_{\mathsf{bb}} & 0 & 0 & 0 \\ \tilde{A}_{\mathsf{ca}}\tilde{C}_{\mathsf{a}} & \tilde{A}_{\mathsf{cb}}\tilde{C}_{\mathsf{b}} & \tilde{A}_{\mathsf{cc}} & 0 & 0 \\ \tilde{A}_{\mathsf{da}}\tilde{C}_{\mathsf{a}} & \tilde{A}_{\mathsf{db}}\tilde{C}_{\mathsf{b}} & 0 & \tilde{A}_{\mathsf{dd}} & 0 \\ \tilde{A}_{\mathsf{ea}}\tilde{C}_{\mathsf{a}} & \tilde{A}_{\mathsf{eb}}\tilde{C}_{\mathsf{b}} & \tilde{B}_{\mathsf{e}}\tilde{A}_{\mathsf{ec}} & \tilde{B}_{\mathsf{e}}\tilde{A}_{\mathsf{ed}} & \tilde{A}_{\mathsf{ee}} \end{bmatrix} + \begin{bmatrix} \tilde{B}_{0\mathsf{a}} \\ \tilde{B}_{0\mathsf{b}} \\ \tilde{B}_{0\mathsf{c}} \\ \tilde{B}_{0\mathsf{d}} \\ \tilde{B}_{0\mathsf{e}} \end{bmatrix} \begin{bmatrix} \tilde{C}_{0\mathsf{a}} & \tilde{C}_{0\mathsf{b}} & \tilde{C}_{0\mathsf{c}} & \tilde{C}_{0\mathsf{d}} & \tilde{C}_{0\mathsf{e}} \end{bmatrix},$$

$$\tilde{B} = P^{-1}BQ = \begin{bmatrix} \tilde{B}_{0\mathsf{a}} & \tilde{B}_{\mathsf{a}} & 0 \\ \tilde{B}_{0\mathsf{b}} & 0 & 0 \\ \tilde{B}_{0\mathsf{c}} & 0 & 0 \\ \tilde{B}_{0\mathsf{d}} & 0 & 0 \\ \tilde{B}_{0\mathsf{e}} & 0 & \tilde{B}_{\mathsf{e}} \end{bmatrix}, \tag{3.2.17}$$

$$\tilde{C} = R^{-1}CP = \begin{bmatrix} \tilde{C}_{0\mathsf{a}} & \tilde{C}_{0\mathsf{b}} & \tilde{C}_{0\mathsf{c}} & \tilde{C}_{0\mathsf{d}} & \tilde{C}_{0\mathsf{e}} \\ \tilde{C}_{\mathsf{a}} & 0 & 0 & 0 & 0 \\ 0 & \tilde{C}_{\mathsf{b}} & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{D} = R^{-1}DQ = \begin{bmatrix} I_{\mathrm{rank}\,D} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here, the arguments of Lemma 3.2.1 remain valid. $\Diamond$

For convenience of readers, the set of partial state introduced in Lemma 3.2.1 and their characteristics are summarized in Table 3.1.

Variable	Characteristics
$\tilde{x}_{\mathsf{a}}$	the chain of integrators with direct input and output
$\tilde{x}_{\mathsf{b}}$	the chain of integrators without a direct input
$\tilde{x}_{\mathsf{c}}$	the stable subsystem without direct input and output
$\tilde{x}_{\mathsf{d}}$	the unstable subsystem without direct input and output
$\tilde{x}_{\mathsf{e}}$	the chain of integrators without a direct output

Table 3.1: The partial state variables introduced in the special coordinate basis decomposition given in Lemma 3.2.1.

3.3 Interconnections between Geometrical Subspaces and Special Coordinate Basis Decomposition

The structurally decomposed form illustrates the system properties more explicitly such as observability (detectability), controllability (stabilizability), invariant zero structure and left and right invertibility. The details are given in [CLS04]. Besides the system properties, there are interconnections between the geometric subspaces and the SCBD. The following shows a part of the interconnections.

Lemma 3.3.1. Let P be the nonsingular matrix given in Lemma 3.2.1 for the system Σ. Then the followings hold.

1. the weakly unobservable subspace $\mathcal{V}^*(\Sigma)$ is spanned by the columns of the matrix $[P_{\mathsf{c}}\, P_{\mathsf{d}}\, P_{\mathsf{e}}]$,

2. the unstable weakly unobservable subspace $\mathcal{V}^{0+}(\Sigma)$ is spanned by the columns of the matrix $[P_{\mathsf{d}}\, P_{\mathsf{e}}]$,

3. the stable weakly unobservable subspace $\mathcal{V}^{-}(\Sigma)$ is spanned by the columns of the matrix $[P_{\mathsf{c}}\, P_{\mathsf{e}}]$, and

4. the controllable weakly unobservable subspace $\mathcal{R}^*(\Sigma)$ is spanned by the columns of the matrix P_{e}.

$\Diamond$

For the proof, refer chapter 5.5 in [CLS04].

Notice that this lemma provides evidence for finding an answer to the motivating question: what is the largest amount of information that can be extracted from the measurements? Recall that, for autonomous plants, the answer is the quotient of the undetectable subspace, which is basically the rest of the smallest amount of information never known by the measurement. Similarly, the answer to that question under the unknown input could be given by the rest of the smallest part of state never known by measurement under the unknown input. In this context, Lemma 3.3.1 plays a key role to identify the portion of the state never known by measurement under the unknown input.

Chapter 4

Distributed State Observer for Linear Systems

In this chapter, we define the problem of distributed estimation, which is the main topic of this dissertation. The purpose of this problem is to allow each agent (sensor) to overcome the limitations of the classical estimations by using its own measurements through the inter-agent communication, thus enabling each agent to estimate the full state of the plant. Obviously it is of significant importance to make effective use of the inter-agent communication. We propose a novel structure of the distributed observer using the agent-wise decomposed diffusive coupling, i.e., a distributed communication protocol, which is the combination of the projection and the diffusive coupling. Its intuitive structure drastically simplifies the observer design process by assigning a sufficiently large coupling gain for each agent.

As pointed out in the introduction, the communication topology of the networked agent system could have various shapes. So it is necessary to study the distributed estimation problem without making assumptions on the plant or the network. In this context, the practicality of a distributed observer can be measured by the size of the class of the plant and communication topology it can accommodate. In this chapter, we investigate the performance of the proposed distributed observer for the general linear time invariant plant over the general directed graph. Furthermore, we obtain an equivalent condition for the existence of the proposed observer for the general linear time invariant plant under the gen-

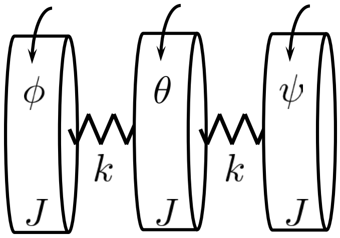


Figure 4.1: The three inertia system.

eral directed graph based on the result from Chapter 2. Lastly, the issues related to the implementation of the proposed distributed observer will be investigated.

Let us start with a motivating example.

Example 4.0.1. Consider a three inertia system monitored by 3 sensors. Denoting ϕ, θ, and ψ as the three inertia's angles, suppose that the sensor's measurements are $y_1 = \phi - \theta$, $y_2 = \theta$, and $y_3 = \theta - \psi$. With state $x := \begin{bmatrix} \phi & \dot{\phi} & \theta & \dot{\theta} & \psi & \dot{\psi} \end{bmatrix}^T$, this system is modeled as $\dot{x} = Ax$ and $y_i = C_i x$ for $i = 1, 2, 3$ with

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k}{J} & 0 & \frac{k}{J} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k}{J} & 0 & -\frac{2k}{J} & 0 & \frac{k}{J} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{k}{J} & 0 & -\frac{k}{J} & 0 \end{bmatrix} \\
C_1 &= \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \\
C_2 &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \\
C_3 &= \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix},
\end{aligned}
$$

where k is the torsional stiffness and J is the moment of inertia. Suppose also that, through the communication, the sensor 1 and 3 can receive information from the sensor 2 and sensor 2 can receive information from both agents. In this case, the communication graph can be modeled as a strongly connected one in Fig. 4.2 and the unweighted adjacency matrix $\mathcal{A}$ and the Laplacian matrix $\mathcal{L}$ are given as

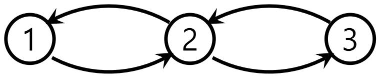


Figure 4.2: The communication graph in Example 4.0.1.

follows:

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathcal{L} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Note that, with the plant dynamics A, none of the individual agent (sensor) is detectable so none of them can estimate the full state of the plant. However, the concatenated (virtual) system $[C_1^T C_2^T C_3^T]^T$ is detectable and the agents can communicate with each other. Therefore, one might expect that there exists an algorithm that enables each agent to estimate the full state. We call that algorithm *distributed observer*. From now on, let us find it.

4.1 Problem Statement

Consider a continuous-time linear time invariant system

$$\dot{x} = Ax, \tag{4.1.1}$$

where $x \in \mathbb{R}^n$ is the state. We assume that the system (4.1.1) is monitored by a network of N agents (sensors) such that the i-th agent measures the partial state $y_i \in \mathbb{R}^{p_i}$ given by

$$y_i = C_i x, \quad i = 1, 2, \cdots, N, \tag{4.1.2}$$

and communicates through the given communication topology, which is represented by a graph $\mathcal{G} = \{\mathcal{N}, \mathcal{E}, \mathcal{A}\}$. Obviously each i-th agent is associated with each i-th node of the graph $\mathcal{G}$. Our goal is to design a distributed observer for the system (4.1.1) and (4.1.2) with a general directed communication network $\mathcal{G}$

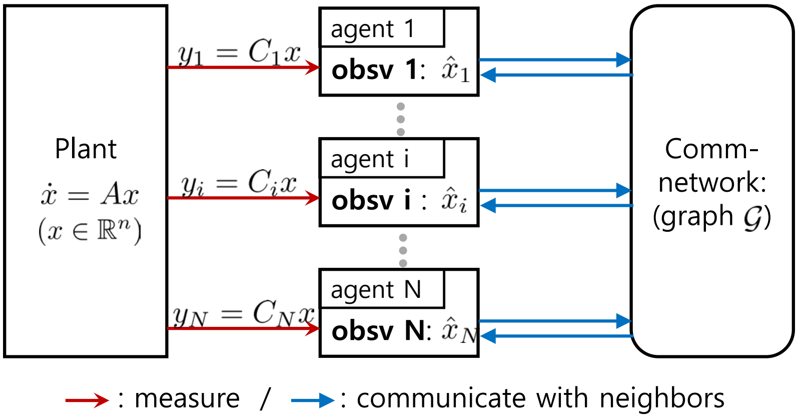


Figure 4.3: The structure of the distributed observer design problem.

consisting of N agents, such that the state estimate $\hat{x}_i(t)$ of each i-th local observer[1] converges to the plant state $x(t)$, i.e.,

$$\lim_{t \to \infty} |\hat{x}_i(t) - x(t)| = 0, \quad \forall i \in \mathcal{N},$$

under the following constraints:

i) *local measurement:* each i-th local observer utilizes its local measurement y_i only, and

ii) *local communication:* each i-th local observer communicates with its neighboring agents.

The structure of the distributed observer is given in Fig. 4.3.

4.2 Detectability and Detectability Decomposition

Considering the distributed observer as an algorithm that enables each agent to reconstruct the full state of the plant by interchanging each sensor's partial information, it is important to explore the partial information that can be estimated by using the measured output y_i only. In this part, we recall useful results in linear system theory on state reconstruction.

[1]The i-th local observer denotes the local observer associated with (and implemented in) the i-th agent.

Let $\chi_A(s)$ be the characteristic polynomial of A, factored as $\chi_A(s) = \chi_A^-(s)\chi_A^{0+}(s)$, where $\chi_A^-(s)$ and $\chi_A^{0+}(s)$ have roots in the open left and the closed right half planes of the complex plane, respectively. Then, the undetectable subspace viewed on each agent is defined as follows.

Notation 1. For $i = 1, 2, \cdots, N$, let us denote $\mathcal{U}_i$ as the undetectable subspace of the pair (A, C_i), which is given by

$$\mathcal{U}_i = \cap_{l=1}^{n} \ker(C_i A^{l-1}) \cap \ker \chi_A^{0+}(A), \tag{4.2.1}$$

where n is the dimension of the square matrix A. $\Diamond$

At Section 5.2 in [TSH01], the undetectable subspace is the minimal in the sense that there exists an asymptotic observer for its quotient, which we call the *detectable part*. Since detectability is an equivalent condition for the existence of asymptotic observers, the detectability of the subsystem governing the detectable part holds naturally as mentioned in page 116 of [TSH01]. For clarity, we state the following lemma, which is an extension of the Kalman decomposition to detectable systems.

Lemma 4.2.1. (Detectability decomposition) Denoting ν_i as the dimension of $\mathcal{U}_i$, the undetectable subspace of (A, C_i), let $U_i \in \mathbb{R}^{n\times\nu_i}$ and $D_i \in \mathbb{R}^{n\times(n-\nu_i)}$ be the matrices whose columns form orthonormal bases of $\mathcal{U}_i$ and its orthogonal complement $\mathcal{U}_i^\perp$, respectively. Then the orthonormal[2] matrix $T_i \in \mathbb{R}^{n\times n}$ defined by

$$T_i := [D_i \, U_i], \tag{4.2.2}$$

satisfies that

$$T_i^T A T_i = \begin{bmatrix} A_{i\mathsf{d}} & 0 \\ A_{i\mathsf{r}} & A_{i\mathsf{u}} \end{bmatrix}, \quad C_i T_i = [C_{i\mathsf{d}} \; 0], \tag{4.2.3}$$

where $A_{i\mathsf{d}} \in \mathbb{R}^{(n-\nu_i)\times(n-\nu_i)}$, $A_{i\mathsf{r}} \in \mathbb{R}^{\nu_i\times(n-\nu_i)}$, $A_{i\mathsf{u}} \in \mathbb{R}^{\nu_i\times\nu_i}$, and $C_{i\mathsf{d}} \in \mathbb{R}^{p_i\times(n-\nu_i)}$. Moreover, the pair $(A_{i\mathsf{d}}, C_{i\mathsf{d}})$ is detectable and the matrix $A_{i\mathsf{u}}$ is unstable. $\Diamond$

[2]The matrix $T_i \in \mathbb{R}^{n\times n}$is called orthonormal if $T_i^T T_i = I_n$.

Proof. See Appendix A.1. □

Lastly, the undetectable subspace for the pair $(A, \mathrm{col}(C_i)_{i\in\mathcal{N}})$ can be described by the set of individual agent's undetectable subspaces as follows.

Lemma 4.2.2. The undetectable subspace of $(A, \mathrm{col}(C_i)_{i\in\mathcal{N}})$ is $\cap_{i=1}^{N}\mathcal{U}_i$. As a result, the pair $(A, \mathrm{col}(C_i)_{i\in\mathcal{N}})$ is detectable if and only if $\cap_{i=1}^{N}\mathcal{U}_i = \{0\}$. ◊

Proof. It is straightforward from the following equality:

$$\cap_{l=1}^{n} \ker(\mathrm{col}(C_i)_{i\in\mathcal{N}} A^{l-1}) = \cap_{i=1}^{N}\{\cap_{l=1}^{n} \ker(C_i A^{l-1})\}.$$

□

We conclude this part by computing the subspaces and the submatrices of the detectability decomposition defined above for the following example.

Example 4.2.1. From Example 4.0.1, for $i = 1, 2, 3$, none of the pairs (A, C_i) is detectable, and the matrices U_1, U_2, and U_3 can be chosen as

$$U_1 = U_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad U_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

And the submatrices for the undetectable part are given by

$$A_{1\mathsf{u}} = A_{3\mathsf{u}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_{2\mathsf{u}} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{J} & 0 \end{bmatrix}.$$

4.3 Distributed Observer with Agent-wise Decomposed Diffusive Coupling

In the previous part, from the each i-th agent's perspective, the plant state can be decompose into the detectable part, which is obtainable solely by the local measurement y_i, and the rest, the undetectable part. Therefore, for each agent to rebuild the full plant state, it is natural to treat the detectable part and the undetectable part differently. To this end, we employ agent-wise decomposed diffusive coupling to design a distributed observer.

Let us propose a distributed observer for the system given by (4.1.1) and (4.1.2) with the communication network $\mathcal{G}$. It consists of N local observers, and each local observer of the i-th agent, say "i-th local observer", has the dynamics

$$\dot{\hat{x}}_i = A\hat{x}_i + L_i(y_i - C_i\hat{x}_i) + \gamma_i M_i \sum_{j\in\mathcal{N}_i} \alpha_{ij}(\hat{x}_j - \hat{x}_i), \tag{4.3.1}$$

where $\mathcal{N}_i$ denotes the neighboring agents of the i-th agent, L_i is the injection gain matrix, M_i is the weighting matrix, and γ_i is the coupling gain of the i-th agent. Here, L_i and M_i are designed as

$$L_i := T_i \begin{bmatrix} L_{i\mathsf{d}} \\ 0 \end{bmatrix}, \quad M_i := T_i \begin{bmatrix} 0 & 0 \\ 0 & I_{\nu_i} \end{bmatrix} T_i^T, \tag{4.3.2}$$

where I_{ν_i} is an identity matrix of size $\nu_i \times \nu_i$ with ν_i being the dimension of the undetectable subspace of (A, C_i). The matrix T_i is the orthonormal coordinate transformation matrix defined in (4.2.2). The matrix $L_{i\mathsf{d}}$ is chosen such that $A_{i\mathsf{d}} - L_{i\mathsf{d}}C_{i\mathsf{d}}$ is Hurwitz.

The greatest feature of the proposed algorithm is the shape of coupling, that is

$$\gamma_i M_i \sum_{j\in\mathcal{N}_i} \alpha_{ij}(\hat{x}_j - \hat{x}_i),$$

which we call *the agent-wise decomposed diffusive coupling*[3]. A more general form of the weighting matrix[4] is given by

$$\bar{M}_i = T_i \begin{bmatrix} \bar{M}_{i\mathsf{d}} & 0 \\ 0 & \bar{M}_{i\mathsf{u}} \end{bmatrix} T_i^T.$$

[3]In literature, the term $\sum_{j\in\mathcal{N}_i} \alpha_{ij}(\hat{x}_j - \hat{x}_i)$ is called *diffusive coupling*[Wie10] or *consensus protocol*[FM04].

[4]The general form of the weighting matrix is employed in [KSC16].

Note that, by using detectable/undetectable parts decomposition given (4.2.2), $\bar{M}_i$ becomes

$$\bar{M}_i = D_i \bar{M}_{i\mathsf{d}} D_i^T + U_i \bar{M}_{i\mathsf{u}} U_i^T,$$

from which we can see that $\bar{M}_{i\mathsf{d}}$ and $\bar{M}_{i\mathsf{u}}$ are weighting factors of the effect of the diffusive coupling between the detectable part and the undetectable part. In other words, the weighting matrix is employed to controls the effect of the diffusive coupling between two decomposed parts, which are, in this case, the detectable part and the undetectable part. From this point of view, the choice of M_i in (4.3.2) is intended to limit the effect of the diffusive coupling only to the undetectable part.

The role of the weighting matrix M_i becomes clearer after decomposing the estimation error into the detectable part and the undetectable part. Let $\eta_i := x - \hat{x}_i$ for $i \in \mathcal{N}$ be the estimation error of the i-th observer. Then the error dynamics of η_i can be written as

$$\begin{aligned} \dot{\eta}_i &= (A - L_i C_i)\eta_i + \gamma_i M_i \sum_{j=1}^{N} \alpha_{ij}(\eta_j - \eta_i) \\ &= (A - L_i C_i)\eta_i - \gamma_i M_i \sum_{j=1}^{N} l_{ij}\eta_j, \end{aligned} \tag{4.3.3}$$

where α_{ij} and l_{ij} are the (i,j)-th entries of the adjacency matrix $\mathcal{A}$ of $\mathcal{G}$ and the Laplacian matrix $\mathcal{L}(\mathcal{G})$, respectively.

Let us decompose the i-th observer's estimation error η_i into the detectable part $\eta_{i\mathsf{d}}$ and the undetectable part $\eta_{i\mathsf{u}}$ such that

$$\begin{bmatrix} \eta_{i\mathsf{d}} \\ \eta_{i\mathsf{u}} \end{bmatrix} := \begin{bmatrix} D_i^T \\ U_i^T \end{bmatrix} \eta_i = T_i^T \eta_i. \tag{4.3.4}$$

Then, based on (4.2.3) and (4.3.2), the error dynamics (4.3.3) becomes

$$\begin{aligned} \dot{\eta}_{i\mathsf{d}} &= (A_{i\mathsf{d}} - L_{i\mathsf{d}} C_{i\mathsf{d}})\eta_{i\mathsf{d}} \\ \dot{\eta}_{i\mathsf{u}} &= A_{i\mathsf{r}}\eta_{i\mathsf{d}} + A_{i\mathsf{u}}\eta_{i\mathsf{u}} - \gamma_i U_i^T \sum_{j=1}^{N} l_{ij}(D_j \eta_{j\mathsf{d}} + U_j \eta_{j\mathsf{u}}). \end{aligned} \tag{4.3.5}$$

Let $\eta_{\mathsf{d}} := \mathrm{col}(\eta_{i\mathsf{d}})_{i\in\mathcal{N}}$ and $\eta_{\mathsf{u}} := \mathrm{col}(\eta_{i\mathsf{u}})_{i\in\mathcal{N}}$ be the concatenated errors of the detectable part and the undetectable part, respectively. Then the whole error dynamics is of the form

$$\begin{aligned}\dot{\eta}_{\mathsf{d}} &= (A_{\mathsf{d}} - L_{\mathsf{d}}H_{\mathsf{d}})\eta_{\mathsf{d}}\\ \dot{\eta}_{\mathsf{u}} &= A_{\mathsf{r}}\eta_{\mathsf{d}} + A_{\mathsf{u}}\eta_{\mathsf{u}} - \Gamma U^T(\mathcal{L}(\mathcal{G})\otimes I_n)(D\eta_{\mathsf{d}} + U\eta_{\mathsf{u}}),\end{aligned} \tag{4.3.6}$$

where $A_{\mathsf{d}} := \mathrm{diag}(A_{i\mathsf{d}})_{i\in\mathcal{N}}$, $L_{\mathsf{d}} := \mathrm{diag}(L_{i\mathsf{d}})_{i\in\mathcal{N}}$, $H_{\mathsf{d}} := \mathrm{diag}(C_{i\mathsf{d}})_{i\in\mathcal{N}}$, $A_{\mathsf{r}} := \mathrm{diag}(A_{i\mathsf{r}})_{i\in\mathcal{N}}$, $A_{\mathsf{u}} := \mathrm{diag}(A_{i\mathsf{u}})_{i\in\mathcal{N}}$, $D := \mathrm{diag}(D_i)_{i\in\mathcal{N}}$, $U := \mathrm{diag}(U_i)_{i\in\mathcal{N}}$ and $\Gamma := \mathrm{diag}(\gamma_i I_{\nu_i})_{i\in\mathcal{N}}$.

Note from (4.3.6) that, as a result of employing agent-wise diffusive coupling, the η_{d} dynamics is coupling-free and the term $\Gamma U^T(\mathcal{L}(\mathcal{G})\otimes I_n)U\eta_{\mathsf{u}}$ emerges in η_{u} dynamics, whose stability makes each agent's reconstruction of the full plant state possible. Since η_{d} is stable and Γ is noting but a diagonal positive matrix, the success of the distributed estimation depends solely on the stabilizability of η_{u} dynamics with the matrix $U^T(\mathcal{L}(\mathcal{G})\otimes I_n)U$, which is addressed in the following lemma.

Lemma 4.3.1. Suppose that the communication network $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ is strongly connected. Then the following statements are equivalent:

(i) the intersection of all undetectable subspaces $\cap_{i\in\mathcal{N}}\mathcal{U}_i$ equals to $\{0\}$,

(ii) the matrix $U^T(\hat{\mathcal{L}}(\mathcal{G})\otimes I_n)U$ is positive definite, and

(iii) the matrix $U^T(\mathcal{L}(\mathcal{G})\otimes I_n)U$ is nonsingular,

where $\hat{\mathcal{L}}(\mathcal{G})$ is the matrix defined in Lemma 2.2.2. ◊

Proof. (i) $\Rightarrow$ (ii). Since $\hat{\mathcal{L}}(\mathcal{G})$ is positive semidefinite by Lemma 2.2.2, it is enough to show that $(U\zeta)^T(\hat{\mathcal{L}}(\mathcal{G})\otimes I_n)(U\zeta) = 0$ where $\zeta := \mathrm{col}(\zeta_i)_{i\in\mathcal{N}}$ and $\zeta_i \in \mathbb{R}^{\nu_i\times 1}$ implies $\zeta = 0$. Note that $\hat{\mathcal{L}}(\mathcal{G})$ is the Laplacian associated with an undirected connected graph and $\ker\hat{\mathcal{L}}(\mathcal{G})$ is the subspace spanned by 1_N. Thus, it follows that $U_i\zeta_i = U_j\zeta_j$ for all i, $j\in\mathcal{N}$. Here, if $\nu_i = 0$, it should be interpreted that $U_i\zeta_i = 0_{n\times 1}$. We have $U_i\zeta_i \in \mathcal{U}_j$ for all $i, j\in\mathcal{N}$, and hence $U_i\zeta_i \in \cap_{j=1}^N\mathcal{U}_j = \{0\}$ for all $i\in\mathcal{N}$. Finally, we have $\zeta_i = 0_{\nu_i}$ for all $i\in\mathcal{N}$ because U_i has full column rank.

(ii) $\Rightarrow$ (iii). Assume to the contrary that $U^T(\mathcal{L}\otimes I_n)U$ is singular. That is, there exists a nonzero vector ζ such that $\zeta^T U^T(\mathcal{L}\otimes I_n)U = 0$. For θ_i's and Θ given in Lemma 2.2.2, ζ can be written by $\mathrm{diag}(\theta_i I_{\nu_i})_{i\in\mathcal{N}}\zeta'$ for another nonzero vector ζ', and it follows that

$$\zeta'^T \mathrm{diag}(\theta_i I_{\nu_i})_{i\in\mathcal{N}} U^T(\mathcal{L}\otimes I_n)U\zeta' = \zeta'^T U^T(\Theta\mathcal{L}\otimes I_n)U\zeta' = 0.$$

Hence, we have $\zeta'^T U^T((\Theta\mathcal{L}(\mathcal{G})+\mathcal{L}^T\Theta)\otimes I_n)U\zeta' = 0$ for some nonzero vector ζ', which implies that $U^T((\Theta\mathcal{L}(\mathcal{G})+\mathcal{L}^T\Theta)\otimes I_n)U$ is not positive definite.

(iii) $\Rightarrow$ (i). Suppose, for the sake of contradiction, that $\cap_{i\in\mathcal{N}}\mathcal{U}_i$ contains a nonzero vector v. Since $v\in\cap_{i\in\mathcal{N}}\mathcal{U}_i\subset\mathcal{U}_i = \mathrm{im}\, U_i$ and U_i has full column rank, there is a nonzero vector $\bar\zeta_i$ such that $U_i\bar\zeta_i = v$ for all $i\in\mathcal{N}$. Then the nonzero vector $\bar\zeta := \mathrm{col}(\bar\zeta_i)_{i\in\mathcal{N}}$ satisfies that $U^T(\mathcal{L}(\mathcal{G})\otimes I_n)U\bar\zeta = U^T(\mathcal{L}(\mathcal{G})\otimes I_n)(1_N\otimes v) = 0$, which implies that $U^T(\mathcal{L}(\mathcal{G})\otimes I_n)U$ is singular. □

From Lemma 4.2.2, the following is straightforward.

Lemma 4.3.2. Suppose that the communication network $\mathcal{G} = (\mathcal{N},\mathcal{E},\mathcal{A})$ is strongly connected. Then the following statements are equivalent:

(i) the pair $(A, \mathrm{col}(C_i)_{i\in\mathcal{N}})$ is detectable,

(ii) the matrix $U^T(\hat{\mathcal{L}}(\mathcal{G})\otimes I_n)U$ is positive definite, and

(iii) the matrix $U^T(\mathcal{L}(\mathcal{G})\otimes I_n)U$ is nonsingular.

◊

Note that in the proof of Lemma 4.3.1, the properties of the undetectable subspace is never used, and hence, considering $\mathcal{U}_i$ as a just subspace of $\mathbb{R}^n$ (not necessarily the undetectable subspace of the pair (A, C_i)) does not change the result. Therefore, Lemma 4.3.1 can be extended to accommodate general subspaces $\mathcal{W}_i$s instead of the $\mathcal{U}_i$s. In this context, the following lemma can be considered to provides the condition for the agent-wise decomposed diffusive coupling to achieve the goal under a different decomposition strategy.

Lemma 4.3.3. Consider a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ and a set of subspaces $\{\mathcal{W}_i \subseteq \mathbb{R}^n \,|\, i \in \mathcal{N}\}$ associated with the graph $\mathcal{G}$ such that $\mathcal{W}_i$ is the subspace of $\mathbb{R}^n$ associated with the i-th node of $\mathcal{N}$. For each $i \in \mathcal{N}$, let W_i be a matrix whose columns form a basis for $\mathcal{W}_i$, and define the matrix W as $\mathrm{diag}(W_i)_{i\in\mathcal{N}}$. If the graph $\mathcal{G}$ is strongly connected, then the following statements are equivalent:

(i) the intersection of all subspaces $\cap_{i\in\mathcal{N}}\mathcal{W}_i$ equals to $\{0\}$,

(ii) the matrix $W^T(\hat{\mathcal{L}}(\mathcal{G}) \otimes I_n)W$ is positive definite, and

(iii) the matrix $W^T(\mathcal{L}(\mathcal{G}) \otimes I_n)W$ is nonsingular.

$\Diamond$

Based on the observations so far, one can expect from (4.3.6) that the sufficiently large coupling gains stabilize the estimation error if the condition of the Lemma 4.3.2 is satisfied. In the followings we will show that this expectation is valid for the strongly connected communication network and extend the result to the general communication network, which is not necessarily strongly connected. As the first step, let us consider the case when the communication network is strongly connected.

Theorem 4.3.4. Suppose that the communication network $\mathcal{G}$ is strongly connected and the pair $(A, \mathrm{col}(C_i)_{i\in\mathcal{N}})$ is detectable. Then the proposed observer of the form (4.3.1) is a distributed observer for the system (4.1.1) and (4.1.2) over $\mathcal{G}$ with the minimum error decay rate $\beta > 0$ if L_i is chosen such that $\mathsf{a}(A_{i\mathsf{d}} - L_{i\mathsf{d}}C_{i\mathsf{d}}) < -\beta$ for each $i \in \mathcal{N}$, and γ_i is chosen such that

$$\gamma_i > \frac{2(\|A\| + \beta)}{\lambda_{\min}(U^T(\hat{\mathcal{L}}(\mathcal{G}) \otimes I_n)U)}, \quad \forall i \in \mathcal{N}. \tag{4.3.7}$$

$\Diamond$

Proof. To guarantee the decay rate $\beta > 0$ of (4.3.6), it is enough to show the stability of the modified error dynamics

$$\begin{aligned} \dot{\eta}_{\mathsf{d}} &= (A_{\mathsf{d}} - L_{\mathsf{d}}H_{\mathsf{d}})\eta_{\mathsf{d}} + \beta\eta_{\mathsf{d}} \\ \dot{\eta}_{\mathsf{u}} &= A_{\mathsf{r}}\eta_{\mathsf{d}} + A_{\mathsf{u}}\eta_{\mathsf{u}} - \Gamma U^T(\mathcal{L}(\mathcal{G}) \otimes I_n)(D\eta_{\mathsf{d}} + U\eta_{\mathsf{u}}) + \beta\eta_{\mathsf{u}}, \end{aligned} \tag{4.3.8}$$

which is obtained by adding $\beta[\eta_{\mathsf{d}}^T\ \eta_{\mathsf{u}}^T]^T$ to the original error dynamics (4.3.6). Note that the condition of L_i guarantees that the minimum decay rate of η_{d} is β. Therefore, it suffices to show that the stability of the following subsystem

$$\dot{\eta}_{\mathsf{u}} = (A_{\mathsf{u}} + \beta I_{(\sum_{i=1}^{N}\nu_i)})\eta_{\mathsf{u}} - \Gamma U^T(\mathcal{L}(\mathcal{G}) \otimes I_n)U\eta_{\mathsf{u}}. \tag{4.3.9}$$

Let $\theta := [\theta_1, \cdots, \theta_N]^T$ be the positive vector obtained by employing Lemma 2.2.2 to $\mathcal{L}(\mathcal{G})$. Consider a Lyapunov function candidate

$$V := \sum_{i=1}^{N} \theta_i \frac{\underline{\gamma}}{\gamma_i} |\eta_{i\mathsf{u}}|^2,$$

where $\underline{\gamma} = \min_{i\in\mathcal{N}} \gamma_i$. Then, the time derivative of V along (4.3.9) becomes

$$\dot{V} = \sum_{i=1}^{N} 2\theta_i \frac{\underline{\gamma}}{\gamma_i} \eta_{i\mathsf{u}}^T (A_{i\mathsf{u}} + \beta I_{\nu_i})\eta_{i\mathsf{u}} - \underline{\gamma}\eta_{\mathsf{u}}^T U^T(\hat{\mathcal{L}} \otimes I_n)U\eta_{\mathsf{u}}. \tag{4.3.10}$$

Since $0 < \theta_i \le 1$ holds from Lemma 2.2.2, it is obtained that $\theta_i \frac{\underline{\gamma}}{\gamma_i} \le 1$. Substituting these inequalities, it holds that

$$\dot{V} \le -\sum_{i=1}^{N} (\lambda_{\min}\underline{\gamma} - 2\|A_{i\mathsf{u}} + \beta I_{\nu_i}\|)|\eta_{i\mathsf{u}}|^2,$$

where $\lambda_{\min} := \lambda_{\min}(U^T(\hat{\mathcal{L}}(\mathcal{G}) \otimes I_n)U)$ is a positive constant by Lemma 4.3.2. Then the condition (4.3.7) yields the stability of (4.3.8), which completes the proof. □

In the following, it is shown that, when the communication network is strongly connected, the detectability of the pair $(A, \mathrm{col}(C_i)_{i\in\mathcal{N}})$ given in Theorem 4.3.4 is also a necessary condition to the existence of the proposed distributed observer.

Theorem 4.3.5. Suppose that the communication network $\mathcal{G}$ is strongly connected. Then there exists a distributed observer of the form (4.3.1) for the system (4.1.1) and (4.1.2) with $\mathcal{G}$ if and only if the pair $(A, \mathrm{col}(C_i)_{i\in\mathcal{N}})$ is detectable. ◊

Proof. Since the 'if' part is given in Theorem 4.3.4, we shall show that the error dynamics of the proposed observer is unstable regardless of the value of γ_i if the pair

$(A, \text{col}(C_i)_{i\in\mathcal{N}})$ is not detectable. The undetectable subspace of $(A, \text{col}(C_i)_{i\in\mathcal{N}})$, denoted as $\mathcal{U}$, is a subspace with dimension $\nu \geq 1$. Now let us construct a matrix P_i whose columns are orthonormal bases of the $\mathcal{U}_i$, the undetectable subspace of (A, C_i), such that

$$P_i := [\bar{U}_i \, U_0] \in \mathbb{R}^{n\times\nu_i}, \tag{4.3.11}$$

where $U_0 \in \mathbb{R}^{n\times\nu}$ and $\bar{U}_i \in \mathbb{R}^{n\times(\nu_i-\nu)}$ are orthonormal bases of $\mathcal{U}$ and $\mathcal{U}^\perp \cap \mathcal{U}_i$, respectively. Denoting U_i as the matrix defined in Lemma 4.2.1, the orthonormality of U_i and P_i implies that there is an orthonormal matrix $R_i \in \mathbb{R}^{\nu_i\times\nu_i}$ such that

$$P_i = U_i R_i. \tag{4.3.12}$$

Since both $\mathcal{U}_i$ and $\mathcal{U}$ are A-invariant, the linear map A restricted on $\mathcal{U}_i$ can be represented by

$$AP_i = P_i \begin{bmatrix} \bar{A}_{i\mathsf{u}} & 0 \\ \bar{A}_{i\mathsf{u}r} & \bar{A}_{\mathsf{u}0} \end{bmatrix}. \tag{4.3.13}$$

By using (4.3.13), (4.3.12), and (4.2.3), it is obtained that

$$P_i \begin{bmatrix} \bar{A}_{i\mathsf{u}} & 0 \\ \bar{A}_{i\mathsf{u}r} & \bar{A}_{\mathsf{u}0} \end{bmatrix} R_i^T = AP_iR_i^T = AU_i = U_iA_{i\mathsf{u}} = P_iR_i^T A_{i\mathsf{u}},$$

which leads to

$$R_i^T A_{i\mathsf{u}} R_i = \begin{bmatrix} \bar{A}_{i\mathsf{u}} & 0 \\ \bar{A}_{i\mathsf{u}r} & \bar{A}_{\mathsf{u}0} \end{bmatrix}. \tag{4.3.14}$$

Let us decompose $\eta_{i\mathsf{u}}$ into $\eta_{i\bar{\mathsf{u}}}$, the error in $\mathcal{U}^\perp \cap \mathcal{U}_i$, and $\eta_{i\mathsf{u}0}$, the error in $\mathcal{U}$, such that

$$\begin{bmatrix} \eta_{i\bar{\mathsf{u}}} \\ \eta_{i\mathsf{u}0} \end{bmatrix} := R_i^T \eta_{i\mathsf{u}}. \tag{4.3.15}$$

Then, the i-th agent's error dynamics (4.3.5) becomes

$$
\begin{aligned}
\dot{\eta}_{i\mathsf{d}} &= (A_{i\mathsf{d}} - L_{i\mathsf{d}}C_{i\mathsf{d}})\eta_{i\mathsf{d}} \\
\dot{\eta}_{i\bar{\mathsf{u}}} &= \bar{A}_{ir1}\eta_{i\mathsf{d}} + \bar{A}_{iu}\eta_{i\bar{\mathsf{u}}} - \gamma_i \bar{U}_i^T \sum_{j=1}^{N} l_{ij}(D_j\eta_{j\mathsf{d}} + \bar{U}_j\eta_{j\bar{\mathsf{u}}} + U_0\eta_{j\mathsf{u}0}) \\
\dot{\eta}_{i\mathsf{u}0} &= \bar{A}_{ir2}\eta_{i\mathsf{d}} + \bar{A}_{iur}\eta_{i\bar{\mathsf{u}}} + \bar{A}_{\mathsf{u}0}\eta_{i\mathsf{u}0} - \gamma_i U_0^T \sum_{j=1}^{N} l_{ij}(D_j\eta_{j\mathsf{d}} + \bar{U}_j\eta_{j\bar{\mathsf{u}}} + U_0\eta_{j\mathsf{u}0}),
\end{aligned}
\tag{4.3.16}
$$

where $[\bar{A}_{ir1}^T \, \bar{A}_{ir2}^T]^T := R_i^T A_{ir}$. Let us define the concatenated errors such that $\eta_{\mathsf{d}} := \mathrm{col}_{i\in\mathcal{N}}(\eta_{i\mathsf{d}})$, $\eta_{\bar{\mathsf{u}}} := \mathrm{col}(\eta_{i\bar{\mathsf{u}}})_{i\in\mathcal{N}}$, and $\eta_{\mathsf{u}0} := \mathrm{col}_{i\in\mathcal{N}}(\eta_{i\mathsf{u}0})$. Then the concatenated error dynamics is of the form

$$
\begin{aligned}
\dot{\eta}_{\mathsf{d}} &= (A_{\mathsf{d}} - L_{\mathsf{d}}H_{\mathsf{d}})\eta_{\mathsf{d}} \\
\dot{\eta}_{\bar{\mathsf{u}}} &= \bar{A}_{\mathsf{r}1}\eta_{\mathsf{d}} + \bar{A}_{\mathsf{u}}\eta_{\bar{\mathsf{u}}} - \Gamma_{\bar{\mathsf{u}}}\bar{U}^T(\mathcal{L}\otimes I_n)(D\eta_{\mathsf{d}} + \bar{U}\eta_{\bar{\mathsf{u}}} + (I_N\otimes U_0)\eta_{\mathsf{u}0}) \\
\dot{\eta}_{\mathsf{u}0} &= \bar{A}_{\mathsf{r}2}\eta_{\mathsf{d}} + \bar{A}_{\mathsf{ur}}\eta_{\bar{\mathsf{u}}} + (I_N\otimes \bar{A}_{\mathsf{u}0})\eta_{\mathsf{u}0} \\
&\quad - \Gamma_{\mathsf{u}0}(I_N\otimes U_0)^T(\mathcal{L}\otimes I_n)(D\eta_{\mathsf{d}} + \bar{U}\eta_{\bar{\mathsf{u}}} + (I_N\otimes U_0)\eta_{\mathsf{u}0}),
\end{aligned}
\tag{4.3.17}
$$

where $\bar{A}_{\mathsf{r}1}$, $\bar{A}_{\mathsf{r}2}$, $\bar{A}_{\mathsf{u}}$, $\bar{A}_{\mathsf{ur}}$, and $\bar{U}$ are the block diagonal matrices of $\bar{A}_{ir1}$, $\bar{A}_{ir2}$, $\bar{A}_{i\mathsf{u}}$, $\bar{A}_{i\mathsf{ur}}$, and $\bar{U}_i$, respectively, $\Gamma_{\bar{\mathsf{u}}} := \mathrm{diag}(\gamma_i I_{(\nu_i-\nu)})_{i\in\mathcal{N}}$, and $\Gamma_{\mathsf{u}0} := \mathrm{diag}(\gamma_i I_\nu)_{i\in\mathcal{N}}$. Now consider the subspace $\mathcal{V}$ defined as

$$
\mathcal{V} := \{[\eta_{\mathsf{d}}^T \, \eta_{\bar{\mathsf{u}}}^T \, \eta_{\mathsf{u}0}^T]^T \mid \eta_{\mathsf{d}} = 0, \eta_{\bar{\mathsf{u}}} = 0, \eta_{\mathsf{u}0} = 1_N\otimes v, v\in\mathbb{R}^\nu\}.
$$

Note that $\mathcal{V}$ is the invariant subspace of the error dynamics (4.3.17). Therefore, the error trajectory starting from $\mathcal{V}$ remains in $\mathcal{V}$ for all $t\geq 0$ and governed by the following dynamics

$$
\eta_{\mathsf{d}} = 0, \; \eta_{\bar{\mathsf{u}}} = 0, \; \dot{\eta}_{\mathsf{u}0} = (I_n\otimes \bar{A}_{\mathsf{u}0})\eta_{\mathsf{u}0}. \tag{4.3.18}
$$

Since $A_{i\mathsf{u}}$ is unstable, it follows from (4.3.14) that $\bar{A}_{\mathsf{u}0}$ is unstable and hence, so is (4.3.18). □

Now we present the main results of this section. Based on the previous analysis, we consider the most general case in which the communication network is given by an arbitrary directed graph. Then, it will be shown that the detectability of every iSCC is an equivalent condition to the existence of the proposed distributed observer as the detectability is equivalent condition to the existence of the conventional Luenberger observer for general linear systems.

Theorem 4.3.6. Let the communication network $\mathcal{G}$ be an arbitrary directed graph with c distinct iSCCs $\bar{\mathcal{G}}_k = (\bar{\mathcal{N}}_k, \bar{\mathcal{E}}_k)$ for $k = 1, \cdots, c$. Then there exists a distributed observer of the form (4.3.1) for the system (4.1.1) and (4.1.2) with $\mathcal{G}$ if and only if every pair $(A, \mathrm{col}(C_i)_{i\in\bar{\mathcal{N}}_k})$ is detectable for $k = 1, \cdots, c$. Moreover, its estimation error decays with the minimum rate of $\beta > 0$ if L_i is chosen such that $\mathsf{a}(A_{i\mathsf{d}} - L_{i\mathsf{d}}C_{i\mathsf{d}}) < -\beta$ for each $i \in \mathcal{N}$, and γ_i is chosen such that

$$\gamma_i > \frac{2(\|A\| + \beta)}{\lambda_{\min}(U_{\bar{\mathcal{N}}_k}^T(\hat{\mathcal{L}}(\bar{\mathcal{G}}_k) \otimes I_n)U_{\bar{\mathcal{N}}_k})}, \quad \text{if } i \in \bar{\mathcal{N}}_k,\ k = 1, \cdots, c \tag{4.3.19}$$

$$\gamma_i > \frac{2(\|A\| + \beta)}{\lambda_{\min}(U_{\bar{\mathcal{N}}_0}^T(\hat{\mathcal{M}}_0 \otimes I_n)U_{\bar{\mathcal{N}}_0})}, \quad \text{if } i \in \bar{\mathcal{N}}_0, \tag{4.3.20}$$

where $\bar{\mathcal{N}}_0 := \mathcal{N} \setminus \cup_{k=1}^c \bar{\mathcal{N}}_k$, $U_{\bar{\mathcal{N}}_k} := \mathrm{diag}(U_i)_{i\in\bar{\mathcal{N}}_k}$ for $k = 0, 1, \cdots, c$, and $\hat{\mathcal{M}}_0$ is the positive definite matrix given in (2.3.1) with $\mathcal{M}_0$ given in (2.3.2). $\diamondsuit$

Proof. Without loss of generality, let us assume that the node index i is re-labeled as same as the one given in Lemma 2.3.2 so that the Laplacian matrix $\mathcal{L}(\mathcal{G})$ has the form given in (2.3.2). For each $k = 0, 1, \cdots, c$, let us denote $\eta_{\bar{\mathcal{N}}_k\mathsf{d}} := \mathrm{col}(\eta_{i\mathsf{d}})_{i\in\bar{\mathcal{N}}_k}$ and $\eta_{\bar{\mathcal{N}}_k\mathsf{u}} := \mathrm{col}(\eta_{i\mathsf{u}})_{i\in\bar{\mathcal{N}}_k}$ be the concatenations of the detectable part and undetectable part of error state of nodes contained in $\bar{\mathcal{N}}_k$, respectively. Under this partition, it follows from (2.3.2) and (4.3.6) that the error dynamics of iSCC $\bar{\mathcal{G}}_k$ is of the form

$$\begin{aligned}
\dot{\eta}_{\bar{\mathcal{N}}_k\mathsf{d}} &= (A_{\bar{\mathcal{N}}_k\mathsf{d}} - L_{\bar{\mathcal{N}}_k\mathsf{d}}H_{\bar{\mathcal{N}}_k\mathsf{d}})\eta_{\bar{\mathcal{N}}_k\mathsf{d}} \\
\dot{\eta}_{\bar{\mathcal{N}}_k\mathsf{u}} &= A_{\bar{\mathcal{N}}_k\mathsf{r}}\eta_{\bar{\mathcal{N}}_k\mathsf{d}} + A_{\bar{\mathcal{N}}_k\mathsf{u}}\eta_{\bar{\mathcal{N}}_k\mathsf{u}} - \Gamma_{\bar{\mathcal{N}}_k}U_{\bar{\mathcal{N}}_k}^T(\mathcal{L}(\bar{\mathcal{G}}_k) \otimes I_n)(D_{\bar{\mathcal{N}}_k}\eta_{\bar{\mathcal{N}}_k\mathsf{d}} + U_{\bar{\mathcal{N}}_k}\eta_{\bar{\mathcal{N}}_k\mathsf{u}}),
\end{aligned} \tag{4.3.21}$$

where $A_{\bar{\mathcal{N}}_k\mathsf{d}}$, $L_{\bar{\mathcal{N}}_k\mathsf{d}}$, $H_{\bar{\mathcal{N}}_k\mathsf{d}}$, $A_{\bar{\mathcal{N}}_k\mathsf{r}}$, $A_{\bar{\mathcal{N}}_k\mathsf{u}}$, $\Gamma_{\bar{\mathcal{N}}_k}$, $U_{\bar{\mathcal{N}}_k}$, and $D_{\bar{\mathcal{N}}_k}$ are the block diagonal matrices of $A_{i\mathsf{d}}$, $L_{i\mathsf{d}}$, $C_{i\mathsf{d}}$, $A_{i\mathsf{r}}$, $A_{i\mathsf{u}}$, $\gamma_i I_{\nu_i}$, U_i, and D_i over $i \in \bar{\mathcal{N}}_k$, respectively. On the other hand, the error dynamics on $\bar{\mathcal{N}}_0$ is

$$\begin{aligned}\dot{\eta}_{\bar{\mathcal{N}}_0\mathsf{d}} &= (A_{\bar{\mathcal{N}}_0\mathsf{d}} - L_{\bar{\mathcal{N}}_0\mathsf{d}}H_{\bar{\mathcal{N}}_0\mathsf{d}})\eta_{\bar{\mathcal{N}}_0\mathsf{d}} \\ \dot{\eta}_{\bar{\mathcal{N}}_0\mathsf{u}} &= A_{\bar{\mathcal{N}}_0\mathsf{r}}\eta_{\bar{\mathcal{N}}_0\mathsf{d}} + A_{\bar{\mathcal{N}}_0\mathsf{u}}\eta_{\bar{\mathcal{N}}_0\mathsf{u}} - \Gamma_{\bar{\mathcal{N}}_0}U_{\bar{\mathcal{N}}_0}^T(\mathcal{M}_0 \otimes I_n)(D_{\bar{\mathcal{N}}_0}\eta_{\bar{\mathcal{N}}_0\mathsf{d}+U_{\bar{\mathcal{N}}_0}\eta_{\bar{\mathcal{N}}_0\mathsf{u}}}) \\ &\quad - \sum_{k=1}^{c}\Gamma_{\bar{\mathcal{N}}_0}U_{\bar{\mathcal{N}}_0}^T(\mathcal{M}_{\mathsf{r}k} \otimes I_n)(D_{\bar{\mathcal{N}}_k}\eta_{\bar{\mathcal{N}}_k\mathsf{u}} + U_{\bar{\mathcal{N}}_k}\eta_{\bar{\mathcal{N}}_k\mathsf{d}}),\end{aligned} \tag{4.3.22}$$

where $A_{\bar{\mathcal{N}}_0\mathsf{d}}$, $L_{\bar{\mathcal{N}}_0\mathsf{d}}$, $H_{\bar{\mathcal{N}}_0\mathsf{d}}$, $A_{\bar{\mathcal{N}}_0\mathsf{r}}$, $A_{\bar{\mathcal{N}}_0\mathsf{u}}$, $\Gamma_{\bar{\mathcal{N}}_0}$, $U_{\bar{\mathcal{N}}_0}$, and $D_{\bar{\mathcal{N}}_0}$ are the block diagonal matrices of $A_{i\mathsf{d}}$, $L_{i\mathsf{d}}$, $C_{i\mathsf{d}}$, $A_{i\mathsf{r}}$, $A_{i\mathsf{u}}$, $\gamma_i I_{\nu_i}$, U_i, and D_i over $i \in \bar{\mathcal{N}}_0$, respectively.

Now let us show the 'if part' of the proof. Note that the equation (4.3.21) is of the form (4.3.6) with the graph $\bar{\mathcal{G}}_k$ and the output matrix $\mathrm{col}(C_i)_{i\in\bar{\mathcal{N}}_k}$. Since the strong connectivity of iSCC $\bar{\mathcal{G}}_k$ and the detectability of $(A, \mathrm{col}(C_i)_{i\in\bar{\mathcal{N}}_k})$ holds, it follows from Theorem 4.3.4 that the condition (4.3.19) implies that $\eta_{\bar{\mathcal{N}}_k\mathsf{d}}$ and $\eta_{\bar{\mathcal{N}}_k\mathsf{u}}$ converge to zero exponentially with the minimum rate of β, for $k = 1, \cdots, c$. To show that $\eta_{\bar{\mathcal{N}}_0\mathsf{d}}$ and $\eta_{\bar{\mathcal{N}}_0\mathsf{u}}$ decay with the minimum rate β, similarly to the proof of Theorem 4.3.4, we will consider the modified error dynamics obtained by adding $\beta\eta_{\bar{\mathcal{N}}_0\mathsf{d}}$ and $\beta\eta_{\bar{\mathcal{N}}_0\mathsf{u}}$ to the first and the second equations of (4.3.22), respectively. Note that the condition of L_i guarantees that the minimum decay rate of $\eta_{\bar{\mathcal{N}}_0\mathsf{d}}$ is β. Combined with the fact that $\eta_{\bar{\mathcal{N}}_k\mathsf{d}}$ and $\eta_{\bar{\mathcal{N}}_k\mathsf{u}}$ also converge to zero exponentially, the minimum decay rate of $\eta_{\bar{\mathcal{N}}_0\mathsf{u}}$ is β if the following subsystem is stable:

$$\dot{\eta}_{\bar{\mathcal{N}}_0\mathsf{u}} = (A_{\bar{\mathcal{N}}_0\mathsf{u}} + \beta I_{\sum_{i\in\bar{\mathcal{N}}_0}} - \Gamma_{\bar{\mathcal{N}}_0}U_{\bar{\mathcal{N}}_0}^T(\mathcal{M}_0 \otimes I_n)U_{\bar{\mathcal{N}}_0})\eta_{\bar{\mathcal{N}}_0\mathsf{u}}. \tag{4.3.23}$$

Let us define $\bar{N}_0 := |\mathcal{N}_0|$ and let $[g_1, g_2, \cdots, g_{\bar{N}_0}]^T$ be the positive vector obtained by employing Lemma 2.3.1 to $\mathcal{M}_0$. Consider a Lyapunov function candidate $V_0 := \sum_{i=1}^{\bar{N}_0} g_i \frac{\underline{\gamma}_0}{\gamma_i}|\eta_{i\mathsf{u}}|^2$, where $\underline{\gamma}_0 := \min_{i\in\bar{\mathcal{N}}_0}\gamma_i$. Then, the time derivative of V_0 along (4.3.23) becomes

$$\dot{V}_0 = \sum_{i\in\bar{\mathcal{N}}_0} 2g_i\frac{\underline{\gamma}_0}{\gamma_i}\eta_{i\mathsf{u}}^T(A_{i\mathsf{u}} + \beta I_{\nu_i})\eta_{i\mathsf{u}} - \underline{\gamma}_0\eta_{\mathsf{u}}^T U_{\bar{\mathcal{N}}_0}^T(\hat{\mathcal{M}}_0 \otimes I_n)U_{\bar{\mathcal{N}}_0}\eta_{\mathsf{u}}. \tag{4.3.24}$$

Since $0 < g_i \leq 1$ holds from Lemma 2.3.1, it is obtained that $g_i\frac{\underline{\gamma}_0}{\gamma_i} \leq 1$. Substituting these inequalities to (4.3.24), it holds that

$$\dot{V}_0 \leq -\sum_{i\in\bar{\mathcal{N}}_0}(\lambda_{\min}\underline{\gamma}_0 - 2\|A_{i\mathsf{u}} + \beta I_{\nu_i}\|)|\eta_{i\mathsf{u}}|^2,$$

where $\lambda_{\min} := \lambda_{\min}(U_{\bar{\mathcal{N}}_0}^T(\hat{\mathcal{M}}_0 \otimes I_n)U_{\bar{\mathcal{N}}_0})$. Here $\lambda_{\min}$ is a positive constant because $\hat{\mathcal{M}}_0$ is positive definite by Lemma 2.3.1 and $U_{\bar{\mathcal{N}}_0}$ has full column rank. Finally the condition (4.3.20) yields the negative definiteness of $\dot{V}_0$, which implies that the decay rate of $\eta_{\bar{\mathcal{N}}_0}$ is larger than or equal to β.

Now let us show the 'only if' part of the proof by assuming that one pair $(A, H_{\bar{\mathcal{G}}_l})$ associated with the l-th iSCC $\bar{\mathcal{G}}_l$ is undetectable, where $1 \leq l \leq c$. Since the $\eta_{\bar{\mathcal{G}}_l}$ dynamics over $\mathcal{G}$ is equal to those over the iSCC $\bar{\mathcal{G}}_l$, which is strongly connected, it follows from Theorem 4.3.5 that there is some initial value that drives the error $\eta_{\bar{\mathcal{G}}_l}$ goes to infinity regardless of the values of γ_is. This completes the proof. □

Remark 4.3.1. The threshold for the coupling gains given in (4.3.19) and (4.3.20) can be computed even tighter by replacing the term $2\|A\|$ with $\max_{i\in\mathcal{N}}\|A_{i\mathsf{u}} + A_{i\mathsf{u}}^T\|$. This can be easily checked from the equation (4.3.10) in the proof of Theorem 4.3.4 and the equation (4.3.24) in the proof of Theorem 4.3.6. Notice that $\|A_{i\mathsf{u}} + A_{i\mathsf{u}}^T\| = 0$, if $A_{i\mathsf{u}}$ is a skew-symmetric matrix. Consequently, if $A_{i\mathsf{u}}$ can be chosen as a skew symmetric matrix, then the threshold of the coupling gains becomes zero under no assigned minimum decay rate, i.e., $\beta = 0$, and it eventually means that the convergence of the estimates can be achieved by putting any positive values to γ_is. As a simple case, if the modes of the plant are all stable except a single harmonic mode, then $A_{i\mathsf{u}}$ can be represented by

$$A_{i\mathsf{u}} = \begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix}$$

for some positive constant w, which is clearly skew-symmetric. Therefore, the positive coupling gain yields the convergence of the estimation. ◊

Remark 4.3.2. Note that Theorem 4.3.6 does not imply that the decay rate β is freely assignable. That is, if the pair $(A, \mathrm{col}(C_i)_{i\in\mathcal{N}})$ is detectable but is

Construction procedure of the local observer for the i-th agent	
STEP 1.	Compute the undetectable subspace $\mathcal{U}_i$ using the definition (4.2.1).
STEP 2.	Compute the matrices U_i, D_i, and the transformation matrix T_i as defined in Lemma 4.2.1
STEP 3.	Compute the matrices $A_{i\mathsf{d}}$ and $C_{i\mathsf{d}}$ from (4.2.3).
STEP 4.	Choose the partial injection gain $L_{i\mathsf{d}}$ so that $\mathsf{a}(A_{i\mathsf{d}} - L_{i\mathsf{d}}C_{i\mathsf{d}}) < \beta$.
STEP 5.	Set L_i and M_i defined in (4.3.2).
STEP 6.	Choose the coupling gain γ_i sufficiently large so that the conditions (4.3.19) and (4.3.20) hold.

Table 4.1: The construction procedure for the i-th local observer.

not observable, then $\mathsf{a}(A_{i\mathsf{d}} - L_{i\mathsf{d}}C_{i\mathsf{d}})$ has a limit regardless of the choice of $L_{i\mathsf{d}}$, which leads to the limit of the error decay rate β. Furthermore, when the pair $(A, \mathrm{col}(C_i)_{i\in\mathcal{N}})$ is observable but the U_i is chosen as the matrix whose columns form an orthonormal basis of undetectable subspace $\mathcal{U}_i$, then the decay rate still has a limit. However, if the pair $(A, \mathrm{col}(C_i)_{i\in\mathcal{N}})$ is observable and the columns of U_i are chosen as an orthonormal basis of the i-th agent's unobservable subspace, not the undetectable subspace, then Theorem 4.3.4 still holds and the decay rate β can be assigned as large as desired. This can be easily checked by noting that Lemma 4.3.3 holds with $\mathcal{W}_i$ being the unobservable subspaces of the pair (A, C_i) and by replacing U_is in the proof of Theorem 4.3.6. $\diamondsuit$

The condition of the coupling gain given in (4.3.19) and (4.3.20) indicates the advantages of the proposed distributed observer in the perspective of implementation. First, as long as it is above the threshold, the coupling gain of each agent may have different value with others. This relaxes the restriction of the distributed observers in [KSC16], where the coupling gains of all agent have a common value. Second, the value of the coupling gain need not be set precisely, which also shows the proposed observer is robust to the computational errors of coupling gains.

The construction procedure of the proposed distributed observer (4.3.1) is summarized in Table 4.1. We conclude this part by illustrating the condition

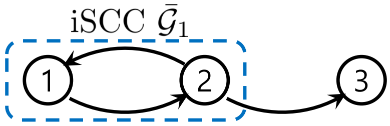


Figure 4.4: The communication graph $\mathcal{G}'$ of Example 4.3.1 is depicted.

given in Theorem 4.3.6 for the following example.

Example 4.3.1. Consider the Example 4.0.1 again with a different communication topology. In this case, the sensor 1 and 3 can receive information from the sensor 2, but, due to the limited transmission range of the sensor 3, the sensor 2 can not receive information from the sensor 3. Then the communication graph can be modeled as the graph $\mathcal{G}'$ given in Fig. 4.4 and its unweighted adjacency matrix $\mathcal{A}'$ and the Laplacian matrix $\mathcal{L}$ are given as follows:

$$\mathcal{A}' = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathcal{L}(\mathcal{G}') = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Note that $\mathcal{G}'$ is not strongly connected. It has one iSCC $\bar{\mathcal{G}}_1 = (\bar{\mathcal{N}}_1, \mathcal{E}_1)$, where $\bar{\mathcal{N}}_1 = \{1, 2\}$ and $\bar{\mathcal{E}}_1 = \{(1,2), (2,1)\}$. Here, the node set $\bar{\mathcal{N}}_0$ defined in Lemma 2.3.2 is given by $\{3\}$. One can also check from the shape of $\mathcal{L}(\mathcal{G}')$ that Lemma 2.3.2 holds with

$$\mathcal{L}(\bar{\mathcal{G}}_1) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \mathcal{M}_0 = 1, \qquad \mathcal{M}_{r1} = \begin{bmatrix} 0 & -1 \end{bmatrix}.$$

Although the communication graph $\mathcal{G}'$ is not strongly connected, the output matrix $\mathrm{col}(C_1, C_2)$ concatenated over the iSCC $\bar{\mathcal{G}}_1$ yields detectable pair with the plant matrix A, so condition given in Theorem 4.3.6 is satisfied.

The simulation results of this example are given in Fig. 4.5. Here, the plant parameter is chosen as $k/J := 1$ and the coupling gains are chosen as $\gamma_1 = \gamma_2 = \gamma_3 = 3$. It shows that every sensor's estimate of the proposed observer converges to the plant state even though the communication graph $\mathcal{G}'$ is not strongly connected.

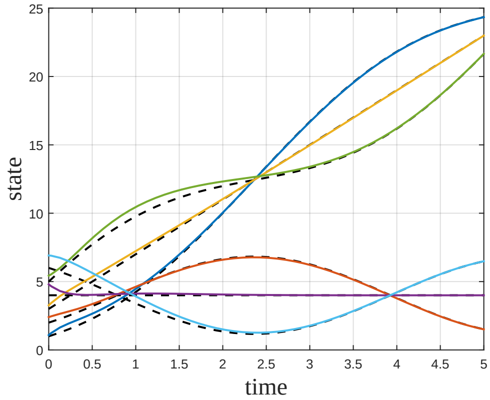
4.4 Summary and Discussion

In this chapter, we addressed the distributed estimation problem, of which the goal is enabling every agent, or sensor node, to reconstruct the plant state by using its own measurements and communicating with the nearby communications through the given topology. As a solution we proposed our distributed observer design scheme; the main idea is to let each agent recover the self-reconstructible portion by itself using its own measurement and accept only insufficient information by the projected diffusive coupling of the neighbors' estimates via the specially designed communication protocol called agent-wise decomposed diffusive coupling. The distributed observer obtained by applying this scheme to autonomous linear time invariant systems was proposed in (4.3.1). It was revealed in Theorem 4.3.6 that the proposed observer can be designed over the general communication topology and its existence condition is nothing but the detectability of the system concatenated over the source components. The limitation of that the assignable converge rate was also analyzed even for the system which is detectable but not observable, so as to claim the proposed scheme's flexibility in performance over the general communication topology. Moreover, we provided the construction procedure in Table 4.1 and it turned out that the construction process is reduced into assigning a sufficiently large value on coupling (scalar) gains, thanks to the proposed structure.

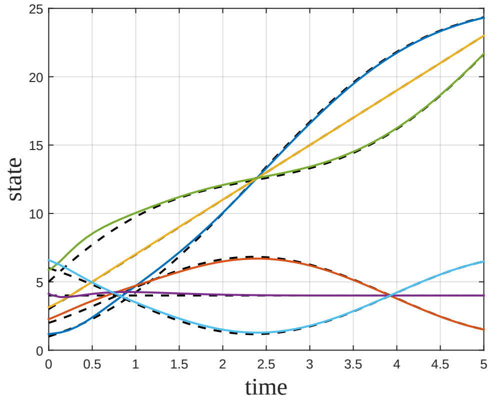
However, to construct the proposed distributed observer, it requires the global information to compute the threshold of the coupling gain. In specific, to compute the value of the denominators of the thresholds in (4.3.7), (4.3.19), and (4.3.20), it is necessary to know the Laplacian matrix and the output matrices of all agents, which is impractical in general.

In some cases, this disadvantage can be avoided if the lower bound of the threshold over all the plausible cases is available. This approach is effective because only the lower threshold is given as a condition to satisfy. The relative sensing network is a very good example because it has a limited number of output matrices a single agent can have (see, for example, [ZM11], [PSF$^+$13], and the references therein). When the upper bound of the number of agents is known, each agent can compute the minimum threshold over all possible combinations of the output matrices and the communication topology. Since there is a way to count the number of agents in the network in a distributed manner [LLKS18], it can be concluded that the distributed construction is possible for the relative sensing network using the proposed scheme.

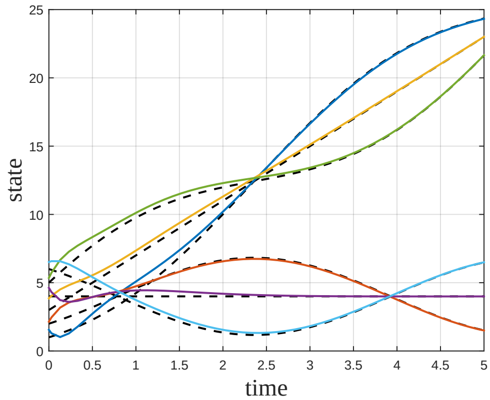
In the next chapter, we will provide a more general method to overcome these limitations so that each local observer can be implemented by the associated agent itself.



(a) Estimate from sensor 1



(b) Estimate from sensor 2



(c) Estimate from sensor 3

Figure 4.5: Simulation results of Example 4.3.1: The plant state $x(t)$ are depicted as black dashed curve and the estimate $\hat{x}_i$ generated by each i-th observer are depicted for $i = 1, 2, 3$ as colored solid curves.

Chapter 5

Adaptive Distributed State Observer for Completely Decentralized Construction

In general, an algorithm that achieves their goal by running in a distributed manner is called a distributed algorithm. Their construction, however, is not distributed but centralized in many cases. Unfortunately, distributed estimation is no exception as revealed in the introduction. The observer construction procedures of [PM17],[ZLL$^+$14],[WM18], and [MS18], for example, involve all agent's subsystems so that the parameters should be computed for the large system which is an aggregation of each agent's local observer. Or the implementation of the algorithm running in a local agent requires global information that is hardly accessible by itself such as the algebraic connectivity of the Laplacian matrix[1] [ZLL$^+$14], the number of agents [KSW16].

For distribute estimation to be effective, it's construction should be done in a decentralized fashion; the implementation of the algorithm running in local agent can be done by the agent itself with the locally accessible information. With the decentralized construction, the algorithm can be implemented independently and practically so that it becomes more robust to the change of environments such as the change of network topologies, or the change of the number of agents due to leaving or joining of agents in the network.

[1]The algebraic connectivity of the Laplacian matrix $\mathcal{L}$ is defined as $\lambda_2(\mathcal{L})$, the second smallest eigenvalue. See Lemma 2.2.1.

Thanks to the agent-wise decomposed diffusive coupling, most of the parameters of the observer proposed in (4.3.1) can be computed by the i-th agent itself, but the coupling gain cannot because its threshold depends on global information such as the value of the Laplacian matrix or the collection of all the undetectable subspaces. In this chapter, we improve the agent-wise decomposed diffusive coupling by employing the adaptive coupling gain so as to achieve the completely decentralized construction of the distributed observer.

5.1 Problem Statement

Consider the plant system (4.1.1) and suppose that a network of N sensors with measurements (4.1.2) and the communication graph $\mathcal{G}$ are given. The previous section's objective is to design a distributed observer such that the state estimate $\hat{x}_i(t)$ of each i-th local observer converges to the plant state $x(t)$, i.e.,

$$\lim_{t\to\infty} |\hat{x}_i(t) - x(t)| = 0, \quad \forall i \in \mathcal{N},$$

under the following constraints:

i) *local measurement:* each i-th local observer utilizes its local measurement y_i only, and

ii) *local communication:* each i-th local observer communicates with its neighboring agents.

Now it is of interest to find a distributed observer that is *completely decentralized constructible*; more specifically, a distributed observer with the additional constraints:

iii) *local construction:* the parameters of the i-th local observer are computed using the local knowledge accessible by the i-th agent, i.e., the plant's system matrix A and the local measurement matrix C_i only.

5.2 Distributed Observer with Adaptive Agent-wise Decomposed Diffusive Coupling

As a consequence of employing the agent-wise decomposed diffusive coupling, according to the construction procedure given in Table 4.1, all parameters of the i-th local observer given in (4.3.1) can be computed by the i-th agent itself except the scalar coupling gain γ_i. Moreover, it has been shown in Theorem 4.3.6 that the sufficiently large coupling gain γ_i yields the successful estimation. Motivated by these facts, we devise an *adaptive agent-wise decomposed diffusive coupling* to yield the completely decentralized construction of distributed observers.

In this part, inspired by the adaptive consensus control technique in [LLDF16], we aim to design an adaptive distributed observer for the system given by (4.1.1) and (4.1.2) over general directed networks under the equivalent condition derived in Theorem 4.3.6.

The adaptive observer of the i-th agent has the form of

$$\dot{\hat{x}}_i = A\hat{x}_i + L_i(y_i - C_i\hat{x}_i) + (\gamma_i + \phi_i)M_i \sum_{j\in\mathcal{N}_i} \alpha_{ij}(\hat{x}_j - \hat{x}_i), \tag{5.2.1}$$

which is the same as (4.3.1) except the fact that γ_i is time-varying and ϕ_i is added to the coupling gain. Here, ϕ_i and the time-varying gain γ_i have the form of:

$$\begin{aligned} \phi_i &= |U_i^T \sum_{j\in\mathcal{N}_i} \alpha_{ij}(\hat{x}_j - \hat{x}_i)|^2, \\ \dot{\gamma}_i &= \phi_i, \quad \gamma_i(0) > 0. \end{aligned} \tag{5.2.2}$$

Remark 5.2.1. It should be noticed that the construction of the i-th adaptive observer (5.2.1) with (5.2.2) requires items 1 to 5 in the Table 4.1 only. Hence, no information other than the knowledge of the pair (A, C_i) is necessary, so the proposed adaptive observer is indeed a distributed algorithm whose design is completely decentralized as asserted in Theorem 5.2.2 below. $\Diamond$

Similar to Lemma 4.3.2, with $\mathcal{M}_0$ and $\bar{\mathcal{N}}_0$ defined in Lemma 2.3.2, it is shown that $U_{\bar{\mathcal{N}}_0}^T(\mathcal{M}_0 \otimes I_n)U_{\bar{\mathcal{N}}_0}$ where $U_{\bar{\mathcal{N}}_0} := \mathrm{diag}(U_i)_{i\in\bar{\mathcal{N}}_0}$ is also nonsingular, which is used in the proof of the theorem below.

Lemma 5.2.1. Let $\mathcal{M}_0$ be the submatrix of the Laplacian matrix $\mathcal{L}(\mathcal{G})$ obtained by selecting rows and columns corresponding to the nodes contained in the set $\bar{\mathcal{N}}_0$ defined in Lemma 2.3.2, and let us denote $U_{\bar{\mathcal{N}}_0}$ as $\text{diag}(U_i)_{i\in\bar{\mathcal{N}}_0}$. Then the matrix $U_{\bar{\mathcal{N}}_0}^T(\mathcal{M}_0 \otimes I_n)U_{\bar{\mathcal{N}}_0}$ is nonsingular. $\diamond$

Proof. Let us show that $U_{\bar{\mathcal{N}}_0}^T(\mathcal{M}_0 \otimes I_n)U_{\bar{\mathcal{N}}_0}v = 0$ leads to $v = 0$. Note that

$$\begin{aligned} 0 &= v^T \text{diag}(g_i I_{\nu_i})_{i\in\bar{\mathcal{N}}_0} U_{\bar{\mathcal{N}}_0}^T(\mathcal{M}_0 \otimes I_n)U_{\bar{\mathcal{N}}_0}v \\ &= v^T U_{\bar{\mathcal{N}}_0}^T(G\mathcal{M}_0 \otimes I_n)U_{\bar{\mathcal{N}}_0}v. \end{aligned}$$

By adding the above equation and its transpose, we have $v^T U_{\bar{\mathcal{N}}_0}^T(\hat{\mathcal{M}}_0 \otimes I_n)U_{\bar{\mathcal{N}}_0}v = 0$, which leads to $v = 0$ because $U_{\bar{\mathcal{N}}_0}^T(\hat{\mathcal{M}}_0 \otimes I_n)U_{\bar{\mathcal{N}}_0}$ is positive definite by Lemma 2.3.1. $\square$

The main result of this part is given as follows.

Theorem 5.2.2. Let the communication network $\mathcal{G}$ be an arbitrary directed graph with c distinct iSCCs $\bar{\mathcal{G}}_k = (\bar{\mathcal{N}}_k, \bar{\mathcal{E}}_k)$ for $k = 1, \cdots, c$. If every pair $(A, \text{col}(C_i)_{i\in\bar{\mathcal{N}}_k})$ is detectable for $k = 1, \cdots, c$, then the proposed observer of the form (5.2.1) with the adaptive law (5.2.2) is a distributed observer for the system (4.1.1) and (4.1.2) with $\mathcal{G}$. Moreover, each time-varying gain γ_i converges to a finite value; more specifically, for $t \geq 0$, it satisfies

$$\begin{aligned} \gamma_i(0) \leq \gamma_i(t) &\leq \rho_1 \\ &+ \rho_2\Big[\sum_i (|U_i^T(x(0) - \hat{x}_i(0))|^2 + |\phi_i(0) + \gamma_i(0)|^2 + |\gamma_i(0) - \rho_1|^2)\Big]^{\frac{1}{2}} \end{aligned} \tag{5.2.3}$$

where ρ_1 and ρ_2 are constants defined by

$$\begin{aligned} \rho_1 &:= 2\frac{6\kappa(R)^2 \max_i \|A - L_i C_i\|^2 + \underline{\lambda} + 1}{\underline{\lambda}\min\{\underline{\theta}, \underline{g}\}} \\ \rho_2 &:= \left[(2 + 18\|\mathcal{L}(\mathcal{G})\|^2/\underline{\lambda}^2 + 3/\underline{\lambda})(\max_i \lambda_{\max}(P_{\mathsf{id}})\rho_1 + \underline{\lambda})\right]^{\frac{1}{2}} \end{aligned} \tag{5.2.4}$$

with

$$R := \begin{bmatrix} I & 0 \\ U^\top(\mathcal{L}\otimes I_n)D & U^\top(\mathcal{L}\otimes I_n)U \end{bmatrix},$$

and, for each $i \in \mathcal{N}$, $P_{i\mathsf{d}}$ being the positive definite matrix such that

$$(A_{i\mathsf{d}} - L_{i\mathsf{d}}C_{i\mathsf{d}})^\top P_{i\mathsf{d}} + P_{i\mathsf{d}}(A_{i\mathsf{d}} - L_{i\mathsf{d}}C_{i\mathsf{d}}) = -I.$$

Here, $\underline{g}$ is the minimum of the diagonal entries of the diagonal matrix G obtained by applying Lemma 2.3.1 to $\mathcal{M}_0$, $\underline{\theta}$ is the minimum of the diagonal entries of the diagonal matrices $\Theta_1, \Theta_2, \cdots, \Theta_c$ obtained by applying Lemma 2.2.2 to the iSCC's Laplacian matrices $\mathcal{L}_1, \cdots, \mathcal{L}_c$, and $\underline{\lambda}$ is the minimum among the eigenvalues of the positive definite matrices $\hat{\mathcal{M}}_0$ and $U_{\bar{\mathcal{N}}_k}^\top(\hat{\mathcal{L}}_k \otimes I_n)U_{\bar{\mathcal{N}}_k}$ for $k = 1, 2, \cdots, c$. $\diamondsuit$

Proof. Without loss of generality, let us assume that the node index is labeled as the one given in Theorem 4.3.6, so that the Laplacian matrix $\mathcal{L}(\mathcal{G})$ is of the form given in (2.3.2), which can be simply rewritten as

$$\mathcal{L}(\mathcal{G}) = \begin{bmatrix} \mathcal{M}_0 & \mathcal{M}_{\mathsf{r}} \\ 0 & \mathcal{L}_+ \end{bmatrix}, \tag{5.2.5}$$

where $\mathcal{M}_{\mathsf{r}} := [\mathcal{M}_1\, \mathcal{M}_2 \cdots \mathcal{M}_c]$ and $\mathcal{L}_+ := \mathrm{diag}(\mathcal{L}(\bar{\mathcal{G}}_k))_{1\le k\le c}$. For each agent $i \in \mathcal{N}$, let us define $\eta_i := x - \hat{x}_i$ as the estimate error and denote $\eta_{i\mathsf{d}}$ and $\eta_{i\mathsf{u}}$ as the detectable part and undetectable part of η_i defined in (4.3.4). Let $\eta_{0\mathsf{d}}$ and $\eta_{0\mathsf{u}}$ be the concatenated errors of $\eta_{i\mathsf{d}}$ and $\eta_{i\mathsf{u}}$ over $i \in \bar{\mathcal{N}}_0$, respectively, and $\eta_{+\mathsf{d}}$ and $\eta_{+\mathsf{u}}$ be the concatenated errors of $\eta_{i\mathsf{d}}$ and $\eta_{i\mathsf{u}}$ over $i \in \mathcal{N} \setminus \bar{\mathcal{N}}_0 = \cup_{k=1}^c \bar{\mathcal{N}}_k$, respectively. Then it can be easily obtained from (5.2.1) that the error dynamics of $\eta_{+\mathsf{d}}$ and $\eta_{+\mathsf{u}}$ is of the form

$$\begin{aligned} \dot{\eta}_{+\mathsf{d}} &= (A_{+\mathsf{d}} - L_{+\mathsf{d}}C_{+\mathsf{d}})\eta_{+\mathsf{d}} \\ \dot{\eta}_{+\mathsf{u}} &= A_{+\mathsf{r}}\eta_{+\mathsf{d}} + A_{+\mathsf{u}}\eta_{+\mathsf{u}} - (\Gamma_+ + \Phi_+)U_+^T(\mathcal{L}_+ \otimes I_n)(D_+\eta_{+\mathsf{d}} + U_+\eta_{+\mathsf{u}}), \end{aligned} \tag{5.2.6}$$

where $A_{+\mathsf{d}}$, $L_{+\mathsf{d}}$, $C_{+\mathsf{d}}$, $A_{+\mathsf{r}}$, $A_{+\mathsf{u}}$, D_+, U_+, and Γ_+ are the block diagonal matrices of $A_{\bar{\mathcal{N}}_k\mathsf{d}}$, $L_{\bar{\mathcal{N}}_k\mathsf{d}}$, $C_{\bar{\mathcal{N}}_k\mathsf{d}}$, $A_{\bar{\mathcal{N}}_k\mathsf{r}}$, $A_{\bar{\mathcal{N}}_k\mathsf{u}}$, $D_{\bar{\mathcal{N}}_k}$, $U_{\bar{\mathcal{N}}_k}$, and $\Gamma_{\bar{\mathcal{N}}_k}$ for $k = 1, 2, \cdots, c$, respectively, and Φ_+ is the block diagonal matrix of $\phi_i I_{\nu_i}$ for all $i \in \mathcal{N}$ in the same order as in $\eta_{+\mathsf{u}}$. On the other hand, the errors of the agents in $\bar{\mathcal{N}}_0$ are governed by

$$\begin{aligned}\dot{\eta}_{0\mathsf{d}} &= (A_{0\mathsf{d}} - L_{0\mathsf{d}} C_{0\mathsf{d}})\eta_{0\mathsf{d}} \\ \dot{\eta}_{0\mathsf{u}} &= A_{0\mathsf{r}}\eta_{0\mathsf{d}} + A_{0\mathsf{u}}\eta_{0\mathsf{u}} - (\Gamma_0 + \Phi_0)U_0^T(\mathcal{M}_0 \otimes I_n)(U_0\eta_{0\mathsf{u}} + D_0\eta_{0\mathsf{d}}) \\ &\quad - (\Gamma_0 + \Phi_0)U_0^T(\mathcal{M}_{\mathsf{r}} \otimes I_n)(D_+\eta_{+\mathsf{d}} + U_+\eta_{+\mathsf{u}}),\end{aligned} \tag{5.2.7}$$

where the subscript 0 is used instead of $\bar{\mathcal{N}}_0$ to simplify notations and $\Phi_0 := \mathrm{diag}(\phi_i I_{\nu_i})_{i \in \bar{\mathcal{N}}_0}$. The adaptive law (5.2.2) becomes

$$\phi_i = |U_i^T \sum_{j \in \mathcal{N}} l_{ij}(D_j\eta_{j\mathsf{d}} + U_j\eta_{j\mathsf{u}})|^2, \quad \dot{\gamma}_i = \phi_i, \quad \forall i \in \mathcal{N}. \tag{5.2.8}$$

Now consider the coordinate transformation

$$\begin{aligned}\xi_{i\mathsf{d}} &:= \eta_{i\mathsf{d}} \\ \xi_{i\mathsf{u}} &:= U_i^T \sum_{j \in \mathcal{N}} l_{ij}(D_j\eta_{j\mathsf{d}} + U_j\eta_{j\mathsf{u}}),\end{aligned} \tag{5.2.9}$$

for each $i = 1, 2, \cdots, N$. Similarly, $\xi_{0\mathsf{d}}$, $\xi_{0\mathsf{u}}$, $\xi_{+\mathsf{d}}$, and $\xi_{+\mathsf{u}}$ denote the concatenations of $\xi_{i\mathsf{d}}$ and $\xi_{i\mathsf{u}}$ in the same order as in $\eta_{0\mathsf{d}}$, $\eta_{0\mathsf{u}}$, $\eta_{+\mathsf{d}}$ and $\eta_{+\mathsf{u}}$, respectively. Then, the coordinate transformation (5.2.9) can be rewritten as

$$\begin{aligned}\xi_{0\mathsf{d}} &= \eta_{0\mathsf{d}} \\ \xi_{+\mathsf{d}} &= \eta_{+\mathsf{d}} \\ \xi_{0\mathsf{u}} &= U_0^T(\mathcal{M}_0 \otimes I_n)D_0\eta_{0\mathsf{d}} + U_0^T(\mathcal{M}_{\mathsf{r}} \otimes I_n)D_+\eta_{+\mathsf{d}} \\ &\quad + U_0^T(\mathcal{M}_0 \otimes I_n)U_0\eta_{0\mathsf{u}} + U_0^T(\mathcal{M}_{\mathsf{r}} \otimes I_n)U_+\eta_{+\mathsf{u}} \\ \xi_{+\mathsf{u}} &= U_+^T(\mathcal{L}_+ \otimes I_n)D_+\eta_{+\mathsf{d}} + U_+^T(\mathcal{L}_+ \otimes I_n)U_+\eta_{+\mathsf{u}}.\end{aligned} \tag{5.2.10}$$

Note that (5.2.10) is the well-defined coordinate transformation because $U_0^T(\mathcal{M}_0 \otimes I_n)U_0$ and $U_+^T(\mathcal{L}_+ \otimes I_n)U_+ = \mathrm{diag}(U_{\bar{\mathcal{N}}_k}^T(\mathcal{L}(\bar{\mathcal{G}}_k) \otimes I_n)U_{\bar{\mathcal{N}}_k})_{1 \le k \le c}$ is nonsingular by Lemmas 5.2.1 and 4.3.2, respectively.

Under the coordinate transformation (5.2.10), the error dynamics of $\xi_{+\mathsf{d}}$ and

$\xi_{+\mathsf{u}}$ is of the form

$$\begin{aligned}\dot{\xi}_{+\mathsf{d}} &= (A_{+\mathsf{d}} - L_{+\mathsf{d}}C_{+\mathsf{d}})\xi_{+\mathsf{d}} \\ \dot{\xi}_{+\mathsf{u}} &= \bar{A}_{+\mathsf{r}}\xi_{+\mathsf{d}} + \bar{A}_{+\mathsf{u}}\xi_{+\mathsf{u}} - U_+^T(\mathcal{L}_+ \otimes I_n)U_+(\Gamma_+ + \Phi_+)\xi_{+\mathsf{u}},\end{aligned} \tag{5.2.11}$$

and, on the other hand, the error dynamics of $\xi_{0\mathsf{d}}$ and $\xi_{0\mathsf{u}}$ is of the form

$$\begin{aligned}\dot{\xi}_{0\mathsf{d}} &= (A_{0\mathsf{d}} - L_{0\mathsf{d}}C_{0\mathsf{d}})\xi_{0\mathsf{d}} \\ \dot{\xi}_{0\mathsf{u}} &= \bar{A}_{0\mathsf{r}}\xi_{0\mathsf{d}} + \bar{A}_{0\mathsf{u}}\xi_{0\mathsf{u}} - U_0^T(\mathcal{M}_0 \otimes I_n)U_0(\Gamma_0 + \Phi_0)\xi_{0\mathsf{u}} \\ &\quad + \bar{A}_{0+\mathsf{r}}\xi_{+\mathsf{d}} + \bar{A}_{0+\mathsf{u}}\xi_{+\mathsf{u}} - U_0^T(\mathcal{M}_\mathsf{r} \otimes I_n)U_+(\Gamma_+ + \Phi_+)\xi_{+\mathsf{u}},\end{aligned} \tag{5.2.12}$$

where

$$\begin{aligned}\begin{bmatrix}\bar{A}_{0\mathsf{u}} & \bar{A}_{0+\mathsf{u}} \\ 0 & \bar{A}_{+\mathsf{u}}\end{bmatrix} &:= [*]\begin{bmatrix}A_{0\mathsf{u}} & 0 \\ 0 & A_{+\mathsf{u}}\end{bmatrix}[*]^{-1}, \\ \begin{bmatrix}\bar{A}_{0\mathsf{r}} & \bar{A}_{0+\mathsf{r}} \\ 0 & \bar{A}_{+\mathsf{r}}\end{bmatrix} &:= [**]\begin{bmatrix}A_{0\mathsf{d}} - L_{0\mathsf{d}}C_{0\mathsf{d}} & 0 \\ 0 & A_{+\mathsf{d}} - L_{+\mathsf{d}}C_{+\mathsf{d}}\end{bmatrix} \\ &\quad + [*]\begin{bmatrix}A_{0\mathsf{r}} & 0 \\ 0 & A_{+\mathsf{r}}\end{bmatrix} - \begin{bmatrix}\bar{A}_{0\mathsf{u}} & \bar{A}_{0+\mathsf{u}} \\ 0 & \bar{A}_{+\mathsf{u}}\end{bmatrix}[**],\end{aligned} \tag{5.2.13}$$

with

$$\begin{aligned}[*] &:= \begin{bmatrix}U_0^T(\mathcal{M}_0 \otimes I_n)U_0 & U_0^T(\mathcal{M}_\mathsf{r} \otimes I_n)U_+ \\ 0 & U_+^T(\mathcal{L}_+ \otimes I_n)U_+\end{bmatrix}, \\ [**] &:= \begin{bmatrix}U_0^T(\mathcal{M}_0 \otimes I_n)D_0 & U_0^T(\mathcal{M}_\mathsf{r} \otimes I_n)D_+ \\ 0 & U_+^T(\mathcal{L}_+ \otimes I_n)D_+\end{bmatrix}.\end{aligned}$$

At the same time, the adaptive law (5.2.8) becomes

$$\phi_i = \xi_{i\mathsf{u}}^T\xi_{i\mathsf{u}}, \quad \dot{\gamma}_i = \phi_i, \quad \forall i \in \mathcal{N}. \tag{5.2.14}$$

Now we are ready to show the stability of the error dynamics and we proceed in three steps. First, we consider the subsystems regarding $\xi_{0\mathsf{u}}$. Let us define $\bar{N}_0 := |\bar{\mathcal{N}}_0|$ and let $[g_1\, g_2\, \cdots\, g_{\bar{N}_0}]^T$ be the positive vector obtained by employing

Lemma 2.3.1 with $\mathcal{M}_0$ and consider a function

$$V_0(\xi_{0\mathsf{u}}) := \frac{1}{2}\sum_{i\in\bar{\mathcal{N}}_0}\left\{g_i(2\gamma_i+\phi_i)\phi_i + g_i(\gamma_i-\gamma^*)^2\right\}, \tag{5.2.15}$$

where γ^* is a positive constant to be determined later. Then the derivative of V_0 along (5.2.12) and (5.2.14) becomes

$$\begin{aligned}
\dot{V}_0 &= \sum_{i\in\bar{\mathcal{N}}_0}\left\{g_i\phi_i\dot{\gamma}_i + g_i(\gamma_i+\phi_i)\dot{\phi}_i + g_i(\gamma_i-\gamma^*)\dot{\gamma}_i\right\}\\
&= \sum_{i\in\bar{\mathcal{N}}_0}\left\{g_i(\gamma_i+\phi_i)\dot{\phi}_i + g_i(\gamma_i+\phi_i-\gamma^*)\dot{\gamma}_i\right\}\\
&= \sum_{i\in\bar{\mathcal{N}}_0}\left\{g_i(\gamma_i+\phi_i)\frac{d}{dt}(\xi_{i\mathsf{u}}^T\xi_{i\mathsf{u}}) + g_i(\gamma_i+\phi_i-\gamma^*)\xi_{i\mathsf{u}}^T\xi_{i\mathsf{u}}\right\}\\
&= \xi_{0\mathsf{u}}^T(\Gamma_0+\Phi_0)G_0\dot{\xi}_{0\mathsf{u}} + \dot{\xi}_{0\mathsf{u}}^T G_0(\Gamma_0+\Phi_0)\xi_{0\mathsf{u}} + \xi_{0\mathsf{u}}^T G_0(\Gamma_0+\Phi_0-\gamma^* I_{\sum\nu_i})\xi_{0\mathsf{u}}\\
&= 2\xi_{0\mathsf{u}}^T(\Gamma_0+\Phi_0)G_0\bar{A}_{0\mathsf{r}}\xi_{0\mathsf{d}} + 2\xi_{0\mathsf{u}}^T(\Gamma_0+\Phi_0)G_0\bar{A}_{0\mathsf{u}}\xi_{0\mathsf{u}}\\
&\quad - \xi_{0\mathsf{u}}^T(\Gamma_0+\Phi_0)U_0^T(\hat{\mathcal{M}}_0\otimes I_n)U_0(\Gamma_0+\Phi_0)\xi_{0\mathsf{u}}\\
&\quad + 2\xi_{0\mathsf{u}}^T(\Gamma_0+\Phi_0)G_0\bar{A}_{0+\mathsf{r}}\xi_{+\mathsf{d}} + 2\xi_{0\mathsf{u}}^T(\Gamma_0+\Phi_0)G_0\bar{A}_{0+\mathsf{u}}\xi_{+\mathsf{u}}\\
&\quad - 2\xi_{0\mathsf{u}}^T(\Gamma_0+\Phi_0)G_0U_0^T(\mathcal{M}_\mathsf{r}\otimes I_n)U_+(\Gamma_++\Phi_+)\xi_{+\mathsf{u}}\\
&\quad + \xi_{0\mathsf{u}}^T G_0(\Gamma_0+\Phi_0-\gamma^* I_{\sum\nu_i})\xi_{0\mathsf{u}},
\end{aligned} \tag{5.2.16}$$

where $G_0 := \mathrm{diag}(g_iI_{\nu_i})_{i\in\bar{\mathcal{N}}_0}$. Let us define $\lambda_0 := \lambda_{\min}(\hat{\mathcal{M}}_0)$, which is positive by Lemma 2.3.1 and apply Young's inequality for squaring cross terms of (5.2.16). Then it follows that

$$\begin{aligned}
\dot{V}_0 &\le \frac{6}{\lambda_0}\|G_0\bar{A}_{0\mathsf{r}}\|^2|\xi_{0\mathsf{d}}|^2 - \frac{\lambda_0}{6}\xi_{0\mathsf{u}}^T(\Gamma_0+\Phi_0)^2\xi_{0\mathsf{u}}\\
&\quad + \xi_{0\mathsf{u}}^T G_0(\Gamma_0+\Phi_0-\gamma^* I_{\sum\nu_i})\xi_{0\mathsf{u}} + \frac{6}{\lambda_0}\|G_0\bar{A}_{0\mathsf{u}}\|^2|\xi_{0\mathsf{u}}|^2\\
&\quad + \frac{6}{\lambda_0}\|G_0\bar{A}_{0+\mathsf{r}}\|^2|\xi_{+\mathsf{d}}|^2 + \frac{6}{\lambda_0}\|G_0\bar{A}_{0+\mathsf{u}}\|^2|\xi_{+\mathsf{u}}|^2\\
&\quad + \frac{6}{\lambda_0}\|G_0U_0^T(\mathcal{M}_\mathsf{r}\otimes I_n)U_+\|^2|(\Gamma_++\Phi_+)\xi_{+\mathsf{u}}|^2\\
&\le \frac{6}{\lambda_0}\|\bar{A}_{0\mathsf{r}}\|^2|\xi_{0\mathsf{d}}|^2 + \xi_{0\mathsf{u}}^T(\Gamma_0+\Phi_0)\xi_{0\mathsf{u}}
\end{aligned}$$

$$
\begin{aligned}
&- \xi_{0\mathsf{u}}^T \left[\frac{\lambda_0}{6}(\Gamma_0 + \Phi_0)^2 + (\gamma^* \underline{g} - \frac{6}{\lambda_0}\|\bar{A}_{0\mathsf{u}}\|^2) I_{\sum \nu_i} \right] \xi_{0\mathsf{u}} \\
&+ \frac{6}{\lambda_0}\|\bar{A}_{0+\mathsf{r}}\|^2 |\xi_{+\mathsf{d}}|^2 + \frac{6}{\lambda_0}\|\bar{A}_{0+\mathsf{u}}\|^2 |\xi_{+\mathsf{u}}|^2 \\
&+ \frac{6}{\lambda_0}\|U_0^T(\mathcal{M}_\mathsf{r} \otimes I_n)U_+\|^2 |(\Gamma_+ + \Phi_+)\xi_{+\mathsf{u}}|^2,
\end{aligned} \tag{5.2.17}
$$

where $\underline{g} := \min_{i \in \bar{\mathcal{N}}_0} g_i > 0$, and the second inequality holds because $0 < g_i \leq 1$ for all $i \in \bar{\mathcal{N}}_0$. Note that $\left[\frac{\lambda_0}{6}(\Gamma_0 + \Phi_0)^2 + (\gamma^* \underline{g} - \frac{6}{\lambda_0}\|\bar{A}_{0\mathsf{u}}\|^2) I_{\sum \nu_i}\right]$ is a diagonal matrix and its diagonal components satisfy that

$$
\begin{aligned}
\frac{\lambda_0}{6}(\gamma_i + \phi_i)^2 + \gamma^* \underline{g} - \frac{6}{\lambda_0}\|\bar{A}_{0\mathsf{u}}\|^2 &= \left\{ \sqrt{\frac{\lambda_0}{6}}(\gamma_i + \phi_i) \right\}^2 + \left\{ \sqrt{\gamma^* \underline{g} - \frac{6}{\lambda_0}\|\bar{A}_{0\mathsf{u}}\|^2} \right\}^2 \\
&\geq 2\sqrt{\frac{\lambda_0 \underline{g}}{6}\gamma^* - \|\bar{A}_{0\mathsf{u}}\|^2} \cdot (\gamma_i + \phi_i) \\
&=: \alpha_0(\gamma^*)(\gamma_i + \phi_i), \quad \forall i \in \bar{\mathcal{N}}_0.
\end{aligned} \tag{5.2.18}
$$

By substituting (5.2.18) to (5.2.17), it is obtained that

$$
\begin{aligned}
\dot{V}_0 \leq\ & \frac{6}{\lambda_0}\|\bar{A}_{0\mathsf{r}}\|^2 |\xi_{0\mathsf{d}}|^2 - (\alpha_0(\gamma^*) - 1)\xi_{0\mathsf{u}}^T(\Gamma_0 + \Phi_0)\xi_{0\mathsf{u}} \\
&+ \frac{6}{\lambda_0}\|\bar{A}_{0+\mathsf{r}}\|^2 |\xi_{+\mathsf{d}}|^2 + \frac{6}{\lambda_0}\|\bar{A}_{0+\mathsf{u}}\|^2 |\xi_{+\mathsf{u}}|^2 \\
&+ \frac{6}{\lambda_0}\|U_0^T(\mathcal{M}_\mathsf{r} \otimes I_n)U_+\|^2 |(\Gamma_+ + \Phi_+)\xi_{+\mathsf{u}}|^2.
\end{aligned} \tag{5.2.19}
$$

As the second step, let us focus on the subsystems regarding $\xi_{+\mathsf{u}}$. Define a positive diagonal matrix $\Theta := \mathrm{diag}(\Theta_k)_{1 \leq k \leq c}$, where Θ_k is the positive diagonal matrix obtained by applying Lemma 2.2.2 to $\mathcal{L}(\bar{\mathcal{G}}_k)$ for $k = 1, \cdots, c$. Let us consider a function

$$
V_+(\xi_{+\mathsf{u}}) := \frac{1}{2}\sum_{i=\bar{N}_0+1}^{N} \theta_{(i-\bar{N}_0)}(2\gamma_i + \phi_i)\phi_i + \theta_{(i-\bar{N}_0)}(\gamma_i - \gamma^*)^2, \tag{5.2.20}
$$

where θ_i is the i-th diagonal component of the positive diagonal matrix Θ. Then the derivative of V_+ along (5.2.11) becomes

$$
\dot{V}_+ = 2\xi_{+\mathsf{u}}^T(\Gamma_+ + \Phi_+)\Theta_+\bar{A}_{+\mathsf{r}}\xi_{+\mathsf{d}} + 2\xi_{+\mathsf{u}}^T(\Gamma_+ + \Phi_+)\Theta_+\bar{A}_{+\mathsf{u}}\xi_{+\mathsf{u}}
$$

$$
\begin{aligned}
&- \xi_{+\mathsf{u}}^T(\Gamma_+ + \Phi_+)U_+^T(\hat{\mathcal{L}}_+ \otimes I_n)U_+(\Gamma_+ + \Phi_+)\xi_{+\mathsf{u}} \\
&+ \xi_{+\mathsf{u}}^T(\Theta_+(\Gamma_+ + \Phi_+ - \gamma^* I_{\sum \nu_i}))\xi_{+\mathsf{u}},
\end{aligned} \tag{5.2.21}
$$

where $\hat{\mathcal{L}}_+ := \Theta_+\mathcal{L}_+ + \mathcal{L}_+^T\Theta_+$ and $\Theta_+ := \mathrm{diag}(\theta_{(i-\bar{N}_0)}I_{\nu_i})_{\bar{N}_0+1\le i\le N}$. Let us define $\lambda_+ := \lambda_{\min}(U_+^T(\hat{\mathcal{L}}_+ \otimes I_n)U_+)$, which is positive because $U_+^T(\hat{\mathcal{L}}_+ \otimes I_n)U_+$ is the block diagonal matrix with the positive definite block $U_{\bar{\mathcal{N}}_k}\hat{\mathcal{L}}(\bar{\mathcal{G}}_k)U_{\bar{\mathcal{N}}_k}$. By squaring cross terms of (5.2.21) and using $\|\Theta_+\| \le 1$, it is obtained that

$$
\begin{aligned}
\dot{V}_+ \le{}& \frac{3}{\lambda_+}\|\bar{A}_{+\mathsf{r}}\|^2|\xi_{+\mathsf{d}}|^2 + \xi_{+\mathsf{u}}^T(\Gamma_+ + \Phi_+)\xi_{+\mathsf{u}} \\
&- \xi_{+\mathsf{u}}^T\left[\frac{\lambda_+}{3}(\Gamma_+ + \Phi_+)^2 + (\gamma^*\underline{\theta} - \frac{3\|\bar{A}_{+\mathsf{u}}\|^2}{\lambda_+})I_{\sum \nu_i}\right]\xi_{+\mathsf{u}},
\end{aligned} \tag{5.2.22}
$$

where $\underline{\theta} := \min_{i\in\mathcal{N}\setminus\bar{\mathcal{N}}_0}\theta_{(i-\bar{N}_0)} > 0$.

Now, as the final step, let us consider the Lyapunov function candidate of the form

$$
V := \beta_{0\mathsf{d}}\xi_{0\mathsf{d}}^T P_{0\mathsf{d}}\xi_{0\mathsf{d}} + V_0 + \beta_{+\mathsf{d}}\xi_{+\mathsf{d}}^T P_{+\mathsf{d}}\xi_{+\mathsf{d}} + \beta_{+\mathsf{u}}V_+, \tag{5.2.23}
$$

where $\beta_{0\mathsf{d}}$, $\beta_{+\mathsf{d}}$, and $\beta_{+\mathsf{u}}$ are positive constants to be determined later, $P_{0\mathsf{d}} = \mathrm{diag}(P_{i\mathsf{d}})_{i\in\bar{\mathcal{N}}_0}$, and $P_{+\mathsf{d}} = \mathrm{diag}(P_{i\mathsf{d}})_{i\in\mathcal{N}\setminus\bar{\mathcal{N}}_0}$. Using (5.2.19) and (5.2.22), the derivative of V along (5.2.11), (5.2.12), and (5.2.14) becomes

$$
\begin{aligned}
\dot{V} \le{}& -(\beta_{0\mathsf{d}} - \frac{6}{\lambda_0}\|\bar{A}_{0\mathsf{r}}\|^2)|\xi_{0\mathsf{d}}|^2 - \left(\beta_{+\mathsf{d}} - \frac{6\|\bar{A}_{0+\mathsf{r}}\|^2}{\lambda_0} - \frac{3\beta_{+\mathsf{u}}\|\bar{A}_{+\mathsf{r}}\|^2}{\lambda_+}\right)|\xi_{+\mathsf{d}}|^2 \\
&- (\alpha_0(\gamma^*) - 1)\xi_{0\mathsf{u}}^T(\Gamma_0 + \Phi_0)\xi_{0\mathsf{u}} + \xi_{+\mathsf{u}}^T(\Gamma_+ + \Phi_+)\xi_{+\mathsf{u}} \\
&- \xi_{+\mathsf{u}}^T\Bigg[(\frac{\beta_{+\mathsf{u}}\lambda_+}{3} - \frac{6\|U_0^T(\mathcal{M}_\mathsf{r} \otimes I_n)U_+\|^2}{\lambda_0})(\Gamma_+ + \Phi_+)^2 \\
&+ (\beta_{+\mathsf{u}}\underline{\theta}\,\gamma^* - \frac{3\beta_{+\mathsf{u}}\|\bar{A}_{+\mathsf{u}}\|^2}{\lambda_+} - \frac{6\|\bar{A}_{0+\mathsf{u}}\|^2}{\lambda_0})I_{\sum \nu_i}\Bigg]\xi_{+\mathsf{u}}.
\end{aligned} \tag{5.2.24}
$$

Pick $\beta_{+\mathsf{u}}$ such as

$$
\beta_{+\mathsf{u}} \ge 18\|U_0^T(\mathcal{M}_\mathsf{r} \otimes I_n)U_+\|^2/(\lambda_+\lambda_0) + 3/\lambda_+ \tag{5.2.25}
$$

so that the coefficient of the diagonal matrix $(\Gamma_+ + \Phi_+)^2$ simply becomes larger than or equal to 1. After following the same steps given in (5.2.18), it is obtained that

$$\dot{V} \le -(\beta_{0\mathsf{d}} - \frac{6}{\lambda_0}\|\bar{A}_{0\mathsf{r}}\|^2)|\xi_{0\mathsf{d}}|^2 - \left(\beta_{+\mathsf{d}} - \frac{6}{\lambda_0}\|\bar{A}_{0+\mathsf{r}}\|^2 - \frac{3\beta_{+\mathsf{u}}}{\lambda_+}\|\bar{A}_{+\mathsf{r}}\|^2\right)|\xi_{+\mathsf{d}}|^2$$
$$- (\alpha_0(\gamma^*) - 1)\xi_{0\mathsf{u}}^T(\Gamma_0 + \Phi_0)\xi_{0\mathsf{u}} - (\alpha_+(\gamma^*) - 1)\xi_{+\mathsf{u}}^T(\Gamma_+ + \Phi_+)\xi_{+\mathsf{u}}, \qquad (5.2.26)$$

where $\alpha_0(\gamma^*)$ is given in (5.2.18) and

$$\alpha_+(\gamma^*) := 2\sqrt{\beta_{+\mathsf{u}}\underline{\theta}\,\gamma^* - \left(\frac{3\beta_{+\mathsf{u}}\|\bar{A}_{+\mathsf{u}}\|^2}{\lambda_+} + \frac{6\|\bar{A}_{0+\mathsf{u}}\|^2}{\lambda_0}\right)}.$$

Let us choose $\beta_{0\mathsf{d}}$ and $\beta_{+\mathsf{d}}$ such than

$$\beta_{0\mathsf{d}} > \frac{6\|\bar{A}_{0\mathsf{r}}\|^2}{\lambda_0}, \quad \beta_{+\mathsf{d}} > \frac{6\|\bar{A}_{0+\mathsf{r}}\|^2}{\lambda_0} + \frac{3\beta_{+\mathsf{u}}\|\bar{A}_{+\mathsf{r}}\|^2}{\lambda_+}. \qquad (5.2.27)$$

Also set γ^* sufficiently large so that $\alpha_0(\gamma^*) > 1$ and $\alpha_+(\gamma^*) > 1$. With these settings, it can be seen from (5.2.26) that $\dot{V} \le 0$, which implies that both the estimation error and the coupling gains are bounded. Moreover, based on (5.2.14), each gain $\gamma_i(t)$ is monotonically nondecreasing. Thus it can be concluded that each time-varying gain $\gamma_i(t)$ converges to some finite value. Also note from (5.2.26) that $\dot{V} = 0$ implies that $\xi_{0\mathsf{d}} = 0$, $\xi_{+\mathsf{d}} = 0$, $\xi_{0\mathsf{u}} = 0$, and $\xi_{+\mathsf{u}} = 0$. Hence, by LaSalle's invariance principle [Kha02], it is obtained that the estimation error converges to zero as time goes to infinity.

Finally to show (5.2.3), let us consider the Lyapunov function V with the parameters chosen as follows: $\beta_{+\mathsf{u}} = 1+18\|\mathcal{L}\|^2/\underline{\lambda}^2+3/\underline{\lambda}$, $\beta_{0\mathsf{d}} = 1/\bar{\lambda}+6\Omega^2/\underline{\lambda}$, $\beta_{+\mathsf{d}} = 1/\bar{\lambda}+6(\beta_{+\mathsf{u}}+1)\Omega^2/\underline{\lambda}$, and $\gamma^* = \rho_1$ given in (5.2.4), where $\bar{\lambda} := \max_i \lambda_{\max}(P_{i\mathsf{d}})$ and $\Omega := \kappa(R)\max_i \|A - L_iC_i\|$. To claim that the chosen parameters yield $\dot{V} \le 0$, let us first show that Ω is a common upper bound of $\|\bar{A}_{0\mathsf{r}}\|$, $\|\bar{A}_{0+\mathsf{r}}\|$, $\|\bar{A}_{+\mathsf{r}}\|$, $\|\bar{A}_{0\mathsf{u}}\|$, $\|\bar{A}_{0+\mathsf{u}}\|$, and $\|\bar{A}_{+\mathsf{u}}\|$. Note that the matrix R represents the coordinate transform (5.2.10), and, hence, (5.2.13) can be rewritten as

$$\begin{bmatrix} A_{\mathsf{d}} - L_{\mathsf{d}}C_{\mathsf{d}} & 0 \\ \bar{A}_{\mathsf{r}} & \bar{A}_{\mathsf{u}} \end{bmatrix} = R \begin{bmatrix} A_{\mathsf{d}} - L_{\mathsf{d}}C_{\mathsf{d}} & 0 \\ A_{\mathsf{r}} & A_{\mathsf{u}} \end{bmatrix} R^{-1} \qquad (5.2.28)$$

where $\bar{A}_{\mathsf{u}}$ and $\bar{A}_{\mathsf{r}}$ are the first and the second matrices defined in (5.2.13), respectively, and A_{d}, L_{d}, C_{d}, A_{r}, and A_{u} are block diagonal matrices of $A_{i\mathsf{d}}$, $L_{i\mathsf{d}}$, $C_{i\mathsf{d}}$, $A_{i\mathsf{r}}$, and $A_{i\mathsf{u}}$ for $i \in \mathcal{N}$. By swapping rows and columns, the matrix in the middle of the right hand side of (5.2.28) can be transformed into the block diagonal matrix of $T_i^T A T_i$ given in (4.2.3). Since the norm of the swapping matrices and the orthonormal matrices T_i is 1, the norm of the right hand side of (5.2.28) is less than or equal to Ω. From the fact that the norm of the submatrix is less than equal to that of the original one, it follows that Ω is a common upper bound. Then note that the conditions (5.2.25) and (5.2.27) are satisfied and two conditions $\alpha_0(\rho_1) > 1$ and $\alpha_+(\rho_1) > 1$ can be easily checked by using the following inequality:

$$\rho_1 \geq \frac{1}{\underline{\lambda}\min\{\underline{\theta}, \underline{g}\}}\left(6\|\bar{A}_{\mathsf{u}}\|^2 + \frac{\underline{\lambda} + 6\|\bar{A}_{\mathsf{u}}\|^2}{\beta_{+\mathsf{u}}} + 2\right).$$

As a result, the Lyapunov function V with the chosen parameters satisfies $\dot{V} \leq 0$. On the other hand, since $\gamma_i(t) \geq \gamma_i(0) > 0$ for all $i \in \mathcal{N}$ and for all $t \geq 0$, it can be directly obtained from (5.2.15) and (5.2.20) that $V \geq \frac{1}{2}\min\{\underline{g}\,,\, \underline{\theta}\}(\gamma_i - \gamma^*)^2$. Combined with $V(0) \geq V(t)$, it is obtained that

$$\gamma_i(0) \leq \gamma_i(t) \leq \gamma^* + \sqrt{\frac{2V(0)}{\min\{\underline{g}\,,\, \underline{\theta}\}}}, \quad \forall t \geq 0, \quad i \in \mathcal{N}. \tag{5.2.29}$$

Now let us find the upper bound of $V(0)$. By substituting chosen $\beta_{+\mathsf{u}}$, $\beta_{0\mathsf{d}}$, and $\beta_{+\mathsf{d}}$, it holds that

$$\begin{aligned} V &\leq \max\{\beta_{0\mathsf{d}}, \beta_{+\mathsf{d}}\} \max_{i\in\mathcal{N}} \lambda_{\max}(P_{i\mathsf{d}}) \sum_{i\in\mathcal{N}} |\xi_{i\mathsf{d}}|^2 \\ &\quad + \max\{1, \beta_{+\mathsf{u}}\} \sum_{i\in\mathcal{N}} (|\phi_i + \gamma_i|^2 + |\gamma_i - \gamma^*|^2) \\ &\leq (\beta_{+\mathsf{d}}\bar{\lambda} + \beta_{+\mathsf{u}}) \sum_{i\in\mathcal{N}} (|\xi_{i\mathsf{d}}|^2 + |\phi_i + \gamma_i|^2 + |\gamma_i - \gamma^*|^2). \end{aligned}$$

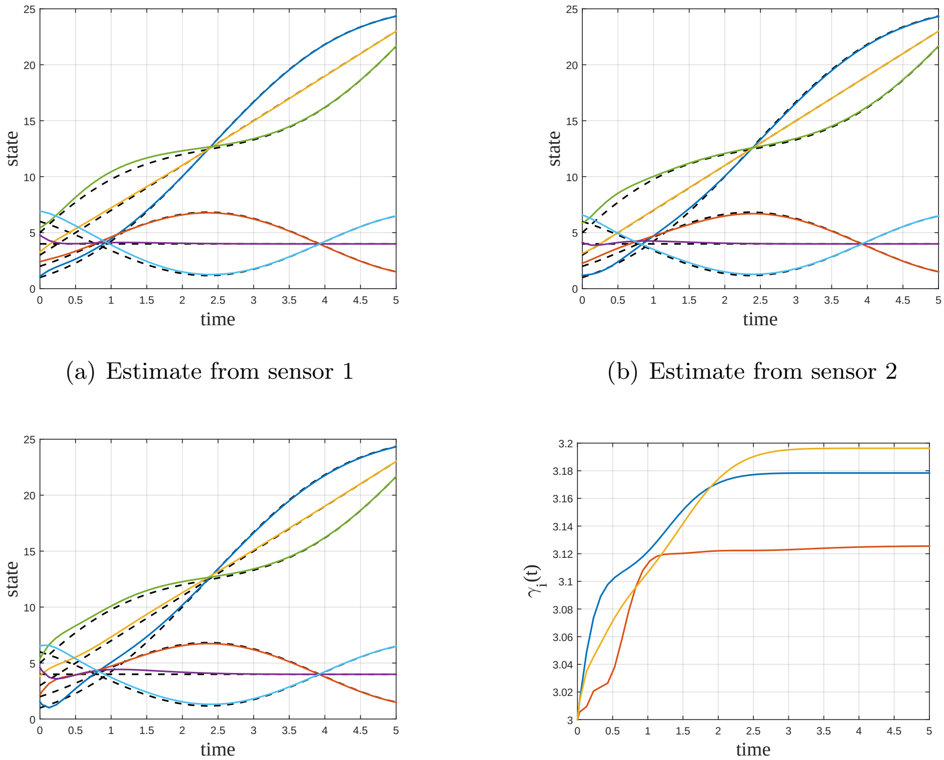
Note that $(\beta_{+\mathsf{d}}\bar{\lambda} + \beta_{+\mathsf{u}}) = (\beta_{+\mathsf{u}} + 1)(6\bar{\lambda}\|\bar{A}_{\mathsf{r}}\|^2\underline{\lambda} + 1)$. By using $\rho_1 \geq 6\|\bar{A}_{\mathsf{r}}\|^2/\underline{\lambda}$ and combining with (5.2.29), the inequality (5.2.3) is obtained. $\square$

Remark 5.2.2. One can notice from the adaptive law (5.2.2) that the coupling gain $\gamma_i(t)$ is monotonically nondecreasing with respect to the time t regardless of the values of the estimation error. Therefore, the noise in the estimates $\hat{x}_i(t)$ causes the coupling gain to diverge even though its magnitude is small, which is a major drawback of the proposed adaptive law. There are several ways to compensate for this shortcoming. Since the adaptive law utilizes the estimate, not the raw measurements, one can use filter to eliminate the high frequency signals, which might contaminated by the noise. Another option is to apply robust adaptive control techniques by adopting the sigma modification or dead zone function (see, for example, [IS95]) to the adaptive law (5.2.2). But there are also some drawbacks to this approach. In exchange for the robustness to the noise, however, the asymptotic convergence of the estimation error might be degraded to the approximate convergence. Furthermore, it is unclear whether the parameters of these modification laws can be chosen in a distributed manner. $\Diamond$

Remark 5.2.3. In [LLDF16], the full-state adaptive law $\dot{\gamma}_i = |\sum_{j\in\mathcal{N}_i}(x_j - x_i)|^2$ is proposed to achieve consensus of the homogeneous agents $\dot{x}_i = Ax_i + Bu_i$ over the strongly connected network. Roughly speaking, based on the homogeneity and the stabilizability of (A, B), the consensus error dynamics is stabilized by the gain constant γ^* larger than λ_2^{-1}, where λ_2 is the algebraic connectivity of $\hat{\mathcal{L}}$. On the other hand, in Theorem 5.2.2, the approach in [LLDF16] is not directly applicable to our problem because each undetectable dynamics $A_{i\mathsf{u}}$ is different with each other and the pair (A, C_i) is not detectable in general. $\Diamond$

Example 5.2.1. Consider the Example 4.3.1 again with a adaptive distributed observer given in (5.2.1) and (5.2.2). The simulation results of this example are given in Fig. 5.1. Here, the plant parameter is chosen as $k/J := 1$ and the initial coupling gains are chosen as $\gamma_1(0) = \gamma_2(0) = \gamma_3(0) = 3$. Notice that, by employing the adaptive coupling gain, each i-th agent can construct its own observer by itself. To be practical, we consider the same plant but the noisy output

$$y_i = C_i x + v_i, \qquad i \in \mathcal{N}, \tag{5.2.30}$$



(a) Estimate from sensor 1

(b) Estimate from sensor 2

(c) Estimate from sensor 3

(d) Adaptive gains

Figure 5.1: Simulation results of Example 5.2.1 with the adaptive Law (5.2.2) for the noise-free measurements: The plant state $x(t)$ are depicted as black dashed curve and the estimate $\hat{x}_i$ generated by each i-th observer are depicted for $i = 1,\ 2,\ 3$ as colored solid curves. Furthermore the adaptive gain $\gamma_i(t)$ are depicted for $i = 1, 2, 3$.

where v_i is the zero mean Gaussian noise with the variance of 0.1. The system parameters and the initial values of coupling gains are same as the former case. The simulation results are given in Fig. 5.2. The estimates $\hat{x}_i$ seems converging to the plant state for each $i = 1, 2, 3$. However, as mentioned in Remark 5.2.2, the coupling gain $\gamma_i(t)$ keep increasing due to the small noise v_i for each $i = 1, 2, 3$. To prevent the divergence of the coupling gains, we employ a dead zone function to the adaptive law such that

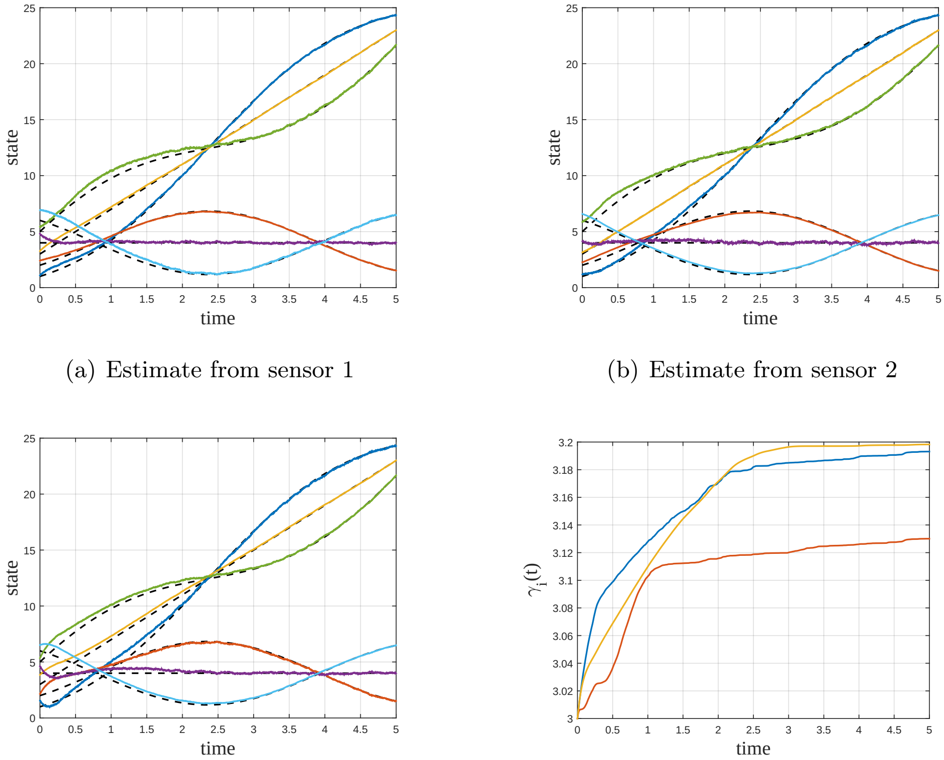
$$\dot{\gamma}_i = \begin{cases} 0, & \text{if } |\phi_i| \leq \delta, \\ \phi_i, & \text{otherwise.} \end{cases} \tag{5.2.31}$$

where δ is chosen as 0.1 in the simulation. The simulation results are given in Fig. 5.3. It is shown that the coupling gain is bounded. Although it is not clear from the figures, the dead zone function usually results in the approximated convergence, not the asymptotic convergence, because the estimation error smaller than the threshold δ will not affect the coupling gain.

5.3 Summary and Discussion

In this chapter, we have further enhanced the results from Chapter 4 by developing the adaptation law, which achieves the completely decentralized construction so that each agent can implement its own local observer by itself. To the best of our knowledge, this is the first decentralized constructible distributed observer, or the first fully distributed observer, in the literature.

Like other adaptive control methods, the proposed method, however, has disadvantages that need to be improved. The first one is monotonically non-decreasing coupling gains. It would be desirable that the coupling gain decreases after the estimation error decreases but this is not the case. In the case of leaving and joining of agents or changing communication topologies, the estimation goes to the plant state eventually but the coupling gain could grow every time the change occurs. As a second drawback of the proposed adaptive scheme, it is not robust to the disturbance or the noise. One feasible solution for improvement is the leakage method proposed in robust control theory [IS95] but it has possibility



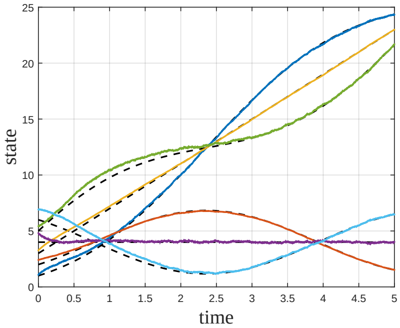
(a) Estimate from sensor 1

(b) Estimate from sensor 2

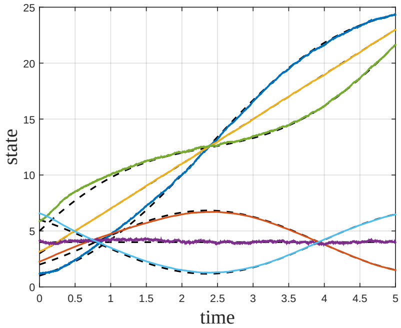
(c) Estimate from sensor 3

(d) Adaptive gains

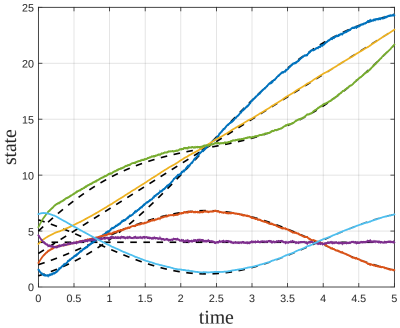
Figure 5.2: Simulation Results of Example 5.2.1 with the adaptive law (5.2.2) for the noisy measurements (5.2.30): The plant state $x(t)$ are depicted as black dashed curve and the estimate $\hat{x}_i$ generated by each i-th observer are depicted for $i = 1, 2, 3$ as colored solid curves. Furthermore the adaptive gain $\gamma_i(t)$ are depicted for $i = 1, 2, 3$.



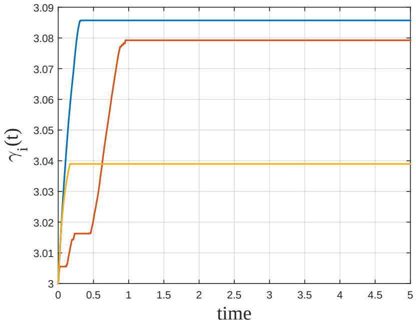
(a) Estimate from sensor 1



(b) Estimate from sensor 2



(c) Estimate from sensor 3



(d) Adaptive gains

Figure 5.3: Simulation results of Example 5.2.1 with the adaptive Law with dead zone function (5.2.31) for the noisy measurements (5.2.30): The plant state $x(t)$ are depicted as black dashed curve and the estimate $\hat{x}_i$ generated by each i-th observer are depicted for $i = 1$, 2, 3 as colored solid curves. Furthermore the adaptive gain $\gamma_i(t)$ are depicted for $i = 1, 2, 3$.

to lose the exact convergence of estimation errors, so there is a trade-off between the robustness and the performance.

Chapter 6

Distributed State Observer for Linear Systems with Input

In the previous chapters, we consider the autonomous plant and introduce a novel structure of the distributed observer which can be implemented in a completely decentralized manner so that each agent can design their own observer by itself. In this chapter, we extend the class of the plant by considering the plant with input and attempt to estimate the state of the plant in a distributed fashion.

It is well-known from the classical linear system theory that, with the known plant input, designing an observer for the plant having input is equivalent to designing an observer for the autonomous plant. In specific, for the plant $\dot{x} = Ax + Bu$ and the output $y = Cx$, one can design a Luenberger observer of the form $\dot{\hat{x}} = A\hat{x} + L(y - C\hat{x}) + Bu$ so that the error dynamics equal to the one without the plant input. This is equally applicable to the case of the proposed distributed observer; if the plant input in known to every agent, then the results given in previous chapters remain valid by adding input term Bu to the every local observer. However, the assumption that every agent knows the plant input is not practical to the networked agent system. What is more practical and more general is, as with the plant state, assuming the plant input is measured in a distributed manner, which accommodates a lot of practical applications such as the large-scale systems with multiple channels or the multi-agent systems with inputs. Besides this setting has not been considered in the literature yet to the best of our knowledge.

This problem setting obviously causes each agent to estimate the plant state under the unknown input, which leads us to consider the geometric approach. This issue is known as the problem of designing an unknown input observer (see, for example, [Bha78] and [CPZ96]), which reconstructs a part of (or entire) plant state despite the unknown input. With the help of the unknown input observer, each agent is capable of identifying the part of the plant state that is obtainable solely by using each agent's own measurements. The rest of the obtainable part, however, is not just unresolvable but also directly affected by the unknown input unlike the case of autonomous plants handled in the previous chapters. Therefore, the main challenge lies on how to enable each agent to recover the part of the state that is not covered by the unknown input observer in spite of the unknown part of input.

In this chapter, the results of the geometric control theory in Chapter 3 is adopted such as the concept of the weakly unobservable subspace and the special coordinate basis decomposition to show that the portion of the plant state that cannot be estimated by the individual agent itself is the substate in the unstable weakly unobservable subspace. To estimate the rest, i.e., the obtainable part, the partial unknown input observer is designed. Finally we show that the agent-wise decomposed diffusive coupling in combinations with the partial unknown input observer yields the practical error convergence under a mild sufficient condition.

6.1 Problem Statement

Consider a continuous-time linear time invariant system

$$\dot{x} = Ax + Bu, \tag{6.1.1}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the input to the plant. We assume that the system (6.1.1) is monitored by a network of N agents such that the i-th agent communicates according to a given network topology $\mathcal{G}$, and measures not only the partial state $y_i \in \mathbb{R}^{p_i}$ but also the partial input $u_i \in \mathbb{R}^{m_i}$ given by

$$y_i = C_i x, \quad u_i = D_i u, \quad i \in \mathcal{N} \tag{6.1.2}$$

where $C_i \in \mathbb{R}^{p_i \times n}$ and $D_i \in \mathbb{R}^{m_i \times m}$.

The problem of interest is to design an algorithm which is executed on each agent and which enables each agent to estimate the full state x of (6.1.1) using the measured partial input and output signals (6.1.2) under the given communication graph $\mathcal{G}$.

Before proceeding with the analysis, let us introduce some notations. Without loss of generality, for each $i \in \mathcal{N}$, the measured input u_i can be transformed so that the rows of the matrix $D_i \in \mathbb{R}^{m_i \times m}$ form a orthonormal basis. Let us define the i-th agent's unknown input u_i^c as

$$u_i^c := D_i^c u, \qquad i \in \mathcal{N}, \tag{6.1.3}$$

where $D_i^c \in \mathbb{R}^{(m-m_i) \times m}$ is a matrix whose rows forms a orthonormal basis for the orthogonal complement of the image spanned by the rows of D_i so that the concatenated matrix $[D_i^T \; (D_i^c)^T]^T \in \mathbb{R}^{m \times m}$ becomes orthonormal. Then the plant input u can be decomposed into the i-th agent's accessible part u_i and its complementary part u_i^c such that $u = D_i^T u_i + (D_i^c)^T u_i^c$ for each $i \in \mathcal{N}$. Based on these observations, we employ the following notations.

Notation 2. For each $i \in \mathcal{N}$, let us define $B_i := BD_i \in \mathbb{R}^{n \times m_i}$ and $B_i^c := BD_i^c \in \mathbb{R}^{n \times (n-m_i)}$ so that the plant input u, the partial input measured by the i-th agent u_i, and its complementary unknown input u_i^c satisfy the following equality:

$$Bu = B_i u_i + B_i^c u_i^c, \qquad i \in \mathcal{N}. \tag{6.1.4}$$

◊

Now we are ready to proceed the distributed observer design. Our approach follows the philosophy of the previous sections; each agent locally estimates the part, called the obtainable part, of the plant state x by using the measured output y_i and the measured input u_i, and the estimation of the rest, called the unobtainable part, relies on the neighboring agent's estimates. As an expected result, all agents' estimates become correct if one agent's unobtainable part is properly compensated by the other's obtainable parts propagated through the network. To begin with, let us find the i-th agent's reconstructible part of the plant state by utilizing y_i and u_i.

6.2 Partial Unknown Input Observer for Individual Agent

In this part, we investigate the plant from the perspective of the individual agent and identify the part of the state that can be estimated solely with the local measurements despite the unknown input. For this purpose, the results about the unstable weakly unobservable subspace and SCBD given in Chapter 3 will be applied.

Notice that it is shown in Section 4.2 that, when the plant is autonomous, i.e., $u \equiv 0$ in (6.1.1), the part of the plant state that can be estimated solely with the local measurement $y_i = C_i x$ is the detectable part of the pair (A, C_i), which is the quotient of the state space $\mathbb{R}^n$ modulo the undetectable subspace $\mathcal{U}_i$. The specific process of reaching that result is given as follows; first, the detectability decomposition from the i-th agent's view point is presented to show that the plant state composed of the part detectable to the i-th agent and the undetectable part, second, it is shown that the substate corresponding to the undetectable part can not be estimated with the local measurement y_i, and lastly, it is shown that a local Luenberger observer is constructed to estimate the detectable part. This part will proceed in the same way but with a different setting.

i) We apply SCBD to the system seen from the i-th agent and decompose the state into several parts to show the relation among the state, the output, and the unknown input.

ii) We explain the unstable weakly unobservable subspace corresponds to the part that can never be estimated using the measurements y_i and u_i.

iii) We construct a partial observer for the rest of the unstable weakly unobservable subspace.

Let us begin. By substituting (6.1.4) into (6.1.1), the system seen from the

i-th agent is given as follows:

$$\begin{aligned} \dot{x} &= Ax + B_i u_i + B_i^c u_i^c, \\ y_i &= C_i x, \end{aligned} \tag{6.2.1}$$

where y_i and u_i are known but u_i^c is unknown. Since u_i is known input, its effect on the estimation error can be eliminated like the effect of the known plant input can be eliminated by adding the input term to the Luenberger observer designed for the autonomous plant. Therefore, the focus should lie on the effect of the unknown input u_i^c to the state x and the output y_i, which explains why we consider the SCBD for the triplet (C_i, A, B_i^c) in the following lemma.

Lemma 6.2.1. (SCBD for the i-th agent) For each $i \in \mathcal{N}$, there is a set of coordinate changes of the form

$$\begin{aligned} x &= P_i \tilde{x} = \begin{bmatrix} P_{i\mathsf{a}} & P_{i\mathsf{b}} & P_{i\mathsf{c}} & P_{i\mathsf{d}} & P_{i\mathsf{e}} \end{bmatrix} \begin{bmatrix} \tilde{x}_{i\mathsf{a}} \\ \tilde{x}_{i\mathsf{b}} \\ \tilde{x}_{i\mathsf{c}} \\ \tilde{x}_{i\mathsf{d}} \\ \tilde{x}_{i\mathsf{e}} \end{bmatrix}, \\ y_i &= R_i \tilde{y}_i = R_i \begin{bmatrix} \tilde{y}_{i\mathsf{a}} \\ \tilde{y}_{i\mathsf{b}} \end{bmatrix}, \quad u_i^c = Q_i \tilde{u}_i^c = Q_i \begin{bmatrix} \tilde{u}_{i\mathsf{a}}^c \\ \tilde{u}_{i\mathsf{e}}^c \end{bmatrix} \end{aligned} \tag{6.2.2}$$

which transforms the system (6.2.1) into

$$\begin{aligned} \dot{\tilde{x}}_i &= \tilde{A}_i \tilde{x}_i + \tilde{B}_i^c \tilde{u}_i^c + \Xi_i u_i \\ \tilde{y}_i &= \tilde{C}_i \tilde{x}_i. \end{aligned} \tag{6.2.3}$$

where

$$
\begin{aligned}
\tilde{A}_i &:= P_i^{-1} A P_i = \begin{bmatrix} \tilde{A}_{i\mathsf{aa}} & \tilde{B}^c_{i\mathsf{a}}\tilde{A}_{i\mathsf{ab}} & \tilde{B}^c_{i\mathsf{a}}\tilde{A}_{i\mathsf{ac}} & \tilde{B}^c_{i\mathsf{a}}\tilde{A}_{i\mathsf{ad}} & \tilde{B}^c_{i\mathsf{a}}\tilde{A}_{i\mathsf{ae}} \\ \tilde{A}_{i\mathsf{ba}}\tilde{C}_{i\mathsf{a}} & \tilde{A}_{i\mathsf{bb}} & 0 & 0 & 0 \\ \tilde{A}_{i\mathsf{ca}}\tilde{C}_{i\mathsf{a}} & \tilde{A}_{i\mathsf{cb}}\tilde{C}_{i\mathsf{b}} & \tilde{A}_{i\mathsf{cc}} & 0 & 0 \\ \tilde{A}_{i\mathsf{da}}\tilde{C}_{i\mathsf{a}} & \tilde{A}_{i\mathsf{db}}\tilde{C}_{i\mathsf{b}} & 0 & \tilde{A}_{i\mathsf{dd}} & 0 \\ \tilde{A}_{i\mathsf{ea}}\tilde{C}_{i\mathsf{a}} & \tilde{A}_{i\mathsf{eb}}\tilde{C}_{i\mathsf{b}} & \tilde{B}^c_{i\mathsf{e}}\tilde{A}_{i\mathsf{ec}} & \tilde{B}^c_{i\mathsf{e}}\tilde{A}_{i\mathsf{ed}} & \tilde{A}_{i\mathsf{ee}} \end{bmatrix}, \\
\tilde{B}^c_i &:= P_i^{-1} B^c_i Q_i^{-1} = \begin{bmatrix} \tilde{B}^c_{i\mathsf{a}} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \tilde{B}^c_{i\mathsf{e}} \end{bmatrix}, \\
\tilde{C}_i &:= R_i^{-1} C_i P_i = \begin{bmatrix} \tilde{C}_{i\mathsf{a}} & 0 & 0 & 0 & 0 \\ 0 & \tilde{C}_{i\mathsf{b}} & 0 & 0 & 0 \end{bmatrix}, \\
\Xi_i &:= P_i^{-1} B_i.
\end{aligned}
\tag{6.2.4}
$$

Here, the transformed system satisfies following properties:

1. the variables $\tilde{x}_{i\mathsf{a}}$, $\tilde{y}_{i\mathsf{a}}$, $\tilde{u}^c_{i\mathsf{a}}$ consist of $\rho_i \geq 1$ parts

$$
\tilde{x}_{i\mathsf{a}} = \begin{bmatrix} \tilde{x}_{i\mathsf{a},1} \\ \vdots \\ \tilde{x}_{i\mathsf{a},\rho_i} \end{bmatrix}, \quad \tilde{y}_{i\mathsf{a}} = \begin{bmatrix} \tilde{y}_{i\mathsf{a},1} \\ \vdots \\ \tilde{y}_{i\mathsf{a},\rho_i} \end{bmatrix}, \quad \tilde{u}^c_{i\mathsf{a}} = \begin{bmatrix} \tilde{u}^c_{i\mathsf{a},1} \\ \vdots \\ \tilde{u}^c_{i\mathsf{a},\rho_i} \end{bmatrix}, \tag{6.2.5}
$$

where $\tilde{u}^c_{i\mathsf{a},k} \in \mathbb{R}$, $\tilde{y}_{i\mathsf{a},k} \in \mathbb{R}$, $\tilde{x}_{i\mathsf{a},k} \in \mathbb{R}^{q_k}$ for $1 \leq k \leq \rho_i$, and the matrices $\tilde{A}_{i\mathsf{aa}}$, $\tilde{B}^c_{i\mathsf{a}}$, and $\tilde{C}_{i\mathsf{a}}$ is of the form

$$
\begin{aligned}
\tilde{A}_{i\mathsf{aa}} &= \mathrm{diag}(\mathsf{A}_{q_1}, \cdots, \mathsf{A}_{q_\rho}) + \tilde{B}^c_{i\mathsf{a}} E_{i\mathsf{a}} + F_{i\mathsf{a}} \tilde{C}_{i\mathsf{a}} \\
\tilde{B}^c_{i\mathsf{a}} &= \mathrm{diag}(\mathsf{B}_{q_1}, \cdots, \mathsf{B}_{q_\rho}) \\
\tilde{C}_{i\mathsf{a}} &= \mathrm{diag}(\mathsf{C}_{q_1}, \cdots, \mathsf{C}_{q_\rho})
\end{aligned}
\tag{6.2.6}
$$

for some matrices $E_{i\mathsf{a}}$ and $F_{i\mathsf{a}}$ with

$$\mathsf{A}_{q_k} := \begin{bmatrix} 0 & I_{q_k-1} \\ 0 & 0 \end{bmatrix}, \quad \mathsf{B}_{q_k} := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{q_k \times 1} \tag{6.2.7}$$

$$\mathsf{C}_{q_k} := \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{1 \times q_k}, \tag{6.2.8}$$

2. the pair $(\tilde{A}_{i\mathsf{bb}}, \tilde{C}_{i\mathsf{b}})$ is observable,

3. the eigenvalues of $\tilde{A}_{i\mathsf{cc}}$ lie in $\mathbb{C}^-$, the eigenvalues of $\tilde{A}_{i\mathsf{dd}}$ lie in $\mathbb{C}^{0+}$,

4. the pair $(\tilde{A}_{i\mathsf{ee}}, \tilde{B}^c_{i\mathsf{e}})$ is controllable, and

5. the unstable weakly unobservable subspace of the triplet (C_i, A, B^c_i), denoted as $\mathcal{U}_i$, is spanned by the columns of $[P_{i\mathsf{d}}\, P_{i\mathsf{e}}]$.

$\Diamond$

Proof. The items 1 to 4 are directly obtained by applying Lemma 3.2.1 to the triplet (C_i, A, B^c_i). The item 5 is the result of Lemma 3.3.1. $\square$

As a result of SCBD, we have a structural map that connects the local output y_i, the plant state x, and the unknown plant input u^c_i. For more detail, see Table 3.1.

In the following, we attempt to find the part of the state that can not be estimated by using the local output y_i and the known input u_i over the unknown input u^c_i. For autonomous plants, the part of the state that can not be estimated using y_i only is represented by the undetectable subspace, which consists of the state that diverges as time goes to infinity but not appear in the output, i.e., $y_i = C_i x = 0$. In a similar fashion, the part of the state that can not be estimated using y_i and u_i with unknown input u^c_i can be characterized by the set of state that leads to the unstable trajectory and that nullifies the output y_i when a specific input is assigned to u^c_i. Notice that the subspace consisting of the state with this property is called unstable weakly unobservable subspace (see Definition 3.1.1) which is, as a result of Lemma 6.2.1, described by the substate $[\tilde{x}^T_{i\mathsf{d}}\, \tilde{x}^T_{i\mathsf{e}}]^T$.

To be specific, we show that the substate $[\tilde{x}_{i\mathsf{d}}^T\, \tilde{x}_{i\mathsf{e}}^T]^T$ can not be reconstructed under a specific plant input even though y_i and u_i are measured. Consider the case when the input is of a specific form such that $u_i = 0$ and $u_i^c = Q_i F_i \tilde{x}_i$, where

$$F_i := \begin{bmatrix} 0 & 0 & 0 & -\tilde{A}_{i\mathsf{ad}} & -\tilde{A}_{i\mathsf{ae}} \\ 0 & 0 & -\tilde{A}_{i\mathsf{ec}} & -\tilde{A}_{i\mathsf{ed}} & F_{i\mathsf{e}} \end{bmatrix}, \tag{6.2.9}$$

and $F_{i\mathsf{e}}$ is designed so that the eigenvalues of $(\tilde{A}_{i\mathsf{ee}} + \tilde{B}_{i\mathsf{e}}^c F_{i\mathsf{e}})$ lie on $\mathbb{C}^{0+}$. This specific input decomposes the system (6.2.3) into two subsystems:

$$\begin{aligned} \begin{bmatrix} \dot{\tilde{x}}_{i\mathsf{a}} \\ \dot{\tilde{x}}_{i\mathsf{b}} \\ \dot{\tilde{x}}_{i\mathsf{c}} \end{bmatrix} &= \begin{bmatrix} \tilde{A}_{i\mathsf{aa}} & \tilde{B}_{i\mathsf{a}}^c \tilde{A}_{i\mathsf{ab}} & \tilde{B}_{i\mathsf{a}}^c \tilde{A}_{i\mathsf{ac}} \\ \tilde{A}_{i\mathsf{ba}} \tilde{C}_{i\mathsf{a}} & \tilde{A}_{i\mathsf{bb}} & 0 \\ \tilde{A}_{i\mathsf{ca}} \tilde{C}_{i\mathsf{a}} & \tilde{A}_{i\mathsf{cb}} \tilde{C}_{i\mathsf{b}} & \tilde{A}_{i\mathsf{cc}} \end{bmatrix} \begin{bmatrix} \tilde{x}_{i\mathsf{a}} \\ \tilde{x}_{i\mathsf{b}} \\ \tilde{x}_{i\mathsf{c}} \end{bmatrix} \\ y_i = R \begin{bmatrix} \tilde{y}_{i\mathsf{a}} \\ \tilde{y}_{i\mathsf{b}} \end{bmatrix} &= \begin{bmatrix} \tilde{C}_{i\mathsf{a}} & 0 & 0 \\ 0 & \tilde{C}_{i\mathsf{b}} & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_{i\mathsf{a}} \\ \tilde{x}_{i\mathsf{b}} \\ \tilde{x}_{i\mathsf{c}} \end{bmatrix}, \end{aligned} \tag{6.2.10}$$

and

$$\begin{bmatrix} \dot{\tilde{x}}_{i\mathsf{d}} \\ \dot{\tilde{x}}_{i\mathsf{e}} \end{bmatrix} = \begin{bmatrix} \tilde{A}_{i\mathsf{dd}} & 0 \\ 0 & \tilde{A}_{i\mathsf{ee}} + \tilde{B}_{i\mathsf{e}}^c F_{i\mathsf{e}} \end{bmatrix} \begin{bmatrix} \tilde{x}_{i\mathsf{d}} \\ \tilde{x}_{i\mathsf{e}} \end{bmatrix} + \begin{bmatrix} \tilde{A}_{i\mathsf{da}} & \tilde{A}_{i\mathsf{db}} \\ \tilde{A}_{i\mathsf{ea}} & \tilde{A}_{i\mathsf{eb}} \end{bmatrix} \begin{bmatrix} \tilde{y}_{i\mathsf{a}} \\ \tilde{y}_{i\mathsf{b}} \end{bmatrix}. \tag{6.2.11}$$

Note that y_i is affected only by the state variables of (6.2.10) and not by the state variables of (6.2.11). Specifically, the initial state of $\tilde{x}_{i\mathsf{a}}(0) = 0$, $\tilde{x}_{i\mathsf{b}}(0) = 0$, $\tilde{x}_{i\mathsf{c}}(0) = 0$, $\tilde{x}_{i\mathsf{d}}(0) \neq 0$, and $\tilde{x}_{i\mathsf{e}}(0) \neq 0$ leads to the zero output $y_i(t) = 0$ for all $t \geq 0$ while $[\tilde{x}_{i\mathsf{d}}^T(t)\, \tilde{x}_{i\mathsf{e}}^T(t)]^T$ diverges as time t goes to infinity.

In summary, there is a plant input signal that makes the partial state $[\tilde{x}_{i\mathsf{d}}^T\, \tilde{x}_{i\mathsf{e}}^T]^T$ invisible from y_i, which, in fact, can be seen as a result of the fifth statement in Lemma 6.2.1, i.e., $[\tilde{x}_{i\mathsf{d}}^T\, \tilde{x}_{i\mathsf{e}}^T]^T$ is the part of the state that never be reconstructed by measuring y_i and u_i only.

So far, we show that the partial state $[\tilde{x}_{i\mathsf{d}}^T\, \tilde{x}_{i\mathsf{e}}^T]^T$ can not be estimated from locally measured output y_i and the known input u_i. In the followings, we will show that the rest part can be reconstructed based on y_i and u_i. Specifically, in the following lemma, we design a *partial observer* for the part of the plant state

$[\tilde{x}_{i\mathsf{a}}^T\, \tilde{x}_{i\mathsf{b}}^T\, \tilde{x}_{i\mathsf{c}}^T]^T$ in order to show that this partial state is reconstructible by utilizing y_i, u_i regardless of the value of the plant input u.

Lemma 6.2.2. (Partial (differential) observer for obtainable part) Let us choose L_i such that $(\tilde{A}_{i\mathsf{bb}} - L_i\tilde{C}_{i\mathsf{b}})$ be Hurwitz and let us denote $\delta_i := \max_k q_k - 1$. Then there exist a set of matrices $G_{i,0}, \cdots, G_{i,\delta_i}$, $H_{i\mathsf{a}}$, $H_{i\mathsf{b}}$, and $H_{i\mathsf{c}}$ such that the following dynamic system

$$\begin{aligned}
\xi_{i\mathsf{a}} &= \sum_{j=0}^{\delta_i} G_{i,j}\tilde{y}_{i\mathsf{a}}^{(j)} + H_{i\mathsf{a}}u_i \\
\dot{\xi}_{i\mathsf{b}} &= \tilde{A}_{i\mathsf{ba}}\tilde{y}_{i\mathsf{a}} + \tilde{A}_{i\mathsf{bb}}\xi_{i\mathsf{b}} - L_i(\tilde{C}_{i\mathsf{b}}\xi_{i\mathsf{b}} - \tilde{y}_{i\mathsf{b}}) + H_{i\mathsf{b}}u_i \\
\dot{\xi}_{i\mathsf{c}} &= \tilde{A}_{i\mathsf{ca}}\tilde{y}_{i\mathsf{a}} + \tilde{A}_{i\mathsf{cb}}\tilde{y}_{i\mathsf{b}} + \tilde{A}_{i\mathsf{cc}}\xi_{i\mathsf{c}} + H_{i\mathsf{c}}u_i,
\end{aligned} \tag{6.2.12}$$

where $\tilde{y}_{i\mathsf{a}}^{(j)}$ stands for the j-th derivative of $\tilde{y}_{i\mathsf{a}}$, is a partial (differential) observer such that the followings hold:

$$\xi_{i\mathsf{a}}(t) = \tilde{x}_{i\mathsf{a}}(t), \quad \forall t \geq 0, \tag{6.2.13}$$

and the estimation errors $\eta_{i\mathsf{b}} := \xi_{i\mathsf{b}} - \tilde{x}_{i\mathsf{b}}$ and $\eta_{i\mathsf{c}} := \xi_{i\mathsf{c}} - \tilde{x}_{i\mathsf{c}}$ are driven by the following stable dynamics

$$\begin{bmatrix} \dot{\eta}_{i\mathsf{b}} \\ \dot{\eta}_{i\mathsf{c}} \end{bmatrix} = \begin{bmatrix} (\tilde{A}_{i\mathsf{bb}} - L_i\tilde{C}_{i\mathsf{b}}) & 0 \\ 0 & \tilde{A}_{i\mathsf{cc}} \end{bmatrix} \begin{bmatrix} \eta_{i\mathsf{b}} \\ \eta_{i\mathsf{c}} \end{bmatrix}, \tag{6.2.14}$$

for all $t \geq 0$. ◊

Proof. For simplicity, let us drop the first subscript i, which indicates the agent number. Note that the upper three part of (6.2.3) can be written as follows:

$$\begin{aligned}
\dot{\tilde{x}}_{\mathsf{a}} &= \tilde{A}_{\mathsf{aa}}\tilde{x}_{\mathsf{a}} + \tilde{B}_{\mathsf{a}}^c w_{\mathsf{a}} + \tilde{B}_{\mathsf{a}}^c\tilde{u}^c + \Xi_{\mathsf{a}}u \\
\dot{\tilde{x}}_{\mathsf{b}} &= \tilde{A}_{\mathsf{ba}}\tilde{y}_{\mathsf{a}} + \tilde{A}_{\mathsf{bb}}\tilde{x}_{\mathsf{b}} + \Xi_{\mathsf{b}}u \\
\dot{\tilde{x}}_{\mathsf{c}} &= \tilde{A}_{\mathsf{ca}}\tilde{y}_{\mathsf{a}} + \tilde{A}_{\mathsf{cb}}\tilde{y}_{\mathsf{b}} + \tilde{A}_{\mathsf{cc}}\tilde{x}_{\mathsf{c}} + \Xi_{\mathsf{c}}u
\end{aligned} \tag{6.2.15}$$

where $w_{\mathsf{a}} := E_{\mathsf{a}}\tilde{x}_{\mathsf{a}} + \tilde{A}_{\mathsf{ab}}\tilde{x}_{\mathsf{b}} + \tilde{A}_{\mathsf{ac}}\tilde{x}_{\mathsf{c}} + \tilde{A}_{\mathsf{ad}}\tilde{x}_{\mathsf{d}} + \tilde{A}_{\mathsf{ae}}\tilde{x}_{\mathsf{e}}$, and Ξ_{a}, Ξ_{b}, and Ξ_{c} denote the submatrices of Ξ_i consisting of the rows corresponding to $\dot{\tilde{x}}_{\mathsf{a}}$, $\dot{\tilde{x}}_{\mathsf{b}}$, and $\dot{\tilde{x}}_{\mathsf{c}}$, respectively.

First, let us show the algebraic equality (6.2.13). To do that, let us focus on the first line of (6.2.15) corresponding to the derivatives of $\tilde{x}_{\mathsf{a}}$. By the first property of Lemma 3.2.1, the variables $\tilde{x}_{\mathsf{a}}$, $\tilde{u}_{\mathsf{a}}$, and $\tilde{y}_{\mathsf{a}}$ can be decomposed into ρ part. In detail, for $1 \leq k \leq \rho$, the k-th part related to the partial state $\tilde{x}_{\mathsf{a},k} := \left[\tilde{x}_{\mathsf{a},k,1}, \cdots, \tilde{x}_{\mathsf{a},k,q_k}\right]^T \in \mathbb{R}^{q_k}$, $\tilde{u}_{\mathsf{a},k} \in \mathbb{R}$, and $\tilde{y}_{\mathsf{a},k} \in \mathbb{R}$ is given as follows

$$\begin{aligned} \dot{\tilde{x}}_{\mathsf{a},k} &= \mathsf{A}_{q_k}\tilde{x}_{\mathsf{a},k} + \mathsf{B}_{q_k}(\tilde{u}^c_{\mathsf{a},k} + w_{\mathsf{a},k}) + F_{\mathsf{a},k}\tilde{y}_{\mathsf{a}} + \Xi_{\mathsf{a},k}u \\ \tilde{y}_{a,k} &= \mathsf{C}_{q_k}\tilde{x}_{a,k} \end{aligned} \tag{6.2.16}$$

where $w_{\mathsf{a},k} \in \mathbb{R}$ is the k-th element of w_{a}, $F_{\mathsf{a},k}$ is the k-th row of the matrix F_{a}, and $\Xi_{\mathsf{a},k}$ is the submatrix of Ξ_{a} consisting of the lows corresponding to $\dot{\tilde{x}}_{a,k}$. Note that, due to the specific form of A_{q_k}, B_{q_k}, and C_{q_k}, (6.2.16) can be rewritten as

$$\begin{bmatrix} \tilde{y}^{(1)}_{\mathsf{a},k} \\ \vdots \\ \tilde{y}^{(q_k-1)}_{\mathsf{a},k} \\ \tilde{y}^{(q_k)}_{\mathsf{a},k} \end{bmatrix} = \begin{bmatrix} \tilde{x}_{\mathsf{a},k,2} \\ \vdots \\ \tilde{x}_{\mathsf{a},k,q_k} \\ \tilde{u}^c_{\mathsf{a},k} + w_{\mathsf{a},k} \end{bmatrix} + \underbrace{F_{\mathsf{a},k}\tilde{y}_{\mathsf{a}} + \Xi_{\mathsf{a},k}u}_{\text{known to } i\text{-th agent}}. \tag{6.2.17}$$

By combining the equality $\tilde{y}_{\mathsf{a},k} = \tilde{x}_{\mathsf{a},k,1}$ and the upper $q_k - 1$ equalities of (6.2.17), $\tilde{x}_{\mathsf{a},k}$ is given as a linear combination of the derivatives of $\tilde{y}_{\mathsf{a},k}$, $\tilde{y}_{\mathsf{a}}$, and u as follows

$$\begin{bmatrix} \tilde{x}_{\mathsf{a},k,1} \\ \tilde{x}_{\mathsf{a},k,2} \\ \vdots \\ \tilde{x}_{\mathsf{a},k,q_k} \end{bmatrix} = \begin{bmatrix} \tilde{y}_{\mathsf{a},k} \\ \tilde{y}^{(1)}_{\mathsf{a},k} \\ \vdots \\ \tilde{y}^{(q_k-1)}_{\mathsf{a},k} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ I_{q_k-1} & 0 \end{bmatrix} (F_{\mathsf{a},k}\tilde{y}_{\mathsf{a}} + \Xi_{\mathsf{a},k}u). \tag{6.2.18}$$

By repeating this process for all $k = 1, \cdots, \rho$, the set of matrices $G_{i,0}$, $G_{i,1}$, $\cdots$, G_{i,δ_i}, and $H_{i\mathsf{a}}$ can be obtained, which yields (6.2.13).

Second, the error dynamics (6.2.14) can be computed from (6.2.12) and (6.2.15) by setting $H_{i\mathsf{b}} = \Xi_{i\mathsf{b}}$ and $H_{i\mathsf{c}} = \Xi_{i\mathsf{c}}$. □

Remark 6.2.1. Note that for $k = 1, 2, \cdots, \rho_i$, the constant q_k is the relative degree of the k-th integral chain that starts from the input $\tilde{u}^c_{i\mathsf{a},k}$ and ends with the output $\tilde{y}_{\mathsf{ia},k}$. Therefore, if the maximum relative degree of every ρ_i input-output chain of $\tilde{x}_{i\mathsf{a}}$ is 1, then $\xi_{i\mathsf{a}}$ can be obtained algebraically without differentiating y_i. Otherwise, the partial observer (6.2.12) requires the differentiation of the output y_i. $\Diamond$

In summary, we show that, by measuring the local output y_i and the local input u_i under the unknown input u_i^c, the partial state $[\tilde{x}_{i\mathsf{a}}^T\, \tilde{x}_{i\mathsf{b}}^T\, \tilde{x}_{i\mathsf{c}}^T]^T$ can be reconstructed while the rest, i.e.,$[\tilde{x}_{i\mathsf{a}}^T\, \tilde{x}_{i\mathsf{b}}^T\, \tilde{x}_{i\mathsf{c}}^T]^T$, are not. Consequently, we can say that $[\tilde{x}_{i\mathsf{a}}^T\, \tilde{x}_{i\mathsf{b}}^T\, \tilde{x}_{i\mathsf{c}}^T]^T$ is the maximum amount of information about the state that can be estimated by using the local measurement y_i and the locally accessible input u_i regardless of the value of the plant input in the coordinate frame given in Lemma 6.2.1.

6.3 Practical Distributed State Observer for Plants with Input

In the previous part, it is shown that the i-th agent itself can estimate the obtainable part of the plant state $[\tilde{x}_{i,\mathsf{a}}^T, \tilde{x}_{i,\mathsf{b}}^T, \tilde{x}_{i,\mathsf{c}}^T]^T$ by utilizing its known input u_i and output y_i. In this part, we propose a distributed observer whose local observer consists of the partial observer and the agent-wise diffusive coupled part and show the error converges to zero in a practical sense.

For each $i \in \mathcal{N}$, let us denote ν_i as the dimension of $\mathcal{U}_i$, the unstable unobservable subspace of the triplet (C_i, A, B_i^c). Set $U_i \in \mathbb{R}^{n\times\nu_i}$ as a matrix whose columns form an orthonormal basis of $\mathcal{U}_i$, the unstable unobservable subspace of the triplet (C_i, A, B_i^c), and $O_i \in \mathbb{R}^{n\times(n-\nu_i)}$ as a matrix whose columns are orthonormal basis of $\mathcal{U}_i^\perp$. Due to their constructions, the concatenated matrix $[U_i\, O_i] \in \mathbb{R}^{n\times n}$ is orthonormal. Moreover, since $\mathcal{U}_i$ is spanned by the columns of $[P_{i\mathsf{d}}\, P_{i\mathsf{e}}]$ according to Lemma 6.2.1, there are nonsingular matrices $S_{i\mathsf{u}} \in \mathbb{R}^{\nu_i\times\nu_i}$ and $S_{i\mathsf{o}} \in \mathbb{R}^{(n-\nu_i)\times(n-\nu_i)}$, and a matrix $S_{i\mathsf{r}} \in \mathbb{R}^{\nu_i\times(n-\nu_i)}$ such that

$$\begin{aligned}\begin{bmatrix} P_{i\mathsf{d}} & P_{i\mathsf{e}} \end{bmatrix} &= U_i S_{i\mathsf{u}},\\ \begin{bmatrix} P_{i\mathsf{a}} & P_{i\mathsf{b}} & P_{i\mathsf{c}} \end{bmatrix} &= O_i S_{i\mathsf{o}} + U_i S_{i\mathsf{r}},\end{aligned} \tag{6.3.1}$$

where $P_{i\mathsf{a}}$, $P_{i\mathsf{b}}$, $P_{i\mathsf{c}}$, $P_{i\mathsf{d}}$, and $P_{i\mathsf{e}}$ are the submatrices of the coordinate change P_i defined in Lemma 6.2.1. Define a nonsingular matrix $T_i := [O_i S_{i\mathsf{o}} \ U_i] \in \mathbb{R}^{n\times n}$ and decompose x_i into the obtainable part $\bar{x}_{i\mathsf{o}} \in \mathbb{R}^{(n-\nu_i)}$ and the unobtainable part $\bar{x}_{i\mathsf{u}} \in \mathbb{R}^{nu_i}$ such that

$$\begin{bmatrix} \bar{x}_{i\mathsf{o}} \\ \bar{x}_{i\mathsf{u}} \end{bmatrix} := T_i^{-1} x = \begin{bmatrix} S_{i\mathsf{o}}^{-1} O_i^T \\ U_i^T \end{bmatrix} x \tag{6.3.2}$$

By applying the coordinate change (6.3.2) to the plant (6.1.1), the dynamics of $\bar{x}_{i\mathsf{u}}$ is given by

$$\dot{\bar{x}}_{i\mathsf{u}} = \bar{A}_{ir}\bar{x}_{i\mathsf{o}} + \bar{A}_{i\mathsf{u}}\bar{x}_{i\mathsf{u}} + U_i^T B u \tag{6.3.3}$$

where $\bar{A}_{ir} \in \mathbb{R}^{\nu_i\times(n-\nu_i)}$ and $\bar{A}_{i\mathsf{u}} \in \mathbb{R}^{\nu_i\times\nu_i}$ denote the lower blocks of $T_i^{-1}AT_i$ corresponding to $\dot{\bar{x}}_{i\mathsf{u}}$ with the appropriate dimension.

Based on theses settings, let us propose a distributed observer for the system given by (6.1.1) and (6.1.2) with the communication network $\mathcal{G}$. It consists of N local observers, one for each agent, and each local observer of the i-th agent, say "i-th local observer", consists of

- the partial observer (6.2.12) that produces the estimate $\xi_i \in \mathbb{R}^{(n-\nu_i)}$ for its obtainable part $\bar{x}_{i\mathsf{o}}$,

- the dynamics using agent-wise decomposed diffusive coupling to estimate the unobtainable part $\bar{x}_{i\mathsf{u}}$, which is given by

$$\dot{\hat{x}}_{i\mathsf{u}} = \bar{A}_{ir}\xi_i + \bar{A}_{i\mathsf{u}}\hat{x}_{i\mathsf{u}} + U_i^T B_i u_i + \gamma_i U_i^T \sum_{j\in\mathcal{N}_i} (\hat{x}_j - \hat{x}_i) \tag{6.3.4}$$

  where $\mathcal{N}_i$ denotes the neighboring agents of the i-th agent, U_i and O_i are the matrices defined in (6.3.1), and γ_i is the scalar coupling gain of the i-th agent, and

- the algebraic computation of the estimate of the plant state $\hat{x}_i$ by combining

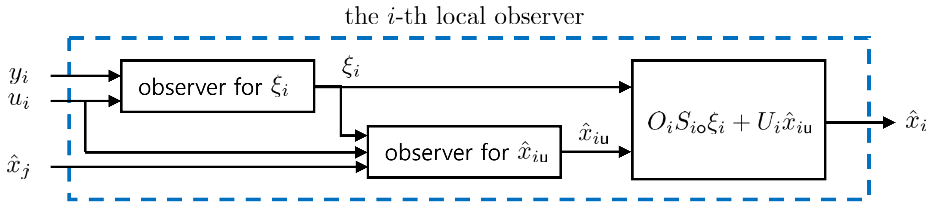


Figure 6.1: The structure of the i-th local observer is illustrated. Here, y_i and u_i are the part of the plant state and the part of the plant input measured by i-th agent, and $\hat{x}_j$ represents the neighboring agent's state estimate received through the communication. The observer for ξ_i indicates the dynamic equation (6.2.12) and one for $\hat{x}_{iu}$ indicates the equation (6.3.4).

the two partial estimates ξ_i and $\hat{x}_{i\mathsf{u}}$:

$$\hat{x}_i = O_i S_{i\mathsf{o}} \xi_i + U_i \hat{x}_{i\mathsf{u}}. \tag{6.3.5}$$

The structure of the i-th local observer is illustrated in Fig. 6.1. Notice that, in (6.3.5), the estimate ξ_i occupies the place where the obtainable partial state $\bar{x}_{i\mathsf{o}}$ is in (6.3.2). As expected from what ξ_i estimates in Lemma 6.2.2, it holds that

$$\bar{x}_{i\mathsf{o}} = \begin{bmatrix} \tilde{x}_{i\mathsf{a}} \\ \tilde{x}_{i\mathsf{b}} \\ \tilde{x}_{i\mathsf{c}} \end{bmatrix}, \tag{6.3.6}$$

which can be checked by looking at the upper part of the following equality:

$$\begin{aligned} \begin{bmatrix} \bar{x}_{i\mathsf{o}} \\ \bar{x}_{i\mathsf{u}} \end{bmatrix} &= T_i^{-1} x = T_i^{-1} P_i \tilde{x}_i = T_i^{-1} [O_i S_{i\mathsf{o}} \, U_i] \begin{bmatrix} I & 0 \\ S_{i\mathsf{r}} & S_{i\mathsf{u}} \end{bmatrix} \tilde{x}_i \\ &= T_i^{-1} T_i \begin{bmatrix} I & 0 \\ S_{i\mathsf{r}} & S_{i\mathsf{u}} \end{bmatrix} \tilde{x}_i = \begin{bmatrix} I & 0 \\ S_{i\mathsf{r}} & S_{i\mathsf{u}} \end{bmatrix} \tilde{x}_i. \end{aligned}$$

Here, in the third equality, the equality (6.3.1) is used in the form of

$$P_i = [O_i S_{i\mathsf{o}} \, U_i] \begin{bmatrix} I & 0 \\ S_{i\mathsf{r}} & S_{i\mathsf{u}} \end{bmatrix}.$$

Let $\eta_i := x - \hat{x}_i$ for $i \in \mathcal{N}$ be the estimation error of the i-th observer and let us decompose η_i into the obtainable part $\eta_{i\mathsf{o}}$ and the unobtainable part $\eta_{i\mathsf{u}}$ such that

$$\begin{bmatrix} \eta_{i\mathsf{o}} \\ \eta_{i\mathsf{u}} \end{bmatrix} := T_i^{-1}\eta_i, \tag{6.3.7}$$

where $\eta_{i\mathsf{o}} \in \mathbb{R}^{n-\nu_i}$ and $\eta_{i\mathsf{u}} \in \mathbb{R}^{\nu_i}$. Then from (6.3.2) and (6.3.5) it holds that

$$\begin{bmatrix} \eta_{i\mathsf{o}} \\ \eta_{i\mathsf{u}} \end{bmatrix} = T_i^{-1}\eta_i = T_i^{-1}x - T_i^{-1}\hat{x}_i = \begin{bmatrix} \bar{x}_{i\mathsf{o}} - \xi_i \\ \bar{x}_{i\mathsf{u}} - \hat{x}_{i\mathsf{u}} \end{bmatrix}. \tag{6.3.8}$$

Given the equality (6.3.6), the behavior of $\eta_{i\mathsf{o}}$ is perfectly described by Lemma 6.2.2, and it is exponentially stable. On the other hand, from (6.3.3) with (6.1.4), (6.3.4), (6.3.7), and (6.3.8), the error dynamics of $\eta_{i\mathsf{u}}$ becomes

$$\begin{aligned} \dot{\eta}_{i\mathsf{u}} &= \dot{\bar{x}}_{i\mathsf{u}} - \dot{\hat{x}}_{i\mathsf{u}} \\ &= \bar{A}_{i\mathsf{r}}\bar{x}_{i\mathsf{o}} + \bar{A}_{i\mathsf{u}}\bar{x}_{i\mathsf{u}} + U_i^T(B_iu_i + B_i^c u_i^c) \\ &\quad - \bar{A}_{i\mathsf{r}}\xi_i - \bar{A}_{i\mathsf{u}}\hat{x}_{i\mathsf{u}} - U_i^T B_i u_i - \gamma_i U_i^T \sum_{j\in\mathcal{N}_i}(\hat{x}_j - \hat{x}_i) \qquad (6.3.9) \\ &= \bar{A}_{i\mathsf{r}}\eta_{i\mathsf{o}} + \bar{A}_{i\mathsf{u}}\eta_{i\mathsf{u}} - \gamma_i U_i^T \sum_{j=1}^{N} l_{ij}(O_j S_{j\mathsf{o}}\eta_{j\mathsf{o}} + U_j\eta_{j\mathsf{u}}) \\ &\quad + U_i^T B_i^c u_i^c. \qquad (6.3.10) \end{aligned}$$

where l_{ij} are the (i, j)-th entries of the Laplacian matrix $\mathcal{L}(\mathcal{G})$. Since $\eta_{i\mathsf{o}}$ exponentially converges to zero according to Lemma 6.2.2, the stability of the estimation error depends on the stability of $\eta_{i\mathsf{u}}$ dynamics.

Let $\eta_{\mathsf{o}} := \mathrm{col}(\eta_{i\mathsf{o}})_{i\in\mathcal{N}}$ and $\eta_{\mathsf{u}} := \mathrm{col}(\eta_{i\mathsf{u}})_{i\in\mathcal{N}}$ be the concatenated errors of the obtainable part and the unobtainable part, respectively. Then the concatenated error dynamics of unobtainable part is of the form

$$\dot{\eta}_{\mathsf{u}} = \bar{A}_{\mathsf{r}}\eta_{\mathsf{o}} + \bar{A}_{\mathsf{u}}\eta_{\mathsf{u}} - \Gamma U^T(\mathcal{L}(\mathcal{G}) \otimes I_n)(OS_{\mathsf{o}}\eta_{\mathsf{o}} + U\eta_{\mathsf{u}}) + U^T B^c u^c, \tag{6.3.11}$$

where $\bar{A}_{\mathsf{r}}$, $\bar{A}_{\mathsf{u}}$, O, U, S_{o}, Γ, and B^c are the block diagonal matrices of $\bar{A}_{i\mathsf{r}}$, $\bar{A}_{i\mathsf{u}}$,

O_i, U_i, $S_{i\mathsf{o}}$, $\gamma_i I_{\nu_i}$, and B_i^c for $i \in \mathcal{N}$, respectively, and $u^c := \mathrm{col}(u_i^c)_{i\in\mathcal{N}}$.

As a result of employing agent-wise diffusive coupling in (6.3.4), the η_{o} dynamics is coupling-free and the term $\Gamma U^T(\mathcal{L}(\mathcal{G})\otimes I_n)U\eta_{\mathsf{u}}$ emerges in η_{u} dynamics. Since η_{o} is stable and Γ is noting but a diagonal positive matrix, the stability of the matrix $U^T(\mathcal{L}(\mathcal{G})\otimes I_n)U$ leads to the stable η_{u} dynamics if there were no input term. Unfortunately, the unknown input can not be eliminated even with the stable $U^T(\mathcal{L}(\mathcal{G})\otimes I_n)U$ but it can be attenuated by assigning appropriately large coupling gains to Γ. To this end, at least the matrix $U^T(\mathcal{L}(\mathcal{G})\otimes I_n)U$ should be stable, and the condition for its stability can be directly obtained from Lemma 4.3.3.

Lemma 6.3.1. Suppose that the communication network $\mathcal{G}$ is strongly connected. Then the matrix $U^T(\hat{\mathcal{L}}(\mathcal{G})\otimes I_n)U$ is positive definite if and only if the intersection of all agent's unstable weakly unobservable subspaces is trivial, i.e.,

$$\cap_{i=1}^N \mathcal{U}_i = \{0\}, \tag{6.3.12}$$

where $\mathcal{U}_i$ is the unstable weakly unobservable subspace of the triplet (C_i, A, B_i^c). $\Diamond$

Now we show that the proposed distributed observer estimates the plant state in a practical sense.

Theorem 6.3.2. Suppose that the plant's input is bounded by $\bar{u}$, i.e., $|u(t)| \le \bar{u}$, for $t \ge 0$, and suppose that the communication network $\mathcal{G}$ is strongly connected and the condition (6.3.12) holds. If the proposed observer consists of (6.2.12), (6.3.4), and (6.3.5) is designed such that, for all $i \in \mathcal{N}$, L_i is chosen to satisfy $\mathsf{a}(\tilde{A}_{i\mathsf{bb}} - L_i\tilde{C}_{i\mathsf{b}}) < -\beta$, $\mathsf{a}(\tilde{A}_{i\mathsf{cc}}) < -\beta$ holds, and γ_i is chosen to satisfy

$$\gamma_i > 4(\max_j \|\bar{A}_{j\mathsf{u}}\| + \beta)/\lambda^*, \quad \forall i \in \mathcal{N}, \tag{6.3.13}$$

where $\lambda^* := \lambda_{\min}(U^T(\hat{\mathcal{L}}(\mathcal{G})\otimes I_n)U) > 0$, then the concatenated estimation error

$\eta := \mathrm{col}(\eta_i)_{i\in\mathcal{N}}$ satisfies that

$$|\eta(t)| \le c|\eta(0)|e^{-\beta t} + \max_i \|B_i^c\|\bar{u}\sqrt{\frac{2N\mathsf{r}}{\underline{\theta}\lambda^*\beta\underline{\gamma}}}, \quad \forall t \ge 0, \tag{6.3.14}$$

for some constant $c > 0$, where $\underline{\gamma} := \min \gamma_i$, $\mathsf{r} := \max \gamma_i / \min \gamma_i$, and $\underline{\theta} := \min \theta_i$ with $\theta := [\theta_1, \theta_2, \cdots, \theta_N]^T$ being the positive vector obtained from Lemma 2.2.2. $\lozenge$

Proof. Note from (6.3.6) and (6.3.8) that $\tilde{x}_{i\mathsf{a}} - \xi_{i\mathsf{a}}$ is an element of $\eta_{i\mathsf{o}}$, but is identically zero according to Lemma 6.2.2. Therefore, without loss of generality, we consider $\eta_{i\mathsf{o}}$ as the variable that consists of $\eta_{i\mathsf{b}}$ and $\eta_{i\mathsf{c}}$ defined in Lemma 6.2.2 and that obeys the error dynamics (6.2.14).

First, let us consider a Lyapunov function candidate

$$V(\eta) := \sum_{i=1}^{N}\left(\mu\eta_{i\mathsf{o}}^T\Phi_{i\mathsf{o}}\eta_{i\mathsf{o}} + \theta_i\frac{\underline{\gamma}}{\gamma_i}|\eta_{i\mathsf{u}}|^2\right), \tag{6.3.15}$$

where, for each $i \in \mathcal{N}$, the i-th agent's estimation error η_i is decomposed into its obtainable part $\eta_{i\mathsf{o}}$ and the unobtainable part $\eta_{i\mathsf{u}}$ by the definition (6.3.7), $\Phi_{i\mathsf{o}}$ is the positive definite matrix satisfying the following Lyapunov equation

$$\Phi_{i\mathsf{o}}\left(\begin{bmatrix}\tilde{A}_{i\mathsf{bb}} - L_i\tilde{C}_{i\mathsf{b}} & 0\\ 0 & \tilde{A}_{i\mathsf{cc}}\end{bmatrix} + \beta I\right) + \left(\begin{bmatrix}\tilde{A}_{i\mathsf{bb}} - L_i\tilde{C}_{i\mathsf{b}} & 0\\ 0 & \tilde{A}_{i\mathsf{cc}}\end{bmatrix} + \beta I\right)^\top \Phi_{i\mathsf{o}} = -I,$$

and μ is a positive constant to be determined. Then, the time derivative of V along (6.2.14) and (6.3.10) becomes

$$\begin{aligned}\dot{V} = &-2\beta\mu\eta_{\mathsf{o}}^T\Phi_{\mathsf{o}}\eta_{\mathsf{o}} - \mu|\eta_{\mathsf{o}}|^2 + 2\eta_{\mathsf{u}}^T\left[\mathrm{diag}(\theta_i\frac{\underline{\gamma}}{\gamma_i}I_{\nu_i})_{i\in\mathcal{N}}\right]\bar{A}_{\mathsf{r}}\eta_{\mathsf{o}}\\ &+ 2\eta_{\mathsf{u}}^T\left[\mathrm{diag}(\theta_i\frac{\underline{\gamma}}{\gamma_i}I_{\nu_i})_{i\in\mathcal{N}}\right]\bar{A}_{\mathsf{u}}\eta_{\mathsf{u}} + 2\eta_{\mathsf{u}}^T\left[\mathrm{diag}(\theta_i\frac{\underline{\gamma}}{\gamma_i}I_{\nu_i})_{i\in\mathcal{N}}\right]U^TB^cu^c\\ &- 2\underline{\gamma}\eta_{\mathsf{u}}^TU^T(\Theta\mathcal{L}\otimes I_n)OS_{\mathsf{o}}\eta_{\mathsf{o}} - \underline{\gamma}\eta_{\mathsf{u}}^TU^T(\hat{\mathcal{L}}\otimes I_n)U\eta_{\mathsf{u}},\end{aligned} \tag{6.3.16}$$

where $\Phi_{\mathsf{o}} := \mathrm{diag}(\Phi_{i\mathsf{o}})_{i\in\mathcal{N}}$. Since $0 < \theta_i \le 1$ holds from Lemma 2.2.2, it is obtained that $\theta_i \frac{\underline{\gamma}}{\gamma_i} \leq 1$. Substituting these inequalities, it continues that

$$\begin{aligned}\dot{V} &\leq -2\beta\mu\eta_{\mathsf{o}}^T\Phi_{\mathsf{o}}\eta_{\mathsf{o}} - \mu|\eta_{\mathsf{o}}|^2 + 2|\eta_{\mathsf{u}}|\|\bar{A}_{\mathsf{r}}\||\eta_{\mathsf{o}}| + 2|\eta_{\mathsf{u}}||B^c u^c| \\ &\quad - (\lambda^*\underline{\gamma} - 2\|\bar{A}_{\mathsf{u}}\|)|\eta_{\mathsf{u}}|^2 + 2\underline{\gamma}|\eta_{\mathsf{u}}|\|\mathcal{L}\|\|S_{\mathsf{o}}\||\eta_{\mathsf{o}}|,\end{aligned}$$

where $\|U\| \leq 1$ and $\|O\| \leq 1$ are used, and λ^* is a positive constant as a result of Lemma 6.3.1. Applying Young's inequality to cross terms leads to

$$\begin{aligned}\dot{V} &\leq -2\beta\mu\eta_{\mathsf{o}}^T\Phi_{\mathsf{o}}\eta_{\mathsf{o}} - \mu|\eta_{\mathsf{o}}|^2 + \frac{\|\bar{A}_{\mathsf{r}}\|^2}{\epsilon}|\eta_{\mathsf{o}}|^2 + \epsilon|\eta_{\mathsf{u}}|^2 \\ &\quad - (\lambda^*\underline{\gamma} - 2\|\bar{A}_{\mathsf{u}}\|)|\eta_{\mathsf{u}}|^2 + \frac{4|B^c u^c|^2}{\lambda^*\underline{\gamma}} + \frac{\lambda^*\underline{\gamma}}{4}|\eta_{\mathsf{u}}|^2 \\ &\quad + \frac{\lambda_{\min}\underline{\gamma}}{4}|\eta_{\mathsf{u}}|^2 + \frac{4\underline{\gamma}\|\mathcal{L}\|^2\|S_{\mathsf{o}}\|^2}{\lambda_{\min}}|\eta_{\mathsf{o}}|^2 \\ &\leq -2\beta\mu\eta_{\mathsf{o}}^T\Phi_{\mathsf{o}}\eta_{\mathsf{o}} - \left(\mu - \frac{\|\bar{A}_{\mathsf{r}}\|^2}{\epsilon} - \frac{4\underline{\gamma}\|\mathcal{L}\|^2\|S_{\mathsf{o}}\|^2}{\lambda^*}\right)|\eta_{\mathsf{o}}|^2 \\ &\quad - \left(\frac{\lambda^*\underline{\gamma}}{2} - 2\|\bar{A}_{\mathsf{u}}\| - \epsilon\right)|\eta_{\mathsf{u}}|^2 + \frac{4N\max_i\|B_i^c\|^2\bar{u}^2}{\lambda^*\underline{\gamma}}.\end{aligned}$$

Set $\epsilon := \lambda^*\underline{\gamma}/2 - 2\|\bar{A}_{\mathsf{u}}\| - 2\beta$, which is positive by the condition (6.3.13). Also let us denote $\sigma_{\max}(S_{\mathsf{o}})$ as the maximum singular value of S_{o} and take

$$\mu := \max\left\{\frac{\|A_{\mathsf{r}}\|^2}{\epsilon} + \frac{4\underline{\gamma}\|\mathcal{L}\|^2\|S_{\mathsf{o}}\|^2}{\lambda^*}, \frac{\sigma_{\max}(S_{\mathsf{o}})^2}{\lambda_{\min}(\Phi_{\mathsf{o}})}\right\}, \tag{6.3.17}$$

so as to obtain

$$\dot{V} \leq -2\beta\mu\eta_{\mathsf{o}}^T\Phi_{\mathsf{o}}\eta_{\mathsf{o}} - 2\beta|\eta_{\mathsf{u}}|^2 + \frac{4N\max_i\|B_i^c\|^2\bar{u}^2}{\lambda^*\underline{\gamma}}.$$

After using $\theta_i \frac{\underline{\gamma}}{\gamma_i} \leq 1$ again, we have

$$\begin{aligned}\dot{V} &\leq -2\beta\mu\eta_{\mathsf{o}}^T\Phi_{\mathsf{o}}\eta_{\mathsf{o}} - 2\beta\left(\sum_{i=1}^N \theta_i\frac{\underline{\gamma}}{\gamma_i}|\eta_{i\mathsf{u}}|^2\right) + \frac{4N\max_i\|B_i^c\|^2\bar{u}^2}{\lambda^*\underline{\gamma}} \\ &= -2\beta V + \frac{4N\max_i\|B_i^c\|^2\bar{u}^2}{\lambda^*\underline{\gamma}}.\end{aligned}$$

From the comparison lemma, the convergence performance of $V(t)$ is given as follows

$$V(t) \leq V(0)e^{-2\beta t} + \frac{2N \max_i \|B_i^c\|^2 \bar{u}^2}{\beta \lambda^* \underline{\gamma}}, \quad \forall t \geq 0. \tag{6.3.18}$$

As a second step, let us show the convergence performance of η using the inequality (6.3.18). Since $\sigma_{\max}(S_{\mathsf{o}}) = \sigma_{\min}(S_{\mathsf{o}}^{-1})^{-1}$ and (6.3.17), the following inequalities hold

$$\begin{aligned} |S_{\mathsf{o}}\eta_{\mathsf{o}}|^2 \leq \mu \eta_{\mathsf{o}}^T \lambda_{\min}(\Phi_{\mathsf{o}}) \sigma_{\min}(S_{\mathsf{o}}^{-1})^2 |S_{\mathsf{o}}\eta_{\mathsf{o}}|^2 &\leq \mu \eta_{\mathsf{o}}^T \Phi_{\mathsf{o}} \eta_{\mathsf{o}} \\ &\leq \mu \lambda_{\max}(\Phi_{\mathsf{o}}) \sigma_{\max}(S_{\mathsf{o}}^{-1})^2 |S_{\mathsf{o}}\eta_{\mathsf{o}}|^2. \end{aligned} \tag{6.3.19}$$

On the other hand, by (6.3.7) and the orthonormality of the matrix $[O_i\, U_i]$ it holds that $|S_{\mathsf{o}}\eta_{\mathsf{o}}|^2 + |\eta_{\mathsf{u}}|^2 = |\eta|^2$. Consequently, substituting this equality, the inequality (6.3.19), and $\frac{\underline{\theta}}{\mathsf{r}} \leq \theta_i \frac{\underline{\gamma}}{\gamma_i} \leq 1$ into (6.3.15) results in the following bound of $V(\eta)$:

$$\frac{\underline{\theta}}{\mathsf{r}} |\eta|^2 \leq V(\eta) \leq \mu \frac{\lambda_{\max}(\Phi_{\mathsf{o}})}{\sigma_{\min}(S_{\mathsf{o}})^2} |\eta|^2, \tag{6.3.20}$$

Here, $\sigma_{max}(S_{\mathsf{o}}^{-1}) = \sigma_{\min}(S_{\mathsf{o}})^{-1}$, (6.3.19), and (6.3.17) are used to obtain the right inequality.

Lastly, from (6.3.18) and (6.3.20), the inequality (6.3.14) follows with the positive constant $c := \mu(\mathsf{r}/\underline{\theta})(\lambda_{\max}(\Phi_{\mathsf{o}})/\sigma_{\min}(S_{\mathsf{o}}))$. This completes the proof. □

Remark 6.3.1. Note from (6.3.14) that the stacked estimation error $\eta(t)$ eventually goes inside the tube with the radius of $\max_i \|B_i^c\| \bar{u} \sqrt{\frac{2N\mathsf{r}}{\underline{\theta}\lambda^* \beta \underline{\gamma}}}$. If the coupling gains are equal, then the ratio r becomes 1. Here the terms $\|B_i^c\|$, $\bar{u}$, N, $\underline{\theta}$, λ^* are given because those are determined by the plant and the communication network. Obviously the minimum of the coupling gain $\underline{\gamma}$ is assigned by the designer. The term β, however, is not always assignable. Since the pair $(\tilde{A}_{i\mathsf{bb}}, \tilde{C}_{i\mathsf{b}})$ is observable, one can freely choose $\mathsf{a}(\tilde{A}_{i\mathsf{bb}} - L_i \tilde{C}_{i\mathsf{b}})$ but $\mathsf{a}(\tilde{A}_{i\mathsf{cc}})$ is given. So as long as the $\tilde{x}_{i\mathsf{c}}$ part exists, the assignable values of β has a limit. It should be noticed that, in any case, the radius of the tube can be decreased as much as required with the sufficiently large coupling gains. ◊

6.4 Summary and Discussion

In this chapter, we considered linear time invariant plants with input, where the plant input is measured in a distributed manner as well as the state is, to accommodate more practical applications such as monitoring the large-scale systems with multiple input/output channels or the localization of the multi-agent systems with inputs. From the individual agent's perspective, this setting naturally led to the sub-problem of identifying the partial state reconstructible by agent itself using its own measurements despite the unknown input. The classical results about unknown input observer and the special coordinate basis decomposition were adopted to reveal the answer, which is the quotient of the unstable weakly unobservable subspace. As a result, a novel form of the distributed observer was proposed by employing the partial unknown input observer and the agent-wise decomposed diffusive coupling. Finally, it was shown in Theorem 6.3.2 that the proposed scheme practically achieves the estimation for the bounded input.

By using the term 'practically', we mean that the estimation goes to the vicinity of the plant state but the steady-state error can be shrunk as small as required by assigning sufficiently large coupling gains. On the other hand, the self-reconstructible part by the individual agent 'exactly converges' to the partial state. Therefore, the results of Theorem 6.3.2 can be extended by re-identifying the self-reconstructible part as the part of the state that can be 'practically' reconstructed by the agent itself by using its own measurements.

Chapter 7

Application to Distributed Multi-robot Localization

Precise localization is one of the main requirements for mobile robot autonomy. To cope with the limitations of self-localization, the error accumulation in case of the long travel, the group localization has gained attention as a solution (see, for example, [RB02], [TL05], [ZR06], [How06], and the references therein.)

Here, we specify the *multi-robot localization problem* which is addresses in [RB02]. First, we state the following assumptions:

i) a group of N independent robots move in an planar (2-dimensional Euclidian) space. The motion of each robot is described by its own equation of motion;

ii) each robot carries proprioceptive and exteroceptive sensing devices in order to propagate and update its own position estimate. These sensors measure the self motion of the robot and monitor the environment for localization features such as landmarks;

iii) each robot also carries exteroceptive sensors that allow it ti detect and identify other robots moving in its vicinity and measure their respective displacement (relative position);

iv) all robots are equipped with communication devices that allow exchange of information within the group.

The problem is to determine a principled way to exploit the information exchanged during the interactions among members of a group. Furthermore, finding a solution that allows distributed processing is called a *distributed multi-robot localization problem*.

In this chapter, we interpret this multi-robot localization problem as a distributed observer design problem and propose a solution by applying the distributed observer design schema described in the previous chapters.

7.1 Problem Statement

There are two cases of interest regarding the convergence performance of the localization;

1. none of the robots has absolute positioning capabilities, and

2. at least one of the robots has absolute positioning capabilities.

Regarding the first case, it is easy to notice that the entire system collected over the whole member of the group is not observable because the fixed relative position has a additional degree of freedom with respect to the global position. In this case, moreover, the redundant degree of freedom is exactly same as one mobile robot's., which becomes the undetectable dynamics seen in the proof of Theorem 4.3.5. As a result, the estimation error diverges as time goes, so, researches focus on reducing the estimation error from the the stochastic point of view compared to the case without employing the algorithm.

Unlike the first case, the second case is observable, so the estimation error converges and it allows more reliable localization. In this chapter, we consider the second case, which is suitable for the distributed observer design. We assume followings.

- We consider a set of N mobile robots and assume that each i-th robot's dynamics is described by the double-integrator model such that

$$\dot{p}_i = v_i \quad \dot{v}_i = a_i \tag{7.1.1}$$

where $p_i := [p_i^x, p_i^y]^T \in \mathbb{R}^2$ is the position, $v_i := [v_i^x, v_i^y]^T \in \mathbb{R}^2$ is the velocity, and $a_i := [a_i^x, a_i^y]^T \in \mathbb{R}^2$ is the acceleration.

- As mentioned above, there are two types of sensing capabilities of robots; absolute and relative. In this problem we suppose that at least one of the robots has a capability of measuring absolute position. Robots also can measure the relative position of nearby robots. For example, it i-th robot is near the j-th robot, then j-th robot measures $p_i - p_j$.

- The communication network among robots are described by the graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$, where $\mathcal{N} = 1, 2, \cdots, N$.

Distributed multi-robot localization problem: design an algorithm that is implemented in each robot to enable every robot recovers its own position in a distributed manner.

This problem can be cast into distributed estimation problem with the plant of the form (6.1.1) with state $x := \mathrm{col}(p_1, v_1, p_2, v_2, \cdots, p_N, v_N)$ and input $u := \mathrm{col}(a_1, a_2, \cdots, a_N)$, where

$$A = I_N \otimes \begin{bmatrix} 0 & I_2 \\ 0 & 0 \end{bmatrix}, \quad B = I_N \otimes \begin{bmatrix} 0 \\ I_2 \end{bmatrix}. \tag{7.1.2}$$

On the other hand, the relative position of the i-th robot from j-th robot can be represented by

$$p_i - p_j = \begin{array}{c} [\, 0 \cdots 0 \quad \underset{i-\mathrm{th}}{I_2} \quad 0 \cdots 0 \quad \underset{j-\mathrm{th}}{-I_2} \quad 0 \cdots 0 \,] x, \end{array} \tag{7.1.3}$$

while the absolute position of i-th robot can be represented by

$$p_i = [\, 0 \cdots 0 \quad \underset{i-\mathrm{th}}{I_2} \quad 0 \cdots 0 \,] x. \tag{7.1.4}$$

In this interpretation each i-th robot corresponds to the i-th agent (sensor) in the distributed estimation settings. Therefore the communication network $\mathcal{G}$ among robots is same as the communication network among agents.

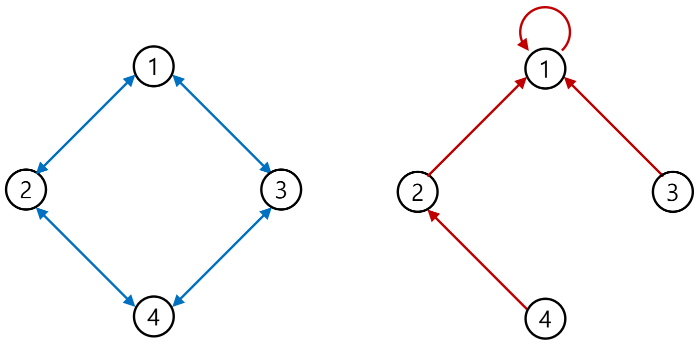


Figure 7.1: The communication network among robots (left) and the robots' absolute and relative position measurements (right) in the considered scenario are illustrated.

7.2 Localization for Robots Moving under Unforced Condition

In this part, we suppose that the robot moves in an unforced condition, that is their acceleration is zero such that

$$\text{(unfored condition:)} \qquad a_i = 0, \quad i = 1, 2, 3, 4, \tag{7.2.1}$$

so that the plant (7.1.2) becomes autonomous.

For simulation we consider the following scenario.

- We consider $N = 4$ robots and the communication topology $\mathcal{G}$ is given by a ring shaped undirected graph. As a result, the Laplacian $\mathcal{L}(G)$ has the following value:

$$\mathcal{L}(\mathcal{G}) = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}. \tag{7.2.2}$$

- We assume that the 1st robot measures its own position, while the others measures only the relative position. In detail, the measurement matrix C_i

of the i-th agent is given as follows

$$C_i = S_i \otimes I_2, \quad i = 1, 2, 3, 4, \tag{7.2.3}$$

Here, the matrices S_is are defined as follows

$$\begin{aligned} S_1 &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \\ S_2 &= \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}, \\ S_3 &= \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}, \\ S_4 &= \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix}, \end{aligned}$$

which represent agent's sensing capabilities.

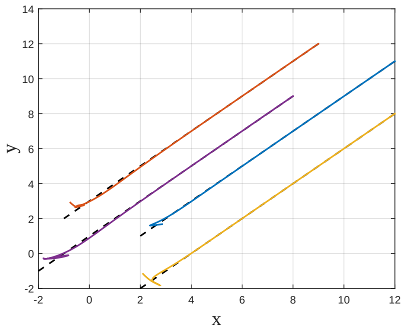
The communication networks and the robots' sensing capabilities are depicted in Fig. 7.1.

7.2.1 Approach using Distributed Observer with Static Coupling Gain

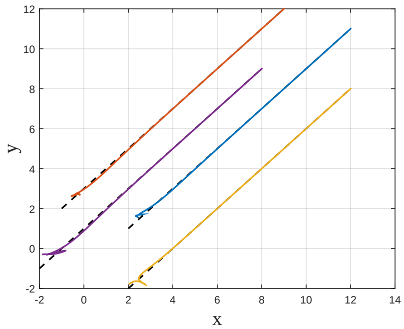
The plant of the form (6.1.1) with state $x := \mathrm{col}(p_1, v_1, p_2, v_2, \cdots, p_N, v_N)$, where A is defined in (7.1.2), the measurements of (7.2.3), and the communication network (7.2.2) are considered. The distributed observer (4.3.1) with the constant coupling gain is constructed and the i-th local observer, which runs in the i-th robot, generates the estimate $\hat{x}_i$. The simulation results are given as follows.

The position of robots $p_1(t)$, $p_2(t)$, $p_3(t)$, and $p_4(t)$ and the estimates of all local observers are depicted in Fig. 7.2. Since the conditions of the Theorem 4.3.6 are met, every estimate of each agent converges to the true position of the all robots as expected.

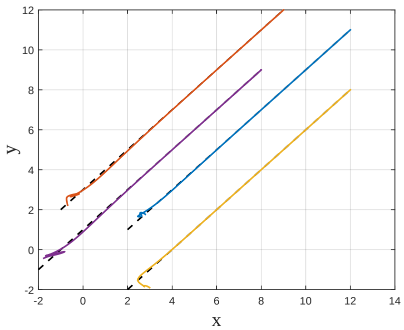
The convergence of the estimate can be measured by checking the concatenated estimation error $\eta := \mathrm{col}(x - \hat{x}_i)_{i \in \mathcal{N}}$, which converges to zero as can be seen from Fig. 7.3.



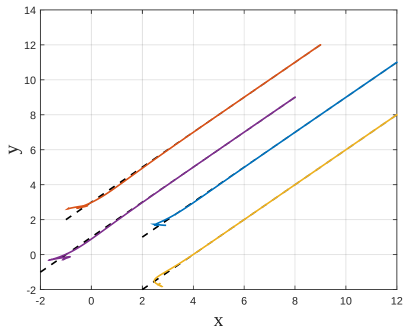
(a) Estimate from robot 1



(b) Estimate from robot 2



(c) Estimate from robot 3



(d) Estimate from robot 4

Figure 7.2: The position of robots $p_1(t)$, $p_2(t)$, $p_3(t)$, and $p_4(t)$ are depicted as black dashed curve and the estimated position of the all local observers generated by each i-th local distributed observers with constant coupling gains of the form (4.3.1) running in each i-th robot are depicted for $i = 1, 2, 3, 4$ as colored solid curves.

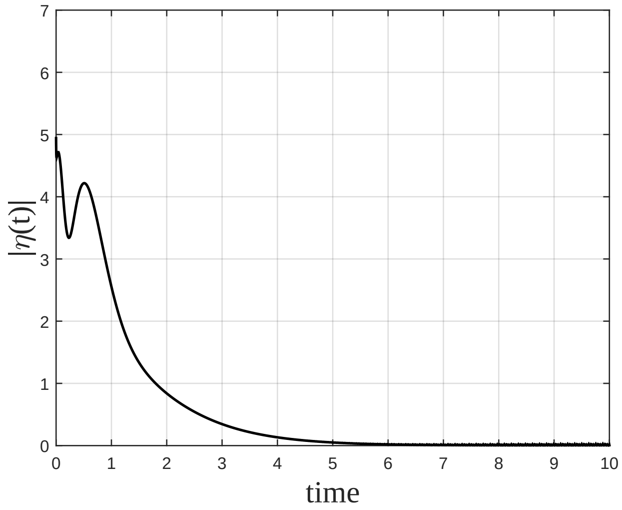


Figure 7.3: The norm of the concatenated estimation $|\eta(t)|$ is depicted.

7.2.2 Approach using Fully Distributed Observer with Adaptive Coupling Gain

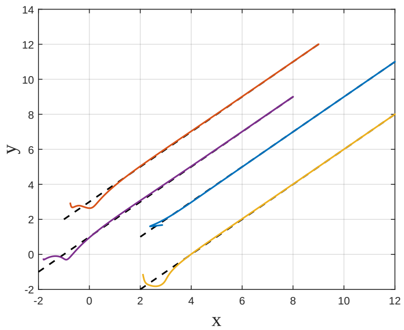
Under the same settings in the previous part, the distributed observer (5.2.1) with the adaptive coupling gain is constructed and the i-th local observer, which runs in the i-th robot, generates the estimate $\hat{x}_i$. The simulation results are given as follows.

The position of robots $p_1(t)$, $p_2(t)$, $p_3(t)$, and $p_4(t)$ and the estimates of all local observers are depicted in Fig. 7.4. Compared to Fig. 7.2, one can notice from Fig. 7.4 that the estimate of adaptive distributed observer converges slowly.

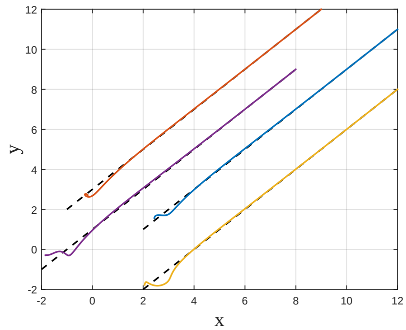
The time-varying coupling gains, γ_i for $i = 1, 2, 3, 4$, are plotted in Fig. 7.5. As mentioned in Theorem 5.2.2, one can confirm that the coupling gain converges to a constant from Fig. 7.5.

7.3 Localization for Robots Moving under Forced Condition

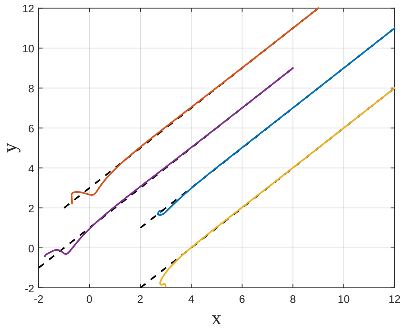
In this part we do not assume unforced condition. The plant is represented by the form (6.1.1), where A and B are given in (7.1.2). The measurements of



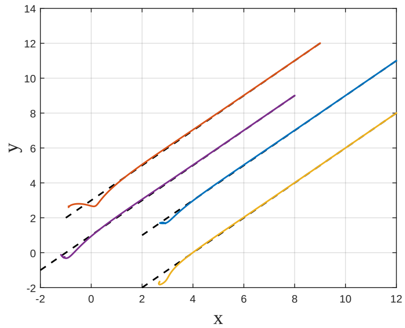
(a) Estimate from robot 1



(b) Estimate from robot 2



(c) Estimate from robot 3



(d) Estimate from robot 4

Figure 7.4: The position of robots $p_1(t)$, $p_2(t)$, $p_3(t)$, and $p_4(t)$ are depicted as black dashed curve and the estimated position of the all local observers generated by each i-th local distributed observers with adaptive coupling gains of the form (5.2.1) running in each i-th robot are depicted for $i = 1, 2, 3, 4$ as colored solid curves.

the i-th agent (robot) are assumed to be two kinds: its own acceleration and the relative or absolute position estimates given in Section 7.2. As a result, the total measurements of i-th agent is of the form (6.1.2) where C_i is same as one in (7.2.3) and D_i is

$$D_i = E_i \otimes I_2$$

with E_i is the matrix whose ii-entry is one and of which the rest are zero. The communication topology are same as those in Section 7.2. In specific, the input

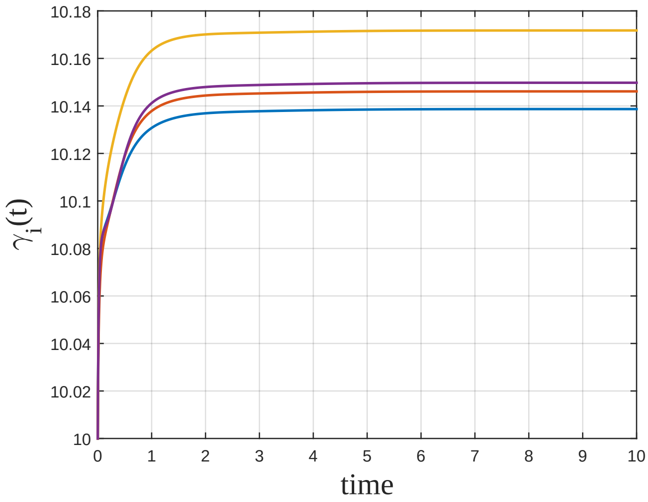


Figure 7.5: The adaptive gains of the local distributed observers, i.e., $\gamma_1(t)$, $\gamma_2(t)$, $\gamma_3(t)$, and $\gamma_4(t)$ in (5.2.2), are depicted.

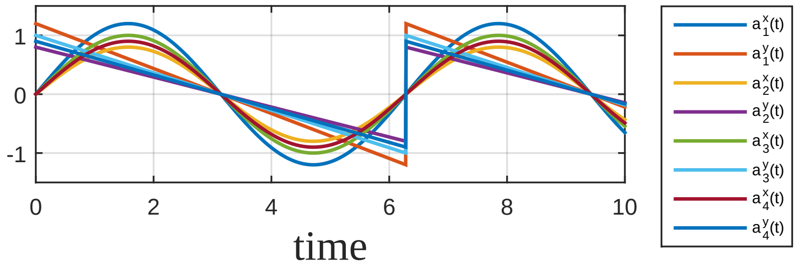


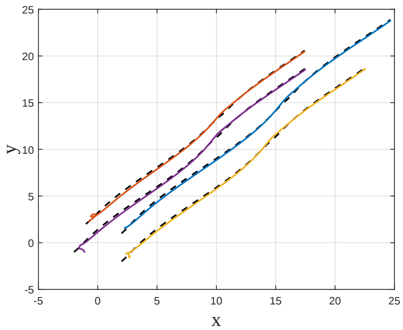
Figure 7.6: The input signals adopted in the simulation.

$u = \mathrm{col}(a_1, a_2, a_3, a_4)$ is the acceleration of the robots and the one illustrated in Fig. 7.6 is adopted in this simulation.

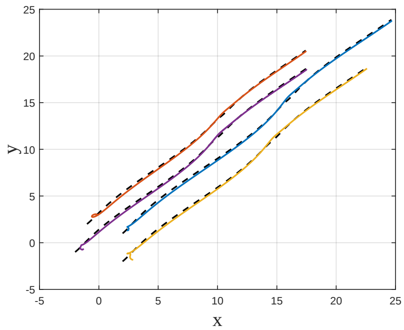
Before providing the main simulation results, we first check the effect of the plant input to the distributed observers with constant coupling gains designed in Section 7.2.1.

The estimation results of applying the distribute observer (4.3.1) with constant coupling gain of 100 to the plant (6.1.1) with input of Fig. 7.6 is depicted in Fig. 7.7.

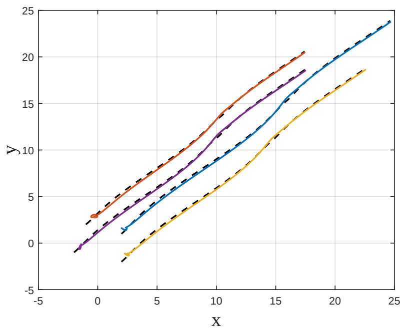
The detailed estimation performance can be seen from Fig. 7.8. Compared to



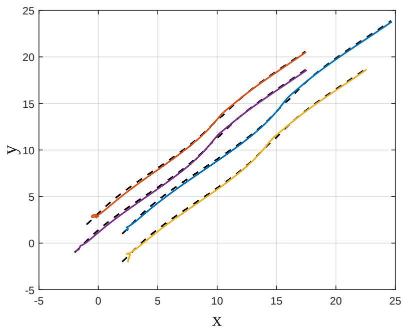
(a) Estimate from robot 1



(b) Estimate from robot 2



(c) Estimate from robot 3



(d) Estimate from robot 4

Figure 7.7: The position of robots $p_1(t)$, $p_2(t)$, $p_3(t)$, and $p_4(t)$ are depicted as black dashed curve and the estimated position of the all local observers generated by each i-th local distributed observers (4.3.1) with constant coupling gains of $\gamma_i = 100$ running in each i-th robot are depicted for $i = 1, 2, 3, 4$ as colored solid curves.

Fig. 7.3 it is clear that the estimation does not converges to the robot positions. It is worth to mention that the coupling gain value is high, specifically 100, which implies that this performance degradation can not be avoided by assigning a large value to the coupling gain.

Now we present the simulation results of the distributed observer consists of (6.2.12), (6.3.4), and (6.3.5) in Chapter 6, which is the main results of this part.

The norm of the concatenated estimation error $|\eta|$ is depicted in 7.10. As time goes it decreases to around 1, which is much smaller than the errors from constant coupling gain distributed observer given in Fig. 7.8. However, the error

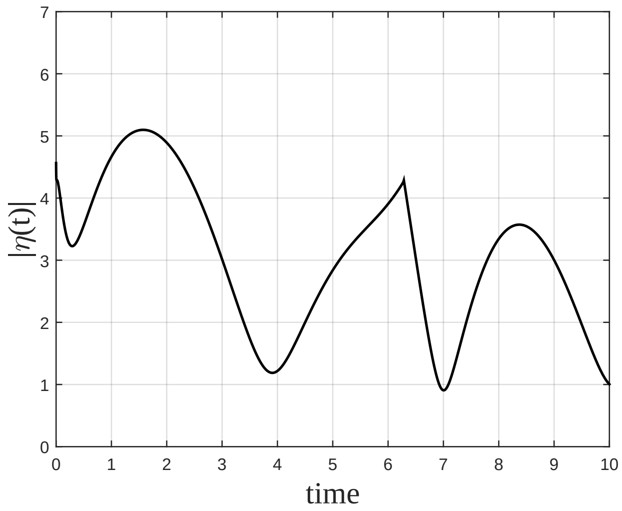


Figure 7.8: The norm of the concatenated estimation $|\eta(t)|$ of the distributed observer with constant coupling gains of 100 is depicted.

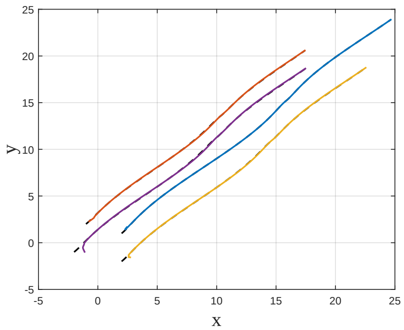
does not converge to zero but remains close to zero after a while.

Finally, the effect of coupling gains on the estimation error can be found from Fig. 7.11. It is clear that the stronger the gain is the smaller the ultimate value of estimation error. However, the strong coupling gain costs in real-life applications, so there is a trade-off.

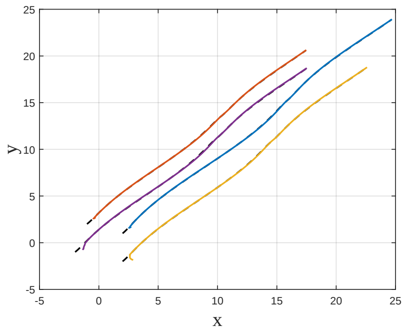
7.4 Summary and Discussion

In this chapter, we provided a solution to the distribute multi-robot localization problem by casting it into the framework of distributed estimation and by applying the proposed scheme in various settings. Unlike previous results that did not specify specific conditions, we provided a clear sufficient condition for the case where at least one agent measures its absolute position.

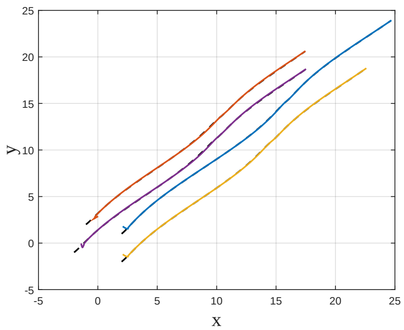
First, we showed that, for the group of mobile robots moving in unforced condition, i.e, with zero acceleration, the condition for designing the fully distributed observer is satisfied with the given settings, so the fully distributed localization can be achieved. Second, we considered more realistic case, where the mobile robots moves with nonzero acceleration. Given that each agent has no informa-



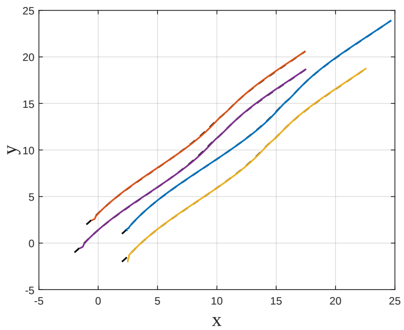
(a) Estimate from robot 1



(b) Estimate from robot 2



(c) Estimate from robot 3



(d) Estimate from robot 4

Figure 7.9: The position of robots $p_1(t)$, $p_2(t)$, $p_3(t)$, and $p_4(t)$ are depicted as black dashed curve and the estimated position of the all local observers generated by each i-th local distributed observer of the form (6.2.12), (6.3.4), and (6.3.5) running in each i-th robot are depicted for $i = 1, 2, 3, 4$ as colored solid curves.

tion about other agents' input, this case was seen as the distributed estimation problem for the plant with input. As expected, the distributed observers designed for autonomous plant in Chapter 4 and Chapter 5 failed while the observer in Chapter 6 achieved the practical estimation.

In addition to the above advantages, there are many improvements to be made. First, the proposed localization method is not scalable; the dimension of the exchanging data increases as the network grows. Since the proposed algorithm provides the estimate of the all robot's position, this is natural, but if the goal is to enable each agent to aware its own position, then the algorithm should be

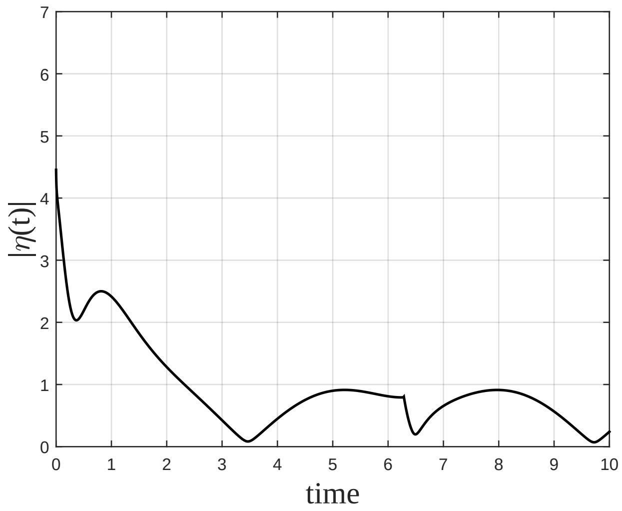
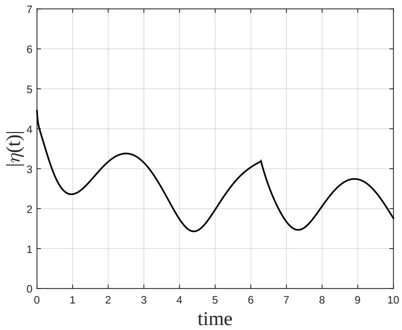


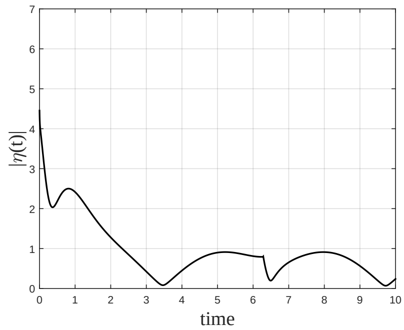
Figure 7.10: The norm of the concatenated estimation $|\eta(t)|$ of the distributed observer of the form (6.2.12), (6.3.4), and (6.3.5) is depicted.

modified. Second, the measuring relative position requires the robots to share a common global coordinate frame. By adopting the results of formation control theory, considering methods of bearing only formation control theory (see, for example, [OPA15] and the references therein) to relax this restriction may yield considerable extension.

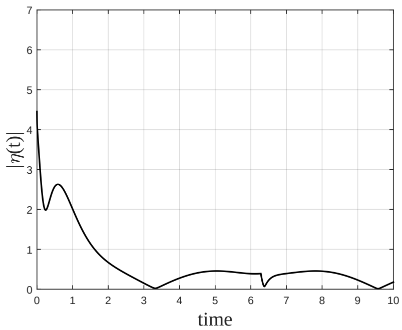
For monitoring purposes, it is often necessary in many cases for each robot to estimate the entire robots' position and the proposed solution could be useful for these cases.



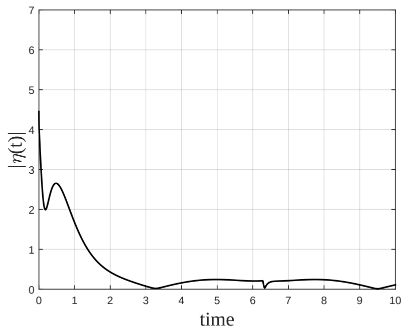
(a) Estimation error with $\gamma_i = 10$



(b) Estimation error with $\gamma_i = 40$



(c) Estimation error with $\gamma_i = 80$



(d) Estimation error with $\gamma_i = 150$

Figure 7.11: The norm of the concatenated estimation $|\eta(t)|$ of the distributed observer of the form (6.2.12), (6.3.4), and (6.3.5) is depicted for multiple coupling gains such as $\gamma_i = 10$, 40, 80, and 150, where γ_i are common for all agents.

Chapter 8

Conclusions

Studies on networked agent systems, including distributed estimation, started to grow rapidly over ten years ago and have evolved in pursuit of the following three directions:

- **the flexibility in performance over various networks** from undirected fixed to randomly varying topologies,

- **the feasibility in implementation** under the constraints on information acquisition or the constraints on computational capability, and

- **the applicability to various systems** from single integrators to general stochastic nonlinear systems.

From this point of view, we summarize the contributions of the results presented in this dissertation.

The first and the most important part of this work is the distributed observer design scheme; the main idea is to let each agent recover the self-reconstructible portion by itself using its own measurement and accept only insufficient information by the projected diffusive coupling of the neighbors' estimates via the specially designed communication protocol called agent-wise decomposed diffusive coupling. The distributed observer obtained by applying this scheme to autonomous linear time invariant systems was proposed in (4.3.1). Its performance under the strongly directed network was shown in Theorem 4.3.4. Moreover, due to the proposed structure, the construction process is reduced into assigning a sufficiently large value on coupling (scalar) gains.

In the rest part of Chapter 4, we investigated the operational flexibility of the proposed scheme over the general directed weighted (communicational) topology. It was revealed in Theorem 4.3.6 that the proposed observer can be designed over the general communication topology and its existence condition is nothing but the detectability of the system concatenated over the source components. In addition, the limitation of that the assignable converge rate was analyzed even for the system which is detectable but not observable, so as to claim the proposed scheme's flexibility in performance over the general communication topology.

In Chapter 5, we enhanced the the feasibility of implementing the proposed scheme by providing a completely decentralized construction method. We devised a specific adaptive law for the coupling gain, which does not impose any knowledge on the global information hardly accessible by the local agent. It was proven in Theorem 5.2.2 that this adaptive distributed observer works under the same condition as the one for the existence of the distributed observer.

In Chapter 6, to widen the applicability of the proposed design scheme, we considered linear time invariant plants with input, where the plant input is measured in a distributed manner as well as the state is. The classical results about unknown input observer and the special coordinate basis decomposition were adopted to reveal the partial state that can be estimated by one agent using its own measurements despite the unknown input. Under the same design philosophy, a distributed observer containing unknown input observer as a component was proposed and Theorem 6.3.2 provided a condition that leads to the practical estimation even for this general linear systems.

So far we have presented a number of novel extensions of the proposed design scheme to enhance the practicality with respect to communication topology, constructional feasibility, system class. Surely this is not the end of the line. In every direction mentioned in the beginning, the results can be further improved or generalized.

In direction of enhancing the flexibility over network, considering the time-varying network may yield promising extension. The norm-based analysis presented in [MLM15] could be applied to solve the distributed estimation problem for continuous-time linear systems in combinations with discrete-time communication, which varies along time.

To improve the constructional feasibility, one may apply this design scheme under more practical communication frames such as event-triggered system [HJT12], or sampled-data system [JY13]. Or as discussed in the summary of Chapter 5, the method of robust adaptive control may be another good direction of extension so as to accomplish more practical fully distributed estimation.

Finally, considering nonlinear plant is also a viable extension. Especially for the class of nonlinear systems described by a linear state space model with additive nonlinearities the similar design scheme can be applied. When the additional nonlinearities satisfies certain conditions such as globally Lipschitz conditions [RH94], sector-bounded [AK01], and incremental quadratic constraints [AC11], the additional nonlinearity could be compensated by the agent-wise decomposed diffusive coupling.

APPENDIX

A.1 Detectability Decomposition

For a vector space $\mathcal{X}$ and its subspace $\mathcal{V}$, there is a natural equivalence relation, which is defined by

$$x \sim y \quad \text{if and only if} \quad x - y \in \mathcal{V}$$

for $x,\ y \in \mathcal{X}$. The *quotient space* $\mathcal{X}/\mathcal{V}$ is a set of equivalent classes, whose elements is denoted as $\bar{x} := \{y \in \mathcal{V} : x \sim y\}$. Note that $\mathcal{X}/\mathcal{V}$ is a vector space with the following well-defined operations:

$$\bar{x} + \bar{y} := \overline{x+y}, \quad c\bar{x} := \overline{cx},$$

where $x,\ y \in \mathcal{X}$ and c is a scalar. Moreover, there is natural mapping $\Pi : \mathcal{X} \to \mathcal{X}/\mathcal{V}$, called the *canonical projection*, defined as $\Pi x = \bar{x}$. Obviously, Π is surjective and $\ker \Pi = \mathcal{V}$.

Let $A : \mathcal{X} \to \mathcal{X}$ be a linear map and let $\mathcal{V} \subseteq \mathcal{X}$ be a A-invariant subspace, i.e., $A\mathcal{V} \subseteq \mathcal{V}$. Then the *quotient map* $\bar{A} : \mathcal{X}/\mathcal{V} \to \mathcal{X}/\mathcal{V}$ is defined by $\bar{A}\bar{x} = \overline{Ax}$. It is easy to check that $\bar{A}$ is well-defined linear map and satisfies $\bar{A}\Pi = \Pi A$ so that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{A} & \mathcal{X} \\ \downarrow_{\Pi} & & \downarrow_{\Pi} \\ \mathcal{X}/\mathcal{V} & \xrightarrow{\bar{A}} & \mathcal{X}/\mathcal{V}. \end{array}$$

Let $\mathcal{V}$ be a subspace of the vector space $\mathcal{X}$ and let $C : \mathcal{X} \to \mathcal{Y}$ be a linear map such that $\mathcal{V} \subset \ker C$. Then we can define the quotient map $\bar{C} : \mathcal{X}/\mathcal{V} \to \mathcal{Y}$ by $\bar{C}\bar{x} = Cx$ for all $\bar{x} \in \mathcal{X}/\mathcal{V}$. Likewise, the map $\bar{C}$ is well-defined linear map and, by definition, it holds that $C = \bar{C}\Pi$.

The following theorem is one of the most important results in linear algebra widely known as the primary decomposition theorem, which can be found on page 23 of [TSH01].

Theorem A.1.1. (Primary decomposition theorem) Let $A : \mathcal{X} \to \mathcal{X}$ be a linear map and suppose that its characteristic polynomial $\chi_A(s)$ is factorized as $\chi_A(s) = p(s)q(s)$, where $p(s)$ and $q(s)$ are monic comprime polynomials. Define $\mathcal{V} := \ker p(A)$ and $\mathcal{W} := \ker q(A)$. Then the following hold:

1. $\mathcal{V} = \operatorname{im} q(A)$, $\mathcal{W} = \operatorname{im} p(A)$,
2. $\mathcal{V} \oplus \mathcal{W} = \mathcal{X}$,
3. $\mathcal{V}$ and $\mathcal{W}$ are A-invariant,
4. $\chi_{A|_\mathcal{V}}(s) = p(s)$, and $\chi_{A|_\mathcal{W}}(s) = q(s)$.

$\Diamond$

Now we are ready to define the undetectable subspace of the linear time invariant system $\dot{x} = Ax$ with the output $y = Cx$.

Definition A.1.1. Let $A : \mathcal{X} \to \mathcal{X}$ and $C : \mathcal{X} \to \mathcal{Y}$ be linear maps, and let us factorize the characteristic polynomial $\chi_A(s) := \det(sI - A)$ as $\chi_A(s) = \chi_A^{0+}(s)\chi_A^{-}(s)$, where $\chi_A^{0+}(s)$ and $\chi_A^{-}(s)$ have roots on the closed right half complex plane $\mathbb{C}^{0+}$ and open left half complex plane $\mathbb{C}^{-}$, respectively. Then, the undetectable subspace $\mathcal{U}$ for the pair (A, C) is defined as

$$\mathcal{U} := \cap_{i=0}^{n-1} \ker CA^i \cap \ker \chi_A^{+}(A),$$

where n is the dimension of $\mathcal{X}$. The pair (A, C) is called detectable if its undetectable subspace $\mathcal{U}$ is trivial ,i.e., $\mathcal{U} = \{0\}$. $\Diamond$

Note from the definition that the detectability is invariant with the similarity transformation, that is, for a nonsingular transformation T, the detectability of the pair (A, C) is equivalent to that of the pair $(T^{-1}AT, CT)$.

Lemma A.1.2. (PBH condition for detectable systems) The pair (A, C) is detectable if and only if

$$\ker \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = 0, \quad \forall \lambda \in \mathbb{C}^{0+}.$$

$\lozenge$

The following result is mentioned on page 116 of [TSH01].

Lemma A.1.3. Let $\mathcal{U}$ be the undetectable subspace of the pair (A, C) and let us denote the quotient map $\bar{A} : \mathcal{X}/\mathcal{U} \to \mathcal{X}/\mathcal{U}$ and $\bar{C} : \mathcal{X}/\mathcal{U} \to \mathcal{Y}$. The pair $(\bar{A}, \bar{C})$ is detectable. $\lozenge$

Proof. Note that both quotient maps $\bar{A}$ and $\bar{C}$ are well defined because $\mathcal{U}$ is A-invariant subspace and $\mathcal{U} \subseteq \ker C$.

To show the detectability of $(\bar{A}, \bar{C})$, let us use Lemma A.1.2 and show the PBH condition is satisfied. Choose $\lambda \in \mathbb{C}^{+}$ and suppose that $(\bar{A} - \lambda I)\bar{x} = 0$ and $\bar{C}\bar{x} = 0$ for some $\bar{x} \in \mathcal{X}/\mathcal{U}$. Then it suffices to show that $\bar{x} = 0$. Using $\Pi x = \bar{x}$, we have

$$(\bar{A} - \lambda I)\Pi x = \Pi(A - \lambda I)x = 0, \quad \bar{C}\Pi x = Cx = 0.$$

Since $\ker \Pi = \mathcal{U}$, the previous equalities are equal to the followings:

$$(A - \lambda I)x \in \mathcal{U}, \quad Cx = 0. \tag{A.1.1}$$

Note that $\mathcal{U} := \cap_{i=0}^{n-1} \ker CA^i \cap \ker \chi_A^{0+}(A)$. Then the inclusion of (A.1.1) yields

$$\begin{aligned} 0 &= C(A - \lambda I)x = CAx - \lambda Cx \\ &\vdots \\ 0 &= CA^{n-2}(A - \lambda I)x = CA^{n-1}x - \lambda CA^{n-2}x \end{aligned}$$

$$0 = CA^{n-1}(A - \lambda I)x = CA^n x - \lambda CA^{n-1}x.$$

Combined with $Cx = 0$ in (A.1.1), it follows that

$$x \in \cap_{i=0}^{n-1} \ker CA^i. \tag{A.1.2}$$

From now on, let us apply Theorem A.1.1 to A with $p(s) = \chi_A^{0+}(s)$ and $q(s) = \chi_A^{-}(s)$ so as to show

$$x \in \ker \chi_A^{0+}(A). \tag{A.1.3}$$

Since $\ker \chi_A^{0+}(A) \oplus \ker \chi_A^{-}(A) = \mathcal{X}$, the vector x can be uniquely written as $x = x^+ + x^-$ for $x^+ \in \ker \chi_A^{0+}(A)$ and $x^- \in \ker \chi_A^{-}(A)$. From $(A - \lambda I)x \in \mathcal{U}$, it follows that

$$\begin{aligned} 0 = \chi_A^{0+}(A)(A - \lambda I)x &= (A - \lambda I)\chi_A^{0+}(A)x^+ + \chi_A^{0+}(A)(A - \lambda I)x^- \\ &= \chi_A^{0+}(A)(A - \lambda I)x^-, \end{aligned}$$

which leads to $(A - \lambda I)x^- \in \ker \chi_A^{0+}(A)$. Since $\ker \chi_A^{-}(A)$ is A-invariant and intersects with $\ker \chi_A^{0+}(A)$ only at the origin, we have $0 = (A - \lambda I)x^-$. Due to $\lambda \in \mathbb{C}^+$, the restriction $(A - \lambda I)|_{\ker \chi_A^{-}(A)}$ is an invertible map, and, hence $x^- = 0$, which leads to (A.1.3). By combining (A.1.2) and (A.1.3), it follows that $x \in \mathcal{U}$, or equivalently, $\Pi x = \bar{x} = 0$. $\square$

Now we are ready to prove Lemma 4.2.1, which is rewritten below for the convenience of readers.

Lemma A.1.4. Let us denote $\mathcal{U}$ as the undetectable subspace of the pair (A, C) with the dimension ν. Let $U \in \mathbb{R}^{n \times \nu}$ and $D \in \mathbb{R}^{n \times (n-\nu)}$ be the matrices whose columns form an orthonormal basis of $\mathcal{U}$ and its orthogonal complement $\mathcal{U}^{\perp}$, respectively. Then the orthonormal matrix $T \in \mathbb{R}^{n \times n}$ defined by $T := [D\, U]$

satisfies that

$$T^T A T = \begin{bmatrix} A_{\mathsf{d}} & 0 \\ A_{\mathsf{r}} & A_{\mathsf{u}} \end{bmatrix}, \quad CT = [C_{\mathsf{d}} \ 0], \tag{A.1.4}$$

where $A_{\mathsf{d}} \in \mathbb{R}^{(n-\nu)\times(n-\nu)}$, $A_{\mathsf{r}} \in \mathbb{R}^{\nu\times(n-\nu)}$, $A_{\mathsf{u}} \in \mathbb{R}^{\nu\times\nu}$, and $C_{\mathsf{d}} \in \mathbb{R}^{p\times(n-\nu)}$. Moreover, the pair $(A_{\mathsf{d}}, C_{\mathsf{d}})$ is detectable and the matrix A_{u} is unstable. $\Diamond$

Proof. For convenience, let us denote v_i as the i-th column of T. Let us view the first equality as

$$AT = T \begin{bmatrix} A_{\mathsf{d}} & 0 \\ A_{\mathsf{r}} & A_{\mathsf{u}} \end{bmatrix}. \tag{A.1.5}$$

First, let us claim that A_{d} is the matrix representation of the quotient map $\bar{A} : \mathcal{X}/\mathcal{U} \to \mathcal{X}/\mathcal{U}$ with respect to the ordered basis $\{\bar{v}_1, \cdots, \bar{v}_{n-\nu}\}$. Consider Av_k, where $1 \leq k \leq n-\nu$. Since the set $\{v_1 \cdots, v_n\}$ forms a basis, there is a set of scalars $a_{1k}, \cdots, a_{nk}$ such that $Av_k = \sum_{i=1}^n a_{ik} v_i$. Note that $[a_{1k} \ \cdots \ a_{nk}]^T$ is the k-th column of the last matrix in (A.1.5). Let us denote $\Pi : \mathcal{X} \to \mathcal{X}/\mathcal{U}$ as the canonical projection. Then the following equality holds:

$$\bar{A}\bar{v}_k = \bar{A}\Pi v_k = \Pi A v_k = \Pi \sum_{i=1}^n a_i v_i = \sum_{i=1}^n a_i \Pi v_i = \sum_{i=1}^{n-\nu} a_i \Pi v_i = \sum_{i=1}^{n-\nu} a_i \bar{v}_i, \tag{A.1.6}$$

where the equality $\ker \Pi = \mathcal{U}$ is used. After repeating this process for $k = 1, \cdots, n-\nu$, the first claim follows.

Second, let us claim that A_{u} is the matrix representation of the restriction $A|_{\mathcal{U}} : \mathcal{U} \to \mathcal{U}$ with respect to the ordered basis $\{v_{n-\nu+1}, \cdots, v_n\}$. Consider Av_k, where $n-\nu+1 \leq k \leq n$. Likewise, there is a set of scalars $a_{1k}, \cdots, a_{nk}$ such that $Av_k = \sum_{i=1}^n a_{ik} v_i$. Since $\mathcal{U}$ is A-invariant subspace and $\{v_{n-\nu+1}, \cdots, v_n\} \subset \mathcal{U}$, we have $a_{1k} = \cdots = a_{(n-\nu)k} = 0$, and, hence, the k-th column of the last matrix in (A.1.5) becomes $[0 \cdots 0 \ a_{(n-\nu+1)k} \cdots a_{nk}]^T$ and $Av_k = \sum_{i=n-\nu+1}^n a_{ik} v_i \in \mathcal{U}$. Note that

$$A|_{\mathcal{U}} v_k = Av_k = \sum_{i=n-\nu+1}^{n} a_{ik} v_i. \tag{A.1.7}$$

After repeating this process for $k = n - \nu + 1, \cdots, n$, the second claim follows.

Third, let us claim that C_{d} is the matrix representation of the quotient map $\bar{C} : \mathcal{X}/\mathcal{U} \to \mathcal{Y}$ with respect to the ordered basis $\{\bar{v}_1, \cdots, \bar{v}_{n-\nu}\}$. Consider Cv_k. For $n-\nu+1 \le k \le n$, it holds that $Cv_k = 0$ because $\mathcal{U} \subset \ker C$ from the definition. For $1 \le k \le n-\nu$, there is a set of scalars $c_{1k}, \cdots, c_{nk}$ such that $Cv_k = \sum_{i=1}^{p} c_{ik} e_i$, where $\{e_1, \cdots, e_p\}$ is the standard basis of $\mathcal{Y}$, and $[c_{1k} \ \cdots \ c_{pk}]^T$ is the k-th column of C_{d}. Note that

$$\bar{C}\bar{v}_k = \bar{C}\Pi v_k = Cv_k = \sum_{i=1}^{p} c_{ik} e_i \tag{A.1.8}$$

After repeating this process for $k = 1, \cdots, n - \nu$, the third claim follows.

Finally, the detectability of the pair $(A_{\mathsf{d}}, C_{\mathsf{d}})$ is a direct consequence of Lemma A.1.3 and the unstability of A_{u} is obvious because its characteristic polynomial $\chi_{A|_{\mathcal{U}}}(s)$ divides $\chi_A^{0+}(s)$. This completes the proof. $\square$

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국문초록

Fully Decentralized Design of Distributed Observer for Linear Time Invariant Systems

선형 시불변 시스템에 대한 분산 관측기의 비집중 설계

본 논문은 연속시간 선형 시불변 시스템에 대해 새로운 형태의 분산 관측기 (distributed observer) 를 제안한다.

단일 센서를 이용해 설계되는 고전적인 관측기와 달리 분산 관측기는 다수의 개체 (센서) 들이 네트워크를 형성하여 분산적으로 플랜트를 측정하는 경우에 이용되며, 각 개체마다 설계된 다수의 국소 (local) 관측기로 구성된다. 각 국소 관측기는 자체 측정값과 통신을 통해 수집한 이웃 개체들의 추정값 (estimate) 을 융합하여 플랜트의 전체 상태변수 (full state) 를 추정한다. 분산 관측은 개체간 협력을 통해 단일 개체의 제한된 성능을 극복함으로써 큰 규모의 개체로 구성된 네트워크 시스템 (networked agent system) 에 대해 향상된 추정 성능을 보인다.

분산 관측에 대한 기존의 연구결과들은, i) 센서 간 통신망의 형태에 대한 유연성, ii) 단일 개체가접근 가능한 정보 만을 이용한 설계 구현, iii) 다양한 형태의 플랜트에 대한 수용성, 세 가지 측면에서 비교적 실용성이 부족하다.

본 논문에서는 새로운 형태의 분산 관측기 설계 기법을 제안한다. 핵심 아이디어는, 플랜트의 전체 상태변수를 추정하기 위하여, 각 개체가 개체 스스로의 측정치만을 이용하여 추정할 수 있는 부분은 스스로 추정하되, 스스로 추정이 불가한 부분은 개체별 분해기반 확산결합 (agent-wise decomposed diffusive coupling) 이라는 특별한 통신 규약을 통해 이웃한 개체로부터 수집한 정보를 이용하는 것이다.

뿐만 아니라, 제안된 분산 관측기 설계 기법의 성능을 위에서 언급한 세 가지 관점에서 분석하고 개선한다.

i) 제약이 없는 가장 일반적인 유향 그래프 (directed graph) 로 표현되는 개체간 네트워크에서 분산 관측기가 설계 가능할 필요충분조건을 제시한다.

ii) 적응 제어 기법을 이용하여 비집중식으로 (decentralized) 설계 가능한 적응 분산 관측기를 제시한다. 제시된 국소 관측기는 네트워크 및 다른 개체에 대한 정보를 필요로 하지 않으며 개체 자체적으로 구현이 가능하다. 따라서 제시된

분산 관측기는 플러그 앤 플레이 (plug and play) 방식으로 작동 가능하여 개체의 드나듦으로 인한 네트워크 변화에 유연하게 동작한다.

iii) 입력을 고려하지 않는 관측기 설계 문제에서 더 나아가, 입력이 있는 플랜트에서 입력 또한 상태변수와 같이 개별 개체에 의해 분산되어 측정되는 상황을 가정한다. 이 때 분산 관측기를 설계하기 위하여 기하학적 제어이론 (geometrical control theory) 기반의 미지 입력 관측기 (unknown input observer) 를 이용한다. 이를 통해 유계 (bounded) 입력에 대해 설계된 분산 관측기의 추정치가 플랜트의 상태에 근사적으로 (approximately) 수렴함을 증명한다.

마지막으로 실용성을 향상시킨 새로운 분산 관측기 설계 기술을 군집 로봇의 위치 인식 문제 (distributed multi-robot localization problem) 에 적용하고 시뮬레이션 결과를 제시함으로써 제안된 설계법의 효과를 검증한다.

주요어 : 분산 추정, 분산 관측기, 다개체 시스템, 센서 네트워크, 비집중 설계, 기하학적 분해, 적응 관측기
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