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# Curvature flows with obstacles

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# Abstract

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Curvature flows are geometric evolutions of a hypersurface moved by curvature quantities such as the mean curvature and the Gauss curvature, which have been applied in material science and image processing. The main difficulty to treat curvature flows is development of singularities in finite time which arises in many case. In this thesis, we would like to propose a method to continue curvature flows for a long time by placing obstacles enclosed by the initial hypersurface. We apply the method to prevent the development of singularities for the mean curvature flow when the initial hypersurface is given by an entire graph and for the Gauss curvature flow when the initial hypersurface is strictly convex and closed. Moreover, we investigate the obstacle problem for the parabolic Monge-Ampère equation which is closely related to the Gauss curvature flow. Our approach is based on the penalization method by allowing the evolution of hypersurface can pass the obstacle, with the property that the more the hypersurface pass the obstacle, the more penalty is imposed on the velocity.

**Key words:** mean curvature flow, Gauss curvature flow, obstacle problem, free boundary problem, Monge-Ampère equation, singularity
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# Contents

# Chapter 1

# Introduction

Curvature flows are geometric evolutions of a hypersurface smoothly immersed in the Euclidean space, moved by curvature quantities such as the mean curvature, the Gauss curvature, and so on. Each flow has been introduced to model physical phenomena [33] or to apply other field of science (material science [63]; image processing [14, 59]). The main difficulty to treat curvature flows is development of singularities in finite time which arises in most case, and understanding singularities is an important problem in geometric analysis. Many mathematicians are devoted to develop methods to continue flows beyond the first singular time, for example, Brakke flow [6], level set flow [16, 30], mean curvature flow with surgery [43], and singularity resolving flow [65]. All these methods are focused on the preservation of some properties of the initial hypersurface.

In this thesis we would like to propose a new method to continue curvature flows for a long time by placing obstacles enclosed by the initial hypersurface. The method is designed for behaviors and phenomena that we would like to impose on the flow. The obstacles are hypersurfaces embedded in the Euclidean space. We impose on the evolutions of hypersurface a condition that it cannot pass the obstacle. When a singularity is developed at some point for the first time, we place an obstacle such that it encloses the point and is enclosed by the curvature flow slightly before the first singular time.

This enable the curvature flow to evolve over the first singular time, and we call this problem *curvature flows with obstacles.*

Our method easily prevents development of the singularity we targeted. On the other hand, an unexpected singularity might be occurred by the obstacle. Since the curvature flows cannot pass the obstacle, a discontinuity of the velocity arises when the flow touches the obstacle. This makes the problem difficult and the optimal regularity of the curvature flow with an obstacle is expected to be at most $C^{1,1}$. The lack of regularity forces us to consider a general concept of solutions, that is, viscosity solutions. Therefore, one of the main goal to justify our method is showing that viscosity solutions for the curvature flow with an obstacle does not develop singularity.

The evolutions of hypersurface under the curvature flows consist of two parts: the coincidence set where the hypersurface touches the obstacle and the non-coincidence set where the hypersurfce does not touch the obstacle. In many case (i.e., convex initial hypersurface), the velocity keeps its sign under the flow, in which case the coincidence set would tend to grow over time. The boundary of the coincidence set (or equivalently of the non-coincidence set) is the so-called *free boundary* which is unknown before we get a solution.

Major difficulties in curvature flows with obstacles occur near the free boundary. Indeed, we will see if the free boundary moves in non-degenerate finite speed, the velocity should be as degenerate as the distance from the free boundary. In PDEs point of view, the curvature flows with obstacles are fully nonlinear non-uniformly parabolic equation in the non-coincidence set, whose ellipticity constant is degenerate at the boundary of the domain which varies in time and is unknown in advance. Thus we do not expect global lower bound of the curvature in the curvature flow with obstacles. Instead, we shall obtain the lower bound of the curvature in terms of the constant depending on the distance from the free boundary and vanishing on the free boundary, which is one of main interests.

The main strategy to deal with the curvature flows with obstacles is an approximation by allowing the evolution of hypersurface can pass the obstacle, with the property that the more the hypersurface pass the obstacle, the more *penalty* is imposed on the velocity. The penalty will be realized by a penalty function which makes the equation holds not on the non-coincidence set but on the whole domain. Then establishing the uniform *a priori* estimates for the approximated problem gives the same estimates for the original problem.

We will apply the method we proposed to the following three cases.

(i) Gauss curvature flow when the initial hypersurface is strictly convex and closed ([55]).

(ii) Mean curvature flow when the initial hypersurface is given by entire graph ([46]).

(iii) Parabolic Monge-Ampère equation ([60]).

Additionally, we consider a free boundary problem arising in composite membrane problem with fractional Laplacian, which could be formulated as two-sided unstable obstacle problem [36].

The Gauss curvature flow, evolution by the Gauss curvature, was introduced by Firey [33] to describe the deformation of shape of stones which is worn down by collision from any random angle. Later, Tso [72] proved that the smooth solution exists uniquely and shrinks to a point when the enclosed volume converges to zero if the initial hypersurface is strictly convex and closed. Furthermore, it was shown in [3] for two dimensional case ($n = 2$) that the contraction is spherical singularity. More generally, the $\alpha$-Gauss curvature flow, the evolution by the Gauss curvature with an exponent $\alpha$, was studied by Chow [20] for the case $\alpha = 1/n$; by Kim and Lee [47] for $1/n \leq \alpha \leq 1$; and by Andrews, Guan, and Ni [4] for $\alpha \geq 1/(n+2)$. They showed the flow converges to a self-similar solution for every $n \geq 2$ and $\alpha \geq 1/(n+2)$ after scaling. In the affine invariant case $\alpha = 1/(n+2)$, the only self-similar solutions are ellipsoid [2, 13]. Also, in the case $\alpha > 1/(n+2)$, Brendle, Choi, and Daskalopoulos [7] proved the only self-similar solutions are round spheres. These flows relate to the parabolic version of Monge-Amère equation since the Gauss curvature is defined by the determinant of the Weingarten map. From the fully nonlinearity of the Gauss curvature, if the initial hypersurface has flat side, then the solution also has flat side for some time unlike the mean curvature flow which is instantly smoothing (see [38] and also [19, 23, 24, 48]).

The first result [55] is concerned with the obstacle problem for the Gauss curvature flow with an exponent $\alpha$. Under the assumption that both the obstacle and the initial hypersurface are strictly convex closed hypersurface and the obstacle is enclosed by the initial hypersurface, the uniform estimates are obtained for several curvatures via penalty method. We also give a heuristic calculation to explain the principal curvature may be zero on the free boundary. In particular, when the hypersurface is two dimensional with $0 < \alpha \leq 1$, we prove that the solution for the Gauss curvature flow with an obstacle exists for all time with bounded principal curvatures $\{\lambda_i\}$ in which the upper bound is uniform and the lower bound depends on the distance from the free boundary. Moreover, we show that there is a finite time $T_*$ such that the solution becomes the obstacle after this time, which is stationary in time.

The mean curvature flow, evolution by the mean curvature, was originally studied by Brakke [6] and has been studied by Huisken, Ecker, Sinestrari, and many others, see [27, 28, 40, 42]. We also refer to the monographs [26, 75] for introductions. The mean curvature flow is a natural generalization of heat equation to the manifold setting in the sense that the position vector $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ satisfies

$$\frac{\partial X}{\partial t} = \Delta_{g(t)} X,$$

where $X(\cdot, t)$ is an immersion of a manifold $M$ and $\Delta_g$ is the Laplace-Beltrami operator on the hypersurface given by $X(\cdot, t)$ with its canonical metric induced by the Euclidean space $\mathbb{R}^{n+1}$. Despite the similarity between the mean curvature flow and the heat equation, there are some important differences.

For a short time, the mean curvature flow behaves like the heat equation with regularizing effects in small-scale; On the other hand, after more time, singularities are developed since the nonlinearities dominate the evolution. We may employ the flow as a tool to produce minimal surfaces, to derive isometric inequalities, or more generally, to classify hypersurfaces by certain curvature conditions.

The second result [46] is concerned with the obstacle problem for evolutions of non-compact complete graphs over an open subset in $\mathbb{R}^n$. Obstacles in this setting are also written graphs over an open subset in $\mathbb{R}^n$, which are non-compact complete and lies above the initial data. We prove that the solution exists for all time with locally bounded principal curvatures $\{\lambda_i\}$.

The parabolic Monge-Ampère equation is parabolic generalization of (elliptic) Monge-Ampère equation and is closely related to the $\alpha$-Gauss curvature flow. In 1976, Krylov [49] suggested three versions of parabolic Monge-Ampère equation:

$$\begin{aligned} -\partial_t u + (\det D^2 u)^{\frac{1}{n}} &= f, \\ [(-\partial_t u) \det D^2 u]^{\frac{1}{n+1}} &= f, \\ [\det(D^2 u - \partial_t u I_n)]^{\frac{1}{n}} &= f, \end{aligned}$$

where $I_n$ denotes the $n \times n$ identity matrix. In this thesis we are concerned with the first form of equation which is relating to the graph representation of the Gauss curvature flow. The Monge-Ampère equation arises in prescribed Gaussian curvature equation [62], optimal transportation [66], and affine geometry [71]. Also, its parabolic version has been applied to image processing [59], where reducing noises and preserving sharp edges are issues to resolving blurring problem. This problem can be controlled by the diffusion driven by the Gauss curvature since its diffusion is slow near edges due to the nondegeneracy of curvature.

The third result [60] is concerned with the obstacle problem for the parabolic Monge-Ampère equation with the forcing term $f(x, t, u, Du)$. We establish existence, uniqueness, and optimal regularity under some structure conditions via the penalization method and a priori estimates. Moreover, we discuss the regularity of the free boundary. As a consequences of our approach, we also obtain the existence and uniqueness of the solution of the Cauchy-Dirichlet problem for the parabolic Monge-Ampère equation with the forcing term $f(x, t, u, Du)$.

The rest of thesis is organized as follows. Chapter 2 describes the notations and conventions used throughout the thesis. In Chapter 3, we study the Gauss curvature flow with an obstacle. In Chapter 4, we discuss the mean curvature flow of entire graphs with an obstacle. Finally, Chapter 5 is devoted to the obstacle problem for the parabolic Monge-Ampère equation.

# Chapter 2

# Preliminaries

We describe some notations and conventions used throughout the thesis. Let $X_0$ be an immersion from $n$-dimensional manifold $M$ into $\mathbb{R}^{n+1}$, and let

$$X(\cdot, t) : M \to \mathbb{R}^{n+1}$$

be a one-parameter family of immersions from $M$ into $\mathbb{R}^{n+1}$. We may take $M$ as a compact closed manifold or a non-compact complete manifold. Let $\vec{\nu}$ be a unit normal vector of $\Sigma_t = X(M, t)$. Given a local coordinate system $\{x^i\}_{i=1}^n$ in $M$, the induced metric and the second fundamental form of $\Sigma_t$ can be computed as

$$g_{ij} = \left\langle \frac{\partial \vec{X}}{\partial x^i}, \frac{\partial \vec{X}}{\partial x^j} \right\rangle \quad \text{and} \quad h_{ij} = -\left\langle \frac{\partial^2 \vec{X}}{\partial x^i \partial x^j}, \vec{\nu} \right\rangle.$$

The connection on $\Sigma_t$ is given by

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left( \frac{\partial}{\partial x^i} g_{jl} + \frac{\partial}{\partial x^j} g_{il} - \frac{\partial}{\partial x^l} g_{ij} \right)$$

and the covariant derivative on $\Sigma_t$ is

$$(\nabla_j v)^i = \frac{\partial}{\partial x^i} v^i + \Gamma_{jk}^i v^k.$$

## 2.1 Second fundamental form and curvatures

The Weingarten map, the differential of the Gauss map, is then defined by

$$h^i_j = g^{ik} h_{kj},$$

where $g^{ij}$ denotes the inverse of the metric. Here we used Einstein's summation convention over repeated indices. The principal curvatures $\lambda_1$, $\lambda_2$, $\cdots$, $\lambda_n$ are the eigenvalues of $h^j_i$, and then the Gauss curvature $K$ and the mean curvature $H$ are given by

$$K = \det(h^j_i) \quad \text{and} \quad H = \mathrm{tr}(h^j_i).$$

We also define the sum of the square $|A|^2 = \sum \lambda_k^2 = \mathrm{tr}((h^2)^j_i)$ and the sum of the inverse $\mathcal{H} = \sum \frac{1}{\lambda_k} = \mathrm{tr}((h^{-1})^j_i)$. From the Gauss equation, we can express the Riemannian curvature tensor, the Ricci tensor, and the scalar curvature as

$$\begin{aligned} R_{ijkl} &= h_{ik}h_{jl} - h_{il}h_{jk}, \\ R_{ij} &= Hh_{ij} - h_{ik}g^{kl}h_{lj}, \\ R &= H^2 - |A|^2, \end{aligned}$$

respectively. The Gauss-Weingarten relations

$$\frac{\partial^2 X}{\partial x^i \partial x^j} = \Gamma^k_{ij}\frac{\partial X}{\partial x^k} - h_{ij}\vec{\nu}$$

gives

$$\Delta_{g(t)} X = g^{ij}\nabla_i\nabla_j X = g^{ij}\left(\frac{\partial^2 X}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial X}{\partial x^k}\right) = -g^{ij}h_{ij}\vec{\nu} = -H\nu.$$

We also deal with the derivative of the second fundamental form.

**Lemma 2.1.1** ([75]). *The following identity holds for $h_{ij}$:*

$$\Delta h_{ij} = \nabla_i \nabla_j H + H h_{ik} g^{kl} h_{lj} - |A|^2 h_{ij}.$$

We also compute the Laplace-Beltrami operator of the outward unit normal vector.

**Lemma 2.1.2.** *Let $\vec{\nu}$ be an outward unit normal vector on a hypersurface $\Sigma_t$ in $\mathbb{R}^{n+1}$, and $\vec{X}$ be a position vector of $\Sigma_t$. Then the Laplace-Beltrami operator of $\vec{\nu}$ is given by*

$$\Delta \vec{\nu} = \nabla H - |A|^2 \vec{\nu}. \tag{2.1.1}$$

*Proof.* First we observe that

$$\nabla_i \vec{\nu} = h_i^k \frac{\partial \vec{X}}{\partial x^k}.$$

since $\left\langle \nabla_i \vec{\nu}, \frac{\partial \vec{X}}{\partial x^k} \right\rangle = h_{ik}$ and $\langle \nabla_i \vec{\nu}, \vec{\nu} \rangle = 0$. Using this, we see that

$$\langle \nabla_i \nabla_j \vec{\nu}, \vec{\nu} \rangle = - \langle \nabla_j \vec{\nu}, \nabla_i \vec{\nu} \rangle = -h_i^k h_j^l g_{kl} = -(h^2)_{ij}.$$

Moreover, by $\nabla_i \nabla_k X = -h_{ik} \vec{\nu}$ and Codazzi equation, we have

$$\left\langle \nabla_i \nabla_j \vec{\nu}, \frac{\partial \vec{X}}{\partial x^k} \right\rangle = \nabla_i h_{jk} - \langle \nabla_j \vec{\nu}, \nabla_i \nabla_k X \rangle = \nabla_k h_{ij}.$$

Thus we arrive at

$$\nabla_i \nabla_j \vec{\nu} = \nabla h_{ij} - (h^2)_{ij} \vec{\nu},$$

which gives the conclusion. $\square$

## 2.2 Auxiliary lemmas

Let us denote $\Box = \alpha K^\alpha (h^{-1})^{ij}\nabla_i\nabla_j$ and recall $\Delta = g^{ij}\nabla_i\nabla_j$. The inner product and the norm induced by $\Box$ and $\Delta$ are

$$\begin{aligned} \langle \nabla A, \nabla B\rangle_\Box &= \alpha K^\alpha (h^{-1})^{ij}\nabla_i A \nabla_j B, & \|\nabla A\|_\Box^2 &= \langle \nabla A, \nabla A\rangle_\Box, \\ \langle \nabla A, \nabla B\rangle &= g^{ij}\nabla_i A\nabla_j B, & \|\nabla A\|^2 &= \langle \nabla A, \nabla A\rangle. \end{aligned}$$

Here we omit the subscript $\Delta$.

We provide an auxiliary lemma which is useful for proving curvature estimates. The proof is a straightforward calculation.

**Lemma 2.2.1.** *Let $A$ and $B$ be smooth functions on $M \times [0, T)$. Assume that $A > 0$ and for a given $\gamma > 0$ define $S = \frac{B}{A^\gamma}$. Then*

$$\begin{aligned} (\partial_t - \Box)S = {} & \frac{1}{A^\gamma}(\partial_t - \Box)B - \frac{\gamma B}{A^{\gamma+1}}(\partial_t - \Box)A \\ & + \frac{2\gamma}{A}\langle \nabla A, \nabla S\rangle_\Box + \frac{\gamma(\gamma-1)B}{A^{\gamma+2}}\|\nabla A\|_\Box^2, \end{aligned}$$

*Moreover, the same holds for $\Delta$ instead of $\Box$.*

# Chapter 3

# Gauss curvature flow with an obstacle

## 3.1 Introduction

In this chapter we study the obstacle problem for the Gauss curvature flow with an exponent $\alpha$, where $0 < \alpha \leq 1$. The precise formulation is as follows. The obstacle in our consideration, denoted by $\mathbf{\Phi}$, is a $C^{1,1}$ strictly convex closed hypersurface in $\mathbb{R}^{n+1}$, and let $\vec{X}_0 : M^n \to \mathbb{R}^{n+1}$ be a smooth immersion of a strictly convex closed $n$-dimensional hypersurface enclosing the obstacle. We consider the evolutions of $\Sigma_0 = \vec{X}_0(M^n)$ under the flow by powers of the Gauss curvature which keep enclosing the obstacle, that is, given an exponent $\alpha$, we consider a one-parameter family of immersions $\vec{X} : M^n \times [0, T) \to \mathbb{R}^{n+1}$ and $\Sigma_t = \vec{X}(M^n, t)$ satisfying

$$
\begin{aligned}
\left\langle \frac{\partial \vec{X}(x,t)}{\partial t}, -\vec{\nu}(x,t) \right\rangle &\leq K^{\alpha}(x,t) && \text{for } (x,t) \in M^n \times [0,T), \\
\frac{\partial \vec{X}(x,t)}{\partial t} &= -K^{\alpha}(x,t)\vec{\nu}(x,t) && \text{if } \vec{X}(x,t) \notin \mathbf{\Phi}, \\
\overline{\mathbf{\Phi}} &\subset \overline{\Sigma_t} && \text{for } 0 \leq t < T, \\
\vec{X}(x,0) &= \vec{X}_0(x) && \text{for } x \in M^n,
\end{aligned}
\tag{GFo}
$$

where $K$ and $\vec{\nu}$ are the Gauss curvature and the outward unit normal vector on $\Sigma_t$, respectively. (Here we use the bar notation to indicate the closed subset of $\mathbb{R}^{n+1}$ enclosed by the given set.)

The Gauss curvature flow was introduced by Firey [33] to describe the deformation of shape of stones which is worn down by collision from any random angle. It is well known that the solution of Gauss curvature flow with an exponent $\alpha$ exists uniquely and has a singularity in finite time, see [72] and [20]. Moreover, the singularity is analyzed for every $\alpha \geq \frac{1}{n+2}$, see [20], [47], [4], [7] and [3]. From the fully nonlinearity of Gauss curvature, if the initial hypersurface has flat side, then the solution also has flat side for some time unlike the mean curvature flow which is instantly smoothing. (See [38].)

Considering the tumbling stone model for the Gauss curvature flow in [33], the Gauss curvature flow with an obstacle can be thought of as the tumbling stone with hard core. Thus, it is not hard to imagine that (GFo) converges to the obstacle in a finite time since the usual tumbling stone disappears in a finite time. However, it is rather clear whether the above model preserves strict convexity. These properties will be described below in mathematical terms.

A discontinuity of the velocity naturally arises from the existence of the obstacle. Thus, the solution of (GFo) has at most Lipschitz regularity in time. This makes us to consider a generalized concept of a solution, that is, a viscosity solution. To introduce this notion, we need the graphical version of (GFo) which can be written as, for a graphical solution $\tilde{u} : \Omega \times [0,T) \to \mathbb{R}$ with an obstacle $\tilde{\varphi} : \Omega \to \mathbb{R}$,

$$\begin{aligned}
\frac{\partial \tilde{u}}{\partial t} &\leq \frac{(\det D^2\tilde{u})^\alpha}{(1+|D\tilde{u}|^2)^{\frac{(n+2)\alpha-1}{2}}} && \text{in } \Omega \times [0,T), \\
\frac{\partial \tilde{u}}{\partial t} &= \frac{(\det D^2\tilde{u})^\alpha}{(1+|D\tilde{u}|^2)^{\frac{(n+2)\alpha-1}{2}}} && \text{in } \{\tilde{u} < \tilde{\varphi}\}, \\
\tilde{u} &\leq \tilde{\varphi} && \text{in } \Omega \times [0,T),
\end{aligned} \tag{3.1.1}$$

where $\Omega$ is a domain in $\mathbb{R}^n$, $D$ represents the usual derivative in Euclidean space, and $\{\tilde{u} < \tilde{\varphi}\}$ denotes the set of points satisfying $\tilde{u}(x,t) < \tilde{\varphi}(x)$ in $\Omega \times [0,T)$. Moreover, (3.1.1) simplifies to the single equation

$$\min\left\{\frac{(\det D^2\tilde{u})^\alpha}{(1+|D\tilde{u}|^2)^{\frac{(n+2)\alpha-1}{2}}} - \frac{\partial \tilde{u}}{\partial t}, \tilde{\varphi} - \tilde{u}\right\} = 0 \quad \text{in } \Omega \times [0,T).$$

Now we can define a viscosity solution of (GFo) as follows:

**Definition 3.1.1** (viscosity solution)**.** A continuous, one parameter family of immersions $\vec{X} : M^n \times [0,T) \to \mathbb{R}^{n+1}$ is a viscosity subsolution (supersolution) of (GFo) if, for any $(x_0,t_0) \in M^n \times [0,T)$ and $\tilde{u}$ which represents $\vec{X}$ locally as a graph near $\vec{X}(x_0,t_0)$ with $(\tilde{x}_0, \tilde{u}(\tilde{x}_0,t_0)) = \vec{X}(x_0,t_0)$ for some $\tilde{x}_0$ after rotation, it holds that

$$\min\left\{\frac{(\det D^2\tilde{\psi}(\tilde{x}_0,t_0))^\alpha}{(1+|D\tilde{\psi}(\tilde{x}_0,t_0)|^2)^{\frac{(n+2)\alpha-1}{2}}} - \frac{\partial \tilde{\psi}(\tilde{x}_0,t_0)}{\partial t}, \tilde{\varphi}(\tilde{x}_0) - \tilde{\psi}(\tilde{x}_0,t_0)\right\} \geq (\leq) 0,$$

whenever $\tilde{\psi}$ is a $C^2$ function satisfying $\tilde{\psi}(\tilde{x}_0,t_0) = \tilde{u}(\tilde{x}_0,t_0)$ and $\tilde{\psi}(\tilde{x},t) \geq (\leq) \tilde{u}(\tilde{x},t)$ for any $\tilde{x}$ in a neighborhood $\tilde{x}_0$ and $t < t_0$. Finally, a continuous, one parameter family of immersions $\vec{X} : M^n \times [0,T) \to \mathbb{R}^{n+1}$ is a viscosity solution of (GFo) if it is both a viscosity subsolution and a viscosity supersolution of (GFo).

With an abuse of terminology, we also say that $\{\Sigma_t = \vec{X}(M,t) : 0 \leq t < T\}$ is a viscosity solution of (GFo) when $\vec{X}$ is a viscosity solution. For more details and properties of viscosity solutions, we refer to [21]. See also [44].

We now state our main results. We proved several estimates for various curvatures. First, the Gauss curvature is bounded so that (GFo) has nonnegative finite speed.

**Proposition 3.1.2.** *For any $\alpha > 0$ and any dimension $n$, the Gauss curvature $K$ of a solution of* (GFo) *satisfies*

$$0 \leq K \leq C \tag{3.1.2}$$

*in* $M^n \times [0, T)$ *where* $C = C(n, \alpha, \mathbf{\Phi}, \vec{X}_0)$.

Notice that, in contrast to the Gauss curvature flow case, $K$ does not have a uniform positive lower bound. It could happen that $K = 0$ on the free boundary, the boundary of $\Sigma_t \cap \mathbf{\Phi}$ in $\Sigma_t$. Later on, we will give a heuristic calculation to explain the reason why such a situation occurs.

Next, the minimum principal curvature $\lambda_{\min}$ is bounded below on the set where the distance from the obstacle is positive. To state this result, let us define the non-coincidence set and the coincidence set by

$$\Omega^M = \{(x,t) \in M^n \times [0,T) : \vec{X}(x,t) \notin \mathbf{\Phi}\},$$
$$\Lambda^M = \{(x,t) \in M^n \times [0,T) : \vec{X}(x,t) \in \mathbf{\Phi}\}.$$

**Proposition 3.1.3.** *For any* $\alpha > 0$ *and any dimension* $n$*, the minimum principal curvature* $\lambda_{\min}$ *satisfies*

$$\lambda_{\min}(x,t) \geq c > 0,$$

*for each* $(x,t) \in \Lambda^M$ *where* $c = c(n, \alpha, \vec{X}_0, \mathbf{\Phi}, d(x, \Lambda^M))$ *is a constant.*

In the following proposition, we consider the 2-dimensional case with $\alpha \leq 1$, where the restrictions arise from a technical reason.

**Proposition 3.1.4.** *For* $0 < \alpha \leq 1$ *and the dimension* $n = 2$*, the mean curvature* $H$ *of the solution of* (GFo) *satisfies*

$$0 < H \leq C,$$

*where* $C = C(n, \alpha, \mathbf{\Phi}, \vec{X}_0)$.

When the dimension $n = 2$, using these uniform estimates, we obtain:

**Theorem 3.1.5.** *Let* $\Sigma_0$ *and* $\mathbf{\Phi}$ *be smooth strictly convex closed surface in* $\mathbb{R}^3$ *such that* $\overline{\mathbf{\Phi}} \subset \overline{\Sigma}_0$. *Assume also that* $0 < \alpha \leq 1$. *Then* *(i) there exists a convex viscosity solution* $\{\Sigma_t = \vec{X}(M,t)\}$ *of* (GFo) *for* $t \in [0,\infty)$*;*

*(ii) the principal curvature of* $\Sigma_t$ *is nonnegative and globally bounded, i.e., the principal curvature* $\lambda_i$ *satisfy*

$$0 \leq \lambda_i \leq C,$$

*where* $C = C(n, \alpha, \mathbf{\Phi}, \vec{X}_0)$ *is a constant;*

*(iii) for each point* $(x,t)$ *in the non-coincidence set* $\Lambda^M$*, the principal curvature of* $\Sigma_t$ *has uniform positive lower bound with dependency on* $d(x,\Lambda)$*, i.e., the principal curvature* $\lambda_i$ *satisfy*

$$0 < c \leq \lambda_i(x,t) \leq C,$$

*where* $C = C(n,\alpha,\mathbf{\Phi},\vec{X}_0)$ *and* $c = c(n,\alpha,\mathbf{\Phi},\vec{X}_0, d(x,\Lambda^M))$ *are constants;*

*(iv) there is a finite time* $T^* = T^*(n,\alpha,\mathbf{\Phi},\vec{X}_0)$ *such that* $\Sigma_t = \mathbf{\Phi}$ *for all* $t \geq T^*$*.*

Our main tool is an approximation using a penalty term. We shall prove that each approximate solution is smooth and several curvatures have uniform bound. For the penalization technique, see [34].

The constraints of dimension $n$ and exponent $\alpha$ will be used in Lemma 3.4.4 to show the uniform upper bound for mean curvature. The main difficulty comes from controlling the third order derivative and the penalty term. There are some ways to overcome the former issue when we think about the Gauss curvature flow with an exponent. For example, in [20], the author consider the quantity $K/H^n$. However, we cannot use this quantity because both $K$ and $H^n$ produce second derivative of penalty term with opposite sign, and one of two sign disturbs having uniform bound.

Finally, we shall explain that why we do not expect the strict positive lower bound in (3.1.2), and that why we could have upper bound in Proposition 3.1.4. To see this, we assume that both $\tilde{u}$ and $\tilde{\varphi}$ in (3.1.1) are rotationally symmetric. Let $|\tilde{x}| = \gamma(t)$ be the equation of free boundary so that

$$\begin{aligned} \tilde{u}(\tilde{x},t) = \tilde{\varphi}(\tilde{x}) &\quad \text{if } |\tilde{x}| \leq \gamma(t), \\ \tilde{u}(\tilde{x},t) < \tilde{\varphi}(\tilde{x}) &\quad \text{if } |\tilde{x}| > \gamma(t), \\ D\tilde{u} = D\tilde{\varphi} &\quad \text{on } |\tilde{x}| = \gamma(t). \end{aligned}$$

From these settings, we obtain

$$\frac{d}{dt}\gamma(t) = \frac{\tilde{u}_t}{\tilde{\varphi}_r - \tilde{u}_r},$$

where the subscript $r$ denotes the derivative in the radial direction, and the denominator is zero on the free boundary. According to [48], the regularity follows from the non-degenerate finite speed of the free boundary, which impose that the numerator $\tilde{u}_t$ is also zero on the free boundary, i.e., $K = 0$ may happen at some point. Next, following the argument in [34, Chapter 1.9], we use $s(\tilde{x}) = t$ for the equation of free boundary instead of $\gamma$ above. Using this, we have an alternative expression of the velocity of the free boundary, that is,

$$\frac{d}{dt}\gamma(t) = \frac{1}{s_r(\tilde{x})}. \tag{3.1.3}$$

In order to use this equation, we consider

$$\tilde{\varphi}(\tilde{x}) - \tilde{u}(\tilde{x},t) = \int_t^{s(\tilde{x})} \tilde{u}_\tau d\tau,$$

so that we imply

$$\begin{aligned}
\tilde{\varphi}_r(\tilde{x}) - \tilde{u}_r(\tilde{x},t) &= \int_t^{s(\tilde{x})} (\tilde{u}_\tau)_r d\tau + \tilde{u}_t(\tilde{x}, s(\tilde{x}))s_r(\tilde{x}) \\
&= \int_t^{s(\tilde{x})} (\tilde{u}_\tau)_r d\tau, \\
\tilde{\varphi}_{rr}(\tilde{x}) - \tilde{u}_{rr}(\tilde{x},t) &= \int_t^{s(\tilde{x})} (\tilde{u}_\tau)_{rr} d\tau + (\tilde{u}_t)_r(\tilde{x}, s(\tilde{x}))s_r(\tilde{x}). \qquad (3.1.4)
\end{aligned}$$

Since $\tilde{u}$ is rotationally symmetric, the second equation in (3.1.1) becomes

$$\frac{\partial \tilde{u}}{\partial t} = \frac{\tilde{u}_r^{(n-1)\alpha}\tilde{u}_{rr}^{\alpha}}{r^{(n-1)\alpha}(1+\tilde{u}_r^2)^{\frac{(n+2)\alpha-1}{2}}},$$

Near the free boundary, this led to $\tilde{u}_t \sim \tilde{u}_{rr}^{\alpha}$ heuristically, and therefore, combining (3.1.3) and (3.1.4), we see that

$$\frac{d}{dt}\gamma(t) \sim \frac{(\tilde{u}_t)_r(\tilde{x}, s(\tilde{x}))}{\tilde{\varphi}_{rr}(\tilde{x})} \sim (\tilde{u}_{rr}^{\alpha})_r.$$

Notice that we have used $\tilde{u}_t = \tilde{u}_{rr} = 0$ and $r \sim \tilde{u}_r \sim \tilde{\varphi}_r \sim \tilde{\varphi}_{rr} \sim 1$ on the free boundary. By considering $K = 0$ at the free boundary points, we may write $\tilde{\varphi} - \tilde{u} = a(r-1)^2 + (r-1)^b$, where $a$ is the constant chosen to satisfy $K = 0$, and $b > 2$. From this, we conclude that $\frac{d}{dt}\gamma(t) \sim 1$ if and only if $(b-2)\alpha - 1 = 0$, in which case $\alpha \in (0, \infty)$. It will also be interesting to consider the case when $\alpha > 1$ or $n \geq 3$ in Proposition 3.1.4, which we leave for a future study.

This chapter is organized as follows: Section 4.2 describes the notations and conventions used throughout the chapter. Section 4.3 has the existence theorem and the evolution equations for the perturbed solutions. Section 4.4 contains the uniform curvature estimates for the perturbed solutions. Section 4.5 has lower bounds for principal curvatures. Finally, section 4.6 contains the proof of Theorem 3.1.5.

## 3.2 Preliminaries

### 3.2.1 Support function

For a strictly convex closed hypersurface $\Sigma$, the outer unit normal vector $\vec{\nu}$ : $\Sigma \to S^n$ is a diffeomorphism. This allow us to reparametrize the hypersurface, namely

$$\vec{X} = \vec{X}(\vec{\nu}^{-1}(z)), \quad z \in S^n.$$

We still denote $\vec{X} \circ \vec{\nu}^{-1}$ by $\vec{X}$ for convenience and we say that $\vec{X}$ is parametrized by $z$-coordinate. Then the support function of the hypersurface $\Sigma$ is defined by

$$u(z) = \left\langle \vec{X}(z), z \right\rangle, \quad z \in S^n. \tag{3.2.1}$$

All information about hypersurface can be recovered from the support function through the relation

$$\vec{X}(z) = \overline{\nabla} u(z) + u(z)z, \quad z \in S^n, \tag{3.2.2}$$

where $\overline{\nabla}$ denotes the Levi-Civita connection of the standard metric $\overline{g}$ on $S^n$. Moreover, the second fundamental form is given by

$$h_{ij} = \overline{\nabla}_i \overline{\nabla}_j u + u\overline{g}_{ij} \quad \text{on} \quad S^n. \tag{3.2.3}$$

On the other hand, the standard metric $\overline{g}$ on $S^n$ can be written as $\overline{g}_{ij} = h_{ik} g^{kl} h_{lj}$ which, together with (3.2.3), implies

$$K = \frac{\det(\overline{g}_{ij})}{\det(\overline{\nabla}_i \overline{\nabla}_j u + u\overline{g}_{ij})}. \tag{3.2.4}$$

We refer the reader to [47] and [75] for the details concerning support function.

Given a one-parameter family of strictly convex closed hypersurface $\Sigma_t$, let $u(\cdot,t)$ be the support function of $\Sigma_t$ for each $t$. We also denote by $\varphi$ the support function of an obstacle $\mathbf{\Phi}$ which is strictly convex closed hypersurface. Then we can restate (GFo) in terms of support function as follows:

$$\begin{aligned}
&-u_t \le \left(\frac{\det(\overline{g}_{ij})}{\det(\overline{\nabla}_i\overline{\nabla}_j u + u\overline{g}_{ij})}\right)^{\alpha} \quad \text{in} \quad S^n \times [0,T),\\
&-u_t = \left(\frac{\det(\overline{g}_{ij})}{\det(\overline{\nabla}_i\overline{\nabla}_j u + u\overline{g}_{ij})}\right)^{\alpha} \quad \text{if} \quad u > \varphi,\\
&\quad u \ge \varphi \quad \text{in} \quad S^n \times [0,T).
\end{aligned} \tag{GFo$_s$}$$

It is also equivalent to the equation

$$\min\left\{u_t + \left(\frac{\det(\overline{g}_{ij})}{\det(\overline{\nabla}_i\overline{\nabla}_j u + u\overline{g}_{ij})}\right)^{\alpha}, u - \varphi\right\} = 0$$

of degenerate type.

### 3.2.2 Obstacle

We denote the strictly convex closed hypersurface $\mathbf{\Phi}$ by the obstacle. For convenience, we parametrize the obstacle by $\mathbf{\Phi}^t = \vec{\nu}_{\mathbf{\Phi}}^{-1}(\vec{\nu}(x,t))$ for each $t$, where $\vec{\nu}_{\mathbf{\Phi}}$ is the outer unit normal vector of $\mathbf{\Phi}$ and $\vec{\nu}(\cdot,t)$ is that of $\Sigma_t$. Thus the obstacle $\mathbf{\Phi}^t : M \to \mathbb{R}^{n+1}$ and the hypersurface $\Sigma_t$ has the same normal at any $x \in M$.

### 3.2.3 Free boundary

Now let us define free boundary $\Gamma$, the non-coincidence set $\Omega$, and the coincidence set $\Lambda$ for the support function $u$ and $\varphi$ as follows:

$$\begin{aligned}
\Omega &= \{(z,t) \in S^n \times [0,T) : u(z,t) > \varphi(z)\},\\
\Lambda &= \{(z,t) \in S^n \times [0,T) : u(z,t) = \varphi(z)\},
\end{aligned} \tag{3.2.5}$$

and

$$\Gamma = \partial\Omega \cap \partial\Lambda. \tag{3.2.6}$$

We also define $\Gamma_t$, $\Omega_t$, and $\Lambda_t$ as the time section of $\Gamma$, $\Omega$, and $\Lambda$, respectively.

## 3.3 Singular perturbation problem

In this and the next section, we shall consider the singular perturbation problem (3.3.2) below. The short-time existence and evolution equation are established here, and then we prove the several uniform bounds for these approximations in the next section.

In our obstacle problem (GFo), the evolving hypersurface cannot pass the obstacle and satisfies the partial differential equation only on the non-coincidence set $\Lambda$ which is unknown before we obtain $\vec{X}$. To solve this difficulty, we will consider the penalized problem which is approximated solution by allowing the hypersurface can pass the obstacle.

Let $\beta$ be the smooth function defined on $\mathbb{R}$ and satisfying

$$\begin{aligned} \beta(0) &= -1, & & \\ \beta(x) &= 0 & &\text{if } x \geq 1, \\ \beta''(x) &= 0 & &\text{if } x < 0, \\ \beta'(x) &\geq 0,\ \beta''(x) \leq 0, & &\text{for all } x \in \mathbb{R}, \end{aligned}$$

and let $K_{\mathbf{\Phi}}$ be the Gauss curvature of the obstacle $\mathbf{\Phi}$. We define the penalty term to be

$$\beta_\delta(x) = A_0 \beta(x/\delta), \tag{3.3.1}$$

where $A_0 = \sup_{\mathbf{\Phi}} K_{\mathbf{\Phi}}^{\alpha} + 1$. Then it is easy to check that $\beta_\delta(x) \leq 0$, $\beta_\delta'(x) \geq 0$, $\beta_\delta''(x) \leq 0$, and $\beta_\delta(0) = -A_0 < -\sup_{\mathbf{\Phi}} K_{\mathbf{\Phi}}^{\alpha}$. Moreover, $\lim_{\delta\to 0} \beta_\delta(x) \to 0$ for $x > 0$ and $\lim_{\delta\to 0} \beta_\delta(x) \to -\infty$ for $x < 0$.

Now let us consider the following penalized problem which approximates (GFo): Given an initial hypersurface $\Sigma_0$ and its immersion $\vec{X}_0 : M^n \to \mathbb{R}^{n+1}$,

$$\begin{aligned}
\frac{\partial}{\partial t}\vec{X}^\delta(x,t) &= -\left[K^\alpha(x,t) + \beta_\delta\left(\left\langle \vec{X}^\delta(x,t) - \mathbf{\Phi}^t(x), \vec{\nu}(x,t)\right\rangle\right)\right]\vec{\nu}(x,t), \\
\vec{X}^\delta(x,0) &= \vec{X}_0(x),
\end{aligned} \tag{3.3.2}$$

for $x \in M$ and $t \in [0,T)$.

The short time existence and the evolution equation will be discussed in the following subsections. Before proceeding further, it is convenient to rewrite (3.3.2) in terms of support function as follows:

$$\begin{aligned}
-\frac{\partial}{\partial t}u^\delta(z,t) &= \left(\frac{\det(\overline{g}_{ij})}{\det(\overline{\nabla}_i\overline{\nabla}_j u^\delta + u^\delta \overline{g}_{ij})}\right)^\alpha + \beta_\delta(u^\delta - \varphi), \\
u^\delta(z,0) &= u_0(z),
\end{aligned} \tag{3.3.3}$$

for $z \in S^n$ and $t \in [0,T)$, where $u_0$ is the support function of $\vec{X}_0$.

### 3.3.1 Short-time existence

We use an existence theorem of Hamilton [37], as in [20], to prove the short-time existence of (3.3.2). The Hamilton's existence theorem is based on the Nash-Moser inverse function theorem.

To do this, we need to compute the principal symbol of the right hand side of (3.3.2), which is obtained by taking the highest order derivatives and replacing $\partial/\partial x^i$ by the Fourier transform variable $\xi_i$ (see [37]). However, since the penalized problem (3.3.2) is only lower order perturbation from the Gauss curvature flow, the desired principal symbol is equal to that of Gauss curvature flow. Note that the proof of Theorem 2.1 in [20] depends only on the principal symbol and integrability condition. Therefore, by taking the same integrability condition, we can obtain the following short time existence result.

**Lemma 3.3.1.** *For any $\alpha > 0$ and dimension $n$, let $\vec{X}_0$ be a smooth strictly convex hypersurface immersion of $M^n$ into $R^{n+1}$. Then there exists a positive $\omega$ such that* (3.3.2) *has a unique smooth solution $\vec{X}^\delta(\cdot, t)$ on $M^n \times [0, \omega)$. Here, $\omega$ may depend on $\vec{X}_0$.*

## 3.3.2 Evolution equations

Under the penalized flow (3.3.2), we can obtain the evolution formula for the geometric quantity of the hypersurface $\Sigma_t$. We denote by $\Box$ the operator $\alpha K^\alpha (h^{-1})^{kl} \nabla_k \nabla_l$. For notational convenience, we will refer to $\beta_\delta \left( \left\langle \vec{X}^\delta - \mathbf{\Phi}, \vec{\nu} \right\rangle \right)$ simply as $\beta_\delta$.

**Lemma 3.3.2.** *Under the flow* (3.3.2)*, the geometric quantities evolve according to*

*(i)* $$\frac{\partial g_{ij}}{\partial t} = -2 \left( K^\alpha + \beta_\delta \right) h_{ij},$$

*(ii)* $$\frac{\partial \vec{\nu}}{\partial t} = \nabla^j \left( K^\alpha + \beta_\delta \right) \frac{\partial \vec{X}}{\partial x^j},$$

*(iii)* $$\begin{aligned} \frac{\partial h_{ij}}{\partial t} &= \nabla_i \nabla_j (K^\alpha + \beta_\delta) - (K^\alpha + \beta_\delta) h_{jk} h_i^k \\ &= \Box h_{ij} + \alpha^2 K^\alpha (h^{-1})^{kl} (h^{-1})^{mn} \nabla_i h_{kl} \nabla_j h_{mn} \\ &\quad - \alpha K^\alpha (h^{-1})^{km} (h^{-1})^{ln} \nabla_i h_{kl} \nabla_j h_{mn} \\ &\quad + \alpha K^\alpha H h_{ij} - (n\alpha + 1) K^\alpha h_{jk} h_i^k + \nabla_i \nabla_j \beta_\delta - \beta_\delta h_{jk} h_i^k, \end{aligned}$$

*(iv)* $$\begin{aligned} \frac{\partial K}{\partial t} &= \Box K + \alpha(\alpha - 1) K^{\alpha - 1} (h^{-1})^{ij} \nabla_i K \nabla_j K + K^{\alpha+1} H \\ &\quad + K (h^{-1})^{ij} \nabla_i \nabla_j \beta_\delta + K^\alpha H \beta_\delta, \end{aligned}$$

*(v)* $$\frac{\partial K^\alpha}{\partial t} = \Box K^\alpha + \alpha K^{2\alpha} H + \Box \beta_\delta + \alpha K^\alpha H \beta_\delta,$$

*(vi)* $$\frac{\partial H}{\partial t} = \Box H + \alpha^2 g^{ij} K^\alpha (h^{-1})^{kl}(h^{-1})^{mn}\nabla_i h_{kl}\nabla_j h_{mn} - \alpha g^{ij}K^\alpha (h^{-1})^{km}(h^{-1})^{ln}\nabla_i h_{kl}\nabla_j h_{mn} + \alpha K^\alpha H^2 + (1-n\alpha)K^\alpha|A|^2 + \Delta\beta_\delta + |A|^2\beta_\delta,$$

*(vii)* $$\frac{\partial|\vec{X}|^2}{\partial t} = \Box|\vec{X}|^2 - 2\alpha K^\alpha \mathcal{H} + 2(n\alpha-1)K^\alpha\langle\vec{X},\vec{\nu}\rangle - 2\beta_\delta\langle\vec{X},\vec{\nu}\rangle,$$

*Proof.* For simplicity, we define $F_\delta = K^\alpha + \beta_\delta$.

(i) Since $\left\langle \vec{\nu}, \frac{\partial\vec{X}}{\partial x^i}\right\rangle = 0$, we have

$$\frac{\partial g_{ij}}{\partial t} = \frac{\partial}{\partial t}\left\langle \frac{\partial\vec{X}}{\partial x^i}, \frac{\partial\vec{X}}{\partial x^j}\right\rangle = \left\langle \frac{\partial}{\partial x^i}(-F_\delta\vec{\nu}), \frac{\partial\vec{X}}{\partial x^j}\right\rangle + \left\langle \frac{\partial\vec{X}}{\partial x^i}, \frac{\partial}{\partial x^j}(-F_\delta\vec{\nu})\right\rangle$$
$$= -F_\delta\left\langle \frac{\partial\vec{\nu}}{\partial x^i}, \frac{\partial\vec{X}}{\partial x^j}\right\rangle - F_\delta\left\langle \frac{\partial\vec{X}}{\partial x^i}, \frac{\partial\vec{\nu}}{\partial x^j}\right\rangle = 2F_\delta\left\langle \vec{\nu}, \frac{\partial^2\vec{X}}{\partial x^i\partial x^j}\right\rangle = -2F_\delta h_{ij}.$$

(ii) From $\langle\vec{\nu},\vec{\nu}\rangle = 1$, we obtain $\left\langle \frac{\partial\vec{\nu}}{\partial t}, \vec{\nu}\right\rangle = 0$ and $\left\langle \frac{\partial\vec{\nu}}{\partial x^i}, \vec{\nu}\right\rangle = 0$. Thus,

$$\frac{\partial\vec{\nu}}{\partial t} = \left\langle \frac{\partial\vec{\nu}}{\partial t}, \frac{\partial\vec{X}}{\partial x^i}\right\rangle g^{ij}\frac{\partial\vec{X}}{\partial x^j} = -\left\langle \vec{\nu}, \frac{\partial(-F_\delta\vec{\nu})}{\partial x^i}\right\rangle g^{ij}\frac{\partial\vec{X}}{\partial x^j} = \frac{\partial F_\delta}{\partial x^i} g^{ij}\frac{\partial\vec{X}}{\partial x^j}.$$

(iii) By the same argument as in (ii), we have

$$\frac{\partial\vec{\nu}}{\partial x^j} = h_{jk}g^{kl}\frac{\partial\vec{X}}{\partial x^l}.$$

This implies that

$$
\begin{aligned}
\frac{\partial h_{ij}}{\partial t} &= \frac{\partial}{\partial t}\left\langle \frac{\partial^2 \vec{X}}{\partial x^i \partial x^j}, -\vec{\nu} \right\rangle = \left\langle \frac{\partial^2(-F_\delta \vec{\nu})}{\partial x^i \partial x^j}, -\vec{\nu} \right\rangle - \left\langle \frac{\partial^2 \vec{X}}{\partial x^i \partial x^j}, \frac{\partial \vec{\nu}}{\partial t} \right\rangle \\
&= \left\langle \frac{\partial}{\partial x^i}\left( \frac{\partial F_\delta}{\partial x^j}\vec{\nu} + F_\delta \frac{\partial \vec{\nu}}{\partial x^j} \right), \vec{\nu} \right\rangle - \left\langle \Gamma_{ij}^k \frac{\partial \vec{X}}{\partial x^k} - h_{ij}\vec{\nu}, \frac{\partial F_\delta}{\partial x^m} g^{mn} \frac{\partial \vec{X}}{\partial x^n} \right\rangle \\
&= \frac{\partial^2 F_\delta}{\partial x^i \partial x^j} + F_\delta \left\langle \frac{\partial}{\partial x^i}\left( h_{jk} g^{kl} \frac{\partial \vec{X}}{\partial x^l} \right), \vec{\nu} \right\rangle - \Gamma_{ij}^k \frac{\partial F_\delta}{\partial x^m} g^{mn} g_{kn} \\
&= \frac{\partial^2 F_\delta}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial F_\delta}{\partial x^k} + F_\delta \left\langle h_{jk} g^{kl} \frac{\partial^2 \vec{X}}{\partial x^i \partial x^l}, \vec{\nu} \right\rangle \\
&= \nabla_i \nabla_j F_\delta - F_\delta h_{jk} h_i^k.
\end{aligned}
$$

For the second equality, we need the following computation.

$$
\begin{aligned}
(h^{-1})^{kl} \nabla_i \nabla_j h_{kl} &= (h^{-1})^{kl} \nabla_i \nabla_k h_{lj} \\
&= (h^{-1})^{kl} (\nabla_k \nabla_i h_{lj} + R_{iklm} h_j^m + R_{ikjm} h_l^m) \\
&= (h^{-1})^{kl} \nabla_k \nabla_l h_{ij} + (h^{-1})^{kl} (h_{il} h_{km} - h_{im} h_{kl}) h_j^m \\
&\quad + (h^{-1})^{kl} (h_{ij} h_{km} - h_{im} h_{kj}) h_l^m \\
&= (h^{-1})^{kl} \nabla_k \nabla_l h_{ij} + h_{im} h_j^m - n h_{im} h_j^m + H h_{ij} - h_{im} h_j^m \\
&= (h^{-1})^{kl} \nabla_k \nabla_l h_{ij} + H h_{ij} - n h_{im} h_j^m.
\end{aligned}
\tag{3.3.4}
$$

On the other hand,

$$
\begin{aligned}
\nabla_i \nabla_j K^\alpha &= \nabla_i \left( \alpha K^\alpha (h^{-1})^{mn} \nabla_j h_{mn} \right) \\
&= \alpha K^\alpha (h^{-1})^{mn} \nabla_i \nabla_j h_{mn} + \alpha^2 K^\alpha (h^{-1})^{kl} (h^{-1})^{mn} \nabla_i h_{kl} \nabla_j h_{mn} \\
&\quad - \alpha K^\alpha (h^{-1})^{km} (h^{-1})^{ln} \nabla_i h_{kl} \nabla_j h_{mn} \\
&= \Box h_{ij} + \alpha K^\alpha (H h_{ij} - n h_{im} h_j^m) \\
&\quad + \alpha^2 K^\alpha (h^{-1})^{kl} (h^{-1})^{mn} \nabla_i h_{kl} \nabla_j h_{mn} \\
&\quad - \alpha K^\alpha (h^{-1})^{km} (h^{-1})^{ln} \nabla_i h_{kl} \nabla_j h_{mn}.
\end{aligned}
$$

Then we get the evolution equation for $h_{ij}$.

$$\begin{aligned}\frac{\partial h_{ij}}{\partial t} &= \nabla_i\nabla_j F_\delta - F_\delta h_{jk}h_i^k \\ &= \nabla_i\nabla_j K^\alpha - K^\alpha h_{jk}h_i^k + \nabla_i\nabla_j\beta_\delta - \beta_\delta h_{jk}h_i^k \\ &= \Box h_{ij} + \alpha^2 K^\alpha (h^{-1})^{kl}(h^{-1})^{mn}\nabla_i h_{kl}\nabla_j h_{mn} \\ &\quad - \alpha K^\alpha (h^{-1})^{km}(h^{-1})^{ln}\nabla_i h_{kl}\nabla_j h_{mn} \\ &\quad + \alpha K^\alpha H h_{ij} - (\alpha n+1)K^\alpha h_{jk}h_i^k + \nabla_i\nabla_j\beta_\delta - \beta_\delta h_{jk}h_i^k.\end{aligned}$$

(iv) By using the previous result,

$$\begin{aligned}\frac{\partial K}{\partial t} &= \frac{\partial}{\partial t}\det(g^{ik}h_{kj}) = \frac{\partial}{\partial t}\frac{\det(h_{ij})}{\det(g_{ij})} = -Kg^{ij}\frac{\partial g_{ij}}{\partial t} + K(h^{-1})^{ij}\frac{\partial h_{ij}}{\partial t} \\ &= 2KHF_\delta + K(h^{-1})^{ij}(\nabla_i\nabla_j F_\delta - F_\delta h_{jk}h_i^k) \\ &= K(h^{-1})^{ij}\nabla_i\nabla_j F_\delta + KHF_\delta \\ &= K(h^{-1})^{ij}\nabla_i(\alpha K^{\alpha-1}\nabla_j K) + K^{1+\alpha}H + K(h^{-1})^{ij}\nabla_i\nabla_j\beta_\delta + KH\beta_\delta \\ &= \Box K + \alpha(\alpha-1)K^{\alpha-1}(h^{-1})^{ij}\nabla_i K\nabla_j K + K^{1+\alpha}H \\ &\quad + K(h^{-1})^{ij}\nabla_i\nabla_j\beta_\delta + KH\beta_\delta.\end{aligned}$$

(v) A direct computation shows $\dfrac{\partial K^\alpha}{\partial t} = \alpha K^{\alpha-1}\dfrac{\partial K}{\partial t} = \Box F_\delta + KHF_\delta$.

(vi)
$$
\begin{aligned}
\frac{\partial H}{\partial t} &= \frac{\partial}{\partial t}(g^{ij}h_{ij}) = -g^{ik}g^{lj}\frac{\partial g_{kl}}{\partial t}h_{ij} + g^{ij}\frac{\partial h_{ij}}{\partial t} \\
&= 2F_\delta g^{ik}g^{lj}h_{kl}h_{ij} \\
&\quad + g^{ij}\big(\Box h_{ij} + \alpha^2 K^\alpha (h^{-1})^{kl}(h^{-1})^{mn}\nabla_i h_{kl}\nabla_j h_{mn} \\
&\quad - \alpha K^\alpha (h^{-1})^{km}(h^{-1})^{ln}\nabla_i h_{kl}\nabla_j h_{mn} \\
&\quad + \alpha K^\alpha H h_{ij} - (\alpha n+1)K^\alpha h_{jk}h_i^k + \nabla_i\nabla_j\beta_\delta - \beta_\delta h_{jk}h_i^k\big) \\
&= 2F_\delta |A|^2 + \Box H + \alpha^2 g^{ij}K^\alpha (h^{-1})^{kl}(h^{-1})^{mn}\nabla_i h_{kl}\nabla_j h_{mn} \\
&\quad - \alpha g^{ij}K^\alpha (h^{-1})^{km}(h^{-1})^{ln}\nabla_i h_{kl}\nabla_j h_{mn} \\
&\quad + \alpha K^\alpha H^2 - (\alpha n+1)K^\alpha |A|^2 + \Delta\beta_\delta - |A|^2\beta_\delta \\
&= \Box H + \alpha^2 g^{ij}K^\alpha (h^{-1})^{kl}(h^{-1})^{mn}\nabla_i h_{kl}\nabla_j h_{mn} \\
&\quad - \alpha g^{ij}K^\alpha (h^{-1})^{km}(h^{-1})^{ln}\nabla_i h_{kl}\nabla_j h_{mn} \\
&\quad + \alpha K^\alpha H^2 - (\alpha n-1)K^\alpha |A|^2 + \Delta\beta_\delta + |A|^2\beta_\delta.
\end{aligned}
$$

(vii) Since

$$
\frac{\partial |\vec{X}|^2}{\partial t} = \frac{\partial}{\partial t}\left\langle \vec{X}, \vec{X}\right\rangle = 2\left\langle \vec{X}, \frac{\partial \vec{X}}{\partial t}\right\rangle = -2F_\delta \left\langle \vec{X}, \vec{\nu}\right\rangle,
$$

and

$$
\begin{aligned}
\Box |\vec{X}|^2 &= 2\alpha K^\alpha (h^{-1})^{kl}\left(\langle \nabla_k\nabla_l \vec{X}, \vec{X}\rangle + \langle \nabla_k \vec{X}, \nabla_l \vec{X}\rangle\right) \\
&= 2\alpha K^\alpha (h^{-1})^{kl}\left(\left\langle \frac{\partial \vec{X}}{\partial x^k \partial x^l} - \Gamma_{kl}^m \frac{\partial \vec{X}}{\partial x^m}, \vec{X}\right\rangle + g_{kl}\right) \\
&= 2\alpha K^\alpha (h^{-1})^{kl}\left(-h_{kl}\langle \vec{\nu}, \vec{X}\rangle + g_{kl}\right) \\
&= 2\alpha K^\alpha \mathcal{H} - 2\alpha n K^\alpha \left\langle \vec{X}, \vec{\nu}\right\rangle,
\end{aligned}
$$

we have

$$
\begin{aligned}
\frac{\partial |\vec{X}|^2}{\partial t} &= \Box |\vec{X}|^2 - 2\alpha K^\alpha \mathcal{H} + 2\alpha n K^\alpha \langle \vec{X}, \vec{\nu}\rangle - 2(K^\alpha + \beta_\delta)\langle \vec{X}, \vec{\nu}\rangle \\
&= \Box |\vec{X}|^2 - 2\alpha K^\alpha \mathcal{H} + 2(\alpha n - 1)K^\alpha \langle \vec{X}, \vec{\nu}\rangle - 2\beta_\delta \langle \vec{X}, \vec{\nu}\rangle.
\end{aligned}
$$

□

We also need the evolution equation in $z$-coordinate. As before, let $\Box_{S^n}$ be the operator $\alpha K^\alpha (h^{-1})^{ij}\overline{\nabla}_i\overline{\nabla}_j$.

**Lemma 3.3.3.** *Under the flow* (3.3.3)*, the geometric quantities evolve according to*

*(i)* $\dfrac{\partial \overline{g}_{ij}}{\partial t} = 0, \quad \overline{\nabla}_k \overline{g}_{ij} = 0,$

*(ii)* $u_t = \Box_{S^n} u + \alpha K^\alpha H u - (n\alpha+1)K^\alpha - \beta_\delta,$

*(iii)* $\dfrac{\partial h_{ij}}{\partial t} = -\overline{\nabla}_i\overline{\nabla}_j F_\delta - F_\delta \overline{g}_{ij},$

*(iv)* $\dfrac{\partial K^\alpha}{\partial t} = \Box_{S^n} F_\delta + \alpha K^\alpha H F_\delta,$

*(v)* $\dfrac{\partial \varphi}{\partial t} = \Box_{S^n}\varphi + \alpha K^\alpha H\varphi - \alpha K^\alpha (h^{-1})^{ij} h^{\mathbf{\Phi}}_{ij}$

*where* $F_\delta = K^\alpha + \beta_\delta$ *and* $h^{\mathbf{\Phi}}$ *is the second fundamental form of* $\mathbf{\Phi}$*.*

*Proof.* The first three assertions follow from (3.2.3). By (3.2.4), we have $\partial_t K^\alpha = -\alpha K^\alpha (h^{-1})^{ij}(h_{ij})_t$. This implies the next assertion. For the last equation, we compute

$$\Box_{S^n}\varphi = \alpha K^\alpha (h^{-1})^{ij}\overline{\nabla}_i\overline{\nabla}_j\varphi = \alpha K^\alpha (h^{-1})^{ij}\left(\left\langle \mathbf{\Phi}, \overline{\nabla}_i\overline{\nabla}_j z\right\rangle - \left\langle \overline{\nabla}_i\overline{\nabla}_j \mathbf{\Phi}, z\right\rangle\right) \tag{3.3.5}$$

$$= \alpha K^\alpha (h^{-1})^{ij}\left(-\overline{g}_{ij}\varphi + h^{\mathbf{\Phi}}_{ij}\right) = -\alpha K^\alpha H\varphi + \alpha K^\alpha (h^{-1})^{ij} h^{\mathbf{\Phi}}_{ij}. \tag{3.3.6}$$

Then $\partial_t \varphi = 0$ yields the conclusion. □

## 3.4 Uniform upper bound for principal curvature

In this section, we are devoted to deriving the uniform upper estimates independent of $\delta$ for Gauss curvature and mean curvature. These estimates will give us a sufficient controls on principal curvature. In fact, if we have $K > 0$ and $H \leq C$, then every principal curvature satisfies

$$0 < \lambda_i \leq H \leq C. \tag{3.4.1}$$

Before we proceed further, we need to show that the solution of penalized problem cannot touch the obstacle and the velocity vector is inward-pointing. For notational convenience, we omit the index $\delta$ from all the geometric quantity, such as $K^\delta$, $H^\delta$, etc., throughout this section.

**Lemma 3.4.1.** *Let $u$ be the solution of* (3.3.3) *in $S^2 \times [0, T)$. Then*

*(i)* $u(z,t) > \varphi(z)$,

*(ii)* $|\beta_\delta(u(z,t) - \varphi(z))| \leq C$,

*(iii)* $K(z,t)^\alpha + \beta_\delta\left(u(z,t) - \varphi(z)\right) > 0$

*for all $(z,t) \in S^2 \times [0,T)$.*

*Proof.* Assume that $\vec{X}$ touches $\mathbf{\Phi}$ for the first time $t_1$ at $x_1$ and let $z_1 = \vec{\nu}(x_1, t_1)$. Clearly $t_1 > 0$. By definition of $(z_1, t_1)$, we have

$$u(z,t) \geq \varphi(z) \tag{3.4.2}$$

for all points $z \in S^2$ and all $t \in [0, t_1]$, with equality for $z = z_1$ and $t = t_1$.This implies

$$\frac{\partial}{\partial t}\left(u(z_1,t) - \varphi(z_1)\right)\Big|_{t=t_1} \leq 0, \tag{3.4.3}$$

hence

$$-K^\alpha - \beta_\delta(u-\varphi) \leq 0 \tag{3.4.4}$$

at $(z_1, t_1)$. Moreover, it also follows from (3.4.2) and (3.2.4) that $K(x_1, t_1) \leq K_{\mathbf{\Phi}}(x_1, t_1)$. Putting these facts together, we obtain

$$0 \leq K^\alpha + \beta_\delta(u-\varphi) \leq K^\alpha_{\mathbf{\Phi}} + \beta_\delta(0) \tag{3.4.5}$$

at $(x_1, t_1)$. This contradicts the fact that $\beta_\delta(0) = -\sup K^\alpha_{\mathbf{\Phi}} - 1$. Consequently, $\vec{X}$ cannot touch $\mathbf{\Phi}$.

We now prove the third assertion. We define

$$A = \{s \in [0,T) | u_t(z,t) < 0 \ for\ z \in S^2 \ and\ t \in [0,s]\}. \tag{3.4.6}$$

Since the initial hypersurface is strictly convex, $0 \in A$. Assume that $s^* = \sup A < T$. Let $Z(t) = \min_{S^n}(-u_t)$. By differentiating (3.3.3), we obtain the evolution equation

$$Z_t \geq (\alpha K^\alpha H - \beta'_\delta) Z,$$

which gives $Z(t) \geq Z(0) e^{\int (\alpha K^\alpha H - \beta'_\delta) dt}$. By continuity, $Z(s^*) > 0$ which is contradict to the definition of $s^*$. This completes the proof. □

We remark that $\beta_\delta$ is bounded independent of $\delta$ since

$$-(\max K^\alpha_{\mathbf{\Phi}} + 1) = \beta_\delta(0) \leq \beta_\delta(u(z,t) - \varphi(z)) \leq 0. \tag{3.4.7}$$

Using this with the following lemma, we can prove the uniform upper bound of Gauss curvature $K$.

**Lemma 3.4.2.** *For any $\alpha > 0$ and dimension $n$, let $u$ be the smooth solution of* (3.3.3) *in $S^n \times [0,T)$. Then there exists a constant $C = C(\alpha, n, \mathbf{\Phi}, \vec{X}_0)$,* *independent of* $\delta$*, such that*

$$0 < F_\delta(z,t) \le C \quad \text{in } S^n \times [0,T)$$

*where* $F_\delta(z,t) = K(z,t)^\alpha + \beta_\delta\left(u(z,t) - \varphi(z)\right)$.

*Proof.* The assertion $F_\delta(z,t) > 0$ follows from Lemma 3.4.1. To prove the uniform upper bound, we use a Tso's trick [72] as in [75]. Let us consider

$$w(z,t) = \frac{F_\delta(z,t)}{u(z,t) - \rho_0} \tag{3.4.8}$$

on $S^n \times [0,T)$, where $u$ is the support function of $\vec{X}$ and $\rho_0 = \dfrac{1}{2}\inf_{S^n} \varphi$. Then, using (i), the denominator remains positive.
We claim that for any $(z,t) \in S^n \times [0,T)$

$$w(z,t) \le \max\left\{ \frac{1}{\rho_0} \sup_{z \in S^n} K(z,0), \frac{(\alpha n + 1)^n}{(\alpha n)^n \rho_0^{n+1}} \right\}. \tag{3.4.9}$$

To prove this, let us consider any time $t_0 \in (0,T)$ and assume that $w$ attains its maximum over $S^n \times [0,t_0]$ at some point $(z_1,t_1)$. If $t_1 = 0$, we have

$$\sup_{(z,t) \in S^2 \times [0,t_0]} w(z,t) \le \frac{F_\delta(z_1,0)}{u(z,0) - \rho_0} \le \frac{1}{\rho_0} \sup_{z \in S^2} K(z,0).$$

Consequently we may assume $t_1 > 0$ and we know that at the maximum point $(z_1,t_1)$ of $w$,

$$w_t \le 0, \quad \overline{\nabla} w = 0, \quad \text{and} \quad \overline{\nabla}^2 w \ge 0. \tag{3.4.10}$$

By Lemma 2.2.1, the evolution equation of $w$ is given by

$$(\partial_t - \Box_{S^n}) w = \frac{(\partial_t - \Box_{S^n}) F_\delta}{u - \rho_0} - \frac{F_\delta (\partial_t - \Box_{S^n})(u - \rho_0)}{(u-\rho_0)^2} + \frac{2\left\langle \overline{\nabla}(u-\rho_0), \overline{\nabla} w \right\rangle_{\Box_{S^n}}}{u - \rho_0},$$

and then, together with (3.4.10), we have

$$0 \le (u-\rho_0)(\partial_t - \Box_{S^n})F_\delta - F_\delta(\partial_t - \Box_{S^n})(u-\rho_0).$$

From the evolution equation of $K^\alpha$ in Lemma 3.3.3 and (3.2.3), we obtain

$$\begin{aligned}0 &\le (u-\rho_0)(\partial_t\beta_\delta + \alpha K^\alpha H F_\delta) + F_\delta(F_\delta + \alpha K^\alpha (h^{-1})^{ij}\overline{\nabla}_i\overline{\nabla}_j u)\\ &= (u-\rho_0)(\beta_\delta' u_t + \alpha K^\alpha H F_\delta) + F_\delta(F_\delta + \alpha K^\alpha (h^{-1})^{ij}(h_{ij} - u\overline{g}_{ij}))\\ &= -(u-\rho_0)\beta_\delta' F_\delta + \alpha(u-\rho_0)K^\alpha H F_\delta + (K^\alpha + \beta_\delta)F_\delta + \alpha K^\alpha F_\delta(n - uH).\end{aligned}$$

Since $\beta_\delta \le 0$ and $\beta_\delta' \ge 0$, we have

$$0 \le K^\alpha F_\delta(\alpha n + 1 - \alpha H\rho_0).$$

Therefore

$$H \le \frac{\alpha n + 1}{\alpha\rho_0},$$

and so

$$\sup_{S^n\times[0,t_0]} w \le \frac{K}{\rho_0} \le \frac{1}{\rho_0}\left(\frac{H}{n}\right)^n \le \left(\frac{\alpha n+1}{\alpha n}\right)^n \rho_0^{-(n+1)}. \tag{3.4.11}$$

Since the right hand side of (3.4.11) is independent of $t_0$, we have (3.4.9). Now using (3.4.8) and (3.4.9), we obtain

$$F_\delta(x,t) = (u(x,t)-\rho_0)w(x,t) \le \sup_{S^2}(u_0(x)-\rho_0)\sup_{S^2\times[0,T)} w(x,t).$$

This bound, together with (3.4.9), completes the proof. □

Lemma 3.4.2 has the following immediate consequence.

**Corollary 3.4.3.** *For any $\alpha > 0$ and any dimension $n$, the Gauss curvature $K$ of the solution of 3.3.2 satisfies*

$$0 < K \le \tilde{C},$$

*where* $\tilde{C} = (C + \sup_{\mathbf{\Phi}} K_{\mathbf{\Phi}} + 1)^{\frac{1}{\alpha}}$ *and* $C$ *is the constant in Lemma 3.4.2.*

*Proof.* By Lemma 3.4.1, we know that $-\sup_{\mathbf{\Phi}} K_{\mathbf{\Phi}} - 1 \leq \beta_\delta(u - \varphi) \leq 0$. This gives the desired result. □

We are now ready to prove the uniform upper bound for mean curvature. Our proof generalizes the quantity used in [3]. When $\alpha = 1$ and $n = 2$, we employ the same quantity but we have to control the effect form the obstacle $\beta_\delta$. For $0 < \alpha < 1$ and $n = 2$, we put an exponent in the denominator.

**Lemma 3.4.4.** *For* $0 < \alpha \leq 1$ *and the dimension* $n = 2$, *let* $\vec{X}$ *be the smooth solution of* (3.3.2) *in* $M \times [0, T)$. *Then we have the estimate*

$$\sup_{M \times [0,T)} H \leq C$$

*for all* $\delta > 0$, *the constant* $C$ *depending only on* $\alpha$, $\mathbf{\Phi}$ *and* $\vec{X}_0$.

*Proof.* By the evolution equation of the $|\vec{X}|^2$ in Lemma 3.3.2, the function $|\vec{X}|$ is decreasing in time. Choose the origin such that $\vec{X}_0$ is contained in a ball of radius $R$ about the origin. Then $D := 2R^2 - |\vec{X}|^2 > 0$. Now, we can consider

$$S = \frac{H}{D^\gamma}$$

where $\gamma$ will be chosen later. Using Lemma 2.2.1 and the evolution equation of $H$ and $|\vec{X}|^2$, we obtain

$$\begin{aligned}(\partial_t - \Box)S = &\frac{1}{D^\gamma}\left(g^{ij}\ddot{K}(\nabla_i h, \nabla_j h) + \alpha K^\alpha H^2 + (1 - n\alpha)K^\alpha |A|^2 + \Delta\beta_\delta + |A|^2\beta_\delta\right)\\ &+ \frac{\gamma H}{D^{\gamma+1}}\left(-2\alpha K^\alpha \mathcal{H} + 2(\alpha n - 1)K^\alpha \left\langle \vec{X}, \vec{\nu}\right\rangle - 2\beta_\delta\langle \vec{X}, \vec{\nu}\rangle\right)\\ &- \frac{2\gamma}{D}\left\langle \nabla|\vec{X}|^2, \nabla S\right\rangle_\Box + \frac{\gamma(\gamma - 1)H}{D^{\gamma+2}}\left\|\nabla|\vec{X}|^2\right\|_\Box^2.\end{aligned}$$

As in Lemma 3.4.2, given $t_0 \in (0, T)$, let $(x_1, t_1)$ achieve the maximum of $S$ over $M \times [0, t_0]$. We may assume $t_1 > 0$. Then, at this maximum point

$(x_1, t_1)$, we have

$$\begin{aligned} 0 \leq & \frac{1}{D^\gamma}\left(g^{ij}\ddot{K}(\nabla_i h, \nabla_j h) + \alpha K^\alpha H^2 + (1-2\alpha)K^\alpha |A|^2\right) \\ & + \frac{\gamma H}{D^{\gamma+1}}\left(-2\alpha K^\alpha \mathcal{H} + 2(2\alpha-1)K^\alpha \left\langle \vec{X}, \vec{\nu}\right\rangle\right) + \frac{\gamma(\gamma-1)H}{D^{\gamma+2}}\left\|\nabla|\vec{X}|^2\right\|_\square^2 \\ & + \frac{\Delta\beta_\delta + (|A|^2 - 2\gamma D^{-1}\left\langle \vec{X}, \vec{\nu}\right\rangle H)\beta_\delta}{D^\gamma}. \end{aligned} \tag{3.4.12}$$

First we will estimate the penalty terms. By a direct computation, we can check that

$$\Delta\beta_\delta = \beta_\delta''\left|\nabla\left\langle \vec{X}(x,t) - \mathbf{\Phi}^t(x), \vec{\nu}(x,t)\right\rangle\right|^2 + \beta_\delta'\Delta\left\langle \vec{X} - \mathbf{\Phi}, \vec{\nu}\right\rangle.$$

The first term on the right is nonpositive since $\beta_\delta'' \leq 0$. For the remaining term, we observe that

$$\Delta\vec{\nu} = \nabla^k H \frac{\partial \vec{X}}{\partial x^k} - |A|^2\vec{\nu}.$$

From this, we have that

$$\begin{aligned} \Delta\left\langle \vec{X} - \mathbf{\Phi}, \vec{\nu}\right\rangle &= g^{ij}\nabla_i\nabla_j\left\langle \vec{X} - \mathbf{\Phi}, \vec{\nu}\right\rangle \\ &= g^{ij}\nabla_i\left\langle \vec{X} - \mathbf{\Phi}, \nabla_j\vec{\nu}\right\rangle \\ &= g^{ij}\left\langle \nabla_i(\vec{X} - \mathbf{\Phi}), \nabla_j\vec{\nu}\right\rangle + \left\langle \vec{X} - \mathbf{\Phi}, \Delta\vec{\nu}\right\rangle \\ &= -\left\langle \Delta(\vec{X} - \mathbf{\Phi}), \vec{\nu}\right\rangle + \left\langle \vec{X} - \mathbf{\Phi}, \nabla^k H\frac{\partial \vec{X}}{\partial x^k} - |A|^2\vec{\nu}\right\rangle \\ &= H - g^{ij}h_{ij}^{\mathbf{\Phi}} + \nabla^k H\left\langle \vec{X} - \mathbf{\Phi}, \frac{\partial \vec{X}}{\partial x^k}\right\rangle - |A|^2\left\langle \vec{X} - \mathbf{\Phi}, \vec{\nu}\right\rangle \end{aligned}$$

At the point $(x_1, t_1)$, we know

$$\nabla^k H = -\frac{\gamma H \nabla^k |\vec{X}|^2}{2R^2 - |\vec{X}|^2},$$

and then

$$\begin{aligned}\Delta \left\langle \vec{X} - \mathbf{\Phi}, \vec{\nu} \right\rangle \leq & \left(1 + \frac{2\gamma}{2R^2 - |\vec{X}|^2} |\vec{X} - \mathbf{\Phi}||\vec{X}|\right) H - \frac{|A|^2}{\mu_{max}} \\ \leq & (1 + 2\gamma) H - \frac{H^2}{2\mu_{\max}}\end{aligned}$$

where $\mu_{\max}$ is the maximum principal curvature of the obstacle. Therefore, if we have

$$H(x_1, t_1) \geq \max\left(2(1 + 2\gamma)\mu_{\max}, \frac{4\gamma}{R}\right),$$

then

$$\frac{\Delta\beta_\delta + (|A|^2 - 2\gamma D^{-1} \left\langle \vec{X}, \vec{\nu} \right\rangle H)\beta_\delta}{D^\gamma} \leq 0.$$

Otherwise, the quantity $S$ has uniform upper bound from the inequality $S \leq (2R^2)^\gamma H$ and the fact that $\gamma$ will be chosen dependent on $\alpha$, $\mathbf{\Phi}$, and $\vec{X}_0$.

Before we dealing with the higher order derivative term, let us recall its expression under the normal coordinate. In the coordinate system, the following formula holds:

$$\begin{aligned}g^{ij} \ddot{K}(\nabla_i h, \nabla_j h) = & \alpha^2 K^\alpha (h^{-1})^{kk} (h^{-1})^{mm} \nabla_i h_{kk} \nabla_i h_{mm} \\ & - \alpha K^\alpha (h^{-1})^{kk} (h^{-1})^{ll} (\nabla_i h_{kl})^2\end{aligned} \tag{3.4.13}$$

To estimate $g^{ij}\ddot{K}(\nabla_i h, \nabla_j h)$, we divide into two cases:

(i) $0 < \alpha < 1$;

(ii) $\alpha = 1$.

For the first case, we could show that

$$g^{ij}\ddot{K}(\nabla_i h, \nabla_j h) \leq \sum_{i=1}^{2} \frac{\alpha\gamma^2 D^{-2}|\nabla_i|\vec{X}|^2|^2 K^\alpha(2\alpha - 1 - 2(1-\alpha)K\lambda_i^{-2})}{4\alpha K + (1-\alpha)H^2} H^2. \tag{3.4.14}$$

at the maximum point $(x_1, t_1)$. In fact, using (3.4.13), we have

$$\begin{aligned} g^{ij}\ddot{K}(\nabla_i h, \nabla_j h) = {} & \alpha(\alpha-1)K^\alpha \lambda_1^{-2}\nabla_1 h_{11}^2 + \alpha(\alpha-1)K^\alpha\lambda_2^{-2}\nabla_2 h_{22}^2 \\ & + 2\alpha^2 K^{\alpha-1}\nabla_1 h_{11}\nabla_1 h_{22} + 2\alpha^2 K^{\alpha-1}\nabla_2 h_{11}\nabla_2 h_{22} \\ & + (-2\alpha K^{\alpha-1} + \alpha(\alpha-1)K^\alpha\lambda_1^{-2})\nabla_2 h_{11}^2 \\ & + (-2\alpha K^{\alpha-1} + \alpha(\alpha-1)K^\alpha\lambda_2^{-2})\nabla_1 h_{22}^2. \end{aligned} \tag{3.4.15}$$

under the normal coordinate system. Moreover, since the point $x_1$ realizes the maximum of $S(\cdot, t)$, we obtain

$$\nabla_i h_{11} + \nabla_i h_{22} = -\frac{\gamma H \nabla_i |\vec{X}|^2}{D} \tag{3.4.16}$$

for $i = 1, 2$. For a moment, we denote the quantity $4\alpha K + (1-\alpha)H^2$ by $Q$. We substitute (3.4.16) to (3.4.15) so that the right hand side of (3.4.15) is

$$\begin{aligned} & -\alpha K^{\alpha-2} Q \sum_{i\neq j} \left( \nabla_i h_{ii} + \frac{\alpha\gamma D^{-1}\nabla_i|\vec{X}|^2 H K^{\alpha-1}(\alpha + 2 + (1-\alpha)K\lambda_j^{-2})}{\alpha K^{\alpha-2} Q} \right)^2 \\ & + \sum_{i=1}^{2} \frac{\alpha\gamma^2 D^{-2}|\nabla_i|\vec{X}|^2|^2 K^\alpha(2\alpha - 1 - 2(1-\alpha)K\lambda_i^{-2})}{Q} H^2. \end{aligned}$$

This implies the assertion.

Then (3.4.12) becomes

$$
\begin{aligned}
0 \leq &\sum_{i=1}^{2} \frac{\alpha\gamma^2 D^{-\gamma-2}|\nabla_i|\vec{X}|^2|^2 K^\alpha(2\alpha-1-2(1-\alpha)K\lambda_i^{-2})}{Q} H^2 \\
&+ D^{-\gamma}((1-\alpha)K^\alpha H^2 - 2(1-2\alpha)K^{\alpha+1}) - 2\alpha\gamma D^{-\gamma-1}K^{\alpha-1}H^2 \\
&+ 2(2\alpha-1)\gamma D^{-\gamma-1}K^\alpha H \left\langle \vec{X}, \vec{\nu} \right\rangle + \sum_{i=1}^{2} \alpha\gamma(\gamma-1)K^\alpha H D^{-\gamma-2}\lambda_i^{-1}|\nabla_i|\vec{X}|^2|^2.
\end{aligned}
\tag{3.4.17}
$$

Now we choose

$$
\gamma = \frac{2(1-\alpha)}{\alpha} K_{max} R^2
$$

so that we obtain

$$
D^{-\gamma}K^{\alpha-1}H^2((1-\alpha)K - 2\alpha\gamma D^{-1}) \leq -(1-\alpha)D^{-\gamma}K^{\alpha-1}H^2 K_{max}. \tag{3.4.18}
$$

Moreover, the first term and the last term become

$$
\begin{aligned}
&\sum_{i=1}^{2} \alpha\gamma D^{-\gamma-2}|\nabla_i|\vec{X}|^2|^2 K^\alpha H \left( (\gamma-1)\lambda_i^{-1} + \gamma H \frac{2\alpha-1-2(1-\alpha)K\lambda_i^{-2}}{Q} \right) \\
&\quad = \sum_{i\neq j} \alpha\gamma K^\alpha H |\nabla_i|\vec{X}|^2|^2 \frac{(\gamma-1)(Q+\gamma H((2\alpha-1)\lambda_i - 2(1-\alpha)\lambda_j)}{\lambda_i D^{\gamma+2} Q} \\
&\quad = \sum_{i=1}^{2} \frac{\alpha\gamma K^\alpha H}{D^{\gamma+2}Q} \times \frac{-(1-\alpha)(\gamma+1)H^2 + (\gamma-1)4\alpha K + \gamma H\lambda_i}{\lambda_i} |\nabla_i|\vec{X}|^2|^2 \\
&\quad = \sum_{i\neq j} \frac{\alpha\gamma K^{\alpha-1}H}{D^{\gamma+2}Q}(\gamma KH - (1-\alpha)(\gamma+1)H^2\lambda_j + (\gamma-1)4\alpha K\lambda_j)|\nabla_i|\vec{X}|^2|^2 \\
&\quad \leq \frac{\alpha\gamma K^{\alpha-1}H}{D^{\gamma+2}Q}(\gamma KH + (\gamma-1)4\alpha KH)(|\vec{X}|^2 - \left\langle \vec{X}, \vec{\nu} \right\rangle^2) \\
&\quad \leq \frac{\alpha\gamma K^\alpha H^2}{D^{\gamma+2}Q}((4\alpha+1)\gamma - 4\alpha)(|\vec{X}|^2 - \left\langle \vec{X}, \vec{\nu} \right\rangle^2) \\
&\quad \leq \frac{\alpha\gamma((4\alpha+1)\gamma - 4\alpha)}{1-\alpha} K^\alpha R^{-2\gamma},
\end{aligned}
\tag{3.4.19}
$$

and the remaining terms are

$$\frac{2(1-2\alpha)K^{\alpha+1}}{D^\gamma} + \frac{2(2\alpha-1)\gamma K^\alpha H\left\langle \vec{X}, \vec{\nu}\right\rangle}{D^{\gamma+1}} \leq 2|2\alpha-1|(K^{\alpha+1}R^{-2\gamma} + \gamma K^\alpha R^{-1}S). \tag{3.4.20}$$

Notice that $D = 2R^2 - |\vec{X}|^2$ and $Q = 4\alpha K + (1-\alpha)H^2$. Putting (3.4.19) and (3.4.20), together with (3.4.18), we obtain

$$0 \leq -C_1S^2 + C_2S + C_3$$

where $C_1$, $C_2$, and $C_3$ are positive constants depending on $\alpha$, $\mathbf{\Phi}$, and $\vec{X}_0$. Thus, $S$ is uniformly bounded on $M \times [0, t_0]$. Since this bound is independent of $t_0$, we have shown $\max_{M\times[0,T)} S \leq C(\alpha, \mathbf{\Phi}, \vec{X}_0)$ and then the conclusion follows from $H \leq (2R^2)^\gamma S$.

It remains to prove when $\alpha = 1$. In this case, we take $\gamma = 1$. Then (3.4.14) becomes

$$g^{ij}\ddot{K}(\nabla_i h, \nabla_j h) \leq (|\vec{X}|^2 - \left\langle \vec{X}, \vec{\nu}\right\rangle^2)S^2.$$

With this and (3.4.12), we have

$$\begin{aligned} 0 \leq & \frac{|\vec{X}|^2 - \left\langle \vec{X}, \vec{\nu}\right\rangle^2}{D}S^2 + \frac{2K^2}{D} - 2S^2 + \frac{2K\left\langle \vec{X}, \vec{\nu}\right\rangle}{D}S \\ \leq & -S^2 + 2K_{\max}S + 2K_{\max}^2R^{-2}. \end{aligned}$$

As in the previous case, we can obtain the uniform upper bound for $S$, i.e.,

$$S \leq K_{\max}(1 + \sqrt{1 + 2R^{-2}}).$$

Again, this bound is independent of $t_0$, we conclude that $\max_{M\times[0,T)} S \leq C(\mathbf{\Phi}, \vec{X}_0)$ and then $H \leq 2R^2S$ proves the lemma. $\square$

**Remark 3.4.5.** The dimension restriction $n = 2$ is used only on Lemma

3.4.4

## 3.5 Lower bound for principal curvature

In this section, we will discuss about the lower bound on the principal curvature. We first prove an upper bound estimate for $\mathcal{H}^\delta$, the sum of the inverse of the principal curvature, which gives the lower bound of principal curvature. This bounds, together with Lemma 3.4.4, yields uniform ellipticity for each $\delta$ so that we can obtain the long time existence of penalized solution. We also show a lower bound on the principal curvature with respect to the distance from the obstacle.

**Lemma 3.5.1.** *For any $\alpha > 0$ and dimension $n$, let $u$ be the solution of* (3.3.3) *in $S^n \times [0,T)$. Then there exists a constant $C = C(n, \mathbf{\Phi}, \vec{X}_0, T)$ such that*

$$\mathcal{H}(z,t) \le C_\delta \quad in \quad S^n \times [0,T).$$

*Proof.* First, we need the evolution equation of $\mathcal{H}$. To see this, note that by definition

$$\begin{aligned}
\mathcal{H}_t = (\overline{g}^{ij} h_{ij})_t &= -\overline{g}^{ij}\overline{\nabla}_i\overline{\nabla}_j(K^\alpha + \beta_\delta) - n(K^\alpha + \beta_\delta) \\
&= \overline{g}^{ij}\overline{\nabla}_i(\alpha K^\alpha (h^{-1})^{kl}\overline{\nabla}_j h_{kl}) - nK^\alpha - \Delta_{S^n}\beta_\delta - n\beta_\delta \\
&= \overline{g}^{ij}\alpha K^\alpha (h^{-1})^{kl}\overline{\nabla}_i\overline{\nabla}_j h_{kl} - \overline{g}^{ij}\alpha^2 K^\alpha (h^{-1})^{mn}(h^{-1})^{kl}\overline{\nabla}_i h_{mn}\overline{\nabla}_j h_{kl} \\
&\quad - \overline{g}^{ij}\alpha K^\alpha (h^{-1})^{km}(h^{-1})^{ln}\overline{\nabla}_i h_{mn}\overline{\nabla}_j h_{kl} \\
&\quad - nK^\alpha - \Delta_{S^n}\beta_\delta - n\beta_\delta
\end{aligned}$$

and, as in (3.3.4),

$$\Box_{S^n}\mathcal{H} = \overline{g}^{kl}\Box_{S^n} h_{kl} = \overline{g}^{kl}\left(\alpha K^\alpha (h^{-1})^{ij}\overline{\nabla}_k\overline{\nabla}_l h_{ij} - \alpha K^\alpha (n\overline{g}_{kl} - h_{kl}H)\right),$$

which implies that

$$
\begin{aligned}
(\partial_t - \Box_{S^n})\mathcal{H} &= -\overline{g}^{ij}\alpha^2 K^\alpha (h^{-1})^{mn}(h^{-1})^{kl}\overline{\nabla}_i h_{mn}\overline{\nabla}_j h_{kl} \\
&\quad - \overline{g}^{ij}\alpha K^\alpha (h^{-1})^{km}(h^{-1})^{ln}\overline{\nabla}_i h_{mn}\overline{\nabla}_j h_{kl} \\
&\quad + \alpha K^\alpha (n^2 - H\mathcal{H}) - nK^\alpha - \Delta_{S^n}\beta_\delta - n\beta_\delta \\
&\leq -\Delta_{S^n}\beta_\delta - n\beta_\delta,
\end{aligned}
$$

by the Cauchy-Schwarz inequality and the fact that the metric and the second fundamental form are positive definite. As in the previous lemmas, for any $t_0 \in (0, T)$, assume that $\mathcal{H} + e^{\gamma(t_0 - t)\delta^{-2}}$ has maximum over $S^n \times [0, t_0]$ at $(x_1, t_1)$ where $t_1 > 0$. The constant $\gamma$ will be chosen later. Then at this point, we have

$$
0 \leq -\Delta_{S^n}\beta_\delta - n\beta_\delta - \frac{\gamma}{\delta^2}. \tag{3.5.1}
$$

It remains to control the penalty terms. From the direct calculation,

$$
\Delta_{S^n}\beta_\delta = \beta_\delta'' \overline{g}^{ij}\overline{\nabla}_i(u-\varphi)\overline{\nabla}_j(u-\varphi) + \beta_\delta' \Delta_{S^n}(u-\varphi).
$$

Using (3.2.2), we have $\vec{X} - \mathbf{\Phi} = \overline{\nabla}(u-\varphi) + (u-\varphi)z$ so that

$$
\overline{g}^{ij}\overline{\nabla}_i(u-\varphi)\overline{\nabla}_j(u-\varphi) = |\vec{X} - \mathbf{\Phi}|^2 - (u-\varphi)^2.
$$

Moreover, by the relation (3.2.3), we obtain

$$
\begin{aligned}
\Delta_{S^n}(u-\varphi) &= \overline{g}^{ij}\overline{\nabla}_i\overline{\nabla}_j(u-\varphi) \\
&= \overline{g}^{ij}(h_{ij} - h_{ij}^{\mathbf{\Phi}} - (u-\varphi)\overline{g}_{ij}) \\
&= \mathcal{H} - \mathcal{H}^{\mathbf{\Phi}} - (u-\varphi)n.
\end{aligned}
$$

Recalling the definition (3.3.1), these facts immediately imply that if we take

$\gamma \geq A_0 \|\beta''\|_\infty (R^2 + n\delta^2)$, then

$$\begin{aligned}-\Delta_{S^n}\beta_\delta - n\beta_\delta - \frac{\gamma}{\delta^2} \leq & A_0\left(-\beta''\frac{R^2}{\delta^2} - \beta'\frac{1}{\delta}(\mathcal{H} - \mathcal{H}^{\mathbf{\Phi}} - (u-\varphi)n) - \frac{\|\beta''\|_\infty R^2}{\delta^2}\right)\\ \leq & -\beta'\frac{1}{\delta}(\mathcal{H} - \mathcal{H}^{\mathbf{\Phi}} - (u-\varphi)n),\end{aligned}$$

and therefore $\mathcal{H} \leq \mathcal{H}^{\mathbf{\Phi}} + (u-\varphi)n$. Finally, we obtain

$$\mathcal{H} \leq \mathcal{H}^{\mathbf{\Phi}} + (u-\varphi)n + e^{\gamma T\delta^{-2}}$$

Since the right hand side does not depend on $t_0$, the desired conclusion follows. □

Lemma 3.5.1, together with the result in Section 4, implies that the linearized operator satisfying uniformly parabolicity, i.e.,

$$C_\delta|\xi|^2 \leq \alpha K(x,t)^\alpha (h^{-1})^{ij}\xi_i\xi_j \leq C|\xi|^2 \tag{3.5.2}$$

on $M \times [0,T)$, where $\xi \in \mathbb{R}^n$, $C = C(n, \vec{X}_0, \mathbf{\Phi})$, and $C_\delta = C_\delta(n, \vec{X}_0, \mathbf{\Phi}, T)$. Then it is a direct consequence that the support function $u$ satisfies a uniformly parabolic equation. We can now apply Krylov-Safonov theory to $u$, as in [72], which implies a $C^{2,\beta}$ estimate and hence smoothness.

Next we establish the long time existence of (3.3.3). From Lemma 3.3.1, we have the unique smooth solution $u$ on $[0,T)$. Take the maximum time $T^*$ that the solution exists, and assume that $T^*$ is finite. However, using the estimates above, the hypersurface $\Sigma_{T^*}$ is smooth and then the solution exists beyond $T^*$ by applying the local existence to $\Sigma_{T^*}$. This is a contradiction and therefore the solution exists on $[0,\infty)$.

Let $\vec{X}^\delta$ be the solution of (3.3.2) for each $\delta$. From Corollary 3.4.3 and Lemma 3.4.1 ((iii)), $\vec{X}^\delta$ is equicontinuous and uniformly bounded. Then, by the Arzela-Ascoli theorem, there is a continuous hypersurface $\vec{X}$ such that $\vec{X}^\delta \to \vec{X}$ uniformly on $M \times [0,T]$ up to subsequence for each $T < \infty$. It is easy to check that $\vec{X}$ is the viscosity solution of (GFo). Moreover, using

Lemma 3.4.4, $\vec{X}$ has $C^{1,1}$ bound with $\vec{X}^\delta \to \vec{X}$ in $C^{1,\beta}$ for some $0 < \beta < 1$. This proves the first and second part of Theorem 3.1.5.

To obtain the third part of Theorem 3.1.5, we need the following local lower curvature bound whose constant depends on the distance from the free boundary. In order to proceed, we have to define cut-off function. In a $z$-coordinate, consider a point $(z_0, t_0)$ such that $2M := u(z_0, t_0) - \varphi(z_0) > 0$. Now define a cut-off function $\psi_\gamma$ by

$$\psi_\gamma(z, t) = (M - u^\delta(z, t) + \varphi(z) - \gamma t)_+.$$

**Lemma 3.5.2.** *For any $\alpha > 0$ and any dimension $n$, let $u^\delta$ be the smooth solution of* (3.3.3) *in $S^n \times [0, T)$. Assume that $\delta < M$. Then we have*

$$\left(\psi_\gamma^{\frac{1}{\alpha}+n-1} \frac{1}{\lambda_{\min}}\right)(z, t) \le M^{\frac{1}{\alpha}+n-1} \sup_{U_0} \frac{1}{\lambda_{\min}(\cdot, 0)},$$

*where $U_0 = \{z : u^\delta(z, 0) - \varphi(z) < M\}$.*

*Proof.* From the definition of the cut-off function $\psi_\gamma$ with Lemma 3.3.3, we obtain

$$\begin{aligned}(\partial_t - \Box_{S^n})\psi_\gamma &= -\alpha K^\alpha H(u - \varphi) + (n\alpha + 1)K^\alpha - \alpha K^\alpha (h^{-1})^{ij} h^{\mathbf{\Phi}}_{ij} - \gamma + \beta_\delta \\ &\le (n\alpha + 1)K^\alpha - \gamma + \beta_\delta\end{aligned}$$

on the support of $\psi_\gamma$. Using the non-positivity of $\beta_\delta$ and Corollary 3.4.3, we also have

$$(\partial_t - \Box_{S^n})\psi_\gamma \le 0$$

if $\gamma \ge \gamma_0$ for some constant $\gamma_0 = \gamma_0(n, \alpha, \max K)$. Thus we get

$$(\partial_t - \Box_{S^n})\psi_\gamma^b \le -b(b-1)\psi_\gamma^{b-2} \left\|\nabla \psi_\gamma\right\|^2_{\Box_{S^n}}, \tag{3.5.3}$$

where $b = \frac{1}{\frac{1}{\alpha}+n-1}$. We next consider the evolution equation for $h_{11}/\ \bar{g}_{11}$.

Observe that

$$(\partial_t - \Box_{S^n})\frac{1}{\overline{g}_{11}} = 0, \tag{3.5.4}$$

and

$$\begin{aligned}(\partial_t - \Box_{S^n})h_{11} = &-\alpha^2 K^\alpha (h^{-1})^{mn}(h^{-1})^{kl}\overline{\nabla}_1 h_{mn}\overline{\nabla}_1 h_{kl}\\ &-\alpha K^\alpha (h^{-1})^{km}(h^{-1})^{ln}\overline{\nabla}_1 h_{mn}\overline{\nabla}_1 h_{kl}\\ &+(n\alpha-1)K^\alpha \overline{g}_{11} - \alpha K^\alpha H h_{11} - \overline{\nabla}_1\overline{\nabla}_1\beta_\delta - \beta_\delta \overline{g}_{11}.\end{aligned} \tag{3.5.5}$$

We are now ready to prove the assertion. Let $t' \in (0,T)$ and assume that the function $\psi_\gamma^b \lambda_{\min}^{-1}$ attains its maximum on $S^n \times [0,t']$ at $(z_1,t_1)$. If $t_1 = 0$, then we get the desired result. Let $t_1 > 0$ and choose a normal coordinate system near $(z_1,t_1)$ so that

$$\overline{g}_{ij}(z_1,t_1) = \delta_{ij}, \quad h_{ij}(z_1,t_1) = \lambda_i^{-1}(z_1,t_1)\delta_{ij}, \quad \lambda_1(z_1,t_1) = \lambda_{\min}(z_1,t_1).$$

Using the similar argument in [18], we can show that for any point,

$$\frac{h_{11}}{\overline{g}_{11}} \le \frac{1}{\lambda_{\min}}$$

so that

$$w = \psi_\gamma^b \frac{h_{11}}{\overline{g}_{11}}$$

attains its maximum at $(z_1,t_1)$. Then by (3.5.3), and (3.5.4), the following holds at $(z_1,t_1)$:

$$\begin{aligned}0 &\le (\partial_t - \Box_{S^n})w\\ &= \frac{h_{11}}{\overline{g}_{11}}(\partial_t - \Box_{S^n})\psi_\gamma^b + \psi_\gamma^b(\partial_t - \Box_{S^n})\frac{h_{11}}{\overline{g}_{11}} - 2\left\langle \nabla\psi_\gamma^b, \nabla\frac{h_{11}}{\overline{g}_{11}}\right\rangle_{\Box_{S^n}}\\ &= -b(b-1)\psi_\gamma^{b-2}\lambda_1^{-1}\left\|\nabla\psi_\gamma\right\|^2_{\Box_{S^n}} + \psi_\gamma^b(\partial_t - \Box_{S^n})h_{11} + 2\psi_\gamma^{-b}\lambda_1^{-1}\left\|\nabla\psi_\gamma^b\right\|^2_{\Box_{S^n}}.\end{aligned} \tag{3.5.6}$$

Notice that we have used

$$\partial_t w(z_1,t_1) \geq 0. \quad \overline{\nabla} w(z_1,t_1) = 0, \quad \overline{\nabla}^2 w(z_1,t_1) \leq 0.$$

Next, the equation (3.5.5) becomes

$$\begin{aligned}(\partial_t - \Box_{S^n})h_{11} = -\alpha^2 K^\alpha \left(\sum_m \lambda_m \overline{\nabla}_1 h_{mm}\right)^2 - \alpha K^\alpha \lambda_m \lambda_n \overline{\nabla}_1 h_{mn}^2 \\ + (n\alpha - 1)K^\alpha - \alpha K^\alpha H \lambda_1^{-1} - \overline{\nabla}_1 \overline{\nabla}_1 \beta_\delta - \beta_\delta.\end{aligned}$$

For the first two terms in the above equation, by the Cauchy-Schwarz inequality,

$$\alpha \left(\sum_m \lambda_m \nabla_1 h_{mm}\right)^2 + \sum_{m\neq 1} \lambda_m^2 \nabla_1 h_{mm}^2 \geq \frac{1}{\frac{1}{\alpha} + n - 1} \lambda_1^2 \nabla_1 h_{11}^2$$

and therefore,

$$\begin{aligned}\alpha^2 K^\alpha \left(\sum_m \lambda_m \nabla_1 h_{mm}\right)^2 + \sum_m \alpha K^\alpha \lambda_m \lambda_n \nabla_1 h_{mn}^2 \\ \geq \left(1 + \frac{1}{\frac{1}{\alpha} + n - 1}\right) \alpha K^\alpha \lambda_1 \sum_m \lambda_m \nabla_m h_{11}^2.\end{aligned}$$

Using $\nabla w(x_1,t_1) = 0$ again, the equation (3.5.6) becomes

$$\begin{aligned}0 \leq & \left(-\frac{b-1}{b} - 1 - \frac{1}{\frac{1}{\alpha} + n - 1} + 2\right) \psi_\gamma^{-b} \lambda_1 \left\|\nabla \psi_\gamma^b\right\|_{\Box_{S^n}}^2 \\ & + \psi_\gamma^b(-\alpha K^\alpha H \lambda_1^{-1} + (n\alpha - 1)K^\alpha) + \psi_\gamma^b(-\overline{\nabla}_1 \overline{\nabla}_1 \beta_\delta - \lambda_1^{-2} \beta_\delta) \\ \leq & - \alpha \psi_\gamma^b K^\alpha\end{aligned}$$

since $b = \frac{1}{\frac{1}{\alpha}+n-1}$ and $u(z_1,t_1) - \varphi(z_1) \geq M > \delta$. This completes the proof. □

## 3.6 Proof of Theorem 3.1.5

In this section, we provide a proof of Theorem 3.1.5. In order to present the proof, we need the comparison principle which is useful to prove the convergence to the obstacle.

**Lemma 3.6.1.** *Let $u$ and $v$ be the solution of* (3.3.3) *with initial condition $u_0$ and $v_0$, respectively. Assume that the hypersurfaces corresponding to $u_0$ and $v_0$ are smooth strictly convex closed hypersurface. If $u_0 \geq v_0$ on $S^n$, then $u \geq v$ in $S^n \times [0, \infty)$.*

*Proof.* For $\varepsilon > 0$, define $w = (u - v)e^{-\gamma t} + \varepsilon$. The constant $\gamma$ will be chosen later. Assume that $w$ achieves zero at $(z_1, t_1)$ for the first time. Clearly, $t_1 > 0$. Then by the simple maximum principle argument and the mean value theorem, we can obtain

$$(\gamma - \alpha(K^*)^\alpha H^* + \beta'_\delta(u^* - \varphi))(u - v) \geq 0,$$

where $K^*$ and $H^*$ are the curvatures corresponding to the support function $u^* = s^*u + (1 - s^*)v$ for some $s^* \in [0, 1]$. At the point $(z_1, t_1)$, $u - v$ is negative, which will derive a contradiction if $\gamma \geq \alpha(K^*)^\alpha H^* + \beta'_\delta(u^* - \varphi)$. This is actually possible since the Gauss curvature and the mean curvature have an upper bound from Corollary 3.4.3 and Lemma 3.5.1. $\square$

We now prove Theorem 3.1.5.

*Proof of Theorem 3.1.5.* The statements $(i)-(iii)$ are proved in the previous section. The last part is to prove the convergence to the obstacle. For any given point on $\mathbf{\Phi}$, we can take the ball $B$ containing the obstacle and touching $\mathbf{\Phi}$ at given point. Since it is well known fact that how the sphere evolves under the Gauss curvature flow, we also take the large ball enclosing $\vec{X}_0$ and evolving to the $B$ in finite time. Note that this ball is also the solution of (GFo), and hence the conclusion (iv) follows from comparison principle. $\square$

# Chapter 4

# Mean curvature flow of entire graphs with an obstacle

## 4.1 Introduction

In this chapter we consider the obstacle problem for the evolution of complete non-compact strictly mean convex graphs by mean curvature. The obstacle $\Phi$ in our consideration is a $C^{1,1}$ complete non-compact strictly convex graphs over an open subset in $\mathbb{R}^n$. Let $\vec{X}$ be a one parameter family of immersions from $M^n \to \mathbb{R}^{n+1}$, where $M^n$ is an $n$-dimensional complete non-compact Riemannian manifold, and the initial hypersurface $\vec{X}_0 : M^n \to \mathbb{R}^{n+1}$ is smooth immersion of a complete non-compact strictly mean convex $n$-dimensional graph enclosing the obstacle. We say that $X$ is a solution of the obstacle problem for the mean curvature flow of complete non-compact hypersurface if

$$\begin{aligned}
\left\langle \frac{\partial \vec{X}(x,t)}{\partial t}, -\vec{\nu}(x,t) \right\rangle \leq H(x,t) \qquad & \text{for } (x,t) \in M^n \times [0,T), \\
\frac{\partial \vec{X}(x,t)}{\partial t} = -H\vec{\nu}(x,t) \quad & \text{if } \vec{X}(x,t) \notin \mathbf{\Phi}, \\
\overline{\mathbf{\Phi}} \subset \overline{\Sigma_t} \qquad & \text{for } 0 \leq t < T, \\
\vec{X}(x,0) = \vec{X}_0(x) \qquad & \text{for } x \in M^n,
\end{aligned} \tag{MCFo}$$

where $H$ and $\vec{\nu}$ are the mean curvature and the outward unit normal vector on $\Sigma_t$, respectively. Here we use the bar notation to indicate the closed subset of $\mathbb{R}^{n+1}$ enclosed by the given set.

We always assume that the initial hypersurface $X_0$ and the obstacle $\Phi$ is given by graphs over an open subset of $\mathbb{R}^n$. Denote by $u_0 : \Omega_0 \to \mathbb{R}$, $u : \Omega \to \mathbb{R}$ and $\varphi : \Omega_\varphi \to \mathbb{R}$ the graph function of $X_0$, $X$, and $\Phi$, respectively. With the graph functions, we could formulate (MCFo) as

$$\begin{aligned} \frac{\partial u}{\partial t} &\leq \sqrt{1+|Du|^2}\,\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) && \text{in } \Omega\times[0,T), \\ \frac{\partial u}{\partial t} &= \sqrt{1+|Du|^2}\,\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) && \text{in } \{u<\varphi\}, \\ u &\leq \varphi && \text{in } \Omega\times[0,T), \\ u(\cdot,0) &= u_0. \end{aligned} \tag{4.1.1}$$

As in the previous chapter, the concept of solution here is a viscosity solution. Now we state our main result in this chapter.

**Theorem 4.1.1.** *Let $\Sigma_0$ and $\Phi$ be complete non-compact graphs over an open subset of $\mathbb{R}^n$ such that $\overline{\Phi} \subset \overline{\Sigma}_0$. Assume also that $\Phi$ is strictly convex and $\Sigma_0$ is strictly mean convex. Then there exists a viscosity solution $u : \Omega \to \mathbb{R}\cup\{\infty\}$ of* (MCFo) *with the local $C^{1,1}$ optimal regularity, where $\Omega \subset \mathbb{R}^{n+1}\times[0,\infty)$ is relatively open and $\Omega$ contains $\Omega_\varphi \times [0,\infty)$.*

## 4.2 Preliminaries

Let $M^n$ be an $n$-dimensional manifold and $X(\cdot,t) : M^n \to \mathbb{R}^{n+1}$ be a one-parameter family of immersions in $\mathbb{R}^{n+1}$. Denote by $\nu$ the outward unit normal vector of $\Sigma_t = X(M,t)$. In a local coordinate system $\{x^i\}_{i=1}^n$, the induced metric and the second fundamental form of $\Sigma_t$ are

$$g_{ij} = \left\langle \frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j} \right\rangle \quad \text{and} \quad h_{ij} = \left\langle \frac{\partial^2 X}{\partial x^i \partial x^j}, -\nu \right\rangle. \tag{4.2.1}$$

We also define the inverse matrix of $\{h_{ij}\}$ by $\{b^{ij}\}$. With the Einstein's summation convention on repeated indices, the Weingarten map is defined by

$$h^i_j = g^{ik} h_{kj}$$

where $g^{ij}$ denotes the inverse matrix of $\{g_{ij}\}$. We call the eigenvalues of $h^i_j$ as the principal curvatures $\lambda_1, \lambda_2, \cdots, \lambda_n$, and then we can define curvatures as follows:

1. $H = \mathrm{tr}(h^i_j) = \sum \lambda_i$ (Mean curvature),

2. $K = \det(h^i_j) = \prod \lambda_i$ (Gauss curvature),

3. $|A|^2 = \mathrm{tr}((h^2)^i_j) = \sum \lambda_i^2$ (sum of square),

4. $\mathcal{H} = \det((h^{-1})^i_j) = \sum \lambda_i^{-1}$ (sum of inverse).

From the Gauss-Weingarten relations, we have

$$\frac{\partial^2 X}{\partial x^i \partial x^j} = \Gamma^k_{ij} \frac{\partial X}{\partial x^k} - h_{ij}\nu \quad \text{and} \quad \frac{\partial \nu}{\partial x^i} = h_{ik} g^{kl} \frac{\partial X}{\partial x^l},$$

where the Christoffel symbol

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left( \frac{\partial}{\partial x^i} g_{jl} + \frac{\partial}{\partial x^j} g_{il} - \frac{\partial}{\partial x^l} g_{ij} \right),$$

and then the Laplace-Beltrami operator of the position vector $X$ can be computed as

$$\Delta_g X = g^{ij} \nabla_i \nabla_j X = g^{ij} \left( \frac{\partial^2 X}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial X}{\partial x^k} \right) = -H\nu.$$

We write $\Delta = \Delta_g$ for simplicity. Recalling the Gauss identity

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk},$$

where $R_{ijkl}$ is the Riemannian curvature tensor, and the Codazzi identity

$$\nabla_i h_{jk} = \nabla_j h_{ik},$$

we can prove the following identity (see [75, Lemma 2.3]):

$$\Delta h_{ij} = \nabla_i \nabla_j H + H(h^2)_{ij} - |A|^2 h_{ij}. \tag{4.2.2}$$

We also compute the Laplace-Beltrami operator of the outward unit normal vector.

### 4.2.1 Obstacles

Let $\Phi$ be a complete non-compact strictly convex hypersurface. We always assume that $\Phi$ can be represented by a graph over an open subset of $\mathbb{R}^n \times \{0\}$. We call $\Phi$ as an obstacle and denote its normal by $\vec{\nu}_\Phi$.

### 4.2.2 Penalization method

Let $\beta$ be the smooth function defined on $\mathbb{R}$ and satisfying

$$\begin{aligned}
\beta(0) &= -1, & & \\
\beta(x) &= 0 & & \text{if } x \geq 1, \\
\beta''(x) &= 0 & & \text{if } x < 0, \\
\beta'(x) &\geq 0,\ \beta''(x) \leq 0, & & \text{for all } x \in \mathbb{R},
\end{aligned}$$

and let $K_{\mathbf{\Phi}}$ be the Gauss curvature of the obstacle $\mathbf{\Phi}$. We define the penalty term to be

$$\beta_\delta(x) = A_0 \beta(x/\delta), \tag{4.2.3}$$

where $A_0 = \sup_{\mathbf{\Phi}} K_{\mathbf{\Phi}}^\alpha + 1$. Then it is easy to check that $\beta_\delta(x) \leq 0$, $\beta'_\delta(x) \geq 0$, $\beta''_\delta(x) \leq 0$, and $\beta_\delta(0) = -A_0 < -\sup_{\mathbf{\Phi}} K_{\mathbf{\Phi}}^\alpha$. Moreover, $\lim_{\delta\to 0} \beta_\delta(x) \to 0$ for $x > 0$ and $\lim_{\delta \to 0} \beta_\delta(x) \to -\infty$ for $x < 0$.

Now let us consider the following penalized problem which approximates (MCFo): Given an initial hypersurface $\Sigma_0$ and its immersion $\vec{X}_0 : M^n \to \mathbb{R}^{n+1}$,

$$\begin{aligned} &\frac{\partial u}{\partial t} = \sqrt{1 + |Du|^2}\, \mathrm{div}\left( \frac{Du}{\sqrt{1+|Du|^2}} \right) + \beta_\delta(\varphi - u) \quad \text{in } \Omega \times [0,T), \\ &u(\cdot, 0) = u_0. \end{aligned} \tag{4.2.4}$$

## 4.3 Evolution equations

In this section, we obtain evolution equations for geometric quantities of the hypersurface $\Sigma_t$ under the flow (4.2.4).

**Lemma 4.3.1.** *Under the flow* (4.2.4)*, we have the following evolution equations.*

*(i)* $\dfrac{\partial g_{ij}}{\partial t} = -2(H + \beta_\delta) h_{ij}$

*(ii)* $\dfrac{\partial u}{\partial t} = \Delta u + v^{-1} \beta_\delta$

*(iii)* $\dfrac{\partial \vec{\nu}}{\partial t} = \nabla (H + \beta_\delta)$

(*iv*) $\dfrac{\partial v}{\partial t} = \Delta v - 2v^{-1}|\nabla v|^2 - v|A|^2 + v^2 \langle \nabla \beta_\delta, \mathbf{e}\rangle$

(*v*) $\dfrac{\partial h_{ij}}{\partial t} = \Delta h_{ij} - 2H(h^2)_{ij} + |A|^2 h_{ij} + \nabla_i \nabla_j \beta_\delta - (h^2)_{ij}\beta_\delta$

(*vi*) $\dfrac{\partial H}{\partial t} = \Delta H + |A|^2 H + \Delta\beta_\delta + |A|^2\beta_\delta$

(*vii*) $\dfrac{\partial |A|^2}{\partial t} = \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4 + 2h^{ij}\nabla_i\nabla_j\beta_\delta + 2C\beta_\delta$

*Proof.* Recall that $F_\delta$ denotes the speed $H + \beta_\delta$ of (3.3.2).

(i) Since $\left\langle \vec{\nu}, \frac{\partial \vec{X}}{\partial x^i}\right\rangle = 0$, we have

$$\begin{aligned}\frac{\partial g_{ij}}{\partial t} &= \left\langle \frac{\partial}{\partial x^i}(-F_\delta \vec{\nu}), \frac{\partial \vec{X}}{\partial x^j}\right\rangle + \left\langle \frac{\partial \vec{X}}{\partial x^i}, \frac{\partial}{\partial x^j}(-F_\delta\vec{\nu})\right\rangle \\ &= -F_\delta \left\langle \frac{\partial \vec{\nu}}{\partial x^i}, \frac{\partial \vec{X}}{\partial x^j}\right\rangle - F_\delta \left\langle \frac{\partial \vec{X}}{\partial x^i}, \frac{\partial \vec{\nu}}{\partial x^j}\right\rangle = -2F_\delta h_{ij}.\end{aligned}$$

(ii) Notice that $u = \left\langle \vec{X}, \mathbf{e}\right\rangle$, $v = \langle -\vec{\nu}, \mathbf{e}\rangle^{-1}$, and $\Delta \vec{X} = -H\vec{\nu}$. Then

$$\frac{\partial u}{\partial t} = \left\langle \frac{\partial \vec{X}}{\partial t}, \mathbf{e}\right\rangle = \langle \Delta X - \beta_\delta \vec{\nu}, \mathbf{e}\rangle = \Delta u + v^{-1}\beta_\delta.$$

(iii) From $\langle \vec{\nu}, \vec{\nu}\rangle = 1$, we obtain $\left\langle \frac{\partial \vec{\nu}}{\partial t}, \vec{\nu}\right\rangle = 0$ and $\left\langle \frac{\partial \vec{\nu}}{\partial x^i}, \vec{\nu}\right\rangle = 0$. Thus,

$$\frac{\partial \vec{\nu}}{\partial t} = g^{ij}\left\langle \frac{\partial \vec{\nu}}{\partial t}, \frac{\partial \vec{X}}{\partial x^i}\right\rangle \frac{\partial \vec{X}}{\partial x^j} = -g^{ij}\left\langle \vec{\nu}, \frac{\partial(-F_\delta \vec{\nu})}{\partial x^i}\right\rangle \frac{\partial \vec{X}}{\partial x^j} = g^{ij}\frac{\partial F_\delta}{\partial x^i}\frac{\partial \vec{X}}{\partial x^j} = \nabla F_\delta.$$

(iv) By a direct computation, we see that

$$\Delta v = g^{ij}\nabla_i\nabla_j v = g^{ij}\nabla_i(-v^2\langle -\nabla_j \nu, \mathbf{e}\rangle) = 2v^{-1}|\nabla v|^2 + v^2\langle \Delta\vec{\nu}, \mathbf{e}\rangle\,.$$

It can now be deduced from Lemma 2.1.2 that

$$\Delta v = 2v^{-1}|\nabla v|^2 + v^2 \langle \nabla H, \mathbf{e}\rangle + v|A|^2$$

which implies

$$\begin{aligned}\frac{\partial v}{\partial t} &= -v^2 \left\langle -\frac{\partial \vec{\nu}}{\partial t}, \mathbf{e}\right\rangle = v^2 \langle \nabla H + \nabla \beta_\delta, \mathbf{e}\rangle \\ &= \Delta v - 2v^{-1}|\nabla v|^2 - v|A|^2 + v^2 \langle \nabla \beta_\delta, \mathbf{e}\rangle .\end{aligned}$$

(v) As in the proof of $(iii)$, it holds that $\frac{\partial \vec{\nu}}{\partial x^i} = h_i^k \frac{\partial \vec{X}}{\partial x^k}$. Using this, we see that

$$\begin{aligned}\frac{\partial h_{ij}}{\partial t} &= \left\langle \frac{\partial^2(-F_\delta \vec{\nu})}{\partial x^i \partial x^j}, -\vec{\nu}\right\rangle + \left\langle \frac{\partial^2 \vec{X}}{\partial x^i \partial x^j}, -\nabla F_\delta\right\rangle \\ &= \frac{\partial^2 F_\delta}{\partial x^i \partial x^j} + F_\delta \left\langle \frac{\partial^2 \vec{\nu}}{\partial x^i \partial x^j}, \vec{\nu}\right\rangle - \left\langle \Gamma_{ij}^k \frac{\partial \vec{X}}{\partial x^k}, g^{mn} \frac{\partial F_\delta}{\partial x^m} \frac{\partial \vec{X}}{\partial x^n}\right\rangle \\ &= \frac{\partial^2 F_\delta}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial \vec{X}}{\partial x^k} - F_\delta \left\langle \frac{\partial \vec{\nu}}{\partial x^i}, \frac{\partial \vec{\nu}}{\partial x^j}\right\rangle \\ &= \nabla_i \nabla_j F_\delta - F_\delta (h^2)_{ij}.\end{aligned}$$

This, together with (4.2.2), yields

$$\frac{\partial h_{ij}}{\partial t} = \Delta h_{ij} - 2H(h^2)_{ij} + |A|^2 h_{ij} + \nabla_i \nabla_j \beta_\delta - (h^2)_{ij} \beta_\delta.$$

(vi) One easily computes

$$\frac{\partial g^{ij}}{\partial t} = 2F_\delta h^{ij} \tag{4.3.1}$$

so that $\frac{\partial H}{\partial t} = g^{ij} \frac{\partial h_{ij}}{\partial t} + 2F_\delta |A|^2$. Then we get the desired result from $(v)$.

(vii) It also follows from (4.3.1) that $\frac{\partial |A|^2}{\partial t} = 2h^{ij} \frac{\partial h_{ij}}{\partial t} + 4F_\delta h^{ij}(h^2)_{ij}$, and observe that $2h^{ij}\Delta h_{ij} = \Delta |A|^2 - 2|\nabla A|^2$. Again, the evolution equation

$(v)$ gives the conclusion. $\square$

## 4.4 Uniform boundedness for $\beta_\delta$

**Lemma 4.4.1.** *Let $u$ be the solution of (4.2.4). Then the solution $u$ does not touch the obstacle, i.e.,*

$$u(x,t) < \varphi(x) \quad \text{for all } x \in \overline{B}_R,\ t \geq 0. \tag{4.4.1}$$

*In particular, the penalty term has the following uniform bound:*

$$-C \leq \beta_\delta(\varphi(x) - u(x,t)) \leq 0, \quad \text{for all } x \in \overline{B}_R,\ t \geq 0, \tag{4.4.2}$$

*where the constant $C$ depends only on $\max_\Phi H_\Phi$.*

*Proof.* The first assertion follows from the simple maximum principle argument. In fact, assume that $(x_0, t_0)$ is a point such that $u(x_0, t_0) = \varphi(x_0)$ for the first time. Since $u = u_0 < \varphi$ on the parabolic boundary of $B_R \times [0, \infty)$, the point $(x_0, t_0)$ should be an interior point and therefore at this point we have

$$\partial_t(\varphi - u) \leq 0, \quad D(\varphi - u) = 0, \quad D^2(\varphi - u) \geq 0. \tag{4.4.3}$$

Applying this to the equation (4.2.4), the inequality

$$0 \leq \frac{\partial u}{\partial t} = a_{ij}(Du)D_{ij}u + \sqrt{1 + |Du|^2}\beta_\delta(\varphi - u) \tag{4.4.4}$$

$$\leq a_{ij}(D\varphi)D_{ij}\varphi + \sqrt{1 + |D\psi|^2}\beta_\delta(0) \tag{4.4.5}$$

holds at $(x_0, t_0)$, which implies

$$H_\Phi(x_0) = \frac{1}{\sqrt{1 + |D\psi(x_0)|^2}} a_{ij}(D\psi(x_0))D_{ij}\psi(x_0) \geq -\beta_\delta(0). \tag{4.4.6}$$

However, this leads to a contradiction as $-\beta_\delta(0) = \max_\Phi H_\Phi + 1$.

To obtain the second assertion, we observe that

$$-\max_{\Phi} H_\Phi - 1 = \beta_\delta(0) \leq \beta_\delta(\varphi - u) \leq \lim_{z\to\infty} \beta_\delta(z) = 0$$

since $\beta_\delta$ is increasing. Taking $C = \max_{\Phi} H_\Phi + 1$ completes the proof. □

## 4.5 Gradient Estimate

In this section, we derive a local gradient estimate for the height function. Consider the cut-off function $\psi_\gamma$ with moving height to get a local gradient estimate:

$$\psi_\gamma := \Big(M - \gamma t - u(p,t)\Big)_+,$$

where we take $\gamma < \frac{M}{T}$ for a fixed $T$. Then one can obtain the evolution equation of $\psi_\gamma$ and recall that of $v$ from Lemma 4.3.1:

$$\frac{\partial}{\partial t}\psi_\gamma = \Delta\psi_\gamma - \gamma - \beta_\delta v^{-1} \tag{4.5.1}$$

$$\frac{\partial}{\partial t} v = \Delta v - 2v^{-1}|\nabla v|^2 - v|A|^2 + v^2 \langle \nabla\beta_\delta, \mathbf{e}\rangle \tag{4.5.2}$$

which gives the following local gradient estimate.

**Lemma 4.5.1.** *Let $\Sigma_0$ be an initial hypersurface and $\Phi$ be an obstacle. Assume that $\Sigma_t$ is a complete strictly mean convex smooth graph solution of (4.2.4) defined on $M^n \times [0,T]$, for some $T > 0$. Suppose also that $\gamma = C_0 + 1$ where $C_0$ is the constant in Lemma 3.4.1. Then one has*

$$v(x,t)\,\psi_\gamma(x,t) \leq M \max\left\{ \sup_{Q_M^\delta} |\langle -\vec{\nu}_\Phi, \mathbf{e}\rangle|^{-1}, \sup_{Q_M} v(\cdot,0)\right\} \tag{4.5.3}$$

*where $Q_M = \{x \in M^n : u(x,0) < M\}$ and $Q_M^\delta = \{x \in M^n : \psi(x) < M + \delta\}$.*

*Proof.* One can find the evolution equation of $Z := \psi_\gamma v$ from (4.5.1) and

(4.5.2):

$$\begin{aligned}\frac{\partial}{\partial t}Z &= \psi_\gamma \frac{\partial}{\partial t}v + v\frac{\partial}{\partial t}\psi_\gamma \\ &= \psi_\gamma\Big[\Delta v - 2v^{-1}|\nabla v|^2 - v|A|^2 + v^2\left\langle \nabla\beta_\delta, \mathbf{e}\right\rangle\Big] \\ &\quad + v\Big[\Delta\psi_\gamma - \gamma - \beta_\delta v^{-1}\Big] \\ &= \Delta Z - 2v^{-1}\langle\nabla Z, \nabla v\rangle - Z|A|^2 - \gamma v \\ &\quad + vZ\left\langle\nabla\beta_\delta, \mathbf{e}\right\rangle - \beta_\delta.\end{aligned} \tag{4.5.4}$$

Moreover, the terms involving penalization become

$$vZ\beta_\delta'(\varphi - u)\left[\nabla\psi\cdot\nabla u - |\nabla u|^2\right] - \beta_\delta(\varphi - u). \tag{4.5.5}$$

Note that the support of $\psi_\gamma$ is compactly supported, which makes $Z$ achieves its maximum on $M^n \times [0,T]$ at some point $(x_0, t_0)$. If we assume $|\nabla u|^2 \leq \nabla\varphi\cdot\nabla u$ at this point, then we obtain $|\nabla u| \leq |\nabla\psi|$ by Cauchy-Schwarz inequality, which is equivalent to $v \leq |\left\langle -\vec{\nu}_\Phi, \mathbf{e}\right\rangle|^{-1}$ at the same point. Then we have

$$Z(x_0,t_0) \leq M \sup_{Q_{M+\delta}} |\left\langle -\vec{\nu}_\Phi, \mathbf{e}\right\rangle|^{-1}. \tag{4.5.6}$$

provided we make the additional assumption $\varphi(x_0) - u(x_0,t_0) \leq \delta$. We now assume $|\nabla u|^2 > \nabla\varphi\cdot\nabla u$ or $\varphi - u > \delta$ at $(x_0,t_0)$ so that $vZ\beta_\delta'(\varphi - u)\left[\nabla\varphi\cdot\nabla u - |\nabla u|^2\right] \leq 0$. Also from the uniform boundedness of $\beta_\delta$, the remaining term in (4.5.5) is bounded by $C_0$, and hence (4.5.5) is bounded by the same constant. If $t_0 > 0$, then from (4.5.4) we have

$$0 \leq -Z|A|^2 - \gamma v + C_0, \tag{4.5.7}$$

which is a contradiction since $\gamma = C_0 + 1$. On the other hand, in the case of

$t_0 = 0$, we obtain

$$Z \leq M \sup_{Q_M} v(\cdot, 0). \tag{4.5.8}$$

This completes the proof. □

## 4.6 Speed estimates

We start with the following computations

**Lemma 4.6.1.** *Let $u$ be a solution of* (4.2.4)*. Then*

$$\Delta u = \langle \Delta X, \mathbf{e} \rangle = \langle -H\nu, \mathbf{e} \rangle = Hv^{-1}, \tag{4.6.1}$$

$$\Delta \psi = D_{ij}\psi \nabla_k X^i \nabla^k X^j + D_i \psi \Delta X^i \tag{4.6.2}$$

$$= D_{ij}\psi \nabla_k X^i \nabla^k X^j - D_i \psi (H\nu^i), \tag{4.6.3}$$

$$\langle \nabla(\psi - u), \mathbf{e} \rangle = \nabla_i(\psi - u) \left\langle \nabla^i X, \mathbf{e} \right\rangle = \langle \nabla\psi, \nabla u \rangle - \|\nabla u\|^2 . \tag{4.6.4}$$

Now we are ready to prove speed estimates

**Lemma 4.6.2.** *Assume that $\Sigma_0$ is a initial hypersurface and $\Phi$ is an obstacle. Let $\Sigma_t$ be a complete strictly mean convex smooth graph solution of* (4.2.4) *on $\mathcal{M}^n \times [0, T)$. Then*

$$\left( \frac{t}{t+1} \right) (H^2 \psi_\gamma^4)(x, t) \leq c(n) M^2 \theta (M^2 + \theta + \mu_{\max}^2) \tag{4.6.5}$$

*where $c(n)$ denotes a dimensional constant and the constant $\theta$ is given by*

$$\theta = \sup\{\max(v^2(x, s), 1 + D\varphi \cdot Du) : u(x, s) < M, s \in [0, t]\}. \tag{4.6.6}$$

*Proof.* We start with the evolution equation of $H^2$ which follows from $(vi)$

in Lemma 4.3.1:

$$\begin{aligned}(\partial_t - \Delta)H^2 &= 2H(|A|^2 H + \Delta\beta_\delta + |A|^2\beta_\delta) - 2\left\|\nabla H\right\|^2 \\ &= -\frac{1}{2}H^{-2}\left\|\nabla H^2\right\|^2 + 2|A|^2H^2 + 2H\Delta\beta_\delta + 2|A|^2 H\beta_\delta.\end{aligned} \tag{4.6.7}$$

Since the sign of the reaction term, $2|A|^2H^2$, in this equation is not good for the maximum principle, we employ the auxiliary function

$$\varphi(v^2) = \frac{v^2}{2\theta - v^2} \tag{4.6.8}$$

following the well-known idea by Caffarelli, Nirenberg, and Spruck in [**?**]. To obtain the evolution equation of $\varphi$ we recall $(iv)$ in Lemma 4.3.1 so that

$$\begin{aligned}(\partial_t - \Delta)(v^2) &= 2v(-2v^{-1}\left\|\nabla v\right\|^2 - v|A|^2 + v^2\left\langle \nabla\beta_\delta, \mathbf{e}\right\rangle) - 2\left\|\nabla v\right\|^2 \\ &= -2v^2|A|^2 - \frac{3}{2}v^{-2}\left\|\nabla v^2\right\|^2 + 2v^3\left\langle\nabla\beta_\delta, \mathbf{e}\right\rangle\end{aligned}$$

and hence, we have

$$\begin{aligned}(\partial_t - \Delta)\varphi &= \varphi'(\partial_t - \Delta)v^2 - \varphi''\left\|\nabla v^2\right\|^2 \\ &= \varphi'\left(-2v^2|A|^2 - \frac{3}{2}v^{-2}\left\|\nabla v^2\right\|^2 + 2v^3\left\langle\nabla\beta_\delta, \mathbf{e}\right\rangle\right) - \varphi''\left\|\nabla v^2\right\|^2 \\ &= -2\varphi' v^2|A|^2 - \left(\frac{3}{2}\varphi' v^{-2} + \varphi''\right)\left\|\nabla v^2\right\|^2 + 2\varphi' v^3\left\langle\nabla\beta_\delta, \mathbf{e}\right\rangle.\end{aligned}$$

This, together with (4.6.7), gives

$$\begin{aligned}(\partial_t - \Delta)(H^2\varphi) &= \varphi(\partial_t - \Delta)H^2 + H^2(\partial_t - \Delta)\varphi - 2\left\langle\nabla H^2, \nabla\varphi\right\rangle \\ &= \varphi\left(-\frac{1}{2}H^{-2}\left\|\nabla H^2\right\|^2 + 2|A|^2H^2 + 2H\Delta\beta_\delta + 2|A|^2H\beta_\delta\right) \\ &\quad + H^2\left(-2\varphi' v^2|A|^2 - \left(\frac{3}{2}\varphi' v^{-2} + \varphi''\right)\left\|\nabla v^2\right\|^2 + 2\varphi' v^3\left\langle\nabla\beta_\delta, \mathbf{e}\right\rangle\right) \\ &\quad - \left\langle\nabla H^2, \nabla\varphi\right\rangle - \frac{1}{\varphi}\left\langle\nabla(H^2\varphi), \nabla\varphi\right\rangle + \frac{H^2}{\varphi}\left\|\nabla\varphi\right\|^2.\end{aligned}$$

Observe that for the first term in the last line

$$-\left\langle \nabla H^2, \nabla\varphi \right\rangle \leq \frac{1}{2}\varphi H^{-2} \left\| \nabla H^2 \right\|^2 + \frac{H^2}{2\varphi} \left\| \nabla\varphi \right\|^2$$

and for the last term in the last line $\left\|\nabla\varphi\right\|^2 = (\varphi')^2 \left\|\nabla v^2\right\|^2$. Using this we arrive at the following inequality:

$$\begin{aligned}(\partial_t - \Delta)(H^2\varphi) \leq & -\frac{1}{\varphi}\left\langle \nabla(H^2\varphi), \nabla\varphi \right\rangle + 2(\varphi - \varphi' v^2)|A|^2 H^2 \\ & - H^2 \left( \frac{3}{2}\varphi' v^{-2} + \varphi'' - \frac{3(\varphi')^2}{2\varphi} \right) \left\|\nabla v^2\right\|^2 \\ & + 2\varphi H \Delta\beta_\delta + 2\varphi' v^3 H^2 \left\langle \nabla\beta_\delta, \mathbf{e} \right\rangle + 2\varphi |A|^2 H \beta_\delta.\end{aligned} \tag{4.6.9}$$

From direct computations, we have

$$\varphi'(v^2) = \frac{2\theta}{(2\theta - v^2)^2}, \quad \varphi''(v^2) = \frac{4\theta}{(2\theta - v^2)^3} = \frac{2}{2\theta - v^2}\varphi'(v^2) \tag{4.6.10}$$

so that

$$\varphi - \varphi' v^2 = \frac{v^2}{2\theta - v^2} - \frac{2\theta v^2}{(2\theta - v^2)^2} = \frac{-v^4}{(2\theta - v^2)^2} = -\varphi^2$$

and

$$\frac{3}{2}\varphi' v^{-2} + \varphi'' - \frac{3(\varphi')^2}{2\varphi} = \varphi' \left( \frac{3}{2v^2} + \frac{2}{2\theta - v^2} - \frac{6\theta}{2v^2(2\theta - v^2)} \right) = \frac{\theta}{(2\theta - v^2)^3}.$$

Using this, the inequality (4.6.9) becomes

$$(\partial_t - \Delta)(H^2\varphi) \leq -\frac{1}{\varphi}\left\langle \nabla(H^2\varphi), \nabla\varphi \right\rangle - \mathcal{A} + \mathcal{B} \tag{4.6.11}$$

where

$$\mathcal{A} = 2\varphi^2 |A|^2 H^2 + \frac{\theta}{(2\theta - v^2)^3} \left\|\nabla v^2\right\|^2 H^2, \tag{4.6.12}$$

$$\mathcal{B} = 2\varphi H \Delta\beta_\delta + 2\varphi' v^3 H^2 \left\langle \nabla\beta_\delta, \mathbf{e} \right\rangle + 2\varphi |A|^2 H \beta_\delta. \tag{4.6.13}$$

Note that $\mathcal{A}$ and $\mathcal{B}$ denote reaction terms and penalty terms, respectively.

Now we proceed to the localized quantities. Recall $\psi_\gamma = (M - u - \gamma t)_+$ and its evolution equation (4.5.1) so that on the support of $\psi_\gamma$,

$$(\partial_t - \Delta)\psi_\gamma^4 = 4\psi_\gamma^3(-\gamma - \beta_\delta v^{-1}) - 12\psi_\gamma^2 \left\|\nabla u\right\|^2.$$

As in the above, this and (4.6.11) gives

$$\begin{aligned}(\partial_t - \Delta)(H^2\varphi\psi_\gamma^4) &= \psi_\gamma^4(\partial_t - \Delta)(H^2\varphi) + H^2\varphi(\partial_t - \Delta)\psi_\gamma^4 - 2\left\langle \nabla(H^2\varphi), \nabla\psi_\gamma^4\right\rangle \\ &= \psi_\gamma^4\left(-\frac{1}{\varphi}\left\langle\nabla(H^2\varphi), \nabla\varphi\right\rangle - \mathcal{A} + \mathcal{B}\right) \\ &\quad + H^2\varphi\big(4\psi_\gamma^3(-\gamma - \beta_\delta v^{-1}) - 12\psi_\gamma^2\left\|\nabla u\right\|^2\big) \\ &\quad - 2\psi_\gamma^{-4}\left\langle\nabla(H^2\varphi\psi_\gamma^4), \nabla\psi_\gamma^4\right\rangle + 2\psi_\gamma^{-4}H^2\varphi\left\|\nabla\psi_\gamma^4\right\|^2.\end{aligned}$$

Observe from $\nabla\varphi = \varphi'\nabla v^2$ and (4.6.10) that

$$\begin{aligned}-\frac{1}{\varphi}\left\langle\nabla(H^2\varphi), \nabla\varphi\right\rangle &= -\frac{1}{\varphi\psi_\gamma^4}\left\langle\nabla(H^2\varphi\psi_\gamma^4), \nabla\varphi\right\rangle + \frac{H^2\varphi'}{\psi_\gamma^4}\left\langle\nabla\psi_\gamma^4, \nabla v^2\right\rangle \\ &\le -\frac{1}{\varphi\psi_\gamma^4}\left\langle\nabla(H^2\varphi\psi_\gamma^4), \nabla\varphi\right\rangle + H^2\theta\left(\frac{\left\|\nabla v^2\right\|^2}{(2\theta - v^2)^3} + \frac{\left\|\nabla\psi_\gamma^4\right\|^2}{\psi_\gamma^8(2\theta - v^2)}\right),\end{aligned}$$

and notice that $\left\|\nabla\psi_\gamma^4\right\|^2 = 16\psi_\gamma^6\left\|\nabla u\right\|^2$. Then we have

$$\begin{aligned}(\partial_t - \Delta)(H^2\varphi\psi_\gamma^4) &= -\left\langle\nabla(H^2\varphi\psi_\gamma^4), \frac{2\nabla\psi_\gamma^4}{\psi_\gamma^4} + \frac{\nabla\varphi}{\varphi}\right\rangle \\ &\quad - 2\varphi^2\psi_\gamma^4|A|^2H^2 - 4\psi_\gamma^3\gamma H^2\varphi + \frac{4H^2\psi_\gamma^2\left\|\nabla u\right\|^2(4\theta + 5v^2)}{2\theta - v^2} \\ &\quad + \psi_\gamma^4\mathcal{B} - 4\psi_\gamma^3 v^{-1}H^2\varphi\beta_\delta.\end{aligned}$$

Setting $\eta := t(t+1)^{-1}$ and $g := H^2\varphi\psi_\gamma^4\eta$, we arrive at

$$\begin{aligned}(\partial_t - \Delta)g \leq g &- \left\langle \nabla g, \frac{2\nabla\psi_\gamma^4}{\psi_\gamma^4} + \frac{\nabla\varphi}{\varphi}\right\rangle \\ &- 2\eta\varphi^2\psi_\gamma^4|A|^2H^2 - 4\eta\psi_\gamma^3\gamma H^2\varphi + \eta\frac{4H^2\psi_\gamma^2\left\|\nabla u\right\|^2(4\theta+5v^2)}{2\theta - v^2} \\ &+ \eta\psi_\gamma^4\mathcal{B} - 4\eta\psi_\gamma^3 v^{-1}H^2\varphi\beta_\delta\end{aligned} \tag{4.6.14}$$

since $\partial_t\eta = (1+t)^{-2} \leq 1$. Now notice that $g$ has a compact support and thus we can take a maximum point $(x_0, t_0)$ of $g$ with $t_0 > 0$. At this point, the inequality (4.6.14) becomes

$$\begin{aligned}0 \leq g &- 2\eta\varphi^2\psi_\gamma^4|A|^2H^2 - 4\eta\psi_\gamma^3\gamma H^2\varphi + \eta\frac{4H^2\psi_\gamma^2\left\|\nabla u\right\|^2(4\theta+5v^2)}{2\theta - v^2} \\ &+ \eta\psi_\gamma^4\mathcal{B} - 4\eta\psi_\gamma^3 v^{-1}H^2\varphi\beta_\delta.\end{aligned} \tag{4.6.15}$$

From now on, every quantity will be considered as the value evaluated at $(x_0, t_0)$.

To proceed further, we define

$$\tilde{\mathcal{A}} = g - 2\eta\varphi^2\psi_\gamma^4|A|^2H^2 - 4\eta\psi_\gamma^3\gamma H^2\varphi + \eta\frac{4H^2\psi_\gamma^2\left\|\nabla u\right\|^2(4\theta+5v^2)}{2\theta - v^2}, \tag{4.6.16}$$

$$\tilde{\mathcal{B}} = \eta\psi_\gamma^4\mathcal{B} - 4\eta\psi_\gamma^3 v^{-1}H^2\varphi\beta_\delta. \tag{4.6.17}$$

Since $n|A|^2 \geq H^2$, $1 \leq v^2 \leq \theta$, and $g = H^2\varphi\psi_\gamma^4\eta$, we obtain

$$\begin{aligned}\tilde{\mathcal{A}} &= g - \frac{2}{n\eta\psi_\gamma^4}g^2 - \frac{4\gamma}{\psi_\gamma}g + \frac{4(4\theta+5v^2)\left\|\nabla u\right\|^2}{v^2\psi_\gamma^2}g \\ &\leq \frac{2g}{n\eta\psi_\gamma^4}\left(\frac{c(n)\eta}{2}(\psi_\gamma^4 + \psi_\gamma^2\theta\left\|\nabla u\right\|^2) - g\right),\end{aligned}$$

where $c(n)$ denotes a dimensional constant. Furthermore, using the facts that

$\psi_\gamma \le M$, $\eta \le 1$, and $\|\nabla u\|^2 = 1 - v^{-2} \le 1$, the last inequality yields

$$\tilde{\mathcal{A}} \le \frac{2g}{n\eta\psi_\gamma^4}\left(\frac{c(n)}{2}(M^4 + M^2\theta) - g\right),$$

which implies $\tilde{\mathcal{A}} < 0$ unless $g(x_0, t_0) \le c(n)(M^4 + M^2\theta)$.

Now we estimate terms including penalty effects. First of all, we may rewrite (4.6.17), substituting (4.6.13), as

$$\tilde{\mathcal{B}} = 2\eta\psi_\gamma^4\varphi H\Delta\beta_\delta + 2\eta\psi_\gamma^4\varphi' v^3 H^2 \langle\nabla\beta_\delta, \mathbf{e}\rangle + 2\eta\psi_\gamma^3\varphi H\left(\psi_\gamma|A|^2 - 2v^{-1}H\right)\beta_\delta.$$

To estimate the lowest order term, we see that

$$\begin{aligned}\psi_\gamma|A|^2 - 2v^{-1}H &\ge \frac{1}{n}\psi_\gamma H^2 - 2v^{-1}H \\ &= \frac{g}{n\psi_\gamma^3\varphi\eta} - \frac{2\sqrt{g}}{v\sqrt{\varphi\eta}\psi_\gamma^2} \\ &= \frac{\sqrt{g}}{n\psi_\gamma^3\varphi\eta}\left(\sqrt{g} - \frac{2n\psi_\gamma\sqrt{\varphi\eta}}{v}\right).\end{aligned}$$

Using the inequalities $v \ge 1$, $\psi_\gamma \le M$, $\varphi \le 1$, and $\eta \le 1$ again, the following holds: if $g(x_0, t_0) \ge 4n^2M^2$, then $\psi_\gamma|A|^2 - 2v^{-1}H \ge 0$. This, together with the fact $\beta_\delta \le 0$, leads us to that

$$2\eta\psi_\gamma^3\varphi H\left(\psi_\gamma|A|^2 - 2v^{-1}H\right)\beta_\delta \le 0$$

unless $g(x_0, t_0) \le 4n^2M^2$. To estimate the highest order term, we notice that

$$\begin{aligned}\Delta\beta_\delta &= g^{ij}\nabla_i\nabla_j\beta_\delta = g^{ij}\nabla_i(\beta_\delta'\nabla_j(\varphi - u)) \\ &= \beta_\delta''\|\nabla(\varphi - u)\|^2 + \beta_\delta'\Delta(\varphi - u).\end{aligned}$$

Since $\beta_\delta'' \le 0$, we have

$$2\eta\psi_\gamma^4\varphi H\Delta\beta_\delta \le 2\eta\psi_\gamma^4\varphi H\beta_\delta'\Delta(\varphi - u). \tag{4.6.18}$$

On the other hand, the remaining term is

$$2\eta\psi_\gamma^4\varphi' v^3 H^2 \langle \nabla\beta_\delta, \mathbf{e}\rangle = 2\eta\psi_\gamma^4\varphi' v^3 H^2 \beta_\delta' \langle \nabla(\varphi - u), \mathbf{e}\rangle \tag{4.6.19}$$

Adding (4.6.18) and (4.6.19) gives that

$$\begin{aligned} &2\eta\psi_\gamma^4\varphi H\Delta\beta_\delta + 2\eta\psi_\gamma^4\varphi' v^3 H^2 \langle \nabla\beta_\delta, \mathbf{e}\rangle & (4.6.20)\\ &\qquad \le 2\eta\psi_\gamma^4 H(\varphi\Delta(\varphi - u) + \varphi' v^3 H \langle \nabla(\varphi - u), \mathbf{e}\rangle)\beta_\delta'. & (4.6.21)\end{aligned}$$

By Lemma 4.6.1, we have

$$\begin{aligned} \varphi\Delta(\varphi - u) &+ \varphi' v^3 H \langle \nabla(\varphi - u), \mathbf{e}\rangle \\ &= \varphi\big((\delta^{ij} - \nu^i\nu^j)D_{ij}\psi - Hv^{-1}D_i\psi D_i u - Hv^{-1}\big) \\ &\quad + \varphi' v H(D\psi\cdot Du - |Du|^2) \\ &\le \varphi(n-1)\mu_{\max} - \frac{(2\theta-1)Hv^3}{(2\theta - v^2)^2} + \frac{(D\psi\cdot Du)Hv^3}{(2\theta - v^2)^2} \\ &= \varphi(n-1)\mu_{\max} - \frac{(2\theta - 1 - D\psi\cdot Du)Hv^3}{(2\theta - v^2)^2}. \end{aligned}$$

From the definition of $\theta$, we see that $\theta \ge 1 + D\psi\cdot Du$. Using this, we deduce

$$\varphi\Delta(\varphi - u) + \varphi' v^3 H \langle \nabla(\varphi - u), \mathbf{e}\rangle \le \varphi\left((n-1)\mu_{\max} - \frac{\theta H v}{2\theta - v^2}\right). \tag{4.6.22}$$

If $(n-1)\mu_{\max} \le \dfrac{\theta H v}{2\theta - v^2}$, by (4.6.20) and (4.6.22), we have

$$2\eta\psi_\gamma^4\varphi H\Delta\beta_\delta + 2\eta\psi_\gamma^4\varphi' v^3 H^2 \langle \nabla\beta_\delta, \mathbf{e}\rangle \le 0.$$

Otherwise, $H \le (2\theta - v^2)(n-1)\mu_{\max}(\theta v)^{-1} \le (n-1)\mu_{\max}$ so that

$$g(x_0, t_0) \le c(n) M^4 \mu_{\max}^2.$$

Now we can claim that

$$g(x_0,t_0) \le c(n)M^2(M^2+\theta+\mu_{\max}^2). \tag{4.6.23}$$

In fact, if (4.6.23) does not hold, we already observed that $\mathcal{A}<0$ and $\mathcal{B}\le 0$, which contradicts to (4.6.15). Finally, we conclude, by noting $\varphi \ge (2\theta)^{-1}$, that

$$(H^2\psi_\gamma^4\eta)(x_0,t_0) \le c(n)M^2\theta(M^2+\theta+\mu_{\max}^2).$$

This completes the proof. □

## 4.7 Estimate for maximum eigenvalue

The purpose of this section is to establish the estimate for the maximum eigenvalue.

**Lemma 4.7.1.** *Let $\Sigma_t$ be a complete strictly mean convex smooth graph solution of* (4.2.4) *on $\mathcal{M}^n\times[0,T]$. Then*

$$\left(\frac{t}{t+1}\right)(\lambda_{\max}^2\psi_\gamma^4)(x,t) \le c(n)M^2\theta(M^2+\theta+\mu_{\max}^2) \tag{4.7.1}$$

*where $c(n)$ denotes a dimensional constant and the constant $\theta$ is given by*

$$\theta = \sup\{\max(v^2(x,s), 1+D\psi\cdot Du) : u(x,s)<M, s\in[0,t]\}. \tag{4.7.2}$$

*Proof.* We shall consider a quantity

$$g_0 = \eta(t)\lambda_{\max}^2\varphi(v^2)\psi_\gamma^4$$

where $\eta(t)=\frac{t}{t+1}$, $\varphi(v^2)=\frac{v^2}{2\theta-v^2}$, and $\psi_\gamma=(M-\gamma t-u)_+$. Observe that $g_0$ is compactly supported since $\psi_\gamma$ has a compact support. Thus we can take a maximum point $(x_0,t_0)$ of $g_0$ over $M^n\times[0,T]$. If $t_0=0$, we have the desired result. Thus we may assume $t_0 > 0$.

Now we define a function

$$g = \eta(t) \left( \frac{h_{1i} g^{ij} h_{j1}}{g_{11}} \right) \varphi(v^2) \psi_\gamma^4$$

in the coordinate chart near $x_0$. We may choose a normal coordinate so that

$$g_{ij} = \delta_{ij}, \quad h_{ij} = \lambda_i \delta_{ij}, \quad \lambda_1 = \lambda_{\max}$$

at the point $(x_0, t_0)$. From the Euler's formula (see Proposition 3.1 in [17]), we see that

$$g \leq g_0, \quad g(x_0, t_0) = g_0(x_0, t_0)$$

Thus $g$ also have a maximum at $(x_0, t_0)$.

Next, observe that $\partial_t g^{ij} = 2(H + \beta_\delta) h^{ij}$ and $\nabla g = 0$. We consider the evolution equation of $h_{1i} g^{ij} h_{j1}/g_{11}$ which follows from $(v)$ in Lemma 4.3.1:

$$\begin{aligned}
(\partial_t - \Delta)(h_{1i} g^{ij} h_{j1}) &= 2h_{1i} g^{ij} (\partial_t - \Delta) h_{j1} + h_{1i} h_{j1} (2(H + \beta_\delta) h^{ij}) \\
&\quad - 2 g^{ij} g^{kl} \nabla_k h_{1i} \nabla_l h_{j1}, \\
(\partial_t - \Delta) \left( \frac{h_{1i} g^{ij} h_{j1}}{g^{11}} \right) &= \frac{2}{g^{11}} h_{1i} g^{ij} (\partial_t - \Delta) h_{j1} + \frac{2(h^3)_{11}}{g^{11}} (H + \beta_\delta) \\
&\quad - \frac{2 g^{ij} g^{kl}}{g^{11}} \nabla_k h_{1i} \nabla_l h_{j1} - \frac{2(H + \beta_\delta)(h^2)_{11} h^{11}}{(g^{11})^2} \\
&= \frac{2}{g^{11}} h_{1i} g^{ij} (-2H (h^2)_{j1} + |A|^2 h_{j1} + \nabla_j \nabla_1 \beta_\delta - (h^2)_{j1} \beta_\delta) \\
&\quad + \frac{2}{g^{11}} (h^3)_{11} (H + \beta_\delta) - \frac{2}{g^{11}} g^{ij} g^{kl} \nabla_k h_{1i} \nabla_l h_{j1} \\
&\quad - \frac{2(H + \beta_\delta)(h^2)_{11} h^{11}}{(g^{11})^2} \\
&= \frac{-2H (h^3)_{11}}{g^{11}} + \frac{2|A|^2 (h^2)_{11}}{g^{11}} - \frac{2H (h^2)_{11} h^{11}}{(g^{11})^2} \\
&\quad - \frac{2}{g^{11}} g^{ij} g^{kl} \nabla_k h_{1i} \nabla_l h_{j1} + \frac{2 h_{1i} g^{ij} \nabla_j \nabla_1 \beta_\delta}{g^{11}} - \frac{2 \beta_\delta (h^2)_{11} h^{11}}{(g^{11})^2}.
\end{aligned}$$

Since the evolution equation of $\varphi(v^2)$ is given by

$$(\partial_t - \Delta)\varphi = -2\varphi' v^2 |A|^2 - \left(\frac{3}{2}\varphi' v^{-2} + \varphi''\right) \left\|\nabla v^2\right\|^2 + 2\varphi' v^3 \left\langle \nabla \beta_\delta, \mathbf{e} \right\rangle .$$

Thus it follows that

$$\begin{aligned}
(\partial_t - \Delta)(Q\varphi) &= \varphi(\partial_t - \Delta)Q + Q(\partial_t - \Delta)\varphi - 2\left\langle \nabla Q, \nabla \varphi \right\rangle \\
&= \varphi\bigg( \frac{-2H(h^3)_{11}}{g^{11}} + \frac{2|A|^2 (h^2)_{11}}{g^{11}} - \frac{2H(h^2)_{11}h^{11}}{(g^{11})^2} \\
&\quad - \frac{2}{g^{11}} g^{ij} g^{kl} \nabla_k h_{1i} \nabla_l h_{j1} + \frac{2h_{1i} g^{ij} \nabla_j \nabla_1 \beta_\delta}{g^{11}} - \frac{2\beta_\delta (h^2)_{11} h^{11}}{(g^{11})^2} \bigg) \\
&\quad + Q\left( -2\varphi' v^2 |A|^2 - \left(\frac{3}{2}\varphi' v^{-2} + \varphi''\right) \left\|\nabla v^2\right\|^2 + 2\varphi' v^3 \left\langle \nabla \beta_\delta, \mathbf{e} \right\rangle \right) \\
&\quad - \left\langle \nabla Q, \nabla \varphi \right\rangle - \frac{1}{\varphi} \left\langle \nabla (Q\varphi), \nabla \varphi \right\rangle + \frac{Q}{\varphi} \left\|\nabla \varphi\right\|^2 .
\end{aligned}$$

Observe that for the first term in the last line

$$-\left\langle \nabla Q, \nabla \varphi \right\rangle \leq \frac{\varphi}{2Q} \left\|\nabla Q\right\|^2 + \frac{Q}{2\varphi} \left\|\nabla \varphi\right\|^2$$

and for the last term in the last line $\left\|\nabla \varphi\right\|^2 = (\varphi')^2 \left\|\nabla v^2\right\|^2$. Using this, we arrive at the following inequality:

$$\begin{aligned}
(\partial_t - \Delta)(Q\varphi) \leq & -\frac{1}{\varphi} \left\langle \nabla (Q\varphi), \nabla \varphi \right\rangle - \frac{2\varphi}{g^{11}} g^{ij} g^{kl} \nabla_k h_{1i} \nabla_l h_{j1} + \frac{\varphi}{2Q} \left\|\nabla Q\right\|^2 \\
& + 2(\varphi - \varphi' v^2)|A|^2 Q - \frac{2\varphi H(h^3)_{11}}{g^{11}} - \frac{2\varphi H (h^2)_{11} h^{11}}{(g^{11})^2} \\
& - Q\left( \frac{3}{2}\varphi' v^{-2} + \varphi'' - \frac{3(\varphi')^2}{2\varphi} \right) \left\|\nabla v^2\right\|^2 \\
& + 2\varphi \frac{h_{1i} g^{ij}}{g^{11}} \nabla_j \nabla_1 \beta_\delta + 2\varphi' v^3 Q \left\langle \nabla \beta_\delta, \mathbf{e} \right\rangle - 2\varphi \frac{(h^2)_{11} h^{11}}{(g^{11})^2} \beta_\delta.
\end{aligned} \tag{4.7.3}$$

From direct computations, we have

$$\varphi - \varphi' v^2 = \frac{v^2}{2\theta - v^2} - \frac{2\theta v^2}{(2\theta - v^2)^2} = \frac{-v^4}{(2\theta - v^2)^2} = -\varphi^2$$

and

$$\frac{3}{2}\varphi' v^{-2} + \varphi'' - \frac{3(\varphi')^2}{2\varphi} = \varphi' \left( \frac{3}{2v^2} + \frac{2}{2\theta - v^2} - \frac{6\theta}{2v^2(2\theta - v^2)} \right) = \frac{\theta}{(2\theta - v^2)^3}.$$

Using this, the inequality (4.7.3) becomes

$$(\partial_t - \Delta)(Q\varphi) \leq -\frac{1}{\varphi} \langle \nabla(Q\varphi), \nabla\varphi \rangle + \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{B} \tag{4.7.4}$$

where

$$\mathcal{A}_1 = -\frac{2\varphi}{g^{11}} g^{ij} g^{kl} \nabla_k h_{1i} \nabla_l h_{j1} + \frac{\varphi}{2Q} \|\nabla Q\|^2 \tag{4.7.5}$$

$$\mathcal{A}_2 = -2\varphi^2 |A|^2 Q - \frac{\theta}{(2\theta - v^2)^3} \left\|\nabla v^2\right\|^2 Q - \frac{2\varphi H (h^3)_{11}}{g^{11}} - \frac{2\varphi H (h^2)_{11} h^{11}}{(g^{11})^2}, \tag{4.7.6}$$

$$\mathcal{B} = 2\varphi \frac{h_{1i} g^{ij}}{g^{11}} \nabla_j \nabla_1 \beta_\delta + 2\varphi' v^3 Q \langle \nabla\beta_\delta, \mathbf{e} \rangle - 2\varphi \frac{(h^2)_{11} h^{11}}{(g^{11})^2} \beta_\delta. \tag{4.7.7}$$

Now we proceed to the localized quantities. Recall $\psi_\gamma = (M - u - \gamma t)_+$ and its evolution equation (4.5.1) so that on the support of $\psi_\gamma$,

$$(\partial_t - \Delta)\psi_\gamma^4 = 4\psi_\gamma^3(-\gamma - \beta_\delta v^{-1}) - 12\psi_\gamma^2 \|\nabla u\|^2.$$

As in the above, this and (4.7.4) gives

$$\begin{aligned}
(\partial_t - \Delta)(Q\varphi\psi_\gamma^4) &= \psi_\gamma^4 (\partial_t - \Delta)(Q\varphi) + Q\varphi(\partial_t - \Delta)\psi_\gamma^4 - 2\left\langle \nabla(Q\varphi), \nabla\psi_\gamma^4 \right\rangle \\
&= \psi_\gamma^4 \left( -\frac{1}{\varphi} \langle \nabla(Q\varphi), \nabla\varphi \rangle + \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{B} \right) \\
&\quad + Q\varphi\big(4\psi_\gamma^3(-\gamma - \beta_\delta v^{-1}) - 12\psi_\gamma^2 \|\nabla u\|^2 \big) \\
&\quad - 2\psi_\gamma^{-4} \left\langle \nabla(Q\varphi\psi_\gamma^4), \nabla\psi_\gamma^4 \right\rangle + 2\psi_\gamma^{-4} Q\varphi \left\|\nabla\psi_\gamma^4\right\|^2.
\end{aligned}$$

Observe from $\nabla\varphi = \varphi'\nabla v^2$ and (4.6.10) that

$$\begin{aligned}
-\frac{1}{\varphi}\left\langle \nabla(Q\varphi), \nabla\varphi \right\rangle &= -\frac{1}{\varphi\psi_\gamma^4}\left\langle \nabla(Q\varphi\psi_\gamma^4), \nabla\varphi \right\rangle + \frac{Q\varphi'}{\psi_\gamma^4}\left\langle \nabla\psi_\gamma^4, \nabla v^2 \right\rangle \\
&\le -\frac{1}{\varphi\psi_\gamma^4}\left\langle \nabla(Q\varphi\psi_\gamma^4), \nabla\varphi \right\rangle + Q\theta\left(\frac{\left\|\nabla v^2\right\|^2}{(2\theta - v^2)^3} + \frac{\left\|\nabla\psi_\gamma^4\right\|^2}{\psi_\gamma^8(2\theta - v^2)}\right),
\end{aligned}$$

and notice that $\left\|\nabla\psi_\gamma^4\right\|^2 = 16\psi_\gamma^6\left\|\nabla u\right\|^2$. Then we have

$$\begin{aligned}
(\partial_t - \Delta)(Q\varphi\psi_\gamma^4) = &-\left\langle \nabla(Q\varphi\psi_\gamma^4), \frac{2\nabla\psi_\gamma^4}{\psi_\gamma^4} + \frac{\nabla\varphi}{\varphi} \right\rangle + \psi_\gamma^4\mathcal{A}_1 \\
&- 2\varphi^2\psi_\gamma^4|A|^2Q - 4\psi_\gamma^3\gamma Q\varphi + \frac{4Q\psi_\gamma^2\left\|\nabla u\right\|^2(4\theta + 5v^2)}{2\theta - v^2} \\
&- \frac{2\psi_\gamma^4\varphi H(h^3)_{11}}{g^{11}} - \frac{2\psi_\gamma^4\varphi H(h^2)_{11}h^{11}}{(g^{11})^2} \\
&+ \psi_\gamma^4\mathcal{B} - 4\psi_\gamma^3 v^{-1}Q\varphi\beta_\delta.
\end{aligned}$$

With $\eta = t(t+1)^{-1}$ and $g = \eta Q\varphi\psi_\gamma^4$, we arrive at

$$\begin{aligned}
(\partial_t - \Delta)g \le Q\varphi\psi_\gamma^4 &- \left\langle \nabla g, \frac{2\nabla\psi_\gamma^4}{\psi_\gamma^4} + \frac{\nabla\varphi}{\varphi} \right\rangle + \eta\psi_\gamma^4\mathcal{A}_1 \\
&- 2\eta\varphi^2\psi_\gamma^4|A|^2Q - 4\eta\psi_\gamma^3\gamma Q\varphi + \frac{4\eta Q\psi_\gamma^2\left\|\nabla u\right\|^2(4\theta + 5v^2)}{2\theta - v^2} \\
&- \frac{2\eta\psi_\gamma^4\varphi H(h^3)_{11}}{g^{11}} - \frac{2\eta\psi_\gamma^4\varphi H(h^2)_{11}h^{11}}{(g^{11})^2} \\
&+ \eta\psi_\gamma^4\mathcal{B} - 4\eta\psi_\gamma^3 v^{-1}Q\varphi\beta_\delta
\end{aligned} \tag{4.7.8}$$

from $\partial_t\eta = (1+t)^{-2} \le 1$. At the point $(x_0, t_0)$, the inequality (4.7.8) becomes

$$\begin{aligned}
0 \le Q\varphi\psi_\gamma^4 - 2\varphi|A|^2 g - 4\eta\psi_\gamma^3\gamma Q\varphi + \eta\frac{4Q\psi_\gamma^2\left\|\nabla u\right\|^2(4\theta + 5v^2)}{2\theta - v^2} - 4gH\lambda_1 \\
+ \eta\psi_\gamma^4\mathcal{B} - 4\eta\psi_\gamma^3 v^{-1}Q\varphi\beta_\delta.
\end{aligned} \tag{4.7.9}$$

since we see

$$\mathcal{A}_1 = -2\varphi \nabla_k h_{1i}^2 + 2\varphi \nabla_k h_{11}^2 \leq 0.$$

From now on, every quantity will be considered as the value evaluated at $(x_0, t_0)$.

To proceed further, we define

$$\begin{aligned}
\tilde{\mathcal{A}} &= Q\varphi\psi_\gamma^4 - 2\varphi|A|^2 g - 4\eta\psi_\gamma^3\gamma Q\varphi + \eta\frac{4Q\psi_\gamma^2 \|\nabla u\|^2 (4\theta + 5v^2)}{2\theta - v^2} - 4gH\lambda_1, \\
\tilde{\mathcal{B}} &= \eta\psi_\gamma^4\mathcal{B} - 4\eta\psi_\gamma^3 v^{-1} Q\varphi\beta_\delta.
\end{aligned}$$

Since $|A|^2 \geq Q$, $1 \leq v^2 \leq \theta$, and $g = Q\varphi\psi_\gamma^4\eta$, we obtain

$$\tilde{\mathcal{A}} \leq \frac{c(n)g}{\eta\psi_\gamma^4}\left(\psi_\gamma^4 + \psi_\gamma^2\theta - g\right)$$

where $c(n)$ denotes a dimensional constant. Furthermore, using the facts that $\psi_\gamma \leq M$, $\eta \leq 1$, and $\|\nabla u\|^2 = 1 - v^{-2} \leq 1$, the last inequality yields

$$\tilde{\mathcal{A}} \leq \frac{g}{\eta\psi_\gamma^4}\left(\frac{c(n)}{2}(M^4 + M^2\theta) - g\right).$$

Now we estimate terms including penalty effects. First of all, we may rewrite $\tilde{\mathcal{B}}$, substituting (4.7.7), as

$$\tilde{\mathcal{B}} = 2\eta\psi_\gamma^4\varphi\lambda_1\nabla_1\nabla_1\beta_\delta + 2\eta\psi_\gamma^4\varphi' v^3 Q\left\langle\nabla\beta_\delta, \mathbf{e}\right\rangle - 2\eta\psi_\gamma^4\varphi\lambda_1^3\beta_\delta - 4\eta\psi_\gamma^3 v^{-1}Q\varphi\beta_\delta.$$

To estimate the lowest order term, we see that

$$-2\eta\psi_\gamma^4\varphi\lambda_1^3\beta_\delta - 4\eta\psi_\gamma^3 v^{-1}Q\varphi\beta_\delta \leq \frac{C_0 g}{\eta\psi_\gamma^4}(g^{1/2} + 1)$$

To estimate the highest order term, we notice that

$$\begin{aligned}\nabla_1\nabla_1\beta_\delta &= \nabla_1(\beta_\delta'\nabla_1(\psi-u))\\ &= \beta_\delta''\left\|\nabla_1(\psi-u)\right\|^2+\beta_\delta'\nabla_1\nabla_1(\psi-u).\end{aligned}$$

Since $\beta_\delta''\le 0$, we have

$$2\eta\psi_\gamma^4\varphi\lambda_1\nabla_1\nabla_1\beta_\delta\le 2\eta\psi_\gamma^4\varphi\lambda_1\beta_\delta'\nabla_1\nabla_1(\psi-u). \tag{4.7.10}$$

On the other hand, the remaining term is

$$2\eta\psi_\gamma^4\varphi' v^3Q\left\langle\nabla\beta_\delta,\mathbf{e}\right\rangle = 2\eta\psi_\gamma^4\varphi' v^3Q\beta_\delta'\left\langle\nabla(\psi-u),\mathbf{e}\right\rangle \tag{4.7.11}$$

Adding (4.7.10) and (4.7.11) gives that

$$\begin{aligned}2\eta\psi_\gamma^4\varphi\lambda_1\nabla_1\nabla_1\beta_\delta+2\eta\psi_\gamma^4\varphi' v^3Q\left\langle\nabla\beta_\delta,\mathbf{e}\right\rangle &\le 2\eta\psi_\gamma^4\lambda_1(\varphi\nabla_1\nabla_1(\psi-u) && (4.7.12)\\ &\quad+\varphi' v^3\lambda_1\left\langle\nabla(\psi-u),\mathbf{e}\right\rangle)\beta_\delta'. && (4.7.13)\end{aligned}$$

Thus we need to estimate

$$\varphi\nabla_1\nabla_1(\psi-u)+\varphi' v^3\lambda_1\left\langle\nabla(\psi-u),\mathbf{e}\right\rangle.$$

For the first term, we have

$$\begin{aligned}\varphi\nabla_1\nabla_1(\psi-u) &= \varphi\nabla_1\left(D\psi\nabla_1(X-u\mathbf{e})-\left\langle\nabla_1X,\mathbf{e}\right\rangle\right)\\ &= \varphi(D^2\psi\nabla_1(X-u\mathbf{e})\nabla_1(X-u\mathbf{e})+D\psi\nabla_1\nabla_1(X-u\mathbf{e})\\ &\quad-\left\langle\nabla_1\nabla_1X,\mathbf{e}\right\rangle)\\ &\le\varphi\left(\mu_{\max}(1-\left\langle\nabla_1X,\mathbf{e}\right\rangle^2)-\lambda_1D\psi(\nu-\left\langle\nu,\mathbf{e}\right\rangle\mathbf{e})-\lambda_1v^{-1}\right)\\ &\le\varphi\left(\mu_{\max}-\lambda_1D\psi Duv^{-1}-\lambda_1v^{-1}\right)\end{aligned}$$

where $\mu_{\max}$ denotes the maximum eigenvalue of $D^2\psi$. To estimate second term, we express $\nabla_iX=a_{ij}E_j$ where $\{E_j\}\cup\{\nu\}$ is orthonormal basis in $\mathbb{R}^{n+1}$.

Also, we extend functions $\psi$ and $u$ to the functions whose space variable is defined on $\mathbb{R}^{n+1}$ by defining $\psi(X) = \psi(X + b\mathbf{e})$ and $u(X) = u(X + b\mathbf{e})$ for $b \in \mathbb{R}$. Then we have

$$\begin{aligned}
\langle \nabla(\psi - u), \mathbf{e} \rangle &= D_\alpha(\psi - u) \nabla_i X^\alpha g^{ij} \nabla_j X^\beta \mathbf{e}^\beta \\
&= D_\alpha(\psi - u)(a_{ik} E_k)^\alpha a^{li} a^{lj} a_{jm} \langle E_m, \mathbf{e} \rangle \\
&= \langle D(\psi - u), E_k \rangle \langle E_k, \mathbf{e} \rangle \\
&= \langle D(\psi - u), \mathbf{e} \rangle - \langle D(\psi - u), \nu \rangle \langle \mathbf{e}, \nu \rangle \\
&= v^{-2} D(\psi - u) \cdot Du.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\varphi \nabla_1 \nabla_1 (\psi - u) + \varphi' v^3 \lambda_1 \langle \nabla(\psi - u), \mathbf{e} \rangle \\
&= \varphi \left( \mu_{\max} - \lambda_1 D\psi Du v^{-1} - \lambda_1 v^{-1} \right) + \varphi' v \lambda_1 D(\psi - u) \cdot Du \\
&= \varphi \mu_{\max} + \frac{(D\psi \cdot Du) \lambda_1 v^3}{(2\theta - v^2)^2} - \frac{(2\theta - 1) \lambda_1 v^3}{(2\theta - v^2)^2} \\
&= \varphi \mu_{\max} - \frac{(2\theta - 1 - D\psi \cdot Du) \lambda_1 v^3}{(2\theta - v^2)^2}.
\end{aligned}$$

From the definition of $\theta$, we see that $\theta \geq 1 + D\psi \cdot Du$. Using this, we deduce

$$\varphi \nabla_1 \nabla_1 (\psi - u) + \varphi' v^3 \lambda_1 \langle \nabla(\psi - u), \mathbf{e} \rangle \leq \varphi \left( \mu_{\max} - \frac{\theta \lambda_1 v^3}{2\theta - v^2} \right). \quad (4.7.14)$$

If $\mu_{\max} \leq \dfrac{\theta H v^3}{2\theta - v^2}$, by (4.7.12) and (4.7.14), we have

$$2\eta \psi_\gamma^4 \varphi \lambda_1 \nabla_1 \nabla_1 \beta_\delta + 2\eta \psi_\gamma^4 \varphi' v^3 Q \langle \nabla \beta_\delta, \mathbf{e} \rangle \leq 0.$$

Otherwise, $H \leq (2\theta - v^2) \mu_{\max} \theta^{-1} v^{-3} \leq \mu_{\max}$ so that

$$g(x_0, t_0) \leq c(n) M^4 \mu_{\max}^2.$$

This completes the proof. $\square$

## 4.8 Proof of Theorem 4.1.1

In this section we finish the proof of Theorem 4.1.1. From Lemma 4.5.1, we can solve the following initial boundary value problem (see [65, 41]):

$$\begin{aligned} \frac{\partial u}{\partial t} &= \sqrt{1+|Du|^2}\,\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) + \beta_\delta(\varphi - u) && \text{in } B_R(0)\times[0,\infty), \\ u &= L && \text{on } \partial B_R(0)\times[0,\infty), \\ u(\cdot,0) &= \min\{u_0, L\} && \text{in } B_R(0). \end{aligned} \tag{4.8.1}$$

Let us denote the solution of (4.8.1) by $u^L$. By Lemma 4.7.1, we obtain uniform bounds for $\left\|u^L\right\|_{C^{0,1;0,1/2}}$ in $B_R(0)\times[0,\infty)$. Using the compactness lemma, Lemma 7.3 in [65], we can obtain a solution $u$ of (4.2.4) such that $u^L$ converge to $u$ uniformly. Now the conclusion follows from the uniform $C^{1;1}$ estimates and the stability property of viscosity solutions.

# Chapter 5

# The obstacle problem for parabolic Monge-Ampère equation

## 5.1 Introduction

### 5.1.1 Backgrounds

The obstacle problem is an example of the free boundary problem, which arises in Stefan problem, option pricing of American option, fluid filtration in porous media, elasto-plasticity, optimal control, and financial mathematics [35, 12]. In [11], Caffarelli established the regularity of the free boundary for the classical obstacle problem. Later, this regularity result has been extended to various class of obstacle problems by many authors [53, 58, 52, 68, 31, 10, 5, 56].

The Monge-Ampère equation is one of the examples of fully nonlinear differential equations but it could be degenerated if the second derivative is degenerate, so we need extra estimates to obtain uniform parabolic operator. It arises in prescribed Gaussian curvature equation [62], optimal transportation [66], and affine geometry [71]. Also, it has been applied to image processing [64, 15, 59], where preserving sharp edges and reducing noises are important to overcome blurring problem. This problem can be resolved by the fact that the diffusion driven by the Gauss curvature is slow near edges due to the degeneracy of curvature.

Krylov suggested three versions of parabolic Monge-Ampère equation in [49]:

$$-u_t + (\det D^2 u)^{\frac{1}{n}} = f, \tag{5.1.1}$$

$$[(-u_t)\det D^2 u]^{\frac{1}{n+1}} = f, \tag{5.1.2}$$

$$[\det(D^2 u - u_t I_n)]^{\frac{1}{n}} = f, \tag{5.1.3}$$

where $I_n$ denotes the $n \times n$ identity matrix. Equation (5.1.1) is related to the graph representation of the Gauss curvature flow (see [55] for instance) and Equation (5.1.2) appears in the Gauss curvature flow represented by its support function [72].

The obstacle problem for (elliptic) Monge-Ampère equation was first considered by [54] and later its generalization to non-convex domains was studied by [74]. In addition, the very recent work by the first and the second authors concerns the obstacle problem for the $\alpha$-Gauss curvature flow in [55]. On the other hand, obstacle type problems with zero lower obstacle for Gauss curvature flow have been considered in [23, 24, 48, 19]. Also, the problem in the Alexandrov sense with zero lower obstacle is researched by [67].

### 5.1.2 Main results

In this chapter, we would like to consider the obstacle problem for the parabolic Monge-Ampère equation of the form (5.1.1). We prove the existence, uniqueness, and optimal regularity ($C^{1,1}$) under some structure conditions via the penalization method and a priori estimates. As a consequence of our approach, we also obtain the existence and uniqueness of the solution of the Cauchy-Dirichlet problem for the parabolic Monge-Ampère equation of the form (5.1.1) with the general forcing term $f(x, t, u, Du)$. Moreover, we discuss the regularity of the free boundary using the method of blowup.

Precisely, we consider the following version of parabolic obstacle problem for the Monge-Ampère equation:

$$\begin{cases} \min\left\{\phi - u, -u_t + \left(\det D^2 u\right)^{\frac{1}{n}} - f(x,t,u,Du)\right\} = 0 & \text{in } \Omega_T, \\ u = g & \text{on } \partial_p\Omega_T. \end{cases} \tag{PMAo}$$

Here $\Omega$ is a strictly convex bounded domain with $\partial\Omega \in C^{3,1}$, the forcing term $f \in C^{2,1}(\overline{\Omega_T} \times \mathbb{R} \times \mathbb{R}^n)$, the boundary data $g \in C^{3,1}(\partial_p\Omega_T)$, and the obstacle function $\phi \in C^{2,1}(\overline{\Omega_T})$ such that the obstacle $\phi$ lies above the boundary data $g$, i.e., $\phi > g$ on $\partial_p\Omega_T$.

To state our main results, we introduce structure conditions on $f$ and assumption on the existence of a subsolution.

(A1) The function $f$ has a lower bound: $f > \left(\min\limits_{\overline{\Omega_T}} \phi_t\right)^-$.

(A2) The function $f = f(x,t,z,p)$ is nondecreasing in $z$.

(A3) The function $f = f(x,t,z,p)$ is convex with respect to $p$.

(A4) There exists a strictly convex subsolution $\underline{u} \in C^2(\overline{\Omega_T})$ satisfying

$$-\underline{u}_t + (\det \underline{u}_{ij})^{\frac{1}{n}} \geq f(x,t,\underline{u},D\underline{u}) \text{ in } \Omega_T \quad \text{and} \quad \underline{u} = g \text{ on } \partial_p\Omega_T.$$

(A5) There exists a nonnegative constant $a$ such that

$$a\left(1 - |f_p|d - \frac{f_z d^2}{2}\right) \geq f_t$$

and

$$\min\left\{\inf_{\partial\Omega\times[0,T]} g_t, \inf_{\Omega\times\{0\}} \left((\det D^2 g)^{\frac{1}{n}} - f(\cdot,g,Dg)\right)\right\} + f > \frac{1}{2}ad^2,$$

where $d$ denotes the minimum radius such that $\Omega \subset B_d$. In particular, $a = 0$ if $f_t \leq 0$.

We will discuss about the conditions in Section 5.1.3 below.

Now we state our first main result:

**Theorem 5.1.1.** *If the assumptions* (A1)-(A5) *hold, then there exists a unique strictly convex viscosity solution* $u$ *of* (PMAo) *with the optimal regularity,* $C^{1,1}(\overline{\Omega_T})$*, satisfying* $u \geq \underline{u}$ *in* $\Omega_T$.

The main strategy to have Theorem 5.1.1 is the penalization method. Since we expect the solution to stay below the obstacle, a discontinuity of the velocity $u_t$ occurs when the solution touches the obstacle. This makes the problem difficult and that is why the optimal regularity of the solution to (PMAo) is expected to be $C^{1,1}$ (see Section 5.2 for the definition). To have the optimal regularity of the solution to (PMAo), we approximate the obstacle problem (PMAo) by allowing the solution can pass the obstacle, with the property such that the more the solution pass the obstacle, the more "*penalty*" is imposed on the velocity $u_t$. This approximation problem is formulated as ($\mathrm{PMAo}_\epsilon$) in Section 5.3. We will prove various a priori estimates for solutions $u^\epsilon$ of the approximation problem ($\mathrm{PMAo}_\epsilon$) in Subsections 5.3.1-5.3.3. The existence of $u^\epsilon$ and Theorem 5.1.1 can be given by a priori $C^{2,\alpha}$-estimates of $u^\epsilon$ and the method of continuity (see Theorem 3.13 in [61] for instance).

We note that all the equations (5.1.1)-(5.1.3) can be viewed as concave operators which are homogeneous of degree one. However, this homogeneity causes some difficulties if we try to obtain interior $C^{1,1}$-estimates since second derivatives of homogeneous operators of degree one must be degenerate in some direction. The equations (5.1.2) and (5.1.3) can make it possible to overcome the difficulties by taking logarithm to both sides, which is not the case for (5.1.1). Thus, the convexity condition (A3) for the forcing term $f(x, t, u, Du)$ is assumed due to the special character of the parabolic operator (5.1.1). Including (A3), general and reasonable structure conditions which have been considered in the literature are supposed to $f(x, t, u, Du)$, see Section 5.1.3 below for details.

As we mentioned above, we also discuss the existence of the following Cauchy-Dirichlet problem:

$$\begin{cases} -\partial_t u + \left(\det D^2 u\right)^{\frac{1}{n}} = f(x,t,u,Du) & \text{in } \Omega_T, \\ \qquad\qquad u = g & \text{on } \partial_p \Omega_T. \end{cases} \tag{PMA}$$

This result will be used to prove the existence of penalization problem (PMAo$_\epsilon$). Our proof will be based on the a priori estimates by using the method of continuity. To deal with the dependence on $u$ and $Du$ of the forcing term, we use a Pogorelov type computation while obtaining interior $C^{1,1}$-estimate. When the forcing term $f$ depends only on $x$ and $t$, this problem has been studied by some authors (see [39, 25, 69]).

**Theorem 5.1.2.** *If the assumptions* (A2)-(A5) *hold, then there exists a unique strictly convex solution $u \in C^3(\overline{\Omega_T})$ to* (PMA) *satisfying $u \geq \underline{u}$ in $\Omega_T$.*

The last result is the free boundary regularity of (PMAo). Since the operator $(\det D^2 u)^{\frac{1}{n}}$ is defined only in the space of positive definite matrix, the reduced problem, the obstacle problem with zero obstacle, is not appropriate in the problem for Monge-Ampère operator. Hence, contrary to Laplacian and fully nonlinear operator [53, 31, 32, 56], we develop the theory for (PMAo) as it is without using the reduced problem.

**Theorem 5.1.3** (Regularity of free boundary)**.** *Let $u \in P_1(M)$ with an obstacle $\phi$ such that*

$$\mathcal{P}\phi - f \geq c > 0 \quad \text{in } Q_1^-.$$

*Let $v := \phi - u$ and suppose*

$$\delta_r(v, X) \geq \epsilon_0 \quad \text{for all } r < 1/4, X \in Q_{1/2}^- \cap \partial N(v). \tag{5.1.4}$$

*Then there is $r_0 = r_0(u,\phi) > 0$ such that $\Gamma(u) \cap Q_{r_0}^-$ is $C^1$ graphs.*

We note that the linearized operator $\mathcal{L}_u = -\partial_t + F_{ij}(D^2u) \cdot \partial_{ij}$ plays an important role throughout Section 5.4 in such as non-degeneracy (Lemma 5.4.5), the classification of the global solutions (Proposition 5.4.7), and the directional monotonicity (Proposition 5.4.10).

### 5.1.3 Discussion on the conditions

The assumption (A1) implies $f > 0$ which is not assumed in Theorem 5.1.2. Unlike (PMA), the assumption (A1) is almost necessary condition in most cases. In fact, when the solution of (PMAo) touches the obstacle, we can deduce

$$(\det D^2u)^{\frac{1}{n}} \geq u_t + f = \phi_t + f \geq f - (\phi_t)^-$$

in the contact set, which ensures the convexity of the solution $u$. Here we used $f = f(\cdot, u, Du)$.

The monotone assumption (A2) is essential for uniqueness assertions. The assumption (A4) has been appeared in many literature, see [8] for instance.

The convexity assumption (A3) is assumed to have the optimal regularity in Theorem 5.1.1. This assumption has been commonly used in the Hessian equation (see [61]). For example, $S_k^{1/k}$ for $1 \leq k < n+1$ and $(S_k/S_m)^{1/(k-m)}$ for $1 \leq m < k \leq n+1$ need the convexity assumption in the gradient variable of $f$, where $S_k = S_k(D^2u, -u_t)$ denotes the elementary symmetric polynomial of degree $k$ in the eigenvalues of $\begin{pmatrix} D^2u & 0 \\ 0 & -u_t \end{pmatrix}$. In case of $S_{n+1}^{1/(n+1)}$, the second form in (5.1.3), the convexity assumption is not needed since it can be considered as an operator which is non-homogeneous and still concave, by taking logarithm to both sides. Our equation, the first form in (5.1.3), is more likely to $S_k^{1/k}$ for $k < n+1$.

The assumption (A5) will be used to show the preservation of convexity, see Lemma 5.3.3. Such conditions also appeared in [39, 69].

### 5.1.4 Notations

| | |
|---|---|
| $Q_r^-(x,t)$ | $B_r(x) \times (t-r^2,t]$ |
| $\Omega_T$ | $\Omega \times (0,T]$ |
| $\Omega(t)$ | $\Omega \times \{t\}$ the time section with respect to $t$ |
| $N(u)$ | $\{(x,t) \in \Omega_T \,\vert\, u(x,t) < \phi(x,t)\}$ the non-coincident set |
| $\Lambda(u)$ | $\{(x,t) \in \Omega_T \,\vert\, u(x,t) = \phi(x,t)\}$ the coincident set |
| $\Gamma(u)$ | $\partial N(u) \cap \Omega_T$ the free boundary |
| $\mathcal{S}^{n\times n}$ | the set of symmetric $n \times n$ matrices |
| $\mathbb{S}^n$ | $\{x \in \mathbb{R}^{n+1} \,\vert\, \vert x\vert = 1\}$ the unit n-sphere |
| $F(\mathcal{M})$ | $(\det \mathcal{M})^{\frac{1}{n}}$ for $\mathcal{M} \in \mathcal{S}^{n\times n}$ |
| $\mathcal{P}u$ | $-\partial_t u + F(D^2u)$ |
| $\mathcal{L}_u$ | $-\partial_t + F_{ij}(D^2u) \cdot \partial_{ij}$ |
| $P_r(M),\ P_\infty(M)$ | see Definitions 5.4.2 and 5.4.3 |
| $\delta_r(u,x),\ \delta_r(u)$ | see Definition 5.4.1 |
| $L_q(\Omega_T), W_q^{2l,l}(\Omega_T)$ | see Section 5.2 |
| $C^k(\overline{\Omega_T}), C^{k,\alpha}(\overline{\Omega_T})$ | see Section 5.2 |

### 5.1.5 Outline

The organization of the chapter is as follows. In Section 5.2, we provide definitions of viscosity solutions and give a proof of the fact that the $n$-th root of the determinant is a concave operator. In Section 5.3, a priori $C^{1,1}$-estimates for the approximation problem (PMAo$_\epsilon$) are established and finish the proof of Theorem 5.1.1 and Theorem 5.1.2. Finally, in Section 5.4, we study the regularity of the free boundary of the obstacle problem.

## 5.2 Preliminaries

We give definitions of function spaces over space-time domain. Also, we introduce the concept of viscosity solutions that is useful to define solutions of obstacle problem for non-divergence form operator. Finally, we prove the concavity of the operator $F$ which we use later in the chapter.

The *Lebesgue space* $L_q(\Omega_T)$ for $q \geq 1$ consists of all measurable functions on $Q_T$ with a finite norm

$$\|u\|_{L^q(\Omega_T)} = \left(\int_0^T \int_\Omega |u(x,t)|^q dxdt\right)^{\frac{1}{q}} \quad \text{and} \quad \|u\|_{L^\infty(\Omega_T)} = \operatorname*{ess\,sup}_{\Omega_T} |u|.$$

The *Sobolev space* $W_q^{2l,l}(\Omega_T)$ for integer $l$ and $q \geq 1$ consists of the elements of $L_q(\Omega_T)$ having generalized derivatives of the form $D_t^r D_x^s$ with any $r$ and $s$ satisfying the inequality $2r + s \leq 2l$. We define its *norm* to be

$$\|u\|_{W_q^{2l,l}(\Omega_T)} = \sum_{j=0}^{2l} \langle\langle u\rangle\rangle_{q,\Omega_T}^j, \quad \text{where } \langle\langle u\rangle\rangle_{q,\Omega_T}^j = \sum_{2r+s=j} \|D_t^r D_x^s u\|_{q,\Omega_T}.$$

Given a nonnegative integer $k$, the function space $C^k(\overline{\Omega_T})$ is the Banach space of all continuous functions on $\Omega_T$ with derivatives of the form $D_x^\gamma D_t^s$ for all $|\gamma| + 2s \leq k$, where

$$\gamma = (\gamma^1, \cdots, \gamma^n), \quad |\gamma| = \gamma^1 + \cdots + \gamma^n, \quad \text{and} \quad D_x^\gamma = \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} \cdots \partial x_n^{\gamma_n}},$$

under the norm

$$\|u\|_{C^k(\overline{\Omega_T})} = \sum_{|\gamma|+2s \leq k} \sup_{(x,t)\in\overline{\Omega_T}} |D_x^\gamma D_t^s u(x,t)| < \infty.$$

Given a nonnegative integer $k$ and $0 < \alpha < 1$, the *Hölder space* $C^{k,\alpha}(\overline{\Omega_T})$ is the Banach space of functions in $C^k(\overline{\Omega_T})$ under the norm

$$\begin{aligned}\|u\|_{C^{k,\alpha}(\overline{\Omega_T})} &= \|u\|_{C^k(\overline{\Omega_T})} \\ &\quad + \sum_{|\gamma|+2s=k} \sup_{(x_1,t_1)\neq(x_2,t_2)\in\overline{\Omega_T}} \frac{|D_x^\gamma D_t^s u(x_1,t_1) - D_x^\gamma D_t^s u(x_2,t_2)|}{(|x_1-x_2|^2 + |t_1-t_2|)^{\alpha/2}} < \infty.\end{aligned}$$

For simplicity of notation, we define

$$\|u\|_k = \|u\|_{C^k(\overline{\Omega_T})} \quad \text{and} \quad \|u\|_{k,\alpha} = \|u\|_{C^{k,\alpha}(\overline{\Omega_T})},$$

and denote $C^{k,1}(\overline{\Omega_T}) = W_\infty^{k+1,(k+1)/2}(\Omega_T)$. For a forcing term $f(x,t,z,p)$, $f \in C^{k,1}(\overline{\Omega_T} \times \mathbb{R} \times \mathbb{R}^n)$ is understood in a natural way.

Now we consider definitions of superjet and subjet. The concept of viscosity solutions to (PMAo) will then follow.

**Definition 5.2.1** (Superjet and Subjet)**.** Let $u$ be an upper (resp. lower) semi-continuous function on $\overline{\Omega_T}$ and $(z,s) \in \Omega_T$. The *superjet* $J^+_{\Omega_T}u(z,s)$ (resp. *subjet* $J^-_{\Omega_T}u(z,s)$) of $u$ at $(z,s)$ is defined to be the set of points $(a,p,X) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^{n\times n}$ such that

$$u(x,t) \leq (\text{resp. } \geq) u(z,s)+a(t-s)+\langle p, x-z\rangle+\frac{1}{2}\langle X(x-z), x-z\rangle+o(|t-s|+|x-z|^2)$$

as $(x,t) \to (z,s)$ in $\Omega_T$.

We also set $J'^-_{\Omega_T}u(z,s) = J^-_{\Omega_T}u(z,s) \cap (\mathbb{R}\times\mathbb{R}^n\times\mathcal{S}^{n\times n}_+)$, where $\mathcal{S}^{n\times n}_+$ denotes the set of positive semi-definite symmetric $n \times n$ matrices.

The following lemma explains the reason why we do not consider $J'^+_{\Omega_T}u(z,s)$.

**Lemma 5.2.2** ([1])**.** *Let $u$ be an upper semi-continuous in $\overline{\Omega_T}$. Then, $u(\cdot,t)$ is convex for each $t \in [0,T]$ if and only if $X \geq 0$ for all $(a,p,X) \in J^+_{\Omega_T}u(z,s)$ and $(z,s) \in \Omega_T$.*

Now we are ready to define viscosity solutions.

**Definition 5.2.3** (Viscosity Solutions)**.** Let $u$ be a function on $\overline{\Omega_T}$ such that $u(\cdot,t)$ is convex in $\Omega$ for each $t \in [0,T]$.

(i) We say that an upper (resp. lower) semi-continuous function $u$ on $\overline{\Omega_T}$ is said to be a *viscosity subsolution* (resp. *supersolution*) of

$$-\partial_t u + (\det D^2 u)^{\frac{1}{n}} = f(x,t,u,Du) \tag{5.2.1}$$

in $\Omega_T$ if for all $(x,t) \in \Omega_T$ and $(a,p,X) \in J^+_{\Omega_T}u(x,t)$ (resp. $J'^-_{\Omega_T}u(x,t)$),

$$-a + (\det X)^{\frac{1}{n}} \geq (\text{resp. } \leq) f(x,t,u(x,t),p).$$

(ii) We say that $u \in C(\overline{\Omega_T})$ is a *viscosity solution* of (5.2.1) if it is both a viscosity subsolution and a viscosity supersolution of (5.2.1).

(iii) A *viscosity solution* of (PMAo) is an upper semi-continuous function on $\overline{\Omega_T}$ such that $u$ is a viscosity subsolution in $\Omega_T$ and a viscosity supersolution in $N(u)$ of (5.2.1), $u \leq \phi$ in $\Omega_T$, and $u = g$ on $\partial_p \Omega_T$.

We close this section by showing that the operator $(\det \mathcal{M})^{\frac{1}{n}}$ is concave.

**Lemma 5.2.4.** *Let $F(\mathcal{M}) = (\det \mathcal{M})^{\frac{1}{n}}$ be an operator defined for $\mathcal{M} \in \mathcal{S}_+^{n\times n}$, where $\mathcal{S}_+^{n\times n}$ is the set of positive semi-definite symmetric $n\times n$ matrices. Then $F$ is a concave operator.*

*Proof.* By continuity of the operator $F$, it suffices to assume $\mathcal{M}$ is positive definite matrix. We observe that the concavity assertion is equivalent to

$$\ddot{F}(\mathcal{N}, \mathcal{N}) = \sum_{i,j,k,l} F^{ij,kl} N_{ij} N_{kl} \leq 0 \quad \text{for any matrix } \mathcal{N} = (N_{ij}). \tag{5.2.2}$$

To show this, we need the formulas

$$F^{ij} = \frac{1}{n} F M^{ij}, \tag{5.2.3}$$

$$F^{ij,kl} = \frac{1}{n^2} F M^{ij} M^{kl} - \frac{1}{n} F M^{ik} M^{jl}, \tag{5.2.4}$$

where $M^{ij}$ is the $(i,j)$-component of the inverse matrix of $\mathcal{M}$. We may assume $\mathcal{M}$ is a diagonal matrix since $F(\mathcal{M}) = F(\mathcal{U}\mathcal{M}\mathcal{U}^{-1})$ for any invertible matrix $\mathcal{U}$. Let us denote this diagonal matrix by $\mathcal{M} = \operatorname{diag}(M_1, M_2, \cdots, M_n)$, where each $M_i$ is positive. Then the left hand side of (5.2.2) becomes

$$\begin{aligned}
\ddot{F}(\mathcal{N}, \mathcal{N}) &= \sum_{i,k} \frac{F}{n^2 M_i M_k} N_{ii} N_{kk} - \sum_{i,j} \frac{F}{n M_i M_j} N_{ij}^2 \\
&= \sum_i \frac{(1-n)F}{n^2 M_i^2} N_{ii}^2 + \sum_{i\neq j} \frac{F}{n^2 M_i M_j} N_{ii} N_{jj} - \sum_{i \neq j} \frac{F}{n M_i M_j} N_{ij}^2 \\
&= -\frac{F}{2n^2} \sum_{i\neq j} \left( \frac{N_{ii}}{M_i} - \frac{N_{jj}}{M_j} \right)^2 - \sum_{i\neq j} \frac{F}{n M_i M_j} N_{ij}^2
\end{aligned}$$

which is clearly nonpositive. This completes the proof. $\square$

## 5.3 The optimal regularity

In this section we will obtain the existence and optimal ($C^{1,1}$) regularity of solutions to (PMAo), as stated in Theorem 5.1.1. Our proof will be based on the uniform a priori estimates for the singular perturbation problem defined below in order to use the method of continuity. Throughout the section, we assume the assumptions in Theorem 5.1.1.

Let us introduce the following singular perturbation problem with a penalty term:

$$\begin{cases} -u_t^\epsilon + \left(\det D^2 u^\epsilon\right)^{\frac{1}{n}} = f(x,t,u^\epsilon,Du^\epsilon) - \beta_\epsilon(\phi - u^\epsilon) & \text{in } \Omega_T, \\ u^\epsilon = g & \text{on } \partial_p\Omega_T, \end{cases} \tag{PMAo$_\epsilon$}$$

for given $\varepsilon > 0$, where $\beta_\epsilon \in C^\infty(\mathbb{R})$ is a so called penalty function satisfying

$$\begin{cases} \beta_\epsilon(z) < 0, \quad \beta_\epsilon'(z) > 0, \quad \beta_\epsilon''(z) \le 0 \text{ for } z < 0, \\ \beta_\epsilon(z) \to -\infty & \text{if } z < 0, \epsilon \to 0, \\ \beta_\delta(z) = 0 & \text{if } z \ge 0, \\ \beta_\epsilon(-\varepsilon) = -1. \end{cases}$$

We begin with the uniform boundedness of $\beta_\epsilon(\phi - u^\epsilon)$ which is important when we deal with convergence.

**Lemma 5.3.1.** *Let $u^\varepsilon$ be a solution of* (PMAo$_\epsilon$). *Then*

$$-C \le \beta_\delta(\phi - u^\varepsilon) \le 0,$$

*where the constant $C = C(\|\phi\|_2, n)$ is independent of $\varepsilon$.*

*Proof.* By the definition of $\beta_\delta$, we see that $\beta_\delta(\phi - u^\varepsilon) \le 0$ on $\overline{\Omega_T}$. To have the lower bound of $\beta_\delta$, we define

$$w(x,t) := \beta_\epsilon(\phi(x,t) - u^\epsilon(x,t))$$

and take $X_0 \in \overline{\Omega_T}$ such that $w(X_0) = \inf_{\overline{\Omega_T}} w < 0$. Since $u^\varepsilon = g < \phi$ on $\partial_p\Omega_T$ and $\beta_\delta(z) = 0$ if $z \geq 0$, we have $X_0 \in \Omega_T$. Thus it follows from the monotone increasing property of $\beta_\delta$ on $(-\infty, 0)$ that $\phi - u_\epsilon$ also has a minimum at $X_0$, and consequently $D^2\phi(X_0) \geq D^2u^\epsilon(X_0)$ and $\partial_t\phi(X_0) \leq \partial_t u^\epsilon(X_0)$. Hence,

$$\begin{aligned} f(x,t,u^\epsilon, Du^\epsilon) - w(X_0) &= -\partial_t u^\epsilon(X_0) + (\det D^2u^\epsilon(X_0))^{\frac{1}{n}} \\ &\leq -\partial_t\phi(X_0) + (\det D^2\phi(X_0))^{\frac{1}{n}} = \mathcal{P}\phi(X_0). \end{aligned}$$

Therefore, we have $w(X_0) \geq -\mathcal{P}\phi(X_0)$. This completes the proof. □

Our next task is to show the uniform $C^1$ estimate which enables us to control the forcing term $f$. It is exploited that the subsolution $\underline{u}$ in the assumption (A4) is also a subsolution of (PMAo$_\epsilon$).

**Lemma 5.3.2.** *Let $u^\varepsilon$ be a solution of* (PMAo$_\epsilon$) *and assume* (A1) *and* (A4). *Then*

$$\|u^\epsilon\|_1 \leq C_0,$$

*for some constant $C_0 = C_0(\|\underline{u}\|_1, \|g\|_1, \Omega_T, n)$ independent of $\varepsilon$.*

*Proof.* We start with a unique function $h$ satisfying

$$\begin{cases} -\partial_t h + \dfrac{1}{n}\Delta h = 0 & \text{in } \Omega_T, \\ h = g & \text{on } \partial_p\Omega_T. \end{cases} \tag{5.3.1}$$

Since $(\det D^2u^\varepsilon)^{\frac{1}{n}} \leq \frac{1}{n}\Delta u^\varepsilon$ and $f - \beta_\delta > 0$, $u^\varepsilon$ is a subsolution of (5.3.1) and $u^\varepsilon \leq h$ in $\overline{\Omega_T}$. On the other hand, since the function $\underline{u} \leq \phi$ and thus $\beta_\delta(\phi - \underline{u}) = 0$, $\underline{u}$ is a subsolution of (PMAo$_\epsilon$) and $u^\varepsilon \geq \underline{u}$ in $\overline{\Omega_T}$. Thus, we have $\underline{u} \leq u^\varepsilon \leq h$ in $\overline{\Omega_T}$ and from the strictly convexity of $u^\varepsilon$, it is easy to show

$$\sup_{\overline{\Omega_T}} |Du^\varepsilon| = \sup_{\partial_p \Omega_T} |Du^\varepsilon|.$$

Furthermore, it follows from $\underline{u} = u^\varepsilon = h$ on $\partial_p \Omega_T$ that for any $x \in \partial\Omega$ and $0 \le t \le T$,

$$D_e \underline{u}(x,t) \le D_e u^\varepsilon(x,t) \le D_e h(x,t),$$

where $e$ denotes the inward unit normal to $\partial\Omega$ at $x$. Thus we can conclude

$$\|u^\varepsilon\|_1 \le C_0(\|\underline{u}\|_1, \|g\|_1, \Omega_T, n).$$

This completes the proof. $\square$

Using Lemma 5.3.2, there exist constants $\mu_1$ and $\mu_2$ (independent of $\varepsilon$) such that

$$0 < \mu_1 = \inf_{\overline{\Omega_T}} f(\cdot, u^\varepsilon, Du^\varepsilon) \le \sup_{\overline{\Omega_T}} f(\cdot, u^\varepsilon, Du^\varepsilon) = \mu_2 < \infty. \tag{5.3.2}$$

From the initial data $g$, there are also constants $\kappa_1$ and $\kappa_2$ such that

$$\begin{aligned} \kappa_1 &= \min\left\{ \inf_{\partial\Omega\times[0,T]} g_t, \inf_{\Omega\times\{0\}} \left((\det D^2 g)^{\frac{1}{n}} - f(\cdot, g, Dg)\right) \right\} \\ &\le \max\left\{ \sup_{\partial\Omega\times[0,T]} g_t, \sup_{\Omega\times\{0\}} \left((\det D^2 g)^{\frac{1}{n}} - f(\cdot, g, Dg)\right) \right\} = \kappa_2. \end{aligned}$$

### 5.3.1 Preservation of convexity and a priori speed estimate

This subsection will be devoted to the proof of a preservation of convexity and a speed bound. We start with the preserving of convexity whose direct consequence is a lower bound for the speed. From the assumptions (A1) and

(A5) in Theorem 5.1.1, we can take a positive constant $\nu$ such that

$$\min\{\phi_t, 0\} + \mu_1 \geq \nu \quad \text{and} \quad \kappa_1 + \mu_1 - \frac{1}{2}ad^2 \geq \nu. \tag{5.3.3}$$

**Lemma 5.3.3.** *Let $u^\varepsilon$ be a solution of* (PMAo$_\epsilon$) *and assume* (A1), (A2), *and* (A5). *Then*

$$(\det D^2u^\varepsilon)^{\frac{1}{n}} = u_t^\varepsilon + f(\cdot, u^\varepsilon, Du^\varepsilon) \geq \nu,$$

*where $\nu$ is the constant defined in* (5.3.3). *In particular, $u_t^\varepsilon \geq \nu - \mu_2$.*

**Remark 5.3.4.** When $f = f(x,t)$ and $\max(f_t)_+ < \infty$, we may take $a = \max(f_t)_+$ as in [39, 69].

*Proof.* Let $\nu$ be a positive constant satisfying (5.3.3). By translation, we can assume $\Omega \subset B_{d/2}$. Let $\mathcal{L}_1 = -\partial_t + F^{ij}\partial_{ij} - f_{p_i}\partial_i - f_z$. Then we have

$$\mathcal{L}_1 u_t^\varepsilon = D_t(f - \beta_\delta(\phi - u^\varepsilon)) - f_{p_i}(u_t^\varepsilon)_i - f_z u_t^\varepsilon = f_t + \beta_\delta'(u^\varepsilon - \phi)_t.$$

For a small constant $b > 0$, consider an auxiliary function $w = \frac{1}{2}a|x|^2 - bt$, where $a$ is the constant in the assumption (A5) of Theorem 5.1.1. Since $\sum_{i=1}^n F^{ii} = \frac{F}{n}\operatorname{tr}((D^2u^\varepsilon)^{-1}) \geq 1$, we obtain

$$\mathcal{L}_1 w \geq a\left(1 - f_{p_i}x_i - f_z|x|^2/2\right) + b \geq a\left(1 - |f_p|d - f_z d^2/2\right) + b \geq f_t + b.$$

Thus, we have shown $\mathcal{L}_1(u_t^\varepsilon - w) \leq \beta_\delta'(u^\varepsilon - \phi)_t - b$.

If $u_t - w$ attains an interior minimum over $\overline{\Omega_T}$, we have $\mathcal{L}_1(u_t^\varepsilon - w) \geq -f_z(u_t^\varepsilon - w)$ at the minimum point. Since both $u_t^\varepsilon - w$ and $u_t^\varepsilon - \phi_t$ are less than or equal to $u_t^\varepsilon - \min\{\phi_t, 0\} + bT$, we obtain

$$b \leq f_z(u_t^\varepsilon - w) + \beta_\delta'(u_t^\varepsilon - \phi_t) \leq (f_z + \beta_\delta')(u_t^\varepsilon - \min\{\phi_t, 0\} + bT)$$

which implies $u_t^\varepsilon \geq \min\{\phi_t, 0\} - bT$ from $f_z + \beta'_\delta \geq 0$. Otherwise, we have

$$u_t^\varepsilon \geq \min g_t - \frac{1}{2}ad^2.$$

In any case, it follows from (5.3.3) that

$$u_t^\varepsilon + f \geq \nu - bT.$$

By taking $b \to 0$, we have the desired result. $\square$

Next, we will show an upper bound for the speed.

**Lemma 5.3.5.** *Let $u^\varepsilon$ be a solution of* (PMAo$_\epsilon$) *and assume* (A2). *Then we have*

$$u_t^\varepsilon \leq e^T \max\left\{ \left( \inf_{v \in \mathcal{A}} \inf_{\Omega_T} f_t(\cdot, v, Dv) \right)^-, \kappa_2, \phi_t \right\}.$$

*Proof.* Let us define $v = e^{-t}u_t^\varepsilon$ and then, it is easy to verify that

$$\mathcal{L}_1 v = e^{-t}(f_t + u_t^\varepsilon - \beta'_\delta(\phi_t - u_t^\varepsilon)), \tag{5.3.4}$$

where $\mathcal{L}_1 = -\partial_t + F^{ij}\partial_{ij} - f_{p_i}\partial_i - f_z$. If a maximum of $v$ is attained on the parabolic boundary $\partial_p\Omega_T$, then we are done. Suppose that $v$ has its positive maximum over $\overline{\Omega_T}$ at $X_0 \in \Omega_T$. Then we have $\mathcal{L}_1 v \leq 0$ at $X_0$, and therefore, (5.3.4) becomes

$$f_t + u_t^\varepsilon - \beta'_\delta(\phi_t - u_t^\varepsilon) \leq 0 \quad \text{at } X_0.$$

At this point, if $\phi_t - u_t^\varepsilon \leq 0$, then $u_t^\varepsilon \leq -f_t$. Otherwise, $u_t^\varepsilon \leq \phi_t$. In any case, we have $0 < v(X_0) \leq u(X_0) \leq \max\{-f_t, \phi_t\}$, and the conclusion follows. $\square$

**Remark 5.3.6.** In case of (PMA), we also have the similar results to Lemma 5.3.3

and Lemma 5.3.5. More precisely, we can obtain

$$u_t + f(\cdot, u, Du) \geq \nu \quad \text{and} \quad u_t \leq e^T \max\left\{ \left( \inf_{v\in\mathcal{A}} \inf_{\overline{\Omega_T}} f_t(\cdot, v, Dv) \right)^-, \kappa_2 \right\}$$

without assumption (A1) in Theorem 5.1.1 by applying the comparison principle directly.

### 5.3.2 A priori interior $C^{1,1}$-estimate

We will prove a priori interior $C^{1,1}$-estimate for $u^\epsilon$. From the results of the previous subsection, Lemmas 5.3.3 and 5.3.5, we note that

$$0 < \nu \leq F \leq \|F\|_0 < \infty, \tag{5.3.5}$$

where

$$\|F\|_0 := \left\|F(D^2u^\epsilon)\right\|_0 = \sup_{(x,t)\in\overline{\Omega_T}} |F(D^2u^\epsilon(x,t))| = \sup_{(x,t)\in\overline{\Omega_T}} \left|\det(D^2u^\epsilon(x,t))^{\frac{1}{n}}\right|$$

is a bounded quantity. We also notice from Lemma 5.3.2 that

$$\|f\|_{1,1}^* := \|f\|_{C^{1,1}(\overline{\Omega_T}\times[-C_0,C_0]\times[-C_0,C_0]^n)} < \infty,$$

and we define $\|f\|_{k,1}^*$ in a similar way.

As we mentioned in the introduction, the operator $\mathcal{P}u$ must be degenerate in some direction so we need to assume the convexity of $f$ in the gradient variable to have $C^{1,1}$ estimate.

**Lemma 5.3.7.** *Let $u^\varepsilon$ be the solution of* (PMAo$_\epsilon$) *and assume* (A1)-(A3) *and* (A5)*. Then we have*

$$\sup_{\overline{\Omega_T}} |D^2u^\varepsilon| \leq C\left(1 + \sup_{\partial_p\Omega_T} |D^2u^\varepsilon|\right),$$

*where the constant $C$ depends only on $n, |\Omega|, \nu, \|F\|_0, \|\phi\|_{2,1}, \|f\|_2^*$, and $\|u\|_1$.*

*Proof.* In this proof, we will use $u$ instead of $u^\varepsilon$ for simplicity. For $\xi \in \mathbb{R}^n$, we set

$$w := u_{\xi\xi} \exp\left\{ \frac{a}{2}|D(u-\phi)|^2 + \frac{b}{2}|x|^2 \right\},$$

where $a$ and $b$ are positive constants to be determined later. Since $\overline{\Omega_T} \times \mathbb{S}^{n-1}$ is compact, $w$ has a maximum over this set. We may assume that the maximum of $w$ is achieved at some point $(x_0, t_0, \xi)$ in $\Omega_T \times \mathbb{S}^{n-1}$. By rotating the coordinates $\{x_1, \cdots, x_n\}$, we also assume that $\xi = e_1 = (1, 0, \cdots, 0)$ and that $D^2u(x_0, t_0)$ is diagonal.

If $u_{11}(x_0, t_0) \leq \phi_{11}(x_0, t_0)$, we are done. So we assume $u_{11}(x_0, t_0) \geq \phi_{11}(x_0, t_0)$. Setting $F(D^2u) = (\det D^2 u)^{\frac{1}{n}}$, we have

$$F^{ij} = \frac{F}{n} u^{ij} \quad \text{and} \quad F^{ij,kl} = \frac{F}{n^2} u^{ij} u^{kl} - \frac{F}{n} u^{ik} u^{jl}. \tag{5.3.6}$$

The linearized operator at $(x_0, t_0)$ is given by

$$\mathcal{L}_u v = -\partial_t v + F^{ij} \partial_{ij} v.$$

Now, we compute that

$$\begin{aligned}
\frac{w_t}{w} &= \frac{u_{11t}}{u_{11}} + a(u-\phi)_k (u-\phi)_{kt}, \\
\frac{w_i}{w} &= \frac{u_{11i}}{u_{11}} + a(u-\phi)_k (u-\phi)_{ki} + b x_i, \\
\frac{w_{ij}}{w} &= \frac{w_i w_j}{w^2} - \frac{u_{11i} u_{11j}}{u_{11}^2} + \frac{u_{11ij}}{u_{11}} \\
&\quad + a(u-\phi)_{ki}(u-\phi)_{kj} + a(u-\phi)_k (u-\phi)_{kij} + b\delta_{ij}.
\end{aligned}$$

Since $F^{ij} \frac{w_i w_j}{w^2} = \frac{F}{n} \frac{1}{w^2} u^{ij} w_i w_j$, we have

$$\begin{aligned}
\frac{\mathcal{L}_u w}{w} \geq & -\frac{F^{ij} u_{11i} u_{11j}}{u_{11}^2} + \frac{\mathcal{L}_u u_{11}}{u_{11}} + a(u-\phi)_k \mathcal{L}_u (u-\phi)_k \\
& + a F^{ij} (u-\phi)_{ki} (u-\phi)_{kj} + b F^{ij}.
\end{aligned} \tag{5.3.7}$$

Recall the equation ($\text{PMAo}_\epsilon$) and differentiate this to get

$$\begin{aligned}\mathcal{L}_u u_k &= D_k f - \beta'_\delta \cdot (\phi - u)_k,\\ \mathcal{L}_u u_{11} &= -F^{ij,kl} u_{ij1} u_{kl1} + D_{11} f - \beta'_\delta \cdot (\phi - u)_{11} - \beta''_\delta \cdot (\phi - u)_1^2.\end{aligned}$$

Since $\beta'_\delta \geq 0$, $\beta''_\delta \leq 0$, and $u_{11}(x_0, t_0) \geq \phi_{11}(x_0, t_0)$, we obtain at $(x_0, t_0)$,

$$\begin{aligned}(u - \phi)_k \mathcal{L}_u u_k &\geq (u - \phi)_k D_k f\\ \mathcal{L}_u u_{11} &\geq -F^{ij,kl} u_{ij1} u_{kl1} + D_{11} f.\end{aligned} \tag{5.3.8}$$

Moreover, we calculate

$$aF^{ij}(u - \phi)_{ki}(u - \phi)_{kj} \geq \frac{aF}{n}(\Delta u - 2\Delta\phi). \tag{5.3.9}$$

By multiplying (5.3.7) by $u_{11}(x_0, t_0)$ and replacing (5.3.8) and (5.3.9) into it, we infer that at $(x_0, t_0)$,

$$\begin{aligned}0 \geq \frac{\mathcal{L}w}{w} \geq &-\frac{F^{ij} u_{11i} u_{11j}}{u_{11}} - F^{ij,kl} u_{ij1} u_{kl1} + D_{11} f + a(u - \phi)_k D_k f u_{11}\\ &- a(u - \phi)_k \mathcal{L}\phi_k u_{11} + \frac{aF}{n}(\Delta u - 2\Delta\phi) u_{11} + \frac{bF}{n} u^{ij} u_{11}.\end{aligned} \tag{5.3.10}$$

The first inequality is obtained from that $w$ has it maximum at $(x_0, t_0)$.

On the other hand, from (5.3.6) and the fact that $D^2 u(x_0, t_0)$ is diagonal, we get

$$\begin{aligned}-\frac{F^{ij} u_{11i} u_{11j}}{u_{11}} - F^{ij,kl} u_{ij1} u_{kl1} &= -\frac{F u_{11i}^2}{n u_{11} u_{ii}} + \frac{F u_{ij1}^2}{n u_{ii} u_{jj}} - \frac{F u_{ii1} u_{jj1}}{n^2 u_{ii} u_{jj}}\\ &= \sum_i \sum_{j \neq 1} \frac{F u_{ij1}^2}{n u_{ii} u_{jj}} - \frac{F}{n^2}\left(\sum_i \frac{u_{ii1}}{u_{ii}}\right)^2 \geq -\frac{F u_{111}^2}{n u_{11}^2},\end{aligned} \tag{5.3.11}$$

by the Cauchy-Schwarz inequality,

$$\frac{F}{n^2}\left(\sum_i \frac{u_{ii1}}{u_{ii}}\right)^2 \le \frac{F}{n}\sum_i \frac{u_{ii1}^2}{u_{ii}^2}.$$

Next, we consider terms involving $f$. By simple computation,

$$\begin{aligned} D_k f &= f_k + f_z u_k + f_{p_i} u_{ik}, \\ D_{11} f &= f_{11} + 2f_{1z}u_1 + f_{zz}u_1^2 + 2f_{1p_i}u_{i1} + 2f_{zp_i}u_1u_{i1} + f_z u_{11} \\ &\quad + f_{p_ip_j}u_{i1}u_{j1} + f_{p_i}u_{i11}. \end{aligned} \tag{5.3.12}$$

Combining (5.3.10), (5.3.11) and (5.3.12),

$$\begin{aligned} 0 \ge &-\frac{Fu_{111}^2}{nu_{11}^2} + f_{p_1p_1}u_{11}^2 + f_{p_i}u_{i11} + a(u-\phi)_k f_{p_k} u_{kk} u_{11} \\ &+ \frac{aF}{n}(\Delta u - 2\Delta\phi)u_{11} + \frac{bF}{n}u^{ij}u_{11} - a(u-\phi)_k \mathcal{L}\phi_k u_{11} \\ &- C - C(1+a)u_{11}, \end{aligned} \tag{5.3.13}$$

where $C$ is a positive constant depending only on $\|u\|_1$, $\|f\|_2^*$, and $\|\phi\|_1$. From the convexity of $u$ and the boundedness of $F$,

$$\frac{aF}{n}(\Delta u - 2\Delta\phi)u_{11} \ge \frac{aFu_{11}^2}{n} - Cau_{11} \tag{5.3.14}$$

where $C$ depends only on $n$, $\|F\|_0$, and $\|\phi\|_2$. On the other hand, if we define

$$b := a\sup_{\overline{\Omega_T}} |(u-\phi)_k\phi_{ijk}|,$$

we see that

$$\frac{bF}{n}u^{ij}u_{11} - a(u-\phi)_k\mathcal{L}\phi_k u_{11} \ge a(u-\phi)_k\phi_t u_{11}. \tag{5.3.15}$$

Combining (5.3.13), (5.3.14), and (5.3.15), we arrive at

$$\begin{aligned}0 \geq & -\frac{Fu_{111}^2}{nu_{11}^2} + \left(\frac{F}{n}a + f_{p_1p_1}\right)u_{11}^2 + f_{p_i}u_{i11} + a(u-\phi)_k f_{p_k}u_{kk}u_{11} \\ & - C - C(1+a)u_{11},\end{aligned}$$

where the constant $C$ depends only on $n$, $\|F\|_0$, $\|\phi\|_{2,1}$, $\|f\|_2^*$, and $\|u\|_1$.

Notice that $w_i = 0$ at $(x_0, t_0)$, which implies

$$0 = \frac{u_{11i}}{u_{11}} + a(u-\phi)_k(u-\phi)_{ki} + bx_i,$$

and therefore,

$$\begin{aligned}f_{p_i}u_{i11} + a(u-\phi)_k f_{p_k}u_{kk}u_{11} &= f_{p_i}u_{11}(a(u-\phi)_k\phi_{ki} - bx_i), \\ -\frac{Fu_{111}^2}{nu_{11}^2} &\geq -\frac{F}{n}a^2(u-\phi)_1^2u_{11}^2 - Ca^2(u_{11}+1),\end{aligned}$$

where $C$ depends only on $n$, $\|F\|_0$, $\|\phi\|_{2,1}$, and $\|u\|_1$. If we define

$$\Theta := \sup_{\overline{\Omega_T}} |D(u-\phi)|^2,$$

we finally obtain that

$$0 \geq \left(-\frac{F\Theta}{n}a^2 + \frac{F}{n}a + f_{p_1p_1}\right)u_{11}^2 - C(1+a^2) - C(1+a+a^2)u_{11}.$$

Notice that

$$-\frac{F\Theta}{n}a^2 + \frac{F}{n}a + f_{p_1p_1} = -\frac{F\Theta}{n}\left(a - \frac{1}{2\Theta}\right)^2 + \frac{F}{4n\Theta} + f_{p_1p_1}$$

and that $f$ is convex with respect to $p$ variable so that $f_{p_1p_1} \geq 0$. Choosing $a = (2\Theta)^{-1}$, we conclude that

$$u_{11} \leq C,$$

where the constant $C$ depends on $n, |\Omega|, \nu, \|F\|_0, \|\phi\|_{3,1}, \|f\|_2^*$, and $\|u\|_1$. This completes the proof. $\square$

### 5.3.3 A priori boundary $C^{1,1}$-estimate

In this subsection, we consider a more general equation without the obstacle, (PMA),

$$\begin{cases} -\partial_t u + \left(\det D^2 u\right)^{\frac{1}{n}} = f(x,t,u,Du) & \text{in } \Omega_T, \\ \qquad\qquad u = g & \text{on } \partial_p \Omega_T. \end{cases}$$

The goal here is to prove the a priori boundary $C^{1,1}$-estimate for a solution of (PMA).

**Proposition 5.3.8.** *Let $u$ be a solution of* (PMA) *and assume* (A2), (A4), *and* (A5)*. Then we have*

$$\sup_{\partial_p \Omega_T} |D^2 u| \le C$$

*where the constant $C$ depends only on $\|f\|_2^*$, $\|g\|_{3,1}$, $\|\underline{u}\|_2$, $\Omega_T$, and $n$.*

We postpone its proof for a moment. Assuming this, we have the following:

**Proposition 5.3.9.** *Let $u^\varepsilon$ be the solution of* (PMAo$_\epsilon$) *and assume* (A1), (A2), (A4), *and* (A5)*. Then we have*

$$\sup_{\partial_p \Omega_T} |D^2 u^\epsilon| \le C$$

*where the constant $C$ depends only on $\|f\|_2^*$, $\|g\|_{3,1}$, $\|\underline{u}\|_2$, $\Omega_T$, and $n$.*

*Proof.* Let $h$ be the solution of (5.3.1) in Lemma 5.3.2. Then, $u^\varepsilon \le h$ in $\overline{\Omega_T}$ and $h = g$ on $\partial_p \Omega_T$. Let $\eta_0 := \frac{1}{2} \inf_{\partial_p \Omega_T} (\phi - h) = \frac{1}{2} \inf_{\partial_p \Omega_T} (\phi - g) > 0$. Now define a set

$$U = \{X \in \Omega_T : \phi - h > \eta_0\}$$

so that $U$ contains a tubular neighborhood of $\partial_p\Omega_T$. Then we deduce

$$\phi - u^\varepsilon \geq \phi - h > \eta_0 \quad \text{in } U$$

and we see that $\beta_\epsilon(\phi - u^\varepsilon) = 0$ in $U$. Thus, $u^\varepsilon$ satisfies

$$\begin{cases} -\partial_t u^\varepsilon + (\det D^2 u^\varepsilon)^{\frac{1}{n}} = f(X, u^\varepsilon, Du^\varepsilon) & \text{in } U, \\ \qquad\qquad\qquad u^\varepsilon = g & \text{on } \partial_p\Omega_T. \end{cases}$$

Now the conclusion follows from Proposition 5.3.8. □

Now we prove Proposition 5.3.8. First, we remark that (5.3.5) is still valid for solutions of (PMA) since we can easily repeat the arguments in Lemma 5.3.3 and Lemma 5.3.5 (see also Remark 5.3.6). Let us take a point $(x_0, t_0) \in \partial_p\Omega_T$. It is enough to consider the case that $t_0 > 0$, since $D^2u(x_0, 0) = D^2g(x_0, 0)$ is controlled by the initial data. Thus, we consider the case $x_0 \in \partial\Omega$ and $t_0 > 0$. We may assume that $x_0$ is the origin and the interior unit normal vector of $\partial\Omega$ at $x_0 = 0$ is $e_n$. In a small neighborhood $U'$ of $0'$ in $\mathbb{R}^{n-1}$, the boundary $\partial\Omega$ is given by a graph $(x', \rho(x'))$ where $x' = (x_1, \cdots, x_{n-1})$, and we can express $\rho$ as

$$x_n = \rho(x') = \frac{1}{2}\sum B_{\alpha\beta}x_\alpha x_\beta + O(|x'|^3), \tag{5.3.16}$$

where $B_{\alpha\beta} = \rho_{\alpha\beta}(0')$ and Greek letters $\alpha$ and $\beta$ go from 1 to $n-1$. We note that since $\Omega$ is strictly convex and bounded, $B_{\alpha\beta}$ is bounded below and above with constants depend only on the boundary of $\Omega$.

We start with the estimate of the second derivates with respect to tangential directions for the solution $u$.

**Lemma 5.3.10.** *Let $u$ be a solution of* (PMA). *Then we have*

$$|\partial_{\alpha\beta}u(0, t_0)| \leq C, \quad \textit{for } 1 \leq \alpha, \beta \leq n-1,$$

*where the constant $C$ depends only on $\partial\Omega$, $\|u\|_1$ and $\|g\|_2$.*

*Proof.* Since $u - g$ vanishes on the lateral boundary $\partial\Omega \times (0, T]$, we see

$$u(x', \rho(x'), t_0) = g(x', \rho(x'), t_0) \quad \text{in } U'. \tag{5.3.17}$$

This implies

$$D_{\alpha\beta}(u-g)(x', \rho(x'), t_0) = 0 \quad \text{for } 1 \leq \alpha, \beta \leq n-1,$$

so that

$$(\partial_\alpha + \rho_\alpha \partial_n)(\partial_\beta + \rho_\beta \partial_n)(u-g)(x', \rho(x'), t_0) = 0.$$

Since $\rho(0') = 0$ and $\rho_\alpha(0') = \rho_\beta(0') = 0$, this gives

$$(u-g)_{\alpha\beta}(0, t_0) = -\rho_{\alpha\beta}(0')(u-g)_n(0, t_0). \tag{5.3.18}$$

Now we have

$$|\partial_{\alpha\beta} u(0, t_0)| \leq |\partial_{\alpha\beta} g(0, t_0)| + \rho_{\alpha\beta}(0')|(u-g)_n(0, t_0)|,$$

and therefore, the conclusion follows. □

We briefly remark that (5.3.18) shows the relation of the normal and tangential derivatives, i.e., the normal derivative is heuristically equal to the second tangential derivatives. We also remark here that this tangential second derivative implied by not the equation but the regularity of domain $\Omega$, the boundary data $g$, and Lipschitz regularity of $u$ up to the boundary.

Now, we claim that $u^\varepsilon_{\xi\xi}$ has a uniform lower bound on the boundary for any tangential direction $\xi$.

**Lemma 5.3.11.** *Let $u$ be a solution of* (PMA) *and assume* (A1), (A2), *and* (A5). *Then there is a uniform constant $c_0 = c_0(\partial\Omega, \|g\|_{3,1}, \nu)$ such that*

$$\sum_{\alpha,\beta<n} u_{\alpha\beta}(0, t_0)\xi_\alpha \xi_\beta \geq c_0 > 0,$$

*for any unit vector* $\xi = (\xi_1, \cdots, \xi_{n-1})$.

*Proof.* We may assume that $\xi = e_1$. In the neighborhood $U' \times (0, T]$ of $(0', t_0)$, we can write

$$g(x', \rho(x'), t_0) = g(0, t_0) + g_\alpha(0, t_0)x_\alpha + \frac{1}{2}\gamma_{\alpha\beta}x_\alpha x_\beta + O(|x'|^3). \qquad (5.3.19)$$

Let $\lambda = \frac{\gamma_{11}}{\rho_{11}(0')} = \frac{\gamma_{11}}{B_{11}}$, $A = (g_1(0, t_0), \cdots, g_{n-1}(0, t_0), \lambda)$,

$$\tilde{u} = u - g(0, t_0) - A \cdot x,$$

and

$$\tilde{f}(x, t, z, p) = f(x, t, z + g(0, t_0) + A \cdot x, p + A).$$

Then, it is easy to verify that

$$-\partial_t \tilde{u} + (\det D^2 \tilde{u})^{\frac{1}{n}} = \tilde{f}(x, t, \tilde{u}, D\tilde{u}).$$

Moreover, from (5.3.17), (5.3.19), and the definitions of $\tilde{u}$ and $\lambda$, we have

$$D_{11}\tilde{u}(0, t_0) = D_{11}u(0, t_0) - \lambda\rho_{11}(0') = D_{11}g(0, t_0) - \lambda B_{11} = \gamma_{11} - \lambda B_{11} = 0, \qquad (5.3.20)$$

which implies

$$u_{11}(0, t_0) = \tilde{u}_{11}(0, t_0) = -\tilde{u}_n(0, t_0)\rho_{11}(0')$$

since $D_{11}\tilde{u}(0, t_0) = \tilde{u}_{11}(0, t_0) + \tilde{u}_n(0, t_0)\rho_{11}(0')$.

It remains to show that $-\tilde{u}_n(0, t_0)$ has a uniform positive lower bound. Hence, we construct a barrier function for $\tilde{u}$. By (5.3.16) and Young's inequality,

$$\begin{aligned} x_1^3 &= \frac{2x_1}{B_{11}} \left( x_n - \frac{1}{2} \sum_{(\alpha,\beta)\neq(1,1)} B_{\alpha\beta} x_\alpha x_\beta + O(|x'|^3) \right) \\ &\leq \frac{2}{B_{11}} x_1 x_n + C \left( \sum_{1<\beta<n} x_\beta^2 + |x|^4 \right) \qquad \text{in } U', \end{aligned} \tag{5.3.21}$$

where $C = C(\partial\Omega)$. On the other hand, we see from (5.3.17), (5.3.19) and (5.3.20) that

$$\tilde{u} = u - g(0,t_0) - A \cdot x = \frac{1}{2} \sum_{(\alpha,\beta)\neq(1,1)} \gamma_{\alpha\beta} x_\alpha x_\beta + O(|x'|^3) \quad \text{in } U'. \tag{5.3.22}$$

Therefore, by (5.3.21), (5.3.22) and Young's inequality, there is $C(\partial\Omega, \|g\|_{3,1})$ such that

$$\tilde{u}|_{\partial\Omega} \leq \sum_{1<j\leq n} a_{1j} x_1 x_j + C \left( \sum_{1<\beta<n} x_\beta^2 + |x|^4 \right),$$

for some constants $a_{1j}$, $1 < j \leq n$.

Consider a barrier function $h$ defined by

$$h = -a x_n + b|x|^2 + \frac{1}{2B} \sum_{1<j\leq n} (a_{1j} x_1 + B x_j)^2,$$

where constants $a$, $b$, and $B$ will be determined below. Denoting the $k \times k$

identity matrix by $I_k$, we see that

$$D^2h = 2bI_n + \begin{pmatrix} \frac{1}{B}(a_{12}^2 + \cdots + a_{1n}^2) & a_{12} & \cdots & a_{1n} \\ a_{12} & & & \\ \vdots & & BI_{n-1} & \\ a_{1n} & & & \end{pmatrix}$$

$$\sim \begin{pmatrix} 2b + (\frac{1}{B} - \frac{1}{2b+B})(a_{12}^2 + \cdots + a_{1n}^2) & 0 & \cdots & 0 \\ a_{12} & & & \\ \vdots & & (2b+B)I_{n-1} & \\ a_{1n} & & & \end{pmatrix}$$

and thus $\det D^2h = (2b+B)^{n-1}2b\left(1 + \frac{1}{B(B+2b)}(a_{12}^2 + \cdots + a_{1n}^2)\right)$. First, we take a constant $B = B(C)$ such that

$$\sum_{1<j\le n} a_{1j}x_1x_j + C\left(\sum_{1<\beta<n} x_\beta^2 + |x|^4\right) \le \frac{1}{2B}\sum_{1<j\le n}(a_{1j}x_1 + Bx_j)^2 \quad \text{on } \partial\Omega.$$

Then, choose small $b = b(\partial\Omega, \|g\|_{3,1}, \nu) > 0$ such that

$$(\det D^2h)^{\frac{1}{n}} < \nu \le \partial_t u + f(x,t,u,Du) = \partial_t\tilde{u} + \tilde{f}(x,t,\tilde{u},D\tilde{u}) = (\det D^2\tilde{u})^{\frac{1}{n}}$$

in $\Omega\times\{t_0\}$, where $\nu$ is the positive constant in Lemma 5.3.3. Since $\Omega$ is strictly convex, there is a small positive constant $a = a(b,\partial\Omega) = a(\partial\Omega, \|g\|_{3,1}, \nu)$ such that $-ax_n + b|x|^2 > 0$ for $x \in \partial\Omega$ and therefore we have

$$\tilde{u} \le h \quad \text{on } \partial\Omega.$$

From the maximum principle, $\tilde{u} \le h$ in $\Omega \times \{t_0\}$ and $\tilde{u}(0,t_0) = h(0,t_0) = 0$. By taking a derivative with respect to $x_n$, we have $-\tilde{u}_n \ge -h_n = a > 0$ at $(0,t_0)$. This completes the proof. □

In the next lemma, we shall estimate the mixed derivative $u_{\alpha n}(0,t_0)$ for $\alpha = 1,2,\cdots,n-1$.

**Lemma 5.3.12.** *Let $u$ be a solution of* (PMA) *and assume* (A1), (A2), (A4), *and* (A5). *Then we have*

$$|\partial_{n\alpha} u(0,t_0)| \le C \quad \text{for } 1 \le \alpha \le n-1,$$

*where the constant* $C = C(\partial\Omega, \|u\|_1, \|f\|_2^*, \|g\|_3, \nu)$.

*Proof.* Let us consider the vector field

$$T_\alpha = \partial_\alpha + \sum_{\beta<n} B_{\alpha\beta}(x_\beta \partial_n - x_n \partial_\beta)$$

and the linearized operators

$$\mathcal{L}_u = -\partial_t + \frac{F}{n} u^{ij} \partial_i \partial_j, \quad \mathcal{L}_0 = \mathcal{L}_u - f_{p_i} \partial_i,$$

where $u^{ij}$ denotes $(i,j)$ component of inverse matrix of $D^2 u$. Recall that $u - g = 0$ in $U' \times [0,t_0]$, where $U'$ is the neighborhood of $0'$ introduced at the beginning of this subsection. Differentiating this with respect to $u - g$ and recalling (5.3.16), we obtain that

$$\begin{aligned} 0 = D_\alpha(u-g) &= (\partial_\alpha + \rho_\alpha \partial_n)(u-g) \\ &= \left( \partial_\alpha + \sum_{\beta=1}^{n-1} B_{\alpha\beta} x_\beta \partial_n \right)(u-g) + O(|x'|^2) \\ &= T_\alpha(u-g) + O(|x'|^2) \quad \text{ in } U' \times [0,t_0]. \end{aligned}$$

Thus, we have

$$|T_\alpha(u-g)| \le C_1 |x'|^2 \quad \text{in} \quad U' \times [0,t_0] \tag{5.3.23}$$

for some $C_1 = C_1(\partial\Omega)$. On the other hand, by a direct computation (or using that our operator is invariant under the rotations), we can get

$$\begin{aligned}
&\mathcal{L}_u(T_\alpha u) = T_\alpha(\mathcal{L}_u u) = T_\alpha f(x,t,u,Du) \le f_{p_i} T_\alpha u_i + C(\partial\Omega, \|f\|_1^*, \|u\|_1),\\
&\mathcal{L}_0(T_\alpha g) \ge -C(\partial\Omega, \|f\|_1^*, \|g\|_3) - C(\partial\Omega, n, \|f\|_2^*, \|g\|_3) \sum u^{ii},
\end{aligned}$$

which yields

$$\mathcal{L}_0(T_\alpha(u-g)) \le C_2\left(1 + \sum u^{ii}\right) \quad \text{in} \quad \Omega_{t_0}, \tag{5.3.24}$$

Cwhere $_2 = C_2(\partial\Omega, \|f\|_2^*, \|g\|_3, \|u\|_1)$.

Next, we shall construct a barrier function for $T_\alpha(u-g)$ in $\Omega^\delta \times [0,t_0]$, where $\Omega^\delta := \{|x| < \delta\} \cap \Omega$. In order to do this, we use

$$w(x', x_n) = x_n - \frac{1}{2}B_{\alpha\beta}x_\alpha x_\beta - \frac{1}{2}Mx_n^2 + \frac{1}{4}\mu|x'|^2,$$

where $\mu$ denotes the minimum eigenvalue of the matrix $B_{\alpha\beta}$ and $M$ will be determined later. We start with estimates on the lateral boundary of $\Omega^\delta \times [0,t_0]$. On $(\partial\Omega^\delta \cap \partial\Omega) \times [0,t_0]$, by (5.3.16),

$$w = \frac{1}{4}\mu|x'|^2 - \frac{1}{2}Mx_n^2 + O(|x'|^3) \ge \frac{1}{4}\mu|x'|^2 - C_3|x'|^3, \quad \text{for } C_3 = C_3(\Omega, M) > 0.$$

If $\delta \le \frac{\mu}{8C_3}$, we have

$$w \ge \frac{1}{8}|x'|^2 \quad \text{on} \quad (\partial\Omega_\delta \cap \partial\Omega) \times [0,t_0]. \tag{5.3.25}$$

On $K_1 \times [0,t_0]$, where $K_1 := \partial\Omega^\delta \cap \Omega \cap \{\frac{1}{4}\mu|x'|^2 \ge Mx_n^2\}$, using $x_n > \rho(x')$ and (5.3.16), we see that

$$\begin{aligned}
w &\ge \rho - \frac{1}{2}B_{\alpha\beta}x_\alpha x_\beta - \frac{1}{2}Mx_n^2 + \frac{1}{4}\mu|x'|^2 = -\frac{1}{2}Mx_n^2 + \frac{1}{4}\mu|x'|^2 + O(|x'|^3)\\
&\ge -C_4|x'|^3 + \frac{1}{8}\mu|x'|^2,
\end{aligned}$$

for some $C_4 = C_4(\partial\Omega)$. If $\delta \leq \frac{\mu}{16C_4}$, we have

$$w \geq \frac{1}{16}\mu|x'|^2 \geq \frac{M\mu}{4(\mu+4M)}\delta^2 \quad \text{on} \quad K_1 \times [0, t_0]. \tag{5.3.26}$$

Finally, on $K_2 \times [0, t_0]$, where $K_2 := \partial\Omega^\delta \cap \Omega \cap \{\frac{1}{4}\mu|x'|^2 \leq Mx_n^2\}$, using $x_n \geq \sqrt{\frac{\mu}{(4M+\mu)}}\delta$, we see that

$$w \geq \sqrt{\frac{\mu}{4M+\mu}}\delta - C_5(\Omega, M, \mu)\delta^2.$$

If $\delta \leq \frac{1}{2C_5}\sqrt{\frac{\mu}{(4M+\mu)}}$, we have

$$w \geq C_5\delta^2 \quad \text{on} \quad K_2 \times [0, t_0]. \tag{5.3.27}$$

From (5.3.23), (5.3.25), (5.3.26), and (5.3.27), it is immediate that if we take sufficiently small constant $\delta > 0$ and a constant $A > 0$ such that $\delta < \min\left(\frac{\mu}{8C_3}, \frac{\mu}{16C_4}, \frac{1}{2C_5}\sqrt{\frac{\mu}{(4M+\mu)}}\right)$ and $A > C_1/\min\left(1/8, \frac{M\mu}{4(\mu+4M)}, C_5\right)$, then

$$Aw \pm T_\alpha(u-g) \geq 0 \quad \text{on} \quad \partial\Omega^\delta \times [0, t_0].$$

Our next task is to show that

$$\mathcal{L}_0(Aw \pm T_\alpha(u-g)) \leq 0 \quad \text{in} \quad \Omega^\delta \times (0, t_0]. \tag{5.3.28}$$

Indeed, from the definition of $w$ and $\mu$, we calculate that

$$\mathcal{L}_u w = \frac{F}{n}u^{ij}\left(-B_{ij} - M\delta_{in}\delta_{jn} + \frac{\mu}{2}\delta_{ij}\right) \leq -\frac{F}{2n}\mu\sum_{i<n}u^{ii} - \frac{F}{n}Mu^{nn}.$$

Using the arithmetic-geometric mean inequality, we have

$$\frac{F}{4n}\mu\sum_{i<n}u^{ii} + \frac{F}{2n}Mu^{nn} \geq \frac{\mu^{(n-1)/n}}{4}(2M)^{1/n},$$

and then we see that if $M \geq \mu/2$,

$$\mathcal{L}_u w \leq -\frac{F}{4n}\mu \sum_{i<n} u^{ii} - \frac{F}{2n} M u^{nn} - \frac{\mu^{(n-1)/n}}{4}(2M)^{1/n} \leq -\frac{F}{4n}\mu \sum u^{ii} - \frac{\mu^{(n-1)/n}}{4}(2M)^{1/n}.$$

Therefore, by the definition of $\mathcal{L}_0$ and $w$,

$$\mathcal{L}_0 w \leq -\frac{F}{4n}\mu \sum u^{ii} - \frac{\mu^{(n-1)/n}}{4}(2M)^{1/n} + C_6(1 + M\delta) \quad \text{in } \Omega^\delta,$$

for some $C_6 = C_6(\|f\|_1^*, \partial\Omega)$. If we choose $M$ so that $\frac{\mu^{(n-1)/n}}{4}(2M)^{1/n} \geq 2C_6 + 1$ and for this $M$ we take $\delta$ so that $M\delta \leq 1$, then we have

$$\mathcal{L}_0 w \leq -\frac{F}{4n}\mu \sum u^{ii} - 1 \leq -C_7\left(1 + \sum u^{ii}\right),$$

for some $C_7 = C_7(\nu, M)$, by (5.3.5). By choosing $A \geq C_2/C_7$ and recalling (5.3.24), we have the desired result, (5.3.28).

In order to finish the proof, we need to prove that

$$Aw \pm T_\alpha(u - g) \geq 0 \quad \text{on} \quad \Omega^\delta \times \{0\}.$$

Since $u - g = 0$ on $\Omega \times \{0\}$, we obtain $T_\alpha(u - g) = 0$ on $\Omega \times \{0\}$. Now it suffices to show that $w \geq 0$ in $\Omega^\delta$, which follows from the maximum principle to $w$ in $\Omega^\delta$ since $w \geq 0$ on $\partial\Omega^\delta$ and $\frac{F}{n} u^{ij}\partial_{ij} w < 0$ in $\Omega^\delta$.

From the comparison principle, we have

$$|T_\alpha(u - g)| \leq Aw \quad \text{in} \quad \overline{\Omega^\delta} \times [0, t_0].$$

Using this and $T_\alpha(u - g)(0, t_0) = w(0, t_0) = 0$, we conclude that

$$|\partial_n T_\alpha(u - g)| \leq A\partial_n w = A \quad \text{at} \quad (0, t_0),$$

which implies

$$|\partial_{n\alpha} u(0,t_0)| \le C(\partial\Omega, \|u\|_1, \|f\|_2^*, \|g\|_3, \nu) \quad \text{for} \quad 1 \le \alpha \le n-1.$$

This completes the proof. □

The last lemma in this section is bound for $u_{nn}(0,t_0)$.

**Lemma 5.3.13.** *Let $u$ be a solution of* (PMA) *and assume* (A1), (A2), (A4), *and* (A5). *Then we have*

$$|\partial_{nn} u(0,t_0)| \le C$$

*where the constant $C = (\partial\Omega, \|u\|_1, \|f\|_2^*, \|g\|_{3,1}, \nu, \|F\|_0)$.*

*Proof.* Denoting the cofactor of $u_{ij}$ by $A^{ij}$, we have

$$\sum_{i=1}^{n} A^{ni} u_{in} = \det D^2 u = (f + u_t)^n,$$

and

$$|A^{nn} u_{nn}| \le (f+u_t)^n + \sum_{i \ne n} |A^{ni} u_{in}| \quad \text{at } (0,t_0). \tag{5.3.29}$$

From Lemma 5.3.11, we see that $A^{nn}(0,t_0)$ has a uniform positive lower bound. By (5.3.5), the first term in the right hand side of (5.3.29) is uniformly bounded (see also the remark at the beginning of the proof of Proposition 5.3.8). Moreover, Lemma 5.3.10 and Lemma 5.3.12 gives the uniform upper bound for the second term of the right hand side of (5.3.29). This completes the proof. □

### 5.3.4 The optimal regularity of the obstacle problem

The results of the previous subsections read as follows:

**Theorem 5.3.14.** *Let* $u^\varepsilon \in C^3(\overline{\Omega_T}) \cap C^4(\Omega_T)$ *be a solution of* (PMAo$_\epsilon$). *Assume that* (A1)-(A5). *Then*

$$\|u^\varepsilon\|_{1,1} \le C,$$

*where the constant* $C$ *depends on* $\|g\|_{3,1}$, $\|f\|_2^*$, $\|\phi\|_{2,1}$, $\|\underline{u}\|_2$, $\Omega_T$, *and* $n$, *and is independent of* $\varepsilon$.

We also have the higher order estimates.

**Proposition 5.3.15.** *Let* $u \in C^{k+1}(\overline{\Omega_T}) \cap C^{k+2}(\Omega_T)$ *be a solution of* (PMAo$_\epsilon$). *Assume that* (A1)-(A5). *Then we have for* $0 < \alpha < 1$,

$$\|u\|_{k,\alpha} \le C_\varepsilon,$$

*where the constant* $C_\varepsilon$ *depends on* $\varepsilon$, $\|g\|_{k+1,1}$, $\|f\|_{k,1}^*$, $\|\phi\|_{k,1}$, $\|\underline{u}\|_2$, $\Omega_T$, *and* $n$.

*Sketch of proof.* From Krylov-Safonov's estimate in [51], we may obtain the Hölder regularity of $\partial_t u$ as in Step 1 of [70, Theorem 2.1]. Observe that our equation is

$$\left(\det D^2 u\right)^{\frac{1}{n}} = f + u_t - \beta_\delta(\phi - u).$$

Since $(\det D^2 u)^{\frac{1}{n}}$ is a concave operator, we have a space Hölder estimate by Evans-Krylov theory (see [29] and [50]). Now the Hölder estimate for $D^2 u$ in $t$ follows from the same argument as in Step 2 of [70, Theorem 2.1]. By standard Schauder theory, we have the desired result. □

Using the method of continuity and the a priori estimates which have been shown above, we can prove the existence of solutions to (PMAo$_\epsilon$) having uniform $C^{1,1}$ bound.

**Lemma 5.3.16.** *Assume that* (A1)-(A5). *There exists a unique solution* $u^\varepsilon \in$

$C^3(\overline{\Omega_T}) \cap C^4(\Omega_T)$ *of* $(\mathrm{PMAo}_\epsilon)$ *for each* $0 < \varepsilon < 1$ *satisfying*

$$u^\varepsilon \geq \underline{u} \text{ in } \overline{\Omega_T} \quad \text{and} \quad \|u^\varepsilon\|_2 \leq C,$$

*where the constant* $C$ *is independent of* $\varepsilon$.

*Proof.* The uniqueness assertion follows from the comparison principle since $f_z \geq 0$. We prove the existence assertion in two cases.

*Case 1.* Assume that $\underline{u} \in C^\infty(\overline{\Omega_T})$. Let us define

$$\underline{f} = -\underline{u}_t + (\det D^2 \underline{u})^{\frac{1}{n}},$$

and $f^\varepsilon(x,t,z,p) = f(x,t,z,p) - \beta_\delta(\phi - z)$. Since $\underline{u}$ is a subsolution of (PMA), we know that $\underline{f} \geq f(\cdot, \underline{u}, D\underline{u}) \geq \mu_1 > 0$. For each $s \in [0,1]$, we consider the Cauchy-Dirichlet problem

$$\begin{aligned} -\partial_t u + \left(\det D^2 u\right)^{\frac{1}{n}} &= s f^\varepsilon(x,t,u,Du) + (1-s)\underline{f}(x,t) \quad \text{in} \quad \Omega_T, \\ u &= g \quad \text{on} \quad \partial_p \Omega_T. \end{aligned} \tag{5.3.30}$$

Let $u \in C^3(\overline{\Omega_T})$ be a solution of (5.3.30) and let $\mathcal{B}$ be the class of solutions $v$ of (5.3.30) such that $v \geq \underline{u}$ in $\Omega_T$. Since $\underline{u}$ is also a subsolution of (5.3.30), it follows from the comparison principle that any strictly convex solution $u \in C^\infty(\overline{\Omega_T})$ satisfies $u \geq \underline{u}$. This gives $u \in \mathcal{B}$, and by the Proposition 5.3.15,

$$\|u\|_{2,\alpha} \leq C_\varepsilon \quad \text{independent of } s.$$

Hence, it is possible to show, by using the method of continuity, that for each $s \in [0,1]$, the equation (5.3.30) has a strictly convex solution in $C^3(\overline{\Omega_T})$. As in Proposition 5.3.15, it follows from the standard regularity theory that $u \in C^4(\Omega_T)$.

*Case 2.* We now consider the case $\underline{u} \in C^2(\overline{\Omega_T})$. Take a sequence of strictly convex functions $\underline{u}_m \in C^\infty(\overline{\Omega_T})$ converging to $\underline{u}$ in $C^2(\overline{\Omega_T})$. Since $\underline{u}$ is a subsolution of (PMA), we may assume that

$$-\partial_t \underline{u}_m + \left(\det D^2 \underline{u}_m\right)^{\frac{1}{n}} \geq (1 - 2^{-m}) f_m(x, t, \underline{u}_m, D\underline{u}_m) \quad \text{in} \quad \Omega_T.$$

Setting $g_m = \underline{u}_m|_{\partial_p \Omega_T}$, we consider Cauchy-Dirichlet problem

$$\begin{aligned} -\partial_t u + \left(\det D^2 u\right)^{\frac{1}{n}} &= (1 - 2^{-m}) f \quad \text{in} \quad \Omega_T, \\ u &= g_m \qquad\qquad\quad \text{on} \quad \partial_p \Omega_T. \end{aligned} \tag{5.3.31}$$

By the result of Case 1 and the fact that $\underline{u}_m$ is a subsolution of (5.3.31), there exists a strictly convex solution $u_m \in C^\infty(\overline{\Omega_T})$ of (5.3.31) satisfying

$$\|u_m\|_{k,\alpha} \leq C(k, \alpha, \|\underline{u}\|_2)$$

for $k \geq 2$ and $0 < \alpha < 1$. Here the constant $C(k, \alpha, \|\underline{u}\|_2)$ also depends on the other known data. Thus we can extract a subsequence converging to a solution of (PMA) in $C^\infty(\overline{\Omega_T})$. □

*proof of Theorem 5.1.1.* We begin with the uniqueness assertion. It was shown in [45] that the comparison principle holds for viscosity subsolutions and supersolutions of equation (5.2.1) if $f_z \geq 0$. Assume that there are two viscosity solutions $u_1$ and $u_2$ of (PMAo) with $u_1(x_1, t_1) < u_2(x_1, t_1)$ for some $(x_1, t_1) \in \Omega_T$. Let $G$ be a connected component of $\{(x, t) \in \Omega_T : u_1(x, t) < u_2(x, t)\}$ containing $(x_1, t_1)$. Since $u_1 < u_2 \leq \phi$ in $G$, it follows that $u_1$ is a viscosity solution of (5.2.1) in $G$. On the other hand, $u_2$ is a viscosity subsolution in $G$. Then by the maximum principle, we have $u_2 \leq u_1$ in $G$, which is a contradiction.

From the uniform $C^{1,1}$ estimate, we can extract a subsequence $u^{\varepsilon_k}$ converging to a function $u \in C^{1,1}(\overline{\Omega_T})$ in $C^{1,\alpha}(\overline{\Omega_T})$ for all $0 < \alpha < 1$. Since $u^\varepsilon \geq \underline{u}$ in $\overline{\Omega_T}$ for any $\varepsilon$, we thus have $u \geq \underline{u}$ in $\overline{\Omega_T}$. Moreover, $u \leq \phi$ follows from the uniform boundedness of $\beta_\delta$, Lemma 5.3.1. To finish the proof, it remains to show that $u$ is the viscosity solution of (PMAo), which is a direct consequence of the stability property in [22]. □

**Remark 5.3.17.** The solution $u$ has, in fact, $C^{3,\alpha}$-regularity in the non-coincidence set $N(u)$ as in the same argument in Proposition 5.3.15.

We close this section by giving the proof of Theorem 5.1.2. Since the main part of proof is simpler and similar to that of Theorem 5.1.1, we do not repeat here.

*proof of Theorem 5.1.2.* Without the penalty term in (PMAo$_\epsilon$), we can obtain the same a priori estimates in this section. Moreover, as we explained in the Remark 5.3.6, the assumption (A1) in Theorem 5.1.1 can be removed. Following the proof of Lemma 5.3.16, we have the desired result. □

## 5.4 Regularity of the free boundary

In this section, we study the regularity of the free boundary $\Gamma(u) = \partial N(u) \cap \Omega_T$ of the solution $u$ to (PMAo). Precisely, we discuss the local regularity of the free boundary $N(u)$ at a free boundary point $X_0 = (x_0, t_0) \in \Gamma(u)$. For simplicity, we set $(x_0, t_0) = (0, 0)$ and consider the problem in a neighborhood of $(0, 0)$:

$$\begin{cases} \mathcal{P}u \geq f(x,t) & \text{in } Q_r^-, \\ \mathcal{P}u = f(x,t) & \text{in } Q_r^- \cap N(u), \\ \quad u \leq \phi & \text{in } Q_r^-, \end{cases} \tag{5.4.1}$$

with $u(0,0) = \phi(0,0) = 0$, $\nabla u(0,0) = \nabla\phi(0,0) = 0$ by subtracting the affine function $\phi(0) + D\phi(0) \cdot x$, and the obstacle $\phi$ such that $\partial_t \phi \in W^{2,1}_\infty(Q_r^-)$ and $D\phi \in W^{2,1}_\infty(Q_r^-)$.

In contrast with the theory for the uniformly elliptic or parabolic fully nonlinear operator in [53, 31, 32, 56], introducing the problem with lower zero obstacle for the Monge-Ampère operator is not appropriate since the modified operator $-\partial_t + G$ (see (5.4.2)) for $F(\mathcal{M}) = \det^{\frac{1}{n}}(\mathcal{M})$ is not the Monge-Ampère operator. Hence, in this section, we deal with the problem (5.4.1) as it is without using the simplified problem with the zero obstacle such as in [53, 9, 31, 32, 56].

First, for a detailed explanation, we briefly introduce the reduced obstacle problem for the uniformly fully nonlinear operator, $F$. The modified operator is defined as follow:

$$G(\mathcal{M}, x) = -F(-\mathcal{M} + D^2\phi(x)) + F(D^2\phi(x)). \tag{5.4.2}$$

Then, for the solution $u$ of the obstacle problem for $F$, $v = \phi - u$ is a solution the *reduced obstacle problem* with the zero obstacle function:

$$-\partial_t v + G(D^2v, x) = \left(-\partial_t\phi + F(D^2\phi) - f\right)\chi_{\{v>0\}}, \qquad v \geq 0 \quad \text{in } Q_1^-. \tag{5.4.3}$$

The operator $G$ is also a uniformly parabolic with the same parabolicity. Hence, to have the regularity of the free boundary, it is enough to discuss the regularity for the reduced problem (5.4.3).

In the case that $F = \det^{\frac{1}{n}}$, $G$ is not the Monge-Ampère operator and defined only in $\{(\mathcal{M}, x) \mid -\mathcal{M} + D^2\phi(x) \geq 0\} \subset \mathcal{S}^{n\times n} \times \mathbb{R}^n$. Hence, to have the regularity theory of the free boundary with the operator $-\partial_t + G$, additional mathematical justification is needed to apply general theories (like maximum principle or regularity theory) to the solution $u$ of PDEs with the operator $G$ although $(D^2v, x)$ is exactly in the subspace. Therefore, instead of $G$ and the reduced form (5.4.3), in this section, we deal with the original problem (5.4.1).

We note that the linearized operator

$$\mathcal{L}_u := -\partial_t + F_{ij}(D^2u)\cdot\partial_{ij}$$

of $\mathcal{P} := -\partial_t + F(\mathcal{M})$, where $F(\mathcal{M}) = (\det\mathcal{M})^{\frac{1}{n}}$ and

$$F_{ij}(D^2u)\cdot\mathcal{M} = \frac{1}{n}F(D^2u)tr((D^2u)^{-1}\mathcal{M})$$

is used throughout this section. Since $F_{ij}(D^2u)$ depends on $D^2u$, $\mathcal{L}_u$ is an operator with continuous coefficient only on $N(u)$. By the optimal regularity of the solution, Theorem 5.1.1, we know that $D^2u$ and $\partial_t u$ are bounded and moreover, by Lemma 5.3.3 and the convergence of $u^\epsilon$ to $u$, we know that

$$(\det D^2u)^{\frac{1}{n}} \geq f + \partial_t u \geq \nu > 0 \qquad \text{in } \Omega_T,$$

and the eigenvalues of $u$ also have a lower bound. Therefore, $\mathcal{L}_u$ is uniformly parabolic in $N(u)$.

In Subsection 5.4.1, we discuss rescalings, blowup functions, and the thickness of $\Lambda(u)$ for the solution $u$ of (5.4.1) and define the solution spaces of the local and global solutions. We note that since it is needed to consider the global solution with the uniform thickness assumption, we introduce a class of global solution including the global solution of the obstacle problem for Monge-Ampère equation, (5.4.1) in $\mathbb{R}^n \times (-\infty, 0]$.

In Subsection 5.4.2, the non-degeneracy of $u \in P_1(M)$ is proved by using (5.4.4) and it is discussed that the blowup function $u_0$ of $u \in P_1(M)$ is a solution of the global solution, $u_0 \in P_\infty(M)$.

In Subsections 5.4.3 and 5.4.4, we will discuss the classification of the global solutions and the directional monotonicity of the local solutions, respectively. We note that since the linearized operator $\mathcal{L}_u$ depends on $D^2u$, in the proof of the classification of the global solutions and the directional monotonicity, we will carefully deal with the case that the $\mathcal{L}_u$ applied on the functions such as $\partial_t\phi, \partial_e\phi$, and $\partial_{ee}\phi$.

Finally, in Subsection 5.4.5, we prove the regularity of the free boundary $\Gamma(u) = \partial\Omega(u) \cap Q_1^-$.

### 5.4.1 Preliminaries

The *rescalings* of the solution $u$ of (5.4.1) and $\phi$ at 0 with $u(0,0) = \phi(0,0) = 0$ and $\nabla u(0,0) = \nabla\phi(0,0) = 0$ for $r > 0$ are

$$u_r(X) := \frac{u(rx, r^2t)}{r^2} \quad \text{and} \quad \phi_r(X) := \frac{\phi(rx, r^2t)}{r^2}, \quad X \in Q_{1/r}^-.$$

By the $W^{2,1}_{\infty}$-regularity of solution $u$ (Theorem 5.1.1), $W^{2,1}_{\infty}$-norm of the rescalings $u_r$ are uniformly bounded. Then, we can extract a limit function which is called a blowup. Specifically, for the solution $u$ of (5.4.1), there exists a sequence $r_i$ and $u_0 \in W^{2,1}_{\infty,loc}(\mathbb{R}^n \times (-\infty, 0])$ such that

$$u_{r_i} \to u_0 \text{ in } W^{2,1}_{p,loc}\left(\mathbb{R}^n \times (-\infty, 0]\right), \qquad \text{for any } n < p < \infty.$$

The limt function $u_0$ is called a *blowup of $u$ at* 0.

**Definition 5.4.1.** We denote by $\delta_r(u, X_0)$ the thickness of $\Lambda(u)$ on $B_r(x^0) \times \{t^0 - r^2\}$, i.e.,

$$\delta_r(u, X^0) := \frac{\mathrm{MD}\left(\Lambda(u) \cap (B_r(x^0) \times \{t^0 - r^2\})\right)}{r},$$

where $\mathrm{MD}(A)$ is the least distance between two parallel hyperplanes containing $A \subset \mathbb{R}^n$. We will use the abbreviation $\delta_r(u)$ for $\delta_r(u, 0)$.

For the convenience of statement, we define a class of local solutions to the problem (5.4.1).

**Definition 5.4.2.** (Local solutions) The class of local solutions $P_r(M)$ ($0 < r < \infty$) consists of all solutions $u$ of (5.4.1) satisfying:

(i) $\|\partial_t u\|_{L^\infty(Q_r^-)} + \|D^2 u\|_{L^\infty(Q_r^-)} \leq M$,

(ii) $0 \in \partial N(u)$,

with $f(0) = 1$, $f \in W^{2,1}_{\infty}(Q_r^-)$, $\partial_t \phi \in W^{2,1}_{\infty}(Q_r^-)$, and $D\phi \in W^{2,1}_{\infty}(Q_r^-)$.

In order to have the classification of global solutions, it is needed to consider global solutions with the uniform thickness assumption, (5.1.4). On the other hand, we know that non-degeneracy of $u \in P_1(M)$ and the uniform thickness assumption on $u$ imply that the blowup function $u_0$ of $u$ satisfies the uniform thickness condition,

$$\delta_r(u_0, X) \geq \epsilon_0 \quad \text{ for all } r > 0, X \in Q^-_{1/2} \cap \partial N_{u_0},$$

where $N_{u_0} = \mathbb{R}^n \times (-\infty, 0] \setminus \Lambda_{u_0}$,

$$\Lambda_{u_0} := \limsup_{r_j \to \infty} \{u_{r_j} = \phi_{r_j}\} \subset \{u_0 = \phi_0\},$$

and $\limsup_{j\to\infty} A_j$ is the set of all limit points of sequences $(x_{j_k}, t_{j_k}) \in A_{j_k}$, see subsection 5.2 of [9] and subsection 6.2 of [57]. Hence, we define the global solution as follows:

**Definition 5.4.3.** (Global solutions) The class of global solutions $P_\infty(M)$ consists of all solutions $u$ of

$$\begin{cases} \mathcal{P}u \geq 1 & \text{in } \mathbb{R}^n \times (-\infty, 0], \\ \mathcal{P}u = 1 & \text{in } \mathbb{R}^n \times (-\infty, 0] \cap N_u, \\ u \leq \phi & \text{in } \mathbb{R}^n \times (-\infty, 0], \end{cases}$$

with the obstacle $\phi$ which satisfying:

(i) $\|\partial_t u\|_{L^\infty(\mathbb{R}^n \times (-\infty,0])} + \|D^2 u\|_{L^\infty(\mathbb{R}^n \times (-\infty,0])} \leq M$,

(ii) $0 \in \partial N_u$,

(iii) $\mathcal{P}\phi = a > 1$ and $\partial_t \phi$ and $D^2\phi$ are constants,

where $N_u$ is an open set such that $N_u \supset \{u < \phi\}$ and $\Lambda_u = \mathbb{R}^n \times (-\infty, 0] \setminus N_u$.

**Remark 5.4.4.** We note that for $u \in P_\infty(M)$, $\mathcal{P}u = \mathcal{P}\phi = a > 1$ a.e. in $\{u = \phi\}$. Hence, $\mathcal{P}u = 1$ in $N_u$ implies $|\{u = \phi\} \setminus \Lambda_u| = |\{u = \phi\} \cap N_u| = 0$.

We note that in the definition of $P_1(M)$, $\partial_t \phi \in W^{2,1}_\infty(Q_r^-)$ and $D\phi \in W^{2,1}_\infty(Q_r^-)$ are assumed. Hence, $\phi \in C^2(Q_r^-)$ and by Taylor expansion, $\phi_0$ is a homogeneous polynomial with homogeneity 1 for $t$ and 2 for $x$, where $\phi_0$ is the blowup of $\phi$ at 0. Especially, we know that $\partial_t \phi_0$ and $D^2\phi_0$ are constant in $\mathbb{R}^n \times (-\infty, 0]$. Therefore, it is assumed that $\partial_t \phi$ and $D^2\phi$ are constants in the definition for the space of global solutions $P_\infty(M)$, where $\phi$ is the obstacle function.

### 5.4.2 Basic results

Now we prove the non-degeneracy by using the linearized operator $\mathcal{L}_u$ and (5.4.4) below. The non-degeneracy implies that the blowup of $\phi - u$ at a free boundary point is not the zero function and the fact that the Lebesgue measure of the free boundary is zero.

By the concavity of $\mathcal{P}$, we have that

$$\mathcal{L}_u(\phi - u) \geq \mathcal{P}\phi - \mathcal{P}u = \mathcal{P}\phi - f \quad \text{in } N(u). \tag{5.4.4}$$

Indeed, from the definition of operators, $\mathcal{P}u = \mathcal{L}_u u = f$ in $N(u)$ and moreover, the arithmetic-geometric mean inequality gives

$$\mathcal{P}\phi \leq \mathcal{L}_u\phi \quad \text{in } \Omega_T.$$

Precisely, by the inequality, we have

$$F(D^2u)^{-1}F(D^2\phi) = \left(\det\left((D^2u)^{-1}D^2\phi\right)\right)^{\frac{1}{n}} \leq \frac{1}{n}\operatorname{tr}\left((D^2u)^{-1}D^2\phi\right)$$

and

$$\mathcal{P}\phi = -\partial_t\phi + F(D^2\phi) \leq -\partial_t\phi + \frac{1}{n}F(D^2u)\operatorname{tr}\left((D^2u)^{-1}D^2\phi\right) = \mathcal{L}_u\phi \quad \text{in } \Omega_T.$$

For $u \in P_1(M)$, by Lemma 5.3.3 and the convergence of $u^\epsilon$ to $u$, we know that

$$F(D^2u) = (\det D^2u)^{\frac{1}{n}} \geq f + \partial_t u \geq \nu > 0 \qquad \text{in } \Omega_T,$$

and the eigenvalues of $D^2u$ have upper and lower bounds, i.e. there is $\epsilon_0 = \epsilon_0(M, \nu) > 0$ such that $\epsilon_0 I \leq D^2u \leq \frac{1}{\epsilon_0}I$.

**Lemma 5.4.5** (Non-degeneracy)**.** *Let $u \in P_1(M)$. If $\mathcal{P}\phi \geq f + c$ in $Q_1^-$ for some positive constant $c$, then*

$$\sup_{X \in Q_r^-(X_0)} (\phi(X) - u(X)) \geq \phi(X_0) - u(X_0) + 2\alpha r^2, \quad X_0 \in \overline{N(u)} \cap Q_1^-,$$

*for any* $Q_r^-(X_0) \subset Q_1^-$, *where* $\alpha := \frac{c}{(1+2(\mu_2+M)/\epsilon_0)}$.

*Proof.* Let $v = \phi - u$ and $X_0 \in N(u) = \{u < \phi\} \cap Q_1^- = \{v > 0\} \cap Q_1^-$ and define an auxiliary function

$$w(x,t) = v(x,t) - v(x_0,t_0) - \alpha\left(|x-x_0|^2 - (t-t_0)\right).$$

Recalling (5.3.2), we have

$$\begin{aligned}\mathcal{L}_u\left(|x-x_0|^2 - (t-t_0)\right) &= 1 + \frac{2}{n}F(D^2u)tr((D^2u)^{-1}) \le 1 + \frac{2}{\epsilon_0}(f + \partial_t u)\\ &\le 1 + \frac{2}{\epsilon_0}(\mu_2 + M) = \frac{c}{\alpha}.\end{aligned} \tag{5.4.5}$$

Then, the inequality (5.4.4) implies

$$\mathcal{L}_u w \ge \mathcal{L}_u(\phi - u) - c \ge \mathcal{P}\phi - f - c \ge 0 \quad \text{in } Q_r^-(X_0).$$

Since $w(X_0) = 0$ and $w(X) \le 0$ on $\partial\{v > 0\} \cap Q_r^-(X_0)$, by the maximum principle, we have

$$w(X_0) = 0 \le \sup_{\{v>0\}\cap\partial_p Q_r^-(X_0)} w$$

and the desired inequality holds for $X_0 \in N(u) \cap Q_1^-$.

For $X_0 \in \partial N(u) \cap Q_1^-$,we will take a sequence of points $X^j \in N(u)$ such that $X^j \to X_0$ as $j \to \infty$. By passing to the limit as $j$ goes to $\infty$, we have the desired inequality for $X_0 \in \overline{N(u)} \cap Q_1^-$. □

**Lemma 5.4.6.** *Let* $u \in P_1(M)$ *with the obstacle function* $\phi$ *such that* $\mathcal{P}\phi \ge f + c$ *in* $Q_1^-$, *for some positive constant* $c$. *Then any blowup* $u_0$ *at* $0$ *is in* $P_\infty(M)$.

*Proof.* Let $u_{r_i}$ be a sequence of the rescalings converging to a blowup $u_0$. The rescaling $u_{r_i}$ satisfies

$$\begin{cases} \mathcal{P}u_{r_i} \geq f(r_i x, r_i^2 t) & \text{in } Q^-_{1/r_i}, \\ \mathcal{P}u_{r_i} = f(r_i x, r_i^2 t) & \text{in } Q^-_{1/r_i} \cap N(u_{r_i}), \\ u_{r_i} \leq \phi_{r_i} & \text{in } Q^-_{1/r_i}, \end{cases} \tag{5.4.6}$$

Take a point $X_0$ in $N_{u_0}$, where $N_{u_0} = \mathbb{R}^n \times (-\infty, 0] \setminus \Lambda_{u_0}$ and $\Lambda_{u_0} = \limsup_{r_j \to \infty}\{u_{r_j} = \phi_{r_j}\}$. Then, there exist $\delta > 0$ and $i_0$ such that $Q^-_\delta(X_0) \subset \{u_{r_i} < \phi_{r_i}\}$ for all $i \geq i_0$ and

$$\begin{aligned} \mathcal{P}u_{r_i}(x,t) &= -\partial_t u_{r_i}(x,t) + (\det D^2 u_{r_i}(x,t))^{\frac{1}{n}} \\ &= -\partial_t u(r_i x, r_i^2 t) + (\det D^2 u(r_i x, r_i^2 t))^{\frac{1}{n}} \\ &= f(r_i x, r_i^2 t) \quad \text{in } Q^-_\delta(X_0). \end{aligned}$$

By the interior uniform $C^{2,\alpha}$ bound, we may assume strong convergence of $u_{r_i}$ to $u_0$ in $C^{2,\beta}(Q^-_\delta(X_0))$, for some $0 < \beta < \alpha$. Thus, we have that

$$-\partial_t u_0(X) + (\det D^2 u_0(X))^{\frac{1}{n}} = f(0) = 1 \quad \text{in } Q^-_\delta(X_0)$$

and

$$-\partial_t u_0(X) + F(D^2 u_0(X)) = 1 \quad \text{in } \mathbb{R}^n \times (-\infty, 0] \cap N_{u_0}.$$

Moreover, we obtain $0 \in \partial N_{u_0}$ by using the non-degeneracy. Therefore, $u_0$ is in $P_\infty(M)$. $\square$

### 5.4.3 Classification of the blowup

In this subsection, we will prove that any global solution $u \in P_\infty(M)$ with the uniform thickness assumption (5.1.4) are of the form

$$v = \phi - u = \frac{c}{2}(x_n^+)^2 \quad \text{in } \mathbb{R}^n \times (-\infty, 0].$$

For the first step, in Proposition 5.4.7 below, we show that $\partial_t v \leq 0$ in

$\mathbb{R}^n \times (-\infty, 0]$ by using the method which is introduced by [9] for the reduced problem (5.4.3) with the heat operator.

In the proof of Proposition 5.4.7, we define limit functions $u_0$ and $\phi_0$ of the rescalings, $u_j$ and $\phi_j$, (5.4.7), respectively, such that $\mathcal{P}u_0 = 1$ in $N_{u_0} \cup Q_1^-$. Then, $\partial_t(\mathcal{P}u_0) = \mathcal{L}_{u_0}\partial_t u_0 = 0$ in $N_{u_0} \cup Q_1^-$. By the fact that $\partial_t \phi$ is constant, $\partial_t \phi_0$ is also constant, $\mathcal{L}_{u_0}\partial_t \phi_0 \equiv 0$ in $\mathbb{R}^n \times (-\infty, 0]$, and

$$\mathcal{L}_{u_0}\partial_t v_0 = \mathcal{L}_{u_0}\partial_t \phi_0 - \mathcal{L}_{u_0}\partial_t u_0 = 0 \quad \text{ in } N_{u_0} \cup Q_1^-(0,0).$$

Thus, we could utilize the maximum principle to $\partial_t v_0$ and have the result of the proposition.

**Proposition 5.4.7.** *Let $u \in P_\infty(M)$. Then,*

$$\partial_t v(x,t) \leq 0 \quad \text{ in } \mathbb{R}^n \times (-\infty, 0].$$

*Proof.* Since $u \in P_\infty(M)$, the function $\partial_t v$ is globally bounded. Suppose that

$$m := \sup_{\mathbb{R}^n \times (-\infty, 0]} \partial_t v > 0$$

and let $(x_j, t_j)$ be a sequence such that

$$\partial_t v(x_j, t_j) \to m = \lim_{j \to \infty} \partial_t v(x,t) > 0.$$

We denote $d_j$ by the supremum of $r$ such that $Q_r^-(x_j, t_j)$ is contained in $N_u \supset N(u) = \{u < \phi\}$. Let $(y_j, s_j) \in \partial_p Q_{d_j}^-(x_j, t_j) \cap \Gamma(u)$, $(\tilde{y}_j, \tilde{s}_j) := \left(\frac{y_j - x_j}{d_j}, \frac{s_j - t_j}{d_j^2}\right)$,

$$u_j(x,t) := \frac{u(d_j x + x_j, d_j^2 t + t_j)}{d_j^2}, \quad \phi_j(x,t) := \frac{\phi(d_j x + x_j, d_j^2 t + t_j)}{d_j^2}, \quad (5.4.7)$$

and

$$v_j := \phi_j - u_j.$$

Since $(y_j, s_j) \in \partial N_u$, $u(y_j, s_j) = \phi(y_j, s_j)$, and $\nabla u(y_j, s_j) = \nabla \phi(y_j, s_j)$, we have that $u_j \le \phi_j$ in $\mathbb{R}^n \times (-\infty, 0]$,

$$u_j(\tilde{y}_j, \tilde{s}_j) = \phi_j(\tilde{y}_j, \tilde{s}_j), \quad \nabla u_j(\tilde{y}_j, \tilde{s}_j) = \nabla \phi_j(\tilde{y}_j, \tilde{s}_j),$$

and

$$\left\| D^2 u_j \right\|_{L^\infty} + \left\| \partial_t u_j \right\|_{L^\infty} \le M.$$

Furthermore, $Q_1^- \subset N_{u_j}$, $\mathcal{P} u_j = 1$ in $N_{u_j} \cup Q_1^-$ and $(\tilde{y}_j, \tilde{s}_j) \in \partial_p Q_1^- \cap \partial N_{u_j}$, where $N_{u_j} := \{(y, s) \,|\, (d_j y + x_j, d_j^2 s + t_j) \in N_u\}$ such that $N_{u_j} \supset \{u_j < \phi_j\}$.

Hence, $u_j$ and $\phi_j$ have at most quadratic growth at infinity and we can extract subsequence of $u_j$ converging to global solutions $u_0$ such that

$$\mathcal{P} u_0 = 1 \text{ in } N_{u_0} \cup Q_1^-, \quad \partial_t v_0 \le m \text{ in } \mathbb{R}^n \times (-\infty, 0], \quad \text{and} \quad \partial_t v_0(0,0) = m,$$

where $v_0 := \phi_0 - u_0$.

Since $\partial_t \phi$ is constant, $\partial_t \phi_0$ is also constant and $\mathcal{L}_{u_0} \partial_t \phi_0 \equiv 0$ in $\mathbb{R}^n \times (-\infty, 0]$. Thus, $\partial_t(\mathcal{P} u_0) = \mathcal{L}_{u_0} \partial_t u_0 = 0$ in $N_{u_0} \cup Q_1^-$ implies

$$\mathcal{L}_{u_0} \partial_t v_0 = \mathcal{L}_{u_0} \partial_t \phi_0 - \mathcal{L}_{u_0} \partial_t u_0 = 0 \quad \text{in } N_{u_0} \cup Q_1^-(0,0).$$

Since $\mathcal{L}_{u_0}$ is uniformly parabolic, by the maximum principle, $\partial_t v_0 \equiv m$ in $Q_1^-$. Furthermore, by the same method, we have that $\partial_t v_0 \equiv m$ in the connected component $\tilde{\Omega}(u_0)$ of $Q_1^- \cup N_{u_0}$, which containing $Q_1^-$ and there is a point $(\tilde{y}_0, \tilde{s}_0) \in \partial_p Q_1^- \cap \partial N_{u_0}$.

Then,

$$v_0(x,t) = mt + f(x) \quad \text{in } (B_1 \times \mathbb{R}^-) \cap \tilde{\Omega}(u_0),$$

where $f(x) := v_0(x, 0) \ge 0$ and $\mathbb{R}^- := (-\infty, 0]$

Then, the free boundary $\partial \tilde{\Omega}(u_0)$ is represented by $t(x) = -f(x)/m$ in $B_1 \times \mathbb{R}^-$. Since

$$\begin{aligned} \nabla v_0 &= \nabla f(x) && \text{in } \{(x,t) \,|\, t > t(x)\} \cap (B_1 \times \mathbb{R}^-), \\ \nabla v_0 &= 0 && \text{on } \{(x,t) \,|\, t = t(x)\} \cap (B_1 \times \mathbb{R}^-), \end{aligned}$$

and $\nabla v_0$ is continuous in $\mathbb{R}^n \times \mathbb{R}^-$, we know that $\nabla f(x) = 0$ in $B_1$. Hence

$$v_0(x,t) = mt + c_0 \quad \text{and} \quad u_0(x,t) = -mt + \phi_0 - c_0 \text{ in } \{(x,t)\,|\,t > t(x)\} \cap (B_1 \times \mathbb{R}^-),$$

for a nonnegative constant $c_0$.

Since $\mathcal{P}\phi_0 = a > 1$,

$$\mathcal{P}u_0 = m + \mathcal{P}\phi_0 > 1 \quad \text{in } \{(x,t)\,|\,t < t(x)\} \cap (B_1 \times \mathbb{R}^-) \subset N_{u_0}$$

and we have a contradiction. □

By using an argument in [57] with rescaled functions such as (5.4.7), we could have that $\partial_t v \geq 0$ in $\mathbb{R}^n \times (-\infty, 0]$. This together with Proposition 5.4.7 implies $\partial_t v \equiv 0$ in $\mathbb{R}^n \times (-\infty, 0]$ and then the results for the elliptic case which is discussed in [54] implies the classification of the global solutions, see the comment prior to Proposition 5.4.8.

Precisely, by $\partial_t v \equiv 0$ in $\mathbb{R}^n \times (-\infty, 0]$, we know that the free boundary $\partial N_u$ is time-invariant and $\partial_t u = \partial_t \phi$ is constant in $\mathbb{R}^n \times (-\infty, 0]$. Hence, for the classification of the global solution, it is enough to consider the classification for the elliptic obstacle problem:

$$\begin{cases} \left(\det D^2\tilde{u}\right)^{\frac{1}{n}} \geq \partial_t u(0,0) + 1 & \text{in } \mathbb{R}^n, \\ \left(\det D^2\tilde{u}\right)^{\frac{1}{n}} = \partial_t u(0,0) + 1 & \text{in } \mathbb{R}^n \cap N(\tilde{u}), \\ \qquad \tilde{u} \leq \tilde{\phi} & \text{in } \mathbb{R}^n, \end{cases} \tag{5.4.8}$$

where $\tilde{u} := u - \partial_t u(0,0)t$, $\tilde{\phi} := \phi - \partial_t u(0,0)t = \phi - \partial_t \phi(0,0)t$, and $N_{\tilde{u}} := N_u(0)$.

Since the linearized operator $F_{ij}(D^2\tilde{u})M = \frac{1}{n}F(D^2\tilde{u})tr((D^2\tilde{u})^{-1}M)$ is uniformly elliptic, the classification of the global solution for the elliptic problem, (5.4.8), is obtained by the method in [53, 31] with the consideration of the special characteristics of the Monge-Ampère operator, $F(D^2u) = (\det D^2u)^{\frac{1}{n}}$, which is discussed in the introduction of this section.

Indeed, for a solution $u$ to $F(D^2u) = (\det D^2u)^{\frac{1}{n}} = c$ in a domain $\Omega \subset \mathbb{R}^n$, the concavity of $F$ implies

$$F_{ij}(D^2u)D^2u_{ee} \geq 0 \quad \text{in } \Omega.$$

On the other hand, since $D^2\phi(x)$ is constant, we have $F_{ij}(D^2u)D^2\phi_{ee} = 0$ and

$$F_{ij}(D^2u)(D^2(\phi - u)_{ee}) \leq 0 \quad \text{in } \Omega. \tag{5.4.9}$$

Then, the method in [53, 31] with (5.4.9) to $v = \phi - u$ implies that $v$ is convex and

$$v = \phi - u = c(x_n^+)^2 \quad \text{in } \mathbb{R}^n \times (-\infty, 0],$$

for a global solution $u$ of (5.4.8) with obstacle $\phi$.

**Proposition 5.4.8.** *[57] Let $u \in P_\infty(M)$ and assume that*

$$\delta_r(v, X) > \epsilon_0 \quad \textit{for all } r > 0, X \in \partial N_u$$

*then,*

$$\partial_t v(x,t) \equiv 0 \quad \textit{in } \mathbb{R}^n \times (-\infty, 0] \quad \textit{and} \quad v := \phi - u = c(x_n^+)^2 \quad \textit{in } \mathbb{R}^n \times (-\infty, 0],$$

*for an appropriate system of coordinates and a positive constant $c$.*

By using Proposition 5.4.8, with the same method as in Lemma 4.1 of [32], we have the following proposition.

**Lemma 5.4.9.** *Let $u \in P_1(M)$ and assume that*

$$\delta_r(v, X) > \epsilon_0 \quad \textit{for all } r > 0, X \in N(u).$$

*Then,*

$$\partial_t v(X) \to 0 \quad \textit{as } X \to (x_0, t_0)$$

*for any $(x_0, t_0) \in \Gamma(u)$.*

### 5.4.4 Directional monotonicity

Now we are ready to prove the directional monotonicity, Proposition 5.4.10. In the proof of the property, we use the $C^{1,\alpha}$ convergence of $u_r$ and $\phi_r$ to $u_0$ and $\phi_0$, respectively, Lemma 5.4.9, and the regularity of $u$, $f$, and $\phi$.

Precisely, for $u \in P_1(M)$ with $u \leq \phi$ in $Q_1^-$, the rescaling $u_r$ at 0 satisfies (5.4.6) in $Q_{1/r_i}^-$ and $u_r$ converges to the blowup function $u_0$ in $C^{1,\alpha}(Q_1^-)$. Then we have

$$\|u_r - u_0\|_{C^{1,\alpha}(Q_1^-)} \quad \text{and} \quad \|\phi_r - \phi_0\|_{C^{1,\alpha}(Q_1^-)} \to 0 \quad \text{as } r \to 0. \tag{5.4.10}$$

On the other hand, by Lemma 5.4.9, $0 \in \Gamma(u)$, the optimal $W^{2,1}_\infty$ regularity of $u$, and $\partial_t \phi \in W^{2,1}_\infty$, we have that

$$\|\partial_t (\phi_r - u_r)\|_{L^\infty(Q_1^-)} = \left\|\partial_t v(rx, r^2 t)\right\|_{L^\infty(Q_1^-)} = \|\partial_t v\|_{L^\infty(Q_r^-)} \to 0 \quad \text{as } r \to 0. \tag{5.4.11}$$

Then, by using (5.4.10) and (5.4.11), for a unit vector $\tilde{e} := (e_x, e_t) \in \mathbb{S}^n$ such that $C\partial_{\tilde{e}} v_0 - v_0 \geq 0$ in $Q_1^-$, we have $C\partial_{\tilde{e}} v_r - v_r \geq -\epsilon_0$ in $Q_1^-$.

Furthermore, since $f \in W^{2,1}_\infty(\Omega_T)$, $\partial_t \phi \in W^{2,1}_\infty(\Omega_T)$, and $D\phi \in W^{2,1}_\infty(\Omega_T)$, for $f_r := f(rx, r^2 t)$, we have

$$\begin{aligned}
\|\partial_i f_r\|_{L^\infty(Q_1^-)} &= r\, \|\partial_i f\|_{L^\infty(Q_r^-)} && \to 0,\\
\|\partial_t f_r\|_{L^\infty(Q_1^-)} &= r^2\, \|\partial_t f\|_{L^\infty(Q_r^-)} && \to 0,\\
\left\|(D^2 + \partial_t)\partial_t \phi_r\right\|_{L^\infty(Q_1^-)} &= r^2 \left\|D^2 \partial_t \phi + \partial_t \partial_t \phi\right\|_{L^\infty(Q_r^-)} && \to 0,\\
\left\|(D^2 + \partial_t)\partial_i \phi_r\right\|_{L^\infty(Q_1^-)} &= r \left\|D^2 \partial_i \phi + \partial_t \partial_i \phi\right\|_{L^\infty(Q_r^-)} && \to 0 \quad \text{as } r \to 0.
\end{aligned} \tag{5.4.12}$$

Thus, by using (5.4.4), we know that the auxiliary function $\hat{w}$ in (5.4.13) is a supersolution. Finally, the maximum principle to the auxiliary function $\hat{w}$ in (5.4.13) implies that $C\partial_{\tilde{e}} v_r - v_r \geq 0$ in $Q_{1/2}^-$ and $\partial_{\tilde{e}} v_r \geq 0$ in $Q_{1/2}^-$.

We note that this argument for the parabolic obstacle problem for the heat operator was introduced in [9, Chapter 13] and [57].

**Proposition 5.4.10.** *(Directional monotonicity) Let $u \in P_1(M)$ and $\mathcal{P}\phi -$*

*$f \geq c > 0$ in $Q_1^-$. Let*

$$v_0(x,t) = \phi_0(x,t) - u_0(x,t) = \frac{1}{2}(x_n^+)^2,$$

*where $u_0$ and $\phi_0$ are blowup functions of $u$ and $\phi$, respectively. Then, for any $\delta \in (0,1]$, there exists $r_\delta = r(\delta, u, \phi) > 0$ such that*

$$\partial_{\tilde{e}} v \geq 0 \quad \text{in } Q_{r_\delta}^- \quad \text{for any } \tilde{e} \in \mathbb{S}^n \text{ such that } \tilde{e} \cdot (e_n, 0) \geq \delta.$$

*Proof.* For any $\delta \in (0,1]$, by direct computation, we know that there is $C_\delta > 0$ such that

$$C_\delta \partial_{\tilde{e}} v_0 - v_0 \geq 0 \text{ in } Q_1^- \quad \text{for any } \tilde{e} \in \mathbb{S}^n \text{ such that } \tilde{e} \cdot (e_n, 0) \geq \delta.$$

By (5.4.10) and (5.4.11), for sufficiently small $r = r(\delta, u, \phi)$,

$$C_\delta \partial_{\tilde{e}} v_r - v_r \geq -\frac{\alpha}{32} \quad \text{in } Q_1^-,$$

where $\alpha$ is the constant in Lemma 5.4.5. We claim that

$$C_\delta \partial_{\tilde{e}} v_r - v_r \geq 0 \quad \text{in } Q_{1/2}^-$$

Suppose that there is a point $Y_0 = (y_0, s_0) \in Q_{1/2}^- \cap \{v_r > 0\}$ such that $C_\delta \partial_{\tilde{e}} v_r(Y_0) - v_r(Y_0) < 0$ and consider

$$\hat{w}(Y) := C_\delta \partial_e v_r(Y) - v_r(Y) + \frac{\alpha}{2}\left(|y - y_0|^2 - (s - s_0)\right), \tag{5.4.13}$$

where $\alpha$ is the constant in Lemma 5.4.5. Recalling (5.4.5), we obtain

$$\frac{\alpha}{2}\mathcal{L}_u\left(|y - y_0|^2 - (s - s_0)\right) = \frac{c}{2}.$$

Then, by $\mathcal{L}_{u_r} \partial_{\tilde{e}} u_r = \partial_{\tilde{e}} \mathcal{P} u_r = \partial_{\tilde{e}}(f(rx, r^2 t))$ in $N(u)$, $\mathcal{L}_{u_r} \partial_{\tilde{e}} \phi_r = -\partial_t \partial_{\tilde{e}} \phi_r +$

$\frac{1}{n}F(D^2u)tr((D^2u)^{-1}D^2\partial_{\tilde{e}}\phi_r)$ in $Q_1^-$, and (5.4.12), we may assume that

$$\|\mathcal{L}_{u_r}\partial_{\tilde{e}}v_r\|_{L^\infty(Q_1^-)} \le \frac{c}{2C_\delta} \quad \text{in } N(u_r)$$

and have

$$\begin{aligned}\mathcal{L}_{u_r}\hat{w}(Y) &= C_\delta\mathcal{L}_{u_r}\partial_{\tilde{e}}v_r(Y) - \mathcal{L}_{u_r}v_r(Y) + \frac{c}{2}\\ &\le c - (\mathcal{P}\phi_r - f_r(ry, r^2s)) \le 0 \qquad \text{in } Q_{1/4}^-(Y_0)\cap N(u_r).\end{aligned}$$

Thus, by the maximum principle for $\hat{w}$ on $Q_{1/4}^-(Y_0)\cap N(u_r)$, we have

$$\inf_{\partial_p Q_{1/4}^-(Y_0)\cap N(u_r)} \hat{w} \le \hat{w}(Y_0) < 0 \quad \text{and} \quad \inf_{\partial_p Q_{1/4}^-(Y_0)\cap N(u_r)} (C\partial_{\tilde{e}}v_r - v_r) < -\frac{\alpha}{32}.$$

Hence, we arrive at a contradiction and have $C_\delta\partial_{\tilde{e}}v_r - v_r \ge 0$ in $Q_{1/2}^-$. By the nonnegativity of $v_r$, we have $\partial_{\tilde{e}}v \ge 0$ in $Q_{r/2}^-$. □

### 5.4.5 Proof of the regularity of the free boundary

**Lemma 5.4.11.** *Let $u \in P_1(M)$ be as in Theorem 5.1.3. Then, there is $r_1' = r_1'(u,\phi) > 0$ such that blowup functions of $v = \phi - u$ at $X \in \Gamma(u)\cap Q_{r_1'}^-$ are half-space functions, i.e., blowups of $v$ at $X$ are of the form $c(x_n^+)^2/2$, up to a rotation, for some constant $c$.*

*Proof.* By Proposition 5.4.8,

$$v_0 = \frac{1}{2}(x_n^+)^2$$

in an appropriate system of coordinates, for any blowup of $v$ at 0.

For $X = (x,t) \in \Gamma(u)\cap Q_{r_1'}^-$ and $t = 0$, by the thickness assumptions (5.1.4) and the same methods as in Proposition 5.4.8, we have that any blowup $v_0$ of $v$ at $X$,

$$v_0(x,t) = \phi_0 - u_0 = c(x\cdot e_x)_+^2 \quad \text{in } \mathbb{R}^n\times(-\infty,0],$$

for $e_x \in \mathbb{S}^{n-1}$, some positive constant $c$.

For the case $X = (x, t) \in \Gamma(u) \cap Q^-_{r'_1}$ and $t < 0$, By using the argument in Proposition 5.4.8 in $\mathbb{R}^n \times \mathbb{R}$ (as in Appendix of [57]), we have that

$$v_0(x, t) = \phi_0 - u_0 = c(x \cdot e_x)^2_+ \quad \text{in } \mathbb{R}^n \times \mathbb{R},$$

for some $e_x \in \mathbb{S}^{n-1}$ and positive constant $c$. □

Now, we are ready to prove the regularity of the free boundaries of $u \in P_1(M)$, Theorem 5.1.3.

*Proof of Theorem 5.1.3.* By the directional monotonicity, Proposition 5.4.10, the free boundary $\Gamma(u) \cap Q^-_{r_\delta/2}$ is a graph $x_n = f(x', t)$ where $f$ is a Lipschitz function with the Lipschitz constant less than $\delta/\sqrt{1-\delta^2}$. Furthermore, in Proposition 5.4.10, $\delta \in (0, 1]$ could be arbitrary small. Hence, we have a tangent plane of $\Gamma(u)$ and the normal vector $(e_n, 0)$ at 0. For any point $Z \in \Gamma(u) \cap Q_{r'_1}$, by Lemma 5.4.11, we know that there is a tangent plane of the free boundary at $Z$ with normal vector $\nu_Z$. By Proposition 5.4.10, for $Z \in \Gamma(u) \cap Q^-_{r_\delta}$, we have $\nu_Z \cdot \tilde{e} \geq 0$, for any $\tilde{e} \cdot (e_n, 0) \geq \delta$. Hence, $\nu_Z$ is close to $e_n$. Therefore, $\Gamma(u) \cap Q^-_{r'_1}$ is $C^1$ at 0.

By the same argument in Propositions 5.4.7 and 5.4.8, for any free boundary point $Z = (z, \tau) \in \Gamma(u) \cap Q^-_{r'_1}$, $\tau < 0$, the blowup of $v$ at $Z$ is of the form $c(x_n)^2_+$ in $\mathbb{R}^n \times \mathbb{R}$ and we have a directional monotonicity for $v$ in $Q_{r'}(Z)$, i.e., we obtain that for some $r' > 0$, for any $\delta \in (0, 1]$ there exists $r_\delta = r(\delta, u, \phi, Z) > 0$ such that

$$\partial_{\tilde{e}} v \geq 0 \quad \text{in } Q^-_{r_\delta} \quad \text{for any } \tilde{e} \in \mathbb{S}^n \text{ such that } \tilde{e} \cdot \nu_Z \geq \delta.$$

Then, by the same argument in the previous paragraph, $\Gamma(u)$ is $C^1$ at $Z$. □

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# 국문초록

곡률 흐름이란 평균 곡률이나 가우스 곡률과 같이 곡률에 대한 양으로 모양이 변하는 초곡면의 기하학적 변화를 말하며, 재료과학 및 영상처리 분야에서 응용되고 있다. 곡률 흐름을 다루는데에 있어 가장 큰 어려움은 유한시간 내에 발생하는 특이점으로, 많은 경우에 이 특이점이 발생한다. 이 학위논문에서는 초기 초곡면의 안쪽에 장애물을 위치시켜 곡률 흐름이 특이점을 넘어 오랜시간 존재하는 방법을 제시하고자 한다. 이 방법을 적용하여, 그래프로 주어진 초기 초곡면에 대한 평균 곡률 흐름 및 닫혀있고 완전 볼록인 초기 초곡면에 대한 가우스 곡률 흐름에 대해 특이점의 발생을 막는다. 또한, 가우스 곡률 흐름과 밀접하게 관련된 포물형 몽쥬-앙페르 방정식에 대한 장애물 문제를 고려한다. 본 연구는 패널티 방법론에 기반하여, 초곡면이 장애물을 통과할 수 있도록 허용하는 대신, 통과한 만큼 패널티를 부과하는 방법을 사용하였다.

**주요어휘:** 평균 곡률 흐름, 가우스 곡률 흐름, 장애물 문제, 자유경계 문제, 몽쥬-앙페르 방정식, 특이점
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