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Linear City Competition with Heterogeneous Product Values*

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We study a model in which a linear city of length 1 exists along the abscissa of a line ($0 \leq x \leq 1$), and consumers are uniformly distributed with density 1 along this interval. There are two firms which sell a kind of goods. Firm 1's goods are valued s_1 to the consumers, and firm 2's goods have intrinsic value of s_2. A consumer purchase one unit of goods if he chooses to do so. We define two kinds of differentiation. One is the differentiation in the location of each firm. Each firm can locate at a point x. The other differentiation is in the product's value to customers. For this differentiation, each firm may try to increase the intrinsic value of its product compared with other firm's products. In our model, this differentiation in product value is assumed to be given such that firm 1 offers higher value than firm 2 ($s_1 > s_2$). A consumer has to suffer a quadratic transportation cost in addition to the price in order to buy a product from a firm. We first study the duopoly competition of two firms which are located at each extreme end of the linear city. And then we deal with the location problem.

Keyword: Hotelling model, location model, differentiation, Nash equilibrium, price competition, Bertrand competition

I. Introduction

We consider a model (originally due to Hotelling [1929]) in which a linear city of length 1 exists along the abscissa of a line ($0 \leq x \leq 1$), and consumers are uniformly distributed with density 1 along this interval. There are two firms which produce distinct and substitute

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goods. Firm 1's goods are valued s_1 to the consumers, and firm 2's goods have intrinsic value of s_2. The goods of each firm are the same except the intrinsic values to the consumers, and we assume $s_1 > s_2$. This is the same situation as firm 1 has competitive advantage of lower unit production cost of $s_1 - s_2$ compared with firm 2 for homogeneous products. Or $s_1 > s_2$ may represent the difference in quality level even though the products serve the same function to the consumers. A consumer purchase one unit of goods if he chooses to do so. We assume that (s_1, s_2) are sufficiently large such that all consumers in the linear city purchase a unit. That is, we assume the whole market is covered by either of the firms.

We define two kinds of differentiation. One is the differentiation in the location of each firm. Each firm can locate at a point $x\,(0 \leq x \leq 1)$. The location may represent a target market for a firm. The location differentiation implies that the target markets are separated apart with a certain distance. The other differentiation is in the product's value to customers. For this differentiation, each firm may try to increase the intrinsic value of its product compared with other firm's products. In our model, this differentiation in product value is assumed to be given such that firm 1 offers higher value than firm 2 $(s_1 > s_2)$.

We suppose that a consumer has to suffer a transportation cost in addition to the price in order to buy a product from a firm. The transportation cost may literally mean logistics cost or represent the disutility from the discrepancy of the consumer and the firm's target market. We assume quadratic transportation costs. That is, a consumer living at the distance x from a firm incurs a transportation cost of tx^2, where t is the unit transportation cost per squared distance. This quadratic transportation rather than linear is assumed for the sake of continuous market share change. And we also assume that the production cost for the goods is 0 without loss of generality.

In our paper, we first study the duopoly competition of two firms which are located at each extreme end of the linear city. And then we deal with the location problem as well.

In Shaked and Sutton (1982), three stage game was studied. In the first stage, firms decide

Figure 1. Linear City

whether or not to enter the industry. In the second stage, each firm chooses the quality of its product. Then in the final stage, each firm chooses its price. They derive the proposition that the only perfect equilibrium is one in which exactly two firms enter; in which they produce distinct products, and earn positive profits at equilibrium. One of the key properties of their model reflects the effect of the lessening of price competition as qualities diverge.

II. Price Competition Model

First we consider the case where firm 1 is located at $x = 0$ and firm 2 at $x = 1$. That is, two firms are already maximally differentiated in location. In this case of fixed location, we study how the firms will compete with prices. The distinct point from Hotelling is that two firms offer goods with different intrinsic values to the consumers. A consumer who is indifferent between the two firms is located at x, where x is given by equating the total costs:

$$s_1 - p_1 - tx^2 = s_2 - p_2 - t(1 - x)^2.$$

This equation implies that the consumer at x has the same residual value of buying either from firm 1 or firm 2. From the full market coverage assumption, we have sufficiently large (s_1, s_2) such that the residual values become non-negative. Denoting $\Delta s = s_1 - s_2$, we can derive

$$x = \frac{\Delta s + p_2 - p_1 + t}{2t}.$$

And thus we have the demands of each firm as $D_1(p_1, p_2) = x$ and $D_2(p_1, p_2) = 1 - x$.

In order for the demand functions to be valid, we need $0 \leq x \leq 1$, and this is equivalent to:

$$p_1 - t - \Delta s \leq p_2 \leq p_1 + t - \Delta s. \tag{1}$$

We have the following profit functions of each firm:

$$\pi^1 = p_1 x = p_1(\frac{\Delta s + p_2 - p_1 + t}{2t}),$$

$$\pi^2 = p_2(1-x) = p_2(\frac{-\Delta s - p_2 + p_1 + t}{2t}).$$

We can derive the reaction functions of each firm by $\frac{\partial \pi^1}{\partial p_1} = 0$ and $\frac{\partial \pi^2}{\partial p_2} = 0$. Thus we have firm 1's reaction function being $R_1(p_2) = \frac{\Delta s + p_2 + t}{2}$. Likewise, we get $R_2(p_1) = \frac{-\Delta s + p_1 + t}{2}$. The effective reaction functions should be adjusted by (1) and $p_1 \geq 0$, $p_2 \geq 0$.

1. Case where $\Delta s \geq 3t$

In this case, we have a monopoly by firm 1. The Nash equilibrium is $(p_1, p_2) = (\Delta s - t, 0)$, and the profits of each firm are $(\pi^1, \pi^2) = (\Delta s - t, 0)$. The case where $\Delta s \geq 3t$ means that we have a large amount of discrepancy in product value compared with the unit transportation cost. In this case, the firm producing goods with higher value dominates the market and the other firm gets 0 profit.

Considering the internet commerce, we note that the transaction costs including search costs, delivery time and costs, and so on decrease as the infrastructure develops. This means that the cost of a consumer purchasing from a firm far away is reduced. For example, using internet, we can easily find an on-line store with some distinct product and purchase the item with low delivery cost. These search and delivery costs are represented by the transportation cost t in our model. Therefore, in the internet commerce where the transportation cost decreases, the differentiated value of a product is rewarded to offer the monopoly. That is, we observe more frequently the phenomenon that the winner takes it all in the e-commerce markets.

2. Case where $0 < \Delta s < 3t$

In this case, we can derive the Nash equilibrium of $(p_1, p_2) = (t + \frac{\Delta s}{3}, t - \frac{\Delta s}{3})$. And the consequent profits are $(\pi^1, \pi^2) = (\frac{1}{2t}(t + \frac{\Delta s}{3})^2, \frac{1}{2t}(t - \frac{\Delta s}{3})^2)$. We can note that firm 1 with

higher intrinsic product value has the privilege of charging higher price and enjoying higher profit than firm 2.

We can derive some implications from the outcome. From the Nash equilibrium prices, we note that $p_1 - p_2 = \frac{2\Delta s}{3}$. That is the price premium for the better goods is $\frac{2\Delta s}{3}$, which is smaller than the difference of intrinsic values, Δs. This phenomenon can be interpreted from the trade-off between two effects. One effect is from the higher profit margin for each unit. And this effect comes from the higher unit price. The other effect comes from larger market share. A firm can enlarge its market share by pricing lower. In our case, the optimal trade-off for firm 1 turns out to charge higher price than firm 2 but not as much as its incremental intrinsic value compared with firm 2. By not charging the whole increment of Δs, firm 1 is better off from the larger demand of its own.

III. Location Model

In the previous section, we considered the case where the firms' locations are predetermined such that maximal differentiation of location is given. Now we study the case where firms can determine their location. The model will consist of two stages. In the first stage, each firm determines its own location considering the other firm's reaction. And then the firms wage price competition in the second stage. As well-known, we solve the model backwards. We solve the second stage problem. And then using the second stage outcome, we derive the first stage optimal solutions.

The optimal location can be derived from the trade-off between two effects. One effect is from the higher profit margin for each unit. And this effect comes from the higher unit price. Higher prices are possible when the two firms are farther away and relax competition. The other effect comes from larger market share. A firm can enlarge its market share by locating near the center of the city and thus pricing lower. That is, the first effect makes the two firms be located far away from each other. And the other effect tries to locate the two firms near the center of the city.

Suppose that firm 1 is located at point $0 \leq a \leq 1$ and firm 2 at point $0 \leq 1 - b \leq 1$.

Without loss of generality, we assume $a \leq 1 - b$, that is $a + b \leq 1$. This means that firm 1 is on the left of firm 2 without loss of generality. Following the analysis of the previous section, we can derive the following demand functions.

$$D_1\left(p_1, p_2\right) = \frac{1+a-b}{2} + \frac{\Delta s + p_2 - p_1}{2t\left(1-a-b\right)},$$

$$D_2\left(p_1, p_2\right) = \frac{1+b-a}{2} + \frac{-\Delta s + p_1 - p_2}{2t\left(1-a-b\right)}.$$

And using the reaction functions, we can derive the Nash equilibrium in prices as follows.

$$p_1^n = \frac{\Delta s + t(b^2 - 4b - 2a - a^2 + 3)}{3},$$

$$p_2^n = \frac{-\Delta s + t(a^2 - 4a - 2b - b^2 + 3)}{3}.$$

We should note that the Nash equilibrium above is the second stage outcome of the game. Using this equilibrium, we can solve the first stage problem of location. The special case of location would be the one where $a = b = 0$, which was dealt with in the previous section. As expected, the Nash equilibrium prices is $(p_1, p_2) = (t + \frac{\Delta s}{3}, t - \frac{\Delta s}{3})$. In the first stage, firm 1 should decide its location of a by trying to maximize the following reduced-form profit function:

$$\pi^1(a,b) = p_1^n(a,b) D_1(a,b, p_1^n(a,b), p_2^n(a,b)).$$

An equilibrium in location is such that firm 1 maximizes $\pi^1(a, b)$ with respect to a, taking b as given, and similarly for firm 2.

By utilizing the envelope theorem, we can derive the following:

$$\frac{d\pi^1}{da} = p_1^n \left(\frac{\partial D_1}{\partial a} + \frac{\partial D_1}{\partial p_2} \frac{\partial p_2^n}{\partial a} \right),$$

$$\frac{d\pi^2}{db} = p_2^n \left(\frac{\partial D_2}{\partial b} + \frac{\partial D_2}{\partial p_1} \frac{\partial p_1^n}{\partial b} \right).$$

We first deal with the location decision of firm 2 and analyze the second derivative above.

$$\frac{\partial D_2}{\partial b} = \frac{1}{2} + \frac{-\Delta s + p_1^n - p_2^n}{2t(1-a-b)^2} = \frac{1}{2} - \frac{\Delta s}{2t(1-a-b)^2} - \frac{-2\Delta s + 2t(a-b)(a+b-1)}{6t(1-a-b)^2}$$

$$= \frac{1}{2} - \frac{\Delta s}{6t(1-a-b)^2} + \frac{a-b}{3(1-a-b)}.$$

From $\frac{\partial D_2}{\partial p_1} = \frac{1}{2t(1-a-b)}$ and $\frac{\partial p_1^n}{\partial b} = \frac{t(2b-4)}{3}$, we get $\frac{\partial D_2}{\partial p_1} \frac{\partial p_1^n}{\partial b} = \frac{b-2}{3(1-a-b)}$. Thus we have

$$\frac{\partial D_2}{\partial b} + \frac{\partial D_2}{\partial p_1} \frac{\partial p_1^n}{\partial b} = -\frac{\Delta s + t(1-a-b)(3b+a+1)}{6t(1-a-b)^2} < 0.$$

Therefore, we know that firm 2 will try to decrease b as much as possible. We thus get firm 2's optimal location becomes $b = 0$. That is, firm 2 chooses to locate at $1 - b = 1$, the right most corner of the linear city.

Using the fact that $b^* = 0$, we now analyze to get the optimal location of firm 1. Suppose that firm 1 is located at a in the linear city.

Let x be the customer who is indifferent to purchasing from firm 1 or firm 2. Then the following equation should be held:

$$s_1 - [p_1 + t(x-a)^2] = s_2 - [p_2 + t(1-x)^2].$$

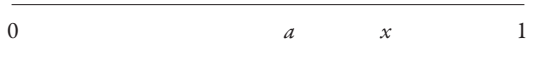

Figure 2. Linear City with firm 1's location of *a*

Here we should note that x can be left of a. That is, the market share of firm 1 can be smaller than a depending on the pricing strategies of each firm.

$$x = \frac{\Delta s - p_1 + p_2}{2t(1-a)} + \frac{a+1}{2}.$$

In order for this x can be valid, we need the condition of $0 \leq x \leq 1$. From this, we can derive the following inequality condition:

$$\Delta s - t(1-a)^2 \leq p_1 - p_2 \leq \Delta s + t(1-a^2). \tag{2}$$

From the derivative of π^1, we get the reaction function of firm 1 as follows:

$$p_1(p_2) = \frac{p_2 + \Delta s + (1-a^2)t}{2}.$$

Likewise, the reaction function of firm 2 becomes:

$$p_2(p_1) = \frac{p_1 - \Delta s + (1-a)^2 t}{2}.$$

From these two reaction functions, we can derive the Nash equilibrium prices as follows:

$$p_1^n = \frac{\Delta s + t(-2a - a^2 + 3)}{3},$$

$$p_2^n = \frac{-\Delta s + t(a^2 - 4a + 3)}{3}.$$

In order to check the validity of the Nash equilibrium prices, we utilize the condition (2). Denoting $k = \frac{\Delta s}{t}$, we can derive the following two inequality conditions from (2).

$$(a+3)(a-1) \leq k,$$

$$k \leq (a-1)(a-3).$$

For $0 \leq a \leq 1$, the first inequality is trivially satisfied. And the second condition is effective with inducing the following range for a:

$$a \leq 2 - \sqrt{1+k} \tag{3}$$

Now using the Nash equilibrium, we can derive the profit function of firm 1:

$$\pi^1(a) = \frac{t(k+3-2a-a^2)^2}{18(1-a)}.$$

Thus we need to find the optimal a which maximizes $g(a) = \frac{(k+3-2a-a^2)^2}{1-a}$. Using the derivative

$$g'(a) = \frac{(k+3-2a-a^2)(k-1-2a+3a^2)}{(1-a)^2}.$$

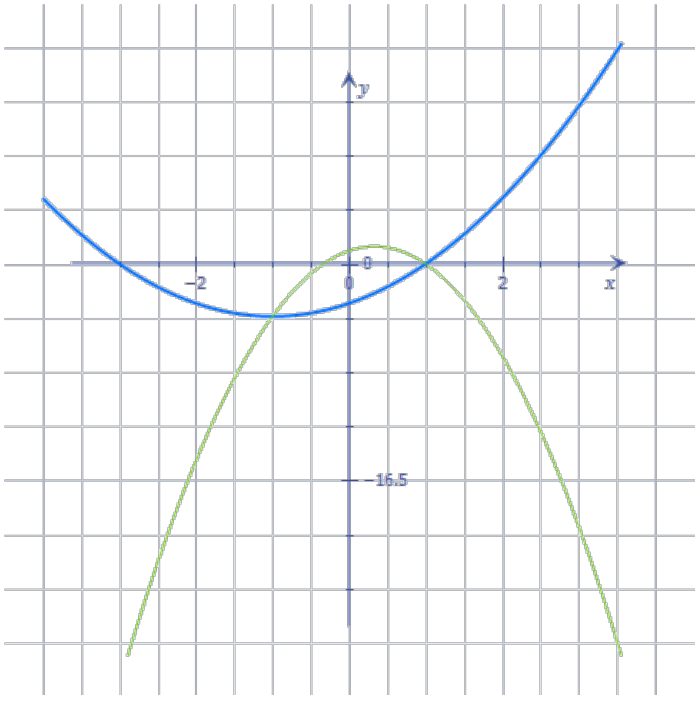

Figure 3. $y = x^2 + 2x - 3$, $y = -3x^2 + 2x + 1$

$$y = x^2 + 2x - 3$$

$$y = -3x^2 + 2x + 1$$

The sign of $g'(a)$ is the same as $(k + 3 - 2a - a^2)(k - 1 - 2a + 3a^2)$. Let $A(a) = k + 3 - 2a - a^2$ and $B(a) = k - 1 - 2a + 3a^2$. Denote $a_1 < a_4$ as the two solutions of $A(a) = 0$. And let $a_2 \leq a_3$ be the two solutions of $B(a) = 0$. The next table shows the signs of $A(a)$, $B(a)$, and $A(a)B(a)$ along the abscissa of a in each case. In case 1 where $0 < k \leq \frac{4}{3}$, we have four real numbers of $a_1 < a_2 \leq a_3 < a_4$. In case 2 where $k > \frac{4}{3}$, we have two solutions of $a_1 < a_4$.

In case 1, the optimal a which maximizes $g(a)$ is either 0 or $2-\sqrt{1+k}$ from (3). We should compare $\pi^1(a = 0)$ with $\pi^1(a = 2-\sqrt{1+k})$ to determine which is bigger. We note that $\pi^1(a=0) = \frac{t(k+3)^2}{18}$ and $\pi^1(a = 2-\sqrt{1+k}) = 2t(\sqrt{1+k}-1)$. In case 1, we have compare $\pi^1(a = 0)$ with $\pi^1(a = 2-\sqrt{1+k})$.

$$\pi^1(a=0) = \frac{t(k+3)^2}{18} < \pi^1(a = 2-\sqrt{1+k}) = 2t(\sqrt{1+k}-1)$$

$$\leftrightarrow (k+3)^2 < 36(\sqrt{1+k}-1)$$

$$\leftrightarrow k^4 + 12k^3 + 126k^2 - 756k + 729 < 0$$

Table 1. Derivatives

Case 1		a1		a2	0	a3	1	a4	
A(a)	−	0	+	+	+	+	+	0	−
B(a)	+	+	+	0	−	0	+	+	+
A(a)B(a)	−	0	+	0	−	0	+	0	−
Case 2									
A(a)	−	0	+	+	+	+	+	0	−
B(a)	+	+	+	+	+	+	+	+	+
A(a)B(a)	−	0	+	+	+	+	+	0	_

$$\leftrightarrow (k-3)(k^3+15k^2+171k-243)<0$$

$$\leftrightarrow \alpha < k < 3,$$

where α is the only real number solution of $k^3 + 15k^2 + 171k - 243 = 0$ and $\alpha \approx 1.26$.

$$y = x^3 + 15x^2 + 171x - 243$$

$$y = x^4 + 12x^3 + 126x^2 - 756x + 729$$

In case 2, we can see that $g(a)$ is maximized at $a^* = 2-\sqrt{1+k}$.
We therefore get the following outcome:

(1) For $0 \le k < \alpha$, we have duopoly with

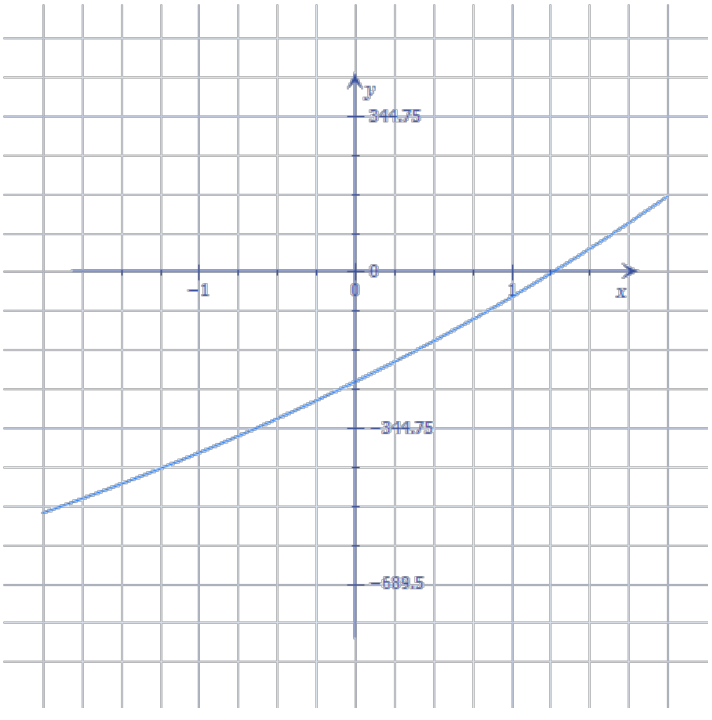

Figure 4. $y = x^3 + 15x^2 + 171x - 243$

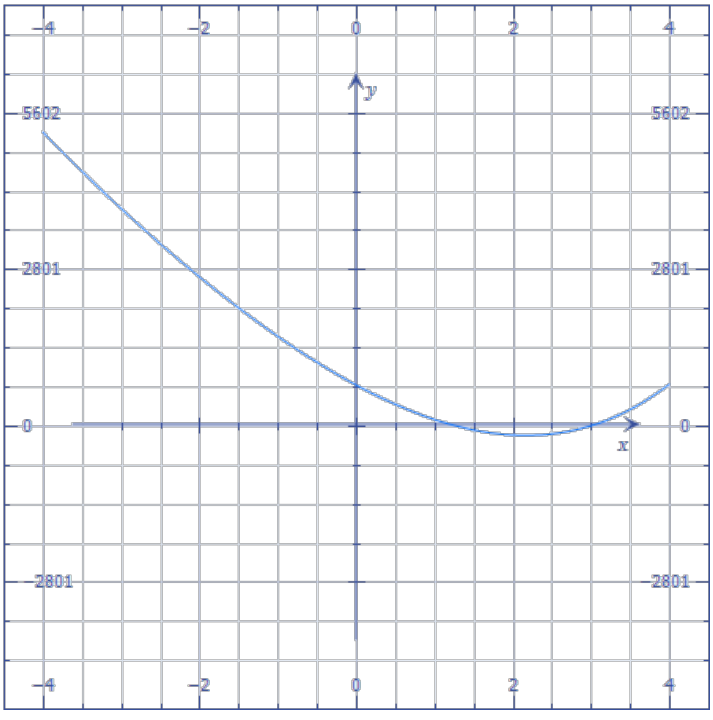

Figure 5. $y = x^4 + 12x^3 + 126x^2 - 756x + 729$

$a^* = 0,$

$$p_1 = t + \frac{\Delta s}{3},\ D_1 = \frac{k}{6} + 0.5 > 0,\ \pi^1 = \frac{(\Delta s + 3t)^2}{18t},$$

$$p_2 = t - \frac{\Delta s}{3},\ D_2 = -\frac{k}{6} + 0.5 > 0,\ \pi^2 = \frac{(3t - \Delta s)^2}{18t}.$$

(2) For $\alpha \le k < 3$, we have monopoly with

$$a^* = 2 - \sqrt{1+k},$$

$$p_1 = 2t(\sqrt{1+k} - 1),\ D_1 = 1,\ \pi^1 = 2t(\sqrt{1+k} - 1),$$

$p_2 = 0,\ D_2 = 0,\ \pi^2 = 0.$

(3) For $3 \le k$, we have monopoly with

$a^* = 0,$

$p_1 = \Delta s - t, D_1 = 1, \pi^1 = \Delta s - t,$

$p_2 = 0, D_2 = 0, \pi^2 = 0.$

IV. Interpretations and Extensions

In cases of (1) and (2), we have $a^* = 0$ but with different meaning. In case (1), we have a duopoly where two firms have positive market shares. However, in cases of (2) and (3), we have a monopoly by firm 1. In those cases, firm 2 does not have positive market share. In case (3), we have so much incremental value of Δs that firm 1 is better off by taking the whole market even at the location of 0.

When the relative value of incremental value to transportation cost ($k = \frac{\Delta s}{t}$) is less than α, firm 1 chooses to locate at 0 by utilizing the maximal differentiation from firm 2, which is located at 1. This maximal differentiation enables firm 1 to charge higher price due to less fierce competition between the two firms. However, when k increases such that $\alpha \leq k < 3$, firm 1 is better off by decreasing the differentiation from firm 2. That is, firm 1 can get benefit by enlarging its market share by locating nearer to frim 2. But firm 1 does not eradicate firm 2 by locating at 1, since firm 1 also suffers by fierce competition. We should note that even in case 2 as k increases the location of firm 1 gets closer to 0, which differentiates further from firm 2. When the incremental value is sufficiently large, firm 1 again goes back to location 0. In this case firm 1 derives the best outcome from higher pricing and drives out firm 2 from the market.

We should note that industrial organization becomes different depending on the value of $k = \frac{s_1 - s_2}{t}$. When $k \geq \alpha$ (cases 2 and 3), only firm 1 prevails in the market, and we thus get monopoly. But this monopoly is sensitive in that firm 2 works as a *potential* entrant to the market. This potential effect is represented in k where intrinsic value of firm 2's product, s_2, is involved. This is different from the monopoly where only one firm exists in the market regardless of the parameters.

When t decreases as in the e-commerce, we have larger k, and thus more chance of observing monopoly. In case 3 where we have huge value of k such that $3 \leq k$, firm 1 can be located at 0 of the maximal differentiation from the potential entrant, firm 2, at 1. Even at 0, firm 1 can maximize its profit repelling firm 2 from the market since it has so much incremental value.

In all the cases, the price difference $p_1 - p_2$ is less than $s_1 - s_2$. That is, firm 1 is better off by making extra charge less than the whole difference of values compared with firm 2's price. This comes from the fact that demand effect from lower price is larger than the price increase effect.

In our model, we assumed market coverage in the sense that all consumers in the linear city purchase a good either from firm 1 or from firm 2. Due to this assumption, the derivation of demand of each firm becomes simpler. There are two conditions for this. The first condition is that the customer x has non-negative residual utility by purchasing from firm 1:

$$s_1 \geq p_1 + t\left[\frac{\Delta s - p_1 + p_2}{2t(1-a)} + \frac{1-a}{2}\right]^2.$$

The other condition should be the extreme customer at 0 would want to buy from firm 1:

$$s_1 \geq p_1 + ta^2.$$

By utilizing the Nash equilibrium prices, we can derive that the market coverage assumption is valid on the area of (s_1, s_2), where $\frac{36s_1}{t} \geq 12(k+3-2a-a^2)+(\frac{k}{1-a}-5a+3)^2$ and $\frac{3s_1}{t} \geq k+3-2a+2a^2$. Roughly speaking, we need sufficiently large value of s_1 compared with t other than the conditions on the incremental value of $\Delta s = s_1 - s_2$. The first inequality comes from the condition that the critical customer at x has non-negative utility by purchasing from either firm. The other condition that customer at 0 should purchase from firm 1 is represented by the second inequality above. If we relax the market coverage condition, we have larger set of (s_1, s_2) for analysis.

Another possible extension of our model would be to study (s_1, s_2) as decision variables.

That is, each firm can choose the quality level of its goods associated with production cost of $c_i(s_i)$. Then we have to solve three stage games. A firm choose its quality level, and then choose the location, and finally choose the price. But in our model where $s_1 > s_2$ are given, we emphasized the fact that each firm produce distinct goods even though they seem to be homogeneous.

Even though homogeneous products are easier to analyze, we see heterogeneous product competition in general. Most products which are taken to be homogeneous are actually heterogeneous when we consider other attributes along with the products. For example, books are homogeneous, but they become heterogeneous when bookstores offer different delivery service or return policy. Therefore we need to analyze the competition for heterogeneous products. In our paper, heterogeneity is considered in (s_1, s_2) and location of the firm. From the outcome of our model, we can conclude that firms are better off by reducing the fierceness of competition when the comparative advantage of $s_1 - s_2$ is small. Otherwise when the comparative advantage of $s_1 - s_2$ is fairly large, the firm with higher value is better off by being a monopolist. And considering the comparative advantage of $s_1 - s_2$, we should note that its relative value with transportation cost t, $\frac{s_1 - s_2}{t}$, is relevant. Therefore when making a decision, a firm should consider t as well as $s_1 - s_2$. The strategic implications of our model are as follows. A firm may try to increase its relative competitiveness, which is $s_1 - s_2$, by improving s_1. Or a firm can choose optimal 'distance' from the enemy for mutual benefit. A firm can choose an optimal strategy considering the investment cost along with it.

V. Bertrand Competition and Location Decision of Firms with Different Intrinsic Values

Here we analyze the case where two firms are located at the same position. That is, the firms are not differentiated other than the fact that their products have different intrinsic values. We assume that the product of firm 1 has value of s_1 to customers. Firm 2's product has value of s_2, and $s_1 > s_2$ is assumed. Assume that two firms are located at $x = 0$ in the linear city of length 1.

We first consider the Bertrand competition of two firms. Let us suppose that firm 1 charge some price p_1. Then firm 2 will respond by pricing $p_2 = p_1 - \Delta s - \varepsilon$ for small ε. This price of p_2 will give firm 2 the whole demand of firm 1. In reaction, firm 1 will reduce its price a little from $p_2 + \Delta s$, and get back the whole demand of firm 2. Following these processes of price cutting, in the Nash equilibrium we will have $p_2 = 0$ and firm 2 will be extinct. And firm 1 will price less than or equal to Δs in the Nash equilibrium. Now we can model the firm 1's optimization model as follows. The marginal customer who would purchase from firm 1 is the x satisfying the equation:

$$s_1 = p + tx^2.$$

We get $x = \sqrt{\frac{s_1 - p}{t}}$. The optimization model would be

$$\text{Max } p * \min\{\sqrt{\frac{s_1 - p}{t}}, 1\}$$

$$\text{s.t. } p \leq \Delta s = s_1 - s_2$$

Depending on whether $s_1 > t$ is satisfied, we can separate the above optimization into two cases with deleting $\frac{1}{\sqrt{t}}$.

(1) $s_1 > t$

$$\text{Max } p\sqrt{s_1 - p}$$

$$\text{s.t. } s_1 - t \leq p \leq \Delta s$$

(2) $s_1 \leq t$

$$\text{Max } p\sqrt{s_1 - p}$$

$$\text{s.t. } 0 \leq p \leq \Delta s$$

By squaring the objective function of (1), we can have equivalent optimization as follows:

(1′) $s_1 > t$

$$\text{Max} \quad m(p) \equiv p^2(s_1 - p)$$
$$\text{s.t.} \quad s_1 - t \leq p \leq \Delta s$$

From $m'(p) = 0$, we get $p = \frac{2s_1}{3}$. Therefore we can deal with three subcases:

(1′-1) $s_1 - t \leq s_1 - s_2 \leq \frac{2s_1}{3}$, $s_1 > t$

$p^* = \Delta s.$

(1′-2) $s_1 - t \leq \frac{2s_1}{3} \leq s_1 - s_2$, $s_1 > t$

$p^* = \frac{2s_1}{3}.$

(1′-3) $\frac{2s_1}{3} \leq s_1 - t \leq s_1 - s_2, s_1 > t$

$p^* = s_1 - t.$

Likewise, we can deal with two subcases of (2):

(2-1) $\frac{2s_1}{3} \leq s_1 - s_2$, $s_1 \leq t$

$p^* = \frac{2s_1}{3}.$

(2-2) $s_1 - s_2 \leq \frac{2s_1}{3}$, $s_1 \leq t$

$p^* = \Delta s.$

Combining the five subcases, we can derive three regions of distinct optimal solutions by denoting $x = \frac{s_1}{t}$, $y = \frac{s_2}{t}$.

$$y = x$$

$$y = \frac{x}{3}$$

$$y = 1$$

$$x = 3$$

(1) Case 1: $\frac{s_1}{3} \le s_2 \le s_1$, $s_2 \le t$

$$p^* = \Delta s,\ D_1 = \sqrt{\frac{s_2}{t}},\ \pi^1 = \Delta s\sqrt{\frac{s_2}{t}}$$

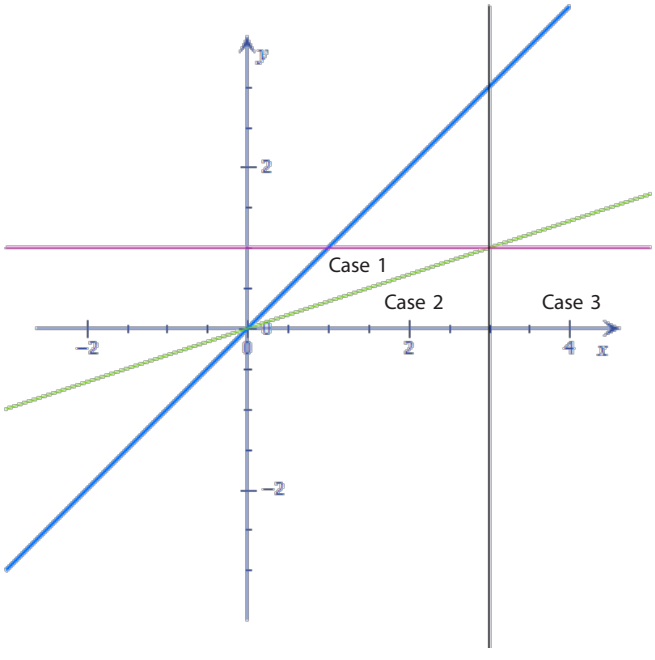

Figure 6. Three Cases

(2) Case 2: $0 \le s_2 \le \frac{s_1}{3}$, $s_1 \le 3t$

$$p^* = \frac{2s_1}{3}, \quad D_1 = \sqrt{\frac{s_1}{3t}}, \quad \pi^1 = \frac{2s_1}{3}\sqrt{\frac{s_1}{3t}}.$$

(3) Case 3: $0 \le s_2 \le t$, $s_1 > 3t$

$p^* = s_1 - t$, $D_1 = 1$, $\pi^1 = s_1 - t$.

We should note that in cases (1) and (2) the demand can be less than 1, which means that some customers do not purchase goods from either firm.

Taking into account $b = 0$, we consider $\frac{d\pi^1}{da}$ to derive optimal location of firm 1, a.

$$\frac{\partial D_1}{\partial a} = \frac{1}{2} + \frac{\Delta s + p_2^n - p_1^n}{2t(1-a-b)^2} = \frac{1}{2} + \frac{\Delta s}{2t(1-a-b)^2} + \frac{-2\Delta s + 2t(a-b)(a+b-1)}{6t(1-a-b)^2}$$

$$= \frac{1}{2} + \frac{\Delta s}{6t(1-a-b)^2} + \frac{b-a}{3(1-a-b)}.$$

$$\frac{\partial D_1}{\partial p_2} = \frac{1}{2t(1-a-b)}$$

$$\frac{\partial p_2^n}{\partial a} = \frac{2t(a-2)}{3}$$

We get $\frac{\partial D_1}{\partial a} + \frac{\partial D_1}{\partial p_2}\frac{\partial p_2^n}{\partial a} = \frac{\Delta s - (3a+b+1)(1-a-b)t}{6t(1-a-b)^2}$. By substituting $b = 0$, we have

$\frac{\partial D_1}{\partial a} + \frac{\partial D_1}{\partial p_2}\frac{\partial p_2^n}{\partial a} = \frac{\Delta s - (3a+1)(1-a)t}{6t(1-a)^2}$. Since $p_1^n \ge 0$, $\frac{d\pi^1}{da} < 0$ is equivalent to

$$\frac{\Delta s}{t} < (3a+1)(1-a).$$

Depending on the value of $\Delta s/t$, we get the following optimal location of firm 1.

(1) Case where $0 < \frac{\Delta s}{t} \leq 1$:

$a = 0$.

(2) Case where $1 < \frac{\Delta s}{t} \leq \frac{4}{3}$

$$a = \frac{1 - \sqrt{4 - 3\frac{\Delta s}{t}}}{3}.$$

(3) Case where $\frac{4}{3} < \frac{\Delta s}{t}$

$a = 1$.

We consider the first case where $0 < \frac{\Delta s}{t} \leq 1$. From the graph, we know firm 1's profit is maximized either at $a = 0$ or at $a = 1$. When we put $a = 0$, we get $p_1 = t + \frac{\Delta s}{3}$ and $D_1 = \frac{1}{2} + \frac{\Delta s}{6t}$ by noting $b = 0$. Therefore $\pi^1 = (t + \frac{\Delta s}{3})(\frac{1}{2} + \frac{\Delta s}{6t})$. When we put $a = 1$, firm 1 becomes a monopolist with pricing $p_1 = \frac{\Delta s}{3}$. And the profit becomes $\pi^1 = \frac{\Delta s}{3}$. By comparing those two profit functions, we can show that

$$\left(1 + \frac{k}{3}\right)(3 + k) > 2k.$$

Therefore, the optimal location for firm 1 becomes $a^* = 0$ in the first case.

The most interesting case is (2). We will analyze the case in detail. Denoting $\frac{\Delta s}{t} = k$, the equation $(3a + 1)(1 - a) = k$ has two solutions. From the sign of $\frac{d\pi^1}{da}$, the optimal location a should be either the smaller solution or 1. The smaller solution is $\alpha = \frac{1 - \sqrt{4 - 3k}}{3}$. We can derive components for firm 1's profit using α.

$$p_1^n(\alpha) = \frac{\Delta s + \frac{16 + 3k + 8\sqrt{1 - 3k}}{9} t}{3} = \frac{4}{9}\Delta s + \frac{16}{27} t + \frac{8}{27} t\sqrt{4 - 3k}.$$

$$p_2^n(\alpha) = \frac{-\Delta s + [\frac{5-3k-2\sqrt{4-3k}}{9} - \frac{4-4\sqrt{4-3k}}{3} + 3]t}{3}$$

$$= -\frac{4}{9}\Delta s + \frac{20}{27}t + \frac{10}{27}t\sqrt{4-3k}.$$

$$D_1(\alpha) = \frac{1+\alpha}{2} + \frac{\Delta s + p_2^n(\alpha) - p_1^n(\alpha)}{2t(1-\alpha)} = \frac{2(4-\sqrt{4-3k})}{9}$$

Therefore we get $\pi^1(\alpha) = t(\frac{7}{9}k + \frac{16}{27} + \frac{16}{27}\sqrt{1-3k})\frac{8-4\sqrt{1-3k}}{9}$. And we need to compare this with $\pi^1(1) = \frac{\Delta s}{3}$. We check whether the following inequality is satisfied.

$$\left(\frac{4}{9}k + \frac{16}{27} + \frac{8}{27}\sqrt{4-3k}\right)\frac{2(4-\sqrt{4-3k})}{9} > \frac{k}{3}$$

This is equivalent to

$$63k + 64 + 8(4-3k)\sqrt{4-3k} > 0$$

For $1 < k = \frac{\Delta s}{t} \le \frac{4}{3}$, we know that $4 - 3k > 0$. Thus, the inequality above is clearly satisfied. Therefore we get the optimal location of firm 1 as $a^* = \alpha$.

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