교육학 석사 학위논문

The competition graphs of multipartite tournaments

(방향 지어진 완전 다분 그래프의 경쟁 그래프)

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The competition graphs of multipartite tournaments

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Abstract

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The *competition graph* $C(D)$ of a digraph D is the (simple undirected) graph, which has the same vertex set as D and has an edge between two distinct vertices u and v if the arcs (u, x) and (v, x) are in D for some vertex $x \in V(D)$. Since Cohen [1] introduced the notion of competition grapsh while studying predator-prey concepts in ecological food webs, there has been a lot of research in the area of competition graphs. Recently, Kim *et al.* [2] studied the competition graph of an oriented complete bipartite graph. In this thesis, we extend their work to study the competition graph of multipartite tournament which is an orientation of a complete multipartite graph. We study multipartite tournaments whose competition graphs are complete. In addition, we characterize connected triangle-free competition graphs of tripartite tournaments. Finally, we study the structure of the competition graphs of multipartite tournaments in the aspect of sink sequences.

Key words: complete graph; multipartite tournament; competition graph; connected triangle-free graph.

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Contents

[1]The material in this chapter is a reprint of the manuscript written by Myungho Choi, Suh-Ryung Kim, and Minki Kwak. The coauthors listed in this manuscript directed and supervised research which forms the basis for the chapter.

Chapter 1

Introduction

1.1 Basic graph terminology

We introduce some basic notions in graph theory. For undefined terms, readers may refer to [5].

Let G be a graph. Two vertices u and v in G are called *adjacent* if there is an edge e in G which connects u and v. Then we say u and v are the *end vertices* of e. Two distinct edges are also called *adjacent* if they have a common end vertex.

Two graphs G and H are said to be *isomorphic* if there exist bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that for every edge $e \in E(G)$, e connects vertices u and v in G if and only if $\phi(e)$ connects vertices $\theta(u)$ and $\theta(v)$ in H. If G and H are isomorphic, then we write $G \cong H$.

Let G (resp. D) be a graph (resp. digraph). A graph H (resp. digraph

E) is a *subgraph* (resp. *subdigraph*) of G (resp. D) if $V(H) \subset V(G)$ (resp. $V(D) \subset V(E)$), $E(H) \subset E(G)$ (resp. $A(D) \subset A(E)$), and we write $H \subset G$ (resp. $E \subset D$). The subgraph resp. digraph of G (resp. D) whose vertex set is X and whose edge set (resp. arc set) consists of all edges (resp. arcs) of G (resp. D) which have both ends in X is called the *subgraph* (resp. *subdigraph*) of G (resp. D) *induced by* X and is denoted by $G[X]$ (resp. $D[X]$). The subgraph induced by $V(G) \setminus X$ (resp. $V(D) \setminus X$) is denoted by $G - X$ (resp. $D - x$). For notational convenience, we write notion $G - v$ (resp. $D - v$) instead of $G - \{v\}$ (resp. $D - \{v\}$) for a vertex v in G (resp. D).

For a vertex v in a digraph D, the *outdegree* of v is the number of vertices D to which v is adjacent, while the *indegree* of v is the number of vertices of D from which v is adjacent. In a digraph D, we call a vertex with indegree 0 and outdegree at least 1 a *source* of D.

A *walk* in a graph G is a sequence of (not necessarily distinct) vertices $v_1, v_2, \ldots, v_l \in V(G)$ such that $v_{i-1}v_i \in E(G)$ for each $2 \leq i \leq l$ and is denoted by $v_1 \to v_2 \to \cdots \to v_l$. If the vertices in a walk are distinct, then the walk is called a *path*. A *cycle* in G is a path $v_1 \to v_2 \to \cdots \to v_k$ together with the edge v_kv_1 where $k \geq 3$.

A *directed walk* in a digraph D is a sequence of (not necessarily distinct) vertices $v_1, v_2, \ldots v_l \in V(D)$ such that $(v_{i-1}, v_i) \in A(D)$ for each $2 \leq i \leq l$ and is denoted by $v_1 \to v_2 \to \cdots \to v_l$. If the vertices in a walk are distinct, then the walk is called a *directed path*. A *directed cycle* is a directed walk formed by a directed path $v_1 \to v_2 \to \cdots \to v_k$ and the arc (v_k, v_1) where $k \geq 1$.

A graph is *bipartite* if its vertex set can be partitioned into two subsets V_1 and V_2 so that every edge has one end in V_1 and the other end in V_2; such a partition (V_1, V_2) is called a *bipartition* of the graph, and V_1 and V_2 are called its *parts*. If a bipartite graph is simple and every vertex in one part is joined to every vertex in the other part, then the graph is called a *complete bipartite graph*. We denote by $K_{m,n}$ a complete bipartite graph with bipartition (V_1, V_2) if $|V_1| = m$ and $|V_2| = n$. Especially, $K_{1,n}$ is called a *star* graph for some positive integer n. A graph that contains no cycles at all is called *acyclic* and a connected acyclic graph is called a *tree*. It is obvious that each star graph is a tree.

1.2 Competition graph and its variants

The *competition graph* $C(D)$ of a digraph D is the (simple undirected) graph, which has the same vertex set as D and has an edge between two distinct vertices u and v if the arcs (u, x) and (v, x) are in D for some vertex $x \in V(D)$. Cohen [1] introduced the notion of competition graphs while studying predator-prey concepts in ecological food webs. Cohen's empirical observation that real-world competition graphs are usually interval graphs had led to a great deal of research on the structure of competition graphs and on the relation between the structure of digraphs and their corresponding competition graphs. For a comprehensive introduction to competition graphs, see [11, 19]. Competition graphs also have applications in coding, regulation of radio transmission, and modeling of complex economic systems (see [22] and [23] for a summary of these applications). For recent work on this topic, see [10, 21, 27, 28].

A variety of generalizations of the notion of competition graph have also been introduced, including the m-step competition graph in [3, 4], the common enemy graph (sometimes called the resource graph) in [20, 26], the competition-common enemy graph (sometimes called the competition-resource graph) in [7, 13–15, 18, 24, 25], the niche graph in [8, 9, 12] and the p-competition graph in [6, 16, 17].

Lundgren and Maybee [20] introduced the common enemy graph. The *common enemy graph* of a digraph D is the graph which has the same vertex set as D and has an edge between two distinct vertices u and v if and only if there exists a common in-neighbor of u and v in D. Their study led Scott [24] to introduce the competition-common enemy graph of D. The *competition-common enemy graph* of a digraph D is the graph which has the same vertex set as D and has an edge between two distinct vertices u and v if and only if there exists a common in-neighbor and a common outneighbor of u and v in D. This graph is fundamentally the intersection of the competition graph and the common enemy graph. That is, two vertices are adjacent if and only if they have both a common prey and a common enemy in D. On the other hand, the niche graph is the union of the competition graph and the common enemy graph. The *niche graph* of a digraph D is the graph which has the same vertex set as D and has an edge between two distinct vertices u and v if and only if there exists a common in-neighbor or a common outneighbor of u and v in D. For a digraph D, let $CE(D)$ be the common enemy graph,

$CCE(D)$ the competition-common enemy graph, and $N(D)$ the niche graph. From the definition of those graphs, we might obtain the relationship among them: $CCE(D) \subset C(D) \subset N(D)$. Another variant of competition graph, the *p-competition graph*, denoted by $C_p(D)$, of a digraph D is the graph which has the same vertex set as D and has an edge between two distinct vertices u and v if and only if there exist p common out-neighbors of u and v in D for a positive integer p. If D happens to be a food web whose vertices are species in some ecosystem with an arc (x, y) if and only if x preys on y, then xy is an edge of $C_p(D)$ if and only if x and y have at least p common prey.

1.3 Multipartite tournament

For a digraph D, the *underlying graph* of D is the graph G such that $V(G) = V(D)$ and $E(G) = \{uv \mid (u, v) \in A(D)\}$. An *orientation* of a graph G is a digraph having no directed 2-cycles, no loops, and no multiple arcs whose underlying graph is G. An *oriented graph* is a graph with an orientation. A *tournament* is an oriented complete graph. A *complete k-partite graph* is a k-partite graph whose vertices can be partitioned into k subsets $V_1, \cdots, V_k$ such that no edge has both endpoints in the same subset, and every possible edge that could connect vertices in different subsets is part of the graph. A *complete multipartite graph* is a graph that is complete k-partite for some k. A complete multipartite graph with partitions of size $|V_1| = n_1, \cdots, |V_k| = n_k$ is denoted by $K_{n_1,\cdots,n_k}$.

In 2016, Kim *et al.* [2] studied the competition graphs of oriented complete bipartite graphs. They characterized graphs that can be represented as the competition graphs of oriented complete bipartite graphs. They also presented the graphs having the maximum number of edges and the graphs having the minimum number of edges among such graphs.

Eoh *et al.* [29] completely characterize the m-step competition graph of a bipartite tournament for any integer $m \geq 2$. In addition, they compute the competition index and the competition period of a bipartite tournament.

Chapter 2

Multipartite tournaments whose competition graphs are complete[1]

In this chapter, we study multipartite tournaments whose competition graphs are complete. The following lemma is immediately true by the definition of the competition graph.

Lemma 2.1. *Let D be a digraph and D' be a subdigraph of D. Then the competition graph of D' is a subgraph of the competition graph of D.*

Lemma 2.2. *Suppose that D is a digraph with at least two vertices whose competition graph is complete. Let D' be a digraph with vertex set $V(D) \cup \{v\}$*

[1]The material in this chapter is a reprint of the manuscript written by Myungho Choi, Suh-Ryung Kim, and Minki Kwak. The coauthors listed in this manuscript directed and supervised research which forms the basis for the chapter.

where v is not a vertex of D. If $A(D) \subset A(D')$ and $N_D^+(u) \subset N_{D'}^+(v)$ for some vertex u in D, then the competition graph of D' is complete.

Proof. Suppose that $A(D) \subset A(D')$ and $N_D^+(u) \subset N_{D'}^+(v)$ for some vertex u in D. Take two vertices x and y in D'. If $x \neq v$ and $y \neq v$, then x and y are adjacent in $C(D')$ by Lemma 2.1. Now we suppose $x = v$. Since $C(D)$ is complete and $|V(D)| \geq 2$, $|N_D^+(u)| \geq 1$. If $y = u$, then $\emptyset \neq N_D^+(u) \subseteq N_{D'}^+(u) \cap N_D^+(u) \subseteq N_{D'}^+(u) \cap N_{D'}^+(v)$ and so x and y are adjacent in $C(D')$. Suppose $y \neq u$. Since y is adjacent to u in $C(D)$, $N_D^+(y) \cap N_D^+(u) \neq \emptyset$. Then $\emptyset \neq N_D^+(y) \cap N_D^+(u) \subseteq N_{D'}^+(y) \cap N_D^+(u) \subseteq N_{D'}^+(y) \cap N_{D'}^+(v)$ and so x and y are adjacent in $C(D')$. Thus we have shown that x and y are adjacent in $C(D')$ in each case and so may conclude that $C(D')$ is complete. $\square$

Lemma 2.3. *Let k and l be positive integers with $l \geq k \geq 3$; $n_1, \ldots, n_k$ be positive integers such that $n_1 \geq \cdots \geq n_k$; $n'_1, \ldots n'_l$ be positive integers such that $n'_1 \geq \cdots \geq n'_l$, $n'_1 \geq n_1$, $n'_2 \geq n_2$, $\ldots$, and $n'_k \geq n_k$. If D is an orientation of $K_{n_1,\ldots,n_k}$ whose competition graph is complete, then there exists an orientation D' of $K_{n'_1,\ldots,n'_l}$ whose competition graph is complete.*

Proof. Suppose that D is an orientation of $K_{n_1,\ldots,n_k}$ whose competition graph is complete. Let V_i be a partite set of D satisfying $|V_i| = n_i$ for each $1 \leq i \leq k$. Then we construct an orientation of $K_{n'_1,n_2,\ldots,n_k}$ whose competition graph is complete in the following way. If $n'_1 = n_1$, then we take D as a desired orientation. Suppose $n'_1 > n_1$. We add a new vertex v to V_1 so that

$$A(D) \subseteq A(D_1) \quad \text{and} \quad N_D^+(u) \subseteq N_{D_1}^+(v)$$

for some vertex u in V_1. Then $C(D_1)$ is complete by Lemma 2.2. We may repeat this process until we obtain a desired orientation $D_{n'_1-n_1}$. Inductively, we obtain an orientation D_t of $K_{n'_1,\ldots,n'_k}$ whose competition graph is complete where $t = (n'_1+\cdots+n'_k)-(n_1+\cdots+n_k)$. If $l = k$, then we are done. Suppose $l > k$. Then we construct a $(k+1)$-partite tournament D_{t+1} by adding a new vertex w to D_t so that $V_{k+1} := \{w\}$ is a partite set of D_{t+1},

$$A(D_t) \subseteq A(D_{t+1}), \quad \text{and} \quad N^+_{D_t}(u) \subseteq N^+_{D_{t+1}}(w).$$

Then D_{t+1} is complete by Lemma 2.2. By applying a similar argument for obtaining a digraph D_1, we may show that there exists an orientation of $K_{n'_1,\ldots,n'_{k+1}}$ whose competition graph is complete. We may repeat this process until we obtain an orientation of $K_{n'_1,\ldots,n'_l}$ whose competition graph is complete. Therefore the statement is true. □

Lemma 2.4. *Suppose that D is an orientation of a multipartite tournament whose competition graph is complete. Then the out-neighbors of each vertex are included in at least two partite sets of D.*

Proof. If there exists a vertex v whose out-neighbors are included in one partite set of D, then v and its out-neighbors cannot have a common prey and so v cannot be adjacent to its out-neighbors in $C(D)$, which is a contradiction. □

The following is immediate consequence of Lemma 2.4.

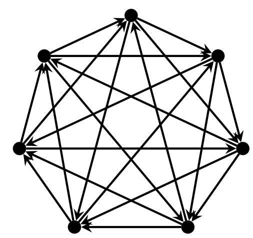


Figure 2.1: The digraph is an orientation of $K_{1,1,1,1,1,1,1}$

Corollary 2.5. *There is no bipartite tournament whose competition graph is complete.*

Proposition 2.6. *Let k be a positive integer grater than or equal to 7. For positive integers $n_1, n_2, \ldots, n_k$, there exists an orientation D of $K_{n_1,\ldots,n_k}$ whose competition graph is complete.*

Proof. The digraph in Figure 2.1 is an orientation of $K_{1,1,1,1,1,1,1}$ whose competition graph is complete. Therefore the statement is true by Lemma 2.3. $\square$

We now completely characterize which partite sizes of a k-partite tournament whose competition graph is complete when $k = 2$ or $k \geq 7$ by Corollary 2.5 and Proposition 2.6. In the following, we study k-partite tournaments whose competition graphs are complete for $3 \leq k \leq 6$.

Proposition 2.7. *Suppose that D is a multipartite tournament whose competition graph is complete. If the out-neighbors of a vertex v are included in exactly two partite sets U and V of D, then $|N^+(v) \cap U| \geq 2$ and $|N^+(v) \cap V| \geq 2$.*

Proof. Suppose that there exists a vertex v whose out-neighbors are included in exactly two partite sets U and V of D. To reach a contradiction, suppose that $|N^+(v) \cap U| < 2$ or $|N^+(v) \cap V| < 2$. Without loss of generality, we may assume $|N^+(v) \cap U| < 2$. Then $N^+(v) \cap U = \{u\}$ for some vertex u in D. Since $C(D)$ is complete, each vertex in $N^+(v) \cap V$ is adjacent to v in $C(D)$. Since $N^+(v) \subseteq U \cup V$, a common prey of v and the vertices in $N^+(v) \cap V$ belongs to $N^+(v) \cap U$. Therefore u is a common prey of v and the vertices in $N^+(v) \cap V$. Since u and v are adjacent in $C(D)$, u and v have a common prey w in D. Since $u \in U$ and $N^+(v) \subseteq U \cup V$, $w \in N^+(v) \cap V$, which is a contradiction. Hence $|N^+(v) \cap U| \geq 2$ and so the statement is true. $\square$

Corollary 2.8. *Suppose that D is a multipartite tournament whose competition graph is complete. Then each vertex has outdegree at least* 3

Proof. Take a vertex v in D. Then the out-neighbors of v belong to at least two partite sets of D by Lemma 2.4. If the out-neighbors of v belong to exactly two partite sets, then v has outdegree at least 4 by Proposition 2.7. If the out-neighbors of v belong to at least three partite sets, then it is obvious that v has outdegree at least 3. Therefore v has outdegree at least 3. Since v was arbitrary chosen, the statement is true. $\square$

Corollary 2.9. *Suppose that D is a multipartite tournament whose competition graph is complete. Then there exist at least* $\max\{4|V(D)| - |A(D)|, 0\}$ *vertices of outdegree* 3 *in D.*

Proof. Let l be the number of vertices of outdegree 3. Since each vertex in

D has outdegree at least 3 by Corollary 2.8,

$$4(|V(D)| - l) + 3l \leq |A(D)|.$$

Therefore $4|V(D)| - |A(D)| \leq l$. Thus the statement is true. $\square$

Lemma 2.10. *If D is a k-partite tournament with 8 vertices whose competition graph is complete for some $k \in \{5, 6\}$ and has at least two vertices of outdegree at least 4, then D is an orientation of $K_{2,2,1,1,1,1}$.*

Proof. Suppose that there exists a k-partite tournament D with 8 vertices whose competition graph is complete for some $k \in \{5, 6\}$ and which has at least two vertices of outdegree at least 4,. Suppose $k = 5$. Then $|A(D)|$ becomes maximum when D is an orientation of $K_{2,2,2,1,1}$, so $|A(D)| \leq 25$. By Corollary 2.9, there exist at least $\max\{4|V(D)| - |A(D)|, 0\}$ vertices of outdegree 3 in D, so at least 7 vertices has outdegree 3 in D, which is a contradiction. Therefore $k = 6$. If D is an orientation of $K_{3,1,1,1,1,1}$, then $|A(D)| \leq 25$ and so, by the same reason, we reach a contradiction. Thus D is an orientation of $K_{2,2,1,1,1,1}$. $\square$

Lemma 2.11. *Suppose that D is an k-partite tournament whose competition graph is complete. If a vertex u has outdegree 3 in D, then the out-neighbors of u form a directed cycle.*

Proof. Each pair of vertices has a common prey in D since $C(D)$ is complete. Let $N^+(u) = \{v_1, v_2, v_3\}$. Then $N^+(u)$ belongs to at least two partite sets by Lemma 2.4. If $N^+(u)$ belongs to exactly two partite sets, then it is a contradiction to a Proposition 2.7. Therefore v_i belongs to distinct partite sets for each $1 \leq i \leq 3$. Since v_1, v_2, v_3 are the only possible prey of u, a common prey of u and v_i belongs to $\{v_1, v_2, v_3\} \setminus \{v_i\}$ for each $1 \leq i \leq 3$ and so v_1, v_2, and v_3 form a directed cycle C. □

Proposition 2.12. *Suppose that D is an k-partite tournament whose competition graph is complete for some integer $4 \leq k \leq 6$ and there exists a vertex u of outdegree 3. Then D contains a subdigraph isomorphic to the digraph D_1 in Figure 2.2 and $|V(D)| \geq 9$. In particular, if $k = 4$, then $|V(D)| \geq 10$.*

Proof. Each pair of vertices has a common prey in D since $C(D)$ is complete. Let $N^+(u) = \{v_1, v_2, v_3\}$. Without loss of generality, we may assume that $C := v_1 \to v_2 \to v_3 \to v_1$ is a cycle of D by Lemma 2.11. Let w_i be a common prey of v_i and v_{i+1} for each $1 \leq i \leq 3$ (identify v_4 with v_1). If $w_j = w_k$ for some distinct $j, k \in \{1, 2, 3\}$, then $\{v_1, v_2, v_3\} \subset N^-(w_j)$ and so w_j does not share a common prey with u, which is a contradiction. Therefore w_1, w_2, and w_3 are all distinct. Thus, so far, we have a subdigraph D_1 with the vertex set $\{u, v_1, v_2, v_3, w_1, w_2, w_3\}$ given in Figure 2.2. Suppose that v_i has outdegree 3 in D for each $1 \leq i \leq 3$. Then the out-neighbors of v_i form a directed 3-cycle for each $1 \leq i \leq 3$ by Lemma 2.11. Thus D_2 given in Figure 2.2 is a subdigraph of D. It is easy to check from D_2 that u is the only possible common prey of each pair of w_1, w_2, and w_3 in D. Therefore $N^-(u) = \{w_1, w_2, w_3\}$ and so D is a 7-partite tournament, which is a contradiction. Thus at least one of v_1, v_2, and v_3 has outdegree at least 4 and so $|V(D)| \geq 8$.

To reach a contradiction, suppose that $|V(D)| = 8$. Then $V(D) = V(D_1) \cup \{x\}$ for some vertex x in D and

$$|N^+(v_i)| = 3 \text{ or } 4 \tag{2.1}$$

for each $1 \leq i \leq 3$. Since x and u must be adjacent and $N^+(u) = \{v_1, v_2, v_3\}$, one of v_1, v_2, v_3 is a common prey of u and x. Therefore $|N^+(v_j)| = 3$ for some $j \in \{1, 2, 3\}$. Without loss of generality, we may assume

$$|N^+(v_1)| = 3.$$

Then

$$N^+(v_1) = \{v_2, w_1, w_3\}.$$

Since the prey of v_1 form a directed cycle by Lemma 2.11, $\{v_1, v_2, v_3, w_1, w_3\}$ forms a 5-tournament, so

$$k \geq 5.$$

Then, since (w_3, v_2) and (v_2, w_1) are arcs of D,

$$(w_1, w_3) \in A(D).$$

If w_2 is a common prey of w_1 and w_3, then, since $N^+(v_1) = \{v_2, w_1, w_3\}$, v_1 and w_2 cannot have a common prey, which is a contradiction. Therefore

$$N^+(w_2) \cap \{w_1, w_3\} \neq \emptyset. \tag{2.2}$$

We first show that $\{v_1, v_2, v_3, w_1, w_2, w_3\}$ forms a tournament. Since D_1 is a subgraph of D, we need show that $\{w_1, w_2, w_3\}$ form a tournament. As we have shown $\{v_1, v_2, v_3, w_1, w_3\}$ is a tournament, it remains to show that w_2 is adjacent to w_1 and w_3.

Suppose, to the contrary, that there is no arc between w_1 and w_2. Then $(w_2, w_3) \in A(D)$ by (2.2). Then the vertices v_1, w_2, w_3 cannot form a directed cycle. Yet, v_1, w_2, w_3 are out-neighbors of v_3, so $|N^+(v_3)| = 4$ by Lemma 2.11 and (2.1). Since x is the only possible prey of v_3 in D, $N^+(v_3) = \{x, v_1, w_2, w_3\}$. Since x is the only possible common prey of w_2 and v_2, $x \in N^+(w_2) \cap N^+(v_2)$. Thus $N^+(v_2) = \{x, v_3, w_1, w_2\}$ and $\{v_1, w_3, x\} \subseteq N^+(w_2)$. Since v_2 and v_3 have outdegree 4, D is an orientation of $K_{2,2,1,1,1,1}$ by Lemma 2.10. Then $\{w_1, w_2\}$ forms a partite set of D. Since u has outdegree 3 in D, $(w_2, u) \in A(D)$ and so $\{u, v_1, w_3, x\} \subseteq N^+(w_2)$. Then v_2, v_3, and w_2 has outdegree at least 4. Therefore there are at most 5 vertices of outdegree 3 in D. By the way, $K_{2,2,1,1,1,1}$ has 8 vertices and 26 arcs, so $4|V(D)| - |A(D)| = 6$. Then there exist at least 6 vertices of outdegree 3 by Corollary 2.9 and we reach a contradiction. Thus there is an arc between w_1 and w_2.

Now we suppose, to the contrary, that there is no arc between w_2 and w_3. Then v_1, w_2, w_3 cannot form a directed cycle. Since they are out-neighbors of v_3, $|N^+(v_3)| = 4$ by Lemma 2.11 and (2.1) and so $N^+(v_3) = \{v_1, w_2, w_3, x\}$. Since there is no arc between w_2 and w_3, there is an arc (w_2, w_1) in D by (2.2). For the same reason, x is the only possible common prey of w_3 and v_2, so $x \in N^+(w_3) \cap N^+(v_2)$. Thus v_2 has outdegree 4 by (2.1). Since v_3 also has outdegree 4, D is an orientation of $K_{2,2,1,1,1,1}$ by Lemma 2.10. Thus $\{w_2, w_3\}$

is a partite set of D. By taking a look at the structure of D determined so far, we may conclude that v_1 a common prey of x and u, so $(x, v_1) \in A(D)$. Since $N^+(v_1) = \{v_2, w_1, w_3\}$, (x, w_1) must be an arc of D in order for v_1 and x are adjacent. Since u is the only possible common prey of x and w_1, there exist arcs (x, u) and (w_1, u). Then $\{x, u, v_1, v_2, v_3, w_1\}$ forms a tournament and we reach a contradiction to the fact that D is an orientation of $K_{2,2,1,1,1,1}$. Therefore $\{v_1, v_2, v_3, w_1, w_2, w_3\}$ forms a tournament. Thus $k = 6$ and u and x are the only vertices that belong to a partite set of size at least 2. Furthermore, since $N^+(u) = \{v_1, v_2, v_3\}$, u cannot form a partite set with v_1, v_2, or v_3 and so u and exactly one of w_1, w_2, and w_3 belong to the same partite set.

Suppose, to the contrary, that $(w_3, w_2) \in A(D)$. Then $(w_2, w_1) \in A(D)$ by (2.2). Therefore $w_1 \to w_2 \to w_3 \to w_1$ forms a cycle. Then, for each pair of w_1, w_2, and w_3, x and u are its only possible common prey. Since u and one of w_1, w_2, and w_3 belong to the same partite set, exactly one pair of w_1, w_2, and w_3 can prey on u. Therefore $\{w_1, w_2, w_3\} \subseteq N^-(x)$. Thus x and exactly one of v_1, v_2, and v_3 belong to the same partite set and so $|N^+(x) \cap \{v_1, v_2, v_3\}| \leq 2$. Hence, for some $j \in \{1, 2, 3\}$, a common prey of x and v_j is contained in $\{u, w_1, w_2, w_3\}$. Since $\{w_1, w_2, w_3\} \subseteq N^-(x)$, u must be a common prey of x and v_j, which contradicts the fact $\{v_1, v_2, v_3\} \subset N^+(u)$. Therefore $(w_3, w_2) \notin A(D)$ and so

$$(w_2, w_3) \in A(D).$$

Thus $N^+(w_3) \subseteq \{u, v_2, x\}$ and so, $N^+(w_3) = \{u, v_2, x\}$ by Corollary 2.8 Then x is the only possible common prey of each pair of v_3 and w_3, and v_2 and w_3. Therefore $N^+(v_3) = \{v_1, w_2, w_3, x\}$ and $N^+(v_2) = \{v_3, w_1, w_2, x\}$ by (2.1). Thus D is an orientation of $K_{2,2,1,1,1,1}$ by Lemma 2.10. Moreover, since $N^+(v_1) = \{v_2, w_3, w_1\}$, w_1 is only possible common prey of x and v_1 and so $(x, w_1) \in A(D)$. Then u must be a common prey of w_1 and w_3. Therefore $\{w_2, u\}$ and $\{x, v_1\}$ are the partite sets of size 2 in D. Since $N^+(w_2) \subseteq \{v_1, w_3, x\}$, w_2 and x has no common prey, which is a contradiction. Therefore we have shown that $|V(D)| \neq 8$ and so $|V(D)| \geq 9$.

To show "particular" part, suppose $k = 4$. Let $V_1, \ldots, V_k$ be partite sets of D. By the above argument, v_1, v_2, and v_3 belong to distinct partite sets. Without loss of generality, we may assume that $u \in V_1$, $v_1 \in V_2$, $v_2 \in V_3$, and $v_3 \in V_4$. Let y_i be a common prey of v_i and w_i in D for each $1 \leq i \leq 3$. If $y_1 = y_2 = y_3$, then $\{v_1, v_2, v_3\} \subseteq N^-(y_1)$, which implies that u and y_1 do not share a common prey, and we reach a contradiction. Therefore at least two of y_1, y_2, and y_3 are distinct. By the above argument, $\{u, w_1, w_2, w_3\} \subset V_1$. Therefore y_i cannot be w_j for each $1 \leq i, j \leq j$. Thus $|V(D)| \geq 9$. Suppose, to the contrary, that $|V(D)| = 9$. Then exactly two of y_1, y_2, and y_3 are the same. Without loss of generality, we may assume $y_1 = y_2$ and $y_1 \neq y_3$. Neither v_1 nor v_2 is a common prey of y_1 and u. Thus v_3 must be a common prey of y_1 and u. Yet, y_1 is a common prey of v_1, v_2, w_1, and w_2, so $y_1 \in V_4$ and we reach a contradiction. Thus $|V(D)| \geq 10$. $\square$

Lemma 2.13. *Let k be a positive integer with $k \geq 3$; $n_1, \ldots, n_k, n'_k, n'_{k+1}$*

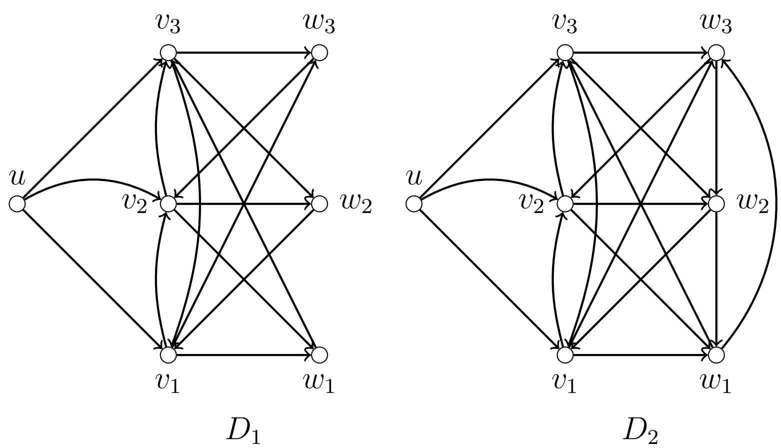


Figure 2.2: The subdigraphs D_1 and D_2 obtained in the proof of Proposition 2.12

be positive integers such that $n_k = n'_k + n'_{k+1}$*. If* D *is an orientation of* $K_{n_1,\ldots,n_{k-1},n_k}$ *whose competition graph is complete, then there exists an orientation* D' *of* $K_{n_1,\ldots,n_{k-1},n'_k,n'_{k+1}}$ *whose competition graph is complete.*

Proof. Suppose that D is an orientation of $K_{n_1,\ldots,n_{k-1},n_k}$ whose competition graph is complete. Let $V_1, \ldots, V_k$ be partite sets of D satisfying $|V_i| = n_i$. We construct a $(k+1)$-partite tournament D' whose competition graph is complete out of D in the following way. We take an n'_k-subset V'_k of V_k and let $V'_{k+1} = V_k \setminus V'_k$. Then $|V'_{k+1}| = n'_{k+1}$. We keep all the arcs in D so that $A(D) \subset A(D')$. To each of the remaining pairs of vertices which has not been ordered yet, we give it an arbitrary order to become an ordered pair. We denote the resulting a $(k+1)$-partite tournament by D'. From the fact that $C(D)$ is complete, $V(D) = V(D')$, and $A(D) \subset A(D')$, we may conclude that $C(D')$ is complete. and so the statement is true. □

The following lemma is obviously true by the definition of competition graph.

Lemma 2.14. *Let D be a digraph whose competition graph is complete. If a vertex v has indegree at most* 1 *in D, then $C(D-v)$ is complete.*

Proposition 2.15. *Let D be a k-partite tournament for some positive integer $k \geq 3$ whose competition graph is complete with the partite sets V_1, …, V_k. Then there exists a k-partite tournament D^* with the partite sets V_1, …, V_k such that $C(D^*)$ is complete and each vertex in D^* has indegree at least* 2*.*

Proof. Suppose that $(t_1, t_2, \ldots, t_l)$ is a sequence, where $t_i \in \{1, 2, \ldots, k\}$ for each $1 \leq i \leq l$, such that there exists a vertex v_i of indegree at most 1 in V_{t_i} in $D - \{v_1, \ldots, v_{i-1}\}$ for each $1 \leq i \leq l$ and $D - \{v_1, \ldots, v_l\}$ has no vertex of indegree at most 1. Such a sequence exists Then $C(D - \{v_1, \ldots, v_l\})$ is complete by Lemma 2.14.

Suppose that there exists a vertex v_1 of indegree at most 1 in V_{t_1} for some $t_1 \in \{1, \ldots, k\}$. Let $D_1 = D - v_1$. Then $C(D_1)$ is complete by Lemma 2.14. By Corollary 2.5, D_1 is not a bipartite tournament. Suppose that there exists a vertex v_2 of indegree at most 1 in V_{t_2} for some $t_2 \in \{1, \ldots, k\}$. Let $D_2 = D_1 - v_2$. Therefore $C(D_2)$ is complete by Lemma 2.14 and so, by Corollary 2.5, D_2 is not a bipartite tournament. We keep repeating this process. Since D has a finite number of vertices, this process terminates to produce digraphs D_1, D_2, …, D_l. Since $C(D_l)$ is complete, the number of partite sets in D_l is at least 3. The fact that the process ended with D_l implies that each vertex in D_l has indegree at least 2. As some of partite sets of D_l are proper subsets of corresponding partite sets of D, we need to add vertices to obtain a desired k-partite tournament. Let X be the partite set of D_{l-1} to which v_l belongs. Then $X \subseteq V_{t_l}$. We consider two cases for X.

Case 1. $X = \{v_l\}$. We take a vertex v' in D_l. Then v' has indegree at least 2. Now we add v_l to D_l so that $\{v_l\}$ is a partite set of D^*_{l-1}, v_l takes the out-neighbors and the in-neighbors of v' as its out-neighbors and in-neighbors, respectively, and the remaining out-neighbors and in-neighbors of v_l are arbitrarily taken. Then the indegree of v_l in D^*_{l-1} is at least 2. Moreover,

$$V(D_l) \cup \{v_l\} = V(D^*_{l-1}), \quad A(D_l) \subset A(D^*_{l-1}), \quad \text{and} \quad N^+_{D_l}(v') \subset N^+_{D^*_{l-1}}(v_l).$$

Case 2. $\{v_l\} \subset X$. Then there exists a vertex v' distinct from v_l in X. Since $D_l = D_{l-1} - v_l$, v' is a vertex of D_l. Now we add v_l to the partite set of D_l where v' belongs so that $\{v_l, v'\}$ is involved in a partite set of D^*_{l-1}, v_l takes the out-neighbors and the in-neighbors of v' as its out-neighbors and in-neighbors, respectively, and the remaining out-neighbors and in-neighbors of v_l are arbitrarily taken. Then the indegree of v_l in D^*_{l-1} is at least 2 since the indegree of v' is at least 2 in D_l. Moreover,

$$V(D_l) \cup \{v_l\} = V(D^*_{l-1}), \quad A(D_l) \subset A(D^*_{l-1}), \quad \text{and} \quad N^+_{D_l}(v') \subset N^+_{D^*_{l-1}}(v_l).$$

In both cases, $C(D^*_{l-1})$ is complete by Lemma 2.2.

Now we add v_{l-1} to D^*_{l-1} and apply an argument similar to the above one to obtain D^*_{l-2} whose competition graph is complete and each vertex in which has indegree at least 2. We may repeat this process until we obtain a k-partite tournament D_0^* whose competition graph is complete and each vertex in which has indegree at least 2. Since we added v_i to the partite set of D_i^* which is included in V_{t_i} for each $1 \leq i \leq l$, it is true that the partite sets of D_0^* are the same as D. Thus D_0^* is a desired k-partite tournament. $\square$

Proposition 2.16. *There is no orientation of $K_{4,1,1,1,1,1}$ whose competition graph is complete.*

Proof. Suppose, to the contrary, that there exists an orientation of $K_{4,1,1,1,1,1}$ whose competition graph is complete. Then, by Proposition 2.15, there exists an orientation D of $K_{4,1,1,1,1,1}$ whose competition graph is complete and each vertex in which has indegree at least 2. Let $V_1, \ldots, V_6$ be partite sets of D with $|V_1| = 4$. By Corollary 2.8, each vertex has outdegree at least 3 in D. Then, since each vertex has indegree at least 2 in D,

$$|N^+(v)| = 3 \quad \text{and} \quad |N^-(v)| = 2 \tag{2.3}$$

for each vertex v in V_1. By Corollary 2.9, there exist at least $\max\{4|V(D)| - |A(D)|, 0\}$ vertices of outdegree 3 in D. Since $4|V(D)| - |A(D)| = 6$, there exist at least 6 vertices of outdegree 3. Thus there exists at least two vertices of outdegree 3 not belonging to V_1. Let u be a vertex of outdegree 3. Without loss of generality, we may assume $u \in V_2$.

Since each out-neighbor of u has indegree at least 3 by by Proposition 2.12, $N^+(u) \cap V_1 = \emptyset$ by (2.3). Moreover, by Lemma 2.11, $N^+(u)$ forms a directed cycle in D. Let $N^+(u) = \{v_1, v_2, v_3\}$ where $v_1 \to v_2 \to v_3 \to v_1$ is a directed cycle in D. Since $N^+(u) \cap V_1 = \emptyset$, we may assume $V_3 = \{v_1\}$, $V_4 = \{v_2\}$, $V_5 = \{v_3\}$, and $V_6 = \{x\}$ and that v_1 is a common prey of x and u. Then $\{u, v_3, x\} \subseteq N^-(v_1)$. Let w_1 be a common prey of v_1 and v_2. Then $w_1 \in V_1$. Therefore, by (2.3), $N^-(w_1) = \{v_1, v_2\}$ and $N^+(w_1) = \{u, v_3, x\}$. Thus $N^+(w_1) \subseteq N^-(v_1)$ and so w_1 and v_1 have no common prey, which is a contradiction. □

Let $V(D) = \{v_1, v_2, \ldots, v_n\}$ and $A = (a_{ij})$ be the *adjacency matrix* of D, that is,

$$a_{ij} = \begin{cases} 1 & \text{if there is an arc } (v_i, v_j) \text{ in } D; \\ 0 & \text{otherwise.} \end{cases}$$

Now we are ready to introduce one of our main theorems.

Theorem 2.17. *Let $n_1, \ldots, n_6$ be positive integers such that $n_1 \geq \cdots \geq n_6$. There exists an orientation D of $K_{n_1,n_2,\ldots,n_6}$ whose competition graph is complete if and only if one of the following holds: (a) $n_1 \geq 5$ and $n_2 = 1$; (b) $n_1 \geq 3$, $n_2 \geq 2$, and $n_3 = 1$; (c) $n_3 \geq 2$.*

Proof. To show the "only if" part, suppose that there exists an orientation D of $K_{n_1,n_2,\ldots,n_6}$ such that $C(D)$ is complete. We suppose $n_3 = 1$.

Case 1. $n_2 = 1$. If $n_1 \leq 4$, then there exists an orientation of $K_{4,1,1,1,1,1}$ whose competition graph is complete by Lemma 2.3, which contradicts Proposition 2.16. Therefore $n_1 \geq 5$.

Case 2. $n_2 \geq 2$. Then $n_1 \geq 2$. Suppose, to the contrary, that $n_1 = 2$. Then $n_2 = 2$, so D is an orientation of $K_{2,2,1,1,1,1}$. Therefore $4|V(D)| - |A(D)| = 6$. By Corollary 2.9, there exists a vertex of outdegree 3 in D'. Therefore

$$A_1 = \begin{pmatrix} 0&0&0&0&0&0&1&0&1&1 \\ 0&0&0&0&0&0&1&1&0&1 \\ 0&0&0&0&0&1&0&1&1&0 \\ 0&0&0&0&0&1&0&1&0&1 \\ 0&0&0&0&0&1&1&0&1&0 \\ 1&1&0&0&0&0&1&1&0&0 \\ 0&0&1&1&0&0&0&1&1&0 \\ 1&0&0&0&1&0&0&0&1&1 \\ 0&1&0&1&0&1&0&0&0&1 \\ 0&0&1&0&1&1&1&0&0&0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0&0&0&1&0&1&0&1&0 \\ 0&0&0&0&1&1&1&0&1 \\ 0&0&0&0&1&1&0&1&1 \\ 0&1&1&0&0&0&0&1&1 \\ 1&0&0&0&0&1&0&0&1 \\ 0&0&0&1&0&0&1&0&1 \\ 1&0&1&1&1&0&0&0&0 \\ 0&1&0&0&1&1&1&0&0 \\ 1&0&0&0&0&0&1&1&0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0&0&0&0&0&0&1&1&1 \\ 0&0&1&0&1&1&0&1&0 \\ 1&0&0&0&1&0&1&1&1 \\ 1&1&0&0&1&0&0&0&1 \\ 1&0&0&0&0&0&1&1&1 \\ 1&0&1&1&0&0&1&0&0 \\ 0&1&0&1&0&0&0&1&0 \\ 0&0&0&1&0&1&0&0&1 \\ 0&1&0&0&0&1&1&0&0 \end{pmatrix}$$

Figure 2.3: The adjacency matrices A_1, A_2, and A_3 which are orientations of $K_{5,1,1,1,1,1}$, $K_{3,2,1,1,1,1}$, and $K_{2,2,2,1,1,1}$, respectively, in the proof of Theorem 2.17

$|V(D)| \geq 9$ by Proposition 2.12, which is a contradiction. Thus $n_1 \geq 3$. Hence the "only if" part is true.

Now we show the "if" part. Let D_1, D_2, and D_3 be the digraphs whose adjacency matrices are A_1, A_2, and A_3, respectively, given in Figure 2.3. It is easy to check that inner product of distinct rows of each matrix is nonzero, so the competition graphs of D_1, D_2, and D_3 are complete. If $n_1 \geq 5$ and $n_2 = 1$, then, by applying Lemma 2.3 to D_1, we obtain an orientation D_1^* of $K_{n_1,n_2,\ldots,n_6}$ whose competition graph is complete.

If $n_1 \geq 3$, $n_2 \geq 2$, and $n_3 = 1$, then, by applying Lemma 2.3 to D_2, we obtain an orientation D_2^* of $K_{n_1,n_2,\ldots,n_6}$ whose competition graph is complete. If $n_3 \geq 2$, then, by applying Lemma 2.3 to D_3, we obtain a desired 6-partite tournament D_3^* whose competition graph is complete. Therefore we have shown that the "if" part is true. □

We completely characterize which partite sets of a k-partite tournament whose competition graph is complete when $k = 6$ by Theorem 2.17. In the following, we study k-partite tournaments whose competition graphs are complete for $3 \leq k \leq 5$.

Proposition 2.18. *Let D be a k-partite tournament whose competition graph is complete for some integer $k \in \{4, 5\}$. Then the number of partite sets having size 1 is at most $k - 3$.*

Proof. Let V_1, V_2, $\ldots$, V_k be a partite set of D. We may assume that $x \in V_1$, $y \in V_2$, and $z \in V_3$.

We suppose $k = 4$. To reach a contradiction, suppose that there are at least 2 partite sets of size 1. Without loss of generality, we may assume $|V_1| = |V_2| = 1$. Then $V_1 = \{x\}$ and $V_2 = \{y\}$. Without loss of generality, we may assume z is a common prey of x and y. Then $N^+(z) \subseteq V_4$, which contradicts Proposition 2.4. Therefore D has at most 1 partite set of size 1.

Suppose $k = 5$. To reach a contradiction, suppose that there are at least 3 partite sets of size 1. Without loss of generality, we may assume $|V_1| = |V_2| = |V_3| = 1$. Then $V_1 = \{x\}$, $V_2 = \{y\}$ and $V_3 = \{z\}$. Suppose that x and y have a common prey w in $V_4 \cup V_5$. Without loss of generality, we may assume $w \in V_4$. Then $N^+(w) \subset V_3 \cup V_5$. By Proposition 2.4, $N^+(w) \cap V_3 \neq \emptyset$ and $N^+(w) \cap V_5 \neq \emptyset$. However, $N^+(w) \cap V_3 = \{z\}$, which contradicts Proposition 2.7. Thus $w \notin V_4 \cup V_5$ and so $w = z$. By symmetry, the only possible common prey of y and z is x. Since $z \in N^+(x)$, x cannot be a prey of z and we reach a contradiction. Therefore D has at most 2 partite sets having size 1. □

By Proposition 2.12, the out-neighbors of a vertex of outdegree 3 in a k-partite tournament whose competition graph is complete for some $k \geq 4$ form a directed cycle and we have the following lemma.

Lemma 2.19. *Let D be a k-partite tournament whose competition graph is complete for some $k \geq 4$. Suppose that a vertex u has outdegree 3. If $N^+(u) \subseteq U \cup V \cup W$ for distinct partite sets U, V and W of D, then $|U| + |V| + |W| \leq |V(D)| - 4$.*

Proof. Suppose that $N^+(u) \subseteq U \cup V \cup W$ for distinct partite sets U, V and W

of D. Since u has outdegree 3, by Proposition 2.12, D contains a subdigraph isomorphic to D_1 given in Figure 2.2. We may assume that the subdigraph is D_1 itself including labels. We may assume $v_1 \in U$, $v_2 \in V$, and $v_3 \in W$. Then $\{u, w_1, w_2, w_3\} \cap (U \cup V \cup W) = \emptyset$. Thus $|V(D) \setminus (U \cup V \cup W)| \geq 4$ and so $|U| + |V| + |W| = |U \cup V \cup W| \leq |V(D)| - 4$. $\square$

Lemma 2.20. *Let D be a 4-partite tournament whose competition graph is complete. Suppose that a vertex u has outdegree 3 in a partite set X. Then any vertex of outdegree 3 is contained in X.*

Proof. Let V_1, V_2, V_3, and V_4 be partite sets of D (we assume $V_1 = X$). By Proposition 2.12, D contains a subdigraph isomorphic to D_1 given in Figure 2.2. We may assume that the subdigraph is D_1 itself including the labels.

Without loss of generality, we may assume $u \in V_1$. Then

$$N^-(u) = V(D) \setminus (V_1 \cup \{v_1, v_2, v_3\}). \tag{2.4}$$

Since v_1, v_2 and v_3 forms a directed cycle in D_1, they belong to distinct partite sets. We may assume that $v_1 \in V_2$, $v_2 \in V_3$ and $v_3 \in V_4$. Then, since D is a 4-partite tournament, $\{w_1, w_2, w_3\} \subset V_1$. Suppose $|N^+(v_j)| = 3$ for some $j \in \{1, 2, 3\}$. Then the out-neighbors of v_j are the ones given in D_1 and so are contained in $V_1 \cup V_2$, $V_1 \cup V_3$, or $V_1 \cup V_4$. Therefore $|N^+(v_j)| \geq 4$ by Proposition 2.7, which is a contradiction. Thus

$$|N^+(v_i)| \geq 4$$

for each $1 \leq i \leq 3$. Now suppose that there is a vertex x of outdegree 3 in $V_2 \cup V_3 \cup V_4$ other than v_1, v_2, and v_3. Without loss of generality, we may assume $x \in V_2$. Since $u \in N^+(x)$,

$$|N^+(x) \cap V_3| = 1 \quad \text{and} \quad |N^+(x) \cap V_4| = 1 \tag{2.5}$$

by Lemma 2.11. Since u and x have a common prey,

$$|N^+(x) \cap \{v_2, v_3\}| \geq 1. \tag{2.6}$$

Let y be a common prey of w_2 and x. Then $y \in V_3 \cup V_4$. To reach a contradiction, suppose that $y \in V_3$. Then $v_2 \notin N^+(x)$ by (2.5). Therefore $v_3 \in N^+(x)$ by (2.6). Thus $N^+(x) = \{u, y, v_3\}$ and so $(v_3, y) \in A(D)$ by (2.4). Moreover, y is the only possible common prey of v_1 and x, so $(v_1, y) \in A(D)$. We hereby have shown that none of v_1, v_2, and v_3 is prey of y. Since $N^+(u) = \{v_1, v_2, v_3\}$, y and u have no common prey, which is a contradiction. Therefore $y \in V_4$. Thus $v_3 \notin N^+(x)$ by (2.5). Hence $v_2 \in N^+(x)$ by (2.6) and so $N^+(x) = \{u, v_2, y\}$. Then v_3 and x have no common prey and we reach a contradiction. Therefore we may conclude that there is no vertex of outdegree 3 in $V_2 \cup V_3 \cup V_4$. □

Corollary 2.21. *There is no orientation of $K_{3,3,2,2}$ whose competition graph is complete.*

Proof. Suppose, to the contrary, that there exists an orientation D of $K_{3,3,2,2}$ whose competition graph is complete. If there exists a vertex u of outdegree

3, then $N^+(u) \subseteq U \cup V \cup W$ for three distinct partite sets U, V, and W by Lemma 2.11 and so, by Lemma 2.19, $|U|+|V|+|W| \leq |V(D)|-4 = 6$, which is impossible. Thus each vertex has outdegree at least 4 and so $|A(D)| \geq 40$. However, $|A(D)| = 9+6+6+6+6+4 = 37$ and we reach a contradiction. $\square$

Corollary 2.22. *There is no orientation of $K_{4,3,2,1}$ whose competition graph is complete.*

Proof. Suppose, to the contrary, that there exists an orientation D of $K_{3,3,2,2}$ whose competition graph is complete. Since $|A(D)| = 12+8+4+6+3+2 = 35$, $4|V(D)| - |A(D)| = 5$ and so, by Corollary 2.9, D has at least 5 vertices of outdegree 3. However, there exist at most 4 vertices of outdegree 3 by Lemma 2.20 and we reach a contradiction. $\square$

Lemma 2.23. *Let D be a multipartite tournament whose competition graph is complete. Suppose that a vertex v has outdegree 4. Then the following are true:*

(1) If the out-neighbors of v are included in exactly two partite sets, then the out-neighbors form an induced directed cycle;

(2) If the out-neighbors of v are included in exactly three partite sets, then the out-neighbors induce the subdigraphs D_1, D_2, or D_3 given in Figure 2.4.

Proof. Let $N^+(u) = \{v_1, v_2, v_3, v_4\}$. Suppose that $N^+(u)$ belongs to exactly two partite sets U and V. Then $|N^+(u) \cap U| = 2$ and $|N^+(u) \cap V| = 2$ by Proposition 2.7. Without loss of generality, we may assume that $\{v_1, v_2\} \subset U$

and $\{v_3, v_4\} \subset V$. Since $N^+(u) = \{v_1, v_2, v_3, v_4\}$ and $\{v_1, v_2\} \subset U$, v_3 or v_4 is a common prey of u and v_1. Without loss of generality, we may assume $(v_1, v_3) \in A(D)$. Then, by applying the same argument to v_3, we may show that $(v_3, v_2) \in A(D)$. By repeating the same argument to v_2 and v_4, we may show that $\{(v_2, v_4), (v_4, v_1)\} \subset A(D)$. Therefore $v_1 \to v_3 \to v_2 \to v_4 \to v_1$ is an induced directed cycle of D.

Now we suppose that $N^+(u)$ belongs to exactly three partite sets U, V, and W. Without loss of generality, we may assume that $\{v_1, v_2\} \subset U$, $v_3 \in V$, and $v_4 \in W$. Since v_1, v_2, v_3 and v_4 are the only prey of u, each vertex in $\{v_1, v_2, v_3, v_4\}$ has a prey in $\{v_1, v_2, v_3, v_4\}$. Therefore the subdigraph of D induced by v_1, v_2, v_3 and v_4 contains no vertex of outdegree 0. Thus it is one of D_1, D_2, or D_3 given in Figure 2.4. □

Corollary 2.24. *Let D be an orientation of K_{n_1,n_2,n_3} with partite sets V_1, V_2, and V_3 whose competition graph is complete for some positive integers n_1, n_2, and n_3. Then, for each $1 \le i \ne j \le 3$ and each $v \in V_i$, $|N_D^+(v) \cap V_j| \ge 2$.*

Proof. It is immediately follows from Lemma 2.4 and Proposition 2.7. □

Theorem 2.25. *Let D be an orientation of K_{n_1,n_2,n_3} whose competition graph is complete for some positive integers n_1, n_2, and n_3. Then $n_i \ge 4$ for each $i \in \{1, 2, 3\}$.*

Proof. Let V_1, V_2, and V_3 be the partite sets of D with $|V_i| = n_i$ for each $i \in \{1, 2, 3\}$. Suppose, to the contrary, that $n_j \le 3$ for some $j \in \{1, 2, 3\}$. Without loss of generality, we may assume that $n_1 \le 3$. Take $v_1 \in V_1$. By

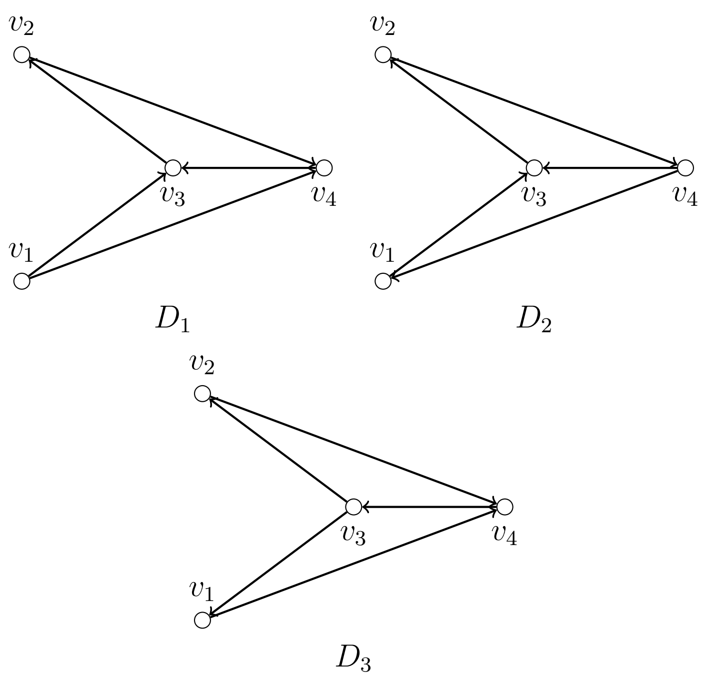


Figure 2.4: There are exactly three orientations D_1, D_2, and D_3 of $K_{2,1,1}$ without vertices of outdegree 0.

Corollary 2.24, there are four vertices u_1, u_2, w_1, and w_2 such that $\{u_1, u_2\} \subseteq N_D^+(v_1) \cap V_2$ and $\{w_1, w_2\} \subseteq N_D^+(v_1) \cap V_3$. By the same corollary, there are two vertices v_2 and v_3 in V_1 such that $\{v_2, v_3\} \subseteq N_D^+(u_1) \cap V_1$. Since $(v_1, u_1) \in A(D)$, v_2 and v_3 are distinct from v_1. Since $n_1 \leq 3$, $V_1 = \{v_1, v_2, v_3\}$. Thus, by Corollary 2.24, $N_D^+(u_2) \cap V_1 = N_D^+(w_1) \cap V_1 = N_D^+(w_2) \cap V_1 = \{v_2, v_3\}$. Since v_1 is adjacent to v_2 in $C(D)$, v_1 and v_2 have a common prey, say x, in V_2 or V_3. Since v_3 is the only possible prey of x in V_1, we reach a contradiction to Corollary 2.24. □

If a graph is the competition graph of a digraph D, then it said to be *competition-realizable through* D. If G is competition-realizable through a k-partite tournament for a graph G and an integer $k \geq 2$, then we say that the pair (G, k) is *competition-realizable* for notational convenience. The following corollary is an immediate consequence of Theorem 2.25.

Corollary 2.26. *For any positive integer* $n \leq 11$, $(K_n, 3)$ *is not competition-realizable.*

Lemma 2.27. *Let* D *be an orientation of* $K_{4,4,4}$ *with partite sets* V_1, V_2, *and* V_3 *whose competition graph is complete. Then, for each* $1 \leq i \neq j \leq 3$ *and each* $u \in V_i$, $|N_D^+(u) \cap V_j| = |N_D^-(u) \cap V_j| = 2$.

Proof. Take distinct i and j in $\{1, 2, 3\}$. Then there are exactly 16 arcs between V_i and V_j. On the other hand, by Corollary 2.24, for each $u \in V_i$ and $v \in V_j$,

$$|N_D^+(u) \cap V_j| \geq 2 \quad \text{and} \quad |N_D^+(v) \cap V_i| \geq 2.$$

Therefore

$$16 = \sum_{u \in V_i} |N_D^+(u) \cap V_j| + \sum_{v \in V_j} |N_D^+(v) \cap V_i| \geq 16.$$

and so $|N_D^+(u) \cap V_j| = |N_D^+(v) \cap V_i| = 2$ for each $u \in V_i$ and $v \in V_j$. Hence $|N_D^+(u) \cap V_j| = |N_D^-(u) \cap V_j| = 2$ for each $u \in V_i$. $\square$

Lemma 2.28. *Let D be an orientation of $K_{4,4,4}$ with partite sets V_1, V_2, and V_3 whose competition graph is complete. Then, for some distinct i and j in $\{1, 2, 3\}$, there is a pair of vertices x and y in V_i such that $N_D^+(x) \cap V_j = N_D^+(y) \cap V_j$.*

Proof. Suppose, to the contrary, that, for each i and j in $\{1, 2, 3\}$,

$$N_D^+(u) \cap V_j \neq N_D^+(v) \cap V_j \tag{2.7}$$

for any pair of vertices u and v in V_i. Fix $i \in \{1, 2, 3\}$ and $u, v \in V_i$. Let z and w be the remaining vertices in V_i. Since $C(D)$ is complete, u and v have a common prey in V_j for some $j \in \{1, 2, 3\} \setminus \{i\}$. By Lemma 2.27 and (2.7), $N_D^+(u) \cap V_j = \{v_1, v_2\}$ and

$$N_D^+(v) \cap V_j = \{v_1, v_3\} \tag{2.8}$$

for distinct vertices v_1, v_2, and v_3 in V_j. Let v_4 be the remaining vertex in V_j. Then, by Lemma 2.27, $N_D^+(v_1) \cap V_i = N_D^-(v_4) \cap V_i = \{z, w\}$, so $N_D^+(v_4) \cap V_i = \{u, v\}$. Therefore v_1 and v_4 cannot have a common prey in V_i. Thus they have a common prey in V_k for $k \in \{1, 2, 3\} \setminus \{i, j\}$. By Lemma 2.27 and

(2.7) again, $N_D^+(v_1) \cap V_k = \{w_1, w_2\}$ and $N_D^+(v_4) \cap V_k = \{w_1, w_3\}$ for distinct vertices w_1, w_2, and w_3 in V_k. Let w_4 be the remaining vertex in V_k. Then, by Lemma 2.27,

$$N_D^-(v_1) \cap V_k = \{w_3, w_4\} \quad \text{and} \quad N_D^-(v_4) \cap V_k = \{w_2, w_4\}.$$

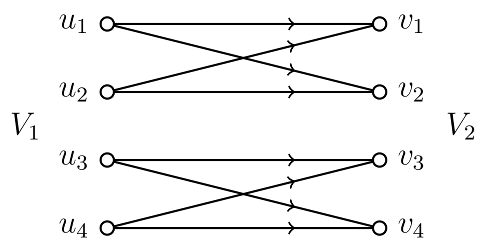
Meanwhile we note that w_2 and u have a common prey in V_j. Since $N_D^+(u) = \{v_1, v_2\}$ and $N_D^+(v_1) \cap V_k = \{w_1, w_2\}$, v_2 is a common prey of w_2 and u. Then $N_D^+(w_2) \cap V_j = \{v_2, v_4\}$. Thus w_2 and v cannot have a common prey in V_j by (2.8). Since w_2 and v belong to V_k and V_i, respectively, they cannot have a common prey in D and we reach a contradiction. □

Theorem 2.29. $(K_{12}, 3)$ *is not competition-realizable.*

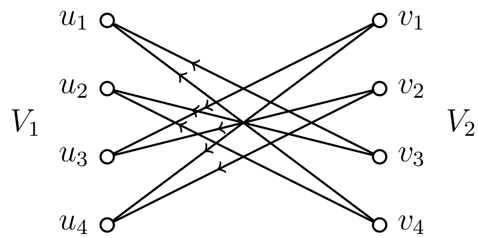
Proof. Suppose, to the contrary, that $(K_{12}, 3)$ is competition-realizable. Then there is an orientation D of K_{n_1,n_2,n_3} with partite sets V_1, V_2, and V_3 for some positive integers n_1, n_2, and n_3 such that $C(D)$ is isomorphic to K_{12}. By Theorem 2.25, $n_1 = n_2 = n_3 = 4$. Then, by Lemma 2.28, for some distinct i and j in $\{1, 2, 3\}$, there is a pair of vertices u_1 and u_2 in V_i such that $N_D^+(u_1) \cap V_j = N_D^+(u_2) \cap V_j = \{v_1, v_2\}$ for vertices v_1 and v_2 in V_j. Without loss of generality, we may assume that $i = 1$ and $j = 2$. Let u_3 and u_4 (resp. v_3 and v_4) be the remaining vertices in V_1 (resp. V_2). Then, by Lemma 2.27,

$$N_D^+(u_3) \cap V_2 = N_D^+(u_4) \cap V_2 = \{v_3, v_4\},$$

$$N_D^+(v_1) \cap V_1 = N_D^+(v_2) \cap V_1 = \{u_3, u_4\},$$



the arcs from V_1 to V_2



the arcs from V_2 to V_1

Figure 2.5: The arcs between V_1 and V_2

$$N_D^+(v_3) \cap V_1 = N_D^+(v_4) \cap V_1 = \{u_1, u_2\}$$

(see Figure 2.5 for an illustraion). Therefore each of the following pairs does not have a common prey in V_2: $\{u_1, u_3\}$; $\{u_1, u_4\}$; $\{u_2, u_3\}$; $\{u_2, u_4\}$. In addition, each of the following pairs does not have a common prey in V_1: $\{v_1, v_3\}$; $\{v_1, v_4\}$; $\{v_2, v_3\}$; $\{v_2, v_4\}$. Then each of these pairs has a common prey in V_3. Let w_1, w_2, w_3, and w_4 be the common preys of $\{u_1, u_3\}, \{u_1, u_4\}, \{u_2, u_3\}$, and $\{u_2, u_4\}$, respectively. Without loss of generality, we may assume that w_1, w_2, w_3, and w_4 are the common preys of $\{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}$, and $\{v_2, v_4\}$, respectively. Then, by Lemma 2.27, w_1 and w_2 are the prey of u_1

and w_3 and w_4 are the prey of v_2 in D, so u_1 and v_2 do not have a common prey in D, which is a contradiction. $\square$

Chapter 3

Tripartite tournaments whose competition graphs are connected and triangle-free

In this chapter, we study tripartite tournaments whose competition graphs are connected and triangle-free.

Lemma 3.1. *Let D be an orientation of K_{n_1,n_2,n_3} whose competition graph has no isolated vertex for some positive integers n_1, n_2, and n_3. Then at least two of n_1, n_2, and n_3 are greater than* 1.

Proof. Suppose, to the contrary, that at most one of n_1, n_2, and n_3 is greater than 1, that is, at least two of n_1, n_2, and n_3 equal 1. Without loss of generality, we may assume that $n_1 = n_2 = 1$. Let $\{u\}, \{v\}$, and V be the partite set of D. Then $|V| = n_3$. Since D is an orientation of K_{n_1,n_2,n_3}, either $(u, v) \in A(D)$ or $(v, u) \in A(D)$. By symmetry, we may assume that $(u, v) \in A(D)$. Since $C(D)$ has no isolated vertex, v is adjacent to some vertex. Since $(u, v) \in A(D)$, v is not adjacent to any vertex in V. Thus u and v are adjacent in $C(D)$ and so u and v have a common prey, say w, in V. Then neither u nor v is a prey of w. Therefore w is isolated in $C(D)$ and we reach a contradiction. □

Lemma 3.2. *Let n_1, n_2, and n_3 be positive integers such that $n_1 \leq n_2 \leq n_3$. If there exists an orientation D of K_{n_1,n_2,n_3} whose competition graph is connected and triangle-free, then $(n_1, n_2, n_3) \in \{(1, 2, 2), (1, 2, 3), (2, 2, 2), (1, 2, 4)\}$. In particular, $d^-_D(v) = 2$ for each v in D if D is an orientation of $K_{1,2,4}$.*

Proof. Suppose that D is an orientation of K_{n_1,n_2,n_3} whose competition graph is connected and triangle-free. Then $|A(D)| = n_1n_2+n_2n_3+n_3n_1$. Since $C(D)$ is triangle-free, for each $v \in V(D)$, $d^-_D(v) \leq 2$. Therefore

$$n_1n_2 + n_2n_3 + n_3n_1 = |A(D)| = \sum_{v \in V(D)} d^-_D(v) \leq 2(n_1 + n_2 + n_3)$$

Thus

$$n_1(n_2 - 2) + n_2(n_3 - 2) + n_3(n_1 - 2) \leq 0 \tag{3.1}$$

and so one of $n_1 - 2$, $n_2 - 2$, and $n_3 - 2$ is nonpositive. Since $n_1 \leq n_2 \leq n_3$, $n_1 - 2 \leq 0$. Suppose $n_1 = 1$. Then, by (3.1), $3 \geq (n_2 - 1)(n_3 - 1)$. Since none of n_2 and n_3 equals 1 by Lemma 3.1, we have $(n_1, n_2, n_3) \in \{(1, 2, 2), (1, 2, 3), (1, 2, 4)\}$. Especially, if $(n_1, n_2, n_3) = (1, 2, 4)$, then $d^-_D(v) =$

2 for each v in D by (3.1) and so the "particular" part is true. If $n_1 = 2$, then $4 \geq n_2 n_3$ by (3.1) and so $(n_1, n_2, n_3) = (2, 2, 2)$.

□

Lemma 3.3. *Let D be an orientation of $K_{1,2,4}$ whose competition graph is triangle-free. Then the number of edges in $C(D)$ is less than 6.*

Proof. Let V_1, V_2, and V_3 be the partite sets of D with $|V_1| = 1$, $|V_2| = 2$ and $|V_3| = 4$. By the "particular" part of Lemma 3.2,

$$d_D^-(v) = 2 \tag{3.2}$$

for each $v \in V(D)$. Then, since $|V(D)| = 7$, the number of edges in $C(D)$ is less than or equal to 7. Moreover, there are only three possible out-neighbors for the vertices in V_3, so there must be two vertices in V_3 having the same in-neighborhood. Thus the number of edges in $C(D)$ is less than 7. Now suppose, to the contrary, that the number of edges in $C(D)$ is 6. Then, there are exactly two vertices in V_3 which share the same in-neighborhood, and if we let w be one of the two vertices, then $N_D^-(x) \neq N_D^-(y)$ for distinct vertices x and y in $V(D) \setminus \{w\}$. Since any 2-subset of $V_1 \cup V_2$ was assigned as the in-neighborhood of a vertex in V_3, the in-neighborhood of a vertex in $V_1 \cup V_2$ must be a 2-subset of V_3, which implies that there are 6 arcs going from V_3 to $V_1 \cup V_2$. However, there are 12 arcs between $V_1 \cup V_2$ and V_3 among which 8 arcs are going from $V_1 \cup V_2$ to V_3, and we reach a contradiction. □

Theorem 3.4. *For a positive integer $n \geq 3$, $(C_n, 3)$ is competition-realizable*

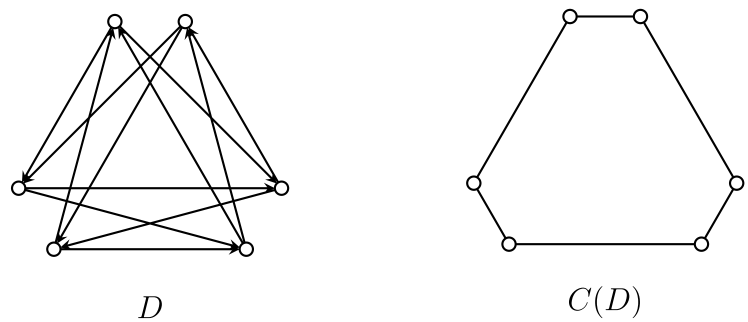


Figure 3.1: A digraph D is an orientation of $K_{2,2,2}$ and $C(D) \cong C_6$.

if and only if $n = 6$.

Proof. Let D be the digraph in Figure 3.1 which is isomorphic to some orientation of $K_{2,2,2}$. It is easy to check that $C(D) \cong C_6$. Therefore the "if" part is true.

Now suppose that $(C_n, 3)$ is competition-realizable. If $n = 3$, then the only possible size of partite sets is $(1, 1, 1)$ which is impossible by Lemma 3.1. Therefore $n \geq 4$. Since C_n is a triangle-free graph without isolated vertex, by Lemma 3.2, $n \in \{5, 6, 7\}$. By the way, by Lemma 3.3, $n \neq 7$. Now suppose that $n = 5$. Then there is an orientation D_1 of $K_{1,2,2}$ such that $C(D_1)$ is cycle of lengths 5. Since $C(D_1)$ is triangle-free, $d^-_{D_1}(v) \leq 2$ for each $v \in V(D_1)$. Moreover, since each edge is a maximal clique and there are 5 edges, each vertex has indegree 2 in D_1. Therefore

$$8 = |A(D)| = \sum_{v \in V(D)} d^-_D(v) = 10$$

and we reach a contradiction. $\square$

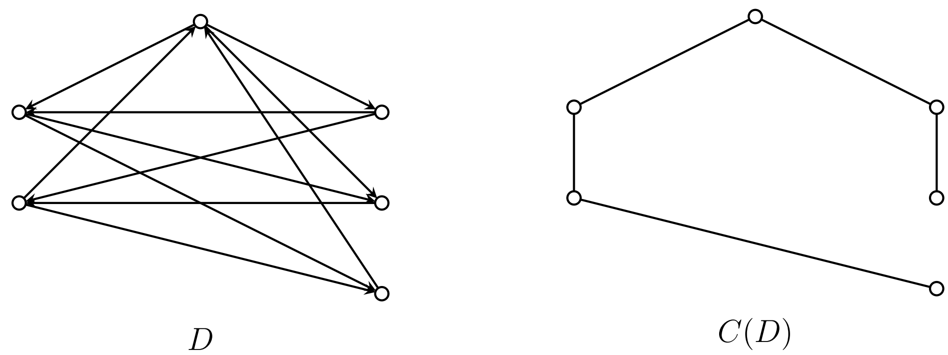


Figure 3.2: A digraph D is an orientation of $K_{1,2,3}$ and $C(D) \cong P_6$.

Corollary 3.5. *Let G be a connected triangle-free of order n. If G is competition-realizable, then $n \in \{5, 6\}$.*

Proof. It immediately follows from Lemma 3.2 and Lemma 3.3. □

Theorem 3.6. *For a positive integer $n \geq 3$, $(P_n, 3)$ is competition-realizable if and only if $n = 6$.*

Proof. Let D be the digraph in Figure 3.2 which is isomorphic to some orientation of $K_{1,2,3}$. It is easy to check that $C(D) \cong P_6$. Therefore the "if" part is true.

Now suppose that $(P_n, 3)$ is competition-realizable. Since P_n is a tree, by Corollary 3.5, $n \in \{5, 6\}$. Suppose that $n = 5$. Then there is an orientation D_1 of $K_{1,2,2}$ such that $C(D_1)$ is a path of length 4. Let V_1, V_2, and V_3 be the partite sets with $|V_1| = 1$ and $|V_2| = |V_3| = 2$. Since $C(D_1)$ is triangle-free, $d^-_{D_1}(v) \leq 2$ for each $v \in V(D_1)$. Since there are 4 edges in $C(D_1)$, there are 4 vertices of indegree 2 in D_1. Since $|A(D_1)| = 8$ and $n = 5$, there exists exactly one vertex of indegree 0 in D_1. Let u be the vertex of indegree 0 in

D_1. If $V_1 = \{u\}$, then $N^+_{D_1}(u) = V_2 \cup V_3$. Otherwise, either $N^+_{D_1}(u) = V_1 \cup V_3$ or $N^+_{D_1}(u) = V_1 \cup V_2$. Therefore $d^+_{D_1}(u) = 3$ or 4. Thus u is incident to at least 3 edges in $C(D_1)$ which is impossible on a path of length at least 4. $\square$

Lemma 3.7. *Let D be an orientation of $K_{1,2,3}$ whose competition graph is connected and triangle-free. Then the following are true:*

(1) There is no vertex of indegree 0 in D;

(2) There are exactly five vertices with indegree 2 in D no two of which share the same in-neighbor.

Proof. Since $C(D)$ is triangle-free, $d^-_D(v) \leq 2$ for each $v \in V(D)$. If there is a vertex of indegree 0 in D, then $11 = \sum_{v \in V(D)} d^-_D(v) \leq 10$ and we reach a contradiction. Therefore the statement (1) is true and so the indegree sequence of D is $(2, 2, 2, 2, 2, 1)$. Meanwhile, since $C(D)$ is connected, the number of edges of $C(D)$ is at least 5. Thus there are exactly five vertices with indegree 2 in D no two of which share the same in-neighbor. $\square$

Theorem 3.8. *Let G be a connected and triangle-free graph with n vertices. Then $(G, 3)$ is competition-realizable if and only if G is isomorphic to a graph belonging to the following set:*

$$\begin{cases} \{G_1, G_2\} & \text{if } n = 5; \\ \{G_3, G_4, P_6, C_6\} & \text{if } n = 6. \end{cases}$$

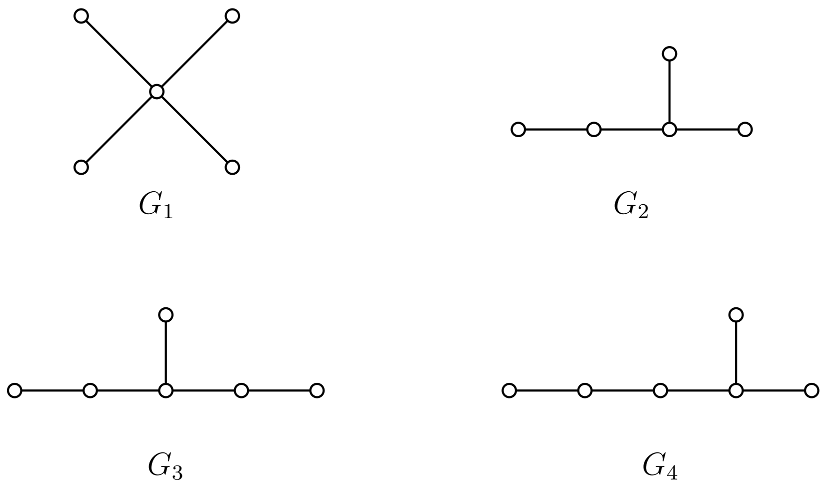


Figure 3.3: Connected triangle-free graphs mentioned in Thorem 3.8
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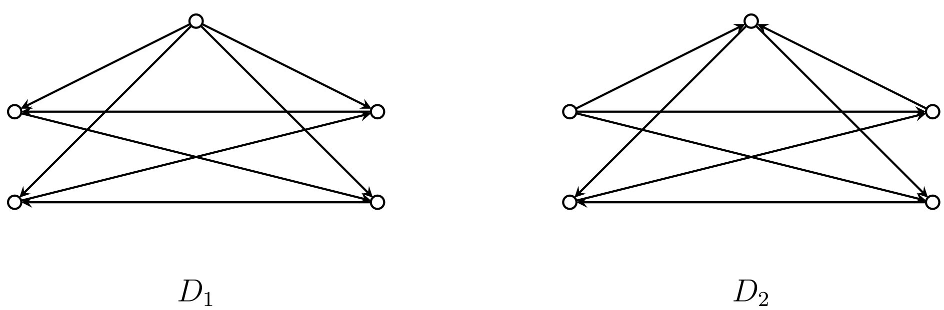
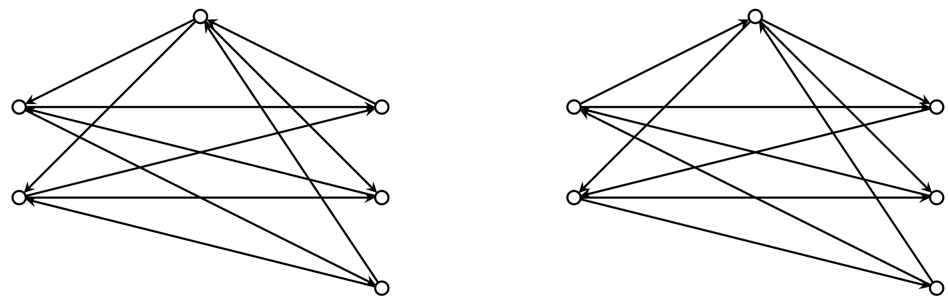


Figure 3.4: The orientations D_1 and D_2 of $K_{1,2,2}$ with $C(D_1) \cong G_1$ and $C(D_2) \cong G_2$.



D_3 D_4

Figure 3.5: The orientations D_3 and D_4 of $K_{1,2,3}$ with $C(D_3) \cong G_3$ and $C(D_4) \cong G_4$.

Proof. Let D be a 3-partite tournament whose competition graph is connected and triangle-free. Then

$$d_D^-(v) \leq 2 \tag{3.3}$$

for each $v \in V(D)$ and $n \in \{5, 6\}$ by Corollary 3.5. Suppose that $n = 5$. If G is a path or a cycle, then, by Theorems 3.4 and 3.6, then G is isomorphic to P_6 or C_6. Now we suppose that G is neither a path nor a cycle.

Case 1. $n = 5$. Then, by Lemma 3.2, D is an orientation of $K_{1,2,2}$. Since $|A(D)| = 8$ and $C(D)$ is connected, by (3.3), there are exactly 4 edges in $C(D)$. Therefore $C(D)$ is isomorphic to G_1 or G_2 in Figure 3.3. Orientations D_1 and D_2 of $K_{1,2,2}$ given in Figure 3.4 have competition graphs isomorphic to G_1 and G_2, respectively, so the "if" part is true.

Case 2. $n = 6$. Then, by Lemma 3.2, D is an orientation of $K_{1,2,3}$ or $K_{2,2,2}$. Suppose that D is an orientation of $K_{2,2,2}$. Then $12 = \sum_{v \in V(D)} d_D^-(v)$.

Therefore, by (3.3), $d_D^-(v) = 2$ for each $v \in V(D)$ and so $d_D^+(v) = 2$ for each $v \in V(D)$. Thus there is no vertex of degree 3 in $C(D)$, which is a contradiction. Hence D is an orientation of $K_{1,2,3}$. By Lemma 3.7, there are exactly 5 edges in $C(D)$. Let V_1, V_2, and V_3 be the partite sets of D with $|V_i| = i$ for each $i \in \{1, 2, 3\}$.

Suppose that there is a vertex w of degree at least 4 in $C(D)$. Then, by (3.3), w has outdegree at least 4 in D. Thus w belongs to V_1 or V_2. If w belongs to V_2, then the indegree of w is 0, which contradicts Lemma 3.7(1). Therefore $V_1 = \{w\}$ and the outdegree of w in D is 4 by Lemma 3.7(1). Then the indegree of each vertex in D except w is exactly 2 by Lemma 3.7(2). If three out-neighbors of w belong to the same partite set, then there exists two pairs of them which shares a common in-neighbor, which contradicts Lemma 3.7(2). Therefore two of the out-neighbors of w belong to V_2 and the remaining out-neighbors belong to V_3. Since the indegree of each vertex in D except w is exactly 2, each vertex in V_2 has exactly one in-neighbor in V_3. Thus there is one vertex in V_3 which is not an in-neighbor of any vertex in V_2. Then w is the only its out-neighbor and so it is isolated in $C(D)$. Hence we have reached a contradiction and so the degree of each vertex of $C(D)$ is at most 3.

Now suppose that there are at least two vertices x and y of degree 3 in $C(D)$. Then, since the number of edges in $C(D)$ is exactly 5, it is easy to see that $C(D)$ is isomorphic to the caterpillar with the spine xy. By (3.3), $d_D^+(x) \geq 3$ and $d_D^+(y) \geq 3$. If x or y belongs to V_3, then $d_D^-(x) = 0$ or $d_D^-(y) = 0$, which contradicts Lemma 3.7(1). Therefore x and y belong to V_1

or V_2. Suppose that $V_2 = \{x, y\}$. Then, since $|V_2 \cup V_3| = 4$, there are at least two vertices which have the same in-neighbors x and y, which contradicts Lemma 3.7(2). Thus one of x and y belongs to V_1 and the other of x and y belongs to V_2. Without loss of generality, we may assume that $V_1 = \{x\}$ and $y \in V_2$. Then, by Lemma 3.7(1), $d_D^-(y) \neq 0$, so $d_D^+(y) = 3$. If $N_D^+(y) = V_3$, then only x and the other vertex in V_2, say y', can be adjacent to y, which is a contradiction. Therefore the out-neighbors of y are x and two vertices in V_3. Since x and y are adjacent in $C(D)$, one of the two vertices in V_3, say z_1, is a common prey of x and y. Moreover, x is a common prey of y and the vertex in $V_3 \setminus \{z_1, z_2\}$ where z_2 is the other out-neighbor of y in V_3. Therefore z_2 is a common prey of y and y'. Then y' is the only possible out-neighbor of x other than z_1 and we reach a contradiction. Thus we have shown that there is exactly one vertex of degree 3 in $C(D)$ and so $C(D)$ is isomorphic to G_3 or G_4 in Figure 3.3. Hence the "only if" part is true. Orientations D_3 and D_4 of $K_{1,2,3}$ given in Figure 3.5 have competition graphs isomorphic to G_3 and G_4, respectively, so the "if" part is true. $\square$

Chapter 4

Structure of competition graphs of multipartite tournaments in the aspect of sink sequences

In this chapter, we study structure of competition graphs of multipartite tournament in the aspect of sink sequences. Eoh *et al.* [29] introduced the notion of *sink elimination index*.

Given a digraph D, we call a vertex of outdegree zero a *sink* in D. Now we define a nonnegative integer $\zeta(D)$ and a squence

$$(W_0, W_1, ..., W_{\zeta(D)})$$

of subsets of $V(D)$ as follows. If $W_0 = V(D)$ or $W_0 = \emptyset$, then let $\zeta(D) = 0$. Otherwise, let $D_1 = D - W_0$ and let W_1 be the set of sinks in D_1. If $W_1 = V(D_1)$ or $W_1 = \emptyset$, then let $\zeta(D) = 1$. Otherwise, let $D_2 = D_1 - W_1$ and let W_2 be the set of sinks in D_2. If $W_2 = V(D_2)$ or $W_2 = \emptyset$, then let $\zeta(D) = 2$. We continue in this way until we obtain $W_k = V(D_k)$ or $W_k = \emptyset$ for some nonnegative integer k. Then we let $\zeta(D) = k$. We call $\zeta(D)$ the *sink elimination index* of D and the sequence $(W_0, W_1, ..., W_{\zeta(D)})$ the *sink sequence* of D and the sequence $(D_0, D_1, ..., D_{\zeta(D)})$ with $D_0 = D$ the *digraph sequence associated with the sink sequence.*

Lemma 4.1. *Let D be a k-partite tournament and W_0 be the set of sinks in D. If W_0 is nonempty, then W_0 is contained in exactly one partite set of D.*

Proof. If $|W_0| = 1$, then it is obviously true. Now, we may assume that $|W_0| \geq 2$. Suppose, to the contrary, that there are distinct partite sets U and V such that $W_0 \cap U \neq \emptyset$ and $W_0 \cap V \neq \emptyset$. Take $u \in W_0 \cap U$ and $v \in W_0 \cap V$. Since D is a k-partite tournament, either $(u, v) \in A(D)$ or $(v, u) \in A(D)$. Then either $d_D^+(u) \geq 1$ or $d_D^+(v) \geq 1$ which contradicts the fact that $d_D^+(u) = d_D^+(v) = 0$. $\square$

Theorem 4.2. *Let D be a k-partite tournament of order n. Suppose that $\zeta(D) \geq 1$ and $W_0 \subseteq U$ for some partite set U of D. Then*

$$C(D) \cong \begin{cases} I_p \dot{\cup} K_{n-p} & \text{if } \ W_0 = U \\ I_p \dot{\cup} H & \text{if } \ W_0 \subsetneq U \text{ and } \zeta(D) \geq 2 \end{cases}$$

where $p = |W_0|$ and H is a graph with $n-p$ vertices containing vertex-disjoint *cliques of sizes $n - |U|$ and $|U| - p$, respectively.*

Proof. Since $\zeta(D) \geq 1$, $W_0 \neq \emptyset$. Let $D[W_0] = D_0$ and $D[V(D) \setminus U] = D^*$. Since D is a k-partite tournament, for any $w \in W_0$, $N_D^+(w) = \emptyset$ and $N_D^-(w) = V(D) \setminus U$ by the definition of W_0. Then

$$C(D_0) \cong I_p, \quad [C(D_0), C(D^*)] = \emptyset, \quad \text{and} \quad C(D^*) \cong K_{n-|U|}.$$

Therefore $C(D) \cong I_p \dot{\cup} H$ where H is a graph with vertex set $V(D) \setminus W_0$ containing $K_{n-|U|}$ as a subgraph. If $W_0 = U$, then $H = C(D^*)$ and so $C(D) \cong I_p \dot{\cup} K_{n-p}$.

Now suppose that $W_0 \subsetneq U$ and $\zeta(D) \geq 2$. Then $\emptyset \neq U \setminus W_0 \subseteq V(D) \setminus W_0$ and $W_1 \neq \emptyset$. Since $|U \setminus W_0| = |U| - p$ and $(U \setminus W_0) \cap (V(D) \setminus U) = \emptyset$, it is sufficient to show that the vertices in $U \setminus W_0$ form a clique in $C(D)$. Consider $D_1 := D - W_0$. Since $U \setminus W_0 \neq \emptyset$, D_1 is still a k-partite tournament with the partite sets same as the ones of D except $U \setminus W_0$ which replaces U in D. Since W_1 is the set of sinks in D_1, W_1 is contained in exactly one partite set, say U', of D_1 by Lemma 4.1. Suppose, to the contrary, that $U' = U \setminus W_0$. Then, since there is no arc in $D[U]$, $d_{D_1}^+(w) = 0$ implies $d_D^+(w) = 0$ for any $w \in W_1$. Therefore $W_1 \subseteq W_0$ and so we reach a contradiction. Thus U' is a partite set of D_1 other than $U \setminus W_0$ and so U' is a partite set of D distinct from U. Therefore, for each $u \in U \setminus W_0$ and $v \in W_1 \subseteq U'$, $(u, v) \in A(D)$. Hence the vertices in $U \setminus W_0$ form a clique in $C(D)$. $\square$

Lemma 4.3. *Let D be a k-partite tournament of order n without a sink whose competition graph has an isolated vertex for some positive integer $k \geq$*

3. *Then the out-neighbor of an isolated vertex is contained in exactly one partite set of D.*

Proof. Take an isolated vertex x in $C(D)$. Suppose, to the contrary, that there are distinct partite sets U and V such that $N_D^+(x) \cap U \neq \emptyset$ and $N_D^+(x) \cap V \neq \emptyset$. Then there exist vertices $u \in N_D^+(x) \cap U$ and $v \in N_D^+(x) \cap V$. Since u and v belong to distinct partite sets, either $(u, v) \in A(D)$ or $(v, u) \in A(D)$. Then x is adjacent to u or v and we reach a contradiction. Therefore $N_D^+(x)$ is contained in one partite set. $\square$

Proposition 4.4. *Let D be a k-partite tournament of order n without a sink for some positive integer $k \geq 3$ and x be an isolated vertex in $C(D)$. Then $C(D)$ contains vertex-disjoint cliques of sizes $|N_D^+(x)|$ and $n - (p + |N_D^+(x)|)$, respectively, where p is the size of the partite set containing x.*

Proof. Let $V_1, V_2, \cdots$, and V_k be the partite sets of D. By Lemma 4.3, $N_D^+(x)$ is contained in exactly one partite set of D. Without loss of generality, we may assume that $x \in V_1$ and $N_D^+(x) \subseteq V_2$. Then $N_D^-(x) = V(D) \setminus (V_1 \cup N_D^+(x))$. Therefore $V(D) \setminus (V_1 \cup N_D^+(x))$ forms a clique in $C(D)$. Meanwhile, since x is an isolated vertex in $C(D)$, $N_D^-(u) = \{x\}$ and so $N_D^+(u) = V(D) \setminus (V_2 \cup \{x\})$ for each $u \in N_D^+(x)$. Thus any vertex in V_3 is a prey of each vertex in $N_D^+(x)$ and so $N_D^+(x)$ forms a clique in $C(D)$. $\square$

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국문초록

유향그래프 D의 경쟁그래프 $C(D)$는 D와 같은 꼭짓점 집합을 갖고 어떤 꼭짓점 x에 대하여 유향변 (u,x)와 (v,x)가 D에 존재하면 2개의 서로 다른 꼭짓점 u와 v 사이의 변을 갖는 방향이 없는 단순 그래프이다. Cohen(1968)은 생태계의 먹이사슬에서 포식자-피식자 개념을 연구하면서 경쟁그래프의 개념을 고안하였고, 경쟁그래프의 분야에서 많은 연구가 있어왔다. 최근, Kim(2016)은 방향 지어진 완전 이분 그래프의 경쟁그래프를 연구하였다. 이 논문에서는 그 결과를 확장하여 완전 다분 그래프에 방향을 부여한 방향지어진 완전 다분 그래프의 경쟁 그래프를 연구하였다. 먼저 방향 지어진 완전 다분 그래프의 경쟁 그래프가 완전 그래프가 되는 경우에 대하여 연구하였다. 또한 경쟁그래프로 연결되고 삼각형을 포함하지 않는 그래프를 갖는 방향 지어진 완전 삼분그래프를 특징화하였다. 마지막으로 방향 지어진 완전 다분 그래프의 경쟁 그래프의 구조를 sink 수열의 관점에서 연구하였다.

주요어휘: 완전그래프, 방향 지어진 완전 다분 그래프, 경쟁 그래프, 삼각형을 포함하지 않는 연결된 그래프.

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