이학박사 학위논문

Analytic multiplier ideals and L^2 extension theorems
(해석적 승수 아이디얼과 L^2 확장 정리)

2019년 8월

서울대학교 대학원

수리과학부

서 호 섭

Analytic multiplier ideals and L^2 extension theorems

(해석적 승수 아이디얼과 L^2 확장 정리)

지도교수 김 다 노

이 논문을 이학박사 학위논문으로 제출함

2019년 4월

서울대학교 대학원

수리과학부

서 호 섭

서 호 섭의 이학박사 학위논문을 인준함

2019년 6월

위 원 장	김 영 훈	(인)
부 위 원 장	김 다 노	(인)
위 원	David Donghoon Hyeon	(인)
위 원	Otto van Koert	(인)
위 원	박 종 도	(인)

Analytic multiplier ideals and L^2 extension theorems

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
to the faculty of the Graduate School of
Seoul National University

by

Hoseob Seo

Dissertation Director : Professor Dano Kim

Department of Mathematical Science
Seoul National University

August 2018

Abstract

Analytic multiplier ideals and L^2 extension theorems

Hoseob Seo

Department of Mathematical Sciences
The Graduate School
Seoul National University

In this thesis, multiplier ideal sheaves and L^2 extension theorems are main themes. Multiplier ideals and its jumping numbers play an important role in algebraic geometry and complex geometry because of its applications. Jumping numbers are deeply studied by Ein, Lazardfeld, Smith and Varolin [ELSV] in the algebraic setting. We extend the study of jumping numbers of multiplier ideals due to [ELSV] from the algebraic case to the case of general plurisubharmonic functions. While many properties from [ELSV] are shown to generalize to the plurisubharmonic case, important properties such as periodicity and discreteness do not hold any more. Previously only two particular examples with a cluster point (i.e. failure of discreteness) of jumping numbers were known, due to Guan-Li and to [ELSV] respectively. We generalize them to all toric plurisubharmonic functions in dimension 2 by characterizing precisely when cluster points of jumping numbers exist and by computing all those cluster points. This characterization suggests that clustering of jumping numbers is a rather frequent phenomenon. In particular, we obtain uncountably many new such examples.

Key words: Jumping Numbers, L^2 Estimates, L^2 Extension Theorem, Analytic Multiplier ideals
Student Number: 2011-20272

Contents

Chapter 1

Introduction

1.1 Multiplier ideals and jumping numbers

Multiplier ideals are important tools in complex algebraic geometry (see e.g. [L], [D11], [S98]). In [ELSV], Ein, Lazarsfeld, Smith and Varolin systematically studied jumping numbers of multiplier ideals in the algebraic case, i.e. for multiplier ideals $\mathcal{J}(cD)$ associated to an effective divisor D or to an ideal sheaf on a smooth complex variety. A jumping number is a coefficient $c > 0$ at which the ideal $\mathcal{J}(cD)$ jumps down to a smaller ideal as the coefficient increases.

Jumping numbers are fundamental invariants of the given singularity : they encode interesting geometric, algebraic and analytic information. When one considers all the nonreduced subschemes defined by $\mathcal{J}(cD)$ as c varies, properties of jumping numbers are important as in recent work of Demailly [D15] on L^2 extension theorems from nonreduced subschemes. Also see the introduction and the references in [ELSV] for earlier works on jumping numbers and several different connections and contexts in mathematics where the jumping numbers naturally arise.

The full strength of the theory of multiplier ideals lies in that one can even take multiplier ideals from a *plurisubharmonic* function, which can be regarded as a limit of divisors and ideals. In [ELSV], the study of jumping numbers could not be extended to general plurisubharmonic functions since the openness conjecture was not yet proved at the time.

Now that the openness conjecture is a theorem (see (2.1.19)) of Guan and Zhou [GZ] (after related works of several authors including [FJ05] in dimension 2 : see the references in [GZ]), one can study the jumping numbers associated to general plurisubharmonic functions: general in the sense that it is not necessarily having the above mentioned algebraic singularities.

More precisely, let X be a complex manifold and let φ be a plurisubharmonic (*psh*) function on X. Let $x \in X$ be a point. From the openness theorem (2.1.19) of [GZ], there exists an increasing sequence of real numbers $\chi_k = \chi_k(\varphi; x)$

$$0 = \chi_0(\varphi; x) < \chi_1(\varphi; x) < \chi_2(\varphi; x) < \cdots \tag{1.1.1}$$

such that the stalks at x of the multiplier ideal sheaves $\mathcal{J}(c\varphi)_x$ are *constant precisely on* the half-open intervals $[\chi_k, \chi_{k+1})$ in the sense that $\mathcal{J}(c\varphi)_x = \mathcal{J}(\chi_k\varphi)_x$ for $c \in [\chi_k, \chi_{k+1})$ and $\mathcal{J}(\chi_{k+1}\varphi)_x \subsetneq \mathcal{J}(\chi_k\varphi)_x$. The real numbers in (1.1.1) are called the jumping numbers of φ at x since they are the points where the multiplier ideals "jump", beginning with the log-canonical threshold χ_1. This is parallel to the algebraic case as in [ELSV].

However, the singularity of a general psh function can be highly complicated and difficult to study compared to the algebraic case, one reason being that one cannot apply a finite number of blow-ups to resolve its singularity. Plurisubharmonic singularity is the topic of intensive current research (see e.g. [D93], [D11], [DH], [DGZ], [R13a], [BFJ]). One important connection of psh singularity to algebraic geometry is that general psh functions are useful to study pseudo-effective line bundles. These line bundles carry singular hermitian metrics with psh local weight functions [D11, (6.2)] instead of usual holomorphic sections. Those psh functions are either not algebraic or often not known to be algebraic. Pseudo-effective line bundles are important in birational geometry and the minimal model program in algebraic geometry (see e.g. [L], [D11], [N04]).

One often expects peculiar behaviors coming from psh singularities which are not seen in the algebraic case. Indeed for jumping numbers, recently Guan and Li [GL] showed by an example (2.4.1) that the set of the above jumping numbers $\{\chi_0, \chi_1, \chi_2, \ldots\}$ can have a cluster point (i.e. limit point) in the psh case (hence we do not need to enumerate the jumping numbers as in (1.1.1) since the jumping numbers greater than a cluster point will not be counted anyway). This is a new phenomenon in the general psh case.

In Chapter 2, along with giving many properties and examples of jumping numbers in the psh case, we first give another psh example (2.2.5) having a cluster point of jumping numbers which is converted from an example of a graded system of ideals in [ELSV]. For this conversion, we define a psh function associated to the graded system of ideals and use an important result Theorem 2.1.21 of S. Boucksom whose proof is given in the appendix. These two examples (2.4.1) and (2.2.5) happen to be toric psh in dimension 2.

In view of the example (2.2.5) where all positive integers are cluster points of the jumping numbers, it is natural to raise the following

Question 1.1.2. Let φ be a psh function on a complex manifold X and let $x \in X$ be a point. Let T be the set of cluster points in the set of jumping numbers $\mathrm{Jump}(\varphi)_x \subset \mathbf{R}_+$. Assume that T is nonempty. Is T infinite? Is T unbounded?

We answer this question positively when φ is toric psh: in that case, T is indeed an infinite and unbounded set of positive real numbers by Corollary 2.4.5.

On the other hand, we completely generalize the above two particular examples (which were all that were known) [ELSV] (2.2.5) and [GL] (2.4.1) as follows.

Theorem 1.1.3 (see Theorem 2.4.8 for the full precise statement)**.** *Let φ be a toric psh function on the unit polydisk $\mathbf{D}^2 \subset \mathbf{C}^2$. Let $P(\varphi)$ be the Newton convex body of φ. Let $\mathrm{Jump}(\varphi)_0$ be the set of jumping numbers of φ at the origin $0 \in \mathbf{D}^2$. We have a complete characterization in terms of $P(\varphi)$ for*

(1) when there exists a cluster point of jumping numbers in $\mathrm{Jump}(\varphi)_0$, and

(2) what are those cluster points.

In particular, we obtain uncountably many new examples of psh functions having a cluster point of jumping numbers.

This characterization (which is in the statement of Theorem 2.4.8) is explained very well by Figure 2.2 in 2.4.2. One can then easily produce so many examples of psh functions with cluster points of jumping numbers, simply by taking Newton convex bodies with the behavior (d) or (h) of Figure 2.2 and then taking corresponding psh functions as in e.g. (2.1.28) in Example 2.1.27. This certainly suggests that clustering of jumping numbers is a rather frequent phenomenon in the singularity of psh functions.

It is worth noting here that for the new examples in Theorem 1.1.3, while we can compute the cluster points of jumping numbers, it will be more difficult or impossible to compute all the jumping numbers themselves. This is unlike the particular case of (2.2.5) where [ELSV] computed all the jumping numbers first and then determined the cluster points from them.

Remark 1.1.4. The new examples in Theorem 1.1.3 also yield new algebraic examples in the above graded system of ideals version of [ELSV] since we can take the graded system of monomial ideals from the Newton convex bodies as in Example 2.1.27.

Theorem 1.1.3 also provides further information on the above example (2.4.1) of Guan and Li [GL] where 1 was shown to be a cluster point of jumping numbers : now we show that all the cluster points are precisely the positive integers in Corollary 2.4.12.

Now turning to Theorem 2.1.21 of S. Boucksom in the appendix, we remark that it answers a natural question on asymptotic multiplier ideals, which may be of independent interest in algebraic geometry.

Theorem 1.1.5 (= Theorem 2.1.21)**.** *Let X be a smooth irreducible complex variety. Let $\mathfrak{a}_\bullet$ be a graded system of ideal sheaves on X. Let $\varphi_{\mathfrak{a}_\bullet} := \log\left(\sum_{k\geq 1}^{\infty} \epsilon_k |\mathfrak{a}_k|^{\frac{1}{k}}\right)$ be a Siu psh function associated to $\mathfrak{a}_\bullet$ with a choice of coefficients $\epsilon_k > 0$ (see (2.1.20)). Then for every real $c > 0$, the analytic multiplier ideal sheaf $\mathcal{J}(c\varphi_{\mathfrak{a}_\bullet})$ of the psh function $c\varphi_{\mathfrak{a}_\bullet}$ is equal to the asymptotic multiplier ideal sheaf $\mathcal{J}(c \cdot \mathfrak{a}_\bullet)$.*

Asymptotic multiplier ideals (see e.g. [L], [Ka99], [ELS01]) were first brought into focus when they replaced the role of Siu psh functions (2.1.20) in casting the crucial analytic proof of invariance of plurigenera by [S98] in algebro-geometric language. In [S98], Siu psh functions appeared as local weight functions of a global hermitian metric, a Siu-type metric, of a line bundle L. This metric can be regarded as the metric with minimal singularities coming from taking all the (multi-)sections of L, hence it is a metric analogue of the stable base locus of L. (See e.g. [K15, Sec.4] for its comparison with Demailly's version of the metric with minimal singularities of L.)

However there remained a natural question asking whether the algebraically defined asymptotic multiplier ideals and the analytic multiplier ideals of Siu psh functions are precisely equal to each other (not only playing analogous roles in the above proof). This is answered positively by Theorem 2.1.21 in the most general setting of the question, i.e. for a graded system of ideals ([ELS01], [L]). Previously a special case was known by [DEL] (see Remark 2.1.23).

The proof of Theorem 2.1.21 uses valuative characterizations of multiplier ideals [BFJ], [BFFU]. The valuative approach to study psh singularities (see [FJ05], [BFJ] and the references therein) plays an increasingly important role recently and it certainly influences our study in this thesis to a great extent.

In this valuative spirit, we define two psh functions φ and ψ to be v-equivalent (in Sec. 2.1) if all their multiplier ideals are equal i.e. $\mathcal{J}(m\varphi) = \mathcal{J}(m\psi)$ for all real $m > 0$. Thus two v-equivalent psh functions have all the same jumping numbers. But there exist many v-equivalent psh functions φ and ψ that look quite different from each other (in particular $\varphi - \psi$ is not locally bounded) as we show in Example 2.1.27 and Proposition 2.1.29.

Finally, in addition to the absense of cluster points, we show that another important property of jumping numbers in the algebraic case shown in [ELSV] fails in the psh case : periodicity. We use an interesting example of Koike [Ko15].

Remark 1.1.6. In a different context of algebraic multiplier ideals on singular varieties, there is an open question asking whether cluster points of jumping numbers ever exist in that algebraic setting (see e.g. [G16, Question 1.3], [U12],

[T10]). See also [BSTZ].

1.2 L^2 extension theorems: toward inversion of adjunction

In Chapter 3, we present ongoing work toward the analytic proof of inversion of adjunction, an important theorem in algebraic geometry. This chapter does not contain original results yet. The starting point of L^2 extension theorem is L^2 estimates for the $\overline{\partial}$ operator. After Andreotti and Vesentini [AV61, AV65], the many techniques for L^2 estimates were developed by Kohn [Ko63], Hörmander [H65], Demailly [D82]. The most elementary version of L^2 estimate is the following theorem.

Theorem 1.2.1 ([H73]). *Let Ω be a pseudoconvex open set in $\mathbf{C}^n$ and φ plurisubharmonic function in Ω. For every $g \in L^2_{p,q}(\Omega, \varphi)$ with $\overline{\partial} g = 0$, there is a solution $u \in L^2_{p,q-1}(\Omega, \mathrm{loc})$ of the equation $\overline{\partial} u = g$ such that*

$$\int_\Omega \frac{|u|^2 e^{-\varphi}}{(1+|z|^2)^2} d\lambda \leq \int_\Omega |g|^2 e^{-\varphi} d\lambda < +\infty$$

where $d\lambda$ is the Lebesgue measure on $\mathbf{C}^n$.

As important and typical applications of L^2 estimates for the $\overline{\partial}$ operator, various version of L^2 extension theorems were established after Ohsawa and Takegoshi [OT87]. We present L^2 extension theorems of Ohsawa-Takegoshi type which are important tools in complex geometry and also in the expected analytic proof of the inversion of adjunction. In particular we will describe briefly the proof of the following L^2 extension theorem in Demailly [D15] and work out explicitly some important details used in [D15] which will be crucial in the analytic proof of the inversion of adjunction.

Theorem 1.2.2. *Suppose that (X, ω) is a weakly pseudoconvex Kähler manifold and ψ is a quasi-psh function on X with neat analytic singularities and* *that there exists $\delta > 0$ such that the curvature tensor*

$$i\Theta(E) + \alpha i\partial\overline{\partial}\psi \otimes \mathrm{Id}_E \geq 0$$

in the sense of Nakano for all $\alpha \in [1, 1+\delta]$. Then, for every smooth section f of $(K_X \otimes E)|_{Y^o}$ on Y^o such that $D''f = 0$ and

$$\int_{Y^o} |f|^2_{\omega,h} \, dV_{Y^o,\omega}[\psi] < +\infty,$$

there exists a section F of $K_X \otimes E$ on X such that $F|_{Y^o} = f$ and

$$\int_X \gamma(\delta\psi)|F|^2_{\omega,h} e^{-\psi} \, dV_{X,\omega} \leq \frac{34}{\delta} \int_{Y^o} |f|^2_{\omega,h} \, dV_{Y^o,\omega}[\psi],$$

where $\gamma : \mathbf{R} \to \mathbf{R}$ is a positive function defined by

$$\gamma(x) = \begin{cases} e^{-x/2} & \text{if } x \geq 0, \\ \dfrac{1}{1+x^2} & \text{if } x < 0. \end{cases}$$

For the inversion of adjunction, we start with the definitions of singularities of pairs and recall some important properties. After that, we reformulate being Kawamata log terminal and being purely log terminal near a hypersurface H in terms of multiplier ideals and adjoint ideals for smooth cases. Finally, we describe the following inversion of adjunction using L^2 extension theorem.

Theorem 1.2.3. *Let X be a smooth variety, D a boundary $\mathbf{Q}$-divisor on X and H a smooth closed subscheme of codimension 1 of X. Then $(X, D + H)$ is plt near H if and only if $(H, D|_H)$ is klt.*

Chapter 2

Multiplier ideals and jumping numbers

Chapter 2 is organized as follows. In Section 2.1, we summarize basic notions of plurisubharmonic functions and multiplier ideal sheaves and we study Siu psh functions associated to a graded system of ideals, which play an important role in this chapter. In Section 2.2, we define jumping numbers, present their basic properties and examples, generalizing counterparts from [ELSV]. We also study more general "mixed" jumping numbers. In Section 2.3, we show that periodicity of jumping numbers as in [ELSV] fails for general psh functions, elaborating on an example of Koike [Ko15]. In Section 2.4, we present and prove the main results on the cluster points of jumping numbers of psh functions.

2.1 Multiplier ideals and plurisubharmonic functions

For excellent general expositions on multiplier ideal sheaves and plurisubharmonic (psh) functions, we refer to [D11] and also to [L] for the algebraic theory of multiplier ideals.

2.1.1 Basic notions on plurisubharmonic functions

In this subsection, we recall briefly basic notions and some properties of plurisubharmonic functions. See [D97] for more details.

Definition 2.1.1. A function $\varphi : U \to [-\infty, +\infty)$ defined on an open subset U of $\mathbf{C}^n$ is said to be *plurisubharmonic(psh)* if

1. φ is upper semicontinuous,

2. for every complex line $L \subset \mathbf{C}^n$, the restriction $\varphi|_L$ of φ is subharmonic on $U \cap L$.

We rephrase the second condition as the following *mean value property*: for every $x \in U$ and $\xi \in \mathbf{C}^n$ with $|\xi| < d(x, \partial U)$,

$$\varphi(x) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(x + e^{i\theta}\xi)\, d\theta.$$

Integrating the above inequality with respect to ξ over $S(0, r)$ gives

$$\mathrm{PSH}(U) \subset \mathrm{SH}(U)$$

where $\mathrm{PSH}(U)$(resp. $\mathrm{SH}(U)$) is the set of all psh(resp. subharmonic) functions on U. For any C^2 function φ, φ is plurisubharmonic on U if and only if for $x \in U$ and $\xi \in \mathbf{C}^n$, by the subharmonicity of φ on each complex line,

$$\sum_{j,k} \frac{\partial^2 \varphi}{\partial z_j \bar{\partial} z_k}(x)\xi_j \bar{\xi}_k \geq 0.$$

This is also true even if φ is not of C^2 in the following sense.

Theorem 2.1.2. *If φ is a psh function on U and φ is not identically equal to $-\infty$ on each connected component of U, then for all $\xi \in \mathbf{C}^n$, the complex hessian of φ*

$$H\varphi(\xi) = \sum_{j,k} \frac{\partial^2 \varphi}{\partial z_j \bar{\partial} z_k} \xi_j \bar{\xi}_k$$

is a positive measure. Conversely, if $\psi \in \mathcal{D}'(U)$ satisfies that $H\psi(\xi)$ is a positive measure for all $\xi \in \mathbf{C}^n$, there exists a unique psh function $\varphi \in L^1_{\text{loc}}(U)$ which is equal to ψ in the sense of distribution.

For a holomorphic function f on U, it can be readily seen that $\log|f|$ has a semi-positive definite complex Hessian in the sense of distribution and therefore $\log|f|$ is plurisubharmonic on U. The following theorem provides various ways to obtain many other psh functions from a psh function on U.

Proposition 2.1.3. *Let $\varphi_1, \ldots, \varphi_p$ be psh function on an open subset $U \subset \mathbf{C}^n$ and $\chi : \mathbf{R}^p \to \mathbf{R}$ be a convex function which is non decreasing in each coordinate variable. Then $\chi(\varphi_1, \ldots, \varphi_p)$ is psh on U. In particular, $\varphi_1 + \cdots + \varphi_p$, $\max\{\varphi_1, \ldots, \varphi_p\}$ and $\log(e^{\varphi_1} + \cdots + e^{\varphi_p})$ is psh on U.*

Note that by this proposition, for holomorphic functions $f_1, \ldots, f_p$ on U, the function φ defined by

$$\varphi = \log(|f_1|^{\alpha_1} + \cdots + |f_p|^{\alpha_p})$$

is plurisubharmonic for arbitrary nonnegative real numbers $\alpha_1, \ldots, \alpha_p$.

Before defining plurisubharmonicity on complex manifolds, we observe the following property.

Proposition 2.1.4. *Let $\varphi : U \to [-\infty, +\infty)$ be a psh function on an open subset $U \subset \mathbf{C}^n$ and $F : V \to \mathbf{C}^n$ a holomorphic map from an open subset $V \subset \mathbf{C}^m$ into U. Then, $\varphi \circ F$ is plurisubharmonic on V.*

With this proposition, it is natural to define psh functions on a complex manifold X by using local coordinates of X.

Definition 2.1.5. Let φ be a function on a complex manifold X. We say that φ is *plurisubharmonic(psh)* on X if for all local coordinates chart $F : U \subset \mathbf{C}^n \to X$, $\varphi \circ F$ is plurisubharmonic on U.

Definition 2.1.6. Let φ be a psh function on an open subset U of a complex manifold X. The *(analytic) multiplier ideal sheaf $\mathcal{J}(\varphi)$* of φ is the sheaf of germs of holomorphic functions f such that $|f|^2 e^{-2\varphi} \in L^1_{\text{loc}}$ with respect to Lebesgue measure in some local coordinates.

One may wonder whether multiplier ideals are coherent or not. To state a theorem stating the coherence of multiplier ideals, let us start with the definition of coherence.

Definition 2.1.7. Let $\mathcal{A}$ be a sheaf of rings on a topological space X and let $\mathcal{F}$ be a sheaf of modules over $\mathcal{A}$. Then, $\mathcal{F}$ is said to be *locally finitely generated* if for every $x_0 \in X$, there are an open neighborhood U of x_0 and sections $F_1, \dots, F_p \in \mathcal{F}(U)$ such that $F_{1,x}, \dots, F_{p,x}$ generate the stalk $\mathcal{F}_x$ for all $x \in U$.

Note that an $\mathcal{A}$-module F is locally finitely generated if and only if the following holds: for every point $x_0 \in X$, there exists a surjective morphism $\mu : \mathcal{A}^{\oplus p}|_U \to \mathcal{F}|_U$ for some open neighborhood U of x_0.

Definition 2.1.8. Let $\mathcal{F}$ be a sheaf of $\mathcal{A}$-modules. For an open subset U of X and sections $F_1, \dots, F_p \in \mathcal{F}(U)$, one can define the sheaf morphism $\mu : \mathcal{A}^{\oplus p}|_U \to \mathcal{F}|_U$ so that for every stalk $\mathcal{F}_x$ with $x \in U$, μ_x is given by

$$(g_1, \dots, g_p) \mapsto \sum g_i F_{i,x} \in \mathcal{F}_x.$$

Then, we call the kernel of μ, denoted by $\mathcal{R}(F_1, \dots, F_p)$, the *sheaf of relations* between $F_1, \dots F_p$.

Definition 2.1.9. A sheaf $\mathcal{F}$ of $\mathcal{A}$-modules on a topological space X is said to be *coherent* if $\mathcal{F}$ is locally finitely generated and $\mathcal{R}(F_1, \dots, F_p)$ is locally finitely generated for any open subset U and any $F_1, \dots, F_p \in \mathcal{F}(U)$.

Every coherent analytic sheaf on a complex manifold X enjoys the *strong Noetherian property.*

Theorem 2.1.10. *Let $\mathcal{F}$ be a coherent analytic sheaf on X and $(\mathcal{F}_k)$ a increasing sequence of coherent subsheaves of $\mathcal{F}$, which means*

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots .$$

Then the sequence $(\mathcal{F}_k)$ is stationary on every compact subset of X.

Every (analytic) multiplier ideal sheaf is coherent. It is proved by using Hörmander's L^2 estimates for $\overline{\partial}$-equations. See [D96] for details.

Theorem 2.1.11. *For any psh function φ on an open subset U, the multiplier ideal sheaf $\mathcal{J}(\varphi)$ of φ is coherent.*

Definition 2.1.12. A psh function φ is said to *have a logarithmic pole of coefficient* γ at a point $x \in X$ if the *Lelong number*

$$\nu(\varphi, x) := \liminf_{z \to x} \frac{\varphi(z)}{\log|z - x|}$$

is non zero and if $\nu(\varphi, x) = \gamma$.

Multiplier ideals and Lelong numbers measure logarithmic poles of psh functions and they are related.

Lemma 2.1.13 ([Sko72])**.** *Let φ be a psh function on an open subset $U \subset X$ and let $x \in U$.*

(1) If $\nu(\varphi, x) < 1$, then $e^{-2\varphi}$ is integrable in a neighborhood of x, in particular $\mathcal{J}(\varphi)_x = \mathcal{O}_{U,x}$.

(2) If $\nu(\varphi, x) \geq n + s$ for some integer $s \geq 0$, then $e^{-2\varphi} \geq C|z - x|^{-2n-2s}$ in a neighborhood of x and $\mathcal{J}(\varphi)_x \subset \mathfrak{m}_{U,x}^{s+1}$, where $\mathfrak{m}_{U,x}$ is the maximal ideal of $\mathcal{O}_{U,x}$.

(3) The zero variety $V(\mathcal{J}(\varphi))$ of $\mathcal{J}(\varphi)$ satisfies

$$E_n(\varphi) \subset V(\mathcal{J}(\varphi)) \subset E_1(\varphi)$$

where $E_c(\varphi) = \{x \in X : \nu(\varphi, x) \geq c\}$ is the c-sublevel set of Lelong numbers of φ.

One can define multiplier ideals of divisor in a purely algebraic way as follows. Let D be a $\mathbf{Q}$-divisor on X. One then can find a log resolution $\mu : Y \to X$ of D so that $\mu^*(D) + \mathrm{excep}(\mu)$ has *simple normal crossing support*, which means that each D_i is smooth and for all $x \in \mathrm{supp}(D)$, there exists a local cooridnate $z_1, \ldots, z_n$ such that $D = \sum_{i=1}^{k} d_i D_i$ is given by the equation

$$z_1^{d_1} \cdots z_k^{d_k} = 0$$

for some $k \leq n$.

Definition 2.1.14. The multiplier ideal sheaf $\mathcal{J}(D)$ associated to D is defined by

$$\mathcal{J}(D) = \mu_* \mathcal{O}_Y(K_{Y/X} - \lfloor \mu^* D \rfloor),$$

where $K_{Y/X}$ is the relative canonical divisor $K_Y - \mu^* K_X$ of Y over X.

To be well-defined, the multiplier ideal sheaf $\mathcal{J}(D)$ must be independent of the choice of a log resolution.

Proposition 2.1.15. *The multiplier ideal sheaf $\mathcal{J}(D)$ is independent of the log resolutions.*

As we have a psh function $\varphi_{\mathfrak{a}}$ associated to a given ideal $\mathfrak{a}$ in Chapter 2, we have a psh function φ_D for a given divisor $D = \sum d_i D_i$ of X. For local coordinates on $U \subset X$, assume that a holomorphic function g_i is a defining functions of D_i for each i. Define a function φ_D by

$$\varphi_D = \sum_i d_i \log |g_i|$$

is plurisubharmonic on U. Note that φ_D then depends on the defining functions g_i but the multiplier ideal $\mathcal{J}(\varphi_D)$ does not depend on g_i's.

Theorem 2.1.16. *The multiplier ideal $\mathcal{J}(\varphi_D)$ of φ_D is equal to $\mathcal{J}(D)$.*

For the valuative characterizations of multiplier ideals, we remind the readers the definition of divisorial valuations(See [BFJ] for more details).

Definition 2.1.17. A *divisorial valuations* on X is a valuation ν such that there exist a smooth birational model $\pi : Y \to X$, a prime divisor E on Y and a constant $a_E > 0$ such that

$$\nu(f) = a_E \operatorname{ord}_E(\pi^* f)$$

for all holomorphic functions f on an open set $U \subseteq X$ such that $\mathrm{c}_X(\nu) \cap U \neq \emptyset$.

We can characterize the multiplier ideal of a psh function φ as follows:

Theorem 2.1.18 ([BFJ])**.** *Let f be a germ of holomorphic function at $x \in X$. Then $f \in \mathcal{J}(\varphi)_x$ if and only if there exists $\epsilon > 0$ such that*

$$\nu(f) \geq (1+\epsilon)\nu(\varphi) - A(\nu)$$

for all divisorial valuation ν on X satisfying $x \in \mathrm{c}_X(\nu)$. Here, $A(\nu)$ is called the log discrepancy of ν and is given by, if $\nu = \mathrm{ord}_E$,

$$A(\nu) = 1 + \mathrm{ord}_E(K_{Y/X})$$

where $\pi : Y \to X$ is a smooth birational model of X.

2.1.2 V-equivalence of psh functions

We have the usual notion of two psh functions φ, ψ having equivalent singularities [D11], i.e. when $\varphi - \psi$ is locally bounded. We will often simply say φ and ψ are equivalent.

As a very convenient and flexible weaker notion (cf. [K19, S 2]), we say that two psh functions φ and ψ are **v-equivalent** and write $\varphi \sim_v \psi$ if the following equivalent conditions hold:

(1) For all $m > 0$, the multiplier ideals are equal : $\mathcal{J}(m\varphi) = \mathcal{J}(m\psi)$.

(2) At every point of all proper modifications over X, the Lelong numbers of φ and ψ coincide. In other words, for every divisorial valuation v centered on X, we have $v(\varphi) = v(\psi)$.

The above equivalence of (1) and (2) is due to [BFJ] together with the following theorem in [GZ].

Theorem 2.1.19 (Qi'an Guan and Xiangyu Zhou [GZ], Openness Theorem)**.** *Let ψ and φ be two psh functions on a complex manifold X. We have the following equality of ideal sheaves*

$$\bigcup_{\epsilon > 0} \mathcal{J}(\psi + \epsilon\varphi) = \mathcal{J}(\psi).$$

We remark that previously some special cases of this theorem (assuming $\psi = c\varphi$ for a constant $c > 0$) were known: when $\dim X = 2$ in [FJ05], when φ is toric psh in [G, (1.15)] and when φ is a Siu psh function (defined below) in [ELSV, (5.4)] together with using Theorem 2.1.21.

2.1.3 Graded systems of ideals and Siu psh functions

Here we show that the algebraic asymptotic multiplier ideal is equal to the analytic multiplier ideal of a Siu psh function associated to the graded system of ideals.

Let X be a smooth irreducible complex variety (or a complex manifold). Let $\mathfrak{a}_\bullet$ be a graded system of ideal sheaves on X [L, Section 2.4.B]. We define a Siu psh function associated to $\mathfrak{a}_\bullet$ following [S98] (see also [BEGZ, p.258], [K15]) by

$$\varphi_{\mathfrak{a}_\bullet} = \log\left(\sum_{k\geq 1}^{\infty} \epsilon_k |\mathfrak{a}_k|^{\frac{1}{k}}\right) = \log\left(\sum_{k\geq 1}^{\infty} \epsilon_k \left(|g_1^{(k)}| + \cdots + |g_m^{(k)}|\right)^{\frac{1}{k}}\right) \tag{2.1.20}$$

on a domain $U \subset X$ where every graded piece $\mathfrak{a}_k$ is an ideal with a choice of a finite number of generators, say $g_1^{(k)}, \ldots, g_m^{(k)}$ (m is not fixed). It is convenient to use the notation $|\mathfrak{a}_k| := |g_1^{(k)}| + \ldots + |g_m^{(k)}|$ as in (2.1.20) with the convention that each time the notation $|\mathfrak{a}_k|$ is used, a specific choice of a finite number of generators is made.

Also ϵ_k's are a choice of positive coefficients such that the infinite series converges. It is known that in general the singularity equivalence class of $\varphi_{\mathfrak{a}_\bullet}$ depends on the choice of coefficients (see [K15]).

In algebraic geometry, the algebraic construction of asymptotic multiplier ideals associated to $\mathfrak{a}_\bullet$ is now standard and has been very useful (see [L, Chap.11]). Following [L, (11.1.15)], the asymptotic multiplier ideal sheaf of $\mathfrak{a}_\bullet$ with coefficient c, $\mathcal{J}(c \cdot \mathfrak{a}_\bullet)$ is defined to be the unique maximal member in the family of ideals $\{\mathcal{J}(\frac{c}{q} \cdot \mathfrak{a}_q)\}_{q\geq 1}$.

Theorem 2.1.21 (Sébastien Boucksom)**.** *For a Siu psh function $\varphi_{\mathfrak{a}_\bullet}$ defined above (on $U \subset X$), we have $\mathcal{J}(c\varphi_{\mathfrak{a}_\bullet}) = \mathcal{J}(c \cdot \mathfrak{a}_\bullet)$ for every real $c > 0$.*

The proof of this theorem is in the appendix. It is interesting to note that both Siu psh functions and asymptotic multiplier ideals first appeared in the context of the celebrated work by Yum-Tong Siu on the invariance of plurigenera [S98] in the general type case and in the subsequent algebraic translation [Ka99], [L, (11.1.A)] (and then also in [ELS01]). We immediately obtain the following

Corollary 2.1.22. *Let $\varphi = \log\left(\sum_{k\geq 1}^{\infty} \epsilon_k |\mathfrak{a}_k|^{\frac{1}{k}}\right)$ and $\tilde{\varphi} = \log\left(\sum_{k\geq 1}^{\infty} \tilde{\epsilon}_k |\mathfrak{a}_k|^{\frac{1}{k}}\right)$ be two Siu psh functions associated to a graded system of ideals $\mathfrak{a}_\bullet$ with all $\epsilon_k, \tilde{\epsilon}_k > 0$. Then for every $c > 0$, we have $\mathcal{J}(c\varphi) = \mathcal{J}(c\tilde{\varphi})$. In particular, φ and $\tilde{\varphi}$ are v-equivalent psh functions.*

Here we remark that for each $\mathfrak{a}_k$, two different choices of generators are allowed for $|\mathfrak{a}_k|$ used in φ and in $\tilde{\varphi}$ respectively, following the previously made convention for $|\mathfrak{a}_k|$.

Remark 2.1.23. Theorem 2.1.21 and Corollary 2.1.22 completely generalize the previously known 'global' special case when the Siu psh functions φ and $\tilde{\varphi}$ are given as local weight functions of two singular hermitian metrics of Siu type of a big line bundle on a projective complex manifold: see the comment before [K15, (1.1)] where [DEL, Theorem 1.11] together with Theorem 2.1.19 was used for this.

Remark 2.1.24. For our purpose in this thesis, it is sufficient to define a Siu psh function only on $U \subset X$ as in (2.1.20) where each ideal $\mathfrak{a}_k$ has a chosen finite set of generators. More generally, when each of $\mathfrak{a}_k$ is an ideal sheaf and X is covered by such U's, there are two options. One is to use the global version of a Siu psh function, namely a singular hermitian metric of a line bundle which is locally of the form (2.1.20). The other is only to define Siu psh functions on each U as above, and in the intersections, different Siu psh functions will be v-equivalent as in Corollary 2.1.22. In either case, Theorem 2.1.21 still makes sense.

Remark 2.1.25. Theorem 2.1.21 has the pleasant consequence that all the types of algebraic multiplier ideals appearing in the standard algebraic theory of multiplier ideals as in [L, Part Three] can now be considered in the uniform setting of psh functions (and singular hermitian metrics of line bundles which have psh local weight functions) : namely, multiplier ideals associated to an effective $\mathbf{Q}$-divisor, to an ideal sheaf, to a linear system and asymptotic multiplier ideals. See Definitions (9.2.1), (9.2.3), (9.2.10), (11.1.2), (11.1.15), (11.1.24) in [L].

2.1.4 Toric psh functions and Newton convex bodies

A psh function $\varphi(z_1, \ldots, z_n)$ is called *toric* (or *multi-circled* in [R13b]) if the value depends only on $|z_1|, \ldots, |z_n|$, i.e., $\varphi(z_1, \ldots, z_n) = \varphi(|z_1|, \ldots, |z_n|)$. (See e.g. [G] and [R13b] for more information on toric psh functions.) Toric psh functions play an important role in this thesis. For the purpose of this thesis, we may assume that the domain of toric psh functions in the following is the unit polydisk $\mathbf{D}^n = D(0,1) \subset \mathbf{C}^n$ (i.e. with the center at the origin and the polyradius $(1, \ldots, 1)$).

It is well known that given a toric psh function $\varphi(z_1, \ldots, z_n)$ on $\mathbf{D}^n$, one has a convex function $\widehat{\varphi}$ on $\mathbf{R}^n_-$, non-decreasing in each variable, associated to φ such that

$$\varphi(z_1, \ldots, z_n) = \widehat{\varphi}(\log |z_1|, \ldots, \log |z_n|).$$

Moreover, there is a naturally associated closed convex subset $P(\varphi) \subset \mathbf{R}^n_{\geq 0}$ satisfying $P(\varphi) + \mathbf{R}^n_{\geq 0} \subset P(\varphi)$ that generalizes the Newton polyhedron associated to a monomial ideal. See [R13b, (3.1)] for its construction where it is called the indicator diagram of φ. We call $P(\varphi)$ the Newton convex body of φ. (This $P(\varphi)$ is the closure of what is called the Newton convex body of φ in [G], see [G, Lemma 1.19].) The key property of $P(\varphi)$ we will use is the following characterization of multiplier ideals.

Proposition 2.1.26. *[G, Theorem A], [R13b, Proposition 3.1]*

Let $\varphi = \varphi(z_1, \ldots, z_n)$ be a toric psh function. Then the multiplier ideal $\mathcal{J}(\varphi)$ is a monomial ideal such that $z_1^{a_1} \ldots z_n^{a_n} \in \mathcal{J}(\varphi)$ if and only if $(a_1 +$

$1, \dots, a_n + 1) \in \operatorname{int} P(\varphi)$.

Now given a closed convex subset $P \subset \mathbf{R}^n_{\geq 0}$ satisfying $P + \mathbf{R}^n_{\geq 0} \subset P$, one can associate a graded system of monomial ideals as in [M02], [ELSV], [JM, S 8], as follows:

Example 2.1.27. Make a choice of an infinite sequence of closed rational convex polyhedra satisfying

(1) $R_1 \subseteq R_2 \subseteq R_3 \subseteq \cdots \subseteq P$,

(2) $R_m + \mathbf{R}^n_{\geq 0} \subset R_m$ for every m,

(3) $P = \bigcup R_m$.

Then $E_k := k \cdot R_k$ satisfies $E_l + E_m \subseteq E_{l+m} \quad$ for all $l, m \geq 1$.

Let $\mathfrak{a}_k$ be the ideal generated by all monomials in $\mathbf{C}[x_1, \dots, x_n]$ whose exponents are contained in E_k. Then the family $\mathfrak{a}_\bullet = (\mathfrak{a}_k)$ is a graded system of monomial ideals. Now take a Siu psh function associated to $\mathfrak{a}_\bullet$:

$$\varphi = \varphi_{\mathfrak{a}_\bullet} = \log \left(\sum_{k \geq 1}^{\infty} \epsilon_k |\mathfrak{a}_k|^{\frac{1}{k}} \right) \tag{2.1.28}$$

where ϵ_k is a choice of a sequence of nonnegative coefficients to make the series converge.

Using the above, we can show that given a Newton convex body, there exist many toric psh functions sharing the same Newton convex body using the following proposition. Note that $\varphi = \varphi_{\mathfrak{a}_\bullet}$ in the above example is a toric psh function.

Proposition 2.1.29. *Let $P \subset \mathbf{R}^n_{\geq 0}$ and $\varphi_{\mathfrak{a}_\bullet}$ be as in Example 2.1.27. The Newton convex body of $\varphi_{\mathfrak{a}_\bullet}$ is equal to P.*

Proof. Let Q be the Newton convex body of $\varphi_{\mathfrak{a}_\bullet}$. If Q is not equal to P, then for some sufficiently large $m > 0$, the multiplier ideals $\mathcal{J}(m\varphi_{\mathfrak{a}_\bullet})$ and $\mathcal{J}(m\mathfrak{a}_\bullet)$ must be different monomial ideals since we have Howald type theorems (cf.[L, 9.3.27]) for $\mathcal{J}(m\varphi_{\mathfrak{a}_\bullet})$ by (2.1.26) and for the asymptotic multiplier ideals $\mathcal{J}(m\mathfrak{a}_\bullet)$ by [JM, Proposition 8.4], respectively. Such difference contradicts Theorem 2.1.21. $\square$

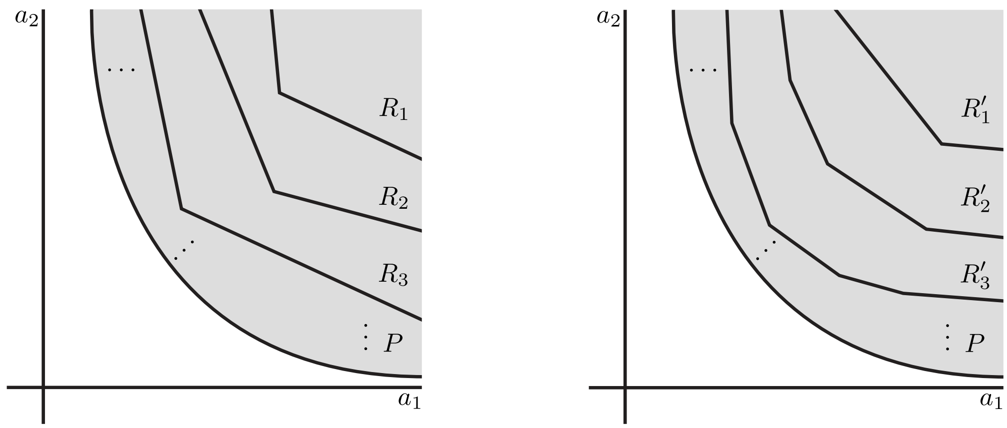

Figure 2.1: Two different choices of a sequence of closed rational convex polyhedra for P in Example 2.1.27

As a consequence of Proposition 2.1.29, we obtain at once so many examples of toric psh functions sharing the same Newton convex body, yet no pair of which is equivalent to each other for at least two reasons. One is from [K15, Theorem 3.5], i.e. from different choices of the coefficients ϵ_k's.

Another is from different choices of a sequence of closed rational convex polyhedra $R_1 \subset R_2 \subset \ldots$ and $R'_1 \subset R'_2 \subset \ldots$ as in Figure 2.1. For reference, we explicitly write out two of those psh functions below again as in (2.1.20). The ideal $\mathfrak{a}'_k$ is the one generated by all monomials whose exponents are contained in $E'_k := k \cdot R'_k$ as in Example 2.1.27.

$$\varphi_{\mathfrak{a}_\bullet} = \log\left(\sum_{k\geq 1}^{\infty} \epsilon_k |\mathfrak{a}_k|^{\frac{1}{k}}\right) = \log\left(\sum_{k\geq 1}^{\infty} \epsilon_k \left(|g_1^{(k)}| + \cdots + |g_m^{(k)}|\right)^{\frac{1}{k}}\right)$$

$$\varphi'_{\mathfrak{a}_\bullet} = \log\left(\sum_{k\geq 1}^{\infty} \epsilon'_k |\mathfrak{a}'_k|^{\frac{1}{k}}\right) = \log\left(\sum_{k\geq 1}^{\infty} \epsilon'_k \left(|{g'}_1^{(k)}| + \cdots + |{g'}_{m'}^{(k)}|\right)^{\frac{1}{k}}\right)$$

Since the multiplier ideals of all these psh functions are determined by the same Newton convex body, they all have the same multiplier ideals $\mathcal{J}(m\varphi)$ for all $m > 0$, i.e. they are v-equivalent : $\varphi_{\mathfrak{a}_\bullet} \sim_v \varphi'_{\mathfrak{a}_\bullet}$.

2.2 Jumping numbers

2.2.1 Definition and basic properties

We define jumping numbers and prove some basic properties.

Definition 2.2.1. Let φ be a psh function on a complex manifold X. A real number $\alpha > 0$ is a **jumping number** of φ at $x \in X$ if the multiplier ideals $\mathcal{J}(c\varphi)$ are constant at x precisely for $c \in [\alpha, \alpha + \delta)$ for some $\delta > 0$ (in the sense of (1.1.1) in the introduction). Denote the set of those jumping numbers by $\text{Jump}(\varphi)_x$.

One may also consider a global version of the set of jumping numbers for a compact complex manifold X defining $\text{Jump}(\varphi)_X := \bigcup_{x \in X} \text{Jump}(\varphi)_x$ (cf. [D15]). Note that $\text{Jump}(\varphi)_x$ and $\text{Jump}(\varphi)_X$ all satisfy DCC due to the openness theorem (2.1.19). Since any set of real numbers satisfying DCC is countable at most, the sets $\text{Jump}(\varphi)_x$ and $\text{Jump}(\varphi)_X$ are countable.

In this thesis, we will concentrate on the study of the jumping numbers at a point as in [ELSV], which then can be applied to study the global version $\text{Jump}(\varphi)_X$. As in [D15], $\text{Jump}(\varphi)_X$ is necessary when one wants to keep track of all the (possibly non-reduced) subschemes associated to $\mathcal{J}(c\varphi)$ for all $c > 0$.

Now for basic properties of jumping numbers, we will first generalize [ELSV, (1.17) and (5.8)] to general psh functions. Let φ be a psh function on a complex manifold X. Let $x \in X$ be a point where the Lelong number of φ, $\nu(\varphi, x) > 0$. Let $c(\varphi, x)$ be the log canonical threshold of φ at x.

Let χ be one of the jumping numbers of φ at x : i.e. $\chi \in \text{Jump}(\varphi)_x$. By the openness theorem (2.1.19), there exists the 'next' jumping number after χ, which we denote by χ' in the following statement.

Proposition 2.2.2. $\chi' \leq \chi + c(\varphi, x)$.

Proof. It is enough to show that $\mathcal{J}\left((\chi + c(\varphi, x))\varphi\right)_x \neq \mathcal{J}(\chi\varphi)_x$. By the subadditivity of multiplier ideals [DEL, Theorem 2.6], we have $\mathcal{J}((\chi + c(\varphi, x))\varphi) \subseteq \mathcal{J}(\chi\varphi) \cdot \mathcal{J}(c(\varphi, x)\varphi)$. Since $\mathcal{J}(c(\varphi, x)\varphi) \neq \mathcal{O}_{X,x}$, the conclusion follows. $\square$

Using (2.1.26), we next find a relation between the jumping numbers of a toric psh function φ at the origin and its Newton convex body. Denote by $C(\varphi)$, the set of positive real numbers c such that $\partial P(c\varphi) \cap \mathbf{Z}^n_{>0}$ is not empty.

Proposition 2.2.3. *Let φ be a toric psh function on the unit polydisk $\mathbf{D}^n \subseteq \mathbf{C}^n$. Then $C(\varphi) \subseteq \mathrm{Jump}(\varphi)_0$ and $\mathrm{Jump}(\varphi)_0$ is the closure of $C(\varphi)$ in $\mathbf{R}$.*

Proof. For $c \in C(\varphi)$, we can find $A + \mathbf{1} = (a_1 + 1, \ldots, a_n + 1) \in \partial P(c\varphi) \cap \mathbf{Z}^n_{>0}$. Since $A + \mathbf{1} \in \operatorname{int} P((c-\epsilon)\varphi)$ for $0 < \epsilon \ll 1$, $z^A \in \mathcal{J}((c-\epsilon)\varphi)$ but $z^A \notin \mathcal{J}(c\varphi)$. Therefore, c is a jumping number of φ at 0.

For the second statement, it is enough to show that the set of jumping numbers which are not cluster points is contained in $C(\varphi)$. If $c > 0$ is a jumping number of φ at 0 and it is not a cluster point of $\mathrm{Jump}(\varphi)_0$, then one can find $z^A = z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n} \in \mathcal{J}((c-\epsilon)\varphi) \backslash \mathcal{J}(c\varphi)$ for sufficiently small $\epsilon > 0$ (where A is independent of ϵ). We have

$$A + \mathbf{1} \in \operatorname{int} P((c-\epsilon)\varphi) \quad \text{but} \quad A + \mathbf{1} \notin \operatorname{int} P(c\varphi).$$

Since the intersection of $\operatorname{int} P((c-\epsilon)\varphi)$ for $0 < \epsilon \ll 1$ is $P(c\varphi)$, we have $\partial P(c\varphi) \cap \mathbf{Z}^n_{>0} \neq \emptyset$. $\square$

On the other hand, the following generalizes [ELSV, Lemma 5.12].

Proposition 2.2.4. *(1) Let φ be a psh function with isolated singularities at $p \in X$. Then the set of its jumping numbers at a point, does not have a cluster point.*

(2) Let $\varphi \in \mathcal{E}(\Omega)$ be a psh function in the Cegrell class. Then the set of its jumping numbers at a point $x \in \Omega$, does not have a cluster point.

A psh function in the Cegrell class is the ultimate generalization of a psh function with isolated singularities : see [C04].

Proof. Let $\mathfrak{m}$ be the maximal ideal of the point p. For $\ell > 0$, the multiplier ideal $\mathcal{J}(\ell\varphi)$ is $\mathfrak{m}$-primary. Thus the colength of $\mathcal{J}(\ell\varphi)$, $\dim \mathcal{O}_X/\mathcal{J}(\ell\varphi)$ gives an upper bound for the possibilities of different ideals $\mathcal{J}(a\varphi)$ with $0 < a < \ell$ since $\dim \mathcal{O}_X/\mathcal{J}(\ell\varphi) \geq \dim \mathcal{O}_X/\mathcal{J}(a\varphi)$.

For (2), the same argument holds since in this case, the stalks of the multiplier ideals $\mathcal{J}(\ell\varphi)$ are either trivial or $\mathfrak{m} = \mathfrak{m}_x$-primary at each point $x \in \Omega$ (see [KR, (3.2)]). □

We note that this proposition can be used to decide whether a psh function belongs to the Cegrell class or not.

Now we give some examples of jumping numbers.

Example 2.2.5 ([ELSV])**.** Consider the convex region $P = \{(x, y) \in \mathbf{R}^2_{>0} : (x-1)(y-1) \geq 1\}$. Apply the construction of Example 2.1.27 to obtain a Siu psh function $\varphi_{\mathfrak{a}_\bullet}$ on $\mathbf{D}^2 \ni (0,0)$. By Theorem 2.1.21, the jumping numbers of $\varphi_{\mathfrak{a}_\bullet}$ are precisely the jumping numbers of $\mathfrak{a}_\bullet$ given in [ELSV, Example 5.10] :

$$\mathrm{Jump}(\varphi_{\mathfrak{a}_\bullet})_{(0,0)} = \left\{ \frac{ef}{e+f} : e, f \geq 1 \right\}. \tag{2.2.6}$$

In this example, all positive integers are cluster points of jumping numbers. On the other hand, the following example illustrates that in general, the jumping numbers of $\varphi+\psi$ may not be simply described in terms of the jumping numbers of φ and ψ.

Example 2.2.7 (Jumping numbers of $\mathcal{J}(c(\varphi + \psi))$)**.** Let $\varphi = \varphi_{\mathfrak{a}_\bullet}$ be as in Example 2.2.5 and let $\psi = \log|z_1|$. The Newton convex body of $c(\varphi + \psi)$ is $P(c(\varphi+\psi)) = c(P(\varphi)+P(\psi))$ by Lemma 2.2.8. The boundary of $P(c(\varphi+\psi))$ is given by the equations $(x-2c)(y-c) = c^2$ and $x > 2c$. Using Proposition 2.2.3, we find that the set of jumping numbers for $\varphi + \psi$ is given by

$$\mathrm{Jump}(\varphi + \psi; 0)_{(0,0)} = \mathrm{cl}\left(\left\{ \frac{(p+2q) - \sqrt{p^2+4q^2}}{2} : p, q \in \mathbf{Z}_{>0} \right\} \right),$$

where $\mathrm{cl}(A)$ is the closure of a subset $A \subset \mathbf{R}$ in $\mathbf{R}$.

The following lemma was used in the above example.

Lemma 2.2.8. *Let φ and ψ be toric psh functions on the polydisk $D(0,r) \subseteq \mathbf{C}^n$. Then the Newton convex body of $\varphi+\psi$ is the Minkowski addition of $P(\varphi)$ and $P(\psi)$, i.e. $P(\varphi+\psi) = P(\varphi) + P(\psi)$.*

Proof. Let f_1 and f_2 be convex functions on $(-\infty, \log r)^n$, non-decreasing in each variable, associated to φ and ψ respectively. We may assume that f_1 and f_2 are defined on $\mathbf{R}^n$ by putting $f_1(x) = f_2(x) = \infty$ whenever $x \notin (-\infty, \log r)^n$. Also replacing f_1 and f_2 by their lower semi-continuous regularizations, we may assume that f_1 and f_2 are lower semi-continuous on $\mathbf{R}^n$. Let $\tilde{f}_1$, $\tilde{f}_2$ and $\widetilde{f_1 + f_2}$ be the Legendre transforms of f_1, f_2 and $f_1 + f_2$. Then by [H, Theorem 2.2.5], $\widetilde{f_1 + f_2}$ is the lower semi-continuous regularization of

$$g(y) = \inf_{y_1+y_2=y} \Big(\tilde{f}_1(y_1) + \tilde{f}_2(y_2)\Big),$$

which implies that $y \in P(\varphi+\psi)$ if and only if there exist $y_1 \in P(\varphi)$ and $y_2 \in P(\psi)$ such that $y_1 + y_2 = y$. ☐

Remark 2.2.9. In the proof of Lemma 2.2.8, we replaced the Newton convex bodies by their interiors so that the lower semi-continuous regularizations of f_1, f_2 and g are identically equal to the original functions on their Newton convex bodies. This does not affect the conclusion since the lower semi-continuous regularization changes values only on the boundary of the Newton convex bodies.

2.2.2 "Mixed" jumping numbers

It is also very natural to consider the following more general type of jumping numbers arising from the multiplier ideals $\mathcal{J}(c\varphi + \psi)$ for psh functions φ and ψ. In particular, this type of jumping numbers is used in a recent work of Demailly [D15] on L^2 extension theorems from non-reduced subschemes defined by multiplier ideal sheaves. These jumping numbers play an important role there to keep track of all the non-reduced subschemes defined by $\mathcal{J}(c\varphi+\psi)$ for all $c > 0$.

Definition 2.2.10. A real number $\alpha > 0$ is a **jumping number of** φ **at** x **with respect to** ψ if the multiplier ideals $\mathcal{J}(c\varphi+\psi)$ are constant at x precisely for $c \in [\alpha, \alpha+\delta)$ for some $\delta > 0$. Denote the set of them by $\mathrm{Jump}(\varphi;\psi)_x$.

Note that $\mathrm{Jump}(\varphi;\psi)_x = \mathrm{Jump}(\varphi)_x$ when ψ is locally bounded, according to our previous notation.

In general, the jumping numbers of $\mathcal{J}(c\varphi)$ and those of $\mathcal{J}(c\varphi+\psi)$ look quite different from each other as seen already in the algebraic case : Let D and F are effective divisors on X. Let $f : X' \to X$ be a log-resolution of $(X, D+F)$. Let $f^*D = \sum r_i E_i$ and $f^*F = \sum s_i E_i$.

Then the jumping numbers of (X, cD) are contained in

$$A := \left\{ \frac{b_j + m}{r_j} \right\}$$

(where $m \geq 1$ are integers). On the other hand, the jumping numbers of $(X, cD+F)$ are contained in

$$B := \left\{ \frac{b_j + m - s_j}{r_j} \right\}.$$

In other words, these sets of numbers are candidate jumping numbers and not all of them are actual jumping numbers (see e.g. [ST07]). In any case, those two sets A and B are rather different set of rational numbers except when $\frac{s_j}{r_j}$ is constant for every j, in which case the one set of jumping numbers are translates of the other.

For general psh functions with not necessarily analytic singularities [D11, (1.10)], s_j and r_j are generalized as the values of divisorial valuations of the psh functions. Thus we are lead to the following proposition.

Proposition 2.2.11. *Suppose that psh functions $r\varphi$ and ψ are v-equivalent for some constant $r > 0$. Then the elements of* $\mathrm{Jump}(\varphi;\psi)_x$ *are 'translates' by $-r$ of the elements in* $\mathrm{Jump}(\varphi;0)_x$.

Proof. We have $(m+r)\varphi \sim_v m\varphi + \psi$. Then for the multiplier ideals, we get (by [BFJ], [GZ]) $\mathcal{J}((m+r)\varphi) = \mathcal{J}(m\varphi+\psi)$ for every real $m > 0$. □

For mixed jumping numbers, we first record the following example where $\mathrm{Jump}(\varphi; 0)_x$ and $\mathrm{Jump}(\varphi; \psi)_x$ are equal while (multiples of) φ and ψ are not v-equivalent.

Example 2.2.12 (Jumping numbers of $\mathcal{J}(\psi + c\varphi)$)**.** Let $\varphi = \varphi_{\mathfrak{a}_\bullet}$ on $\mathbf{C}^2$ be as in Example 2.2.5. Let $\psi = \log |z_1|$. Since $\psi + c\varphi$ is toric psh, we can use Proposition 2.1.26 to determine $\mathcal{J}(\psi + c\varphi)$. The Newton convex body of $\psi + c\varphi$ is given by

$$P(\psi + c\varphi) = \{(x, y) \in \mathbf{R}^2_{\geq 0} : (x - 1 - c)(y - c) \geq c^2, x \geq 1 + c\}.$$

By Proposition 2.2.3, a positive real number c is an element of $\mathrm{Jump}(\varphi; \psi)_{(0,0)}$ if and only if there exist two positive integers p and q such that (p, q) is on the boundary of $P(\psi + c\varphi)$. Therefore we have

$$\mathrm{Jump}(\varphi; \psi)_{(0,0)} = \left\{ \frac{ef}{e+f} : e, f \geq 1 \right\} = \mathrm{Jump}(\varphi; 0)_{(0,0)}.$$

In the following example, $\mathrm{Jump}(\varphi; \psi)_x$ are translates by -1 of $\mathrm{Jump}(\varphi; 0)_x$ while (multiples of) φ and ψ are not v-equivalent. Note that v-equivalence of toric psh functions is determined by the Newon convex bodies due to (2.1.26).

Example 2.2.13 (Jumping numbers of $\mathcal{J}(\psi + c\varphi)$)**.** Consider two convex sets P_1 and P_2 given by

$$P_1 = \{(x, y) \in \mathbf{R}^2_{\geq 0} : xy \geq 1\}, \quad P_2 = \{(x, y) \in \mathbf{R}^2_{>0} : (x - 1)(y - 1) \geq 1\}.$$

From Proposition 2.1.29, there exist psh functions φ and ψ such that $P(\varphi) = P_1$ and $P(\psi) = P_2$. Then we have

$$\mathrm{Jump}(\varphi)_{(0,0)} = \left\{ \sqrt{k} : k \in \mathbf{Z}_{>0} \right\} \quad \text{and} \quad \mathrm{Jump}(\psi)_{(0,0)} = \left\{ \frac{ef}{e+f} : e, f \in \mathbf{Z}_{>0} \right\}.$$

By Lemma 2.2.8, we have

$$P(\psi + c\varphi) = \{(x, y) \in \mathbf{R}^2_{\geq 0} : (x-1)(y-1) = (c+1)^2\}$$

for $c > 0$. Therefore we obtain

$$\mathrm{Jump}(\varphi; \psi)_{(0,0)} = \left\{ \sqrt{k} - 1 : k \in \mathbf{Z}_{>0}, k \geq 2 \right\}.$$

2.3 Periodicity of jumping numbers fails

In [ELSV, Proposition A, Proposition 1.12], they showed that in the algebraic case, jumping numbers have a periodic behavior in the algebraic case. In [ELSV], a specific definition of periodicity was not given. The following is a general definition of periodicity which generalize the phenomenon of [ELSV] Proposition 1.12 and Proposition A for general psh functions.

Definition 2.3.1. The set of jumping numbers $J := \mathrm{Jump}(\varphi; \psi)_x$ (or more generally, a set of nonnegative real numbers) is said to have a **period** $c > 0$ ($c \in \mathbf{R}$) if the following holds: for every $\alpha \in J$, there exists an integer $m_0 = m_0(\alpha, J) \geq 1$ such that $m \geq m_0$ implies $\alpha + mc \in J$.

Note that there can be possibly more than one period for $\mathrm{Jump}(\varphi; \psi)_x$ as easily seen from the following example which is a psh version of [ELSV, Example 1.9, Example 5.1 (iii)] : *diagonal ideals*.

Example 2.3.2. For positive real numbers $m_1, \dots, m_n$, consider the psh function $\varphi = \log(|z_1|^{m_1} + \cdots + |z_n|^{m_n})$. The jumping numbers of φ at the origin $0 \in \mathbf{C}^n$ consist of

$$\mathrm{Jump}(\varphi)_0 = \left\{ \frac{e_1 + 1}{m_1} + \cdots + \frac{e_n + 1}{m_n} : e_1, \dots, e_n \in \mathbf{Z}_{\geq 0} \right\}.$$

Now we will show by an example that the periodicity of jumping numbers does not hold for general psh functions. Takayuki Koike [Ko15] gave an example of a psh function φ with the following set of jumping numbers at a point. The psh function φ appears as a local weight function of a singular hermitian metric for a holomorphic line bundle on some smooth projective variety X (given by [N04]) of dimension 4 (see [Ko15, Theorem 1.1]).

Example 2.3.3. Let $x \in X$ and φ be as in [Ko15, Theorem 1.1] so that

$$\operatorname{Jump}(\varphi)_x = \left\{ \frac{1}{2}(p + \sqrt{2a^2p^2 - q^2}) : p, q \in \mathbf{Z}, p > q \geq 0,\ p \equiv q \mod 2 \right\}$$

where $a \geq 2$ is a fixed integer.

It was mentioned in [Ko15, p.299] that "it seems difficult to expect the periodicity property" for this example. We confirm that it is indeed the case.

Proposition 2.3.4. *For the psh function φ at x in Example 2.3.3, the set of jumping numbers $\operatorname{Jump}(\varphi)_x$ does not have a period.*

Proof. It suffices to show that $J := 2\operatorname{Jump}(\varphi)_x$ has no periods. Assume that J has a period c. Then there exist positive integers α and m such that α, $\alpha + mc$ and $\alpha + 2mc$ are in J since we can take

$$\alpha = p_1 + \sqrt{2a^2p_1^2 - q_1^2}$$

where $p_1 = z$ and $q_1 = a|x - y|$ for some $x, y, z \in \mathbf{Z}_{>0}$ satisfying $x^2 + y^2 = z^2$. Since such a Pythagorean triple (x, y, z) can be written as

$$x = 2nk + k^2, \quad y = 2n^2 + 2nk, \quad z = 2n^2 + 2nk + k^2$$

for positive integers k, n, we have

$$\frac{z}{|x - y|} = \frac{\left(\frac{k}{n}\right)^2 + \frac{2k}{n} + 2}{\left|\left(\frac{k}{n}\right)^2 - 2\right|}.$$

Therefore we get $p_1 > q_1$ when we take $\frac{k}{n}$ sufficiently close to $\sqrt{2}$. Also $p_1 \equiv q_1 (\text{mod } 2)$ holds when we take k even. Then by the definition of J, we can write

$$\alpha + mc = p_2 + \sqrt{2a^2p_2^2 - q_2^2},$$
$$\alpha + 2mc = p_3 + \sqrt{2a^2p_3^2 - q_3^2}$$

for some p_i, q_i. Thus we have

$$2p_2 - p_3 - \alpha = \sqrt{2a^2p_3^2 - q_3^2} - 2\sqrt{2a^2p_2^2 - q_2^2}. \tag{2.3.5}$$

Since the left hand side of (2.3.5) is an integer, the right hand side is also an integer. Observe that if $\sqrt{a}-\sqrt{b}$ is an integer for two integers a and b, then both $\sqrt{a}$ and $\sqrt{b}$ are integers. With this observation, $\sqrt{2a^2p_3^2 - q_3^2}$ and $2\sqrt{2a^2p_2^2 - q_2^2}$ are also integers, which implies that c is rational.

Now take a positive integer n such that nc is an integer. Let $\beta \in J$ be irrational. There exists a positive integer m_0 such that $\beta + mc \in J$ for all $m \geq m_0$. For $m \geq m_0$, we have $\beta + (mn)c \in J$. Hence for some $p,\ q,\ r,\ s$, we have

$$\beta = p + \sqrt{2a^2p^2 - q^2},$$
$$\beta + (mn)c = r + \sqrt{2a^2r^2 - s^2}.$$

This implies that $\sqrt{2a^2p^2 - q^2} - \sqrt{2a^2r^2 - s^2}$ is an integer and thus $\sqrt{2a^2p^2 - q^2}$ and $\sqrt{2a^2r^2 - s^2}$ are integers as well. This contradicts to irrationality of β. Hence the period c cannot exist. $\square$

Before closing this section, let us consider again Example 2.2.5. If a period c exists in Example 2.2.5, it must be an integer. A possible way to show that a period cannot exist in this case is as follows: consider jumping numbers $\frac{e}{e+1}$ converging to 1. If a period c exists, $\frac{e}{e+1} + c$ must be all in the set of jumping numbers. They are all having difference $\frac{1}{e+1}$ from the integer c. It reduces to the following arithmetic question whose negative answer will imply that a period cannot exist in this case.

Question 2.3.6. Is it true that for every integer $e, c \geq 1$, one can find integers

$r, s \geq 1$ such that the following holds

$$\frac{e}{e+1} + c = \frac{rs}{r+s}.$$

2.4 Cluster points of jumping numbers

In this section, we will study the 'bad' behaviour of jumping numbers in the general psh case given by cluster points. Recently Guan and Li [GL] gave the following example of a psh function with non-analytic singularities such that it has a cluster point of jumping numbers:

$$\varphi(z_1, z_2) = \log|z_1| + \sum_{k=1}^{\infty} a_k \log(|z_1| + \left|\frac{z_2}{k}\right|^{b_k}) =: \log|z_1| + \psi \tag{2.4.1}$$

which is toric psh, where $a_k = M^{-k}, b_k = M^{2k} \quad (M \geq 2)$.

Theorem 2.4.2 (Guan and Li, [GL, Theorem 1.1])**.** *In the set of jumping numbers* $\mathrm{Jump}(\varphi)_0$ *at the origin, there exists a sequence* $(c_k)_{k\geq 1}$ *converging to* 1*, i.e.* 1 *is a cluster point of jumping numbers.*

On the other hand, a cluster point of jumping numbers was known in the context of graded system of ideals from [ELSV, Example 5.10]. Thanks to Theorem 2.1.21, this gives another example of a psh function with a cluster point of jumping numbers as noted in Example 2.2.5.

In this section, we will completely generalize these two particular examples in Theorem 2.4.8.

2.4.1 Cluster points of jumping numbers for toric psh functions

We first note the following general property.

Proposition 2.4.3. *Let c be a cluster point of the set of jumping numbers* $\mathrm{Jump}(\varphi;\psi)_x$*. Then c itself is also a jumping number in* $\mathrm{Jump}(\varphi;\psi)_x$*.*

Proof. Suppose not. Then there exists $\epsilon > 0$ such that for $d \in (c-\epsilon, c+\epsilon)$, the multiplier ideals $\mathcal{J}(d\varphi + \psi)$ are constant. This contradicts to c being a cluster point. $\square$

When a psh function is toric, we answer Question 1.1.2 by the following

Theorem 2.4.4. *Let φ be a toric psh function on the unit polydisk $\mathbf{D}^n \subset \mathbf{C}^n$. If c is a jumping number at the origin $0 \in \mathbf{D}^n$, then for each integer $m \geq 1$, mc is also a jumping number in* $\mathrm{Jump}(\varphi)_0$.

We first note that for the individual examples of toric psh functions in Example 2.2.5, Example 2.3.2, Example 2.3.3, the validity of this theorem can be easily checked.

Proof. Let $P(c\varphi)$ be the Newton convex body of $c\varphi$. Assume that c is a jumping number of φ at 0. By Proposition 2.2.3, it is enough to show that $mC(\varphi) \subset C(\varphi)$. If $c \in C(\varphi)$, we have, since $\partial P(c\varphi) \cap \mathbf{Z}^n_{>0} \neq \emptyset$,

$$\partial P((mc)\varphi) \cap \mathbf{Z}^n_{>0} \supseteq m \cdot (\partial P(c\varphi) \cap \mathbf{Z}^n_{>0}) \neq \emptyset,$$

which implies that $mc \in C(\varphi)$. $\square$

Note that the property in Theorem 2.4.4 describes an intrinsic property satisfied by toric psh functions. This can be used to show that a particular psh function is never toric in any coordinates (as in Example 2.4.6). The definition of toric psh as in Section 2.1.4 depends on the choice of coordinates. An immediate consequence of Theorem 2.4.4 is the following

Corollary 2.4.5. *Let φ be a toric psh function on $\mathbf{D}^n$. If c is a cluster point of jumping numbers at 0, then for each integer $m \geq 1$, mc is also a cluster point of* $\mathrm{Jump}(\varphi)_0$.

We remark that the statement of Theorem 2.4.4 does not hold for non-toric psh functions. Consider the following example of Morihiko Saito [S93, Example 3.5] with computation from [CL, Example 6.1]. (See also e.g. [ELSV, S 2], [BS05], [S07] for more relations between jumping numbers and Bernstein-Sato polynomials.)

Example 2.4.6. Let $f = x^5 + y^4 + x^3y^2$ on $\mathbf{C}^2$. The multiplier ideals $\mathcal{J}(c\varphi)$ for $\varphi = \log|f|$ at $(0,0) \in \mathbf{C}^2$ are given by

$$\mathcal{J}(c\varphi) = \begin{cases} \mathcal{O}_{\mathbf{C}^2,(0,0)} & \text{if } 0 \leq c < \frac{9}{20}, \\ (x,y) & \text{if } \frac{9}{20} \leq c < \frac{13}{20}, \\ (x^2,y) & \text{if } \frac{13}{20} \leq c < \frac{7}{10}, \\ (x^2,xy,y^2) & \text{if } \frac{7}{10} \leq c < \frac{17}{20}, \\ (x^3,xy,y^2) & \text{if } \frac{17}{20} \leq c < \frac{9}{10}, \\ (x^3,x^2y,y^2) & \text{if } \frac{9}{10} \leq c < \frac{19}{20}, \\ (x^3,x^2y,xy^2,y^3) & \text{if } \frac{19}{20} \leq c < 1, \end{cases}$$

where all the ideals are in $\mathcal{O}_{\mathbf{C}^2,(0,0)}$ and $\mathcal{J}(c\varphi) = (f) \cdot \mathcal{J}((c-1)\varphi)$ for $c \geq 1$.

If the statement of Theorem 2.4.4 holds for this φ, then $\frac{27}{20}$ should also be a jumping number of φ, which is contradiction. Thus the statement of Theorem 2.4.4 does not hold for non-toric psh functions.

2.4.2 Cluster points of jumping numbers for toric psh functions in dimension 2

When we restrict the dimension to $n = 2$, we can completely determine the cluster points of jumping numbers for toric psh functions.

Let $\mathrm{pr}_i : \mathbf{R}^2 \to \mathbf{R}$ be the projection to the i-th coordinate for $i = 1, 2$. Set $x_0 := \inf \mathrm{pr}_1(P(\varphi))$ and $y_0 := \inf \mathrm{pr}_2(P(\varphi))$. Define $f, g : \mathbf{R} \to [0, +\infty]$ as follows:

$$f(x) = \begin{cases} +\infty & \text{if } x < x_0, \\ \inf\{y \in \mathbf{R}_{\geq 0} : (x,y) \in P(\varphi)\} & \text{if } x \geq x_0. \end{cases}$$

$$g(y) = \begin{cases} +\infty & \text{if } y < y_0, \\ \inf\{x \in \mathbf{R}_{\geq 0} : (x,y) \in P(\varphi)\} & \text{if } y \geq y_0. \end{cases}$$

Note that the epigraph $\{(x,t) \in \mathbf{R}^2 : x \geq x_0, t \geq f(x)\}$ of f is equal to $P(\varphi)$, thus f is a convex function (see [H, §2.1]). Similarly, g is also a convex function. From the inclusion $P(\varphi) + \mathbf{R}^2_{\geq 0} \subseteq P(\varphi)$, we conclude that f and g are decreasing and

$$\lim_{x \to +\infty} f(x) = y_0 \quad \text{and} \quad \lim_{y \to +\infty} g(y) = x_0.$$

From this, we have

Lemma 2.4.7. *Let φ be a toric psh function on $\mathbf{D}^2$. Then, for every $\epsilon > 0$, the lines $x = x_0 + \epsilon$ and $y = y_0 + \epsilon$ intersect with the interior of $P(\varphi)$ but the lines $x = x_0$ and $y = y_0$ do not intersect with the interior of $P(\varphi)$. We call the lines $y = y_0$, $x = x_0$ the horizontal asymptote and the vertical asymptote of $P(\varphi)$, respectively.*

Recall that $P(\varphi)$ is always closed from its definition. It is easy to see (from $P(\varphi) + \mathbf{R}^2_{\geq 0} \subseteq P(\varphi)$) that the horizontal asymptote $\{(t, y_0) : t \in \mathbf{R}\}$ either is disjoint from $P(\varphi)$ (as in (f), (h) of Figure 2.2) or has a half-line on it contained in the boundary of $P(\varphi)$ (as in (e), (g) of Figure 2.2). Similarly for the vertical asymptote as well.

Theorem 2.4.8. *Assume that $n = 2$ in the setting of Theorem 2.4.4. The set of jumping numbers* $\mathrm{Jump}(\varphi)_0$ *has at least one (thus infinitely many, according to Corollary 2.4.5) cluster point if and only if at least one of the following (1) and (2) holds:*

1. *$x_0 > 0$ and $\{(x_0, t) : t \in \mathbf{R}\} \cap P(\varphi) = \emptyset$,*
2. *$y_0 > 0$ and $\{(t, y_0) : t \in \mathbf{R}\} \cap P(\varphi) = \emptyset$.*

Moreover, the set of cluster points of jumping numbers is precisely equal to

$$\left\{ \frac{k}{m} : k \in \mathbf{Z}_{>0},\, m \in S \right\}$$

where S is a subset of $\{x_0, y_0\}$ satisfying the following (a) and (b):

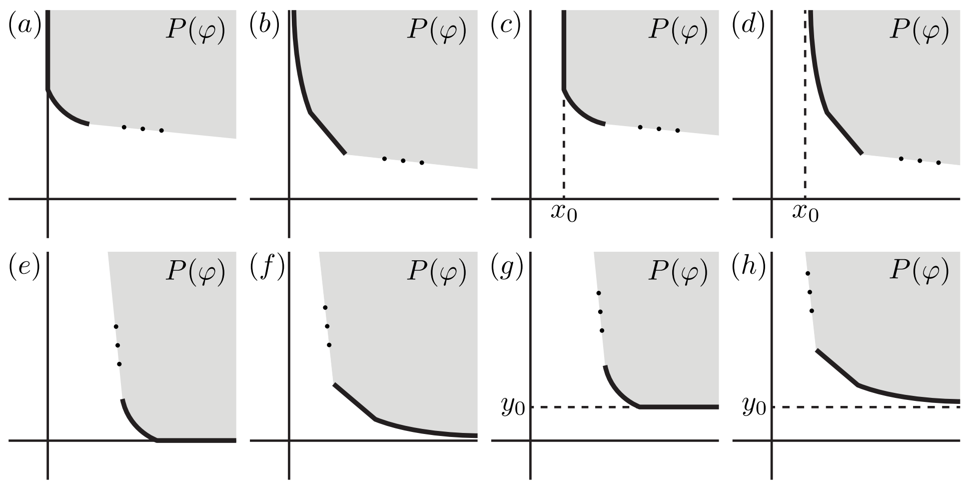

Figure 2.2: All the possible asymptotic behaviors of the boundary of $P(\varphi)$ with respect to the two axes.

(a) $x_0 \in S$ *if and only if (1) holds,*

(b) $y_0 \in S$ *if and only if (2) holds.*

Remark 2.4.9. All the possible asymptotic behaviors of the boundary of $P(\varphi)$ are listed in Figure 2.2. Theorem 2.4.8 exactly says that the jumping numbers of φ at 0 has a cluster point if and only if either (d) or (h) holds. In other words, there does not exist a cluster point if and only if, for the Newton convex body $P(\varphi)$, one of (a,b,c) holds *and* one of (e,f,g) holds.

Remark 2.4.10. One can also easily see that this theorem explicitly generalizes Proposition 2.2.4 in this dimension 2 toric psh case.

Proof of Theorem 2.4.8. Assume that c is a cluster point of jumping numbers of φ at 0. In view of Proposition 2.1.26, we then can find a sequence (ϵ_k) of positive real numbers and a sequence (A_k) in $\mathbf{Z}^2_{\geq 0}$ such that

1. $\epsilon_k > \epsilon_{k+1}$ for every $k \geq 1$ and $\epsilon_k \to 0$,

2. $A_k + \mathbf{1} \in \operatorname{int} P((c - \epsilon_k)\varphi)$ but $A_k + \mathbf{1} \notin \operatorname{int} P((c - \epsilon_{k+1})\varphi)$ for every $k \geq 1$.

Note that $A_k + \mathbf{1} \notin \operatorname{int} P(c\varphi)$ for every $k \geq 1$ and therefore the first coordinate and the second coordinate of A_k cannot increase simultaneously when k increases. By the second property of (A_k), for each $k \geq 1$, we have either $\mathrm{pr}_1(A_k) < \mathrm{pr}_1(A_{k+1})$ or $\mathrm{pr}_2(A_k) < \mathrm{pr}_2(A_{k+1})$ but not both. One can find a subsequence (A_{k_j}) of (A_k) such that either

1. $(\mathrm{pr}_1(A_{k_j}))$ is a constant sequence and $(\mathrm{pr}_2(A_{k_j}))$ is a strictly increasing sequence, or

2. $(\mathrm{pr}_2(A_{k_j}))$ is a constant sequence and $(\mathrm{pr}_1(A_{k_j}))$ is a strictly increasing sequence.

In the case of (1), $cx_0 = \mathrm{pr}_1(A_{k_1}) + 1$ is a positive integer and the line $x = cx_0$ is the vertical asymptote of $P(c\varphi)$. Similarly, if (A_{k_j}) satisfies (2), then $cy_0 = \mathrm{pr}_2(A_{k_1}) + 1$ is a positive integer and the line $y = cy_0$ is the horizontal asymptote of $P(c\varphi)$. Note that $x = cx_0$ in the case of (1) and $y = cy_0$ in the case of (2) do not intersect with $P(c\varphi)$.

Conversely, without loss of generality, we may assume that (1) in the statement holds. We want to show that $\frac{k}{x_0}$ is a cluster point of jumping numbers for every positive integer k. Since the line $x = k$ is the vertical asymptote of $P(\frac{k}{x_0}\varphi)$ which does not meet $P(\frac{k}{x_0}\varphi)$. One can take a sequence (ϵ_j) of positive real numbers and a sequence (α_j) of positive integers such that

1. $\epsilon_j > \epsilon_{j+1}$ for every $j \geq 1$ and $\epsilon_j \to 0$,

2. $\alpha_j < \alpha_{j+1}$ for every $j \geq 1$,

3. $B_j \in \operatorname{int} P((\frac{k}{x_0} - \epsilon_j)\varphi)$ but $B_j \notin \operatorname{int} P((\frac{k}{x_0} - \epsilon_{j+1})\varphi)$ for every $j \geq 1$,

where $B_j = (k, \alpha_j)$. This means that there is a sequence (λ_j) of jumping numbers such that

$$\frac{k}{x_0} - \epsilon_j < \lambda_j \leq \frac{k}{x_0} - \epsilon_{j+1}$$

for each j. Since λ_j converges to $\frac{k}{x_0}$, we conclude that $\frac{k}{x_0}$ is a cluster point of jumping numbers. The last assertion follows immediately from the above argument. □

As a corollary of Theorem 2.4.8, we have

Corollary 2.4.11. *In the setting of Theorem 2.4.4 with $n = 2$, the set of all the cluster points in* $\mathrm{Jump}(\varphi)_0$ *is discrete. In other words, there is no cluster point of cluster points of jumping numbers in this case.*

Now we can strengthen Theorem 2.4.2 of [GL] as follows.

Corollary 2.4.12. *Let φ be the psh function (2.4.1) of [GL] in Theorem 2.4.2. The set T of cluster points of* $\mathrm{Jump}(\varphi)_0$ *is precisely equal to the set of positive integers $\mathbf{Z}_{>0}$.*

Proof. From Theorem 2.4.4 and Theorem 2.4.2, we know that $T \supset \mathbf{Z}_{>0}$. The other inclusion can be shown by proving that $P(\varphi)$ intersects with the line $y = 0$ in view of Theorem 2.4.11 and 1 is the smallest cluster point of jumping numbers. Let $\widehat{\varphi}$ be the convex function associated to φ. Then the gradient of $\widehat{\varphi}$ at (x, y) is

$$\nabla\widehat{\varphi}(x,y) = \left(1 + \sum_{k=1}^{\infty} \frac{M^{-k}e^{x}}{e^{x} + k^{-b_k}e^{b_k y}}, \sum_{k=1}^{\infty} \frac{M^{k}k^{-b_k}e^{b_k y}}{e^{x} + k^{-b_k}e^{b_k y}}\right)$$

and this is contained in $P(\varphi)$(see [G, p. 1015]). Therefore, we have

$$\nabla\widehat{\varphi}(0,y) = \left(1 + \sum_{k=1}^{\infty} \frac{M^{-k}}{1 + k^{-b_k}e^{b_k y}}, \sum_{k=1}^{\infty} \frac{M^{k}e^{b_k y}}{k^{b_k} + e^{b_k y}}\right) \in P(\varphi)$$

and observe that $\nabla\widehat{\varphi}(0, y)$ converges to a point $(\alpha, 0)$, where $\alpha = 1+\frac{1}{M-1}$, when y tends to $-\infty$. Moreover, since $\alpha \leq 2$ for $M \geq 2$ and $P(\varphi)$ does not equal to $(\alpha, 0)+\mathbf{R}^2_{\geq 0}$, the point $(\alpha, 2) \in \mathbf{R}^2$ should be in the interior of $P(\varphi)$. Therefore the point $(2, 2)$ is contained in the interior of $P(\varphi)$. We have $(1, 1) \in \operatorname{int} P(\frac{1}{2}\varphi)$ and thus $1 \in \mathcal{J}(\frac{1}{2}\varphi)$. We conclude that $\mathcal{J}(c\varphi) = \mathcal{O}_{\mathbf{C}^2,0}$ whenever $0 \leq c \leq \frac{1}{2}$ and hence 1 is the smallest cluster point of jumping numbers. $\square$

It is also easy to see that for this example (2.4.1), we have the case (e) and then necessarily (d) (since cluster points exist) of Figure 2.2.

Remark 2.4.13. In Example 2.2.7, the vertical asymptote of $P(\varphi+\psi)$ is $x=2$ and the horizontal asymptote of $P(\varphi+\psi)$ is $y=1$. These asymptotes do not intersect with $P(\varphi+\psi)$. By Theorem 2.4.8, the set of all cluster points of jumping numbers is $\{\frac{k}{2} : k \in \mathbf{Z}_{>0}\}$. This can be also checked by direct computation using $C(\varphi+\psi)$ (see (2.2.3)).

2.5 Appendix by Sébastien Boucksom

Theorem 2.1.21 states that given a graded system of ideals $\mathfrak{a}_\bullet$, its asymptotic multiplier ideals are equal to the analytic multiplier ideals of a Siu psh function $\varphi = \varphi_{\mathfrak{a}_\bullet}$ (on $U \subset X$) associated to $\mathfrak{a}_\bullet$.

Proof of Theorem 2.1.21. Before using the valuative characterizations of multiplier ideal sheaves, we will first prove that $v(\varphi) = v(\mathfrak{a}_\bullet)$ for every divisorial valuation v with nonempty center on $U \subset X$.

We know that $v(\mathfrak{a}_\bullet) = \inf_k \frac{1}{k} v(\mathfrak{a}_k) = \lim_k \frac{1}{k} v(\mathfrak{a}_k)$ from [JM, Lemma 2.3]. Now following [BEGZ, p.258], let $\phi_k := \frac{1}{k!} \log |\mathfrak{a}_{k!}|$ (in the notation of (2.1.20)). Since $v(\varphi) \le v(\phi_k)$ for every k, it follows that $v(\varphi) \le v(\mathfrak{a}_\bullet)$.

On the other hand, by adding constants, we can arrange that φ is the increasing limit of ϕ_k. Then we have $v(\varphi) \ge \limsup_k v(\phi_k) = v(\mathfrak{a}_\bullet)$.

Now the theorem will follow from valuative characterization of multiplier ideals of both sides of $\mathcal{J}(c\varphi) = \mathcal{J}(c \cdot \mathfrak{a}_\bullet)$, i.e. [BFJ, Theorem 5.5] and [BFFU, Theorem 4.1 (b)] respectively. The former states that $f \in \mathcal{J}(c\varphi)$ if and only if there exists $\epsilon > 0$ such that $v(f) \ge (1+\epsilon)v(c\varphi) - A(v)$ for all divisorial valuations with center on $U \subset X$.

The latter states that $f \in \mathcal{J}(c \cdot \mathfrak{a}_\bullet)$ if and only if for every normalizing subscheme $N \subset X$ containing 0 and every $0 < \delta \ll 1$, $v(f) \ge cv(\mathfrak{a}_\bullet) - A(v) + \delta v(\mathcal{I}_N)$ for all divisorial valuations with center on $U \subset X$.

Since $\mathcal{J}(c \cdot \mathfrak{a}_\bullet) = \mathcal{J}(\frac{c}{m}\mathfrak{a}_m)$ for some $m \ge 1$, we have $\mathcal{J}(c \cdot \mathfrak{a}_\bullet) \subset \mathcal{J}(c\varphi)$. For the other direction of inclusion, suppose that there exists $\epsilon > 0$ such that $v(f) \ge (1+\epsilon)v(c\varphi) - A(v)$ holds. Choose δ sufficiently small so that

$\delta v(\mathcal{I}_N) = \delta < \epsilon v(c\varphi)$. Then we have

$$v(f) \geq (1+\epsilon)v(c\varphi) - A(v) > cv(\mathfrak{a}_\bullet) - A(v) + \delta v(\mathcal{I}_N)$$

since $v(c\varphi) = cv(\varphi) = cv(\mathfrak{a}_\bullet)$. This completes the proof. $\square$

Chapter 3

L^2 extension theorems: toward inversion of adjunction

In this chapter, as mentioned in the introduction, we present ongoing work toward the analytic proof of inversion of adjunction, an important theorem in algebraic geometry. This chapter *does not contain original results* yet. We present L^2 extension theorems of Ohsawa-Takegoshi type which are important tools in complex geometry and also in the expected analytic proof of the inversion of adjunction. In particular we will follow the proof of the L^2 extension theorem in Demailly [D15] and work out explicitly some important details used in [D15] which will be crucial in the analytic proof of the inversion of adjunction. In Section 3.2, we deal with singularities of pairs and the relations between singularities and multiplier ideals. We will focus on inversion of adjunction and Kollár's theorem that uses an L^2 extension theorem to prove the inversion of adjunction.

3.1 L^2 estimates for the $\overline{\partial}$ operator

3.1.1 Basic notions

Let (X, ω) be a hermitian manifold of dimension n with a hermitian form ω and let E a holomorphic hermitian vector bundle of rank r over X. A connection

$\nabla : \mathcal{C}^\infty_\bullet(X,E) \to \mathcal{C}^\infty_\bullet(X,E)$ is a linear differential operator of order 1 satisfying

$$\nabla(\alpha \wedge \beta) = (\nabla\alpha) \wedge \beta + (-1)^{|\alpha|}\alpha \wedge (\nabla\beta).$$

For the given holomorphic hermitian vector bundle (E,h) with a hermitian metric h, there is a unique connection D, which is called the *Chern connection*, compatible with the hermitian structure of (E,h) such that D is uniquely decomposed by $D = D' + D''$, where

$$D' : \mathcal{C}^\infty_{p,q}(X,E) \to \mathcal{C}^\infty_{p+1,q}(X,E) \quad \text{and} \quad D'' = \overline{\partial}.$$

For an explicit formula of D, write $De_j = \sum_i a_{ij} \otimes e_i$ with smooth 1-forms a_{ij} for each j. Then for two smooth E-valued forms α, β on an open subset U of X which can be written as

$$\alpha = \sum_{j=1}^r \alpha_j \otimes e_j \in \mathcal{C}^\infty_{p,q}(U,E), \quad \beta = \sum_{j=1}^r \beta_j \otimes e_j \in \mathcal{C}^\infty_{p',q'}(U,E),$$

we have

$$\begin{aligned} D\alpha &= \sum_{j=1}^r d\alpha_j \otimes e_j + (-1)^{|\alpha|}\sum_{i,j} \alpha_j \wedge a_{ij} \otimes e_i, \\ D\beta &= \sum_{j=1}^r d\beta_j \otimes e_j + \sum_{i,j} a_{ij} \wedge \beta_j \otimes e_i. \end{aligned}$$

If h is given by a matrix (h_{ij}), then $\{\alpha,\beta\} = \sum_{k,l} h_{kl}\alpha_k \wedge \overline{\beta}_l$ and hence

$$\begin{aligned} &\sum_{k,l} \left(dh_{kl} \wedge \alpha_k \wedge \overline{\beta}_l + h_{kl} d\alpha_k \wedge \overline{\beta}_l + (-1)^{|\alpha|} h_{kl}\alpha_k \wedge d\overline{\beta}_l\right) \\ &\qquad = \sum_{j,k} h_{jk} d\alpha_j \wedge \overline{\beta}_k + (-1)^{|\alpha|}\sum_{i,j,k} h_{ik}\alpha_j \wedge a_{ij} \wedge \overline{\beta}_k \\ &\qquad\qquad + (-1)^{|\alpha|}\sum_{j,k} h_{jk}\alpha_j \wedge \overline{d\beta}_k + (-1)^{|\alpha|}\sum_{i,j,k} h_{ki}\alpha_k \wedge \overline{a}_{ij} \wedge \overline{\beta}_j. \end{aligned}$$

We can simplify the above identity and rephrase in the form of matrices as follows.

$$\alpha^t \wedge (dh - A^t h - h\overline{A}) \wedge \overline{\beta} = 0, \tag{3.1.1}$$

where α^t is the transpose of α and $A = (a_{ij})$. Since (3.1.1) is equivalent to $dh - A^t h - h\overline{A} = 0$ and $D'' = \overline{\partial}$, we conclude that

$$D' = \partial + \overline{h}^{-1}\partial\overline{h}.$$

With this observation, the following proposition can be easily shown.

Proposition 3.1.2. *For a connection ∇ on a complex vector bundle E, $\nabla^2 : \mathcal{C}^\infty_p(X, E) \to \mathcal{C}^\infty_{p+2}(X, E)$ is globally defined* Hom(E, E)*-valued* $(1, 1)$*-form on X. In particular, if a hermitian metric h on a holomorphic vector bundle E is given and ∇ is the Chern connection, then ∇^2 is called the Chern curvature form, denoted by $\Theta_h(E)$. Moreover,*

$$i\Theta_h(E) = i\overline{\partial}(\overline{h}^{-1}\partial\overline{h}) \in \mathcal{C}^\infty_{1,1}(X, \mathrm{Herm}(E, E)).$$

Remark 3.1.3. In the case where the rank of E is 1, the (not necessarily smooth) hermitian metric h is equal to $e^{-\varphi}$ on a small neighborhood U. The Chern curvature $\Theta_h(E)$ on U is, then, given by

$$\Theta_h(E) = \partial\overline{\partial}\varphi.$$

Here, the derivatives are taken in the sense of currents if φ is not of C^2.

Now, the metric ω on X induces a norm on $\wedge^p T^*X$ for each $p \geq 1$. Let $(e_1, \ldots, e_n)$ be a local orthonormal frame of TX and $(e_1^*, \ldots, e_n^*)$ a its dual. It is natural to define an inner product on $\wedge^p T^*X$ for all $p \geq 1$ so that the vectors $(e_{i_1} \wedge \cdots \wedge e_{i_p})_{i_1 < \cdots < i_p}$ is an orthonormal frame for $\wedge^p T^*X$. Then, these inner products are independent of the choice of orthonormal frames.

Similarly, the complexified cotangent bundle $\Omega^{\bullet,\bullet} := \wedge^\bullet(\mathbf{C} \otimes TX)^*$ possesses the induced inner product. Choose a coordinate $(z_1, \ldots, z_n)$ near a point

$x \in X$ so that the fundamental form ω at x can be written by

$$\omega = \frac{i}{2} \sum_{i=1}^{n} dz_i \wedge d\overline{z}_i$$

As the above definition of the induced inner product on $\wedge^{p,q} T^* X$, we have an orthonormal basis $(dz_I \wedge d\overline{z}_J)_{|I|=p,|J|=q}$ for $\wedge^{p,q} T^* X$, where I and J is index sets whose entries are in increasing order.

Now, let us define basic operators on a complex manifold.

Definition 3.1.4 (Lefschetz operator)**.** The *Lefschetz operator* L is an operator of type $(1, 1)$ on $\Omega^{\bullet,\bullet}$ defined by

$$L\alpha = \omega \wedge \alpha,$$

where ω is the fundamental form of (X, ω).

Since the complexified cotangent spaces $\Omega^{\bullet,\bullet}$ has an inner product and L is a bounded operator on $\Omega^{\bullet,\bullet}$, one can get the adjoint of L, denoted by Λ. The operator Λ is called the *dual Lefschetz operator* and is of type $(-1, -1)$. Note that L and Λ can be naturally extended to $\Omega^{\bullet,\bullet} \otimes E$. Another important operator is the *Hodge $*$-operator*.

Definition 3.1.5 (complex Hodge operator)**.** For every $\beta \in (\Omega^{p,q} \otimes E)_x$, the E-valued differential form $*_E \beta$ of bidegree $(n-q, n-p)$ is uniquely determined to be satisfying the following for all $\alpha \in (\Omega^{p,q} \otimes E)_x$,

$$\langle \alpha, \beta \rangle dV_x = \{\alpha, *_E \beta\},$$

where $dV = \omega^n/n!$ is the volume form of (X, ω). The operator $*_E$ is called the Hodge operator. If there is no risk of confusion, we then denote by $*$ the Hodge operator.

Note that it can be readily seen that the Hodge operator $*_E$ is an isometry on each fiber $(\Omega^{p,q} \otimes E)_x$. Using this fact, we can prove the following lemma.

Lemma 3.1.6. *On $\Omega^{p,q} \otimes E$,*

$$*_E *_E = (-1)^{(p+q)} Id.$$

Proof. For $\alpha, \beta \in \Omega^{p,q} \otimes E$, by the definition of the Hodge operator, we have

$$\begin{aligned}\langle \alpha, \beta \rangle dV_x &= \overline{\langle *_E \beta, *_E \alpha \rangle} dV_x \\ &= \overline{\{*_E \beta, *_E *_E \alpha\}} \\ &= (-1)^{(p+q)(2n-p-q)} \{*_E *_E \alpha, *_E \beta\} \\ &= (-1)^{(p+q)} \langle *_E *_E \alpha, \beta \rangle.\end{aligned}$$

Therefore, we conclude that $*_E *_E = (-1)^{(p+q)}\mathrm{Id}$ on $\Omega^{p,q} \otimes E$.

$\square$

For the later purposes, we write down the formula for L, Λ and $*$ in local coordinates $(z_1, \ldots, z_n)$. To do this, we prove the following lemma.

Lemma 3.1.7. *Let $\eta = \sum \eta_i dz_i$ be a $(1,0)$-form in $\mathbf{C}^n$ and consider η as the operator defined by $\eta(\alpha) = \eta \wedge \alpha$. Then, we have*

$$\eta^* = \sum \eta_i \frac{\partial}{\partial z_i} \lrcorner$$

where $\partial/\overline{\partial} z_i \lrcorner$ is the contraction with respect to $\partial/\overline{\partial} z_i$.

Proof. Assume that $(dz_1, \ldots, dz_n, d\overline{z}_1, \ldots, d\overline{z}_n)$ is an orthonormal basis for $\Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1}$ at a given point. It is enough prove the result when $\gamma = dz_i$ and $\gamma = d\overline{z}_i$. Let α and β be forms of bidegree (p,q) and $(p+1,q)$ respectively. Using this basis, we can write

$$\alpha = \sum_{I,J}{}' \alpha_{I,J} dz_I \wedge d\overline{z}_J, \quad \beta = \sum_{I',J'}{}' \beta_{I',J'} dz_{I'} \wedge d\overline{z}_{J'},$$

where $\sum'$ is the summation running over indices I and J with increasing order. Since $dz_i \wedge \alpha = \sum' \epsilon(I,i) \alpha_{I,J} dz_{I \cup i} \wedge d\overline{z}_J$, where $\epsilon(I,i)$ is the signature of the permutation (I, i) and $I \cup i$ is the ordered pair of the entries of I and i in increasing order. Since

$$\langle dz_i \wedge \alpha, \beta \rangle = \sum_{|I|=p,|J|=q} \epsilon(I,i)\alpha_{I\cup i,J}\overline{\beta}_{I\cup i,J},$$

one can easily know that

$$\langle dz_i \wedge \alpha, \beta \rangle = \langle \alpha, \frac{\partial}{\partial z_i} \lrcorner \beta \rangle.$$

$\square$

Let γ be a real $(1,1)$-form, that is, $\gamma = \overline{\gamma}$. One can always find local coordinates $(z_1, \ldots, z_n)$ near a point $x \in X$ so that the following holds at x.

$$\omega = \frac{i}{2}\sum_{i=1}^{n} dz_i \wedge d\overline{z}_i, \quad \gamma = \frac{i}{2}\sum_{i=1}^{n} \gamma_i dz_i \wedge d\overline{z}_i.$$

For two endomorphisms A and B on $\mathcal{C}^\infty_{p,q}(X, E)$, define the *graded commutator* of A and B by

$$[A, B] = AB - (-1)^{\deg A \cdot \deg B} BA.$$

Proposition 3.1.8. *For every differential form $\alpha \in \Omega^{p,q}$,*

$$[L, \Lambda]\alpha = (p + q - n)\alpha.$$

3.1.2 Kähler identity and Bochner-Kodaira-Nakano inequality

For L^2 estimates and L^2 extension theorems, we need commutator relations related to the Chern connection and the Lefschetz operator. Let us begin with the L^2 space of differential forms. Note that we have an induced inner product $\langle \bullet, \bullet \rangle_x$ on each fiber of $\Omega^{\bullet,\bullet} \otimes E$. Denote by $L^2_{p,q}(X, E)$ the space of E-valued (p, q)-form on X with measurable coefficients. And we have an inner product

$\langle\langle \bullet, \bullet \rangle\rangle$ on $L^2_{p,q}(X, E)$ defined by

$$\langle\langle \alpha, \beta \rangle\rangle = \int_X \langle \alpha, \beta \rangle_x \, dV_X,$$

for all $\alpha, \beta \in L^2_{p,q}(X, E)$, where $dV_X = \omega^n/n!$ is the volume form on X. Then, $L^2_{p,q}(X, E)$ is a Hilbert space with the inner product. There exists the *formal adjoint* δ of the Chern connection D with respect to the inner product $\langle\langle \bullet, \bullet \rangle\rangle$. This adjoint operator δ is of order -1 and can be written as the sum of δ' and δ'', which are the adjoint operators of D' and D'' respectively, that is $\delta = \delta' + \delta''$.

Lemma 3.1.9. *Let δ' and δ'' be formal adjoints of ∂ and $\overline{\partial}$ respectively. Then,*

$$\delta' = - * \overline{\partial} *, \quad \delta'' = - * \partial * .$$

Proof. Let α be a smooth E-valued (p, q)-form with compact support on X and let β be a smooth E-valued $(p+1, q)$-form with compact support on X. Then, using the Hodge operator, we have

$$\begin{aligned} \langle\langle \partial\alpha, \beta \rangle\rangle &= \int_X \langle \partial\alpha, \beta \rangle_x \, dV_X \\ &= \int_X \partial\alpha \wedge \overline{*\beta} \\ &= \int_X \alpha \wedge \Big((-1)^{p+q+1} \overline{\overline{\partial} * \beta} \Big). \end{aligned}$$

The last equality follows from an integration by parts. By Lemma 3.1.6,

$$\int_X \alpha \wedge \big((-1)^{p+q+1} \partial \overline{*\beta} \big) = \int_X \alpha \wedge \overline{* \big(- * \overline{\partial} * \beta \big)} = \int_X \langle \alpha, - * \overline{\partial} * \beta \rangle dV_X.$$

Therefore, we have

$$\langle\langle \partial\alpha, \beta \rangle\rangle = \langle\langle \alpha, - * \overline{\partial} * \beta \rangle\rangle,$$

which implies that $\delta' = - * \overline{\partial} *$. Similarly, it is easily shown that

$$\delta'' = - * \partial * .$$

□

Let (X, ω) be a Kähler manifold of dimension n, which means that its fundamental form ω is closed. Then, we have the following commutator relations.

Theorem 3.1.10. *If a vector bundle E is a trivial bundle $X \times \mathbf{C}$ with the standard euclidean norm, then the following relations hold, which are called Kähler identities.*

$$[\delta'', L] = i\partial, \quad [\delta', L] = -i\overline{\partial},$$
$$[\Lambda, \overline{\partial}] = -i\delta', \quad [\Lambda, \partial] = i\delta''.$$

Now, we want to generalize these relations to a arbitrary hermitian manifold. Let (X, ω) be a hermitian manifold. In this case, Theorem 3.1.10 should be modified with extra terms since the fundamental form ω involves the terms of order 1. Let τ be the operator defined by $\tau = [\Lambda, \partial\omega]$. We call $\partial\omega$ the *torsion form* of ω and τ the *torsion operator*.

Theorem 3.1.11. *For the trivial vector bundle $E = X \times \mathbf{C}$ with the standard euclidean norm, then the following relations hold.*

$$[\delta'', L] = i(\partial + \tau), \quad [\delta', L] = -i(\overline{\partial} + \overline{\tau}), \quad [\Lambda, \overline{\partial}] = -i(\delta' + \tau^*), \quad [\Lambda, \partial] = i(\delta'' + \overline{\tau}^*),$$

where τ^* is the formal adjoint of the operator τ.

Note that Theorem 3.1.11 can be easily extended to the cases where E is not trivial. The proof of the following theorem is similar with the proof of Theorem 3.1.11, we omit the proof.

Theorem 3.1.12. *Let (X, ω) be a hermitian manifold and let (E, h) be a hermitian holomorphic bundle over X. For the Chern connection D_E of E and* *the torsion operator* $\tau = [\Lambda, \partial\omega]$*, the following equalities hold:*

$$[\delta''_E, L] = i(D'_E + \tau), \quad [\delta'_E, L] = -i(D''_E + \overline{\tau}),$$
$$[\Lambda, D''_E] = -i(\delta'_E + \tau^*), \quad [\Lambda, D'_E] = i(\delta''_E + \overline{\tau}^*).$$

Now, let us define the holomorphic Laplace operator Δ' and the antiholomorphic Laplace operator Δ'' on $\mathcal{C}^\infty_{\bullet,\bullet}X, E$.

Definition 3.1.13. Let D_E be the Chern connection of a hermitian holomorphic bundle (E, h) over a hermitian manifold (X, ω). Then, the operator Δ' defined by

$$\Delta'_E = D'_E\delta'_E + \delta'_E D'_E$$

is called the *holomorphic Laplace operator*. Similarly, the operator Δ'' defined by

$$\Delta''_E = D''_E\delta''_E + \delta''_E D''_E$$

is called the *antiholomorphic Laplace operator*.

We omit the subscript E in the operators if there is no risk of confusion.

Theorem 3.1.14 (Bochner-Kodaira-Nakano identity)**.**

$$\Delta'' = \Delta' + [i\Theta(E), \Lambda] + [D', \tau^*] - [D'', \overline{\tau}^*].$$

Corollary 3.1.15 (Akizuki-Nakano identity)**.** *If* (X, ω) *is Kähler, then we have*

$$\Delta'' = \Delta' + [i\Theta(E), \Lambda].$$

In particular, if E *is the trivial bundle* $X \times \mathbf{C}$ *with the standard euclidean norm, we have*

$$\Delta'' = \Delta' = \frac{1}{2}\Delta,$$

where $\Delta = dd^* + d^*d$ *is the Laplace operator.*

Since two terms in Bochner-Kodaira-Nakano identity involve derivatives of order 1, it is not good for L^2 estimates. For the purpose of resolving this, define the operator Δ'_τ by

$$\Delta'_\tau = [D' + \tau, \delta' + \tau^*].$$

Then, Δ'_τ is a positive and self-adjoint operator and $\Delta'_\tau - \Delta'$ is an operator of order 0.

Theorem 3.1.16.

$$\Delta'' = \Delta'_\tau + [i\Theta(E), \Lambda] + T_\omega,$$

where T_ω is an operator of order 0 defined by

$$T_\omega = \left[\Lambda, [\Lambda, \frac{i}{2}\partial\overline{\partial}\omega]\right] - [\partial\omega, (\partial\omega)^*].$$

3.1.3 Positivity of vector bundles

Let E be a hermitian holomorphic vector bundle of rank r over a hermitian manifold X of dimension n. In local coordinates $(z_1, \ldots, z_n)$ of X, take an (smooth) orthonormal frame $(e_1, \ldots, e_r)$ of E. The curvature tensor $i\Theta(E)$ then can be written as

$$i\Theta(E) = i \sum c_{jk\lambda\mu} dz_j \wedge d\overline{z}_k \otimes e^*_\lambda \otimes e_\mu.$$

Since $i\Theta$ is hermitian in the sense of $c_{jk\lambda\mu} = \overline{c}_{kj\mu\lambda}$ and therefore, $i\Theta(E)$ can be considered as a hermitian form θ_E on $TX \otimes E$ defined by

$$\theta_E = \sum c_{jk\lambda\mu}(dz_j \otimes e^*_\lambda) \otimes (\overline{dz_k \otimes e^*_\mu}).$$

Explicitly, for $u = \sum u_{j\lambda} \frac{\partial}{\partial z_j} \otimes e_\lambda$ and $v = \sum v_{j\lambda} \frac{\partial}{\partial z_j} \otimes e_\lambda$, we have

$$\theta_E(u, v) = \sum c_{jk\lambda\mu} u_{j\lambda} \overline{v}_{k\mu}.$$

Definition 3.1.17. Denote by θ_E the hermitian form induced from the curvature tensor $i\Theta(E)$. A hermitian holomorphic vector bundle E of rank r is said to be

(a) *positive in the sense of Nakano* if the hermitian form θ_E is positive definite on $TX \otimes E$.

(b) *semi-positive in the sense of Nakano* if the hermitian form θ_E is semi-positive definite on $TX \otimes E$.

(c) *positive in the sense of Griffiths* if for all $\xi \in T_x X$ and $e \in E_x$,

$$\theta_E(\xi \otimes e, \xi \otimes e) > 0.$$

(d) *semi-positive in the sense of Griffiths* if for all $\xi \in T_x X$ and $e \in E_x$,

$$\theta_E(\xi \otimes e, \xi \otimes e) \geq 0.$$

3.1.4 L^2 estimates for the $\overline{\partial}$ operator

In this subsection, wa want to talk about solving the following first order partial diffenrential equation,

$$\overline{\partial} u = g$$

for a given (p, q)-form g.

Theorem 3.1.18. *Let H_1 and H_2 be Hilbert spaces and let $T : H_1 \to H_2$ be a densely defined operator. Then, the adjoint T^* of T is also closed and densely defined operator on H_2.*

Now, consider Hilbert spaces H_1, H_2 and H_3 and a chain of two closed and densely defined operators $T : H_1 \to H_2$ and $S : H_2 \to H_3$, that is, $S \circ T = 0$. There is an essential theorem for L^2 estimates for $\overline{\partial}$ in functional analysis.

Theorem 3.1.19. $\operatorname{Im} T = \operatorname{Ker} S$ *if and only if there exists a positive constant C such that*

$$\|T^* x\|_1^2 + \|Sx\|_3^2 \geq C^{-1} \|x\|_2^2 \tag{3.1.20}$$

for all $x \in \operatorname{Dom} S \cap \operatorname{Dom} T^*$.

Proof. Since the image of T is contained in the kernel of S, it is enough to show that
$$\operatorname{Ker} S \subseteq \operatorname{Im} T.$$
This is equivalent to solving the following equation:
$$Tu = v$$
for all v with $Sv = 0$. For $x \in \operatorname{Dom} T^*$, since $H_2 = \operatorname{Ker} S \oplus (\operatorname{Ker} S)^\perp$, we write
$$x = x_1 + x_2,$$
where $x_1 \in \operatorname{Ker} S$ and $x_2 \in (\operatorname{Ker} S)^\perp \subset \operatorname{Ker} T^*$ and thus $x_1 \in \operatorname{Dom} T^*$. Since $v \in \operatorname{Ker} S$, we have
$$\langle x, v\rangle_2 = \langle x_1, v\rangle_2.$$
Applying the Cauchy-Schwarz inequality gives, since $\|T^* x_1\|_1^2 \geq C\|x_1\|_2^2$,
$$|\langle x, v\rangle_2|^2 = |\langle x_1, v\rangle_2|^2 \leq \|x_1\|_2^2\|v\|_2^2 \leq C\|v\|_2^2\|T^* x\|_1^2,$$
which implies that the operator defined on $\operatorname{Im} T^*$ defined by
$$T^* x \mapsto \langle x, v\rangle_2$$
is a continuous operator. Using Hahn-Banach theorem and Riesz representation theorem, we can find $u \in H_1$ such that
$$\langle T^* x, u\rangle_1 = \langle x, v\rangle_2$$
for all $x \in \operatorname{Dom} T^*$. Thus $v = (T^*)^* u = Tu$ and $\operatorname{Ker} S = \operatorname{Im} T$. $\square$

Remark 3.1.21.

(a) In the proof of Theorem 3.1.19, we can get a norm estimate of u for fixed v. Since Hahn-Banach theorem and Riesz representation theorem preserve the norm of an operator, we have

$$\|u\|_1^2 \le C\|v\|_2^2.$$

In this way, we have norm estimates in solving $\overline{\partial}$ equations.

(b) For a given $v \in \operatorname{Ker} S$, the following inequality is enough for Theorem 3.1.19 instead of (3.1.20):

$$|\langle x, v\rangle_2|^2 \le C\left(\|T^*x\|_1^2 + \|Sx\|_3^2\right)$$

for all $x \in \operatorname{Dom} S \cap \operatorname{Dom} T^*$. In this case, for a solution $u \in H_1$ for $Tu = v$, we have

$$\|u\|_2^2 \le C.$$

Theorem 3.1.22. *Let (X, ω) be a complete hermitian manifold. Then, the set $\mathcal{D}_{p,q}(X, E)$ of smooth E-valued (p, q)-forms with compact support on X is dense in $\operatorname{Dom} D'$, $\operatorname{Dom} \delta'$ and $\operatorname{Dom} D' \cap \operatorname{Dom} \delta'$ respectively with respect to the graph norms.*

$$u \mapsto \|u\| + \|D'u\|, \quad u \mapsto \|u\| + \|\delta'u\|, \quad u \mapsto \|u\| + \|D'u\| + \|\delta'u\|.$$

The same conclusion holds when D' and δ' is replaced by D'' and δ'' respectively.

3.1.5 General L^2 estimate theorem for the $\overline{\partial}$ operator

The starting point of L^2 estimates is the Bochner-Kodaira-Nakano identity:

$$\Delta'' = \Delta'_\tau + [i\Theta(E), \Lambda] + T_\omega.$$

In our situation, the operators T and S in Theorem 3.1.19 would be the $(0,1)$-part D'' of the Chern connection. For an E-valued (p, q)-form $\alpha \in \operatorname{Dom} \overline{\partial} \cap$

Dom δ'', we have

$$\langle\langle \Delta''\alpha, \alpha\rangle\rangle = \langle\langle (D''\delta'' + \delta''D'')\alpha, \alpha\rangle\rangle = \|D''\alpha\|^2 + \|\delta''\alpha\|^2,$$
$$\langle\langle \Delta'_\tau\alpha, \alpha\rangle\rangle = \|(D' + \tau)\alpha\|^2 + \|(D' + \tau)^*\alpha\|^2,$$

which are always nonnegative real numbers. Put the operator $A_{E,\omega} := [i\Theta(E), \Lambda]+T_\omega$ and then

$$\|D''\alpha\|^2 + \|\delta''\alpha\|^2 \geq \langle\langle A_{E,\omega}\alpha, \alpha\rangle\rangle.$$

Therefore, in the view of Theorem 3.1.19, it is enough to show that, for all $\alpha \in \operatorname{Dom} D'' \cap \operatorname{Dom} \delta''$,

$$\|\alpha\|^2 \leq C\langle\langle A_{E,\omega}\alpha, \alpha\rangle\rangle.$$

If $g \in L^2_{p,q}(X, E)$ is specified, it is enough to prove that, for all $\alpha \in \operatorname{Dom} D'' \cap \operatorname{Dom} \delta''$,

$$|\langle\langle \alpha, g\rangle\rangle|^2 \leq C\langle\langle A_{E,\omega}\alpha, \alpha\rangle\rangle.$$

Suppose that $A_{E,\omega}$ is semi-positive definite on $\Omega^{p,q} \otimes E$ and let g be an E-valued (p, q)-form g with L^2 coefficients such that $D''g = 0$. On each fiber $\Omega^{p,q}_x \otimes E_x$, defined a real number, denoted by $\langle A^{-1}_{E,\omega}g, g\rangle(x)$, by the smallest non-negative(possibly $+\infty$) real number a such that

$$|\langle u, g(x)\rangle|^2 \leq a\langle A_{E,\omega}u, u\rangle$$

for all $u \in \Omega^{p,q}_x \otimes E_x$. We obtain the following theorem immediately.

Theorem 3.1.23 (General L^2 estimate for $\overline{\partial}$)**.** *Suppose that (X, ω) is a complete hermitian manifold and (E, h) is a hermitian holomorphic vector bundle over X. For any E-valued (p, q)-form g with L^2 coefficients such that $D''g = 0$ and*

$$\int_X \langle A^{-1}_{E,\omega}g, g\rangle \, dV_X < +\infty,$$

there exists an E-valued $(p, q-1)$-form f with L^2 coefficients such that $D''f = g$ and

$$\|f\|^2 \leq \int_X \langle A^{-1}_{E,\omega}g, g\rangle \, dV_X.$$

Definition 3.1.24. A complex manifold X is said to be *weakly pseudoconvex* if there exists a smooth psh function rho such that

$$X_c := \{x \in X : \rho(x) < c\}$$

is relatively compact for all real number c.

Note that such a function are called an *exhaustion* of X. Since the completeness of a hermitian manifold is essential for L^2 estimates.

Theorem 3.1.25. *For any weakly pseudoconvex Kähler manifold (X, ω), there exsits a complete Kähler metric $\widehat{\omega}$.*

When X is Kähler, the situation becomes more simpler than the case where X is not Kähler because the Akizuki-Nakano identity

$$\Delta'' = \Delta' + [i\Theta(E), \Lambda]$$

holds.

Theorem 3.1.26. *Let (X, ω) be a (non necessarily complete) Kähler manifold and (E, h) is a hermitian holomorphic vector bundle of rank r over X which is semi-positive definite in the sense of Nakano. Suppose that X possesses a complete Kähler metric $\widehat{\omega}$. Then, for any $g \in L^2_{n,q}(X, E)$ such that $D''g = 0$ and*

$$\int_X \langle A_q^{-1} g, g\rangle \, dV_\omega < +\infty,$$

where $A_q = [i\Theta(E), \Lambda] = i\Theta(E)\Lambda$ is an operator on $L^2_{n,q}(X, E)$, there exsits $f \in L^2_{n,q-1}(X, E)$ such that $D''f = g$ and

$$\|f\|^2 \leq \int_X \langle A_q^{-1} g, g\rangle \, dV_\omega.$$

Before ending this subsection, let us state a different version of the inequality involving with the curvature tensor and Laplace operators and L^2 estimate obtained from the new inequality, which was proved by Ohsawa[Oh01]. The following lemma is a slightly modified version by Demailly[D15].

Lemma 3.1.27. *Let η and λ be positive smooth functions on X. Then for every $u \in \mathcal{C}^\infty_{p,q}(X,E)$ with compact support,*

$$\begin{aligned}\|(\eta+\lambda)^{\frac{1}{2}}\delta''u\|^2 + \|\eta^{\frac{1}{2}}D''u\|^2 + \|\lambda^{\frac{1}{2}}D'u\|^2 + 2\|\lambda^{-\frac{1}{2}}\partial\eta\wedge u\|^2 \\ \geq \langle\!\langle [i\left(\eta\Theta(E) - \partial\overline{\partial}\eta - \lambda^{-1}\partial\eta\wedge\overline{\partial}\eta\right), \Lambda]u, u\rangle\!\rangle.\end{aligned}$$

In the following two theorems, a Kähler manifold (X,ω) possesses a complete Kähler metric and ω need not to be complete.

Theorem 3.1.28. *Let (E,h) be a hermitian holomorphic vector bundle over X. Suppose that smooth, bounded and positive function η and λ on X are given so that*

$$B := [i\left(\eta\Theta(E) - \partial\overline{\partial}\eta - \lambda^{-1}\partial\eta\wedge\overline{\partial}\eta\right), \Lambda_\omega]$$

is positive definite on $\Omega^{n,q}_x \otimes E_x$ for all $x \in X$ and some $q \geq 1$. Then, for every $g \in L^2_{n,q}(X,E)$ such that $D''g = 0$ and

$$\int_X \langle B^{-1}g, g\rangle\, dV_\omega < +\infty,$$

there exists $f \in L^2_{n,q-1}(X,E)$ such that $D''f = g$ and

$$\int_X (\eta+\lambda)^{-1}|f|^2\, dV_\omega \leq \int_X \langle B^{-1}g, g\rangle\, dV_\omega.$$

Theorem 3.1.29. *Suppose that $B + \epsilon I > 0$ for some $\epsilon > 0$, where I is the identity endomorphism. For every $g \in L^2_{n,q}(X,E)$ such that $D''g = 0$ and*

$$M(\epsilon) = \int_X \langle (B+\epsilon I)^{-1}g, g\rangle\, dV_\omega < +\infty,$$

there exists $f_\epsilon \in L^2_{n,q-1}(X,E)$ and $g_\epsilon \in L^2_{n,q}(X,E)$ such that $D''f_\epsilon = g - g_\epsilon$ and

$$\int_X (\eta+\lambda)^{-1}|f_\epsilon|^2\, dV_\omega + \frac{1}{\epsilon}\int_X |g_\epsilon|^2\, dV_\omega \leq M(\epsilon).$$

Here, we call f_ϵ an approximate solution and g_ϵ a correcting term. If g is *smooth, then f_ϵ and g_ϵ can be taken smooth.*

3.1.6 L^2 extension theorems

Extending a holomorphic function defined on a subset to an ambient space have been studied from the beginning of complex analysis. We are interested the case when a subset is a closed submanifold or a closed subvariety. In this case, many holomorphic functions can be easily extended with L^2 norm estimates. After Hörmander's L^2 estimates, many mathematicians contributed to develope various versions of L^2 extension theorems.

In this subsection, we mainly deal with Demailly's L^2 extension theorem and fill the details which is necessary for the theorem to be more complete.

Definition 3.1.30. Let X be a complex manifold and let $\psi : X \to [-\infty, +\infty)$ be a quasi-plurisubharmonic(quasi-psh) function on X. We say that ψ *has neat analaytic singularities* if for every $x \in X$, there exist an open neighborhood U, a smooth function w on U and holomorphic functions $g_1, \ldots, g_k$ on U such that

$$\psi = c \log \left(\sum_{i=1}^{k} |g_i|^2 \right) + w,$$

where c is a nonnegative real number.

Definition 3.1.31. Let ψ be a quasi-psh function on a complex manifold X and Y be the zero variety of the multiplier ideal $\mathcal{J}(\psi)$ of ψ. We say that ψ *has log canonical singularities along* Y if $\mathcal{J}((1-\epsilon)\psi) = \mathcal{O}_X|_Y$ for every $\epsilon > 0$.

Note that if ψ has log canonical singularities along $Y = \mathcal{J}(\psi)$, then $\mathcal{J}(\psi)$ is reduced and hence Y is a reduced variety. Let (X, ω) be a Kähler manifold of dimension n and E a hermitian holomorphic vector bundle of rank r. Let Y^o be the set of regular points of Y. Now we want to define a measure $dV_{Y^o,\omega}[\psi]$ on Y^o. To do this, it is enough to define a positive linear functional L on the set $\mathcal{C}_c(Y^o)$ of continuous functions with compact support. For each $g \in \mathcal{C}_c(Y^o)$, define $L(g)$ by

$$L(g) = \limsup_{t \to -\infty} \int_{\{t < \psi(x) < t+1\}} \tilde{g} e^{-\psi} \, dV_{X,\omega},$$

where $\tilde{g}$ is a compactly supported continuous extension of g to X. Note that $L(g)$ is independent of the choice of an extension $\tilde{g}$. Obviously, L is a positive linear functional on $\mathcal{C}_c(Y^o)$ and therefore there exists a measure $dV_{Y^o,\omega}[\psi]$ on Y^o such that

$$L(g) = \int_{Y^o} g\, dV_{Y^o,\omega}[\psi]$$

for all $g \in \mathcal{C}_c(Y^o)$. Demailly proved a version of L^2 extension theorem with the measure $dV_{Y^o,\omega}[\psi]$ in [D15].

Theorem 3.1.32. *Suppose that (X, ω) is a weakly pseudoconvex Kähler manifold and ψ is a quasi-psh function on X with neat analytic singularities and that there exists $\delta > 0$ such that the curvature tensor*

$$i\Theta(E) + \alpha i\partial\overline{\partial}\psi \otimes \mathrm{Id}_E \geq 0$$

in the sense of Nakano for all $\alpha \in [1, 1+\delta]$. Then, for every smooth section f of $(K_X \otimes E)|_{Y^o}$ on Y^o such that $D''f = 0$ and

$$\int_{Y^o} |f|^2_{\omega,h}\, dV_{Y^o,\omega}[\psi] < +\infty,$$

there exists a section F of $K_X \otimes E$ on X such that $F|_{Y^o} = f$ and

$$\int_X \gamma(\delta\psi)|F|^2_{\omega,h} e^{-\psi}\, dV_{X,\omega} \leq \frac{34}{\delta} \int_{Y^o} |f|^2_{\omega,h}\, dV_{Y^o,\omega}[\psi],$$

where $\gamma : \mathbf{R} \to \mathbf{R}$ is a positive function defined by

$$\gamma(x) = \begin{cases} e^{-x/2} & \text{if } x \geq 0, \\ \dfrac{1}{1+x^2} & \text{if } x < 0. \end{cases}$$

See [D15] for the whole proof of Theorem 3.1.32. Since Demailly omitted the details in the limit process of the last part of the proof, we will give the sketch of the proof and the details of the limit process.

Sketch of the proof. Let f be a section of $(K_X \otimes E)|_{Y^o}$ on Y^o such that $D''f =$

0. To make X complete Kähler, we replace X by a relatively compact weakly pseudoconvex domain $X_c = \{x \in X : \rho(x) < c\}$. Then there exists a complete Kähler metric $\widehat{\omega}$ on $X_c \backslash Y$. Using a partition of unity, one can obtain a smooth extension $\tilde{f} \in C^\infty(X, K_X \otimes E)$ of f such that

(1) $\tilde{f}|_Y = f$ on Y,

(2) $D''\tilde{f} = 0$ on Y,

(3) $|D''\tilde{f}|^2_{\omega,h} e^{-\psi}$ is locally integrable near Y.

Apply Theorem 3.1.28 to solve the equation

$$D''u_t = v_t := D''(\theta(\psi - t)\tilde{f}), \quad t \in (-\infty, -1]$$

where $\theta : [-\infty, +\infty) \to [0, 1]$ is defined so that θ is smooth non increasing such that $\theta(s) = 1$ fo $s \in [-\infty, \epsilon/3]$, $\theta(s) = 0$ for $s \in [1-\epsilon/3, +\infty)$ and $|\theta'| \le 1+\epsilon$. For simplicity, assume first that $D''\tilde{f} = 0$, that is, a smooth extension $\tilde{f}$ is holomorphic. Then

$$v_t = D''(\theta(\psi - t)\tilde{f}) = \theta'(\psi - t)\overline{\partial}\psi \wedge \tilde{f}$$

is supported in $W_t = \{x \in X : t < \psi(x) < t + 1\}$. Now, take

$$\begin{aligned} \eta_t &= 1 - \delta\chi_t(\psi), \\ \lambda_t &= \pi(1 + \delta^2\psi^2), \end{aligned}$$

where a function $\chi_t : (-\infty, 0] \to \mathbf{R}$ is chosen so that χ_t satisfies certain conditions. Then setting

$$\begin{aligned} R_t &= i\eta_t(\Theta(E) + \partial\overline{\partial}\psi) - i\partial\overline{\partial}\eta_t - \lambda_t^{-1} i\partial\eta_t \wedge \overline{\partial}\eta_t, \\ B_t &= [R_t, \Lambda_\omega], \end{aligned}$$

we have the following estimate: for every $(n, 0)$-form u,

$$\langle B_t^{-1}(\overline{\partial}\psi \wedge u), \overline{\partial}\psi \wedge u\rangle \le \frac{8}{(1-\epsilon)\delta}|u|^2 \quad \text{on } W_t.$$

In particular, for $v_t = \theta'(\psi - t)\overline{\partial}\psi \wedge \tilde{f}$, we have

$$\langle B_t^{-1} v_t, v_t \rangle \leq \frac{8(1+\epsilon)^2}{(1-\epsilon)\delta} |\tilde{f}|^2.$$

For a sufficiently small $\epsilon > 0$, By Theorem 3.1.28, one can find a solution u_t such that $D'' u_t = v_t$ on $X_c \backslash Y$ and

$$\int_{X_c \backslash Y} (1 + \delta^2 \psi^2)^{-1} |u_t|^2_{\omega,h} e^{-\psi} \, dV_\omega \leq \frac{34}{\delta} \int_{W_t} |\tilde{f}|^2_{\omega,h} e^{-\psi} \, dV_\omega.$$

Then the function $F_t = \theta(\psi - t)\tilde{f} - u_t$ is the extension of $\tilde{f}$. For any $\alpha > 0$, we have

$$|F_t|^2 \leq (1+\alpha)|u_t|^2 + (1+\alpha^{-1})|\theta(\psi - t)|^2 |\tilde{f}|^2.$$

Therefore, we obtain the following inequality:

$$\begin{aligned}
&\int_{X_c \backslash Y} (1+\delta^2\psi^2)^{-1}(1+\alpha^2\psi^2)^{-(n-1)/2} |F_t|^2_{\omega,h} e^{-\psi} \, dV_\omega \\
&\leq \frac{34(1+\alpha)}{\delta} \int_{X_c \cap W_t} |\tilde{f}|^2_{\omega,h} e^{-\psi} \, dV_\omega \\
&+ (1+\alpha^{-1}) \int_{X_c \cap \{\psi < t+1\}} (1+\delta^2\psi^2)^{-1}(1+\alpha^2\psi^2)^{-(n-1)/2} |\tilde{f}|^2_{\omega,h} e^{-\psi} \, dV_\omega.
\end{aligned}$$

Since the second integral in the the right hand side converges to 0 as t tends to $-\infty$, we can find a weak limit G_α of $(F_t)_t$ such that

$$\begin{aligned}
\int_{X_c \backslash Y} (1+\delta^2\psi^2)^{-1}(1+\alpha^2\psi^2)^{-(n-1)/2} |G_\alpha|^2_{\omega,h} e^{-\psi} \, dV_\omega& \\
\leq \frac{34(1+\alpha)}{\delta} \int_{X_c \cap Y} |f|^2_{\omega,h} dV_{Y,\omega}[\psi]&.
\end{aligned}$$

Note that since F_t is an extension of f for all t, a weak limit G_t also can be chosen as an extension of f. Now, letting α tend to 0, we can find again a weak limit H_c of $(G_\alpha)_\alpha$ such that H_c is an extension of f on X_c and

$$\int_{X_c \setminus Y} (1+\delta^2\psi^2)^{-1} |H_c|^2_{\omega,h} e^{-\psi}\, dV_\omega \leq \frac{34}{\delta} \int_{X_c \cap Y} |f|^2_{\omega,h}\, dV_{Y,\omega}[\psi].$$

Repeat this procedure as letting c tend to $+\infty$ to obtain a weak limit F of $(H_c)_c$, we have an extension F of f on X and the norm estimate

$$\int_X (1+\delta^2\psi^2)^{-1} |F|^2_{\omega,h} e^{-\psi}\, dV_\omega \leq \frac{34}{\delta} \int_Y |f|^2_{\omega,h}\, dV_{Y,\omega}[\psi]$$

as desired. □

3.2 Singularities of pairs

In this section, we briefly introduce the contents in [Kol97].

3.2.1 Basic concepts of singularities of pairs

Let X be a normal variety and $D = \sum d_i D_i$ be an effective $\mathbf{Q}$-divisor such that $K_X + D$ is $\mathbf{Q}$-Cartier, where K_X is the canonical divisor of X. We call such a pair (X, D) a *log pair*. Assume that $f : Y \to X$ is a proper birational morphism from a normal variety Y to X. Then, we can write

$$K_X \equiv f^*(K_X + D) + \sum a(E, D)E,$$

where E are distinct prime divisors and $a(E, D) \in \mathbf{Q}$. Note that if E is nonexceptional if and only if E is the strict transform $f_*^{-1} D_i$ of D_i for some i and $a(E, D) = -d_i$.

Definition 3.2.1. In the above situation, $a(E, D)$ is called the discrepancy of E with respect to (X, D). We say that a divisor E on X' is *exceptional over* X if there exists a proper birational morphism f from X' to X where E is an exceptional divisor of f. Denote the infimum of $a(E, D)$ for all exceptional divisors E over X by $\operatorname{discrep}(X, D)$, which is called the *discrepancy* of (X, D), and the infimum of $a(E, D)$ for all divisors E over X with nonempty center by totaldiscrep(X, D), which is called the *total discrepancy* of (X, D).

If there is an exceptional divisor E such that $a(E, D) < -1$, one then can find a sequence (f_α) of birational morphisms such that $\min_i (a(E_{\alpha,i}, D))$ tends to $+\infty$ as $\alpha \to +\infty$. If E is an exceptional divisor obtained by blowing up a smooth locus of codimension 2 which is not contained in the support of D, then $a(E, D) = 1$ and so discrep$(X, D) \leq 1$. We then have the following proposition. See [KM] for details.

Proposition 3.2.2. *Either* $\operatorname{discrep}(X, D) = -\infty$*, or*

$$-1 \leq \operatorname{totaldiscrep}(X, D) \leq \operatorname{discrep}(X, D) \leq 1.$$

There are basic classes of singularities of pairs determined by discrep(X, D) and totaldiscrep(X, D).

Definition 3.2.3. We say that (X, D) is

$$\begin{cases} \textit{termianl} \\ \textit{canonical} \\ \textit{Kawamata log terminal(klt)} \\ \textit{purely log terminal(plt)} \\ \textit{log canonical(lc)} \end{cases} \text{if } \operatorname{discrep}(X, D) \begin{cases} > 0, \\ \geq 0, \\ > -1 \text{ and } \lfloor D \rfloor = 0, \\ > -1, \\ \geq -1. \end{cases}$$

Note that (X, D) is klt if and only if totaldiscrep$(X, D) > -1$.

By Proposition 3.2.2, if (X, D) has singularities listed above except klt, then the coefficients of $D = \sum d_i D_i$ satisfies that $0 \leq d_i \leq 1$ for all i. In this case, we call D a boundary divisor. On the other hand, if (X, D) is klt, we have $0 \leq d_i < 1$ for all i. The following two lemmas help us determine the singularities of a pair.

Lemma 3.2.4. *Let* $f : Y \to X$ *be a proper birational morhpism and wirte* $K_Y + D' \equiv f^*(K_X + D)$*. Then*

(1) $a(E, D) = a(E, D')$ *for every divisor* E *over* X*.*

(2) (X,D) *is klt(resp. lc) if and only if* (Y,D') *is klt(resp. lc)*

(3) (X,D) *is plt if and only if* (Y,D') *is plt and* $a(E,D) > -1$ *for every exceptional divisor* E *of* f.

(4) (X,D) *is terminal(resp. canonical) if and only if* (Y,D') *is terminal(resp. canonical) and* $a(E,D) > 0$*(resp.* $a(E,D) \geq 0$*) for every exceptional divisor* E *of* f.

Lemma 3.2.5. *Let* X *be a smooth scheme and* $D = \sum d_i D_i$ *a divisor with simple normal crossing support. Assume that* $d_i \leq 1$*. Then for a (not necesssarily closed) point* $x \in X$ *and a divisor* E *over* X *with* $\mathrm{c}_X(E) = x$,

(1) $a(E,D) \geq \operatorname{codim}(x,X) - 1 - \sum_{x \in D_j} d_j$,

(2) $\operatorname{totaldiscrep}(X,D) = \min\{0, -d_i\}$,

(3) $\operatorname{discrep}(X,D) = \min\{1, 1-d_i, 1-d_i-d_j : D_i \cap D_j \neq \emptyset\}$.

In the definition of singularities of pairs, to determine how worse a pair (X,D) has, one have to compute $a(E,D)$ for all exceptional divisors E over X. For a pair (X,D) where X is a normal variety, one can always find a proper birational morphism $f : Y \to X$ such that Y is smooth and $f^{-1}(K_X + D) \cup \operatorname{excep}(f)$ is a divisor with simple normal crossing. We call f a log resolution of D. Using Lemma 3.2.2 and a *log resolution* of (X,D), we need only finite number of computations to determine what singularities (X,D) has.

Theorem 3.2.6. *Let* X *be a smooth variety and* $D = \sum d_i D_i$ *is a simple normal crossing divisor on* X*. Then* (X,D) *is*

$$\begin{cases} \textit{termianl} & & d_i < 1 \textit{ and } d_i + d_j < 1 \textit{ if } D_i \cap D_j \neq \emptyset, \\ \textit{canonical} & & d_i \leq 1 \textit{ and } d_i + d_j \leq 1 \textit{ if } D_i \cap D_j \neq \emptyset, \\ \textit{klt} & \textit{if and only if} & d_i < 1, \\ \textit{plt} & & d_i \leq 1 \textit{ and } d_i + d_j < 2 \textit{ if } D_i \cap D_j \neq \emptyset, \\ \textit{lc} & & d_i \leq -1. \end{cases}$$

3.2.2 Singularities of pairs and multiplier ideals

From now, we assume for simplicity that X is a smooth variety and D is a effective $\mathbf{Q}$-divisor. See 2.1 to recall the algebraic definition of multiplier ideals.

Let $\mu : Y \to X$ be a log resolution of D. By Lemma 3.2.2, (X, D) is klt if and only if (Y, D') is klt, where $D' = \mu^*(K_X + D) - K_Y$. Suppose that

$$\mu^* K_X = K_Y + \sum k_j E_j,$$
$$\mu^* D = \sum d_i D'_i + \sum e_j E_j,$$

where each D'_i is the strict transform of D_i and each E_i is an exceptional divisor of μ. Then, we have

$$D' = \sum d_i D'_i + \sum (e_j + k_j) E_j.$$

Note that $\sum k_j E_j$ is given by the *Jacobian* of μ. More explicitly, take a $(n, 0)$-form η_X on X and a $(n, 0)$-form η_Y on Y so that $\eta_X \wedge \overline{\eta}_X$(resp. $\eta_Y \wedge \overline{\eta}_Y$) is a smooth volume form on X(resp. Y). Then $\mu^* \eta_X$ is a section of $\mu^* K_X$ and we have $\mu^* \eta_X = J_\mu \cdot \eta_Y$ for some holomorphic function J_μ and

$$\operatorname{div}(J_\mu) = \sum k_j E_j.$$

Moreover, since μ is a log resolution, D' has simple normal crossing support. Thus we may assume that after linear change of coordinates, a defining function of D_i(resp. E_j) is x_i(resp. y_j) on an open neighborhood U of x_0 with local coordinates $(x_1, \ldots, y_1, \ldots, z_1, \ldots)$. By Theorem 3.2.2, (Y, D') is klt at x_0 if and only if $d_i < 1$ for all i and $e_j + k_j < 1$ for all j, which is equivalent to the following condition:

$$\int_U e^{-2\varphi_{D'}} \eta_Y \wedge \overline{\eta}_Y < +\infty.$$

Observing that $e^{-2\varphi_{D'}} \eta_Y \wedge \overline{\eta}_Y$ is equal to $\mu^*(e^{-2\varphi_D} \eta_X \wedge \overline{\eta}_X)$, we conclude that by the change of variables (Y, D') is klt at x_0 if and only if for a sufficiently small neighborhood V of $\mu(x_0)$,

$$\int_V e^{-2\varphi_D}\,\eta_X \wedge \overline{\eta}_X < +\infty.$$

Since $\eta_X \wedge \overline{\eta}_X$ is a smooth volume form on V, we have the following theorem.

Theorem 3.2.7. *A pair (X, D) is klt if and only if $e^{-2\varphi_D}$ is locally integrable with respect to the Lebesgue measure in any local coordinates, or equivalently*

$$\mathcal{J}(\varphi_D) = \mathcal{O}_X.$$

Now, to consider the case when (X, D) is plt, let us define an algebraic adjoint ideal sheaf.

Definition 3.2.8. Let $\mathfrak{a}$ be a nonzero ideal sheaf of $\mathcal{O}_X$, c a positive real number and H a smooth closed subscheme of X of codimension 1 such that $\mathfrak{a}$ is not contained in the ideal sheaf $\mathcal{J}_H$ of H. Let $\mu : Y \to X$ be a log resolution of $\mathfrak{a}$ such as $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ for some effective divisor F and $F + \mu^*H + K_{Y/X} + \mathrm{excep}(\mu)$ has simple normal crossing support. The *(algebraic) adjoint ideal* $\mathrm{Adj}(\mathfrak{a}^c, H)$ of $\mathfrak{a}^c$ along H is defined by

$$\mathrm{Adj}(\mathfrak{a}^c, H) = \mu_*\mathcal{O}_Y(K_{Y/X} - \lfloor c \cdot F \rfloor - \mu^*H + H')$$

where H' is the strict transform of H.

Similarly, one can define the adjoint ideal sheaf of D along H for an effective divisor D.

Definition 3.2.9. Let D be a effective divisor and H a smooth closed subscheme of codimension 1 such that any prime divisors appearing in D is not contained in H. Let $\mu : Y \to X$ be a log resolution of D so that $K_{Y/X} + \mu^*(D + H) + \mathrm{excep}(\mu)$ has simple normal crossing support. The *(algebraic) adjoint ideal* $\mathrm{Adj}(D, H)$ of D along H is defined by

$$\mathrm{Adj}(D, H) = \mu_*\mathcal{O}_Y(K_{Y/X} - \lfloor \mu^*D \rfloor - \mu^*H + H').$$

Note that if $\mathfrak{a}$ is a principal ideal sheaf generated by local defining functions of $m \cdot D$, where m is a positive integer so that $m \cdot D$ is an integral divisor, then we have obviously,

$$\mathrm{Adj}(\mathfrak{a}^{\frac{1}{m}}, H) = \mathrm{Adj}(D, H).$$

Guenancia introduced the definition of analytic adjoint ideal $\mathrm{Adj}(\varphi, H)$ for a psh function φ and proved that the algebraic adjoint ideal sheaf of $\mathfrak{a}$ along H is equal to the analytic adjoint ideal sheaf of $\varphi_{\mathfrak{a}}$ along H in [G].

Definition 3.2.10. A positive $(1,1)$-form ω on $X_0 := X \backslash H$ is said to be *H-Poincaré* if for all sufficiently small open set $U \subset X$, there exists local coordinates $(z_1, \ldots, z_n)$ such that $U \cap H = \{(z_1, \ldots, z_n) : z_1 = 0\}$ and a positive constant C such that

$$C^{-1}\omega \leq \frac{i}{2}\left(\frac{dz_1 \wedge d\overline{z}_1}{|z_1|^2(\log|z_1|)^2} + \sum_{j=2}^{n} dz_j \wedge d\overline{z}_j\right) \leq C\omega.$$

In this case, we says that the volume form $dV_\omega = \frac{\omega^n}{n!}$ is H-Poincaré. Note that dV_ω is H-Poincaré if and only if locally dV_ω is equal to

$$\frac{1}{|z_1|^2(\log|z_1|)^2} dV$$

where dV is the Lebesgue measure in local coordinates. Using a partition of unity, one can always take a H-Poincaré $(1,1)$-form ω_P and its associated volume form dV_{ω_P}.

Definition 3.2.11. Let φ be a psh function on a complex manifold X and H a complex submanifold of codimension 1. The *analytic adjoint ideal sheaf* $\mathrm{Adj}(\varphi, H)$ of φ along H is the ideal subsheaf of $\mathcal{O}_X$ of germs of holomorphic functions $f \in \mathcal{O}_{X,x}$ such that $|f|^2 e^{-2(1+\epsilon)\varphi}$ is integrable with respect to dV_{ω_P} near x for sufficiently small $\epsilon > 0$.

Theorem 3.2.12 ([G])**.** *Let $\mathfrak{a}$ be an ideal sheaf of $\mathcal{O}_X$ and D be a effective $\mathbf{Q}$-divisor. Then we have*

$$\mathrm{Adj}(\mathfrak{a}^c, H) = \mathrm{Adj}(\varphi_{\mathfrak{a}^c}, H), \quad \mathrm{Adj}(D, H) = \mathrm{Adj}(\varphi_D, H).$$

Now, let H be a complex submanifold of codimension 1 such that H is not contained in the support of D.

Definition 3.2.13. A log pair (X, D) is said to be *purely log terminal near* H if (X, D) is plt at x for all $x \in H$.

We adapt Lemma and Theorem for the above new definition.

Lemma 3.2.14. *Let $f : Y \to X$ be a proper birational morphism and write $K_Y + D' \equiv f^*(K_X + D)$. Then (X, D) is plt near H if and only if (Y, D') is plt along the strict transform H' of H and $a(E, D) > -1$ for every exceptional divisor E of f such that $f(E) \cap H \neq \emptyset$.*

Theorem 3.2.15. *Let X be a smooth variety and $D = \sum d_i D_i$ is a simple normal crossing divisor on X. Then (X, D) is plt near H if and only if $d_i \leq 1$ if $D_i \cap H \neq \emptyset$ and $d_i + d_j < 2$ if $D_i \cap D_j \cap H \neq \emptyset$.*

Consider a log resolution $\mu : Y \to X$ of $D + H$ so that $K_{Y/X} + \mu^*(D + H) + \mathrm{excep}(\mu)$ has simple normal crossing support. We then write

$$\begin{aligned} \mu^* K_X &= K_Y + \sum k_j E_j, \\ \mu^* H &= H' + \sum h_j E_j, \\ \mu^* D &= \sum d_j D'_j + \sum e_j E_j, \end{aligned}$$

where H' and D'_i's are the strict transforms of H and D_i's respectively and E_i's are exceptional divisors of μ. With this notations, we have

$$K_{Y/X} - \lfloor \mu^* D \rfloor - \mu^* H + H' = -\sum \lfloor d_j \rfloor D'_j - \sum (\lfloor e_j \rfloor + k_j + h_j) E_j.$$

By rearrangement of indices, we may assume that $D'_j \cap H \neq \emptyset$ if and only if

$j \leq M$ and similarly, $E_j \cap H \neq \emptyset$ if and only if $j \leq N$. Therefore,

$$K_{Y/X} - \lfloor \mu^* D \rfloor - \mu^* H + H' = -\sum_{j \leq M} \lfloor d_j \rfloor D'_j - \sum_{j \leq N} (\lfloor e_j \rfloor + k_j + h_j) E_j \\ - \sum_{j > M} \lfloor d_j \rfloor D'_j - \sum_{j > N} (\lfloor e_j \rfloor + k_j + h_j) E_j. \tag{3.2.16}$$

In view of Lemma 3.2.14, on the other hand, we have

$$D' = H' + \sum_{j \leq M} d_j D'_j + \sum_{j \leq N} (e_j + k_j + h_j) E_j \\ + \sum_{j > M} d_j D'_j + \sum_{j > N} (e_j + k_j + h_j) E_j.$$

Since $D'_j (j = 1, \ldots, M)$ and $E_j (j = 1, \ldots, N)$ intersect with H', a pair (Y, D') is plt near H if and only if $d_j < 1$ for all $j \leq M$ and $e_j + k_j + h_j < 1$ for all $j \leq N$. Therefore, (Y, D') is plt near H if and only if $\lfloor d_j \rfloor < -1$ for all $j \leq M$ and $\lfloor e_j \rfloor + k_j + h_j < -1$. As a consequence, we have the following theorem.

Theorem 3.2.17. *Let H be a smooth submanifold of X of codimension 1 such that H is not contained in the support of D and $(X, D + H)$ is a log pair. Then $(X, D + H)$ is plt near H if and only if*

$$\mathrm{Adj}(D, H)|_H = \mathcal{O}_X|_H.$$

Equivalently, for each $x \in H$, there exists $\epsilon = \epsilon(x) > 0$ such that $e^{-2(1+\epsilon)\varphi_D}$ is integrable near x with respect to the H-Poincaré volume form dV_{ω_P}.

3.2.3 Inversion of adjunction

Let X be a smooth complex variety and D an effective $\mathbf{Q}$-divisor on X. Then singularities of a pair (X, D) discussed in the previous subsection has 'the *adjunction property*'. For example, if H is an irreducible Cartier divisor such that H is not contained in the support of D, then the fact that (X, D) is klt implies that $(H, D|_H)$ is klt. Like this, a properties that concludes something about H from something about X is called *adjunction*. Then the following question is naturally arose: if $(H, D|_H)$ is klt, then what can we say about singularities of a pair (X, D)? A phenomenon that some property holding on H implies a similar property on X is called *inversion of adjunction*.

Theorem 3.2.18. *Let X be normal and $H \subset X$ an irreducible Cartier divisor. Let D be an effective $\mathbf{Q}$-divisor and suppose that $K_X + H + D$ is $\mathbf{Q}$-Cartier. Then, $(X, H + D)$ is plt near H if and only if $(H, D|_H)$ is klt.*

Assume in addition that D is $\mathbf{Q}$-Cartier and H is klt. Then $(X, H + D)$ is lc near H if and only if $(H, D|_H)$ is lc.

Theorem 3.2.18 was proved in [Kol92] and the proof is purely algebraic. The crucial tools used in the proof of Theorem 3.2.18 are the following two theorems.

Theorem 3.2.19. *Let $\mu : Y \to X$ be a birational morphism between projective varieties, Y smooth. Let L be a line bundle on Y and suppose that $L \equiv M + \sum d_i D_i$ where*

(1) M is nef,

(2) $\sum D_i$ has simple normal crossing support,

(3) $0 \leq d_i < 1$ and $d_i \in \mathbf{Q}$ for all i.

Then $R^i f_*(\omega_Y \otimes M) = 0$ for all $i > 0$. In particular, $R^i f_* \omega_Y = 0$ for all $i > 0$.

Theorem 3.2.20. *Let $\mu : Y \to X$ be a proper and birational morphism, Y smooth, X normal and let $D = \sum d_i D_i$ be a $\mathbf{Q}$-divisor with simple normal crossing support on Y such that $\mu_* D$ is effective and $-(K_Y + D)$ is μ-nef. Write*

$$A = \sum_{d_i < 1} d_i D_i, \quad \text{and} \quad F = \sum_{d_i \geq 1} d_i D_i$$

Then $\mathrm{supp} F = \mathrm{supp} \lfloor F \rfloor$ is connnected in a neighborhood of any fiber of μ.

In this subsection, we introduce the following version of inversion of adjunction proved *analytically* in [Kol97]. The proof depends on Ohsawa-Takegoshi L^2 extension theorem [OT87].

Theorem 3.2.21. *Let X be a complex manifold, D an effective $\mathbf{Q}$-divisor on X and H a complex submanifold of codimension 1 such that H is not contained in the support of D. Then, (X, D) is klt near H if and only if $(H, D|_H)$ is klt.*

The 'only if' part can be easily shown. For the 'if' part, suppose that $(H, D|_H)$ is klt. As we have shown in the previous subsection, it is equivalent to the local integrability of $e^{\varphi_D}|_H$. For $x \in H$, one can always find a neighborhood U of x such that U is a bounded pseudoconvex domain and $U \cap H$ is a hyperplane in $\mathbf{C}^n$. Since $e^{\varphi_D}|_H$ is integrable near x, the following theorem can be applied to extending a holomorphic function $f \equiv 1$. In this way, we obtain the local integrability of e^{φ_D} on U and hence (X, D) is klt at x.

Theorem 3.2.22. *Let U be a bounded pseudoconvex domain in $\mathbf{C}^n$ and $\varphi : U \to [-\infty, \infty)$ a plurisubharmonic function and H a complex hyperplane. Then there exists a positive constant C depending only on the diameter of U such that for every holomorphic function f on $U \cap H$ satisfying*

$$\int_{U\cap H} |f|^2 e^{-2\varphi}\, dV_{n-1} < +\infty$$

where dV_{n-1} denotes the $(2n-2)$-dimensional Lebesgue measure, there exists a holomorphic function F on U satisfying $F|_{U\cap H} = f$ and

$$\int_U |F|^2 e^{-2\varphi}\, dV_n \le C \int_{U\cap H} |f|^2 e^{-2\varphi}\, dV_{n-1}.$$

We can analytically obtain inversion of adjunction for plt. To do this, recall the following L^2 extension theorem [D01].

Theorem 3.2.23 (Ohsawa-Takegoshi-Marnivel L^2 extension theorem)**.** *Let X be a bounded pseudoconvex open subset of $\mathbf{C}^n$ and $Y \subset X$ a complex submanifold of codimension r defined by a section s of a hermitian holomorphic bundle* *with bounded curvature tensor. Suppose that s is everywhere transverse to the zero section, and that the inequality $|s| \leq e^{-1}$ holds on X. Then there exists a constant $C > 0$ depending only on E such that for all psh function φ on X and all holomorphic functions f on Y satisfying $\int_Y |f|^2 |\Lambda^r(ds)|^{-2} e^{-2\varphi} dV_Y < +\infty$, there exists a holomorphic function F on X such that $F|_Y = f$ and*

$$\int_X \frac{|F|^2 e^{-2\varphi}}{|s|^2 \log^2 |s|} dV_X \leq C \int_Y \frac{|f|^2 e^{-2\varphi}}{|\Lambda^r(ds)|^2} dV_Y.$$

Locally, let s be a section of the trivial line bundle $X \times \mathbf{C}$ given by $s = z_1$. Then $Y = H$ and all hypotheses hold after shrinking an open neighborhood of $y \in H$ if necessary. Since $|ds|^{-1}$ is bounded near y, the integrability of $e^{-2\varphi}$ near an open neighborhood of y in Y implies the integrability of $\frac{e^{-2\varphi}}{|s|^2 \log^2 |s|}$ near y in X. Thanks to Demailly's openness theorem, we conclude that there exist $\epsilon > 0$ and an open set V of X containing y such that

$$\int_V \frac{e^{-2(1+\epsilon)\varphi}}{|s|^2 \log^2 |s|} dV_X < +\infty.$$

Therefore we obtain the following theorem.

Theorem 3.2.24. *$(X, D + H)$ is plt near H if and only if $(H, D|_H)$ is klt.*

Generally, the case when X is just normal and H is given as a Weil divisor on X is more complicated since the adjunction formula require an additional term which is called the *different* of D on the normalization of H.

Bibliography

[AV61] A. Andreotti, E. Vesentini, *Sopra un teorema di Kodaira*, Ann. Scuola Norm. Sup. Pisa **15** (1961), 283–309.

[AV65] A. Andreotti, E. Vesentini, *Carleman estimates for the Laplace-Beltrami equation on complex manifolds*, Publ. Math. I.H.E.S. **25** (1965), 81-130.

[BEGZ] S. Boucksom, P. Eyssidieux, V. Guedj and A. Zeriahi, *Monge-Ampère equations in big cohomology classes*, Acta Math. **205** (2010), 199 - 262.

[BFFU] S. Boucksom, T. de Fernex, C. Favre and S. Urbinati, *Valuation spaces and multiplier ideals on singular varieties*, Recent advances in algebraic geometry, 29 - 51, London Math. Soc. Lecture Note Ser., 417, Cambridge Univ. Press, Cambridge, 2015.

[BFJ] S. Boucksom, C. Favre and M. Jonsson, *Valuations and plurisubharmonic singularities*, Publ. Res. Inst. Math. Sci **44** (2008), no.2, 449-494.

[BSTZ] M. Blickle, K. Schwede, S. Takagi and W. Zhang, *Discreteness and rationality of F-jumping numbers on singular varieties*, Math. Ann. 347 (2010), no. 4, 917–949.

[BS05] N. Budur and M. Saito, *Multiplier ideals, V-filtration, and spectrum*, J. Algebraic Geom. 14 (2005), no. 2, 269–282.

[C04] U. Cegrell, *The general definition of the complex Monge-Ampère operator*, Ann. Inst. Fourier (Grenoble) **54** (2004), no. 1, 159-179.

[CKM88] H. Clemens, J. Kollár and S. Mori, *Higher Dimensional Complex Geometry*, Asterisque 166, 1988.

[CL] C. Berkesch and A. Leykin, *Algorithms for Bernstein-Sato polynomials and multiplier ideals*, ISSAC 2010—Proceedings of the 2010 International Symposium on Symbolic and Algebraic Computation, 99–106, ACM, New York, 2010.

[D82] J.-P. Demailly, *Estimations L^2 pour l'opérateur $\overline{\partial}$ d'un fibré vectoriel holomorphe semipositif au dessus d'une variété kählérienne compléte*, Ann. Sci. École Norm. Sup. **15** (1982), 457-511.

[D97] J.-P. Demailly, *Complex analytic and differential geometry*, book available at http://www-fourier.ujf-grenoble.fr/ demailly/manuscripts/agbook.pdf.

[D93] J.-P. Demailly, *Monge-Ampère operators, Lelong numbers and intersection theory*, Complex analysis and geometry, 115-193, Univ. Ser. Math., Plenum, New York, 1993.

[D96] J.-P. Demailly, *L^2 vanishing theorems for positive line bundles and adjunction theory*, Transcendental Methods in Algebraic Geometry (Cetraro, 1994), Lect. Notes in Math., vol. 1464, Springer, Berlin, 1996, pp. 1-97.

[D01] J.-P. Demailly, *Multiplier ideal sheaves and analytic methods in algebraic geometry*, in *Vanishing Theorems and Effective Results in Algebraic Geometry*, ICTP Lecture Notes 6 (2001), pp. 1-148.

[D11] J.-P. Demailly, *Analytic methods in algebraic geometry*, Surveys of Modern Mathematics, 1. International Press, Somerville, MA; Higher Education Press, Beijing, 2012. (Also July 2011 version from the author's website.)

[D15] J.-P. Demailly, *Extension of holomorphic functions defined on non reduced analytic subvarieties*, The legacy of Bernhard Riemann after one hundred and fifty years. Vol. I, 191 - 222, Adv. Lect. Math., 35.1, Int. Press, Somerville, MA, 2016.

[DEL] J.-P. Demailly, L.Ein and R.Lazarsfeld, *A subadditivity property of multiplier ideals*, Michigan Math. J. **48** (2000), 137 -156.

[DGZ] S. Dinew, V. Guedj and A. Zeriahi, *Open problems in pluripotential theory*, Complex Var. Elliptic Equ. 61 (2016), no. 7, 902-930.

[DH] J.-P. Demailly and H.H. Pham, *A sharp lower bound for the log canonical threshold*, Acta Math. 212 (2014), no. 1, 1-9.

[ELSV] L. Ein, R. Lazarsfeld, K. Smith and D. Varolin, *Jumping coefficients of multiplier ideals*, Duke Math. J. 123 (2004), no. 3, 469-506.

[ELS01] L. Ein, R. Lazarsfeld and K. Smith, *Uniform bounds and symbolic powers on smooth varieties*, Invent. Math. 144 (2001), no. 2, 241–252.

[FJ05] C. Favre and M. Jonsson, *Valuations and multiplier ideals*, J. Amer. Math. Soc. 18 (2005), no. 3, 655–684.

[G16] P. Graf, *The jumping coefficients of non-*$\mathbf{Q}$*-Gorenstein multiplier ideals*, J. Algebra 450 (2016), 323–348.

[GL] Q. Guan and Z. Li, *Cluster points of jumping coefficients and equisingularties of plurisubharmonic functions*, arXiv:1604.03406.

[GZ] Q. Guan and X. Zhou, *A proof of Demailly's strong openness conjecture*, Ann. of Math. (2) **182** (2015), no. 2, 605-616.

[G] H. Guenancia, *Toric plurisubharmonic functions and analytic adjoint ideal sheaves*, Math. Z. **271** (2012), 1011 - 1035.

[H] L. Hörmander, *Notions of Convexity*, Modern Birkhäuser Classics, Birkhäuser Boston, 2007.

[H65] L. Hörmander, *L^2 estimate and existence theorems for the $\overline{\partial}$ operator*, Acta. Math. **113** (1965), 89-152.

[H73] L. Hörmander, *An introduction to Complex Analysis in several variables*, 2nd edition, vol. 7, North-Holland Mathematical Library, Elsevier Science, 1973.

[JM] M. Jonsson and M. Mustaţă, *Valuations and asymptotic invariants for sequences of ideals*, Ann. Inst. Fourier (Grenoble) 62 (2012), no. 6, 2145-2209.

[Ka99] Y. Kawamata, *Deformations of canonical singularities*, J. Amer. Math. Soc. 12 (1999), no. 1, 85–92.

[K15] D. Kim, *Equivalence of plurisubharmonic singularities and Siu-type metrics*, Monatsh. Math. **178** (2015), no. 1, 85-95.

[K19] D. Kim, *Canonical bundle formula and semipositivity*, preprint, 2019.

[Ko63] J. J. Kohn, *Harmonic integrals on strongly pseudo-convex manifolds I*, Ann. Math. (2) **78** (1963), 112-148.

[Kol92] J. Kollár, *Flips and Abundance for Algebraic Threefolds*, vol. 211, Astérisque, 1992.

[Kol97] J. Kollár, *Singularities of pairs: in "Algebraic Geometry-Santa Cruz 1995"*, Proc. Symp. Pure Math. **62** (1997), 221–287.

[KM] J. Kollár, S. Mori, *Birational Geometry of Algebraic Varieties*, Cambridge Tracts in Math., vol. 134, Cambridge University Press, Cambridge, 1998.

[KR] D. Kim and A. Rashkovskii, *Higher Lelong numbers and convex geometry*, arXiv:1803.07948.

[Ko15] T. Koike, *Minimal singular metrics of a line bundle admitting no Zariski decomposition*, Tohoku Math. J. (2) 67 (2015), no. 2, 297-321.

[L] R. Lazarsfeld, *Positivity in Algebraic Geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. 48 - 49. Springer-Verlag, Berlin, 2004.

[M58] C. B. Morrey, *The analytic embedding of abstract real analytic manifolds*, Ann. Math. (2) **68** (1958), 159-201.

[M02] M. Mustaţă, *On multiplicities of graded sequences of ideals*, J. Algebra 256 (2002), no. 1, 229-249.

[N04] N. Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs, 14. Mathematical Society of Japan, Tokyo, 2004.

[Oh01] T. Ohsawa, *On the extension of L^2 holomorphic functions, V: Effects of generalization*, Nagoya Math. J. **161** (2001), 1-21, erratum: Nagoya Math. J. **163** (2001), 229.

[OT87] T. Ohsawa, K. Takegoshi, *On the extension of L^2 holomorphic functions*, Math. Zeitschrift **195** (1987), 197-204.

[R13a] A. Rashkovskii, *Analytic approximations of plurisubharmonic singularities*, Math. Z. 275 (2013), no. 3-4, 1217-1238.

[R13b] A. Rashkovskii, *Multi-circled singularities, Lelong numbers, and integrability index*, J. Geom. Anal. 23 (2013), no. 4, 1976-1992.

[S93] M. Saito, *On b-function, spectrum and rational singularity*, Math. Ann. 295 (1993), no. 1, 51–74.

[S07] M. Saito, *Multiplier ideals, b-function, and spectrum of a hypersurface singularity*, Compos. Math. 143 (2007), no. 4, 1050-1068.

[S98] Y.-T. Siu, *Invariance of plurigenera*, Invent. Math., **134** (1998), 661 - 673.

[Sko72] H. Skoda, Sous-ensembles analytiques d'ordre fini ou infini dans $\mathbf{C}^n$, Bull. Soc. Math. France, **100** (1972), 353-408

[ST07] K. Smith and H. Thompson, *Irrelevant exceptional divisors for curves on a smooth surface*, Algebra, geometry and their interactions, 245-254, Contemp. Math., 448, Amer. Math. Soc., Providence, RI, 2007.

[T10] K. Tucker, *Jumping numbers on algebraic surfaces with rational singularities*, Trans. Amer. Math. Soc. 362 (2010), no. 6, 3223–3241.

[Ta10] S. Takagi, *Adjoint ideals along closed subvarieties of higher codimension*, J. reine angew. Math. **641** (2010), 145-162.

[U12] S. Urbinati, *Discrepancies of non-$\mathbf{Q}$-Gorenstein varieties*, Michigan Math. J. 61 (2012), no. 2, 265–277.

국문초록

해석적 승수 아이디얼과 L^2 확장 정리

이 논문에서는 승수 아이디얼 층과 L^2 확장 정리에 대하여 다룬다. 승수 아이디얼과 승수 아이디얼의 뛈수는 그 응용으로 인해 대수기하학과 복소기하학에서 중요한 역할을 한다. 대수적 승수 아이디얼의 뛈수는 이미 Ein, Lazarsfeld, Smith 그리고 Varolin에 의해 [ELSV]에서 심도 깊게 연구된 바 있다. 이 논문에서 뛈수에 대한 연구를 대수적 상황에서 다중버금조화함수의 경우로 확장하여 [ELSV]에서 보여진 다양한 성질들 중 다중버금조화함수의 경우로 일반화 될 수 있는 성질과 일반화 될 수 없는 성질들에 대하여 논한다. 특히 기존에 Guan-Li와 Ein-Lazarsfeld-Smith-Varolin에 의해 뛈수가 집적점을 가지는 승수 아이디얼의 예가 두 가지만 알려져 있는데 차원이 2인 경우에 원환다중버금조화함수의 승수 아이디얼의 뛈수가 언제 집적점을 가지는지 규명하고 집적점을 가진다면 정확한 값을 가지는 방법을 논한다. 이를 통해 해석적 승수 아이디얼의 뛈수가 집적점을 가지는 경우가 빈번하게 발생하는 현상임을 밝히고 승수 아이디얼의 뛈수가 집적점을 가지는 다중버금조화함수를 무수히 많이 얻어낼 수 있음을 밝힌다.

주요어휘: L^2 추정, L^2 확장 정리, 뛈수, 해석적 승수 아이디얼
학번: 2011-20272

감사의 글

대학원에서 공부하는 시간 동안 부족한 저를 항상 따뜻하고 자상하게 지도해 주시고 학업과 생활을 걱정해 주시고 배려해 주신 김다노 선생님께 말로 다 할 수 없는 깊은 감사를 드립니다. 김다노 선생님께서 학문을 대하는 열정적인 모습과 연구자의 자세를 본받기 위해서 항상 노력하겠습니다. 또한 바쁘신 와중에도 학위 논문 심사를 맡아주시고 아낌없는 조언을 해 주신 김영훈 선생님, 현동훈 선생님, Otto van Koert 선생님, 그리고 먼 곳에서 오셔서 따뜻한 조언을 해 주신 박종도 선생님께 깊은 감사의 인사를 드립니다. 지금까지 공부하는데 많은 가르침을 주신, 미처 언급하지 못한 많은 선생님들께도 깊은 감사의 인사를 드립니다.

대학원에서 공부한 시간만큼 함께 공부했던 좋은 사람들에게도 감사의 인사를 드립니다. 입학 전 수업부터 함께 많은 수업을 같이 들으며 많은 조언과 도움을 준 이기현 형, 시시콜콜한 이야기들을 하고 같이 야식을 먹으며 많은 시간을 보냈던 고광현 형, 또한 항상 모르는 것들을 물을 때마다 친절하게 알려준 김윤환 형, 그리고 다른 많은 사람들에게도 깊은 감사의 인사를 드립니다.

지금까지 저를 길러주시고 제가 하려는 것들을 항상 묵묵히 지원해주신 어머니와 먼저 하늘나라로 가신 아버지, 항상 감사드리고 사랑합니다. 또한 제가 하는 일들에 열정적인 응원을 보내주신 장인어른과 장모님에게도 깊은 감사와 사랑한다는 말씀을 드리고 싶습니다. 제가 하는 일을 항상 응원해주고 물심양면으로 지원해주고 지칠 때 든든한 버팀목이 되어주며 저보다 더 고생한 아내 엄지은에게 고맙고 사랑한다고 말하고 싶습니다.

이 자리에 있기까지 많은 분들이 주신 가르침과 조언, 그리고 도움을 잊지 않고 마음에 새겨 앞으로 하는 모든 일에 최선을 다하겠습니다.