이학 박사 학위논문

# Heat kernel estimates for jump processes with application

## (점프 확률과정의 확률밀도함수 추정치와 그 응용)

2020년 8월


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# Heat kernel estimates for jump processes with application

A dissertation
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# Abstract

In this thesis, we study heat kernel estimates for a class of Markov processes and its applications. We first consider heat kernel estimates for Hunt processes in metric measure spaces corresponds to symmetric Dirichlet forms. Next we will study the Levi's method to obtain heat kernel estimates for non-symmetric nonlocal operators concern with jump processes. The last part of this thesis is devoted to the applications of heat kernel estimates. We deals with the boundary regularity estimates for nonlocal operators with kernels of variable orders. Then, the laws of iterated logarithms for Markov processes will be introduced. Heat kernel and its estimates plays an important role in both problems.

**Key words:** Markov process, heat kernel estimate, Dirichlet form, nonlocal operator, laws of iterated logarithm, Green function
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# Contents

# Chapter 1

# Introduction

The heat kernel provides an important link between probability theory and partial differential equation. In probability theory, the heat kernel of an operator $\mathcal{L}$ is the transition density $p(t,x,y)$ (if it exists) of the Markov process $X$, which possesses $\mathcal{L}$ as its infinitesimal generator. In the field of partial differential equation, it is called the fundamental solution of the heat equation $\partial_t u = \mathcal{L}u$. However, except in a few special cases, obtaining an explicit expression of $p(t,x,y)$ is usually impossible. Thus finding sharp estimates of $p(t,x,y)$ is a fundamental issue both in probability theory and partial differential equation.

Although heat kernels for diffusion processes have been studied for over a century, heat kernel estimates for discontinuous Markov processes have only been studied in recent years. After pioneering works such as [14, 30, 64], obtaining sharp two-sided estimates of heat kernels for various classes of discontinuous Markov processes has become an active topic in modern probability theory (see [1, 5, 6, 8, 9, 16, 10, 17, 23, 24, 25, 28, 22, 31, 32, 34, 36, 35, 43, 44, 45, 47, 50, 51, 48, 52, 53, 54, 55, 61, 62, 63, 67, 68, 70, 71, 72, 73, 78, 88, 89] and references therein). Moreover, heat kernel estimates for Markov processes on metric measure spaces provide information on not only the behaviour of the corresponding processes but also intrinsic properties such as walk dimension of underlying space ([5, 6, 48, 68]).

Let us consider a symmetric pure jump Markov process on a general metric measure space $(M, d, \mu)$ that satisfies volume doubling conditions (see Section 1.1 for the settings). In [31], the authors investigated heat kernel estimates for symmetric discontinuous Markov processes (on a large class of metric measure spaces) whose jumping intensities are comparable to radially symmetric functions of variable order. In particular, the heat kernel estimates therein cover the class of symmetric Markov processes $X = (X_t, \mathbb{P}^x, x \in M, t \geq 0)$, without diffusion part, whose jumping kernels $J(x, y)$ satisfy the following conditions:

$$J(x,y) \asymp \frac{1}{V(x, d(x,y))\psi(d(x,y))}, \quad x, y \in M, \tag{1.0.1}$$

where $B(x,r) = \{y \in M : d(x,y) < r\}$ and $V(x,r) = \mu(B(x,r))$ are open ball and its volume, and $\psi$ is a strictly increasing function on $[0,\infty)$ satisfying

$$c_1(R/r)^{\beta_1} \leq \psi(R)/\psi(r) \leq c_2(R/r)^{\beta_2}, \quad 0 < r < R < \infty \tag{1.0.2}$$

with $0 < \beta_1 \leq \beta_2 < 2$. Here we have used the notation $f \asymp g$ if the quotient $f/g$ remains bounded between two positive constants. We say that $\psi$ is *the rate function* since $\psi$ gives the growth of jump intensity according to its size. Under the assumptions (1.0.1), (1.0.2) and $V(x,r) \asymp \widetilde{V}(r)$ for strictly increasing function $\widetilde{V}$, the transition density $p(t,x,y)$ of Markov process has the following estimates: for any $t > 0$ and $x, y \in M$,

$$p(t,x,y) \asymp \left( \frac{1}{V(x, \psi^{-1}(t))} \wedge \frac{t}{V(x, d(x,y))\psi(d(x,y))} \right). \tag{1.0.3}$$

(See [31, Theorem 1.2]. See also [32] where the extra condition $V(x,r) \asymp \widetilde{V}(r)$ is removed). We call a function $\Phi$ *the scale function* for $p(t,x,y)$ if $\Phi(d(x,y)) = t$ provides the borderline for $p(t,x,y)$ to have either near-diagonal estimates or off-diagonal estimates. Observe that by (1.0.3) we have $p(t,x,x) \asymp p(t,x,y)$ for $y \in B(x, \psi^{-1}(t))$. Thus, $\psi$ is the scale function in (1.0.3) so that the rate function and the scale function coincide when

$0 < \beta_1 \le \beta_2 < 2$.

Moreover, we see that (1.0.1) is equivalent to (1.0.3) since $p(t, x, y)/t \to J(x, y)$ weakly as $t \to 0$. Thus, for a large class of pure jump symmetric Markov processes on metric measure space satisfying volume doubling properties, (1.0.1) is equivalent to (1.0.3) under the condition (1.0.2).

One of major problems in this field was to obtain heat kernel estimates for jump processes on metric measure space without the restriction $\beta_2 < 2$. When the process $X$ is a subordinate Brownian motion, Ante Mimica [70] established the heat kernel estimates for the case that $\beta_2$ may not be strictly below 2. Also, [89] partially generalized to Lévy processes. In [3, 2], we study heat kernel estimates for general jump processes without imposing $\beta_2 < 2$. In particular, we obtained that the rate function and scale function may not be comparable. [3] deals with processes in $\mathbb{R}^d$, and [2] is for metric measure spaces. The results will be introduced in Chapter 2.

Since we highly rely on the symmetric Dirichlet form theory to obtain heat kernel estimate, the symmetricity of the process is indispensable. In [92], the authors obtained the heat kernel estimates for the operator $\triangle + b \cdot \nabla$ with Hölder continuous drift $b$, which may be nonsymmetric, by using Levi's method. In this paper, the heat kernel of nonsymmetric operator, which is the solution of heat equation, is constructed by the heat kernel of symmetric operator and its perturbation. [34] generalized this result to nonsymmetric $\alpha$-stable like processes in $\mathbb{R}^d$, in other words the jumping kernel $J(x, y) \asymp \frac{1}{|x-y|^{d+\alpha}}$. The methods in [34] are quite robust and have been applied to nonsymmetric and non-convolution operators (see [19, 24, 35, 36, 61, 55, 53] and references therein). In Chapter 3, we consider the case that $J(x, y)$ decays exponentially or subexponentially when $|x - y|$ goes to $\infty$ and we obtain sharp two-sided estimates for the heat kernel. This chapter is based on [60]. In the last part, my ongoing research project related to this topic will be introduced.

Besides heat equation, there are many applications of heat kernel estimate. Parabolic and elliptic Harnack inequalities are the most well-known consequences of heat kernel estimate (see [10, 12, 26, 33, 79] and references therein). Chapter 4 contains various applications of heat kernel estimate including Green function estimate and laws of iterated logarithms. This chapter is based on the my papers ([3, 2, 58]) and preprints ([37, 38]). In particular, [58] obtains boundary decaying rate for the Poisson equation with respect to nonlocal operators which is infinitesimal generator for a class of isotropic Lévy processes. The result will be introduced in Section 4.1.

## 1.1 Basic settings and notations

In this section, we gather some definitions and concepts which will be used throughout this thesis. First we define *the weak scaling condition*, which describes the polynomial growth rate of the function.

**Definition 1.1.1.** Let $g:(0,\infty)\to(0,\infty)$ and $\beta, C>0$.
(1) For $a\in(0,\infty]$, we say that $g$ satisfies $L_a(\beta,C)$ (resp. $L^a(\beta,C)$) if $g(R)/g(r)\geq C(R/r)^\beta$ for all $r\leq R<a$ (resp. $a\leq r\leq R$). We also say that the condition $L_a(\beta,C,g)$ (resp. $L^a(\beta,C,g)$) hold. In particular, we write $L_\infty(\beta_1,c)$ or $U_\infty(\beta_2,C)$ as $L(\beta_1,c)$ or $U(\beta_2,C)$.
(2) For $a\in[0,\infty)$, we say that $g$ satisfies $U_a(\beta,C)$ (resp. $U^a(\beta,C)$) if $g(R)/g(r)\leq C(R/r)^\beta$ for all $r\leq R<a$ (resp. $a\leq r\leq R$). We also say that the condition $U_a(\beta,C,g)$ (resp. $U^a(\beta,c,g)$) hold.

The following lemmas are useful tools to deal with weak scaling conditions.

**Lemma 1.1.2.** *Let $g:(0,\infty)\to(0,\infty)$ be a non-decreasing function. Then,*
*(1) For any $a\in(0,\infty]$ and $\beta,c>0$, $L_a(\beta,c,g)$ implies* $\lim_{r\to0}g(r)=0$.
*(2) For any $a\in[0,\infty)$ and $\beta,c>0$, $U^a(\beta,c,g)$ implies* $\lim_{r\to\infty}g(r)=\infty$.

**Lemma 1.1.3.** ([2, Remark A.1]) Let $g:(0,\infty)\to(0,\infty)$ be a non-decreasing function and $a\in(0,\infty)$. Then, $L_a(\beta,c,g)$ implies $L_b(\beta,c(a/b)^\beta,g)$ for any $a<b$. Similarly, $L^a(\beta,c,g)$ implies $L^b(\beta,c(a^{-1}b)^\beta,g)$ for any $b<a$.

Let $g^{-1}(s):=\inf\{r\geq0:g(r)>s\}$ be the generalized inverse function of $g$.

**Lemma 1.1.4.** ([2, Remark A.2]) Let $g : [0, \infty) \to [0, \infty)$ be a nondecreasing function with $g(0) = 0$ and $g(\infty) = \infty$. Then, for $\beta > 0$ and $c, C > 0$,

(1) If $g$ satisfies $L_a(\beta, c)$ (resp. $U_a(\beta, C)$), then $g^{-1}$satisfies $U_{g(a)}(\frac{1}{\beta}, c^{-1/\beta})$ (resp. $L_{g(a)}(\frac{1}{\beta}, C^{-1/\beta})$).

(2) If $g$ satisfies $L^a(\beta, c)$ (resp. $U^a(\beta, C)$), then $g^{-1}$satisfies $U^{g(a)}(\frac{1}{\beta}, c^{-1/\beta})$ (resp. $L^{g(a)}(\frac{1}{\beta}, C^{-1/\beta})$).

**Lemma 1.1.5.** ([2, Lemma 3.7]) Let $g : (0, \infty) \to (0, \infty)$ be a non-decreasing function satisfying $U(\beta, C)$. Then, for any $t > 0$,

$$C^{-1}t \leq g(g^{-1}(t)) \leq Ct, \tag{1.1.1}$$

where $g^{-1}$ is the generalized inverse function of $g$.

Throughout this thesis, we consider Euclidean space $\mathbb{R}^d$ or metric measure space $(M, d, \mu)$, where $(M, d)$ is a locally compact separable metric space, and $\mu$ is a positive Radon measure on $M$ with full support. As mentioned above, we denote $B(x, r) := \{y \in M : d(x, y) < r\}$ and $V(x, r) := \mu(B(x, r))$ an open ball in $M$ and its volume, respectively. We introduce local versions of *volume doubling properties* for the metric measure space $(M, d, \mu)$, whose original version is in [32].

**Definition 1.1.6.** (i) We say that $(M, d, \mu)$ satisfies the *volume doubling property* VD$(d_2)$ if there exists a constant $C_\mu \geq 1$ such that

$$\frac{V(x, R)}{V(x, r)} \leq C_\mu \left(\frac{R}{r}\right)^{d_2} \quad \text{for all } x \in M \text{ and } 0 < r \leq R. \tag{1.1.2}$$

(ii) We say that $(M, d, \mu)$ satisfies the *reverse volume doubling property* RVD$(d_1)$ if there exist constants $d_1 > 0$, $c_\mu > 0$ such that

$$\frac{V(x, R)}{V(x, r)} \geq c_\mu \left(\frac{R}{r}\right)^{d_1} \quad \text{for all } x \in M \text{ and } 0 < r \leq R.$$

It is obvious that $\mathbb{R}^d$ satisfies both VD($d_2$) and RVD($d_1$). Note that $V(x,r) > 0$ for every $x \in M$ and $r > 0$ since $\mu$ has full support on $M$. Also, under VD($d_2$), we have from (1.1.2) that for all $x \in M$ and $0 < r \le R$,

$$\frac{V(x,R)}{V(y,r)} \le \frac{V(y,d(x,y)+R)}{V(y,r)} \le C_\mu \left(\frac{d(x,y)+R}{r}\right)^{d_2}.$$

*Notations* : Throughout this thesis, the constants $C_i$, $\alpha_i$, $\beta_i$, $\gamma_i$ and $\delta_i$ for $i \in \mathbb{N}$, $\bar{C}$, $c_L$, $C_L$, $\widetilde{C}_L$, $c_U$, $C_U$, $d_1$, $d_2$ will retain throughout the section, whereas $c$, $C$, $\varepsilon$, $\eta$ and $\theta$ represent constants having insignificant values that may be changed from one appearance to another. All these constants are positive finite. The labeling of the constants $c_1$, $c_2$, $\dots$ begins anew in the proof of each result. $c_i = c_i(a,b,c,\dots)$, $i = 0,1,2,\dots$, denote generic constants depending on $a,b,c,\dots$. Recall that we use the notation $f \asymp g$ if the quotient $f/g$ remains bounded between two positive constants. Define $a \wedge b = \min\{a,b\}$, $a \vee b := \max\{a,b\}$. Also, for any point $x$ and set $D$, define $\delta_D(x) := \mathrm{dist}(x, D^c)$ for the distance between $x \in D$ and $D^c$. For $d \ge 1$, let $\omega_d = \int_{\mathbb{R}^d} \mathbf{1}_{\{|y|\le 1\}} dy$ be the volume of $d$-dimensional ball. Let $\lceil a \rceil := \sup\{n \in \mathbb{Z} : n \le a\}$. In any set $S$, we define $diag := \{(x,x) : x \in S\}$. Let $\mathbb{R}_+ = (0,\infty)$ and $\mathbb{R}^n_+ := \{x = (x_1,...,x_n) \in \mathbb{R}^n \mid x_n > 0\}$ be the upper half plane.

# Chapter 2

# Heat kernel estimates for symmetric Dirichlet form on metric measure space

In this chapter, we study the transition densities of pure-jump symmetric Markov processes in $\mathbb{R}^d$ or a metric measure space $(M, d, \mu)$ equipped with volume doubling condition, whose rate function enjoys weak scaling condition. Under some mild assumptions on rate functions, we can establish sharp two-sided estimates of the transition densities for such processes.

Recall that if the rate function $\psi$ of jumping kernel of a symmetric Markov process $X$ satisfies $L(\beta_1, c, \psi)$ and $U(\beta_2, C, \psi)$ for some $0 < \beta_1 \le \beta_2 < 2$, the scale function for such Markov process coincides with the rate function. On the other hand, in [3, 2, 70], new forms of heat kernel estimates for symmetric jump Markov processes in Euclidean spaces were obtained without the condition $\beta_2 < 2$. In particular, the results in [3, 2, 70] cover Markov processes with high intensity of small jumps. In this case, unlike [32], the rate function and the scale function may not be comparable and the heat kernel estimates are written in a more general form.

This chapter consists of two sections. In the first section we study the heat kernel estimates in [3], which deals with Euclidean space. In Section

2.2 we will consider the processes on metric measure spaces. This section is based on [2]. Heat kernel estimates for Markov processes on metric measure spaces provide information on not only the behaviour of the corresponding processes but also intrinsic properties such as walk dimension of underlying space ([5, 6, 48, 68]). The result covers metric measure spaces whose walk dimension is bigger than 2 such as Sierpinski gasket and Sierpinski carpet (See Subsection 2.2.6).

## 2.1 Symmetric jump processes on Euclidean space

Throughout this subsection, we will assume that $\psi : (0,\infty) \to (0,\infty)$ is a non-decreasing function satisfying $L(\beta_1, C_L)$, $U(\beta_2, C_U)$, and

$$\int_0^1 \frac{s}{\psi(s)}\, ds < \infty. \tag{2.1.1}$$

Denote $diag = \{(x,x) : x \in \mathbb{R}^d\}$. Assume that $J : \mathbb{R}^d \times \mathbb{R}^d \setminus diag \to [0,\infty)$ is a symmetric function satisfying

$$\frac{\bar{C}^{-1}}{|x-y|^d \psi(|x-y|)} \le J(x,y) \le \frac{\bar{C}}{|x-y|^d \psi(|x-y|)} \tag{2.1.2}$$

for all $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \setminus diag$, with some $\bar{C} \ge 1$. Note that (2.1.1) combined with (2.1.2) and $L(\beta_1, C_L)$ on $\psi$ is a natural assumption to ensure that

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \big(|x-y|^2 \wedge 1\big) J(x,y)dy \le c \left( \int_0^1 \frac{sds}{\psi(s)} + \int_1^\infty \frac{ds}{s\psi(s)} \right) < \infty \tag{2.1.3}$$

For $u, v \in L^2(\mathbb{R}^d, dx)$, define

$$\mathcal{E}(u,v) := \int_{\mathbb{R}^d \times \mathbb{R}^d} (u(x)-u(y))(v(x)-v(y))J(x,y)dxdy \tag{2.1.4}$$

and $\mathcal{F} = \{f \in L^2(\mathbb{R}^d) : \mathcal{E}(f,f) < \infty\}$. By applying the lower scaling assumption $L(\beta_1, C_L)$ on $\psi$, (2.1.2) and (2.1.3) to [81, Theorem 2.1] and [82, Theorem 2.4], we observe that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d, dx)$. Thus, there is a Hunt process $X$ associated with $(\mathcal{E}, \mathcal{F})$, starting from quasi-everywhere point in $\mathbb{R}^d$. Moreover, by (2.1.3) and [69, Theorem 3.1], $X$ is conservative.

We define our scale function by

$$\Phi(r) := \frac{r^2}{2\int_0^r \frac{s}{\psi(s)}\,ds}.$$

In general, the function $\Phi$ is strictly increasing, and is less than $\psi$ (see (2.1.10)-(2.1.12) below). However, these two functions may not be comparable unless $\beta_2 < 2$. We remark here that the function $\Phi$ has been observed as the correct scale function (see [50, 51, 60, 70, 79]).

**Theorem 2.1.1.** *Let $\psi$ be a non-decreasing function satisfying $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$. Assume that conditions* (2.1.1) *and* (2.1.2) *hold. Then, there is a conservative Feller process $X = (X_t, \mathbb{P}^x, x \in \mathbb{R}^d, t \geq 0)$ associated with $(\mathcal{E}, \mathcal{F})$ that can starts from every point in $\mathbb{R}^d$. Moreover, $X$ has a jointly continuous transition density function $p(t,x,y)$ on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ with the following estimates: there exist $a_U, C, \delta_1 > 0$ such that*

$$p(t,x,y) \leq \frac{C}{\Phi^{-1}(t)^d} \wedge \left( \frac{C\,t}{|x-y|^d \psi(|x-y|)} + \frac{C}{\Phi^{-1}(t)^d} e^{-\frac{a_U |x-y|^2}{\Phi^{-1}(t)^2}} \right) \tag{2.1.5}$$

*and*

$$p(t,x,y) \geq \frac{C^{-1} \mathbf{1}_{\{|x-y| \leq \delta_1 \Phi^{-1}(t)\}}}{\Phi^{-1}(t)^d} + \frac{C^{-1}\,t}{|x-y|^d \psi(|x-y|)} \mathbf{1}_{\{|x-y| \geq \delta_1 \Phi^{-1}(t)\}}. \tag{2.1.6}$$

Using our scale function $\Phi$, we define for $a > 0$,

$$\mathscr{K}(s) := \sup_{b \leq s} \frac{\Phi(b)}{b} \text{ and } \mathscr{K}_\infty(s) := \begin{cases} \sup_{a \leq b \leq s} \frac{\Phi(b)}{b}, & s \geq a, \\ a^{-2}\Phi(a)s, & 0 < s < a. \end{cases} \tag{2.1.7}$$

If $\Phi$ satisfies $L_a(\delta, \widetilde{C}_L)$ with $\delta > 1$, then $\mathscr{K}(0) = 0$ and $\mathscr{K}$ is non-decreasing. Thus, the generalized inverse $\mathscr{K}^{-1}(t) := \inf\{s \geq 0 : \mathscr{K}(s) > t\}$ is well defined on $[0, \sup_{b<\infty} \frac{\Phi(b)}{b})$.

If $\Phi$ satisfies $L^a(\delta, \widetilde{C}_L)$ with $\delta > 1$, $\mathscr{K}_\infty$ and the generalized inverse $\mathscr{K}_\infty^{-1}$ are well-defined and non-decreasing on $[0, \infty)$. Some properties of $\mathscr{K}$ and $\mathscr{K}_\infty$ are shown in Subsection 2.1.1.

**Theorem 2.1.2.** *Let $\psi$ be a non-decreasing function satisfying $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$. Assume that conditions* (2.1.1) *and* (2.1.2) *hold, and $\Phi$ satisfies $L_a(\delta, \widetilde{C}_L)$ or $L^a(\delta, \widetilde{C}_L)$ for some $a > 0$ and $\delta > 1$. Then, the following estimates hold:*
*(1) When $\Phi$ satisfies $L_a(\delta, \widetilde{C}_L)$: For every $T > 0$, there exist positive constants $c_1 = c_1(T, a, \delta, \beta_1, \beta_2, \widetilde{C}_L, C_L, C_U) \geq 1$ and $a_U \leq a_L$ such that for any $(t, x, y) \in (0, T) \times \mathbb{R}^d \times \mathbb{R}^d$,*

$$c_1^{-1}\left(\frac{1}{\Phi^{-1}(t)^d} \wedge \left(\frac{t}{|x-y|^d\psi(|x-y|)} + \frac{1}{\Phi^{-1}(t)^d} e^{-\frac{a_L|x-y|}{\mathscr{K}^{-1}(t/|x-y|)}}\right)\right) \tag{2.1.8}$$

$$\leq p(t,x,y) \leq c_1\left(\frac{1}{\Phi^{-1}(t)^d} \wedge \left(\frac{t}{|x-y|^d\psi(|x-y|)} + \frac{1}{\Phi^{-1}(t)^d} e^{-\frac{a_U|x-y|}{\mathscr{K}^{-1}(t/|x-y|)}}\right)\right).$$

*Moreover, if $\Phi$ satisfies $L(\delta, \tilde{C}_L)$, then* (2.1.8) *holds for all $t \in (0, \infty)$.*
*(2) When $\Phi$ satisfies $L^a(\delta, \widetilde{C}_L)$: For every $T > 0$, there exist positive constants $c_2 = c_2(T, a, \delta, \beta_1, \beta_2, \widetilde{C}_L, C_L, C_U) \geq 1$ and $a'_U \leq a'_L$ such that for any $(t, x, y) \in [T, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$,*

$$c_2^{-1}\left(\frac{1}{\Phi^{-1}(t)^d} \wedge \left(\frac{t}{|x-y|^d\psi(|x-y|)} + \frac{1}{\Phi^{-1}(t)^d} e^{-\frac{a'_L|x-y|}{\mathscr{K}_\infty^{-1}(t/|x-y|)}}\right)\right)$$
$$\leq p(t,x,y) \leq c_2\left(\frac{1}{\Phi^{-1}(t)^d} \wedge \left(\frac{t}{|x-y|^d\psi(|x-y|)} + \frac{1}{\Phi^{-1}(t)^d} e^{-\frac{a'_U|x-y|}{\mathscr{K}_\infty^{-1}(t/|x-y|)}}\right)\right).$$

*In particular, if $\delta = 2$, then $\mathscr{K}_\infty^{-1}(t) \asymp t$ for $t \geq T$.*

A non-negative $C^\infty$ function $\phi$ on $(0, \infty)$ is called a Bernstein function if $(-1)^n\phi^{(n)}(\lambda) \le 0$ for every $n \in \mathbb{N}$ and $\lambda > 0$. The exponent $(r/\Phi^{-1}(t))^2$ in (2.1.5) is not comparable to $r/\mathscr{K}^{-1}(t/r)$ in general (see Lemma 2.1.9 and Corollary 2.1.26 below). However, the following corollary indicates that we can replace $r/\mathscr{K}^{-1}(t/r)$ with a simpler function $(r/\Phi^{-1}(t))^2$ if we additionally assume that $r \mapsto \Phi(r^{-1/2})^{-1}$ is a Bernstein function.

**Corollary 2.1.3.** *Let $\psi$ be a non-decreasing function satisfying $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$. Assume that conditions* (2.1.1) *and* (2.1.2) *hold, $\Phi$ satisfies $L_a(\delta, \widetilde{C}_L)$ some $a > 0$ and $\delta > 1$, and $r \mapsto \Phi(r^{-1/2})^{-1}$ is a Bernstein function. Then, for any $T > 0$, there exist positive constants $c \ge 1$ and $a_U \le a_L$ such that for all $(t,x,y) \in (0,T) \times \mathbb{R}^d \times \mathbb{R}^d$,*

$$c^{-1}\left(\frac{1}{\Phi^{-1}(t)^d} \wedge \left(\frac{t}{|x-y|^d\psi(|x-y|)} + \frac{1}{\Phi^{-1}(t)^d} e^{-a_L \frac{|x-y|^2}{\Phi^{-1}(t)^2}}\right)\right) \tag{2.1.9}$$

$$\le p(t,x,y) \le c\left(\frac{1}{\Phi^{-1}(t)^d} \wedge \left(\frac{t}{|x-y|^d\psi(|x-y|)} + \frac{1}{\Phi^{-1}(t)^d} e^{-a_U \frac{|x-y|^2}{\Phi^{-1}(t)^2}}\right)\right).$$

*Moreover, if $\Phi$ satisfies $L(\delta, C_L)$ with $\delta > 1$,* (2.1.9) *holds for all $t \in (0, \infty)$.*

### 2.1.1 Basic properties of scale functions

In this subsection, we will observe some elementary properties of scale functions $\psi$, $\Phi$ and $\mathscr{K}$. This is based on [3, Subsection 2.1 and 2.2]. Since $\psi$ is non-decreasing and $\lim_{r\to 0}\psi(r) = 0$ by $L(\beta_1, C_L)$ for $\psi$, we have that

$$\Phi(r) = \frac{r^2}{2\int_0^r \frac{s}{\psi(s)}\,ds} < \frac{r^2}{2\int_0^r \frac{s}{\psi(r)}\,ds} = \psi(r). \tag{2.1.10}$$

Thus, under (2.1.2), we obtain that for any $x, y \in \mathbb{R}^d$,

$$J(x,y) \le \frac{\bar{C}}{|x-y|^d\Phi(|x-y|)}. \tag{2.1.11}$$

Since $(1/\Phi(r))' = \frac{4}{r\psi(r)} - \frac{4}{r\Phi(r)} < 0$, $r \mapsto \Phi(r)$ is strictly increasing. Note that, since $r^2/\Phi(r)$ is increasing in $r$, we have that for any $0 < r \le R$,

$$\Phi(R)/\Phi(r) \le (R/r)^2. \tag{2.1.12}$$

The following two lemmas will be used several times. In particular, the second lemma shows that the scaling index for $\Phi$ is always in $(0, 2]$.

**Lemma 2.1.4.** *Assume that $\psi$ satisfies $L(\beta, c)$ and $U(\widehat{\beta}, C)$. Then, for any $x \in \mathbb{R}^d$ and $r > 0$, $\int_r^\infty (s\psi(s))^{-1} ds \asymp 1/\psi(r)$.*

**Lemma 2.1.5.** *Let $a \in (0, \infty]$, $0 < \beta \le \widehat{\beta}$, $0 < c \le 1 \le C$.*

*(1) If $\psi$ satisfies $U_a(\widehat{\beta}, C)$, then $\Phi$ satisfies $U_a(\widehat{\beta} \wedge 2, C)$.*

*(2) If $\psi$ satisfies* (2.1.1) *and $L_a(\beta, c)$, then $\beta < 2$ and $\Phi$ satisfies $L_a(\beta, c)$.*

We remark here that the comparability of $\psi$ and $\Phi$ is equivalent to that the index of the weak upper scaling condition is strictly less than 2 (see [16, Corollaries 2.6.2 and 2,6,4]). Next we establish some basic properties of $\mathscr{K}$ and $\mathscr{K}_\infty$ defined in (2.1.7).

**Lemma 2.1.6.** *If $\Phi$ satisfies $L_a(\delta, \widetilde{C}_L)$ with $\delta > 1$ and $a \in (0, \infty]$, then $\Phi(t)/t \le \mathscr{K}(t) \le \widetilde{C}_L^{-1}\Phi(t)/t$ for $t < a$, and*

$$\widetilde{C}_L^2 \, (t/s)^{\delta-1} \le \mathscr{K}(t)/\mathscr{K}(s) \le \widetilde{C}_L^{-1} t/s, \quad \textit{for } \; s \le t < a. \tag{2.1.13}$$

**Lemma 2.1.7.** *(1) For any $t > 0$, $\widetilde{\Phi}_a(t) \le \Phi(t)$ and for $t \ge c > 0$, $\widetilde{\Phi}_a(t) \ge ((c/a)^2 \wedge 1)\Phi(t)$. (2) For $0 < s < t$, $\widetilde{\Phi}_a(t)/\widetilde{\Phi}_a(s) \le t^2/s^2$. (3) Suppose $\Phi$ satisfies $L^a(\delta, \widetilde{C}_L)$ with some $\delta \le 2$. Then, $\widetilde{\Phi}_a$ satisfies $L(\delta, \widetilde{C}_L)$.*

**Lemma 2.1.8.** *Let $a \in (0, \infty)$. If $\Phi$ satisfies $L^a(\delta, \widetilde{C}_L)$ with $\delta > 1$, then $\widetilde{\Phi}_a(t)/t \le \mathscr{K}_{\infty,a}(t) \le \widetilde{C}_L^{-1}\widetilde{\Phi}_a(t)/t$ for $t > 0$, and*

$$\widetilde{C}_L^2 (t/s)^{\delta-1} \le \mathscr{K}_{\infty,a}(t)/\mathscr{K}_{\infty,a}(s) \le \widetilde{C}_L^{-1} t/s, \quad \textit{for } \; t > s > 0. \tag{2.1.14}$$

*Moreover, for any $c_1 > 0$, there exists $c_2 = c_2(c_1, a, \delta, \widetilde{C}_L) \geq 1$ such that for any $t \geq c_1$,*

$$c_2^{-1} \sup_{c_1 \leq b \leq t} \Phi(b)/b \leq \mathscr{K}_{\infty,a}(t) \leq c_2 \sup_{c_1 \leq b \leq t} \Phi(b)/b.$$

We also have some inequalities between $\Phi^{-1}$ and $\mathscr{K}^{-1}$, and between $\Phi^{-1}$ and $\mathscr{K}_\infty^{-1}$.

**Lemma 2.1.9.** *(1) Suppose $\Phi$ satisfies $L_a(\delta, \widetilde{C}_L)$ with $\delta > 1$ and for some $a > 0$. For any $T > 0$ and $b > 0$ there exists a constant $c_1 = c_1(b, \widetilde{C}_L, a, \delta, T) > 0$ such that*

$$\Phi^{-1}(t) \leq c_1 \mathscr{K}^{-1}\left(\frac{t}{b\Phi^{-1}(t)}\right) \quad \text{for all } t \in (0, T), \tag{2.1.15}$$

*and there exists a constant $c_2 = c_2(a, \widetilde{C}_L, \delta, T) \geq 1$ such that for every $t, r > 0$ satisfying $t < \Phi(r) \wedge T$,*

$$(r/\Phi^{-1}(t))^2 \leq r/\mathscr{K}^{-1}(t/r) \leq c_2 \left(r/\Phi^{-1}(t)\right)^{\delta/(\delta-1)}. \tag{2.1.16}$$

*Moreover, if $a = \infty$, then (2.1.15) and (2.1.16) hold with $T = \infty$. In other words, (2.1.15) holds for all $t < \infty$ and (2.1.16) holds for $t < \Phi(r)$.*

*(2) Suppose $\Phi$ satisfies $L^1(\delta, \widetilde{C}_L)$ with $\delta > 1$. For any $T > 0$ and $b > 0$ there exists a constant $c_3 = c_3(T, b, \widetilde{C}_L, \delta) \geq 1$ such that for $t \geq T$,*

$$\Phi^{-1}(t) \leq c_3 \mathscr{K}_\infty^{-1}\left(\frac{t}{b\Phi^{-1}(t)}\right),$$

*and for any $T > 0$ there exists a constant $c_4 = c_4(a, \widetilde{C}_L, \delta, T) \geq 1$ such that for every $t, r > 0$ satisfying $T \leq t \leq \Phi(r)$,*

$$c_4^{-1} \left(r/\Phi^{-1}(t)\right)^2 \leq r/\mathscr{K}_\infty^{-1}(t/r) \leq c_4 \left(r/\Phi^{-1}(t)\right)^{\delta/(\delta-1)}. \tag{2.1.17}$$

### 2.1.2 Near-diagonal estimates and preliminary upper bound

Here we will prove (weak) Poincaré inequality with respect to our jumping kernel $J$.

**Lemma 2.1.10.** *For $r > 0$, let $g : (0, r] \to \mathbb{R}$ be a continuous and non-increasing function satisfying $\int_0^r sg(s)ds \geq 0$ and $h : [0, r] \to [0, \infty)$ be a subadditive measurable function with $h(0) = 0$, i.e., $h(s_1)+h(s_2) \geq h(s_1+s_2)$, for $0 < s_1, s_2 < r$ with $s_1 + s_2 < r$. Then, $\int_0^r h(s)g(s)ds \geq 0$.*

By applying the above lemma, we have the following (weak) Poincaré inequality.

**Proposition 2.1.11.** *There exists $C > 0$ such that for every bounded and measurable function $f$, $x_0 \in \mathbb{R}^d$ and $r > 0$,*

$$\frac{C}{r^d \Phi(r)} \int_{B(x_0,r)\times B(x_0,r)} (f(y) - f(x))^2 dxdy \leq \int_{B(x_0,3r)\times B(x_0,3r)} (f(y) - f(x))^2 J(x, y)dxdy. \tag{2.1.18}$$

**Proof.** Denote $B(r) := B(x_0, r)$. For $0 < s < 2r$, let

$$h(s) := s^{-d} \int_{B(3r-s)} \int_{|z|=s} \big(f(x + z) - f(x)\big)^2 \sigma(dz)dx,$$

where $\sigma$ is surface measure of the ball. We observe that the left hand side of (2.1.18) is bounded above by

$$\begin{aligned} &\frac{c_1}{r^d \Phi(r)} \int_{B(r)} \int_0^{2r} \int_{|z|=s} \big(f(x + z) - f(x)\big)^2 \sigma(dz)dsdx \\ &\leq \frac{c_1}{r^d \Phi(r)} \int_0^{2r} h(s)s^d ds \leq \frac{2^{d+2}c_1}{\Phi(2r)} \int_0^{2r} h(s)ds, \end{aligned}$$

where the last inequality follows from (2.1.12). On the other hand, the right hand side of (2.1.18) is bounded below by

$$c_2 \int_{B(2r)} \int_{B(3r-|z|)} (f(x + z) - f(x))^2 \frac{1}{|z|^d \psi(|z|)} dxdz = c_3 \int_0^{2r} h(s) \frac{1}{\psi(s)} ds.$$

Let $g(s) = \frac{1}{\psi(s)} - \frac{1}{\Phi(r)}$. Then, $g(s)$ is continuous, non-increasing and $\int_0^r sg(s)ds = \int_0^r \frac{s}{\psi(s)} - \frac{s}{\Phi(r)}ds = 0$. Also, for $s_1, s_2 > 0$ with $s_1 + s_2 := s < 2r$,

$$
\begin{aligned}
h(s) &= \int_{|\xi|=1} \int_{B(3r-s)} \frac{\big(f(x+s\xi) - f(x)\big)^2}{s} dx\sigma(d\xi) \\
&\le \int_{|\xi|=1} \int_{B(3r-s)} \frac{\big(f(x+s\xi) - f(x+s_2\xi)\big)^2}{s_1} + \frac{\big(f(x+s_2\xi) - f(x)\big)^2}{s_2} dx\sigma(d\xi) \\
&\le \int_{|\xi|=1} \int_{B(x_0+s_2\xi, 3r-s)} \frac{\big(f(x+s_1\xi) - f(x)\big)^2}{s_1} dx\sigma(d\xi) + h(s_2) \le h(s_1) + h(s_2),
\end{aligned}
$$

where the first inequality follows from $\frac{(b_1+b_2)^2}{s} \le \frac{b_1^2}{s_1} + \frac{b_2^2}{s_2}$. Thus, the functions $g$ and $h$ satisfy the assertions of Lemma 2.1.10. Therefore, by Lemma 2.1.10 we have $\int_0^r h(s)\frac{1}{\Phi(r)}ds \le \int_0^r h(s)\frac{1}{\psi(s)}ds$, which implies (2.1.18). $\square$

Using Proposition 2.1.11 and

**Corollary 2.1.12.** *There exists a constant $C > 0$ such that for any bounded $f \in \mathcal{F}$ and $r > 0$,*

$$
\frac{1}{r^d} \int_{\mathbb{R}^d} \int_{B(x,r)} (f(x) - f(y))^2 dydx \le C\Phi(r)\mathcal{E}(f,f). \tag{2.1.19}
$$

**Proof.** Fix $r > 0$ and let $\{x_n\}_{n\in\mathbb{N}}$ be a countable set in $\mathbb{R}^d$ satisfying $\bigcup_{n=1}^{\infty} B(x_n, r) = \mathbb{R}^d$ and $\sup_{y\in\mathbb{R}^d} |\{n : y \in B(x_n, 6r)\}| \le M$. Then by , the left hand side of (2.1.19) is bounded above by

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{r^d} \int_{B(x_n,2r)\times B(x_n,2r)} (f(x) - f(y))^2 dydx \\
&\le c_1 \sum_{n=1}^{\infty} \Phi(r) \int_{B(x_n,6r)\times B(x_n,6r)} (f(x) - f(y))^2 J(x,y) dydx \\
&\le c_1 M\Phi(r) \int_{\mathbb{R}^d} \int_{B(x,12r)} (f(x) - f(y))^2 J(x,y) dydx \le c_1 M\Phi(r)\mathcal{E}(f,f).
\end{aligned}
$$

This finishes the proof. $\square$

Using (2.1.12) and (2.1.19) and following [31, Section 3], we obtain Nash's inequality for $(\mathcal{E}, \mathcal{F})$ and the near-diagonal upper bound of $p(t, x, y)$ in terms of $\Phi$.

**Theorem 2.1.13.** *There is a positive constant $c > 0$ such that for every $u \in \mathcal{F}$ with $\|u\|_1 = 1$, we have $\vartheta(\|u\|_2^2) \le c\,\mathcal{E}(u, u)$ where $\vartheta(r) := r/\Phi(r^{-1/d})$.*

Recall that $X$ is the Hunt process corresponding to our Dirichlet form $(\mathcal{E}, \mathcal{F})$ defined in (2.1.4) with jumping kernel $J$ satisfying (2.1.2). By using our Nash's inequality Theorem (2.1.13) and [9, Theorem 3.1], $X$ has a density function $p(t, x, y)$ with respect to Lebesgue measure, which is quasi-continuous, and that the upper bound estimate holds quasi-everywhere.

**Lemma 2.1.14.** *There is a properly exceptional set $\mathcal{N}$ of $X$, a positive symmetric kernel $p(t, x, y)$ defined on $(0, \infty) \times (\mathbb{R}^d \setminus \mathcal{N}) \times (\mathbb{R}^d \setminus \mathcal{N})$, and positive constants $C$ depending on $\bar{C}$ in (2.1.2) and $\beta_1, C_L$, such that $\mathbb{E}^x[f(X_t)] = \int_{\mathbb{R}^d} p(t, x, y) f(y) dy$, and $p(t, x, y) \le C\Phi^{-1}(t)^{-d}$ for every $x, y \in \mathbb{R}^d \setminus \mathcal{N}$ and for every $t > 0$. Moreover, for every $t > 0$, and $y \in \mathbb{R}^d \setminus \mathcal{N}$, $x \mapsto p(t, x, y)$ is quasi-continuous on $\mathbb{R}^d$.*

Moreover, following the proof of [31, Theorem 3.2] with the above Nash-type inequality we obtain

**Theorem 2.1.15.** *There exists a constant $C > 0$ such that for any $t > 0$ and $x, y \in \mathbb{R}^d \setminus \mathcal{N}$,*

$$p(t, x, y) \le C\Big(\frac{1}{\Phi^{-1}(t)^d} \wedge \frac{t}{\Phi(|x - y|)|x - y|^d}\Big). \qquad (2.1.20)$$

The upper bound in Theorem 2.1.15 may not be sharp. However, using the main results in [32, 33], there are several important consequences which are induced from (2.1.20).

**Lemma 2.1.16.** *There exists a constant $C > 0$ such that $\mathbb{P}^x(\tau_{B(x,r)} \le t) \le Ct/\Phi(r)$ for any $r > 0$ and $x \in \mathbb{R}^d \setminus \mathcal{N}$.*

Again, by [33, Theorem 1.19] we have the interior near-diagonal lower bound of $p^B(t, x, y)$ (and parabolic Harnack inequality).

**Lemma 2.1.17.** *There exist $\varepsilon \in (0,1)$ and $c_1 > 0$ such that for any $x_0 \in \mathbb{R}^d$, $r > 0$, $0 < t \le \Phi(\varepsilon r)$ and $B = B(x_0, r)$, $p^B(t,x,y) \ge c_1 \Phi^{-1}(t)^{-d}$ for all $x, y \in B(x_0, \varepsilon\Phi^{-1}(t))$.*

**Lemma 2.1.18.** *For any $r > 0$ and $x \in \mathbb{R}^d$, $\mathbb{E}^x[\tau_{B(x,r)}] \asymp \Phi(r)$.*

### 2.1.3 Off-diagonal estimates

Recall that for $\rho > 0$, $(\mathcal{E}^\rho, \mathcal{F})$ is $\rho$-truncated Dirichlet form of $(\mathcal{E}, \mathcal{F})$. Also, the Hunt process associated with $(\mathcal{E}^\rho, \mathcal{F})$ is denoted by $X^\rho$, and $p^\rho(t,x,y)$ is the transition density function of $X^\rho$.

For any open set $D \subset \mathbb{R}^d$, let $\{P_t^D\}$ and $\{Q_t^{\rho,D}\}$ be the semigroups of $(\mathcal{E}, \mathcal{F}_D)$ and $(\mathcal{E}^\rho, \mathcal{F}_D)$, respectively. We write $\{Q_t^{\rho,\mathbb{R}^d}\}$ as $\{Q_t^\rho\}$ for simplicity. We also use $\tau_D^\rho$ to denote the first exit time of the process $\{X_t^\rho\}$ in $D$.

**Lemma 2.1.19** ([32, Lemma 5.2])**.** *There exist constants $c, C_1, C_2 > 0$ such that for any $t, \rho > 0$ and $x, y \in \mathbb{R}^d$,*

$$p^\rho(t,x,y) \le c\Phi^{-1}(t)^{-d} \exp\Big(C_1 \frac{t}{\Phi(\rho)} - C_2 \frac{|x-y|}{\rho}\Big).$$

**Proof.** Note that by Lemma 2.1.5, $\Phi$ satisfies $U(\beta_2 \wedge 2, C_U)$ and $L(\beta_1, C_L)$. By Theorem 2.1.14, (2.1.11), and Lemma 2.1.18, the assumptions of [32, Lemma 5.2] are satisfied. Thus, the lemma follows. □

Also, from (2.1.2) we obtain the relation between $p^{(\rho)}(t,x,y)$ and $p(t,x,y)$

**Lemma 2.1.20.** *([11, Lemma 3.1] There exists a constant $c > 0$ such that for any $t, \rho > 0$ and $x, y \in \mathbb{R}^d$,*

$$p(t,x,y) \le p^{(\rho)}(t,x,y) + \frac{ct}{|x-y|^d \psi(|x-y|)}.$$

The following lemma is a key to obtain upper bound of transition density function and will be used in several times.

**Lemma 2.1.21.** *Let $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function satisfying that $t \mapsto f(r,t)$ is non-increasing for all $r > 0$ and that $r \mapsto f(r,t)$ is non-decreasing for all $t > 0$. Fix $T \in (0,\infty]$. Suppose that the following hold: (i) For each $b > 0$, $\sup_{t \le T} f(b\Phi^{-1}(t),t) < \infty$ (resp., $\sup_{t \ge T} f(b\Phi^{-1}(t),t) < \infty$); (ii) there exist $\eta \in (0,\beta_1]$, $a_1 > 0$ and $c_1 > 0$ such that*

$$\mathbb{P}^x(|X_t - x| > r) \le c_1(\psi^{-1}(t)/r)^\eta + c_1 \exp\big(-a_1 f(r,t)\big) \tag{2.1.21}$$

*for all $t \in (0,T)$ (resp. $t \in [T,\infty)$) and $r > 0$, $x \in \mathbb{R}^d$.*

*Then, there exist constants $k, c > 0$ such that*

$$p(t,x,y) \le \frac{c\,t}{|x-y|^d \psi(|x-y|)} + c\,\Phi^{-1}(t)^{-d} \exp\Big(-a_1 k f(|x-y|/(16k),t)\Big)$$

*for all $t \in (0,T)$ (resp. $t \in [T,\infty)$) and $x,y \in \mathbb{R}^d$.*

**Proof.** Since the proofs for the case $t \in (0,T)$ and the case $t \in [T,\infty)$ are similar, we only prove for $t \in (0,T)$. For $x_0 \in \mathbb{R}^d$, let $B(r) = B(x_0,r)$. By the strong Markov property, (2.1.21), and the fact that $t \mapsto f(r,t)$ is non-increasing, we have that for $x \in B(r/4)$ and $t \in (0,T/2)$,

$$\begin{aligned}
&\mathbb{P}^x(\tau_{B(r)} \le t) \le \mathbb{P}^x(X_{2t} \in B(r/2)^c) + \mathbb{P}^x(\tau_{B(r)} \le t, X_{2t} \in B(r/2)) \\
&\le \mathbb{P}^x(X_{2t} \in B(x,r/4)^c) + \sup_{z \in B(r)^c, s \le t} \mathbb{P}^z(X_{2t-s} \in B(z,r/4)^c) \\
&\le c_1(4\psi^{-1}(2t)/4)^\eta + c_1 \exp\big(-a_1 f(r/4,2t)\big).
\end{aligned}$$

From this and Lemma 1.1.4, we have that for $x \in B(r/4)$ and $t \in (0,T/2)$,

$$1 - P_t^B \mathbf{1}_B(x) = \mathbb{P}^x(\tau_B \le t) \le c_2\Big(\frac{\psi^{-1}(t)}{r}\Big)^\eta + c_1 \exp\Big(-a_1 f(r/4,2t)\Big). \tag{2.1.22}$$

By [47, Proposition 4.6] and Lemma 2.1.4, letting $\rho = r$ we have

$$\Big|P_t^{B(r)}\mathbf{1}_{B(r)}(x) - Q_t^{r,B(r)}\mathbf{1}_{B(r)}(x)\Big| \le 2t \operatorname*{ess\,sup}_{z \in \mathbb{R}^d} \int_{B(z,r)^c} J(z,y)\,dy \le \frac{c_3 t}{\psi(r)}.$$

Combining this with (2.1.22), we see that for all $x \in B(r/4)$ and $t \in (0, T/2)$,

$$\begin{aligned} \mathbb{P}^x(\tau^r_{B(r)} \le t) &= 1 - Q_t^{r,B(r)} \mathbf{1}_{B(r)}(x) \le 1 - P_t^{B(r)} \mathbf{1}_{B(r)}(x) + \frac{c_3 t}{\psi(r)} \\ &\le c_2(\psi^{-1}(t)/r)^\eta + c_1 \exp\big( - a_1 f(r/4, 2t)\big) + c_3(t/\psi(r)) =: \phi_1(r,t). \end{aligned} \tag{2.1.23}$$

Applying [32, Lemma 7.11] with $r = \rho$ to (2.1.23), we see that for $t \in (0, T/2)$,

$$\int_{B(x,2kr)^c} p^r(t,x,y)dy = Q_t^r \mathbf{1}_{B(x,2kr)^c}(x) \le \phi_1(r,t)^k. \tag{2.1.24}$$

Let $k = \lceil (\beta_2 + d)/\eta \rceil$. For $t \in (0,T)$ and $x, y \in \mathbb{R}^d$ satisfying $4k\Phi^{-1}(t) \ge |x - y|$, by using that $r \mapsto f(r,t)$ is non-decreasing and the assumption $(i)$, we have $f(|x-y|/(16k), t) \le f(\Phi^{-1}(t)/4, t) \le M < \infty$. Thus, by Theorem 2.1.14,

$$p(t,x,y) \le c_4 e^{a_1 k M} \Phi^{-1}(t)^{-d} \exp\Big( - a_1 k f(|x-y|/(16k), t)\Big). \tag{2.1.25}$$

For the remainder of the proof, assume $t \in (0,T)$ and $4k\Phi^{-1}(t) < |x - y|$, and let $r = |x - y|$ and $\rho = r/(4k)$. By (2.1.24) and Lemmas 1.1.4, 2.1.5 and 2.1.19, we have

$$\begin{aligned} p^\rho(t,x,y) &= \int_{\mathbb{R}^d} p^\rho(t/2,x,z) p^\rho(t/2,z,y) dz \\ &\le \Big( \int_{B(x,r/2)^c} + \int_{B(y,r/2)^c} \Big) p^\rho(t/2,x,z) p^\rho(t/2,y,z) dz \\ &\le \Big( \sup_{z \in \mathbb{R}^d} p^\rho(t/2,z,y) \Big) \int_{B(x,2k\rho)^c} p^\rho(t/2,x,z) dz \\ &\quad + \Big( \sup_{z \in \mathbb{R}^d} p^\rho(t/2,x,z) \Big) \int_{B(y,2k\rho)^c} p^\rho(t/2,y,z) dz \\ &\le c_5 \Phi^{-1}(t)^{-d} \phi_1(\rho, t/2)^k. \end{aligned} \tag{2.1.26}$$

Note that $k\beta_1 \ge k\eta \ge \beta_2 + d$, and $\rho \ge \Phi^{-1}(t) > \psi^{-1}(t)$. Thus, by $L(\beta_1, C_L)$

on $\psi$ and using Lemmas 1.1.4 and 2.1.5,

$$\begin{aligned}
&\Phi^{-1}(t)^{-d}\Big(\big(\psi^{-1}(t)/\rho\big)^{\eta k}+\big(t/\psi(\rho)\big)^{k}\Big)\le \frac{c_6}{r^d}\frac{\psi^{-1}(t)^d}{\Phi^{-1}(t)^d}\big(\psi^{-1}(t)/r\big)^{\beta_2}\\
&\le \frac{c_6}{r^d}\left[\psi^{-1}(t)/\psi^{-1}(\psi(r))\right]^{\beta_2}\le \frac{c_7 t}{r^d\psi(r)}.
\end{aligned}$$

Applying this to (2.1.26) we have

$$\begin{aligned}
p^\rho(t,x,y)&\le c_8\Phi^{-1}(t)^{-d}\Big(\big(\frac{\psi^{-1}(t)}{\rho}\big)^{\eta k}+\Big(-a_1kf(\rho/4,t)\Big)+\big(\frac{t}{\psi(\rho)}\big)^{k}\Big)\\
&\le \frac{c_9 t}{r^d\psi(r)}+c_8\Phi^{-1}(t)^{-d}\exp\Big(-a_1kf(r/(16k),t)\Big).
\end{aligned}$$

Thus, by Lemma 2.1.20 and $U(\beta_2, C_U)$ on $\psi$, we have

$$\begin{aligned}
p(t,x,y)&\le p^\rho(t,x,y)+\frac{c_{10}t}{\rho^d\psi(\rho)} &&(2.1.27)\\
&\le \frac{c_{11}t}{|x-y|^d\psi(|x-y|)}+c_{11}\Phi^{-1}(t)^{-d}\exp\Big(-a_1kf(r/(16k),t)\Big).
\end{aligned}$$

Now the lemma follows immediately from (2.1.25) and (2.1.27). □

The following inequality will be used several times in the proofs of this section: For any $c_0>0$ and $\alpha\in(0,1)$, there exists $c_1=c_1(c_0,\alpha)>0$ such that $2n\le \frac{c_0}{2d}2^{n(1-\alpha)}+c_1$ holds for every $n\ge 0$. Thus, for any $n\ge 0$ and $\kappa\ge 1$,

$$\begin{aligned}
&2^{nd}\exp(-c_02^{n(1-\alpha)}\kappa)\le 2^{-nd}\exp(2nd-c_02^{n(1-\alpha)}\kappa)\\
&\le e^{c_1d}2^{-nd}\exp\big(\frac{c_0}{2}2^{n(1-\alpha)}-c_02^{n(1-\alpha)}\kappa\big)\le e^{c_1d}2^{-nd}\exp(-\frac{c_0}{2}\kappa). \qquad (2.1.28)
\end{aligned}$$

Recall that, without loss of generality, whenever $\Phi$ satisfies the weak lower scaling property at infinity with index $\delta>1$, we have assumed that $\Phi$ satisfies $L^1(\delta,\widetilde{C}_L)$ instead of $L^a(\delta,\widetilde{C}_L)$.

We are now ready to prove the sharp upper bound of $p(t,x,y)$, which is the most delicate part of [3]. Since the proof of Theorem 2.1.1 is easier, we only provide the proof of the upper bound of Theorem 2.1.2 in this thesis.

**Theorem 2.1.22.** *(1) Assume that $\Phi$ satisfies $L_a(\delta, \widetilde{C}_L)$ with $\delta > 1$. Then for any $T > 0$, there exist constants $a_U > 0$ and $c > 0$ such that for every $x, y \in \mathbb{R}^d$ and $t < T$,*

$$p(t,x,y) \le \frac{c\,t}{|x-y|^d \psi(|x-y|)} + \frac{c}{\Phi^{-1}(t)^d} \exp\Big( - \frac{a_U |x-y|}{\mathscr{K}^{-1}(t/|x-y|)} \Big). \tag{2.1.29}$$

*Moreover, if $\Phi$ satisfies $L(\delta, \widetilde{C}_L)$, then* (2.1.29) *holds for all $t < \infty$.*
*(2) Assume that $\Phi$ satisfies $L^1(\delta, \widetilde{C}_L)$ with $\delta > 1$. Then for any $T > 0$, there exist constants $a'_U > 0$ and $c' > 0$ such that for every $x, y \in \mathbb{R}^d$ and $t \ge T$,*

$$p(t,x,y) \le \frac{c'\,t}{|x-y|^d \psi(|x-y|)} + \frac{c'}{\Phi^{-1}(t)^d} \exp\Big( - \frac{a'_U |x-y|}{\mathscr{K}_\infty^{-1}(t/|x-y|)} \Big).$$

**Proof.** Take $\theta = \frac{\beta_1(\delta-1)}{2\delta d + \delta\beta_1 + \beta_1}$ and $\widetilde{C}_0 = \big(\frac{2C_1}{C_2 \widetilde{C}_L^2}\big)^{1/(\delta-1)}$, where $C_1$ and $C_2$ are constants in Lemma 2.1.19. Without loss of generality, we may and do assume that $\widetilde{C}_0 \ge 1$. Note that $\theta$ satisfies $\frac{\delta(d+\beta_1)}{\delta-1}\frac{\theta}{1+\theta} = \frac{\beta_1}{2}$ and $\theta < \delta - 1$. Let $\alpha \in (d/(d+\beta_1), 1)$.

(1) Again we will show that there exist $a_1 > 0$ and $c_1 > 0$ such that for any $t \le T$ and $r > 0$,

$$\int_{B(x,r)^c} p(t,x,y)\,dy \le c_1 (\psi^{-1}(t)/r)^{\beta_1/2} + c_1 \exp\Big( - \frac{a_1 r}{\mathscr{K}^{-1}(t/r)} \Big). \tag{2.1.30}$$

When $r \le \widetilde{C}_0 \Phi^{-1}(t)$ using (2.1.15) we have for $t \le T$

$$\int_{B(x,r)^c} p(t,x,y) dy \le 1 \le e^{c_2} \exp\Big( - \frac{r}{\mathscr{K}^{-1}(t/r)} \Big). \tag{2.1.31}$$

The proof of case $r > \widetilde{C}_0 \frac{\Phi^{-1}(t)^{1+\theta}}{\psi^{-1}(t)^\theta}$ is exactly same as the corresponding part in the proof of (2.1.5) in Theorem 2.1.1.

Now consider the case $\widetilde{C}_0 \Phi^{-1}(t) < r \le \widetilde{C}_0 \Phi^{-1}(t)^{1+\theta}/\psi^{-1}(t)^\theta$. In this case, there exists $\theta_0 \in (0, \theta]$ such that $r = \widetilde{C}_0 \Phi^{-1}(t)^{1+\theta_0}/\psi^{-1}(t)^{\theta_0}$. Define $\rho = \mathscr{K}^{-1}(t/r)$ and $\rho_n = \widetilde{C}_0 2^{n\alpha}\rho$ for integer $n \ge 0$. Note that for $t \le T$ and

$\widetilde{C}_0\Phi^{-1}(t) < r$, we have $t \leq T \wedge \Phi(r)$. Thus, by (2.1.16)

$$\rho \leq \rho_0 = \widetilde{C}_0\rho \leq \widetilde{C}_0\Phi^{-1}(t)^2/r \leq \widetilde{C}_0\Phi^{-1}(T)\Phi^{-1}(t)/r \leq \Phi^{-1}(T). \tag{2.1.32}$$

By Lemma 1.1.3, we may assume that $\Phi^{-1}(T) < a$. Thus, by (2.1.32), Lemma 2.1.6, the condition $L_a(\delta, \widetilde{C}_L)$ on $\Phi$, and the definition of $\widetilde{C}_0$, we have

$$\begin{aligned}
&C_1\frac{t}{\Phi(\rho_n)} - C_2\frac{2^n r}{\rho_n} \leq C_1\frac{\Phi(\rho)}{\Phi(\rho_0)}\frac{t}{\Phi(\rho)} - \frac{C_2}{\widetilde{C}_0}\frac{2^{n(1-\alpha)}r}{\rho} \\
&\leq \frac{C_2 r}{\widetilde{C}_0\rho}\Big(\frac{\widetilde{C}_0C_1}{C_2\widetilde{C}_L}\frac{\Phi(\rho)}{\Phi(\rho_0)} - 2^{n(1-\alpha)}\Big) \leq \frac{C_2 r}{\widetilde{C}_0\rho}\Big(\frac{\widetilde{C}_0^{1-\delta}C_1}{C_2\widetilde{C}_L^2} - 2^{n(1-\alpha)}\Big) \\
&= \frac{C_2 r}{\widetilde{C}_0\rho}\Big(\frac{1}{2} - 2^{n(1-\alpha)}\Big) \leq -c_3 2^{n(1-\alpha)}\frac{r}{\rho}.
\end{aligned} \tag{2.1.33}$$

Combining (2.1.33) and 2.1.19, we have that

$$\begin{aligned}
&\int_{B(x,r)^c} p(t,x,y)dy \leq \sum_{n=0}^{\infty}\int_{B(x,2^{n+1}r)\setminus B(x,2^n r)} p(t,x,y)dy \\
&\leq c_4\sum_{n=0}^{\infty}\Big(\frac{2^n r}{\Phi^{-1}(t)}\Big)^d \exp\Big(-c_5\frac{2^{n(1-\alpha)}r}{\rho}\Big) + c_4\sum_{n=0}^{\infty}\Big(\frac{2^n r}{\rho_n}\Big)^d\frac{t}{\psi(\rho_n)} := I_1 + I_2.
\end{aligned}$$

We first estimate $I_1$. Note that by (2.1.16), $r/\rho \geq (r/\Phi^{-1}(t))^2 \geq \widetilde{C}_0^2$. Using this, (2.1.16), and (2.1.28) we have

$$I_1 \leq c_6\sum_{n=0}^{\infty}(r/\rho)^{d/2}2^{nd}\exp(-c_5 2^{n(1-\alpha)}r/\rho) \leq c_7\exp\Big(-2^{-2}c_5 r/\rho\Big)$$

We next estimate $I_2$. By using (2.1.16), $r = \widetilde{C}_0\Phi^{-1}(t)^{1+\theta_0}/\psi^{-1}(t)^{\theta_0}$, $\psi^{-1}(t) \leq \Phi^{-1}(t)$, and $\theta_0 \leq \theta < \delta - 1$, we have

$$\frac{\Phi^{-1}(t)}{c_8\rho} \leq \Big(\frac{r}{\Phi^{-1}(t)}\Big)^{1/(\delta-1)} = \widetilde{C}_0^{1/(\delta-1)}\Big(\frac{\Phi^{-1}(t)}{\psi^{-1}(t)}\Big)^{\theta_0/(\delta-1)} \leq \widetilde{C}_0^{1/(\delta-1)}\frac{\Phi^{-1}(t)}{\psi^{-1}(t)}.$$

Thus, we have $\rho_n > \rho \geq C_3^{-1}\widetilde{C}_0^{-1/(\delta-1)}\psi^{-1}(t)$. Using this, $L(\beta_1, C_L)$ condition on $\psi$, and (2.1.16),

$$I_2 \le c_9\Big(\frac{r}{\rho}\Big)^{d+\beta_1}\Big(\frac{\psi^{-1}(t)}{r}\Big)^{\beta_1} \le c_{10}\Big(\frac{\Phi^{-1}(t)}{r}\Big)^{-\frac{\delta}{\delta-1}(d+\beta_1)}\Big(\frac{\psi^{-1}(t)}{r}\Big)^{\beta_1}.$$

= Since $\psi^{-1}(t) \le \Phi^{-1}(t) < r = \widetilde{C}_0\Phi^{-1}(t)^{1+\theta_0}/\psi^{-1}(t)^{\theta_0}$ and $\theta_0 \le \theta$, using $\frac{\delta(d+\beta_1)}{\delta-1}\frac{\theta}{1+\theta} = \frac{\beta_1}{2}$, we have

$$\Big(\frac{\Phi^{-1}(t)}{r}\Big)^{-\frac{\delta}{\delta-1}(d+\beta_1)} \le \widetilde{C}_0^{\delta(d+\beta_1)/(\delta-1)}\Big(\frac{\psi^{-1}(t)}{r}\Big)^{-\beta_1/2},$$

which implies $I_2 \le c_{11}(\psi^{-1}(t)/r)^{\beta_1/2}$. Using estimates of $I_1$ and $I_2$ and combining (2.1.31) and Lemma 2.1.21 we obtain (2.1.30).

Let $f(r,t) := \frac{r}{\mathscr{K}^{-1}(t/r)}$. Then, by (2.1.32) and Lemma 2.1.9, we see that $f(r,t)$ satisfies the condition in Lemma 2.1.21. Thus, by Lemma 2.1.21, we obtain

$$p(t,x,y) \le \frac{c_{12}\, t}{|x-y|^d\psi(|x-y|)} + c_{12}\,\Phi^{-1}(t)^{-d}\exp\Big(-\frac{c_{13}|x-y|}{\mathscr{K}^{-1}(c_{14}t/|x-y|)}\Big).$$

Since $|x-y| \ge \widetilde{C}_0\Phi^{-1}(t) \ge c_{15}t^{1/\delta} \ge c_{15}T^{-1+1/\delta}t$, we can apply (2.1.13) and get $\mathscr{K}^{-1}(c_{18}t/|x-y|) \le c_{21}\mathscr{K}^{-1}(t/|x-y|)$. We have proved the first claim of the theorem.

(2) The proof of the second claim is similar to the proof of the first claim. We skip the proof.

Combining Theorems 2.1.14 and 2.1.22 and Lemma 2.1.9, we get the desired upper bounds of $p(t,x,y)$. □

We now prove the lower bound in (2.1.6).

**Proposition 2.1.23.** *There exist constants $\delta_1 \in (0,1/2)$ and $C_3 > 0$ such that*

$$p(t,x,y) \ge C_3\frac{\mathbf{1}_{\{|x-y|\le\delta_1\Phi^{-1}(t)\}}}{\Phi^{-1}(t)^d} + \frac{C_3 t}{|x-y|^d\psi(|x-y|)}\mathbf{1}_{\{|x-y|\ge\delta_1\Phi^{-1}(t)\}} \tag{2.1.34}$$

**Proof.** Let $\delta_1 = \varepsilon/2 < 1/2$ where $\varepsilon$ is the constant in Lemma 2.1.17. Then by Lemma 2.1.17, for all $|x-y| \le \delta_1 \Phi^{-1}(t)$,

$$p(t,x,y) \ge p^{B(x,\Phi^{-1}(t)/\varepsilon)}(t,x,y) \ge c_0 \Phi^{-1}(t)^{-d}. \tag{2.1.35}$$

Thus, we have (2.1.34) when $|x-y| \le \delta_1 \Phi^{-1}(t)$.

By Lemma 2.1.16 we have $\mathbb{P}^x(\tau_{B(x,r)} \le t) \le c_1 t/\Phi(r)$ for any $r > 0$ and $x \in \mathbb{R}^d$. Let $\delta_2 := (C_L/2)^{1/\beta_1}\delta_1 \in (0, \delta_1)$ so that $\delta_1 \Phi^{-1}((1-b)t) \ge \delta_2 \Phi^{-1}(t)$ holds for all $b \in (0, 1/2]$. Then choose $\lambda \le c_1^{-1} C_U^{-1}(2\delta_2/3)^{\beta_2}/2 < 1/2$ small enough so that $c_1 \lambda t/\Phi(2\delta_2\Phi^{-1}(t)/3) \le \lambda c_1 C_U (2\delta_2/3)^{-\beta_2} \le 1/2$. Thus we have $\lambda \in (0, 1/2)$ and $\delta_2 \in (0, \delta_1)$ (independent of $t$) such that

$$\delta_1 \Phi^{-1}((1-\lambda)t) \ge \delta_2 \Phi^{-1}(t), \quad \text{for all } t > 0 \tag{2.1.36}$$

and

$$\mathbb{P}^x(\tau_{B(x,2\delta_2\Phi^{-1}(t)/3)} \le \lambda t) \le 1/2, \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d. \tag{2.1.37}$$

For the remainder of the proof we assume that $|x-y| \ge \delta_1 \Phi^{-1}(t)$. Since, using (2.1.35) and (2.1.36),

$$\begin{aligned}
p(t,x,y) &\ge \int_{B(y,\delta_1\Phi^{-1}((1-\lambda)t))} p(\lambda t, x, z) p((1-\lambda)t, z, y) dz \\
&\ge \inf_{z \in B(y,\delta_1\Phi^{-1}((1-\lambda)t))} p((1-\lambda)t, z, y) \int_{B(y,\delta_1\Phi^{-1}((1-\lambda)t))} p(\lambda t, x, z) dz \\
&\ge c_0 \Phi^{-1}(t)^{-d} \mathbb{P}^x(X_{\lambda t} \in B(y, \delta_2 \Phi^{-1}(t))),
\end{aligned}$$

it suffices to prove

$$\mathbb{P}^x(X_{\lambda t} \in B(y, \delta_2\Phi^{-1}(t))) \ge c_2 \frac{t\Phi^{-1}(t)^d}{|x-y|^d \psi(|x-y|)}. \tag{2.1.38}$$

Using the strong Markov property, Lévy system, the lower bound of $J(x,y)$, (2.1.2), and (2.1.37), the proof of (2.1.38) is standard. (See [32, Proposition 5.4(ii)].) We omit the details. $\square$

By using the properties of $\mathscr{K}$ and $\mathscr{K}_\infty$, we give the lower bound of

$p(t,x,y)$ under $L_a(\delta,\widetilde{C}_L)$ or $L^a(\delta,\widetilde{C}_L)$ on $\Phi$ with $\delta>1$. See [89, Lemmas 3.1–3.2] for similar bound for Lévy processes.

**Proposition 2.1.24.** *Suppose $\Phi$ satisfies $L_a(\delta,\widetilde{C}_L)$ with $\delta>1$ and for some $a>0$. For $T>0$ there exist $C>0$ and $a_L>0$ such that for any $t\le T$ and $x,y\in\mathbb{R}^d$,*

$$p(t,x,y)\ge C\Phi^{-1}(t)^{-d}\exp\left(-a_L\frac{|x-y|}{\mathscr{K}^{-1}(t/|x-y|)}\right). \qquad (2.1.39)$$

*Moreover, if $a=\infty$, then* (2.1.39) *holds for all $t<\infty$.*

**Proof.** Let $r=|x-y|$. By Proposition 2.1.23 and Lemma 1.1.3, without loss of generality, we assume that $\delta_1\Phi^{-1}(t)\le r$ and $a\ge\delta_1\Phi^{-1}(T)$ where $\delta_1$ is the constants in Proposition 2.1.23. Let $k=\lceil 3r\delta_1^{-1}/\mathscr{K}^{-1}(3^{-1}\delta_1 t/r)\rceil$. Note that by (2.1.16), $\mathscr{K}^{-1}(t/r)\le\Phi^{-1}(t)^2/r\le\delta_1\Phi^{-1}(t)\le\delta_1\Phi^{-1}(T)\le a$. Thus by (2.1.13) we have $\mathscr{K}^{-1}(t/r)\le\widetilde{C}_L^{-1}(3/\delta_1)\mathscr{K}^{-1}(3^{-1}\delta_1 t/r)$. Since $3^{-1}\delta_1 t/r\le 3^{-1}\delta_1\Phi(r/\delta_1)/r\le 3^{-1}\mathscr{K}(r/\delta_1)$, we see that $\mathscr{K}^{-1}(3^{-1}\delta_1 t/r)\le\frac{r}{\delta_1}$, hence

$$3\le k\le\frac{4r}{\delta_1\mathscr{K}^{-1}(3^{-1}\delta_1 t/r)}\le\frac{12\widetilde{C}_L^{-1}r}{\delta_1^2\mathscr{K}^{-1}(t/r)}. \qquad (2.1.40)$$

On the other hand, by Lemma 2.1.6 and our choice of $k$ we have

$$\Phi\left(\frac{3r}{\delta_1 k}\right)\frac{\delta_1 k}{r}\le 3\mathscr{K}\left(\frac{3r}{\delta_1 k}\right)\le\frac{\delta_1 t}{r}.$$

Thus, we obtain $\frac{r}{k}\le\frac{\delta_1}{3}\Phi^{-1}(t/k)$. Let $z_l=x+\frac{l}{k}(y-x)$, $l=0,1,\cdots,k-1$. For $\xi_l\in B(z_l,\frac{\delta_1}{3}\Phi^{-1}(\frac{t}{k}))$ and $\xi_{l-1}\in B(z_{l-1},\frac{\delta_1}{3}\Phi^{-1}(\frac{t}{k}))$, $|\xi_l-\xi_{l-1}|\le|\xi_l-z_l|+|z_l-z_{l-1}|+|z_{l-1}-\xi_{l-1}|\le\delta_1\Phi^{-1}(t/k)$. Thus by Proposition 2.1.23, $p(\frac{t}{k},\xi_{l-1},\xi_l)\ge C_3\Phi^{-1}(t/k)^{-d}$. Using the semigroup property and (2.1.40), we get

$$\begin{aligned} & p(t,x,y) \\ & \geq \int_{B(z_{k-1},\frac{\delta_1}{3}\Phi^{-1}(\frac{t}{k}))} \cdots \int_{B(z_1,\frac{\delta_1}{3}\Phi^{-1}(\frac{t}{k}))} p(\tfrac{t}{k},x,\xi_1)\cdots p(\tfrac{t}{k},\xi_{k-1},y)d\xi_1\cdots d\xi_{k-1} \\ & \geq C_5^k \Phi^{-1}(\tfrac{t}{k})^{-dk} \prod_{l=1}^{k-1} \Big| B(z_l,\tfrac{\delta_1}{3}\Phi^{-1}(\tfrac{t}{k}))\Big| = c_2 c_3^k \Phi^{-1}(\tfrac{t}{k})^{-dk} \Big(\frac{\delta_1}{3}\Phi^{-1}(\tfrac{t}{k})\Big)^{d(k-1)} \\ & \geq c_2 \Big(\frac{c_3\delta_1^d}{3^d}\Big)^k \Phi^{-1}(t)^{-d} \geq c_2\Phi^{-1}(t)^{-d} e^{-C_4 k} \geq c_2 \Phi^{-1}(t)^{-d} e^{-c_4\frac{r}{\mathscr{K}^{-1}(t/r)}}. \end{aligned} \tag{2.1.41}$$

This finishes the proof. Here we record that the constant $C_4$ in (2.1.41) depends only on $d$ and constants $\delta_1, C_3$ in (2.1.34). □

Next one is infinite version of lower heat kernel estimates. Since the proof is similar, we skip it.

**Proposition 2.1.25.** *Suppose $\Phi$ satisfies $L^1(\delta, \widetilde{C}_L)$ with $\delta > 1$. For any $T > 0$ and $\theta > 0$ satisfying $\frac{1}{\delta} + \theta(\frac{1}{\delta} - \frac{1}{\beta_2}) \leq 1$, there exist $c_1$, $c_2 > 0$ and $a'_L > 0$ such that for $(t,x,y) \in [T,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ satisfying $\delta_1\Phi^{-1}(t) < |x-y| \leq c_1\Phi^{-1}(t)^{1+\theta}/\psi^{-1}(t)^{\theta}$,*

$$p(t,x,y) \geq c_2\Phi^{-1}(t)^{-d} \exp\Big(-a'_L \frac{|x-y|}{\mathscr{K}_\infty^{-1}(t/|x-y|)}\Big),$$

*where $\delta_1$ is the constant in Proposition 2.1.23.*

**Proof of Theorem 2.1.2.** The both upper bounds of $p(t,x,y)$ in Theorem 2.1.2 follows from Theorems 2.1.14 and 2.1.22 and Lemma 2.1.9. The lower bound in (2.1.8) is a direct consequence of Propositions 2.1.23 and 2.1.24.

By Propositions 2.1.23 and 2.1.25, to complete the proof of Theorem 2.1.2, it is enough to show that for $t \geq T$ and $(c_1\Phi^{-1}(t)^{1+\theta}/\psi^{-1}(t)^{\theta}) \vee \delta_1\Phi^{-1}(t) < r$,

$$p(t,x,y) \geq C\Phi^{-1}(t)^{-d} \exp\Big(-a'_L \frac{r}{\mathscr{K}_\infty^{-1}(t/r)}\Big), \tag{2.1.42}$$

where $c_1$ and $\theta$ are the constants in Proposition 2.1.25.

By Proposition 2.1.23 we have that,

$$p(t,x,y) \ge \frac{c_2 t}{r^d \psi(r)} \ge c_3 \Phi^{-1}(t)^{-d} \exp\Big(-\frac{a_2 r^2}{\Phi^{-1}(t)^2}\Big).$$

By (2.1.14) and Lemma 1.1.4, $\mathscr{K}_\infty$ satisfies $L(\delta-1, \widetilde{C}_L^{-2/(\delta-1)})$. Using this property and (2.1.17) (note that $r \ge \delta_1 \Phi^{-1}(t)$ and $T \le t$), we get

$$\Big(\frac{\delta_1^{-1} r}{\Phi^{-1}(t)}\Big)^2 \le c_4 \frac{\delta_1^{-1} r}{\mathscr{K}_\infty^{-1}(\delta_1 t/r)} \le c_4 \frac{\widetilde{C}_L^{-2/(\delta-1)} \delta_1^{-\delta/(\delta-1)} r}{\mathscr{K}_\infty^{-1}(t/r)}.$$

Thus, (2.1.42) holds. □

**Proof of Corollary 2.1.3.** Since the upper bound is a direct consequence of Theorem 2.1.1, we show the lower bound in (2.1.9). Let $r = |x-y|$ and $\phi(s) := \Phi(s^{-1/2})^{-1}$. Since $\Phi$ satisfies $L_a(\delta, \widetilde{C}_L)$, $\phi$ satisfies $L^{1/a^2}(\delta/2, \widetilde{C}_L)$. Let $Z$ be a subordinate Brownian motion whose Laplace exponent is $\phi$ and $p^Z(t, |z-w|)$ be its transition density. Then, by [70, Proposition 3.5] and Theorem 2.1.2 , for any $T>0$, there exist positive constants $\widetilde{a}_L$, $a_U$, $c_1$ and $c_2$ such that for all $(t,x,y) \in (0,T) \times \mathbb{R}^d \times \mathbb{R}^d$,

$$c_1 \exp\left(-\frac{\widetilde{a}_L r^2}{\Phi^{-1}(t)^2}\right) \le \frac{p^Z(t,r)}{\Phi^{-1}(t)^{-d}} \le c_2 \exp\left(-\frac{a_U r}{\mathscr{K}^{-1}(t/r)}\right) + \frac{c_2 t \Phi^{-1}(t)^d}{r^d \psi(r)}.$$

Let $a_L \ge a_U$ be a constant in Theorem 2.1.2 and $A := a_L/a_U \ge 1$. Then, for all $t \in (0,T)$ $s>0$,

$$\begin{aligned} c_1 \exp\left(-\frac{\widetilde{a}_L A^2 s^2}{\Phi^{-1}(t)^2}\right) &\le c_2 \exp\left(-\frac{a_U A s}{\mathscr{K}^{-1}(t/As)}\right) + \frac{c_2 t \Phi^{-1}(t)^d}{(As)^d \psi(As)} \\ &\le c_2 \exp\left(-\frac{a_L s}{\mathscr{K}^{-1}(t/s)}\right) + \frac{c_3 t \Phi^{-1}(t)^d}{s^d \psi(s)}. \end{aligned}$$

Thus, by Theorem 2.1.2, we obtain the desired results. □

### 2.1.4 Examples

In this section, we will use the notation $f(\cdot) \simeq g(\cdot)$ at $\infty$ (resp. 0) if $\frac{f(t)}{g(t)} \to 1$ as $t \to \infty$ (resp. $t \to 0$). We denote $\mathcal{R}_0^\infty$ (resp. $\mathcal{R}_0^0$) by the class of slowly varying functions at $\infty$ (resp. 0). For $\ell \in \mathcal{R}_0^\infty$, we denote $\Pi_\ell^\infty$ (resp. $\Pi_\ell^0$) by the class of real-valued measurable function $f$ on $[c, \infty)$ (resp. $(0, c)$) such that for all $\lambda > 0$, $f(\lambda \cdot) - f(\cdot) \simeq \log \lambda \ell(\cdot)$ at $\infty$ (resp. 0) $\Pi_\ell^\infty$ (resp. $\Pi_\ell^0$) is called de Haan class at $\infty$ (resp. 0) determined by $\ell$.

For $\ell \in \mathcal{R}_0^\infty$ (resp. $\mathcal{R}_0^0$), we say $\ell_\#$ is de Bruijn conjugate of $\ell$ if both $\ell(t)\ell_\#(t\ell(t)) \simeq 1$ and $\ell_\#(t)\ell(t\ell_\#(t)) \simeq 1$ at $\infty$ (resp. 0). Note that $|f| \in \mathcal{R}_0^\infty$ if $f \in \Pi_\ell^\infty$ (see [16, Theorem 3.7.4]).

In the following corollary and examples $a_i = a_{i,L}$ or $a_i = a_{i,U}$ depending on whether we consider lower or upper bound.

**Corollary 2.1.26.** *Let $T \in (0, \infty)$ and $\psi$ be a non-decreasing function that satisfies $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$.*
*(1) Let $\ell \in \mathcal{R}_0^0$ be such that $\int_0^1 \frac{\ell(s)}{s} ds < \infty$ and $f(s) := \int_0^s \frac{\ell(t)}{t} dt \in \Pi_\ell^0$ satisfies $f(sf^\gamma(s)) \simeq f(s)$ at 0 for $\gamma = 1/2, 1$. Suppose that $\psi(s) \asymp \frac{s^2}{\ell(s)}$ for $s < 1$. Then for $t < T$,*

$$p(t,x,y) \asymp \frac{1}{(tf(t^{1/2}))^{d/2}} \wedge \left( \frac{t}{|x-y|^d \psi(|x-y|)} + \frac{1}{(tf(t^{1/2}))^{d/2}} e^{-\frac{a_1 |x-y|^2}{tf(t/|x-y|)}} \right).$$

*Furthermore, if $f(s^2) \asymp f(s)$ for $s < 1$, then for $t < T$,*

$$p(t,x,y) \asymp \frac{1}{(tf(t))^{d/2}} \wedge \left( \frac{t}{|x-y|^d \psi(|x-y|)} + \frac{1}{(tf(t))^{d/2}} e^{-a_2 \frac{|x-y|^2}{tf(t)}} \right).$$

*(2) Assume that $\ell \in \mathcal{R}_0^\infty$ satisfies $\int_1^\infty \frac{\ell(t)}{t} dt = \infty$.*

*Suppose that $\psi(s) \asymp \frac{s^2}{\ell(s)}$ for $s > 1$ and $f \in \Pi_\ell^\infty$ satisfies $f(sf^\gamma(s)) \simeq f(s)$ at $\infty$ for $\gamma = 1/2, 1$. Then for $t > T$,*

$$p(t,x,y) \asymp \frac{1}{(tf(t^{1/2}))^{d/2}} \wedge \left( \frac{t\ell(|x-y|)}{|x-y|^{d+2}} + \frac{1}{(tf(t^{1/2}))^{d/2}} e^{-\frac{a_3 |x-y|^2}{tf(t/|x-y|)}} \right).$$

*Furthermore, if $f(s^2) \asymp f(s)$ for $s > 1$, then for $t > T$,*

$$p(t,x,y) \asymp \frac{1}{(tf(t))^{d/2}} \wedge \left( \frac{t\ell(|x-y|)}{|x-y|^{d+2}} + \frac{1}{(tf(t))^{d/2}} e^{-a_4 \frac{|x-y|^2}{tf(t)}} \right).$$

**Proof.** Let $r = |x-y|$ and $\delta_1 > 0$ be the constant in Proposition 2.1.23.

(1) By [16, Corollary 2.3.4], $(f^\gamma)_\# \simeq 1/f^\gamma$ at 0. Thus, using [16, Theorem 3.6.8], we have for $s < T$,

$$\Phi(s) \asymp \frac{s^2}{f(s)}, \quad \Phi^{-1}(s) \asymp s^{1/2} f^{1/2}(s^{1/2}), \quad \mathscr{K}_\infty^{-1}(s) \asymp sf(s).$$

Therefore, by Theorem 2.1.2(1) and Theorem 2.1.1, we obtain the first claim and the upper bound in the second claim.

For the lower bound in the second claim, choose small $\theta > 0$ such that $\frac{1}{2} + \theta(\frac{1}{2} - \frac{1}{\beta_1}) =: \varepsilon_1 < 1$. Note that $f(s) \asymp f(s^2)$ for $s < 1$ implies $f(s^b) \asymp f(s)$ for all $b > 0$ since $f$ is non-decreasing. Since the last term in the heat kernel estimates dominate other terms only in the case $\delta_1 \Phi^{-1}(t) < r \le \delta_1 \frac{\Phi^{-1}(t)^{1+\theta}}{\psi^{-1}(t)^\theta}$, it suffices to show $f(t/r) \ge cf(t)$ for this case. Using (2.1.12) and $L(\beta_1, C_L)$ for $\psi$ we have $\Phi^{-1}(t)/\psi^{-1}(t) \le c_1 t^{\frac{1}{2} - \frac{1}{\beta_1}}$ for $t \le T$. Thus we have $f(t/r) \ge f(c_2 t^{1-\varepsilon_1}) \asymp f(t)$ for every $t \le T$ and $\delta_1 \Phi^{-1}(t) < r \le \delta_1 \frac{\Phi^{-1}(t)^{1+\theta}}{\psi^{-1}(t)^\theta}$.

(2) Similarly, $(f^\gamma)_\# \simeq 1/f^\gamma$ at $\infty$ by [16, Corollary 2.3.4]. Thus, using [16, (1.5.8), Theorem 3.7.3], we have that for $s > T$,

$$\Phi(s) \asymp s^2/f(s), \quad \Phi^{-1}(s) \asymp s^{1/2} f^{1/2}(s^{1/2}), \quad \mathscr{K}_\infty^{-1}(s) \asymp sf(s).$$

Note that $\psi(r) \asymp \frac{r^2}{\ell(r)}$ when $r > \delta_1 \Phi^{-1}(t)$ since $r > \delta_1 \Phi^{-1}(t) \ge \delta_1 \Phi^{-1}(T)$. Since the second term in the heat kernel estimates dominate only in the case $r > \delta_1 \Phi^{-1}(t)$, the first claim and upper bound in the second one follow from Theorem 2.1.2(2) and Theorem 2.1.1.

Now choose small $\theta' > 0$ such that $\frac{1}{\delta} + \theta'(\frac{1}{\delta} - \frac{1}{\beta_2}) =: \varepsilon_2 < 1$. Without loss of generality we can assume that $f$ is non-decreasing since $f(s) \asymp \int_1^s \frac{\ell(t)}{t} dt$. Now $f(s) \asymp f(s^2)$ for $s > 1$ implies $f(s^b) \asymp f(s)$ for all $b > 0$. Similarly, using $L^a(\delta, \widetilde{C}_L)$ for $\Phi$ and $U(\beta_2, C_U)$ for $\psi$ we have $\frac{\Phi^{-1}(t)}{\psi^{-1}(t)} \le c_3 t^{\frac{1}{\delta} - \frac{1}{\beta_2}}$ so $f(t/r) \ge$

$f(c_4 t^{1-\varepsilon_2}) \asymp f(t)$ for every $t \geq T$ and $r \leq \delta_1 \frac{\Phi^{-1}(t)^{1+\theta'}}{\psi^{-1}(t)^{\theta'}}$. This finishes the proof. $\square$

## 2.2 Symmetric jump processes on MMS

In this section, we will deal with several different types of heat kernel estimates on metric measure space so we consider different assumptions in each case to obtain our results. First, under the assumption that the lower scaling index of the scale function is strictly bigger than 1, we establish an upper bound of heat kernel and its stability which generalize Theorem 2.1.22. As in Section 2.1, the scale function is less than the rate function. Since $M$ may not satisfy chain condition in general, upper bounds and lower bounds in a generalized version of Theorem 2.1.2 may have different forms. To obtain sharp two-sided estimates, we further assume that metric measure space satisfies chain condition. Under the same assumption on the scale function and the chain condition, in Theorems 2.2.11, 2.2.14 and 2.2.15 we establish a sharp heat kernel estimates and their stability.

For the extension of heat kernel estimates in Section 2.1 and the corresponding stability result, we assume that underlying space admits conservative diffusion process whose transition density has a general sub-Gaussian bounds in terms of an increasing function $F$ (see Definition 2.2.2). The function $F$ serves as a generalization of walk dimension for underlying space. Note that in Theorem 2.1.1, $(d(x,y)/\Phi^{-1}(t))^2$ appears in the exponential term of the off-diagonal part and the order 2 is the walk dimension of Euclidean space. It is shown in [48] that the general sub-Gaussian bounds for diffusion is equivalent to the conjunction of elliptic Harnack inequality and estimates of mean exit time for diffusion process if volume double property holds a priori. Diffusion processes on Sierpinski gasket and generalized Sierpinski carpets satisfy our assumption ([6, 13]). See also [8, 10, 43, 71, 78] for studies on stability of (sub-)Gaussian type heat kernel estimates for diffusion processes on metric measure spaces. Under the general sub-Gaussian bound assumption on diffusion with $F$, we can define scale function explicitly by using the rate function and $F$ (see (2.2.18)). It is worth mentioning that we do not assume neither that the chain condition nor the lower scaling index of the scale function being strictly bigger than 1 in Theorem 2.2.17. Note that, GHK$(\Phi, \psi)$ in Theorem 2.2.17 is not sharp in general. Without the chain condition, even the transition density of diffusion may not have the sharp two-sided bounds. However, if the upper scaling index $\beta_2$ of the rate function is strictly less than the walk dimension, our heat kernel estimates GHK$(\Phi, \psi)$ is equivalent to (1.0.3).

### 2.2.1 Settings and Main results

Recall that $(M, d)$ be a locally compact separable metric space, and $\mu$ be a positive Radon measure on $M$ with full support and $\mu(M) = \infty$. We also assume that every ball in $(M, d)$ is relatively compact. Note that $V(x, r) > 0$ for every $x \in M$ and $r > 0$ since $\mu$ has full support on $M$. It is easy to see that under VD$(d_2)$, we have

$$\frac{V(x,R)}{V(y,r)} \le \frac{V(y, d(x,y)+R)}{V(y,r)} \le C_\mu \left(\frac{d(x,y)+R}{r}\right)^{d_2} \tag{2.2.1}$$

for all $x \in M$ and $0 < r \le R$. We introduce several conditions on metric measure space to establish stabilities of heat kernel estimates.

**Definition 2.2.1.** We say that a metric space $(M, d)$ satisfies the *chain condition* Ch$(A)$ if there exists a constant $A \ge 1$ such that, for any $n \in \mathbb{N}$ and $x, y \in M$, there is a sequence $\{z_k\}_{k=0}^n$ of points in $M$ such that $z_0 = x, z_n = y$ and

$$d(z_{k-1}, z_k) \le A\frac{d(x,y)}{n} \quad \text{for all} \quad k = 1, \dots, n.$$

**Definition 2.2.2.** For a strictly increasing function $F : (0, \infty) \to (0, \infty)$, we say that a metric measure space $(M, d, \mu)$ satisfies the condition Diff$(F)$ if there exists a conservative symmetric diffusion process $Z = (Z_t)_{t\ge0}$ on $M$ such that the transition density $q(t, x, y)$ of $Z$ with respect to $\mu$ exists and it satisfies the following estimates: there exist constants $c > 0$ and $a_0 > 1$ such that for all $t > 0$ and $x, y \in M$,

$$\frac{c^{-1}}{V(x, F^{-1}(t))} \mathbf{1}_{\{F(d(x,y)) \leq t\}} \leq q(t, x, y) \leq \frac{c \exp\big( - a_0 F_1(d(x, y), t)\big)}{V(x, F^{-1}(t))} \tag{2.2.2}$$

where the function $F_1$ is defined as

$$F_1(r, t) := \sup_{s>0} \left[ \frac{r}{s} - \frac{t}{F(s)} \right]. \tag{2.2.3}$$

Throughout this subsection, we will assume that $\psi : [0, \infty) \to [0, \infty)$ is a non-decreasing function satisfying $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$ for some $0 < \beta_1 \leq \beta_2$. Note that $\psi(0) = 0$ by Lemma 1.1.2. We also assume that there exists a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(M, \mu)$, which is given by

$$\mathcal{E}(u, v) := \int_{M\times M\setminus diag} (u(x) - u(y))(v(x) - v(y))J(x, y)\mu(dx)\mu(dy) \tag{2.2.4}$$

for $u, v \in \mathcal{F}$, where $J$ is a symmetric and positive Borel measurable function on $M \times M \setminus diag$. In terms of Beurling-deny formula in [41, Theorem 3.2], the above Dirichlet form has jump part only.

**Definition 2.2.3.** We say $\mathrm{J}_\psi$ holds if there exists a constant $\bar{C} > 1$ so that for every $x, y \in M$,

$$\frac{\bar{C}^{-1}}{V(x, d(x, y))\psi(d(x, y))} \leq J(x, y) \leq \frac{\bar{C}}{V(x, d(x, y))\psi(d(x, y))}. \tag{2.2.5}$$

We say that $\mathrm{J}_{\psi,\leq}$ (resp. $\mathrm{J}_{\psi,\geq}$) if the upper bound (resp. lower bound) in (2.2.5) holds.

Associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(M; \mu)$ is a $\mu$-symmetric Hunt process $X = \{X_t, t \geq 0; \mathbb{P}^x, x \in M \setminus \mathcal{N}\}$. Here $\mathcal{N}$ is a properly exceptional set for $(\mathcal{E}, \mathcal{F})$ in the sense that $\mu(\mathcal{N}) = 0$ and $\mathbb{P}^x(X_t \in \mathcal{N}$ for some $t > 0) = 0$ for all $x \in M \setminus \mathcal{N}$. This Hunt process is unique up to a properly exceptional set (see [41, Theorem 4.2.8].) We fix $X$ and $\mathcal{N}$, and write $M_0 = M \setminus \mathcal{N}$.

For a set $A \subset M$ and process $X$, define the exit time $\tau_A = \inf\{t > 0 : X_t \in A^c\}$. Let $\mathcal{F}' := \{u + b : u \in \mathcal{F}, b \in \mathbb{R}\}$.

**Definition 2.2.4.** Let $U \subset M$ be an open set, $A$ be any Borel subset of $U$ and $\kappa \geq 1$ be a real number. A $\kappa$-cutoff function of pair $(A, U)$ is any function $\varphi \in \mathcal{F}$ such that $0 \leq \varphi \leq \kappa$ $\mu$-a.e. in $M$, $\varphi \geq 1$ $\mu$-a.e. in $A$ and $\varphi = 0$ $\mu$-a.e. in $U^c$. We denote by $\kappa$-cutoff$(A, U)$ the collection of all $\kappa$-cutoff function of pair $(A, U)$. Any 1-cutoff function will be simply referred to as a cutoff function.

**Definition 2.2.5** (c.f. [45, Definition 1.11])**.** For a non-negative function $\phi$, we say that Gcap$(\phi)$ holds if there exist constants $\kappa \geq 1$ and $C > 0$ such that for any $u \in \mathcal{F}' \cap L^\infty$ and for all $x_0 \in M$ and $R, r > 0$, there exists a function $\varphi \in \kappa$-cutoff$(B(x_0, R), B(x_0, R + r))$ such that

$$\mathcal{E}(u^2\varphi, \varphi) \leq \frac{C}{\phi(r)} \int_{B(x_0,R+r)} u^2 d\mu.$$

**Definition 2.2.6.** For a non-negative function $\phi$, we say that $\mathrm{E}_\phi$ holds if there is a constant $c > 1$ such that

$$c^{-1}\phi(r) \leq \mathbb{E}^x[\tau_{B(x,r)}] \leq c\phi(r) \quad \text{for all } x \in M_0,\ r > 0.$$

We say that $\mathrm{E}_{\phi,\leq}$ (resp. $\mathrm{E}_{\phi,\geq}$) holds if the upper bound (resp. lower bound) in the inequality above holds.

**Remark 2.2.7.** Suppose VD$(d_2)$, RVD$(d_1)$ and $\mathrm{J}_{\psi,\geq}$ hold. Let $x \in M_0$ and $r > 0$. By the Lévy system in [32, Lemma 7.1] and $\mathrm{J}_{\psi,\geq}$, we have that for

$t > 0$,

$$\begin{aligned}
1 &\geq \mathbb{P}^x(X_{\tau_{B(x,r)}} \in B(x,2r)^c) \geq \mathbb{E}^x\left[\int_0^{\tau_{B(x,r)}} \int_{B(x,2r)^c} J(X_s,y)\mu(dy)ds\right] \\
&\geq \mathbb{E}^x[\tau_{B(x,r)}] \inf_{z \in B(x,r)} \int_{B(x,2r)^c} J(z,y)\mu(dy) \\
&\geq \bar{C}^{-1}\mathbb{E}^x[\tau_{B(x,r)}] \int_{B(x,2r)^c} \frac{1}{V(d(x,y))\psi(d(x,y))}\mu(dy).
\end{aligned}$$

By RVD($d_1$), there exists a constant $c_1 > 1$ such that $V(x, c_1 r) \geq 2V(x,r)$ for any $x \in M$ and $r > 0$. Using this and $U(\beta_2, C_U, \psi)$ we obtain

$$\int_{B(x,2r)^c} \frac{1}{V(d(x,y))\psi(d(x,y))}\mu(dy) \geq \frac{V(x,2c_1r) - V(x,2r)}{V(x,2c_1r)} \frac{1}{\psi(2c_1r)} \geq \frac{c_2}{\psi(r)}.$$

Combining two estimates, we obtain

$$\mathbb{E}^x[\tau_{B(x,r)}] \leq c\psi(r), \quad x \in M_0,\ r > 0,$$

which implies $\mathrm{E}_{\psi,\leq}$.

By Remark 2.2.7, we expect that our scale function with respect to the process $X$, which is comparable to the exit time $\mathbb{E}^x[\tau_{B(x,r)}]$, is smaller than $\psi$.

Let $\Phi : (0,\infty) \to (0,\infty)$ be a non-decreasing function satisfying $L(\alpha_1, c_L)$ and $U(\alpha_2, c_U)$ with some $0 < \alpha_1 \leq \alpha_2$ and $c_L, c_U > 0$ and

$$\Phi(r) < \psi(r), \quad \text{for all} \quad r > 0. \tag{2.2.6}$$

By the virtue of Remark 2.2.7, the assumption (2.2.6) is quite natural for the scale function. For any $c > 1$, (2.2.6) can be relaxed to the condition $\Phi(r) \leq c\psi(r)$. Recall that $\alpha_2$ is the global upper scaling index of $\Phi$. If $\Phi$ satisfies $L_a(\delta, \widetilde{C}_L)$, then we have $\alpha_2 \geq \delta$. Indeed, if $\delta > \alpha_2$, then for any

$0 < r \le R < a$, we have

$$\widetilde{C}_L\big(\frac{R}{r}\big)^{\delta} \le \frac{\Phi(R)}{\Phi(r)} \le c_U\big(\frac{R}{r}\big)^{\alpha_2},$$

which is contradiction by letting $r \to 0$. Also, we define a function $\Phi_1 : (0,\infty) \times (0,\infty) \to \mathbb{R}$ by

$$\Phi_1(r,t) := \sup_{s>0}\left\{\frac{r}{s} - \frac{t}{\Phi(s)}\right\}. \tag{2.2.7}$$

For $a_0, t, r > 0$ and $x \in M_0$, we define

$$\mathcal{G}(a_0,t,x,r) := \frac{t}{V(x,r)\psi(r)} + \frac{1}{V(x,\Phi^{-1}(t))}\exp\left(-a_0\,\Phi_1(t,r)\right),$$

where $\Phi^{-1}$ is the generalized inverse function of $\Phi$, i.e., $\Phi^{-1}(t) := \inf\{s \ge 0 : \Phi(s) > t\}$ (with the convention $\inf\emptyset = \infty$).

**Definition 2.2.8.** (i) We say that $\mathrm{HK}(\Phi,\psi)$ holds if there exists a heat kernel $p(t,x,y)$ of the semigroup $\{P_t\}$ associated with $(\mathcal{E},\mathcal{F})$, which has the following estimates: there exist $\eta, a_0 > 0$ and $c \ge 1$ such that for all $t > 0$ and $x, y \in M_0$,

$$\begin{aligned}&\frac{c^{-1}}{V(x,\Phi^{-1}(t))}\mathbf{1}_{\{d(x,y)\le\eta\Phi^{-1}(t)\}} + \frac{c^{-1}t}{V(x,d(x,y))\psi(d(x,y))}\mathbf{1}_{\{d(x,y)>\eta\Phi^{-1}(t)\}}\\&\le p(t,x,y) \le c\left(\frac{1}{V(x,\Phi^{-1}(t))} \wedge \mathcal{G}\big(a_0,t,x,d(x,y)\big)\right).\end{aligned}$$

(ii) We say $\mathrm{UHK}(\Phi,\psi)$ holds if the upper bound in above estimate holds.

(iii) We say $\mathrm{UHKD}(\Phi)$ holds if there is a constant $c > 0$ such that for all $t > 0$ and $x \in M_0$,

$$p(t,x,x) \le \frac{c}{V(x,\Phi^{-1}(t))}.$$

(vi) We say that $\mathrm{SHK}(\Phi,\psi)$ holds if there exists a heat kernel $p(t,x,y)$

of the semigroup $\{P_t\}$ associated with $(\mathcal{E}, \mathcal{F})$, which has the following estimates: there exist $a > 0$ such that for all $t > 0$ and $x, y \in M_0$,

$$p(t,x,y) \asymp \frac{1}{V(x, \Phi^{-1}(t))} \wedge \mathcal{G}\big(a, t, x, d(x,y)\big).$$

(v) Assume that $(M, d, \mu)$ satisfies VD$(d_2)$, RVD$(d_1)$ and Diff$(F)$. We say that GHK$(\Phi, \psi)$ holds if there exists a heat kernel $p(t,x,y)$ of the semigroup $\{P_t\}$ associated with $(\mathcal{E}, \mathcal{F})$, which has the following estimates: there exists $0 < a_U$, $0 < \eta$ and $c \geq 1$ such that for all $t > 0$ and $x, y \in M_0$,

$$\begin{aligned} &\frac{c^{-1}}{V(x, \Phi^{-1}(t))} \mathbf{1}_{\{d(x,y) \leq \eta \Phi^{-1}(t)\}} + \frac{c^{-1} t}{V(x, d(x,y)) \psi(d(x,y))} \mathbf{1}_{\{d(x,y) \geq \eta \Phi^{-1}(t)\}} \\ &\leq p(t,x,y) \qquad\qquad (2.2.8) \\ &\leq \frac{c}{V(x, \Phi^{-1}(t))} \wedge \Big( \frac{c\,t}{V(x, d(x,y)) \psi(d(x,y))} + \frac{c e^{-a_U F_1(d(x,y), F(\Phi^{-1}(t)))}}{V(x, \Phi^{-1}(t))} \Big). \end{aligned}$$

(vi) We say GUHK$(\Phi, \psi)$ holds if the upper bound in (2.2.8) holds.

**Remark 2.2.9.** For strictly increasing and continuous function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ satisfying $L(\alpha_1, c_L)$ and $U(\alpha_2, c_U)$ and for any $C > 1$, the condition HK$(\Phi, C\Phi)$ is equivalent to the existence of heat kernel $p(t,x,y)$ such that for all $t > 0$ and $x, y \in M_0$,

$$p(t,x,y) \asymp \frac{1}{V(x, \Phi^{-1}(t))} \wedge \frac{t}{V(x, d(x,y)) \Phi(d(x,y))}. \qquad (2.2.9)$$

This shows that if $\Phi \asymp \psi$, then the condition HK$(\Phi, \psi)$ is equivalent to (2.2.9).

From now on, we denote HK$(\Phi, C\Phi)$(resp. UHK$(\Phi, C\Phi)$) by HK$(\Phi)$ (resp. UHK$(\Phi)$). By Remark 2.2.9, the condition HK$(\Phi)$ is equivalent to the condition HK$(\Phi)$ of [32].

Let $\mathcal{F}_b$ be the collection of bounded functions in $\mathcal{F}$.

**Definition 2.2.10.** We say that the (weak) Poincaré inequality PI($\Phi$) holds if there exist constants $C > 0$ and $\kappa \geq 1$ such that for any ball $B_r := B(x, r)$ with $x \in M_0$, $r > 0$ and for any $f \in \mathcal{F}_b$,

$$\int_{B_r} (f - \bar{f}_{B_r})^2 d\mu \leq C\Phi(r) \int_{B_{\kappa r} \times B_{\kappa r}} (f(y) - f(z))^2 J(y, z)\mu(dy)\mu(dz),$$

where $\bar{f}_{B_r} = \frac{1}{\mu(B_r)} \int_{B_r} f d\mu$ is the average value of $f$ on $B_r$.

Recall that we always assume that $\psi : [0, \infty) \to [0, \infty)$ is a non-decreasing function satisfying $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$ for some $0 < \beta_1 \leq \beta_2$.

For the function $\Phi$ satisfying (2.2.6) and $L^a(\delta, \widetilde{C}_L)$ with $\delta > 1$, we define

$$\widetilde{\Phi}(s) := c_U^{-1} \frac{\Phi(a)}{a^{\alpha_2}} s^{\alpha_2} \mathbf{1}_{\{s<a\}} + \Phi(s)\mathbf{1}_{\{s \geq a\}}. \tag{2.2.10}$$

Note that for $s \leq a$ we have, $\frac{\widetilde{\Phi}(s)}{\Phi(s)} = c_U^{-1} \frac{s^{\alpha_2}}{a^{\alpha_2}} \frac{\Phi(a)}{\Phi(s)} \leq 1$. Thus,

$$\widetilde{\Phi}(r) \leq \Phi(r) < \psi(r), \quad r > 0. \tag{2.2.11}$$

Also, $L(\delta, \widetilde{C}_L, \widetilde{\Phi})$ holds. Indeed, for any $0 < r \leq a \leq R$,

$$\frac{\widetilde{\Phi}(R)}{\widetilde{\Phi}(r)} = \frac{\widetilde{\Phi}(R)}{\widetilde{\Phi}(a)} \frac{\widetilde{\Phi}(a)}{\widetilde{\Phi}(r)} \geq \widetilde{C}_L \big(\frac{R}{a}\big)^{\delta} \big(\frac{a}{r}\big)^{\delta} = \widetilde{C}_L \big(\frac{R}{r}\big)^{\delta}.$$

The other cases are straightforward. By the same way as (2.2.7), let us define

$$\widetilde{\Phi}_1(r, t) := \sup_{s>0} \left[ \frac{r}{s} - \frac{t}{\widetilde{\Phi}(s)} \right]. \tag{2.2.12}$$

The following are the main results of this subsection.

**Theorem 2.2.11.** *Assume that the metric measure space $(M, d, \mu)$ satisfies* VD($d_2$)*, and the process $X$ satisfies* $\mathrm{J}_{\psi,\leq}$, UHKD($\Phi$) *and* $\mathrm{E}_\Phi$*, where $\psi$ is a non-decreasing function satisfying $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$, and $\Phi$ is a non-decreasing function satisfying* (2.2.6)*, $L(\alpha_1, c_L)$ and $U(\alpha_2, c_U)$, where $0 < \beta_1 \leq \beta_2$ and $0 < \alpha_1 \leq \alpha_2$.*

*(i) Suppose that $\Phi$ satisfies $L_a(\delta, \widetilde{C}_L)$ with some $a > 0$ and $\delta > 1$. Then, for any $T \in (0, \infty)$, there exist constants $a_U > 0$ and $c > 0$ such that for any $t < T$ and $x, y \in M_0$,*

$$p(t, x, y) \le \frac{c\,t}{V(x, d(x, y))\psi(d(x, y))} + \frac{c}{V(x, \Phi^{-1}(t))} \exp\big( - a_U \Phi_1(d(x, y), t)\big). \tag{2.2.13}$$

*Moreover, if $\Phi$ satisfies $L(\delta, \widetilde{C}_L)$, then (2.2.13) holds for all $t < \infty$.*

*(ii) Suppose that $\Phi$ satisfies $L^a(\delta, \widetilde{C}_L)$ with some $a > 0$ and $\delta > 1$. Then, for any $T \in (0, \infty)$ there exist constants $a_U > 0$ and $c > 0$ such that for any $t \ge T$ and $x, y \in M_0$,*

$$p(t, x, y) \le \frac{c\,t}{V(x, d(x, y))\psi(d(x, y))} + \frac{c}{V(x, \Phi^{-1}(t))} \exp\Big(-a_U \widetilde{\Phi}_1(d(x, y), t)\Big). \tag{2.2.14}$$

**Theorem 2.2.12.** *Assume that $(M, d, \mu)$ satisfies $\mathrm{RVD}(d_1)$ and $\mathrm{VD}(d_2)$. Let $\psi$ be a non-decreasing function satisfying $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$, and $\Phi$ be a non-decreasing function satisfying (2.2.6), $L(\alpha_1, c_L)$ and $U(\alpha_2, c_U)$ with $1 < \alpha_1 \le \alpha_2$. Then the following are equivalent:*
(1) $\mathrm{UHK}(\Phi, \psi)$ *and* $(\mathcal{E}, \mathcal{F})$ *is conservative.*
(2) $\mathrm{J}_{\psi,\le}$, $\mathrm{UHK}(\Phi)$ *and* $(\mathcal{E}, \mathcal{F})$ *is conservative.*
(3) $\mathrm{J}_{\psi,\le}$, $\mathrm{UHKD}(\Phi)$ *and* $\mathrm{E}_\Phi$.

See [32, Definitions 1.5 and 1.8] for the definitions of $\mathrm{FK}(\Phi)$, $\mathrm{CSJ}(\Phi)$ and $\mathrm{SCSJ}(\Phi)$.

**Corollary 2.2.13.** *Under the same settings as Theorem 2.2.12, each equivalent condition in above theorem is also equivalent to the following:*
(4) $\mathrm{FK}(\Phi)$, $\mathrm{J}_{\psi,\le}$ *and* $\mathrm{SCSJ}(\Phi)$.
(5) $\mathrm{FK}(\Phi)$, $\mathrm{J}_{\psi,\le}$ *and* $\mathrm{CSJ}(\Phi)$.
(6) $\mathrm{FK}(\Phi)$, $\mathrm{J}_{\psi,\le}$ *and* $\mathrm{Gcap}(\Phi)$.

**Theorem 2.2.14.** *Assume that the metric measure space $(M, d, \mu)$ satisfies $\mathrm{Ch}(A)$, $\mathrm{RVD}(d_1)$ and $\mathrm{VD}(d_2)$. Suppose that the process $X$ satisfies $\mathrm{J}_\psi$, $\mathrm{E}_\Phi$ and $\mathrm{PI}(\Phi)$, where $\psi$ is a non-decreasing function satisfying $L(\beta_1, C_L)$ and*

*$U(\beta_2, C_U)$, and $\Phi$ is a non-decreasing function satisfying* (2.2.6)*, $L(\alpha_1, c_L)$ and $U(\alpha_2, c_U)$.*

*(i) Suppose that $L_a(\delta, \widetilde{C}_L, \Phi)$ holds with $\delta > 1$. Then, for any $T \in (0, \infty)$, there exist constants $c > 0$ and $a_L > 0$ such that for any $x, y \in M_0$ and $t \in (0, T]$,*

$$p(t,x,y) \geq \frac{c}{V(x,\Phi^{-1}(t))} \wedge \left( \frac{ct}{V(x,d(x,y))\psi(d(x,y))} + \frac{c}{V(x,\Phi^{-1}(t))} \exp\left(-a_L \Phi_1(d(x,y),t)\right) \right) \tag{2.2.15}$$

*Moreover, if $L(\delta, \widetilde{C}_L, \Phi)$ holds, then* (2.2.15) *holds for all $t \in (0, \infty)$.*

*(ii) Suppose that $L^a(\delta, \widetilde{C}_L, \Phi)$ holds with $\delta > 1$. Then, for any $T \in (0, \infty)$, there exist constants $c > 0$ and $a_L > 0$ such that for any $x, y \in M_0$ and $t \geq T$,*

$$p(t,x,y) \geq \frac{c}{V(x,\Phi^{-1}(t))} \wedge \left( \frac{ct}{V(x,d(x,y))\psi(d(x,y))} + \frac{c}{V(x,\Phi^{-1}(t))} \exp\left(-a_L \widetilde{\Phi}_1(d(x,y),t)\right) \right). \tag{2.2.16}$$

**Theorem 2.2.15.** *Under the same settings as Theorem 2.2.12, the following are equivalent:*
(1) $\mathrm{HK}(\Phi, \psi)$.
(2) $\mathrm{J}_\psi$, $\mathrm{PI}(\Phi)$, $\mathrm{UHK}(\Phi)$ *and $(\mathcal{E}, \mathcal{F})$ is conservative.*
(3) $\mathrm{J}_\psi$, $\mathrm{PI}(\Phi)$ *and* $\mathrm{E}_\Phi$.
*If we further assume that $(M, d)$ satisfies $\mathrm{Ch}(A)$ for some $A \geq 1$, then the following is also equivalent to others:*
(4) $\mathrm{SHK}(\Phi, \psi)$.

By Theorem 2.2.15 and Corollary 2.2.13, we also obtain that

**Corollary 2.2.16.** *Under the same settings as Theorem 2.2.12, each equivalent condition in Theorem 2.2.15 is also equivalent to the following:*
(5) $\mathrm{J}_\psi$, $\mathrm{PI}(\Phi)$ *and* $\mathrm{SCSJ}(\Phi)$.
(6) $\mathrm{J}_\psi$, $\mathrm{PI}(\Phi)$ *and* $\mathrm{CSJ}(\Phi)$.
(7) $\mathrm{J}_\psi$, $\mathrm{PI}(\Phi)$ *and* $\mathrm{Gcap}(\Phi)$.

We now consider a metric measure space that allows conservative diffusion process which has the transition density with respect to $\mu$ satisfying $\mathrm{Diff}(F)$. In this case, we can find $\Phi$ explicitly from $F$ and $\psi$.

From now on, let $F$ be a strictly increasing function satisfying $L(\gamma_1, c_F^{-1})$ and $U(\gamma_2, c_F)$ with some $1 < \gamma_1 \leq \gamma_2$, and we assume that $\psi : (0,\infty) \to (0,\infty)$ is a non-increasing function which satisfies $L(\beta_1, C_L)$, $U(\beta_2, C_U)$ and that

$$\int_0^1 \frac{dF(s)}{\psi(s)} < \infty. \tag{2.2.17}$$

Recall that the function $F_1(r,t) = \sup_{s>0}\left[\frac{r}{s} - \frac{t}{F(s)}\right]$ has defined in (2.2.3). Consider

$$\Phi(r) := \frac{F(r)}{\int_0^r \frac{dF(s)}{\psi(s)} ds}, \quad r > 0. \tag{2.2.18}$$

Then $\Phi$ is strictly increasing function satisfying (2.2.6) and $U(\gamma_2, C_U)$. Also, there is $\widetilde{c} > 0$ such that $L(\alpha_1, \widetilde{c}, \Phi)$ holds. (see Section 2.2.5).

**Theorem 2.2.17.** *Assume that the metric measure space $(M, d, \mu)$ satisfies* $\mathrm{RVD}(d_1)$ *and* $\mathrm{VD}(d_2)$*. Assume further that* $\mathrm{Diff}(F)$ *holds for a strictly increasing function $F : (0,\infty) \to (0,\infty)$ satisfying $L(\gamma_1, c_F^{-1})$ and $U(\gamma_2, c_F)$ with $1 < \gamma_1 \leq \gamma_2$. Let $\psi$ be a non-decreasing function satisfying $L(\beta_1, C_L)$, $U(\beta_2, C_U)$ and* (2.2.17)*, and $\Phi$ be the function defined in* (2.2.18)*.*

*(i)* $\mathrm{J}_\psi$ *is equivalent to* $\mathrm{GHK}(\Phi, \psi)$*. Moreover, both equivalent conditions imply* $\mathrm{PI}(\Phi)$ *and* $\mathrm{E}_\Phi$*.*

*(ii) If we further assume that $(M, d)$ satisfies* $\mathrm{Ch}(A)$ *for some $A \geq 1$ and that $\Phi$ satisfies $L(\alpha_1, c_L)$ with $\alpha_1 > 1$, then* $\mathrm{J}_\psi$ *is also equivalent to* $\mathrm{SHK}(\Phi, \psi)$*.*

Finally, we now state local estimates of heat kernels.

**Corollary 2.2.18.** *Assume that the metric measure space $(M, d, \mu)$ satisfies* $\mathrm{RVD}(d_1)$ *and* $\mathrm{VD}(d_2)$*. Assume further that* $\mathrm{Diff}(F)$ *holds for a strictly increasing function $F : (0,\infty) \to (0,\infty)$ satisfying $L(\gamma_1, c_F^{-1})$ and $U(\gamma_2, c_F)$*

*with* $1 < \gamma_1 \le \gamma_2$*. Let* $\psi$ *be a non-decreasing function satisfying* $L(\beta_1, C_L)$*,* $U(\beta_2, C_U)$ *and* (2.2.17)*, and* $\Phi$ *be the function defined in* (2.2.18)*. Suppose that the process* $X$ *satisfies* $\mathrm{J}_\psi$*.*

*(i) Assume that* $L_a(\delta, \widetilde{C}_L, \Phi)$ *holds with some* $\delta > 1$ *and* $a > 0$*. Then, for any* $T \in (0, \infty)$*, there exist constants* $0 < a_U \le a_L$ *and* $c > 0$ *such that* (2.2.13) *and* (2.2.15) *holds for all* $t \in (0, T]$ *and* $x, y \in M_0$*.*

*(ii) Assume that* $L^a(\delta, \widetilde{C}_L, \Phi)$ *holds with some* $\delta > 1$ *and* $a > 0$*. Then, for any* $T \in (0, \infty)$*, there exist constants* $0 < a_U \le a_L$ *and* $c > 0$ *such that* (2.2.14) *and* (2.2.16) *holds for all* $t \in [T, \infty)$ *and* $x, y \in M_0$*.*

### 2.2.2 Preliminary

Consider a non-decreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $L(\alpha_1, c_L)$ and $U(\alpha_2, c_U)$ with some $0 < \alpha_1 \le \alpha_2$. Recall that $\phi^{-1}(t) := \inf\{s \ge 0 : \phi(s) > t\}$ is the generalized inverse function of $\phi$. We further assume that $L_a(\delta, \widetilde{C}_L, \phi)$ holds for some $a > 0$ and $\delta > 1$. We define

$$\mathcal{T}(\phi)(r,t) := \sup_{s>0} \left[ \frac{r}{s} - \frac{t}{\phi(s)} \right], \quad r, t > 0. \tag{2.2.19}$$

Note that from $L(\alpha_1, c_L, \phi)$ and $L_a(\delta, \widetilde{C}_L, \phi)$, we obtain $\lim_{s\to\infty} \phi(s) = \infty$ and $\lim_{s\to 0} \frac{\phi(s)}{s} = 0$, respectively. This concludes that $\mathcal{T}(\phi)(r,t) \in [0, \infty)$ for all $r, t > 0$. Also, comparing the definitions in (2.2.7) and (2.2.19), we see that $\mathcal{T}(\Phi) = \Phi_1$ and $\mathcal{T}(F) = F_1$. for instance. It immediately follows from the definition of $\mathcal{T}(\phi)$ that for any $c, r, t > 0$,

$$\mathcal{T}(\phi)(cr, ct) = c\mathcal{T}(\phi)(r,t).$$

We first observe when the supremum in (2.2.19) occurs.

**Lemma 2.2.19.** *Let* $\delta_1 := \frac{1}{\delta - 1}$*. For any* $T \in (0, \infty)$*, there exists constant*

$b \in (0,1)$ *such that for any* $r > 0$, $t \in (0, T]$ *with* $r \geq 2c_U \phi^{-1}(t)$,

$$\mathcal{T}(\phi)(r,t) = \sup_{s \in [br^{-\delta_1}\phi^{-1}(t)^{\delta_1+1}, 2\phi^{-1}(t)]} \left[\frac{r}{s} - \frac{t}{\phi(s)}\right] \geq \frac{r}{2\phi^{-1}(t)}. \tag{2.2.20}$$

*Moreover, if* $L(\delta, \widetilde{C}_L, \phi)$ *holds,* (2.2.20) *holds for all* $t \in (0, \infty)$.

**Lemma 2.2.20.** *(i) For any* $T > 0$ *and* $c_1, c_2 > 0$, *there exists a constant* $c > 0$ *such that for any* $r > 0$ *and* $t \in (0, T]$ *with* $r \geq 2c_U\phi^{-1}(t)$,

$$\mathcal{T}(\phi)(c_1 r, c_2 t) \leq c\mathcal{T}(\phi)(r,t). \tag{2.2.21}$$

*(ii) For any* $T > 0$ *and* $c_3 > 0$, *there exists a constant* $\tilde{c} > 0$ *such that for any* $t \in (0, T]$ *and* $r \leq c_3\phi^{-1}(t)$,

$$\mathcal{T}(\phi)(r,t) \leq \tilde{c}. \tag{2.2.22}$$

*Moreover, if* $L(\delta, \widetilde{C}_L, \phi)$ *holds, both* (2.2.21) *and* (2.2.22) *hold for all* $t \in (0, \infty)$.

### 2.2.3 Stability of upper heat kernel estimates

In this section we prove Theorems 2.2.11 and 2.2.12, and Corollary 2.2.13. Throughout this subsection, we assume that the function $\psi$ satisfies $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$ with $0 < \beta_1 \leq \beta_2$, and $\Phi$ satisfies (2.2.6), $L(\alpha_1, c_L)$ and $U(\alpha_2, c_U)$, with $0 < \alpha_1 \leq \alpha_2$.

Assume that there exists regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ defined in (2.2.4) satisfying $\mathrm{J}_{\psi,\leq}$. Let $X$ be the Hunt process corresponds to $(\mathcal{E}, \mathcal{F})$. Fix $\rho > 0$ and define a bilinear form $(\mathcal{E}^\rho, \mathcal{F})$ by

$$\mathcal{E}^\rho(u,v) = \int_{M \times M \setminus diag} (u(x) - u(y))(v(x) - v(y))\mathbf{1}_{\{d(x,y) \leq \rho\}} J(x,y)\mu(dx)\mu(dy).$$

Clearly, the form $\mathcal{E}^\rho(u,v)$ is well defined for $u, v \in \mathcal{F}$, and $\mathcal{E}^\rho(u,u) \leq \mathcal{E}(u,u)$ for all $u \in \mathcal{F}$. Let $J_\rho(x,y) = J(x,y)\mathbf{1}_{\{d(x,y) > \rho\}}$. Since $\psi$ satisfies $L(\beta_1, C_L)$

and $U(\beta_2, C_U)$, for all $u \in \mathcal{F}$,

$$\begin{aligned}\mathcal{E}(u,u) - \mathcal{E}^\rho(u,u) &= \int (u(x)-u(y))^2 J_\rho(x,y)\mu(dx)\mu(dy) \\ &\le 4\int_M u^2(x)\,\mu(dx)\int_{B(x,\rho)^c} J(x,y)\,\mu(dy) \le \frac{c_0\|u\|_2^2}{\psi(\rho)}.\end{aligned}$$

Thus, $\mathcal{E}_1(u,u) := \mathcal{E}(u,u) + \|u\|_2^2$ is equivalent to $\mathcal{E}_1^\rho(u,u) := \mathcal{E}^\rho(u,u) + \|u\|_2^2$ for every $u \in \mathcal{F}$, which implies that $(\mathcal{E}^\rho, \mathcal{F})$ is also a regular Dirichlet form on $L^2(M, d\mu)$. We call $(\mathcal{E}^\rho, \mathcal{F})$ the $\rho$-truncated Dirichlet form. The Hunt process associated with $(\mathcal{E}^\rho, \mathcal{F})$ which will be denoted by $X^\rho$ can be identified in distribution with the Hunt process of the original Dirichlet form $(\mathcal{E}, \mathcal{F})$ by removing those jumps of size larger than $\rho$. Let $p^\rho(t,x,y)$ and $\tau_D^\rho$ be the transition density and exit time of $X^\rho$ correspond to $(\mathcal{E}^\rho, \mathcal{F})$, respectively.

For any open set $D \subset M$, $\mathcal{F}_D$ is defined to be the $\mathcal{E}_1$-closure in $\mathcal{F}$ of $\mathcal{F} \cap C_c(D)$. Let $\{P_t^D\}$ and $\{P_t^{\rho,D}\}$ be the semigroups of $(\mathcal{E}, \mathcal{F}_D)$ and $(\mathcal{E}^\rho, \mathcal{F}_D)$, respectively.

**Lemma 2.2.21.** *Assume* $\mathrm{VD}(d_2)$, $\mathrm{J}_{\psi,\le}$ *and* $\mathrm{E}_\Phi$. *Then, there is a constant* $c > 0$ *such that for any* $\rho > 0$, $t > 0$ *and* $x \in M_0$,

$$\mathbb{E}^x \int_0^t \frac{1}{V(X_s^\rho, \rho)} ds \le \frac{ct}{V(x,\rho)} \left(1 + \frac{t}{\Phi(\rho)}\right)^{d_2+1}.$$

**Proof.** Following the proof of [32, Proposition 4.24], using $\mathrm{J}_{\psi,\le}$ we have

$$\begin{aligned}&\mathbb{E}^x\left[\int_0^t \frac{1}{V(X_s^\rho,\rho)}ds\right] \\ &= \sum_{k=1}^\infty \mathbb{E}^x\left[\int_0^t \frac{1}{V(X_s^\rho,\rho)}ds; \tau^\rho_{B(x,(k-1)\rho)} \le t < \tau^\rho_{B(x,k\rho)}\right] := \sum_{k=1}^\infty I_k.\end{aligned} \tag{2.2.23}$$

When $t < \tau^\rho_{B(x,k\rho)}$, we have $d(X_s^\rho, x) \le k\rho$ for all $s \le t$. This along with

VD($d_2$) yields that for all $k \geq 1$ and $s \leq t < \tau^{\rho}_{B(x,k\rho)}$,

$$\frac{1}{V(X^{\rho}_s,\rho)} \leq \frac{c_1 k^{d_2}}{V(X^{\rho}_s,2k\rho)} \leq \frac{c_1 k^{d_2}}{\inf\limits_{d(z,x)\leq k\rho} V(z,2k\rho)} \leq \frac{c_1 k^{d_2}}{V(x,\rho)}. \tag{2.2.24}$$

On the other hand, by [32, Corollary 4.22], there exist constants $c_i > 0$ with $i = 2,3,4$ such that for all $t,\rho > 0$, $k \geq 1$ and $x \in M_0$,

$$\mathbb{P}^x(\tau^{\rho}_{B(x,k\rho)} \leq t) \leq c_2 \exp\big(-c_3 k + c_4 \frac{t}{\Phi(\rho)}\big). \tag{2.2.25}$$

Let $k_0 = \lceil \frac{2c_4}{c_3}\frac{t}{\Phi(\rho)} \rceil + 1$. Using (2.2.24) and definition of $k_0$, we have

$$\sum_{k=1}^{k_0} I_k \leq \sum_{k=1}^{k_0} \frac{c_1 k^{d_2} t}{V(x,\rho)} \leq \frac{c_5 k_0^{d_2+1} t}{V(x,\rho)} \leq \frac{c_6 t}{V(x,\rho)} \left(1 + \frac{t}{\Phi(\rho)}\right)^{d_2+1}.$$

On the other hand, for any $k_0 < k$, using (2.2.24) and (2.2.25) with $k_0 = \lceil \frac{2c_4}{c_3}\frac{t}{\Phi(\rho)} \rceil + 1$ we have

$$I_k \leq \frac{c_1 k^{d_2} t}{V(x,\rho)} \mathbb{P}^x(\tau^{\rho}_{B(x,k\rho)} \leq t) \leq \frac{c_1 c_2 k^{d_2} t}{V(x,\rho)} \exp\left(-\frac{c_3}{2}k\right).$$

Thus, we conclude

$$\sum_{k=k_0+1}^{\infty} I_k \leq \frac{c_1 c_2 t}{V(x,\rho)} \sum_{k=k_0+1}^{\infty} k^{d_2} e^{-\frac{c_3}{2}k} := \frac{c_5 t}{V(x,\rho)}.$$

From above two estimates, we obtain $\sum_{i=1}^{\infty} I_k \leq \frac{c_6 t}{V(x,\rho)}(1 + \frac{t}{\Phi(\rho)})^{d_2+1}$. Combining this with (2.2.23), we obtain the desired estimate. □

In the next lemma, we obtain a priori estimate for the upper bound of heat kernel.

**Lemma 2.2.22.** *Assume* VD($d_2$), $\mathrm{J}_{\psi,\leq}$, UHKD($\Phi$) *and* $\mathrm{E}_\Phi$. *Then, there are* *constants* $c > 0$ *and* $C_1, C_2 > 0$ *such that for any* $\rho > 0$, $t > 0$ *and* $x, y \in M_0$,

$$p(t,x,y) \le \frac{c}{V(x,\Phi^{-1}(t))}\Big(1+\frac{d(x,y)}{\Phi^{-1}(t)}\Big)^{d_2} \exp\big(C_1\frac{t}{\Phi(\rho)} - C_2\frac{d(x,y)}{\rho}\big) + \frac{ct}{V(x,\rho)\psi(\rho)}\Big(1+\frac{t}{\Phi(\rho)}\Big)^{d_2+1}.$$

**Proof.** Recall that $X_t^\rho$ and $p^\rho(t,x,y)$ are the Hunt process and heat kernel correspond to $(\mathcal{E}^\rho, \mathcal{F})$, respectively. Using [11, Lemma 3.1, (3.5)] and [9, Lemma 3.6], we have for $t > 0$ and $x, y \in M_0$,

$$p(t,x,y) \le p^\rho(t,x,y) + \mathbb{E}^x\left[\int_0^t \int_M J_\rho(X_s^\rho, z)p(t-s,z,y)\mu(dz)ds\right] \tag{2.2.26}$$

Also, using symmetry of heat kernel, $\mathrm{J}_{\psi,\le}$ and Lemma 2.2.21 we obtain

$$\begin{aligned}&\mathbb{E}^x\left[\int_0^t \int_M J(X_s^\rho, z)\mathbf{1}_{\{d(z,X_s^\rho)\ge\rho\}}(z)\, p(t-s,z,y)\mu(dz)ds\right]\\ &\le c_1\mathbb{E}^x\left[\int_0^t \frac{1}{V(X_s^\rho,\rho)\psi(\rho)}ds\right] \le \frac{c_1 t}{V(x,\rho)\psi(\rho)}\left(1+\frac{t}{\Phi(\rho)}\right)^{d_2+1}.\end{aligned} \tag{2.2.27}$$

Combining the estimates in [32, Lemma 5.2] and Lemma 2.2.21, we conclude the proof. Note that since $\mathrm{J}_{\psi,\le}$ and (2.2.6) imply $\mathrm{J}_{\Phi,\le}$, the conditions in [32, Lemma 5.2] are satisfied. □

**Lemma 2.2.23.** *Assume* $\mathrm{VD}(d_2)$, $\mathrm{J}_{\psi,\le}$, $\mathrm{UHKD}(\Phi)$ *and* $\mathrm{E}_\Phi$. *Let* $T > 0$ *and* $f : \mathbb{R}_+\times\mathbb{R}_+ \to \mathbb{R}_+$ *be a measurable function satisfying that* $t \mapsto f(r,t)$ *is non-increasing for all* $r > 0$ *and that* $r \mapsto f(r,t)$ *is non-decreasing for all* $t > 0$. *Suppose that the following hold: (i) For each* $b > 0$, $\sup_{t\le T} f(b\Phi^{-1}(t),t) < \infty$ *(resp.,* $\sup_{t\ge T} f(b\Phi^{-1}(t),t) < \infty$*); (ii) there exist* $\eta \in (0,\beta_1]$, $a_1 > 0$ *and* $c > 0$ *such that*

$$\mathbb{P}^x\big(d(x,X_t) > r\big) \le c(\psi^{-1}(t)/r)^\eta + c\exp\big(-a_1 f(r,t)\big) \tag{2.2.28}$$

*for all* $t \in (0,T]$ *(resp.* $t \in [T,\infty)$*),* $r > 0$ *and* $x \in M_0$.

*Then, there exist constants $k \in \mathbb{N}, c_0 > 0$ such that*

$$
\begin{aligned}
p(t,x,y) \le & \frac{c_0\, t}{V(x,d(x,y))\psi(d(x,y))} \\
&+ \frac{c_0}{V(x,\Phi^{-1}(t))}\left(1+\frac{d(x,y)}{\Phi^{-1}(t)}\right)^{d_2} \exp\Big(-a_1 k f\big(d(x,y)/(16k),t\big)\Big)
\end{aligned}
$$

*for all $t \in (0,T)$ (resp. $t \in [T,\infty)$) and $x,y \in M_0$.*

**Proof.** Since the proofs for cases $t \in (0,T]$ and $t \in [T,\infty)$ are similar, we only prove for $t \in (0,T]$. For $x_0 \in M_0$, let $B(r) = B(x_0,r) \cap M_0$. By the strong Markov property, (2.2.28), and the fact that $t \mapsto f(r,t)$ is non-increasing, we have that for $x \in B(r/4)$ and $t \in (0,T/2]$,

$$
\begin{aligned}
\mathbb{P}^x(\tau_{B(r)} \le t) &\le \mathbb{P}^x(X_{2t} \in B(r/2)^c) + \mathbb{P}^x(\tau_{B(r)} \le t, X_{2t} \in B(r/2)) \\
&\le \mathbb{P}^x(X_{2t} \in B(x,r/4)^c) + \sup_{z \in B(r)^c, s \le t} \mathbb{P}^z(X_{2t-s} \in B(z,r/4)^c) \\
&\le c(4\psi^{-1}(2t)/r)^\eta + c\exp\big(-a_1 f(r/4,2t)\big).
\end{aligned}
$$

From this with $L(\beta_1, C_L, \psi)$ and Lemma 1.1.4, we have that for $x \in B(r/4)$ and $t \in (0,T/2]$,

$$
1 - P_t^{B(r)}\mathbf{1}_{B(r)}(x) = \mathbb{P}^x(\tau_{B(r)} \le t) \le c_1\left(\frac{\psi^{-1}(t)}{r}\right)^\eta + c\exp\Big(-a_1 f(r/4,2t)\Big). \tag{2.2.29}
$$

By [47, Proposition 4.6] and [32, Lemma 2.1], we have

$$
\left|P_t^{B(r)}\mathbf{1}_{B(r)}(x) - P_t^{r,B(r)}\mathbf{1}_{B(r)}(x)\right| \le 2t \sup_{z \in M} \int_{B(z,r)^c} J(z,y)\,dy \le \frac{c_3 t}{\psi(r)}.
$$

Combining this with (2.2.29), we see that for all $x \in B(r/4)$ and $t \in (0,T/2]$,

$$
\begin{aligned}
&\mathbb{P}^x(\tau^r_{B(r)} \le t) = 1 - P_t^{r,B(r)}\mathbf{1}_{B(r)}(x) \le 1 - P_t^{B(r)}\mathbf{1}_{B(r)}(x) + \frac{c_3 t}{\psi(r)} \\
&\le c_2(\psi^{-1}(t)/r)^\eta + c_1\exp\big(-a_1 f(r/4,2t)\big) + c_3(t/\psi(r)) =: \phi(r,t). \qquad (2.2.30)
\end{aligned}
$$

Applying [2, Lemma 3.3] with $r = \rho$ to (2.2.30), we see that for any $t \in$

$(0, T/2]$, $x \in B(r/4)$ and $k \in \mathbb{N}$,

$$\int_{B(x,2kr)^c} p^r(t,x,y)\mu(dy) = P_t^r \mathbf{1}_{B(x,2kr)^c}(x) \le \phi(r,t)^k. \tag{2.2.31}$$

Let $k := \lceil \frac{\beta_2+2d_2}{\eta} \rceil + 1$. For $t \in (0,T]$ and $x, y \in M_0$ satisfying $4k\Phi^{-1}(t) \ge d(x,y)$, by using that $r \mapsto f(r,t)$ is non-decreasing and the assumption $(i)$, we have $f(d(x,y)/(16k),t) \le f(\Phi^{-1}(t)/4,t) \le C < \infty$. Thus, using [32, Lemma 5.1],

$$p(t,x,y) \le \frac{c_5 e^{a_1 kC}}{V(x,\Phi^{-1}(t))} \exp\Big( - a_1 k f(d(x,y)/(16k),t)\Big). \tag{2.2.32}$$

For the remainder of the proof, assume $t \in (0,T]$ and $4k\Phi^{-1}(t) < d(x,y)$. Also, denote $r = d(x,y)$ and $\rho = r/(4k)$. Using [32, Lemma 5.2], (2.2.31) and (2.2.1), we have

$$\begin{aligned}
p^\rho(t,x,y) &= \int_M p^\rho(t/2,x,z)p^\rho(t/2,z,y)\mu(dz) \\
&\le \Big( \int_{B(x,r/2)^c} + \int_{B(y,r/2)^c} \Big) p^\rho(t/2,x,z)p^\rho(t/2,z,y)\mu(dz) \\
&\le \frac{c_7}{V(x,\Phi^{-1}(t))} \left(1 + \frac{r}{\Phi^{-1}(t)}\right)^{d_2} \phi(\rho,t/2)^k \\
&\le \frac{c_8}{V(x,\Phi^{-1}(t))} \Big(\frac{r}{\Phi^{-1}(t)}\Big)^{d_2} \phi(\rho,t/2)^k.
\end{aligned} \tag{2.2.33}$$

Note that $k\beta_1 \ge k\eta \ge \beta_2 + 2d_2$ and $\rho \ge \Phi^{-1}(t) > \psi^{-1}(t)$. Thus, by $L(\beta_1, C_L, \psi)$ we obtain

$$\Big(\frac{\psi^{-1}(t)}{\rho}\Big)^{\eta k} + \Big(\frac{t}{\psi(\rho)}\Big)^k \le \Big(\frac{\psi^{-1}(t)}{\rho}\Big)^{\beta_2+2d_2} + c_9 \Big(\frac{\psi^{-1}(t)}{\rho}\Big)^{k\beta_1} \le c_{10} \Big(\frac{\psi^{-1}(t)}{r}\Big)^{\beta_2+2d_2}.$$

Applying this to (2.2.33) and using $\mathrm{VD}(d_2)$ and $U(\beta_2, C_U, \psi)$ we have

$$
\begin{aligned}
&p^\rho(t,x,y)\\
&\le \frac{c_{11}}{V(x,\Phi^{-1}(t))}\Big(\frac{r}{\Phi^{-1}(t)}\Big)^{d_2}\Big(\Big(\frac{\psi^{-1}(t)}{\rho}\Big)^{\eta k} + \exp\Big(-a_1 k f(\rho/4,t)\Big) + \Big(\frac{t}{\psi(\rho)}\Big)^k\Big)\\
&\le \frac{c_{12}}{V(x,\Phi^{-1}(t))}\Big(\frac{r}{\Phi^{-1}(t)}\Big)^{d_2}\Big(\Big(\frac{\psi^{-1}(t)}{r}\Big)^{\beta_2+2d_2} + \exp\big(-a_1 k f(\rho/4,t)\big)\Big)\\
&\le \frac{c_{13}t}{V(x,r)\psi(r)} + \frac{c_{13}}{V(x,\Phi^{-1}(t))}\left(1+\frac{r}{\Phi^{-1}(t)}\right)^{d_2}\exp\Big(-a_1 k f(r/(16k),t)\Big).
\end{aligned}
$$

Thus, by (2.2.26), (2.2.27) and $U(\beta_2, C_U, \psi)$, we conclude that for any $t \in (0,T]$ and $x, y \in M_0$ with $4k\Phi^{-1}(t) < d(x,y)$,

$$
\begin{aligned}
p(t,x,y) &\le p^\rho(t,x,y) + \frac{c_{14}t}{V(x,\rho)\psi(\rho)}\left(1+\frac{t}{\Phi(\rho)}\right)^{d_2+1}\\
&\le \frac{c_{15}e^{-a_1 k f\left(\frac{d(x,y)}{16k},t\right)}}{V(x,\Phi^{-1}(t))}\left(1+\frac{d(x,y)}{\Phi^{-1}(t)}\right)^{d_2} + \frac{c_{15}t}{V(x,d(x,y))\psi(d(x,y))}.
\end{aligned}
\tag{2.2.34}
$$

Here in the second inequality we have used $\Phi(\rho) \ge t$. Now the conclusion follows from (2.2.32) and (2.2.34). □

**Lemma 2.2.24.** *Suppose* $\mathrm{VD}(d_2)$, $\mathrm{J}_{\psi,\le}$, $\mathrm{UHKD}(\Phi)$ *and* $\mathrm{E}_\Phi$. *Then, there exist constants* $a_0, c > 0$ *and* $N \in \mathbb{N}$ *such that*

$$
p(t,x,y) \le \frac{c\,t}{V(x,d(x,y))\psi(d(x,y))} + c\,V(x,\Phi^{-1}(t))^{-1}\exp\left(-\frac{a_0 d(x,y)^{1/N}}{\Phi^{-1}(t)^{1/N}}\right),
\tag{2.2.35}
$$

*for all* $t > 0$ *and* $x, y \in M_0$.

**Proof.** Let $N := \lceil \frac{\beta_1+d_2}{\beta_1} \rceil + 1$, and $\eta := \beta_1 - (\beta_1 + d_2)/N > 0$. We first claim that there exist $a_1 > 0$ and $c_1 > 0$ such that for any $t, r > 0$ and $x \in M_0$,

$$
\int_{\{y:d(x,y)\ge r\}} p(t,x,y)\mu(dy) \le c_1\left(\frac{\psi^{-1}(t)}{r}\right)^{\eta} + c_1\exp\left(-\frac{a_1 r^{1/N}}{\Phi^{-1}(t)^{1/N}}\right).
\tag{2.2.36}
$$

When $r \le \Phi^{-1}(t)$, we immediately obtain (2.2.36) by letting $c = \exp(a_1)$.

Thus, we will only consider the case $r > \Phi^{-1}(t)$. Fix $\alpha \in (d_2/(d_2+\beta_1), 1)$ and define for $n \in \mathbb{N}$,

$$\rho_n = \rho_n(r,t) = 2^{n\alpha} r^{1-1/N} \Phi^{-1}(t)^{1/N}.$$

Since $r > \Phi^{-1}(t)$ , we have $\Phi^{-1}(t) < \rho_n \le 2^n r$. In particular, $t \le \Phi(\rho_n)$. Thus, using Lemma 2.2.22 with $\rho = \rho_n$, we have that for every $t > 0$ and $x, y \in M_0$ with $2^n r \le d(x,y) < 2^{n+1} r$,

$$\begin{aligned}
p(t,x,y) &\le \frac{c_2}{V(x,\Phi^{-1}(t))} \left( \frac{2^{n+1} r}{\Phi^{-1}(t)} \right)^{d_2+1} \exp\left( C_1 \frac{t}{\Phi(\rho_n)} - C_2 \frac{d(x,y)}{\rho_n} \right) \\
&\quad + \frac{c_2 t}{V(x,\rho_n)\psi(\rho_n)} \left( 1 + \frac{t}{\Phi(\rho_n)} \right)^{d_2+1} \\
&\le \frac{c_3}{V(x,\Phi^{-1}(t))} \left( \frac{2^n r}{\Phi^{-1}(t)} \right)^{d_2+1} \exp\left( -C_2 \frac{2^n r}{\rho_n} \right) + \frac{c_3 t}{V(x,\rho_n)\psi(\rho_n)} \\
&= \frac{c_3}{V(x,\Phi^{-1}(t))} \left( \frac{2^n r}{\Phi^{-1}(t)} \right)^{d_2+1} \exp\left( -C_2 \frac{2^{n(1-\alpha)} r^{1/N}}{\Phi^{-1}(t)^{1/N}} \right) + \frac{c_3 t}{V(x,\rho_n)\psi(\rho_n)}.
\end{aligned}$$

Using the above estimate and $\mathrm{VD}(d_2)$ we get that

$$\begin{aligned}
\int_{B(x,r)^c} p(t,x,y)\mu(dy) &= \sum_{n=0}^{\infty} \int_{B(x,2^{n+1}r)\setminus B(x,2^n r)} p(t,x,y)\mu(dy) \\
\le c_3 \sum_{n=0}^{\infty} & \frac{V(x,2^{n+1}r)}{V(x,\Phi^{-1}(t))} \left( \frac{2^n r}{\Phi^{-1}(t)} \right)^{d_2+1} \exp\left( -C_2 \frac{2^{n(1-\alpha)} r^{1/N}}{\Phi^{-1}(t)^{1/N}} \right) \\
&+ c_3 \sum_{n=0}^{\infty} \frac{V(x,2^{n+1}r)}{V(x,\rho_n)} \frac{t}{\psi(\rho_n)} =: I_1 + I_2.
\end{aligned}$$

We first estimate $I_1$. Observe that for any $d_0 \ge 1$, there exists $c_1 = c_1(c_0, \alpha) > 0$ such that $2n \le \frac{c_0}{2d_0} 2^{n(1-\alpha)} + c_1$ holds for every $n \ge 0$. Thus, for any $n \ge 0$

and $\kappa \geq 1$,

$$\begin{aligned}
&2^{nd_0} \exp\left(-C_2 2^{n(1-\alpha)}\kappa\right) \leq 2^{-nd_0} \exp\left(2nd_0 - C_2 2^{n(1-\alpha)}\kappa\right) \\
&\leq 2^{-nd_0} \exp\left(\left(\frac{C_2}{2d_0} 2^{n(1-\alpha)} + c_1\right) d_0 - C_2 2^{n(1-\alpha)}\kappa\right) \\
&\leq 2^{-nd_0} \exp\left(\frac{C_2}{2} 2^{n(1-\alpha)}\kappa + c_1 d_0 - C_2 2^{n(1-\alpha)}\kappa\right) = e^{c_1 d_0} 2^{-nd_0} \exp\left(-\frac{C_2}{2}\kappa\right).
\end{aligned} \tag{2.2.37}$$

Using $\Phi^{-1}(t) < r$, $\mathrm{VD}(d_2)$, (2.2.37), and the fact that

$$\sup_{1\leq s} s^{2d_2+1} \exp\left(-\frac{C_2}{4} s^{1/N}\right) := c_4 < \infty,$$

we obtain

$$\begin{aligned}
I_1 &= c_3 \sum_{n=0}^{\infty} \frac{V(x, 2^{n+1}r)}{V(x, \Phi^{-1}(t))} \left(\frac{2^n r}{\Phi^{-1}(t)}\right)^{d_2+1} \exp\left(-C_2 \frac{2^{n(1-\alpha)} r^{1/N}}{\Phi^{-1}(t)^{1/N}}\right) \\
&\leq c_4 \sum_{n=0}^{\infty} \left(\frac{r}{\Phi^{-1}(t)}\right)^{2d_2+1} 2^{n(2d_2+1)} \exp\left(-C_2 \frac{2^{n(1-\alpha)} r^{1/N}}{\Phi^{-1}(t)^{1/N}}\right) \\
&\leq c_5 \exp\left(-\frac{C_2 r^{1/N}}{4\Phi^{-1}(t)^{1/N}}\right).
\end{aligned} \tag{2.2.38}$$

We next estimate $I_2$. Note that by (2.2.6) and $t < \Phi(\rho_n)$, we have $\psi^{-1}(t) \leq \Phi^{-1}(t) \leq \rho_n$. Thus, using $\mathrm{VD}(d_2)$ and $L(\beta_1, C_L, \psi)$ for the first line and using $\alpha(d_2 + \beta_1) > d_2$ for the second line, we obtain

$$\begin{aligned}
&I_2 = c_3 \sum_{n=0}^{\infty} \frac{V(x, 2^{n+1}r)}{V(x, \rho_n)} \frac{\psi(\psi^{-1}(t))}{\psi(\rho_n)} \leq c_6 \sum_{n=0}^{\infty} \left(\frac{2^n r}{\rho_n}\right)^{d_2} \left(\frac{\psi^{-1}(t)}{\rho_n}\right)^{\beta_1} \\
&= c_6 \left(\frac{\Phi^{-1}(t)}{r}\right)^{-\frac{d_2+\beta_1}{N}} \left(\frac{\psi^{-1}(t)}{r}\right)^{\beta_1} \sum_{n=0}^{\infty} 2^{n(d_2 - \alpha(d_2+\beta_1))} \\
&:= c_7 \left(\frac{\Phi^{-1}(t)}{r}\right)^{-\frac{d_2+\beta_1}{N}} \left(\frac{\psi^{-1}(t)}{r}\right)^{\beta_1} \leq c_7 \left(\frac{\psi^{-1}(t)}{r}\right)^{\beta_1 - \frac{d_2+\beta_1}{N}} = c_7 \left(\frac{\psi^{-1}(t)}{r}\right)^{\eta}.
\end{aligned}$$

Thus, by above estimates of $I_1$ and $I_2$, we obtain (2.2.36).

By $\eta < \beta_1$ and (2.2.36), assumptions in Lemma 2.2.23 hold with $f(r,t) := \big(r/\Phi^{-1}(t)\big)^{1/N}$. Thus, by Lemma 2.2.23, we have

$$\begin{aligned} p(t,x,y) \le & \frac{c_8 t}{V(x,d(x,y))\psi(d(x,y))} \\ & + \frac{c_8}{V(x,\Phi^{-1}(t))}\big(1+\frac{d(x,y)}{\Phi^{-1}(t)}\big)^{d_2} \exp\Big[-a_1 k(\frac{d(x,y)}{16k\Phi^{-1}(t)})^{1/N}\Big]. \end{aligned}$$

Using the fact that $\sup_{s>0}(1+s)^{d_2}\exp(-cs^{1/N}) < \infty$ for every $c>0$, we conclude (2.2.35). $\square$

**Lemma 2.2.25.** *Suppose* $\mathrm{VD}(d_2)$*,* $\mathrm{J}_{\psi,\le}$*,* $\mathrm{UHKD}(\Phi)$ *and* $\mathrm{E}_\Phi$*. Then, for any* $\theta > 0$ *and* $c_0, c_1 \ge 1$*, there exists* $c > 0$ *such that for any* $x \in M_0$*,* $t > 0$ *and* $r \ge c_0 \frac{\Phi^{-1}(c_1 t)^{1+\theta}}{\psi^{-1}(c_1 t)^\theta}$*,*

$$\int_{B(x,r)^c} p(t,x,y)\mu(dy) \le c\left(\frac{\psi^{-1}(t)}{r}\right)^{\beta_1}.$$

**Proof.** Denote $t_1 = c_1 t$ and let $a_0, N$ be the constants in Lemma 2.2.24. By (2.2.6) we have that for any $y \in M_0$ with $d(x,y) > r$, there exists $\theta_0 \in (\theta, \infty)$ satisfying $d(x,y) = c_0\Phi^{-1}(t_1)^{1+\theta_0}/\psi^{-1}(t_1)^{\theta_0}$. Note that there exists a positive constant $c_2 = c_2(\theta)$ such that for any $s > 0$,

$$s^{-d_2-\beta_2-\beta_2/\theta} \ge c_2 \exp(-a_0 s^{1/N}). \tag{2.2.39}$$

Also, since $c_0 \ge 1$ we have

$$\psi^{-1}(t_1) \le c_0\psi^{-1}(t_1) \le c_0\Phi^{-1}(t_1) < d(x,y) = c_0\Phi^{-1}(t_1)^{1+\theta_0}/\psi^{-1}(t_1)^{\theta_0}.$$

Thus, using $\mathrm{VD}(d_2)$ and $U(\beta_2, C_U, \psi)$ for the first inequality and (2.2.39) for the second, we have

$$\begin{aligned}
\frac{t}{V(x,d(x,y))\psi(d(x,y))} &= \frac{c_1^{-1}}{V(x,c_0\Phi^{-1}(t_1))}\frac{V(x,c_0\Phi^{-1}(t_1))}{V(x,d(x,y))}\frac{\psi(\psi^{-1}(t_1))}{\psi(d(x,y))}\\
&\geq \frac{c_1^{-1}}{V(x,c_0\Phi^{-1}(t_1))}C_\mu^{-1}\big(\frac{c_0\Phi^{-1}(t_1)}{d(x,y)}\big)^{d_2}C_U^{-1}\big(\frac{\psi^{-1}(t_1)}{d(x,y)}\big)^{\beta_2}\\
&= \frac{c_1^{-1}c_0^{-\beta_2}C_\mu^{-1}C_U^{-1}}{V(x,c_0\Phi^{-1}(t_1))}\left(\frac{\psi^{-1}(t_1)}{\Phi^{-1}(t_1)}\right)^{d_2\theta_0}\left(\frac{\psi^{-1}(t_1)}{\Phi^{-1}(t_1)}\right)^{(1+\theta_0)\beta_2}\\
&= \frac{c_1^{-1}c_0^{-\beta_2}C_U^{-1}C_\mu^{-1}}{V(x,c_0\Phi^{-1}(t_1))}\left(\left(\frac{\Phi^{-1}(t_1)}{\psi^{-1}(t_1)}\right)^{\theta_0}\right)^{-d_2-\beta_2-\beta_2/\theta_0}\\
&\geq \frac{c_2c_1^{-1}c_0^{-\beta_2}C_U^{-1}C_\mu^{-1}c_U^{-1}}{V(x,c_0\Phi^{-1}(t_1))}\exp\left(-\frac{a_0\Phi^{-1}(t_1)^{\theta_0/N}}{\psi^{-1}(t_1)^{\theta_0/N}}\right)\\
&= \frac{c_2c_1^{-1}c_0^{-\beta_2}C_U^{-1}C_\mu^{-1}c_U^{-1}}{V(x,c_0\Phi^{-1}(t_1))}\exp\left(-\frac{a_0 d(x,y)^{1/N}}{\Phi^{-1}(t_1)^{1/N}}\right).
\end{aligned}$$

Applying Lemma 1.1.4 for $L(\alpha_1, c_L, \Phi)$, we have $U(1/\alpha_1, c_L^{-1/\alpha_1}, \Phi^{-1})$, which yields $\Phi^{-1}(t) \leq \Phi^{-1}(t_1) \leq c_L^{-1/\alpha_1}c_1^{1/\alpha_1}\Phi^{-1}(t)$. Thus, using this and VD($d_2$) again, we have

$$\frac{1}{V(x,c_0\Phi^{-1}(t_1))} \geq C_\mu^{-1}(c_0c_L^{-1/\alpha_1}c_1^{1/\alpha_1})^{d_2}\frac{1}{V(x,\Phi^{-1}(t))}$$

Thus, by Lemma 2.2.24 and above two estimates, we have that for every $y \in M_0$ with $d(x,y) > r$,

$$\begin{aligned}
p(t,x,y) &\leq \frac{c\,t}{V(x,d(x,y))\psi(d(x,y))} + \frac{c}{V(x,\Phi^{-1}(t))}e^{-a_0\frac{d(x,y)^{1/N}}{\Phi^{-1}(t)^{1/N}}}\\
&\leq \frac{c_2\,t}{V(x,d(x,y))\psi(d(x,y))}.
\end{aligned}$$

Using this, [32, Lemma 2.1] and $L(\beta_1, C_L, \psi)$ with the fact that $r > c_0\psi^{-1}(c_1t)$ which follows from (2.2.6), we conclude that

$$\int_{B(x,r)^c} p(t,x,y)\mu(dy) \leq c_3\left(\frac{\psi^{-1}(t)}{r}\right)^{\beta_1}.$$

This proves the lemma. □

Now we are ready to prove the first main result in this section.

*Proof of Theorem 2.2.11.* Note that under the condition $L^a(\delta, \widetilde{C}_L, \Phi)$, we have $\alpha_2 \geq \delta \vee \alpha_1$. Take

$$\theta := \frac{(\delta-1)\beta_1}{\delta(2d_2+\beta_1)+(\beta_1+2\alpha_2+2d_2\alpha_2)} \in (0, \delta-1)$$

and $C_0 = \frac{4c_U}{C_2}$, where $C_1$ and $C_2$ are the constants in Lemma 2.2.22. Without loss of generality, we may and do assume that $C_1 \geq 2$ and $C_2 \leq 1$. Let $\alpha$ be a number in $(\frac{d_2}{d_2+\beta_1}, 1)$.

(i) We will show that there exist $a_1 > 0$ and $c_1 > 0$ such that for any $t \leq T$, $x \in M_0$ and $r > 0$,

$$\int_{B(x,r)^c} p(t,x,y)\,\mu(dy) \leq c_1 \left(\frac{\psi^{-1}(t)}{r}\right)^{\beta_1/2} + c_1 \exp\left(-a_1 \Phi_1(r,t)\right). \quad (2.2.40)$$

Firstly, since $\Phi_1(r,t)$ is increasing on $r$, by (2.2.21) and (2.2.22) we have that for $r \leq C_0\Phi^{-1}(C_1 t)$,

$$\Phi_1(r,t) \leq \Phi_1(C_0\Phi^{-1}(C_1t), t) \leq c_2\Phi_1(C_0\Phi^{-1}(C_1t), C_1t) \leq c_3.$$

Here for the second inequality, $C_0 \geq 2c_U$ yields the condition in (2.2.21). Thus, for any $x \in M_0$ and $r \leq C_0\Phi^{-1}(C_1t)$ we have

$$\int_{B(x,r)^c} p(t,x,y)\mu(dy) \leq 1 \leq e^{a_1c_3}\exp\left(-a_1\Phi_1(r,t)\right).$$

Also, when $r > C_0\Phi^{-1}(C_1t)^{1+\theta}/\psi^{-1}(C_1t)^{\theta}$, (2.2.40) immediately follows from Lemma 2.2.25 and the fact that $r > \psi^{-1}(t)$.

Now consider the case $C_0\Phi^{-1}(C_1t) < r \leq C_0\Phi^{-1}(C_1t)^{1+\theta}/\psi^{-1}(C_1t)^{\theta}$. In this case, there exists $\theta_0 \in (0, \theta]$ such that $r = C_0\Phi^{-1}(C_1t)^{1+\theta_0}/\psi^{-1}(C_1t)^{\theta_0}$ by (2.2.6). Since $C_0 = \frac{4c_U}{C_2}$, applying Lemma 2.2.19 with the constant $C_1T$

we have $\rho \in [b(C_2r/2)^{-\delta_1}\Phi^{-1}(C_1t)^{\delta_1+1}, 2\Phi^{-1}(C_1t)]$ such that

$$\Phi_1(C_2r/2, C_1t) - \frac{C_0C_2}{8} \le \frac{C_2r}{2\rho} - \frac{C_1t}{\Phi(\rho)} \le \Phi_1(C_2r/2, C_1t),$$

where $\delta_1 = \frac{1}{\delta-1}$. Also, let $\rho_n = 2^{n\alpha}\rho$ for $n \in \mathbb{N}_0$. Then, we have

$$\frac{C_2 2^n r}{\rho_n} = \frac{C_2r}{2\rho} + \frac{C_2r}{\rho}(2^{n(1-\alpha)} - \frac{1}{2}) \ge \frac{C_2r}{2\rho} + \frac{C_2r}{4\Phi^{-1}(C_1t)}2^{n(1-\alpha)}.$$

Using this, $r > C_0\Phi^{-1}(C_1t)$ and (2.2.21) with $U(1/\alpha_1, c_L^{-1/\alpha_1}, \Phi^{-1})$, which follows from $L(\alpha_1, c_L, \Phi)$ and Lemma 1.1.4, yield that for any $n \in \mathbb{N}_0$,

$$\begin{aligned}
\frac{C_1t}{\Phi(\rho_n)} - \frac{C_2 2^n r}{\rho_n} &\le \frac{C_1t}{\Phi(\rho)} - \frac{C_2r}{2\rho} - \frac{C_2r}{4\Phi^{-1}(C_1t)}2^{n(1-\alpha)} \\
&\le -\Phi_1(C_2r/2, C_1t) + \frac{C_0C_2}{8} - \frac{C_2r}{4\Phi^{-1}(C_1t)}2^{n(1-\alpha)} \\
&\le -\Phi_1(C_2r/2, C_1t) - \frac{C_2r}{8\Phi^{-1}(C_1t)}2^{n(1-\alpha)} \qquad (2.2.41) \\
&\le -\Phi_1(C_2r/2, C_1t) - \frac{C_2}{8}(c_L/C_1)^{1/\alpha_1}2^{n(1-\alpha)}\frac{r}{\Phi^{-1}(t)} \\
&\le -c_4\Phi_1(r,t) - c_5 2^{n(1-\alpha)}\frac{r}{\Phi^{-1}(t)}.
\end{aligned}$$

Combining (2.2.41) and Lemma 2.2.22 with $\rho = \rho_n$, we have that for $t \in (0, T]$ and $y \in B(x, 2^{n+1}r)/B(x, 2^n r)$,

$$\begin{aligned}
&p(t,x,y) - \frac{c_6 t}{V(x,\rho_n)\psi(\rho_n)}\left(1 + \frac{t}{\Phi(\rho_n)}\right)^{d_2+1} \\
&\le \frac{c_7}{V(x,\Phi^{-1}(t))}\left(1 + \frac{2^{n+1}r}{\Phi^{-1}(t)}\right)^{d_2} \exp\left(C_1\frac{t}{\Phi(\rho_n)} - C_2\frac{2^n r}{\rho_n}\right) \\
&\le \frac{c_7}{V(x,\Phi^{-1}(t))}\left(1 + \frac{2^{n+1}r}{\Phi^{-1}(t)}\right)^{d_2} \exp\left(-c_4\Phi_1(r,t) - c_5\frac{2^{n(1-\alpha)}r}{\Phi^{-1}(t)}\right) \\
&\le \frac{c_7}{V(x,\Phi^{-1}(t))} \exp\left(-c_4\Phi_1(r,t) - \frac{c_5}{2}\frac{2^{n(1-\alpha)}r}{\Phi^{-1}(t)}\right), \qquad (2.2.42)
\end{aligned}$$

where for the last inequality we have used the fact that $\frac{r}{\Phi^{-1}(t)} > 1$ and

$$\sup_{n\in\mathbb{N}}\sup_{s>1}(1+2^{n+1}s)^{d_2}\exp\big[-\frac{c_5}{2}2^{n(1-\alpha)}s\big]<\infty.$$

With estimates in (2.2.42), we get that

$$\begin{aligned}\int_{B(x,r)^c}p(t,x,y)\mu(dy) &\le \sum_{n=0}^{\infty}\int_{B(x,2^{n+1}r)\setminus B(x,2^n r)}p(t,x,y)\mu(dy)\\ &\le c_8\sum_{n=0}^{\infty}\frac{V(x,2^n r)}{V(x,\Phi^{-1}(t))}\exp\left(-c_4\Phi_1(r,t)-\frac{c_5}{2}\frac{2^{n(1-\alpha)}r}{\Phi^{-1}(t)}\right)\\ &\quad+c_8\sum_{n=0}^{\infty}\frac{tV(x,2^n r)}{V(x,\rho_n)\psi(\rho_n)}\left(1+\frac{t}{\Phi(\rho_n)}\right)^{d_2+1}\\ &:= c_8(I_1+I_2).\end{aligned}$$

Using $r > C_0\Phi^{-1}(C_1 t) \ge \Phi^{-1}(t)$, we obtain upper bound of $I_1$ by following the calculations in (2.2.38). Thus, we have

$$I_1 \le c_9\exp\left(-a_2\Phi_1(r,t)\right).$$

Next, we estimate $I_2$. Since $r = C_0\Phi^{-1}(C_1t)^{1+\theta_0}/\psi^{-1}(C_1t)^{\theta_0}$, $\psi^{-1}(t) < \Phi^{-1}(t)$ and $\theta_0 \le \theta < 1/\delta_1$, we obtain

$$\frac{\Phi^{-1}(C_1t)}{\rho}\le\frac{\Phi^{-1}(C_1t)b^{-1}(C_2r/2)^{\delta_1}}{\Phi^{-1}(C_1t)^{\delta_1+1}}\le b^{-1}(C_0C_2/2)^{\delta_1}\frac{\Phi^{-1}(C_1t)}{\psi^{-1}(C_1t)}.$$

Thus, $\frac{b\psi^{-1}(C_1t)}{(C_0C_2/2)^{\delta_1}} \le \rho \le r$. Using this, VD($d_2$), (1.1.1), $L(\beta_1, C_L, \psi)$ and

$U(\alpha_2, c_U, \Phi)$ we have

$$
\begin{aligned}
I_2 &= \sum_{n=0}^{\infty} \frac{V(x,2^n r)}{V(x,\rho_n)} \frac{t}{\psi(\rho_n)} \left(1+\frac{t}{\Phi(\rho_n)}\right)^{d_2+1} \\
&\le c_{10} \sum_{n=0}^{\infty} \frac{V(x,2^n r)}{V(x,\rho_n)} \frac{C_1 t}{\psi(b^{-1}(C_0C_2/2)^{\delta_1}\rho_n)} \frac{\psi(b^{-1}(C_0C_2/2)^{\delta_1}\rho_n)}{\psi(\rho_n)} \left(\frac{C_1 t}{\Phi(\rho/2)}\right)^{d_2+1} \\
&\le c_{11} \sum_{n=0}^{\infty} \big(\frac{2^n r}{\rho_n}\big)^{d_2} \big(\frac{\psi^{-1}(C_1t)}{\rho_n}\big)^{\beta_1} \big(\frac{\Phi^{-1}(C_1t)}{\rho/2}\big)^{\alpha_2(d_2+1)} \\
&\le c_{12} \sum_{n=0}^{\infty} 2^{n(d_2-\alpha(d_2+\beta_1))} \big(\frac{r}{\rho}\big)^{d_2} \big(\frac{\psi^{-1}(C_1t)}{\rho}\big)^{\beta_1} \big(\frac{\Phi^{-1}(C_1t)}{\rho}\big)^{\alpha_2(d_2+1)} \\
&= c_{13} r^{d_2} \psi^{-1}(C_1t)^{\beta_1} \Phi^{-1}(C_1t)^{\alpha_2(d_2+1)} \rho^{-d_2-\beta_1-\alpha_2(d_2+1)}.
\end{aligned}
$$

Since $br^{-\delta_1}\Phi^{-1}(C_1t)^{\delta_1+1} \le \rho$, we conclude that

$$
I_2 \le c_{13} b^{-d_2-\beta_1-\alpha_2(d_2+1)} \big(\frac{\psi^{-1}(C_1t)}{r}\big)^{\beta_1} \big(\frac{\Phi^{-1}(C_1t)}{r}\big)^{-\delta_1(d_2\alpha_2+\alpha_2+\beta_1+d_2)-(d_2+\beta_1)}. \tag{2.2.43}
$$

Using $r = C_0\Phi^{-1}(C_1t)^{1+\theta_0}/\psi^{-1}(C_1t)^{\theta_0}$, we have $C_0\psi^{-1}(C_1t) < C_0\Phi^{-1}(C_1t) < r$. Since $\theta_0 \le \theta$, we have

$$
\frac{C_0\Phi^{-1}(C_1t)}{r} = \left(C_0 \frac{\psi^{-1}(C_1t)}{r}\right)^{\theta_0/(1+\theta_0)} \ge \left(C_0 \frac{\psi^{-1}(C_1t)}{r}\right)^{\theta/(1+\theta)}.
$$

By using $\theta = \frac{(\delta-1)\beta_1}{\delta(2d_2+\beta_1)+(\beta_1+2\alpha_2+2d_2\alpha_2)} > 0$, we have

$$
\begin{aligned}
&\left(\frac{\Phi^{-1}(C_1t)}{r}\right)^{-\delta_1(d_2\alpha_2+\alpha_2+\beta_1+d_2)-(d_2+\beta_1)} \\
&\le c_{14} \left(\frac{\psi^{-1}(C_1t)}{r}\right)^{\frac{\theta}{1+\theta}\left[-\delta_1(d_2\alpha_2+\alpha_2+\beta_1+d_2)-(d_2+\beta_1)\right]} = c_{14} \left(\frac{\psi^{-1}(C_1t)}{r}\right)^{-\beta_1/2}.
\end{aligned}
$$

Therefore, using (2.2.43) we obtain

$$
I_2 \le c_{15} \left(\frac{\psi^{-1}(t)}{r}\right)^{\beta_1/2}.
$$

By the estimates of $I_1$ and $I_2$, we arrive

$$\int_{B(x,r)^c} p(t,x,y)dy \le c_7(I_1+I_2) \le c_9\left(\frac{\psi^{-1}(t)}{r}\right)^{\beta_1/2} + c_{15}\exp\big(-a_2\Phi_1(r,t)\big).$$

Combining all the cases, we obtain (2.2.40). Thus the assertions on Lemma 2.2.23 holds with $f(r,t) := \Phi_1(r,t)$. Thus, using Lemma 2.2.23 we have constants $k, c_0 > 0$ such that

$$p(t,x,y) \le \frac{c_0\, t}{V(x,d(x,y))\psi(d(x,y))} + \frac{c_0 e^{-a_2 k\Phi_1(d(x,y)/(16k),t)}}{V(x,\Phi^{-1}(t))}\left(1+\frac{d(x,y)}{\Phi^{-1}(t)}\right)^{d_2} \tag{2.2.44}$$

for all $t \in (0,T]$ and $x,y \in M_0$. Recall that $2c_U > 0$ is the constant in Lemma 2.2.20 with $\phi = \Phi$. When $d(x,y) \le 32c_U k\Phi^{-1}(t)$, using UHKD($\Phi$) and (2.2.1) we have

$$p(t,x,y) \le p(t,x,x)^{1/2}p(t,y,y)^{1/2} \le \frac{c_{16}}{V(x,\Phi^{-1}(t))}.$$

Thus, by (2.2.22) and $d(x,y)/16k \le 2c_U\Phi^{-1}(t)$ we have

$$p(t,x,y) \le \frac{c_{16}}{V(x,\Phi^{-1}(t))} \le \frac{c_{16}e^{a_2c_{17}k}}{V(x,\Phi^{-1}(t))}\exp\big(-a_2k\Phi_1(d(x,y)/(16k),t)\big),$$

which yields (2.2.13) for the case $d(x,y) \le 32c_U k\Phi^{-1}(t)$. Also, for $r > 32c_U k\Phi^{-1}(t)$ with $0 < t \le T$, using (2.2.41) with $n = 0$ and (2.2.21) we have

$$c_4\Phi_1(r,t) + c_5\frac{r}{\Phi^{-1}(t)} \le \Phi_1(C_2r, C_1t) \le c_{17}\Phi_1(r/16k,t).$$

Therefore, using (2.2.44) we obtain

$$p(t,x,y) \le \frac{c_0\, t}{V(x,d(x,y))\psi(d(x,y))} + \frac{c_{19}}{V(x,\Phi^{-1}(t))}\exp\big(-a_3\Phi_1(d(x,y),t)\big),$$

where we have used $\sup_{s>0}(1+s)^{d_2}\exp(-c_{18}s) < \infty$ for the last line. Combining two cases, we obtain (2.2.13).

(ii) Again we will show that there exist $a_1 > 0$ and $c_1 > 0$ such that for any

$t \geq T$, $x \in M_0$ and $r > 0$,

$$\int_{B(x,r)^c} p(t,x,y)\,\mu(dy) \leq c_1 \left( \frac{\psi^{-1}(t)}{r} \right)^{\beta_1/2} + c_1 \exp\left( -a_1 \widetilde{\Phi}_1(r,t) \right). \quad (2.2.45)$$

Note that using (2.2.11), the proof of (2.2.45) for the case $r \leq C_0 \Phi^{-1}(C_1 t)$ and $r > C_0 \frac{\Phi^{-1}(C_1 t)^{1+\theta}}{\psi^{-1}(C_1 t)^{\theta}}$ are the same as that for (i).

Without loss of generality we may assume $a = \Phi(T)$. Then for $t \geq T$, we have $\Phi^{-1}(t) = \widetilde{\Phi}^{-1}(t)$. Applying this and (2.2.11) for Lemma 2.2.22, we have for any $t \geq T$,

$$\begin{aligned} p(t,x,y) &\leq \frac{c_1 e^{C_1 \frac{t}{\Phi(\rho)} - C_2 \frac{d(x,y)}{\rho}}}{V(x, \Phi^{-1}(t))} \big(1 + \frac{d(x,y)}{\Phi^{-1}(t)}\big)^{d_2} + \frac{c_1 t}{V(x,\rho)\psi(\rho)} \big(1 + \frac{t}{\Phi(\rho)}\big)^{d_2+1} \\ &\leq \frac{c_1 e^{C_1 \frac{t}{\widetilde{\Phi}(\rho)} - C_2 \frac{d(x,y)}{\rho}}}{V(x, \widetilde{\Phi}^{-1}(t))} \big(1 + \frac{d(x,y)}{\widetilde{\Phi}^{-1}(t)}\big)^{d_2} + \frac{c_1 t}{V(x,\rho)\psi(\rho)} \big(1 + \frac{t}{\widetilde{\Phi}(\rho)}\big)^{d_2+1}. \end{aligned}$$

Since $L(\delta, \widetilde{C}_L, \widetilde{\Phi})$ holds, following the proof of (i) we have for any $t > 0$ and $r > 0$,

$$\int_{B(x,r)^c} p(t,x,y)\mu(dy) \leq c_1 \big( \frac{\psi^{-1}(t)}{r} \big)^{\beta_1/2} + c_1 \exp(-a_1 \widetilde{\Phi}_1(r,t)).$$

Since the assumptions in Lemma 2.2.23 follows from (2.2.22) and the fact that $\Phi^{-1}(t) = \widetilde{\Phi}^{-1}(t)$ for $t \geq T$, we obtain that for any $t \geq T$ and $x, y \in M$,

$$p(t,x,y) \leq \frac{c_0 t}{V(x, d(x,y))\psi(d(x,y))} + \frac{c_0 e^{-a_U \widetilde{\Phi}_1(d(x,y),t)}}{V(x, \Phi^{-1}(t))} \big(1 + \frac{d(x,y)}{\Phi^{-1}(t)}\big)^{d_2}.$$

Here in the last term we have used (2.2.21). With the aid of $L(\delta, \widetilde{C}_L, \widetilde{\Phi})$, The remainder is same as the proof of (i). □

Now we give the proof of Theorem 2.2.12 and Corollary 2.2.13.

*Proof of Theorem 2.2.12.* Since $\mathrm{J}_{\psi,\leq}$ implies $\mathrm{J}_{\Phi,\leq}$, [32, Theorem 1.15] yields that (2) implies (3), and (3) implies the conservativeness of $(\mathcal{E}, \mathcal{F})$. Thus, by Theorem 2.2.11, (3) implies (1). It remains to prove that (1) implies (2). By

(2.2.6) and Remark 2.2.9, UHK$(\Phi, \psi)$ implies UHK$(\Phi)$. Also, following the proof of [32, Proposition 3.3], we easily prove that UHK$(\Phi, \psi)$ also implies $\mathrm{J}_{\psi,\leq}$. □

*Proof of Corollary 2.2.13.* By (2.2.6), $\mathrm{J}_{\psi,\leq}$ implies $\mathrm{J}_{\Phi,\leq}$. Thus, [32, Theorem 1.15] implies the equivalence between the condition in Theorem 2.2.12(3) and Corollary 2.2.13(4) and (5). We now prove the equivalence between the condition in Theorem 2.2.12(3) and Corollary 2.2.13(6). To do this, we will use the results in [32, 45].

Suppose that $\mathrm{J}_{\psi,\leq}$, UHKD$(\Phi)$ and $\mathrm{E}_\Phi$ hold. By [32, Proposition 7.6], we have FK$(\Phi)$. Since we have $\mathrm{E}_\Phi$, the condition $\mathrm{EP}_{\Phi,\leq,\varepsilon}$ in [32, Definition 1.10] holds by [32, Lemma 4.16]. Since $\mathrm{EP}_{\Phi,\leq,\varepsilon}$ implies the condition $(S)$ in [45, Definition 2.7] with $r < \infty$ and $t < \delta\Phi(r)$, we can follow the proof of [45, Lemma 2.8] line by line (replace $r^\beta$ to $\Phi(r)$) and obtain Gcap$(\Phi)$.

Now, suppose that FK$(\Phi)$, $\mathrm{J}_{\psi,\leq}$ and Gcap$(\Phi)$ hold. Then, by [32, Lemma 4.14], we have $\mathrm{E}_{\Phi,\leq}$. To obtain $\mathrm{E}_{\Phi,\geq}$, we first show that [32, Lemma 4.15] holds under our conditions. i.e., by using Gcap$(\Phi)$ instead of CSJ$(\Phi)$, we derive the same result in [32, Lemma 4.15]. To show [32, Lemma 4.15], we give the main steps of the proof only. Recall that for any $\rho > 0$, $(\mathcal{E}^\rho, \mathcal{F})$ is $\rho$-truncated Dirichlet form. For $\rho$-truncated Dirichlet form, we say $\mathrm{AB}^\rho_\zeta(\Phi)$ holds if the inequality [45, (2.1)] holds with $R' < \infty$, $\Phi(r\wedge\rho)$ and $J(x,y)\mathbf{1}_{\{d(x,y)<\rho\}}$ instead of $R' < \overline{R}$, $r^\beta$ and $j$ respectively. Then, by VD$(d_2)$, $\mathrm{J}_{\psi,\leq}$, [32, Lemma 2.1] and (2.2.6), we can follow the proof of [45, Lemma 2.4] line by line (replace $r^\beta$ to $\Phi(r)$) to obtain $\mathrm{AB}_\zeta(\Phi)$. To get $\mathrm{AB}_{1/8}(\Phi)$, we use the proof of [45, Lemma 2.9]. Here, we take different $r_n, s_n, b_n, a_n$ from the one in the proof of [45, Lemma 2.9]. Let $\lambda > 0$ be a constant which will be chosen later. Take $s_n = cre^{-n\lambda/2\alpha_2}$ for $n \geq 1$, where $c = c(\lambda)$ is chosen so that $\sum_{n=1}^\infty s_n = r$ and $\alpha_2$ is the upper scaling index of $\Phi$. Let $r_n = \sum_{k=1}^n s_k$ for $n \geq 1$ and $r_0 = 0$. We also take $b_n = e^{-n\lambda}$ for $n \geq 0$ and $a_n = b_{n-1} - b_n$ for $n \geq 1$. (c.f. [32, Proposition 2.4].) With these $r_n, s_n, b_n, a_n$, we can follow the proof of [45, Lemma 2.9] line by line and obtain $\mathrm{AB}_{1/8}(\Phi)$ by choosing small $\lambda > 0$. Moreover, using the argument in the proof of [32, Proposition 2.3], we also obtain $\mathrm{AB}^{\rho}_{1/8}(\Phi)$ which yields [32, (4.8)] for $\rho$-truncated Dirichlet form. Thus, we get [32, Corollary 4.12]. For open subsets $A, B$ of $M$ with $A \subset B$, and for any $\rho > 0$, define $\mathrm{Cap}^{\rho}(A,B) = \inf\{\mathcal{E}^{\rho}(\varphi,\varphi) : \varphi \in \mathrm{cutoff}(A,B)\}$. By $\mathrm{Gcap}(\Phi)$ with $u = 1$(c.f. [45, Definition 1.13] and below), we have

$$\mathrm{Cap}^{\rho}(B(x,R), B(x,R+r)) \leq \mathrm{Cap}(B(x,R), B(x,R+r)) \leq c\frac{V(x,R+r)}{\Phi(r\wedge\rho)},$$

which implies the inequalities in [32, Proposition 2.3(5)]. Having this and [32, Corollary 4.12] at hand, we can follow the proof and get the result of [32, Lemma 4.15]. Now $\mathrm{E}_{\Phi,\geq}$ follows from the proof of [32, Lemma 4.17]. Since we have $\mathrm{E}_{\Phi}$, $\mathrm{UHKD}(\Phi)$ holds by [32, Theorem 4.25]. □

### 2.2.4 Stability of heat kernel estimates

In this section, we prove Theorems 2.2.14 and 2.2.15. Throughout this subsection, we will assume that the metric measure space $(M, d, \mu)$ satisfies $\mathrm{VD}(d_2)$ and $\mathrm{RVD}(d_1)$, and the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ and the corresponding Hunt process satisfy $\mathrm{J}_{\psi}$, $\mathrm{PI}(\Phi)$ and $\mathrm{E}_{\Phi}$, where $\psi$ is non-decreasing function satisfying $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$, and $\Phi$ is non-creasing function satisfying (2.2.6), $L(\alpha_1, c_L)$ and $U(\alpha_2, c_U)$.

From $\mathrm{J}_{\psi}$ and $\mathrm{VD}(d_2)$, we immediately see that there is a constant $c > 0$ such that for all $x, y \in M_0$ with $x \neq y$,

$$J(x,y) \leq \frac{c}{V(x,r)} \int_{B(x,r)} J(z,y)\,\mu(dz) \quad \text{for every } 0 < r \leq d(x,y)/2. \quad (\mathrm{UJS})$$

(See [28, Lemma 2.1]).

For any open set $D \subset M$, let $\mathcal{F}_D := \{u \in \mathcal{F} : u = 0 \text{ q.e. in } D^c\}$. Then, $(\mathcal{E}, \mathcal{F}_D)$ is also a regular Dirichlet form. We use $p^D(t,x,y)$ to denote the transition density function corresponding to $(\mathcal{E}, \mathcal{F}_D)$.

Note that $(\mathcal{E}, \mathcal{F})$ is a conservative Dirichlet form by [32, Lemma 4.21]. Thus, by [32, Theorem 1.15], we see that $\mathrm{CSJ}(\Phi)$ defined in [32] holds. Thus, by $\mathrm{J}_{\psi,\le}$, PI($\Phi$), CSJ($\Phi$) and (UJS) with (2.2.6), we have (7) in [33, Theorem 1.20].

Therefore, by [33, Theorem 1.20], UHK($\Phi$) and the following joint Hölder regularity hold for parabolic functions. We refer [33, Definition 1.13] for the definition of parabolic functions. Note that, by a standard argument, we now can take the continuous version of parabolic functions (for example, see [45, Lemma 5.12]). Let $Q(t,x,r,R) := (t,t+r) \times B(x,R)$.

**Theorem 2.2.26.** *There exist constants $c > 0$, $0 < \theta < 1$ and $0 < \epsilon < 1$ such that for all $x_0 \in M$, $t_0 \ge 0$, $r > 0$ and for every bounded measurable function $u = u(t,x)$ that is parabolic in $Q(t_0, x_0, \Phi(r), r)$, the following parabolic Hölder regularity holds:*

$$|u(s,x) - u(t,y)| \le c \left( \frac{\Phi^{-1}(|s-t|) + d(x,y)}{r} \right)^{\theta} \sup_{[t_0, t_0+\Phi(r)]\times M} |u|$$

*for every $s,t \in (t_0, t_0 + \Phi(\epsilon r))$ and $x,y \in B(x_0, \epsilon r)$.*

Since $p^D(t,x,y)$ is parabolic, from now on, we assume $\mathcal{N} = \emptyset$ and take the joint continuous versions of $p(t,x,y)$ and $p^D(t,x,y)$. (c.f., [45, Lemma 5.13]).)

Again, by [33, Theorem 1.20] we have the interior near-diagonal lower bound of $p^B(t,x,y)$ and parabolic Harnack inequality.

**Theorem 2.2.27.** *There exist $\varepsilon \in (0,1)$ and $c_1 > 0$ such that for any $x_0 \in M$, $r > 0$, $0 < t \le \Phi(\varepsilon r)$ and $B = B(x_0, r)$,*

$$p^B(t,x,y) \ge \frac{c_1}{V(x_0, \Phi^{-1}(t))}, \quad x,y \in B(x_0, \varepsilon\Phi^{-1}(t)).$$

**Proposition 2.2.28.** *Suppose* VD($d_2$), RVD($d_1$), $\mathrm{J}_\psi$, PI($\Phi$) *and* $\mathrm{E}_\Phi$. *Then, there exists $\eta > 0$ and $C_3 > 0$ such that for any $t > 0$,*

$$p(t,x,y) \ge C_3 V(x, \Phi^{-1}(t))^{-1}, \quad x,y \in M \text{ with } d(x,y) \le \eta\Phi^{-1}(t), \quad (2.2.46)$$

*and*

$$p(t,x,y) \geq \frac{C_3 t}{V(x,d(x,y))\psi(d(x,y))}, \quad x,y \in M \textit{ with } d(x,y) \geq \eta\Phi^{-1}(t).$$

**Proof.** The proof of the proposition is standard. For example, see [32, Proposition 5.4].

Let $\eta = \varepsilon/2 < 1/2$ where $\varepsilon$ is the constant in Theorem 2.2.27. Then by Theorem 2.2.27,

$$p(t,x,y) \geq p^{B(x,\Phi^{-1}(t)/\varepsilon)}(t,x,y) \geq \frac{c_0}{V(x,\Phi^{-1}(t))} \qquad \text{for all } d(x,y) \leq \eta\Phi^{-1}(t). \tag{2.2.47}$$

Note that in the beginning of this section we have mentioned that UHK($\Phi$) holds under $\mathrm{J}_{\psi,\leq}$, PI($\Phi$) and $\mathrm{E}_\Phi$. Thus, by [32, Lemma 2.7] and UHK($\Phi$), we have

$$\mathbb{P}^x(\tau_{B(x,r)} \leq t) \leq \frac{c_1 t}{\Phi(r)}, \quad r>0,\ t>0,\ x \in M.$$

Let $\eta_1 := (C_L/2)^{1/\beta_1}\eta \in (0,\eta)$ so that $\eta\Phi^{-1}((1-b)t) \geq \eta_1\Phi^{-1}(t)$ holds for all $b \in (0,1/2]$. Then choose $\lambda \leq c_1^{-1}C_U^{-1}(2\eta_1/3)^{\beta_2}/2 < 1/2$ small enough so that $\frac{c_1\lambda t}{\Phi(2\eta_1\Phi^{-1}(t)/3)} \leq \lambda c_1 C_U (2\eta_1/3)^{-\beta_2} \leq 1/2$. Thus we have $\lambda \in (0,1/2)$ and $\eta_1 \in (0,\eta)$ (independent of $t$) such that

$$\eta\Phi^{-1}((1-\lambda)t) \geq \eta_1\Phi^{-1}(t), \quad \text{for all } t>0, \tag{2.2.48}$$

and

$$\mathbb{P}^x(\tau_{B(x,2\eta_1\Phi^{-1}(t)/3)} \leq \lambda t) \leq 1/2, \quad \text{for all } t>0 \text{ and } x \in M. \tag{2.2.49}$$

For the remainder of the proof we assume that $d(x,y) \geq \eta\Phi^{-1}(t)$. Since, using (2.2.47) and (2.2.48),

$$\begin{aligned}
p(t,x,y) &\geq \int_{B(y,\eta\Phi^{-1}((1-\lambda)t))} p(\lambda t,x,z)p((1-\lambda)t,z,y)\mu(dz)\\
&\geq \inf_{z\in B(y,\eta\Phi^{-1}((1-\lambda)t))} p((1-\lambda)t,z,y)\int_{B(y,\eta\Phi^{-1}((1-\lambda)t))} p(\lambda t,x,z)\mu(dz)\\
&\geq \frac{c_0}{V(y,\Phi^{-1}(t))}\mathbb{P}^x(X_{\lambda t}\in B(y,\eta_1\Phi^{-1}(t))),
\end{aligned}$$

it suffices to prove

$$\mathbb{P}^x(X_{\lambda t}\in B(y,\eta_1\Phi^{-1}(t)))\geq c_2\frac{tV(y,\Phi^{-1}(t))}{V(x,d(x,y))\psi(d(x,y))}. \tag{2.2.50}$$

For $A\subset M$, let $\sigma_A := \inf\{t>0 : X_t\in A\}$. Using (2.2.49) and the strong Markov property we have

$$\begin{aligned}
&\mathbb{P}^x(X_{\lambda t}\in B(y,\eta_1\Phi^{-1}(t)))\\
&\geq \mathbb{P}^x(\sigma_{B(y,\eta_1\Phi^{-1}(t)/3)}\leq\lambda t)\inf_{z\in B(y,\eta_1\Phi^{-1}(t)/3)}\mathbb{P}^z(\tau_{B(z,2\eta_1\Phi^{-1}(t)/3)}>\lambda t)\\
&\geq \frac{1}{2}\mathbb{P}^x(\sigma_{B(y,\eta_1\Phi^{-1}(t)/3)}\leq\lambda t)\\
&\geq \frac{1}{2}\mathbb{P}^x\Big(X_{(\lambda t)\wedge\tau_{B(x,2\eta_1\Phi^{-1}(t)/3)}}\in B(y,\eta_1\Phi^{-1}(t)/3)\Big).
\end{aligned}$$

Since $d(x,y)>\eta_1\Phi^{-1}(t)$, clearly $B(y,\eta_1\Phi^{-1}(t)/3)\subset\overline{B(x,2\eta_1\Phi^{-1}(t)/3)}^c$.

Thus by (2.2.5), Lévy system and (2.2.49), we have

$$
\begin{aligned}
&\mathbb{P}^x\Big(X_{(\lambda t)\wedge\tau_{B(x,2\eta_1\Phi^{-1}(t)/3)}} \in B(y,\eta_1\Phi^{-1}(t)/3)\Big)\\
&\quad = \mathbb{E}^x\left[\sum_{s\le(\lambda t)\wedge\tau_{B(x,2\eta_1\Phi^{-1}(t)/3)}} \mathbf{1}_{\{X_s\in B(y,\eta_1\Phi^{-1}(t)/3)\}}\right]\\
&\quad \ge \mathbb{E}^x\left[\int_0^{(\lambda t)\wedge\tau_{B(x,2\eta_1\Phi^{-1}(t)/3)}} ds\int_{B(y,\eta_1\Phi^{-1}(t)/3)} J(X_s,u)\mu(du)\right]\\
&\quad \ge c_3\mathbb{E}^x[(\lambda t)\wedge\tau_{B(x,2\eta_1\Phi^{-1}(t)/3)}]V(y,\eta_1\Phi^{-1}(t)/3)\frac{1}{V(x,d(x,y))\psi(d(x,y))}\\
&\quad \ge c_4(\lambda t)\mathbb{P}^x(\tau_{B(x,2\eta_1\Phi^{-1}(t)/3)}\ge\lambda t)\,(\eta_1/3)^{d_1}\,\frac{V(y,\Phi^{-1}(t))}{V(x,d(x,y))\psi(d(x,y))}\\
&\quad \ge c_4 2^{-1}\lambda(\eta_1/3)^{d_1}\frac{tV(y,\Phi^{-1}(t))}{V(x,d(x,y))\psi(d(x,y))},
\end{aligned}
$$

where in the third inequality we used the fact that

$$
d(X_s,u)\le d(X_s,x)+d(x,y)+d(y,u)\le d(x,y)+\eta_1\Phi^{-1}(t)\le 2d(x,y).
$$

Thus, combining the above two inequality, we have proved (2.2.50). $\square$

Recall that for $A\ge 1$, we call that the chain condition(Ch($A$)) holds for the metric measure space $(M,d)$ if for any $n\in\mathbb{N}$ and $x,y\in M$, there is a sequence $\{z_k\}_{k=0}^n$ of points in $M$ such that $z_0=x, z_n=y$ and

$$
d(z_{k-1},z_k)\le A\frac{d(x,y)}{n}\quad\text{for all}\quad k=1,\dots,n.
$$

**Lemma 2.2.29.** *Assume* Ch($A$), VD($d_2$) *and* RVD($d_1$). *We further assume that there exist $\eta>0$ and $c>0$ such that* (2.2.46) *holds with the function $\Phi$ satisfying $L_a(\delta,\widetilde{C}_L)$ with $a>0$ and $\delta>1$. Then, for any $T>0$ and $C>0$, there exists a constant $c_1>0$ such that for any $t\in(0,T]$ and $x,y\in M$ with $d(x,y)\le C\Phi^{-1}(t)$,*

$$
p(t,x,y)\ge\frac{c_1}{V(x,\Phi^{-1}(t))}. \tag{2.2.51}
$$

*In particular, if $L(\delta, \widetilde{C}_L, \Phi)$ holds, then we may take $T = \infty$.*

**Proof.** Without loss of generality we may and do assume $a = \Phi^{-1}(T)$. Fix $t > 0$ and $x, y \in M$ with $d(x,y) \le C\Phi^{-1}(t)$. Let $N := \lceil \big(\frac{3AC}{\eta}\big)^{\frac{\delta}{\delta-1}} \widetilde{C}_L^{-\frac{1}{\delta-1}} \rceil + 1 \in \mathbb{N}$. Then, by $\mathrm{Ch}(A)$ there exists a sequence $\{z_k\}_{k=0}^N$ of the points in $M$ such that $z_0 = x$, $z_N = y$ and $d(z_k, z_{k+1}) \le A\frac{d(x,y)}{N}$ for all $k = 0, \dots, N-1$. Note that by Lemma 1.1.4 and $L_{\Phi^{-1}(T)}(\delta, \widetilde{C}_L, \Phi)$ we have $U_T(1/\delta, \widetilde{C}_L^{-1/\delta}, \Phi^{-1})$. Using this and the definition of $N$, we have

$$A\frac{d(x,y)}{N} \le \frac{AC\Phi^{-1}(t)}{N} \le \frac{AC}{N}\widetilde{C}_L^{-1/\delta} N^{1/\delta}\Phi^{-1}(t/N) \le \frac{\eta}{3}\Phi^{-1}(t/N).$$

For $k = 1, \dots, N$, let $B_k := B(z_k, \eta\Phi^{-1}(t/N)/3)$. Then, for any $0 \le k \le N-1$, $\xi_k \in B_k$ and $\xi_{k+1} \in B_{k+1}$. So we have

$$d(\xi_k, \xi_{k+1}) \le d(\xi_k, z_k) + d(z_k, z_{k+1}) + d(z_{k+1}, \xi_{k+1}) \le \eta\Phi^{-1}(t/N).$$

Thus, by (2.2.46) and (2.2.1) with $\xi_{k+1} \in B_{k+1}$, we have for any $k = 0, \dots, N$, $\xi_k \in B_k$ and $\xi_{k+1} \in B_{k+1}$,

$$\begin{aligned} &p(t/N, \xi_k, \xi_{k+1}) \ge \frac{c_1}{V(\xi_{k+1}, \Phi^{-1}(t/N))} \\ &\ge \frac{c_1 C_\mu^{-1}}{V(z_{k+1}, \Phi^{-1}(t/N))}\Big(\frac{\Phi^{-1}(t/N)}{d(z_{k+1}, \xi_{k+1}) + \Phi^{-1}(t/N)}\Big)^{d_2} \ge \frac{c_2}{V(z_{k+1}, \Phi^{-1}(t/N))}. \end{aligned}$$

Using above estimates and VD($d_2$), we conclude

$$\begin{aligned}
p(t,x,y) &= \int_M \dots \int_M p(t/N,x,\xi_1)\dots p(t/N,\xi_{N-1},y)\mu(d\xi_1)\dots\mu(d\xi_{N-1}) \\
&\geq \int_{B_1} \dots \int_{B_{N-1}} \prod_{k=0}^{N-1} \frac{c_2}{V(z_k,\Phi^{-1}(t/N))}\mu(d\xi_1)\mu(d\xi_2)\dots\mu(d\xi_{N-1}) \\
&\geq c_2^N \prod_{k=1}^{N-1} V(z_k,\eta\Phi^{-1}(t/N)/3) \prod_{k=0}^{N-1} V(z_k,\Phi^{-1}(t/N))^{-1} \\
&\geq c_2^N c_3^{N-1} \prod_{k=1}^{N-1} V(z_k,\Phi^{-1}(t/N)) \prod_{k=0}^{N-1} V(z_k,\Phi^{-1}(t/N))^{-1} \\
&= c_4 V(x,\Phi^{-1}(t/N))^{-1} \geq c_4 V(x,\Phi^{-1}(t))^{-1}.
\end{aligned}$$

This proves the lemma. □

**Proposition 2.2.30.** *Assume that the metric measure space $(M,d)$ satisfies* Ch($A$)*,* VD($d_2$) *and* RVD($d_1$)*. We further assume that there exists $\eta > 0$ and $c > 0$ such that* (2.2.46) *holds.*

*(i) Suppose that $L_a(\delta,\widetilde{C}_L,\Phi)$ holds with $\delta > 1$. Then, for any $T \in (0,\infty)$, there exist constants $c > 0$ and $a_L > 0$ such that for any $x,y \in M$ and $t \in (0,T]$,*

$$p(t,x,y) \geq cV(x,\Phi^{-1}(t))^{-1} \exp\left(-a_L\Phi_1(d(x,y),t)\right). \tag{2.2.52}$$

*Moreover, if $L(\delta,\widetilde{C}_L,\Phi)$ holds, then* (2.2.52) *holds for all $t \in (0,\infty)$.*

*(ii) Suppose that $L^a(\delta,\widetilde{C}_L,\Phi)$ holds with $\delta > 1$. Then, for any $T \in (0,\infty)$, there exist constants $c > 0$ and $a_L > 0$ such that for any $x,y \in M$ and $t \geq T$,*

$$p(t,x,y) \geq cV(x,\Phi^{-1}(t))^{-1} \exp\big(-a_L\widetilde{\Phi}_1(d(x,y),t)\big). \tag{2.2.53}$$

**Proof.** (i) Without loss of generality we may and do assume that $a = \Phi^{-1}(T)$. Note that by (2.2.51), we have a constant $c_1 > 0$ such that for any $t \in (0,T]$

and $x, y \in M$ with $d(x,y) \le 2c_U \Phi^{-1}(t)$,

$$p(t,x,y) \ge \frac{c_1}{V(x, \Phi^{-1}(t))}. \tag{2.2.54}$$

Note that if $t \in (0,T]$ and $d(x,y) \le 2c_U\Phi^{-1}(t)$, (2.2.52) immediately follows from (2.2.54) since $\Phi_1(d(x,y),t) \ge 0$. Now we consider $x, y \in M$ and $t \in (0,T]$ with $d(x,y) > 2c_U\Phi^{-1}(t)$. Let $r := d(x,y)$ and $\theta := \frac{\widetilde{C}_L c_U}{2A} \wedge 2$. Define

$$\varepsilon = \varepsilon(t,r) := \inf\{s > 0 : \frac{\Phi(s)}{s} \ge \theta \frac{t}{r}\}.$$

Note that by (1.1.1) and $\theta \le 2$, we have

$$\frac{\Phi(\Phi^{-1}(t))}{\Phi^{-1}(t)} \ge \frac{c_U^{-1} t}{(2c_U)^{-1} r} \ge \theta \frac{t}{r},$$

which implies $\varepsilon(t,r) \le \Phi^{-1}(t)$. Also, using $\lim_{s\to 0} \frac{\Phi(s)}{s} = 0$ we have $\varepsilon(t,r) > 0$. Observe that by the definition of $\varepsilon$, we have a decreasing sequence $\{s_n\}$ converging to $\varepsilon$ satisfying $\frac{\Phi(s_n)}{s_n} \ge \theta \frac{t}{r}$ for all $n \in \mathbb{N}$. Using $U(\alpha_2, c_U, \Phi)$ we have

$$c_U (\frac{\varepsilon}{s_n})^{\alpha_2 - 1} \frac{\Phi(\varepsilon)}{\varepsilon} \ge \frac{\Phi(s_n)}{s_n} \ge \theta \frac{t}{r} \quad \text{for all } n \in \mathbb{N}$$

Letting $n \to \infty$ we obtain

$$\frac{\theta t}{c_U r} \le \frac{\Phi(\varepsilon)}{\varepsilon}. \tag{2.2.55}$$

By a similar way, using $L_{\Phi^{-1}(T)}(\delta, \widetilde{C}_L, \Phi)$ and $\frac{\Phi(s)}{s} \le \frac{\theta t}{r}$ for any $s < \varepsilon$ we have

$$\frac{\Phi(\varepsilon)}{\varepsilon} \le \frac{\theta t}{\widetilde{C}_L r}. \tag{2.2.56}$$

Also, (2.2.55) yields that

$$\Phi_1(2c_U r, \theta t) \ge \frac{2c_U r}{\varepsilon} - \frac{\theta t}{\Phi(\varepsilon)} \ge \frac{r}{\varepsilon}\big(2c_U - \frac{\varepsilon}{\Phi(\varepsilon)} \frac{\theta t}{r}\big) \ge \frac{c_U r}{\varepsilon}.$$

Thus, using Lemma 2.2.20(i) with the fact that $r \ge 2c_U \Phi^{-1}(t)$, we have a constant $c_1 > 0$ satisfying

$$\frac{r}{\varepsilon} \leq c_U^{-1}\Phi_1(2c_U r, \theta t) \leq c_1\Phi_1(r,t). \tag{2.2.57}$$

Define $N = N(t,r) := \lceil \frac{3Ar}{2c_U\varepsilon} \rceil + 1$. Since $r \geq 2c_U\Phi^{-1}(t) \geq 2c_U\varepsilon$, we have $N \geq \lceil 3A \rceil + 1 \geq 4$. Observe that by $\frac{3Ar}{2c_U\varepsilon} \leq N \leq \frac{2Ar}{c_U\varepsilon}$ and (2.2.56) with $\theta \leq \frac{\widetilde{C}_L c_U}{2A}$,

$$\Phi(\frac{3Ar}{2c_U N}) \leq \Phi(\varepsilon) \leq \frac{\varepsilon\theta t}{r\widetilde{C}_L} \leq \frac{2A\theta}{c_U\widetilde{C}_L}\frac{t}{N} \leq \frac{t}{N}.$$

This implies $\frac{Ar}{N} \leq \frac{2}{3}c_U\Phi^{-1}(\frac{t}{N})$. On the other hand, since $(M,d)$ satisfies $\mathrm{Ch}(A)$, we have a sequence $\{z_l\}_{l=0}^N$ of points in $M$ such that $z_0 = x$, $z_N = y$ and $d(z_{l-1}, z_l) \leq A\frac{r}{N}$ for any $l \in \{1, \ldots, N\}$.
Thus for any $\xi_l \in B(z_l, \frac{2}{3}c_U\Phi^{-1}(\frac{t}{N}))$ and $\xi_{l-1} \in B(z_{l-1}, \frac{2}{3}c_U\Phi^{-1}(\frac{t}{N}))$ we have

$$\begin{aligned} d(\xi_l, \xi_{l-1}) &\leq d(\xi_l, z_l) + d(z_l, z_{l-1}) + d(z_{l-1}, \xi_{l-1}) \\ &\leq \frac{2}{3}c_U\Phi^{-1}(t/N) + \frac{Ar}{N} + \frac{2}{3}c_U\Phi^{-1}(t/N) \\ &\leq 2c_U\Phi^{-1}(t/N). \end{aligned}$$

Therefore, using semigroup property and (2.2.51) with $N \leq \frac{2Ar}{c_U\varepsilon}$ and (2.2.57) we have

$$\begin{aligned} &p(t,x,y) \\ &\geq \int_{B(z_{N-1}, \frac{\eta}{3}\Phi^{-1}(t/N))} \cdots \int_{B(z_1, \frac{\eta}{3}\Phi^{-1}(t/N))} p(\tfrac{t}{N}, x, \xi_1) \cdots p(\tfrac{t}{N}, \xi_{N-1}, y) d\xi_1 \cdots d\xi_{N-1} \\ &\geq c_2^N \prod_{l=0}^{N-1} V(z_l, \Phi^{-1}(t/N))^{-1} \prod_{l=1}^{N-1} V(z_l, \Phi^{-1}(t/N)) = c_3 c_4^N V(x, \Phi^{-1}(t/N))^{-1} \\ &\geq c_3 \left(\frac{c_4 C^{d_1}}{3^{d_1}}\right)^N V(x, \Phi^{-1}(t))^{-1} \geq c_3 V(x, \Phi^{-1}(t))^{-1} \exp(-c_5 N) \qquad (2.2.58) \\ &\geq c_3 V(x, \Phi^{-1}(t))^{-1} \exp(-c_6\frac{r}{\varepsilon}) \geq c_3 V(x, \Phi^{-1}(t))^{-1} \exp(-c_7\Phi_1(r,t)). \end{aligned}$$

This concludes (2.2.52). Now assume that $\Phi$ satisfies $L(\delta, \widetilde{C}_L)$. Note that the case $d(x,y) \le 2c_U\Phi^{-1}(t)$ is same, since we have (2.2.54) for every $t > 0$. Also, by the similar way we obtain $0 < \varepsilon(t,r) \le \Phi^{-1}(t)$, (2.2.55) and (2.2.56) for all $t \in (0,\infty)$ and $r > 2c_U\Phi^{-1}(t)$. Following the calculations in (2.2.58) again, we conclude (2.2.52) for every $t > 0$ and $x, y \in M$ with $d(x,y) > 2c_U\Phi^{-1}(t)$.

(ii) Without loss of generality, we may assume $a = \Phi(T)$. Then, it suffices to prove

$$p(t,x,y) \ge cV(x, \widetilde{\Phi}^{-1}(t))^{-1} \exp(-a_L\widetilde{\Phi}_1(d(x,y),t)), \quad t \ge T, \ x, y \in M.$$

Indeed, $\Phi^{-1}(t) = \widetilde{\Phi}^{-1}(t)$ for $t \ge T$. Note that for the proof of (2.2.52) with $T = \infty$, we only used near-diagonal estimate in (2.2.46) and $L(\delta, \widetilde{C}_L, \Phi)$ with semigroup property. Since $L(\delta, \widetilde{C}_L, \widetilde{\Phi})$ holds, (2.2.53) follows from (2.2.46) and (2.2.11). □

*Proof of Theorem 2.2.14.* Combining Proposition 2.2.28 and Proposition 2.2.30 we obtain our desired result. Note that the conditions in Proposition 2.2.30 follows from Proposition 2.2.28. □

*Proof of Theorem 2.2.15.* First we assume (2). Using Theorem 2.2.12 we obtain $\mathrm{UHK}(\Phi, \psi)$. Also, by $\mathrm{UHK}(\Phi)$, $\mathrm{J}_{\psi,\le}$ and the conservativeness of $(\mathcal{E}, \mathcal{F})$ with Theorem 2.2.12 we have $\mathrm{E}_\Phi$. Now, the lower bound of $\mathrm{HK}(\Phi, \psi)$ follows from Proposition 2.2.28. Therefore, (2) implies both (1) and (3).

Now we assume (1). The implication (1) $\Rightarrow$ $\mathrm{J}_\psi$ is the same as that in the proof of Theorem 2.2.12. Since $\mathrm{UHK}(\Phi)$ holds, using [33, Theorem 1.20 (3) $\Rightarrow$ (7)] we obtain $\mathrm{PI}(\Phi)$. The conservativeness follows from [32, Proposition 3.1].

Applying [33, Theorem 1.20] and [32, Lemma 4.21], respectively, we easily see that (3) with (2.2.6) implies $\mathrm{UHK}(\Phi)$ and the conservativeness of $(\mathcal{E}, \mathcal{F})$.

If we further assume $\mathrm{Ch}(A)$, Theorem 2.2.14 yields that (3) $\Rightarrow$ (4). Also, (4) $\Rightarrow$ (1) is straightforward. □

### 2.2.5 HKE and stability on metric measure space with sub-Gaussian estimates for diffusion process

In this section, we consider a metric measure space having sub-Gaussian estimates for diffusion process. We will obtain equivalence relation similar to Theorems 2.2.12 and 2.2.15 without assuming that the the index of local weak lower scaling conditions is strictly bigger than 1.

Recall that we always assume that $\psi : (0, \infty) \to (0, \infty)$ is a nondecreasing function which satisfies $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$. We also recall that if $\mathrm{Diff}(F)$ holds, there exists conservative symmetric diffusion process on $M$ such that the transition density $q(t, x, y)$ for the symmetric diffusion process $Z = (Z_t)_{t\geq 0}$ on $M$ with respect to $\mu$ exists and satisfies the estimates in (2.2.2). Throughout this section, we assume $\mathrm{VD}(d_2)$ and $\mathrm{Diff}(F)$ for the metric measure space $(M, d, \mu)$, where $F : (0, \infty) \to (0, \infty)$ is strictly increasing function satisfying (2.2.17), $L(\gamma_1, c_F^{-1})$ and $U(\gamma_2, c_F)$ with $1 < \gamma_1 \leq \gamma_2$, that is,

$$c_F^{-1}\left(\frac{R}{r}\right)^{\gamma_1} \leq \frac{F(R)}{F(r)} \leq c_F\left(\frac{R}{r}\right)^{\gamma_2}, \quad 0 < r \leq R \tag{2.2.59}$$

with some constants $1 < \gamma_1 \leq \gamma_2$ and $c_F \geq 1$. Note that, by Lemma 1.1.4, $F^{-1}$ satisfies $L(1/\gamma_2, c_F^{-1/\gamma_2})$ and $U(1/\gamma_1, c_F^{1/\gamma_1})$. Define $\Phi(r) = F(r)/\int_0^r \frac{dF(s)}{\psi(s)}$ as (2.2.18).

Since $\psi$ is non-decreasing and $\lim_{s\to 0} \psi(s) = 0$, we easily observe that

$$\psi(r) = \frac{F(r)}{\int_0^r \frac{dF(s)}{\psi(r)}} > \frac{F(r)}{\int_0^r \frac{dF(s)}{\psi(s)}} = \Phi(r), \quad r > 0,$$

and

$$\frac{\Phi(R)}{\Phi(r)} = \frac{F(R)}{F(r)} \cdot \frac{\int_0^r \frac{dF(s)}{\psi(s)}}{\int_0^R \frac{dF(s)}{\psi(s)}} \leq \frac{F(R)}{F(r)}, \quad 0 < r \leq R. \tag{2.2.60}$$

Thus, $\Phi$ satisfies $U(\gamma_2, c_F)$, and (2.2.6) holds for functions $\Phi$ and $\psi$. Recall that $F_1 = \mathcal{T}(F)$. Note that $F_1(r, t) \in (0, \infty)$ for every $r, t > 0$ under (2.2.59).

Here we record [48, Lemma 3.19] for the next use. Since $F$ is strictly increasing and satisfying (2.2.59), we have that for any $r, t > 0$,

$$F_1(r,t) \geq \left(\frac{F(r)}{t}\right)^{\frac{1}{\gamma_1 - 1}} \wedge \left(\frac{F(r)}{t}\right)^{\frac{1}{\gamma_2 - 1}} \geq \left(\frac{F(r)}{t}\right)^{\frac{1}{\gamma_2 - 1}} - 1. \tag{2.2.61}$$

**Lemma 2.2.31.** *$\Phi$ is strictly increasing. Moreover, $L(\alpha_1, c_L, \Phi)$ holds for some $\alpha_1, c_L > 0$.*

**Proof.** Since $\psi$ is non-decreasing, we may observe that for any $0 \leq a < b$,

$$\frac{F(b) - F(a)}{\psi(b)} \leq \int_a^b \frac{dF(s)}{\psi(s)} \leq \frac{F(b) - F(a)}{\psi(a)},$$

regarding $\frac{1}{\psi(0)} = \infty$. Thus, there exists $a_* \in (a,b)$ such that $\int_a^b \frac{dF(s)}{\psi(s)} = \frac{F(b)-F(a)}{\psi(a_*)}$. For any $r < R$, let $r_* \in (0,r)$ and $R_* \in (r,R)$ be the constants satisfying

$$\int_0^r \frac{dF(s)}{\psi(s)} = \frac{F(r)}{\psi(r_*)} \quad \text{and} \int_r^R \frac{dF(s)}{\psi(s)} = \frac{F(R) - F(r)}{\psi(R_*)}.$$

Then, since $\psi$ is non-decreasing,

$$\Phi(R) = \frac{F(R)}{\frac{F(r)}{\psi(r_*)} + \frac{F(R)-F(r)}{\psi(R_*)}} \geq \frac{F(R)}{\frac{F(r)}{\psi(r_*)} + \frac{F(R)-F(r)}{\psi(r_*)}} = \psi(r_*) = \Phi(r).$$

Thus, $\Phi$ is also non-decreasing. Now suppose that the equality of above inequality holds. Then, since $F(R) - F(r) > 0$, we have $\psi(r_*) = \psi(R_*)$, which implies that $\psi(r_*) = \psi(r)$ since $\psi$ in non-decreasing. Thus, we conclude $\int_0^r \frac{dF(s)}{\psi(s)} = \frac{F(r)}{\psi(r)}$, which is contradiction since $\lim_{s \to 0} \psi(s) = 0$. Therefore, $\Phi$ is strictly increasing.

Using $L(\gamma_1, c_F^{-1}, F)$ and $L(\beta_1, C_L, \psi)$, there is a constant $C > 1$ such that

$$F(Cr) \geq 4F(r) \quad \text{and} \quad \psi(Cr) \geq 4\psi(r), \quad \text{all} \quad r > 0. \tag{2.2.62}$$

For $r > 0$, let $r_1 \in (0,r)$, $r_2 \in (r, Cr)$, $r_3 \in (Cr, C^2 r)$ be the constants satisfying $\int_0^r \frac{dF(s)}{\psi(s)} = \frac{F(r)}{\psi(r_1)}$, $\int_r^{Cr} \frac{dF(s)}{\psi(s)} = \frac{F(Cr)-F(r)}{\psi(r_2)}$ and $\int_{Cr}^{C^2r} \frac{dF(s)}{\psi(s)} = \frac{F(C^2r)-F(Cr)}{\psi(r_3)}$. Then,

$$\Phi(r) = \frac{F(r)}{\int_0^r \frac{dF(s)}{\psi(s)}} = \psi(r_1)$$

and

$$\Phi(C^2r) = \frac{F(C^2r)}{\int_0^{C^2r} \frac{dF(s)}{\psi(s)}} \geq \frac{F(C^2r)}{\frac{F(r)}{\psi(r_1)} + \frac{F(Cr)-F(r)}{\psi(r_1)} + \frac{F(C^2r)-F(Cr)}{\psi(r_3)}}.$$

By (2.2.62) and the fact that $r_1 \leq r \leq Cr \leq r_3$, we have

$$\frac{\psi(r_1)}{\psi(r_3)} \leq \frac{1}{4} \quad \text{and} \quad \frac{F(Cr)}{F(C^2r)} \leq \frac{1}{4}.$$

Therefore, for any $r > 0$ we have

$$\frac{\Phi(C^2r)}{\Phi(r)} \geq \frac{F(C^2r)}{F(Cr) + \frac{\psi(r_1)}{\psi(r_3)}(F(C^2r) - F(Cr))} \geq 2. \tag{2.2.63}$$

Using (2.2.63) we easily prove that $L(\alpha_1, c_L, \Phi)$ holds with $\alpha_1 = \frac{\log 2}{2\log C}$ and $c_L = \frac{1}{2}$. □

With the functions $\psi$ and $F$, let $\phi$ be the function defined by

$$\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda t}) \frac{dt}{t\psi(F^{-1}(t))}. \tag{2.2.64}$$

Note that by (2.2.17), $L(\beta_1, C_L, \psi)$ and $U(\gamma_2, c_F, F)$,

$$\int_0^\infty (1 \wedge t) \frac{dt}{t\psi(F^{-1}(t))} = \int_0^1 \frac{dF(s)}{\psi(s)} + \int_{F(1)}^\infty \frac{dt}{t\psi(F^{-1}(t))} < \infty.$$

Thus, there exists a subordinator $S = (S_t, t > 0)$ which is independent of $Z$ and whose Laplace exponent is $\phi$. Then, the process $Y$ defined by $Y_t := Z_{S_t}$ is pure jump process whose jump kernel is given by

$$\mathcal{J}_\psi(x,y) = \int_0^\infty q(t,x,y) \frac{1}{t\psi(F^{-1}(t))} dt.$$

Also, the transition density $p^Y(t,x,y)$ of $Y$ can be written by $p^Y(t,x,y) = \int_0^\infty q(s,x,y)\mathbb{P}(S_t \in ds)$. Then, from the sub-Gaussian estimates (2.2.2), we obtain the following lemma.

**Lemma 2.2.32.** ([2, Lemma 4.2]) *$\mathcal{J}_\psi$ satisfies* (2.2.5)*. In other words,* $\mathrm{J}_\psi$ *holds for the process $Y$.*

**Lemma 2.2.33.** *There exists $c > 0$ such that for any $\lambda > 0$,*

$$\frac{1}{2\Phi(F^{-1}(\lambda^{-1}))} \le \phi(\lambda) \le \frac{c}{\Phi(F^{-1}(\lambda^{-1}))}. \tag{2.2.65}$$

**Proof.** Using (2.2.64) and (2.2.18),

$$\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda t}) \frac{dt}{t\psi(F^{-1}(t))} \ge \frac{1}{2\Phi(F^{-1}(\lambda^{-1}))},$$

and by (2.2.18) and (2.2.6),

$$\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda t}) \frac{dt}{t\psi(F^{-1}(t))} \le \frac{c}{\Phi(F^{-1}(\lambda^{-1}))}.$$

From the above two inequalities we conclude the lemma. □

Let us define

$$\mathcal{E}_\psi(f,f) := \iint_{M\times M} \big(f(x) - f(y)\big)^2 \mathcal{J}_\psi(x,y)\mu(dx)\mu(dy), \quad f \in L^2(M,\mu),$$

and $\{Q_t, t > 0\}$ be the transition semigroup with respect to $Z$ on $L^2(M,\mu)$, thus

$$Q_t f(x) = \int_M q(t,x,y) f(y)\mu(dy).$$

Following the proof of [27, Lemma 2.3], we obtain a consequence of $\mathrm{Diff}(F)$.

**Lemma 2.2.34.** *Assume that the metric measure space $(M,d,\mu)$ satisfies* $\mathrm{VD}(d_2)$, $\mathrm{RVD}(d_1)$ *and* $\mathrm{Diff}(F)$*. Then, there exist $c_1, \theta > 0$ and $\varepsilon \in (0,1)$*

*such that*

$$q^{B(x_0,r)}(t,x,y) \geq \frac{c_1}{V(x_0,F^{-1}(t))}$$

*for all* $x_0 \in M_0$, $r>0$, $x,y \in B(x_0,\varepsilon F^{-1}(t))$ *and* $t \in (0,\theta F(r)]$.

**Proof.** Assume $\theta \in (0,1]$. Let $x_0 \in M_0$, $r>0$ and denote $B_r := B(x_0,r)$. Using (2.2.2), for any $x,y \in B(x_0,\varepsilon F^{-1}(t))$ and $t \in (0,\theta F(r)]$,

$$\begin{aligned}
&q^{B_r}(t,x,y) = q(t,x,y) - \mathbb{E}^x[q(t-\tau_{B_r}, Z_{\tau_{B_r}}, y) : \tau_{B_r} < t] \\
&\geq \frac{c^{-1}}{V(x_0,F^{-1}(t))} - \mathbb{E}^x[\frac{c}{V(x_0,F^{-1}(t-\tau_{B_r}))} e^{-a_0F_1(d(Z_{\tau_{B_r}},y),t-\tau_{B_r})} : \tau_{B_r} < t] \\
&\geq \frac{c^{-1}}{V(x_0,F^{-1}(t))} - \mathbb{E}^x[\frac{c}{V(x_0,F^{-1}(t-\tau_{B_r}))} e^{-a_0F_1((1-\varepsilon)r,t-\tau_{B_r}} : \tau_{B_r} < t] \\
&\geq \frac{c^{-1}}{V(x_0,F^{-1}(t))} - c\mathbb{P}^x(\tau_{B_r} < t) \sup_{0<s\leq t} \frac{1}{V(x_0,F^{-1}(s))} e^{-a_0F_1((1-\varepsilon)r,s)} \\
&\geq \frac{c^{-1}}{V(x_0,F^{-1}(t))} - c \sup_{0<s\leq t} \frac{1}{V(x_0,F^{-1}(s))} \exp\big(-a_0F_1((1-\varepsilon)r,s)\big).
\end{aligned}$$

By (2.2.61), we also have

$$\begin{aligned}
&\sup_{0<s\leq t} \frac{1}{V(x_0,F^{-1}(s))} \exp\big(-a_0F_1((1-\varepsilon)r,s)\big) \\
&\leq \sup_{0<s\leq t} \frac{e^{a_0}}{V(x_0,F^{-1}(s))} \exp\big(-a_0(\frac{F((1-\varepsilon)r)}{s})^{\frac{1}{\gamma_2-1}}\big) \\
&= \sup_{0<s\leq t} \frac{e^{a_0}}{V(x_0,F^{-1}(t))} \frac{V(x_0,F^{-1}(t))}{V(x_0,F^{-1}(s))} \exp\big(-a_0(\frac{t}{s})^{\frac{1}{\gamma_2-1}}(\frac{F((1-\varepsilon)r)}{t})^{\frac{1}{\gamma_2-1}}\big) \\
&\leq \frac{e^{a_0}C_\mu c_F^{d_2/\gamma_1}}{V(x_0,F^{-1}(t))} \sup_{0<s\leq t} (\frac{t}{s})^{d_2/\gamma_1} \exp\left(-a_0 c_F^{-1/(\gamma_2-1)}\left(\frac{(1-\varepsilon)^{\gamma_2}}{\theta}\right)^{\frac{1}{\gamma_2-1}}(\frac{t}{s})^{\frac{1}{\gamma_2-1}}\right) \\
&= \frac{e^{a_0}C_\mu c_F^{d_2/\gamma_1}}{V(x_0,F^{-1}(t))} \sup_{1\leq u} u^{d_2/\gamma_1} \exp\left(-a_0 c_F^{-1/(\gamma_2-1)}\left(\frac{(1-\varepsilon)^{\gamma_2}}{\theta}\right)^{\frac{1}{\gamma_2-1}} u^{\frac{1}{\gamma_2-1}}\right) \\
&:= \frac{C(\theta)e^{a_0}C_\mu c_F^{d_2/\gamma_1}}{V(x_0,F^{-1}(t))}.
\end{aligned}$$

Since $\lim_{\theta\to 0} C(\theta) = 0$, there is $\theta > 0$ such that $C(\theta) \leq \frac{1}{2c^2e^{a_0}C_\mu c_F^{d_2/\gamma_1}}$. With this, we obtain

$$q^{B_r}(t,x,y) \geq \frac{c^{-1}}{2V(x_0, F^{-1}(t))}.$$

This concludes the lemma. □

**Lemma 2.2.35.** *Suppose that the metric measure space $(M,d,\mu)$ satisfies* $\mathrm{VD}(d_2)$*,* $\mathrm{RVD}(d_1)$ *and* $\mathrm{Diff}(F)$ *where $F:(0,\infty)\to(0,\infty)$ strictly increasing function satisfying* (2.2.17)*, $L(\gamma_1, c_F^{-1})$ and $U(\gamma_2, c_F)$ with $1<\gamma_1\leq\gamma_2$. There exist constants $c>0$ and $\hat{\varepsilon}\in(0,1)$ such that for any $x_0\in M_0$, $r>0$, $0<t\leq\Phi(\hat{\varepsilon}r)$ and $x,y\in B(x_0,\hat{\varepsilon}\Phi^{-1}(t))$,*

$$p^{Y,B(x_0,r)}(t,x,y) \geq \frac{c}{V(x_0,\Phi^{-1}(t))}.$$

**Proof.** Recall that we have defined $Y_t = Z_{S_t}$, where $S_t$ is a subordinator independent of $Z$ and whose Laplace exponent is the function $\phi$ in (2.2.64). Also, by (2.2.65) we have $\frac{c_1^{-1}}{\Phi(F^{-1}(\lambda^{-1}))} \leq \phi(\lambda) \leq \frac{c_1}{\Phi(F^{-1}(\lambda^{-1}))}$. Take $\lambda$ by $F(\Phi^{-1}(c_1^{-1}t^{-1}))^{-1}$ and $F(\Phi^{-1}(c_1t^{-1}))^{-1}$, and using the fact that $\Phi$ and $F$ are strictly increasing we obtain that for any $t>0$

$$F(\Phi^{-1}(c_1^{-1}t)) \leq \phi^{-1}(t^{-1})^{-1} \leq F(\Phi^{-1}(c_1t)). \tag{2.2.66}$$

By [70, Proposition 2.4], there exist $\rho, c_2>0$ such that

$$\mathbb{P}\Big(\frac{1}{2\phi^{-1}(t^{-1})} \leq S_t \leq \frac{1}{\phi^{-1}(\rho t^{-1})}\Big) \geq c_2. \tag{2.2.67}$$

Choose $\hat{\varepsilon}>0$ such that

$$\hat{\varepsilon}\Phi^{-1}(t) \leq \varepsilon F^{-1}\Big(\frac{1}{2}F(\Phi^{-1}(c_1^{-1}t))\Big) \quad \text{and} \quad F(\Phi^{-1}(c_1\rho^{-1}\Phi(\hat{\varepsilon}r))) \leq \theta F(r),$$

where $\varepsilon\in(0,1)$ and $\theta$ are the constants in Lemma 2.2.34. Then, by (2.2.66), we see that for $0<t\leq\Phi(\hat{\varepsilon}r)$ and $s\in[\frac{1}{2\phi^{-1}(t^{-1})}, \frac{1}{\phi^{-1}(\rho t^{-1})}]$, we have

$$s \leq \frac{1}{\phi^{-1}(\rho t^{-1})} \leq F(\Phi^{-1}(c_1\rho^{-1}t)) \leq F(\Phi^{-1}(c_1\rho^{-1}\Phi(\hat{\varepsilon}r))) \leq \theta F(r)$$

and

$$\hat{\varepsilon}\Phi^{-1}(t) \le \varepsilon F^{-1}(\frac{1}{2}F(\Phi^{-1}(c_1^{-1}t))) \le \varepsilon F^{-1}\big(\frac{1}{2\phi^{-1}(t^{-1})}\big) \le \varepsilon F^{-1}(s).$$

Thus, by [87, Proposition 3.1], Lemma 2.2.34, (2.2.67), (2.2.66), VD($d_2$), (2.2.59) and $U(1/\alpha_1, c_L^{-1/\alpha_1}, \Phi^{-1})$, we see that for $0 < t \le \Phi(\hat{\varepsilon}r)$ and $x, y \in B(x_0, \hat{\varepsilon}\Phi^{-1}(t))$

$$\begin{aligned} p^{Y,B(x_0,r)}(t, x, y) &\ge \int_0^\infty q^{B(x_0,r)}(s, x, y)\mathbb{P}(S_t \in ds) \\ &\ge \int_{\frac{1}{2\phi^{-1}(t^{-1})}}^{\frac{1}{\phi^{-1}(\rho t^{-1})}} q^{B(x_0,r)}(s, x, y)\mathbb{P}(S_t \in ds) \\ &\ge \frac{c_3}{V(x_0, F^{-1}(\phi^{-1}(\rho t^{-1})^{-1}))}\mathbb{P}\big(\frac{1}{2\phi^{-1}(t^{-1})} \le S_t \le \frac{1}{\phi^{-1}(\rho t^{-1})}\big) \\ &\ge \frac{c_2 c_3}{V(x_0, F^{-1}(F(\Phi^{-1}(c_1\rho^{-1}t))))} \ge \frac{c_4}{V(x_0, \Phi^{-1}(t))}. \end{aligned}$$

This finishes the lemma. □

**Theorem 2.2.36.** *Suppose that the metric measure space $(M, d, \mu)$ satisfies* VD($d_2$)*,* RVD($d_1$) *and* Diff($F$) *where $F : (0, \infty) \to (0, \infty)$ strictly increasing function satisfying* (2.2.17)*, $L(\gamma_1, c_F^{-1})$ and $U(\gamma_2, c_F)$ with $1 < \gamma_1 \le \gamma_2$. Assume that $X$ is a Markov process on $(M, d)$ satisfying* $\mathrm{J}_\psi$*. Then, there exists a constant $c > 0$ such that*

$$p(t, x, y) \le c\left(\frac{1}{V(x, \Phi^{-1}(t))} \wedge \frac{t}{V(x, d(x, y))\Phi(d(x, y))}\right)$$

*for all $t > 0$ and $x, y \in M$. Moreover,* $\mathrm{E}_\Phi$ *and* PI($\Phi$) *holds for $X$.*

**Proof.** Note that from Lemma 2.2.32 and Lemma 2.2.35, the condition (4) in [33, Theorem 1.20] holds for the process $Y$. In particular, using [33, Theorem 1.20], the conditions CSJ($\Phi$) and PI($\Phi$) holds for the process $Y$. Since the jump kernel of $X$ and $Y$ are comparable by Lemma 2.2.32, the conditions PI($\Phi$) and CSJ($\Phi$) also hold for $X$. In particular, the process $X$ satisfies condition (7) in [33, Theorem 1.20]. Now, using [33, Theorem 1.20] again we obtain $\mathrm{E}_\Phi$ and UHK($\Phi$). This completes the proof. □

*Proof of Theorem 2.2.17.* Assume $\mathrm{J}_\psi$. Then, with Theorem 2.2.36 and the fact that $\mathcal{T}(F) = F_1$, the proof of the upper bound in GHK($\Phi, \psi$) follows similarly as the proof of Theorem 2.2.11 (see [2, Theorem 4.5]). Also, using Theorem 2.2.36 and Lemma 2.2.35 we have $\mathrm{E}_\Phi$ and PI($\Phi$). Since all conditions in Proposition 2.2.28 holds, we obtain the lower bound of GHK($\Phi, \psi$).
Also, following the proof of [32, Proposition 3.3], we obtain that GHK($\Phi, \psi$) implies $\mathrm{J}_\psi$. □

*Proof of Corollary 2.2.18.* Using Theorem 2.2.17, $X$ satisfies $\mathrm{J}_\psi$, PI($\Phi$) and $\mathrm{E}_\Phi$. Thus, the conclusion follows from Theorems 2.2.11 and 2.2.14. □

### 2.2.6 Examples

We give some examples which are covered by our results. Throughout this section, $(M, d, \mu)$ is a metric measure space satisfying Ch($A$), VD($d_2$), RVD($d_1$) and Diff($F$), where the function $F : (0, \infty) \to (0, \infty)$ is strictly increasing function satisfying $L(\gamma_1, c_F^{-1})$ and $U(\gamma_2, c_F)$ with some constants $1 < \gamma_1 \le \gamma_2$.

Typical examples of metric measure spaces satisfying the above conditions are unbounded Sierpinski gasket and unbounded Sierpinski carpet in $\mathbb{R}^d$ with $n \ge 2$. First, let us check that unbounded Sierpinski gasket in $\mathbb{R}^d$ satisfies the above conditions. Let $(M_{SG}, d_{SG}, \mu_{SG})$ be the unbounded Sierpinski gasket in $\mathbb{R}^2$, which was introduced in [13]. Here $d_{SG}(x, y)$ denotes the length of the shortest path in $M_{SG}$ from $x$ to $y$, and $\mu_{SG}$ is a multiple of the $d_f$-dimensional Hausdorff measure on $M_{SG}$ with $d_f = \log 3/\log 2$ (see [13, Lemma 1.1]). By [13, (1.13)], $d_{SG}(x, y)$ is comparable to $|x - y|$ which is the Euclidean distance, which implies Ch($A$). Also, by [13, Theorem 1.5], Diff($F$) holds for $F(r) = r^{d_w}$ with $d_w = \log 5/\log 2 > 2$. Since $d_{SG}(x, y) \asymp |x - y|$ and $M_{SG}$ is subset of $\mathbb{R}^2$, all metric balls in $(M_{SG}, d_{SG})$ are precompact. Since $M_{SG}$ is unbounded, by [48, Corollary 7.6], we have VD($d_2$). Moreover, by [46, Corollary 5.3], we also have RVD($d_1$) since $M_{SG}$ is connected. Thus, we see that $(M_{SG}, d_{SG}, \mu_{SG})$ satisfies Ch($A$), VD($d_2$), RVD($d_1$) and Diff($F$). This result also holds for unbounded Sierpinski gaskets constructed in $n$-dimensional ($n \geq 3$) Euclidean space with different $d_f(n) > 0$ and $d_w(n) > 1$ (see [13, Section 10]). Now, let $(M_{SC}, |\cdot|, H(d_f))$ be the unbounded generalized Sierpinski carpet constructed in $\mathbb{R}^d$, which was introduced in [7]. Then, by [7, Hypotheses 2.1], $(M_{SC}, |\cdot|)$ satisfies Ch($A$) and connected. Also, by [7, Theorem 1.3], Diff($F$) holds for $F(r) = r^{d_w}$ with $d_w \geq 2$. Moreover, by [7, Remark 2.2], $|x-y| \asymp d_{SC}(x,y)$ for all $x, y \in M_{SC}$, where $d_{SC}(x,y)$ is the length of the shortest path in $M_{SC}$ from $x$ to $y$. Thus, by the same argument as in the unbounded Sierpinski gasket case, we see that $(M_{SC}, |\cdot|, H(d_f))$ satisfies Ch($A$), VD($d_2$), RVD($d_1$) and Diff($F$).

Let $X$ be the symmetric pure-jump Hunt process on $(M, d, \mu)$, which is associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ in (2.2.4) satisfying $\mathrm{J}_\psi$ and $p(t,x,y)$ be the transition density of $X$. In this section, we will use the notation $f(\cdot) \simeq g(\cdot)$ at $\infty$ (resp. 0) if $\frac{f(t)}{g(t)} \to 1$ as $t \to \infty$ (resp. $t \to 0$).

**Example 2.2.37.** Suppose $F$ is differentiable function satisfying $F(s) \asymp sF'(s)$ and $F(s)\mathbf{1}_{\{s<1\}} \asymp s^\gamma(\log\frac{1}{s})^\kappa \mathbf{1}_{\{s<1\}}$ for $\gamma > 1$ and $\kappa \in \mathbb{R}$. Suppose further that $\psi : (0,\infty) \to (0,\infty)$ is a non-decreasing function which satisfies $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$. Define $f_{\alpha,\beta}(s) := (\log\frac{1}{s})^{1-\alpha}(\log\log\frac{1}{s})^{-\beta}$ and $D := \{(a,b) \in \mathbb{R}^2 : a > 1, b \in \mathbb{R}\} \cup \{(1,b) \in \mathbb{R}^2 : b > 1\}$. Then, we observe that for $(\alpha,\beta) \in D$, $\ell_{\alpha,\beta}(s) := sf'_{\alpha,\beta}(s) \asymp (\log\frac{1}{s})^{-\alpha}(\log\log\frac{1}{s})^{-\beta}$. In particular, $\ell_{\alpha,\beta} \in \mathcal{R}_0^0$ and $f_{\alpha,\beta}(s) = \int_0^s \ell_{\alpha,\beta}(u)u^{-1}du$. Assume that for $(\alpha,\beta) \in D$

$$\psi(\lambda) \asymp F(\lambda)(\log\frac{1}{\lambda})^{-\alpha}(\log\log\frac{1}{\lambda})^{-\beta}, \quad 0 < \lambda < 2^{-4}.$$

Then, there exist $T = T(\alpha - \kappa, \beta) \leq 2^{-4}$ such that for $s \leq T$, $f_{\alpha-\kappa,\beta}$ is monotone and satisfies $(f_{\alpha-\kappa,\beta})(s^\gamma) \asymp (f_{\alpha-\kappa,\beta})(s)$. Thus, by the above observation we have the following heat kernel estimates for $t < T$:

$$
\begin{aligned}
p(t,x,y) \asymp \; & \frac{1}{V(x, t^{1/\gamma}(f_{\alpha-\kappa,\beta})(t)^{1/\gamma})} \wedge \Big( \frac{t}{V(x,d(x,y))\psi(d(x,y))} \\
& + \frac{1}{V(x, t^{1/\gamma}(f_{\alpha-\kappa,\beta})(t)^{1/\gamma})} \exp\left( -a_1 \left( \frac{d(x,y)^\gamma}{(f_{\alpha-\kappa,\beta})(t)} \right)^{1/(\gamma-1)} \right) \Big).
\end{aligned}
$$

**Example 2.2.38.** Suppose $F$ is differentiable function satisfying $F(s) \asymp sF'(s)$ and $F(s)\mathbf{1}_{\{s>1\}} \asymp s^{\gamma'}(\log s)^\kappa \mathbf{1}_{\{s>1\}}$ for $\gamma' > 1$ and $\kappa \in \mathbb{R}$. Suppose further that $\psi : (0,\infty) \to (0,\infty)$ is a non-decreasing function which satisfies (2.2.17), $L(\beta_1, C_L)$, $U(\beta_2, C_U)$ and $\psi(r)\mathbf{1}_{\{r>16\}} \asymp F(r)(\log r)^\beta \mathbf{1}_{\{r>16\}}$ for $\beta \in \mathbb{R}$. Let $\ell(s) = (\log s)^{-\beta}$. Then for $\beta \le 1$, $\int_{16}^\infty \frac{\ell(s)}{s} ds = \infty$. For $s > 16$, let

$$
f(s) = \begin{cases} \frac{1}{1-\beta}(\log s)^{1-\beta} & \text{if} \quad \beta < 1, \\ \log\log s & \text{if} \quad \beta = 1. \end{cases}
$$

Then, there exists $T = T(\beta,\kappa) \ge 16$ such that for $s \ge T$, $f(s)/(\log s)^\kappa$ is monotone and $f(s)/(\log s)^\kappa \asymp f(s^{\gamma'})/(\log s^{\gamma'})^\kappa$. Thus we have the following heat kernel estimates for $t \ge T$:

(i) If $\beta < 1$:

$$
\begin{aligned}
p(t,x,y) \asymp \; & \frac{1}{V(x, t^{1/\gamma'}(\log t)^{(1-\beta-\kappa)/\gamma'})} \\
& \wedge \Bigg( \frac{t}{V(x,d(x,y))d(x,y)^{\gamma'}(\log(1+d(x,y)))^{\beta+\kappa}} \\
& \quad + \frac{1}{V(x, t^{1/\gamma'}(\log t)^{(1-\beta-\kappa)/\gamma'})} \exp\left( -a_2 \left( \frac{d(x,y)^{\gamma'}}{t(\log t)^{1-\beta-\kappa}} \right)^{\frac{1}{\gamma'-1}} \right) \Bigg),
\end{aligned}
$$

(ii) If $\beta = 1$:

$$
\begin{aligned}
p(t,x,y) \asymp \; & \frac{1}{V(x, t^{1/\gamma'}(\log t)^{-\kappa/\gamma'}(\log\log t)^{1/\gamma'})} \\
\wedge & \left( \frac{t}{V(x,d(x,y))d(x,y)^{\gamma'}(\log(1+d(x,y)))^{1+\kappa}} \right. \\
+ & \left. \frac{1}{V(x, t^{1/\gamma'}(\log t)^{-\kappa/\gamma'}(\log\log t)^{1/\gamma'})} \exp\left( -a_3 \left( \frac{d(x,y)^{\gamma'}}{t(\log\log t)(\log t)^{-\kappa}} \right)^{\frac{1}{\gamma'-1}} \right) \right).
\end{aligned}
$$

**Example 2.2.39.** Recall that $\gamma_1, \gamma_2 > 1$ are the constants in (2.2.59). Suppose $F$ is differentiable function such that there exists $c > 0$ satisfying $\gamma_1 F(s) \le sF'(s) \le cF(s)$ for all $s > 0$. Let $T > 0$ and $\psi(r) = r^\alpha \mathbf{1}_{\{r \le 1\}} + r^\beta \mathbf{1}_{\{r>1\}}$, where $\alpha < \gamma_1 \le \gamma_2 < \beta$. Then, by Corollary 2.2.17, we see that for $t \le T$,

$$
p(t,x,y) \asymp \frac{1}{V(x,t^{1/\alpha})} \wedge \frac{t}{V(x,d(x,y))\psi(d(x,y))}. \tag{2.2.68}
$$

Indeed, for $d(x,y) < 1$, (2.2.68) follows from Theorem 2.2.36. If $d(x,y) \ge 1$, then $\frac{t}{V(x,d(x,y))\psi(d(x,y))}$ dominates the upper bound of off-diagonal term in (2.2.8).

On the other hand, by the condition $\gamma_2 < \beta$, we have $\int_0^\infty \frac{dF(s)}{\psi(s)} \le c + c\int_1^\infty \frac{s^{\gamma_2}}{s^{1+\beta}} ds < \infty$. Thus, for $r > 1$, $\Phi(r)$ defined in (2.2.18) is comparable to $F(r)$ and $\Phi(r)/r \asymp F(r)/r \asymp F'(r)$. Now we see that for $t > T$,

$$
p(t,x,y) \asymp \frac{1}{V(x,F^{-1}(t))} \wedge \left( \frac{t}{V(x,d(x,y))d(x,y)^\beta} + \frac{e^{-a_5 \frac{d(x,y)}{F'(t/d(x,y))}}}{V(x,F^{-1}(t))} \right).
$$

Recall that $\mathscr{K}(t) := \sup_{0<s\le t} \frac{\Phi(s)}{s}$ appeared in Section 2.1. The following lemma yields that Theorem 2.1.2 is a special case of Corollary 2.2.18.

**Lemma 2.2.40.** *Suppose $\Phi$ is non-decreasing function satisfying $L(\alpha_1, c_L)$, $U(\alpha_2, c_U)$ and $L_a(\delta, \widetilde{C}_L)$ for $\delta > 1$. Let $T \in (0,\infty)$. Then, there exists a* *constant $c > 1$ such that for any $t \in (0, T]$ and $r \geq 2c_U^2 \Phi^{-1}(t)$,*

$$c^{-1}\Phi_1(r,t) \leq \frac{r}{\mathscr{K}^{-1}(t/r)} \leq c\Phi_1(r,t). \tag{2.2.69}$$

*Moreover, if $L(\delta, \widetilde{C}_L)$ holds, then* (2.2.69) *holds for any $t \in (0,\infty)$ and $r \geq 2c_U^2\Phi^{-1}(t)$.*

**Proof.** Without loss of generality we may and do assume $a = \Phi^{-1}(T)$. Note that $\alpha_2 > 1$. Let $R_0 := \Phi^{-1}(T)$ and $c_1 = \widetilde{C}_L^{-1}$ so that $L_{R_0}(\delta, c_1^{-1}, \Phi)$ and Lemma 2.1.6 hold. Denote $\varepsilon := \frac{1}{\alpha_2 - 1}$. Since $r \geq 2c_U^2\Phi^{-1}(t)$, we have

$$c_U^{2\varepsilon}\frac{\Phi^{-1}(t)^{1+\varepsilon}}{r^\varepsilon} \leq \Phi^{-1}(t) \leq R_0.$$

It follows from Lemma 2.1.6, Lemma 1.1.5 and $U(\alpha_2, c_U, \Phi)$ that

$$\begin{aligned}
\mathscr{K}\left(c_U^{2\varepsilon}\frac{\Phi^{-1}(t)^{1+\varepsilon}}{r^\varepsilon}\right) &\geq c_U^{-2\varepsilon}\frac{r^\varepsilon}{\Phi^{-1}(t)^{1+\varepsilon}}\Phi\left(\Phi^{-1}(t)c_U^{2\varepsilon}\frac{\Phi^{-1}(t)^\varepsilon}{r^\varepsilon}\right)\\
&\geq c_U^{-1-2\varepsilon}\frac{r^\varepsilon t}{\Phi^{-1}(t)^{1+\varepsilon}}\frac{\Phi(\Phi^{-1}(t)c_U^{2\varepsilon}\frac{\Phi^{-1}(t)^\varepsilon}{r^\varepsilon})}{\Phi(\Phi^{-1}(t))}\\
&\geq c_U^{-2-2\varepsilon}\frac{t}{r}\frac{r^{1+\varepsilon}}{\Phi^{-1}(t)^{1+\varepsilon}}\left(c_U^{2\varepsilon}\frac{\Phi^{-1}(t)^\varepsilon}{r^\varepsilon}\right)^{\alpha_2} = \frac{t}{r}.
\end{aligned}$$

Thus,

$$\rho := \mathscr{K}^{-1}(\frac{t}{r}) \leq c_U^{2\varepsilon}\frac{\Phi^{-1}(t)^{1+\varepsilon}}{r^\varepsilon} \leq 2^{-\varepsilon}\Phi^{-1}(t) \leq R_0.$$

By Lemma 2.1.6, $\mathscr{K}$ satisfies $U_{R_0}(\alpha_2 - 1, c_1c_U)$ and $L_{R_0}(\delta - 1, c_1^{-1}\widetilde{C}_L)$. Thus, using Lemma 1.1.5 we have

$$(c_1c_U)^{-1}\frac{t}{r} \leq \mathscr{K}(\rho) \leq c_1c_U\frac{t}{r}. \tag{2.2.70}$$

Using (2.2.70) and Lemma 2.1.6, we have

$$(c_1c_U)^{-1}\frac{t}{r} \leq \mathscr{K}(\rho) \leq c_1\frac{\Phi(\rho)}{\rho}.$$

Then, letting $c_2 = c_1^2 c_U$, the above inequality and (2.2.21) imply that there exists $c_3 > 0$ such that

$$c_3 \Phi_1(r,t) \geq \Phi_1(2c_2 r, t) \geq \frac{2c_2 r}{\rho} - \frac{t}{\Phi(\rho)} \geq \frac{c_2 r}{\rho} = \frac{c_2 r}{\mathscr{K}^{-1}(t/r)}.$$

This proves the second inequality in (2.2.69). For the first one, we take a $s > 0$ such that

$$0 \leq \frac{r}{s} - \frac{t}{\Phi(s)} \leq \Phi_1(r,t) \leq 2\Big(\frac{r}{s} - \frac{t}{\Phi(s)}\Big). \tag{2.2.71}$$

Since $\Phi_1(r,t) \geq 0$, we have $\Phi(s)/s \geq t/r$. Using this, Lemma 2.1.6 and (2.2.70) we have

$$\frac{\Phi(\rho)}{\rho} \leq \mathscr{K}(\rho) \leq c_1 c_U \frac{t}{r} \leq c_1 c_U \frac{\Phi(s)}{s}. \tag{2.2.72}$$

Thus, if $s < \rho \leq R_0$, using $L_{R_0}(\delta, c_1^{-1}, \Phi)$ and (2.2.72)

$$c_1^{-1} \Big(\frac{\rho}{s}\Big)^{\delta - 1} \leq \frac{\Phi(\rho)}{\rho} \Big/ \frac{\Phi(s)}{s} \leq c_1 c_U.$$

Thus, we conclude that there is $c_4 > 0$ such that $s > c_4 \rho$. Using this and (2.2.71), we have

$$\Phi_1(r,t) \leq 2\frac{r}{s} \leq 2c_4^{-1} \frac{r}{\rho} = 2c_4^{-1} \frac{r}{\mathscr{K}^{-1}(t/r)}.$$

When $L(\delta, \widetilde{C}_L, \Phi)$ holds, we may take $R_0 = \infty$ and $c_1 = \widetilde{C}_L^{-1}$. Then, the proof is same as above since Lemma 2.1.6 holds for all $r > 0$ and (2.2.21) holds for all $t > 0$ and $r \geq 2c_U^2 \Phi^{-1}(t)$. This completes the proof. $\square$

# Chapter 3

# Heat kernel estimates for nonsymmetric nonlocal operators

Let $d \geq 1$, $\mathbb{R}^d$ be the $d$-dimensional Euclidean space. Define

$$\mathcal{L}^\kappa f(x) := \lim_{\varepsilon \downarrow 0} \mathcal{L}^{\kappa,\varepsilon} f(x) := \lim_{\varepsilon \downarrow 0} \int_{\{z \in \mathbb{R}^d : |z| > \varepsilon\}} (f(x+z) - f(x))\, \kappa(x,z) J(|z|) dz \tag{3.0.1}$$

where $\kappa : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ is a Borel function satisfying the following conditions: there exist positive constants $\kappa_0, \kappa_1, \kappa_2$ and $\delta \in (0,1)$ such that

$$\kappa_0 \leq \kappa(x,z) \leq \kappa_1, \quad \kappa(x,z) = \kappa(x,-z) \quad \text{for all } x, z \in \mathbb{R}^d \tag{3.0.2}$$

and

$$|\kappa(x,z) - \kappa(y,z)| \leq \kappa_2 |x-y|^\delta \qquad \text{for all } x, y, z \in \mathbb{R}^d. \tag{3.0.3}$$

In [34], Zhen-Qing Chen and Xicheng Zhang studied the operator $\mathcal{L}^\kappa$ and its heat kernel when $J(r) = r^{-d-\alpha}$, $r > 0$ and $\alpha \in (0,2)$. They proved the existence and uniqueness of the heat kernel and its sharp two-sided estimates, cf. [34, Theorem 1.1] for details. The methods in [34] are quite robust and have been applied to non-symmetric and non-convolution operators (see [19, 24, 35, 36, 61, 55, 53] and references therein). In particular, [61] studied the operator $\mathcal{L}^\kappa$ and its heat kernel when $J$ is comparable to jumping kernels of subordinate Brownian motions and its Lévy exponent satisfying a weak lower scaling condition at infinity. In this chapter we introduce the result in [60], which deals with the case that $J(r)$ decays exponentially or subexponentially when $r \to \infty$ and we obtain sharp two-sided estimates for the heat kernel of $\mathcal{L}^\kappa$.

## 3.1 Jump processes with exponentially decaying kernel

In this section, we study the transition densities for a large class of non-symmetric Markov processes whose jumping kernels decay exponentially or subexponentially. We obtain their upper bounds which also decay at the same rate as their jumping kernels. When the lower bounds of jumping kernels satisfy the weak upper scaling condition at zero, we also establish lower bounds for the transition densities, which are sharp.

Again we consider the operator in (3.0.1) where $\kappa : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ is a Borel function satisfying (3.0.2) and (3.0.3).

We assume that $J : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and non-increasing function satisfying that there exist a continuous and strictly increasing function $\psi : [0, 1] \to \mathbb{R}_+$ with $\psi(0) = 0$, and constants $b > 0$, $0 < \beta \le 1$ and $a \ge 1$ such that

$$\frac{a^{-1}}{r^d\psi(r)} \le J(r) \le \frac{a}{r^d\psi(r)}, \quad 0 < r \le 1 \quad \text{and} \quad J(r) \le a\exp(-br^\beta), \quad r > 1. \tag{3.1.1}$$

In addition, we assume that $J$ is differentiable in $\mathbb{R}_+$ and

$$r \longmapsto -\frac{J'(r)}{r} \quad \text{is non-increasing in } \mathbb{R}_+. \tag{3.1.2}$$

Our main assumption on $\psi$ is the following weak lower scaling condition at zero: there exist $\alpha_1 \in (0,2]$ and $a_1 > 0$ such that

$$a_1\Big(\frac{R}{r}\Big)^{\alpha_1} \le \frac{\psi(R)}{\psi(r)}, \quad 0 < r \le R \le 1. \tag{3.1.3}$$

Since we allow $\alpha_1$ to be 2, to guarantee that $J$ is to be a Lévy density, we also need the following integrability condition for $\psi$ near zero:

$$\int_0^1 \frac{s}{\psi(s)} ds := C_0 < \infty. \tag{3.1.4}$$

The monotonicity of $J(r)$ and (3.1.4) ensure the existence of an isotropic unimodal Lévy process in $\mathbb{R}^d$ with the Lévy measure $J(|x|)dx$, which is infinite because of (3.1.3) and the lower bound in (3.1.1).

Our goal is to obtain estimates of the heat kernel for $\mathcal{L}^\kappa$. First we introduce the function $\mathscr{G}(t,x)$ which plays an important role for the estimates of heat kernel. Let us define the function $\Phi$ and $\theta$ as

$$\Phi(r) := \begin{cases} \dfrac{r^2}{2\int_0^r \frac{s}{\psi(s)} ds}, & 0 < r \le 1 \\ \Phi(1) r^2, & r > 1 \end{cases} \tag{3.1.5}$$

and

$$\theta(r) := \begin{cases} \dfrac{1}{r^d \Phi(r)}, & r \le 1, \\ \exp(-br^\beta)\mathbf{1}_{\{0<\beta<1\}} + r^{-d-1}\exp(-\frac{b}{5}r)\mathbf{1}_{\{\beta=1\}}, & r > 1. \end{cases}$$

Note that we define $\Phi(r) := \Phi(1)r^2$ for $r > 1$ since $\psi$ is defined in $(1,\infty)$. and we will study heat kernel estimates for small time.

By (3.1.4), $\int_0^r \frac{s}{\psi(s)} ds$ is integrable so that $\Phi$ is well-defined. Note that $\Phi(1) = \left(2\int_0^1 \frac{s}{\psi(s)} ds\right)^{-1} = (2C_0)^{-1}$ is determined by $C_0$. Note that as in Chapter 2, $\Phi$ is a strictly increasing function in $\mathbb{R}_+$ and $\lim_{r\downarrow 0} \Phi(r) = 0$. (See Subsection 2.1.1) So, there exists an inverse function $\Phi^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$. For

$t > 0$ and $r > 0$, define $\mathscr{G}(t, r)$ by

$$\mathscr{G}(t, r) = \mathscr{G}^{(d)}(t, r) := \frac{1}{t\Phi^{-1}(t)^d} \wedge \theta(r).$$

By an abuse of notation we also define

$$\mathscr{G}(t, x) = \mathscr{G}^{(d)}(t, x) := \frac{1}{t\Phi^{-1}(t)^d} \wedge \theta(|x|), \quad t > 0,\, x \in \mathbb{R}^d, \tag{3.1.6}$$

so $\mathscr{G}(t, x) = \mathscr{G}(t, |x|)$. Note that the definition of $\theta(r)$ for $\beta = 1$ is simply technical and it is harmless for readers to regard $\theta(r)$ as $\frac{1}{r^d\Phi(r)}\mathbf{1}_{\{r\le 1\}} + \exp(-\frac{b}{6}r)\mathbf{1}_{\{r>1\}}$ as the upper bound of heat kernel for $\beta = 1$ in Theorems 3.1.1-3.1.3 below.

Let us compare $\mathscr{G}$ with the following function defined by

$$\tilde{\mathscr{G}}(t, x) = \tilde{\mathscr{G}}(t, |x|) := \frac{1}{t\Phi^{-1}(t)^d} \wedge \frac{1}{|x|^d\Phi(|x|)}. \tag{3.1.7}$$

By [61, Proposition 2.1] and our Lemma 3.1.13 below we see that $\tilde{\mathscr{G}}$ is the function used for the upper heat kernel estimates in [61] (see Remark 3.1.6 for details). It is easy to see that $\mathscr{G}(t, x) \le c\tilde{\mathscr{G}}(t, x)$ (see Lemma 3.1.5 below). Here is our main result.

**Theorem 3.1.1.** *Let $\mathcal{L}^\kappa$ be the operator in* (3.0.1). *Assume that jumping kernel $J$ satisfies* (3.1.1) *and* (3.1.2), *that $\psi$ satisfies* (3.1.3) *and* (3.1.4), *and that $\kappa$ satisfies* (3.0.2) *and* (3.0.3). *Then, there exists a unique jointly continuous function $p^\kappa(t, x, y)$ on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ solving*

$$\partial_t p^\kappa(t, x, y) = \mathcal{L}^\kappa p^\kappa(t, \cdot, y)(x), \quad x \ne y, \tag{3.1.8}$$

*and satisfying the following properties:*

*(i) (Upper bound) For every $T \ge 1$, there is a constant $c_1 > 0$ such that for all $t \in (0, T]$ and $x, y \in \mathbb{R}^d$,*

$$p^\kappa(t, x, y) \le c_1 t\mathscr{G}(t, x - y). \tag{3.1.9}$$

*(ii) (Fractional derivative) For any $x, y \in \mathbb{R}^d$ with $x \neq y$, the map $t \mapsto \mathcal{L}^\kappa p^\kappa(t, \cdot, y)(x)$ is continuous, and for each $T \geq 1$, there exists a constant $c_2 > 0$ such that for all $t \in (0, T]$, $\varepsilon \in [0, 1]$ and $x, y \in \mathbb{R}^d$,*

$$|\mathcal{L}^{\kappa,\varepsilon} p^\kappa(t, \cdot, y)(x)| \leq c_2 \tilde{\mathscr{G}}(t, x-y). \tag{3.1.10}$$

*(iii) (Continuity) For any bounded and uniformly continuous function $f : \mathbb{R}^d \to \mathbb{R}$,*

$$\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p^\kappa(t, x, y) f(y) dy - f(x) \right| = 0. \tag{3.1.11}$$

*Furthermore, such unique function $p^\kappa(t, x, y)$ satisfies the following lower bound: for every $T \geq 1$, there exists a constant $c_3, c_4 > 0$ such that for all $t \in (0, T]$,*

$$p^\kappa(t, x, y) \geq c_3 \begin{cases} \Phi^{-1}(t)^{-d}, & |x-y| \leq c_4 \Phi^{-1}(t) \\ tJ(|x-y|), & |x-y| > c_4 \Phi^{-1}(t) \end{cases} \tag{3.1.12}$$

*The constants $c_i$, $i = 1, \dots, 4$, depend only on $d, T, a, a_1, \alpha_1, b, \beta, C_0, \delta, \kappa_0, \kappa_1$ and $\kappa_2$.*

The upper bound of the fractional derivative of $p^\kappa$ in (3.1.10), which is a counterpart of [61, (1.12)], will be used to prove the uniqueness of heat kernel.

We emphasize here that unlike [61, (1.21)] we obtain (3.1.12) without any upper weak scaling condition on $\psi$. The estimates in (3.1.9) and (3.1.12) in Theorem 3.1.1 are not sharp in general. However, when the jumping kernel $J$ satisfies

$$J(r) \geq a_1 \exp(-b_1 r^{\beta_1}), \quad r > 1, \tag{3.1.13}$$

and $\psi$ satisfies upper weak scaling condition at zero, that is,

$$\frac{\psi(R)}{\psi(r)} \leq a_2 \big(\frac{R}{r}\big)^{\alpha_2}, \quad 0 < t \leq R \leq 1 \tag{3.1.14}$$

where $a_2 > 0$ and $\alpha_2 \in (0, 2)$, then the lower bound in (3.1.12) is comparable to that in [49, Theorem 1.2], which is lower heat kernel estimates for symmetric Hunt process with exponentially decaying jumping kernel. Note that $\psi$ is comparable to $\Phi$ under (3.1.3) and (3.1.14). Therefore, under additional assumptions (3.1.14) and (3.1.13) we have the following corollary.

**Corollary 3.1.2.** *Let $\mathcal{L}^\kappa$ be the operator in* (3.0.1)*. Assume that jumping kernel $J$ satisfies* (3.1.1)*,* (3.1.2) *and* (3.1.13)*, that $\psi$ satisfies* (3.1.3) *and* (3.1.14)*, and that $\kappa$ satisfies* (3.0.2) *and* (3.0.3)*. Then, the heat kernel $p^\kappa(t,x,y)$ for $\mathcal{L}^\kappa$ satisfies the following estimates: for every $T \geq 1$, there is a constant $c_1 > 0$ such that for all $t \in (0,T]$ and $x, y \in \mathbb{R}^d$,*

$$\begin{aligned}
&c_1^{-1}\left(\psi^{-1}(t)^{-d} \wedge \frac{t}{|x-y|^d\psi(|x-y|)}\right)\\
&\leq p^\kappa(t,x,y) \leq c_1\left(\psi^{-1}(t)^{-d} \wedge \frac{t}{|x-y|^d\psi(|x-y|)}\right), \quad |x-y| \leq 1,
\end{aligned}$$

*and*

$$c_1^{-1} t \exp(-b_1|x-y|^{\beta_1}) \leq p^\kappa(t,x,y) \leq c_1 t\theta(|x-y|), \quad |x-y| > 1.$$

*The constant $c_1$ depend on $d, T, a, a_1, a_2, \alpha_1, \alpha_2, b, b_1, \beta, \beta_1, C_0, \delta, \kappa_0, \kappa_1$ and $\kappa_2$.*

Comparing to [61], Corollary 3.1.2 provides further precise heat kernel estimates for the operator (3.0.1) with exponential decaying function $J$. We remark here that, when $\beta > 1$, the estimates of $p^\kappa(t,x,y)$ are different and so the result in Corollary 3.1.2 does not hold even for symmetric Lévy processes. See [28, 89]. We will address this interesting case somewhere else.

More properties of the heat kernel $p^\kappa(t,x,y)$ are listed in the following theorems.

**Theorem 3.1.3.** *Suppose that the assumptions of Theorem 3.1.1 are satisfied.*

*(1) (Conservativeness) For all* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p^\kappa(t,x,y)\,dy = 1\,.$$

*(2) (Chapman-Kolmogorov equation) For all* $s,t>0$ *and* $x,y \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p^\kappa(t,x,z)p^\kappa(s,z,y)\,dz = p^\kappa(t+s,x,y)\,.$$

*(3) (Hölder continuity) For every* $T \ge 1$ *and* $\gamma \in (0,\alpha_1) \cap (0,1]$, *there is a constant* $c_1>0$ *such that for all* $0<t\le T$ *and* $x,x',y \in \mathbb{R}^d$ *with either* $x \ne y$ *or* $x' \ne y$,

$$|p^\kappa(t,x,y) - p^\kappa(t,x',y)| \le c_1|x-x'|^\gamma t\,\Phi^{-1}(t)^\gamma(\mathscr{G}(t,x-y) \vee \mathscr{G}(t,x'-y))\,. \tag{3.1.15}$$

*The constant* $c_1$ *depends only on* $d,T,a,a_1,\alpha_1,b,\beta,C_0,\gamma,\delta,\kappa_0,\kappa_1$ *and* $\kappa_2$.

For $t>0$, define the operator $P_t^\kappa$ by

$$P_t^\kappa f(x) = \int_{\mathbb{R}^d} p^\kappa(t,x,y)f(y)\,dy\,, \quad x \in \mathbb{R}^d\,, \tag{3.1.16}$$

where $f$ is a nonnegative (or bounded) Borel function on $\mathbb{R}^d$, and let $P_0^\kappa = \mathrm{Id}$. Then by Theorems 3.1.3, $(P_t^\kappa)_{t\ge0}$ is a Feller semigroup with the strong Feller property. Let $C_b^{2,\varepsilon}(\mathbb{R}^d)$ be the space of bounded twice differentiable functions in $\mathbb{R}^d$ whose second derivatives are uniformly Hölder continuous.

**Theorem 3.1.4.** *(1) (Generator) Let* $\varepsilon>0$. *For any* $f \in C_b^{2,\varepsilon}(\mathbb{R}^d)$, *we have*

$$\lim_{t\downarrow0}\frac{1}{t}\left(P_t^\kappa f(x) - f(x)\right) = \mathcal{L}^\kappa f(x)\,, \tag{3.1.17}$$

*and the convergence is uniform. (2) (Analyticity) The semigroup* $(P_t^\kappa)_{t\ge0}$ *of* $\mathcal{L}^\kappa$ *is analytic in* $L^p(\mathbb{R}^d)$ *for every* $p \in [1,\infty)$.

Note that we defined the function $\mathscr{G}(t,x)$ from the conditions on $J$ directly, while in [61] the function $\rho(t,x)$ is defined by the characteristic exponent of an isotropic unimodal Lévy process with jumping kernel $J(x)dx$. The reason is that, in our situation, it is more convenient than using characteristic exponent to describe exponential decaying jumping kernel. See Remark 3.1.6 below for the connections between two definitions.

In this section, we denote $\mathrm{diam}(A) = \sup\{|x-y| : x, y \in A\}$ and $\sigma(dz) = \sigma_d(dz)$ be a uniform measure in the sphere $\{z \in \mathbb{R}^d : |z| = 1\}$. For a function $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$, we define $f(t, x \pm z) = f(t, x+z) + f(t, x-z)$ and

$$\delta_f(t, x; z) := f(t, x+z) + f(t, x-z) - 2f(t, x) = f(t, x \pm z) - 2f(t, x). \quad (3.1.18)$$

### 3.1.1 Preliminaries

We start from the fact that our main results hold for all $t \le T$, while the definition of $\mathscr{G}$ in (3.1.6) is independent of $T$. To make our proofs simpler, we introduce a family of auxiliary functions which will be used mostly in proofs. Let $T \ge \Phi(1)$ and define $\mathscr{G}_T : (0, T] \times (0, \infty) \to (0, \infty)$ by

$$\mathscr{G}_T(t, r) = \begin{cases} \dfrac{1}{t\Phi^{-1}(t)^d}, & r \le \Phi^{-1}(t), \\ \dfrac{1}{r^d \Phi(r)}, & \Phi^{-1}(t) < r \le \Phi^{-1}(T), \\ C_T \exp(-br^\beta)\mathbf{1}_{0<\beta<1} + \frac{C_T}{r^{d+1}} \exp(-\frac{b}{5}r)\mathbf{1}_{\beta=1}, & r > \Phi^{-1}(T), \end{cases}$$

where $C_T := T^{-1}\Phi^{-1}(T)^{-d} e^{b\Phi^{-1}(T)^\beta}\mathbf{1}_{\beta<1} + T^{-1}\Phi^{-1}(T) e^{\frac{b}{4}\Phi^{-1}(T)}\mathbf{1}_{\beta=1}$. Note that $r \mapsto \mathscr{G}_T(t, r)$ is continuous and non-increasing (due to such choice of $C_T$).

Recall that $\tilde{\mathscr{G}}(t, r)$ is defined in (3.1.7). In the following lemma we show that $\mathscr{G}_T$ and $\mathscr{G}(t, x)$ are comparable and less than $\tilde{\mathscr{G}}(t, r)$.

**Lemma 3.1.5.** *(a) Let $T \ge \Phi(1)$. Then, there exists a constant $c_1 = c_1(T) > 0$ such that*

$$c_1^{-1}\mathscr{G}_T(t, r) \le \mathscr{G}(t, r) \le c_1 \mathscr{G}_T(t, r) \quad (3.1.19)$$

*for any $t \in (0, T]$ and $r > 0$.*

*(b) There exists a constant $c_2 > 0$ such that*

$$\mathscr{G}(t,r) \le c_2 \tilde{\mathscr{G}}(t,r). \tag{3.1.20}$$

*for any $t > 0$ and $r > 0$. The constant $c_1$ depends on $d, b, T, \Phi^{-1}(T), \beta$ and $C_0$, and $c_2$ depends on $d, b, \beta$ and $C_0$.*

**Proof.** (a) Define

$$\theta_T(r) := \begin{cases} r^{-d}\Phi(r)^{-1}, & r \le \Phi^{-1}(T), \\ C_T \exp(-br^\beta)\mathbf{1}_{0<\beta<1} + C_T r^{-d-1}\exp(-\frac{b}{5}r)\mathbf{1}_{\beta=1}, & r > \Phi^{-1}(T). \end{cases}$$

Note that $r \mapsto \theta_T(r)$ is strictly decreasing and $\theta_T(\Phi^{-1}(t)) = \frac{1}{t\Phi^{-1}(t)^d}$ for any $0 < t \le T$. Thus we can obtain

$$\theta_T(r) \le \frac{1}{t\Phi^{-1}(t)^d} \quad \text{if and only if} \quad t \le \Phi(r). \tag{3.1.21}$$

By (3.1.21) we have

$$\mathscr{G}_T(t,r) = \frac{1}{t\Phi^{-1}(t)^d} \wedge \theta_T(r). \tag{3.1.22}$$

Let

$$M_T := \begin{cases} \sup_{1\le r\le \Phi^{-1}(T)} \frac{1}{r^d\Phi(r)}\exp(br^\beta) & \text{for} \quad 0<\beta<1, \\ \sup_{1\le r\le \Phi^{-1}(T)} \frac{r}{\Phi(r)}\exp(\frac{b}{5}r) & \text{for} \quad \beta = 1 \end{cases}$$

and

$$m_T := \begin{cases} \inf_{1\le r\le \Phi^{-1}(T)} \frac{1}{r^d\Phi(r)}\exp(br^\beta) & \text{for} \quad 0<\beta<1, \\ \inf_{1\le r\le \Phi^{-1}(T)} \frac{r}{\Phi(r)}\exp(\frac{b}{5}r) & \text{for} \quad \beta = 1 \end{cases}$$

Then, for $0 < \beta < 1$,

$$\theta(r) = \begin{cases} \frac{1}{r^d\Phi(r)} = \theta_T(r), & r \le 1, \\ \exp(-br^\beta) \ge M_T^{-1}\frac{1}{r^d\Phi(r)} = M_T^{-1}\theta_T(r), & 1 < r \le \Phi^{-1}(T), \\ \exp(-br^\beta) \le m_T^{-1}\theta_T(r), & 1 < r \le \Phi^{-1}(T), \\ \exp(-br^\beta) = C_T^{-1}\theta_T(r), & r > \Phi^{-1}(T) \end{cases}$$

and for $\beta = 1$,

$$\theta(r) = \begin{cases} \frac{1}{r^d\Phi(r)} = \theta_T(r), & r \le 1, \\ \frac{1}{r^{d+1}}\exp(-\frac{b}{5}r) \ge M_T^{-1}\frac{1}{r^d\Phi(r)} = M_T^{-1}\theta_T(r), & 1 < r \le \Phi^{-1}(T), \\ \exp(-br^\beta) \le m_T^{-1}\theta_T(r), & 1 < r \le \Phi^{-1}(T), \\ \frac{1}{r^{d+1}}\exp(-\frac{b}{5}r) = C_T^{-1}\theta_T(r), & r > \Phi^{-1}(T). \end{cases}$$

Thus, for any $0 < \beta \le 1$ and $r > 0$,

$$(1 \wedge M_T^{-1} \wedge C_T^{-1})\theta_T(r) \le \theta(r) \le (1 \vee m_T^{-1} \vee C_T^{-1})\theta_T(r).$$

Using this and (3.1.22) we arrive (3.1.19).
(b) Clearly we have $\tilde{\mathscr{G}}(t,r) = \mathscr{G}(t,r)$ for $r \le 1$. For any $r > 1$ and $0 < \beta < 1$ we have

$$\tilde{\mathscr{G}}(t,r) = \frac{1}{r^d\Phi(r)} \ge \left(\sup_{s\ge 1} s^d\Phi(s)\exp(-bs^\beta)\right)^{-1}\exp(-br^\beta) = c(\beta)\mathscr{G}(t,r).$$

Similarly, for $r > 1$ and $\beta = 1$

$$\tilde{\mathscr{G}}(t,r) = \frac{1}{r^{d+2}\Phi(1)} \ge \left(\sup_{s\ge 1}\frac{\Phi(s)}{s}\exp(-\frac{b}{5}s)\right)^{-1}\frac{1}{r^{d+1}}e^{-\frac{b}{5}r} = c(1)\mathscr{G}(t,r).$$

Combining above estimates with (3.1.19) we arrive (3.1.20). □

In the following remark we will see that our $\tilde{\mathscr{G}}(t,x)$ and the function $\rho(t,x)$ in [61] are comparable.

**Remark 3.1.6.** Let $r(t,r) := \phi^{-1}(t^{-1})^d \wedge [t\phi(r^{-1})r^{-d}]$ as in [61], where $\phi$ is the characteristic exponent with respect to the Lévy process whose jumping kernel is $J(|y|)dy$. By Lemma 3.1.13 below we have $\phi(r^{-1})^{-1} \asymp \Phi(r)$ for all $r > 0$, which implies that $r(t,r)/t \asymp \tilde{\mathscr{G}}(t,r)$ for all $r > 0$. Thus, by [61, Proposition 2.1] we conclude that $\tilde{\mathscr{G}}(t,x)$ is comparable to the function $\rho(t,x)$ in [61].

### 3.1.2 Basic scaling inequalities.

Let $\Phi$ be the function in (3.1.5). Note that $L(\alpha_1, a_1, \Phi)$ and $U(2,1,\Phi)$ hold using $L_1(\alpha_1, a_1, \psi)$, (2.1.12) and Lemma 2.1.5. Also, by Lemma 1.1.4 we have that $L(1/2, 1, \Phi^{-1})$ and $U(1/\alpha_1, a_1^{-1/\alpha_1}, \Phi^{-1})$ hold. Now we introduce some scaling properties of $\mathscr{G}$ which will be used throughout this section.

**Lemma 3.1.7.** ([60, Lemma 2.5]) *Let $T \geq 1$ and $\varepsilon > 0$. Then, there exist constants $c_1, c_2 > 0$ such that for any $0 < t \leq T$, $x \in \mathbb{R}^d$ and $z \in \mathbb{R}^d$ satisfying $\Phi(|z|) \leq t$,*

$$\mathscr{G}(\varepsilon t, x) \leq c_1 \mathscr{G}(t,x) \tag{3.1.23}$$

*and*

$$\mathscr{G}(t, x+z) \leq c_2 \mathscr{G}(t,x), \tag{3.1.24}$$

*where $c_1$ depends only on $d, a_1, \alpha_1, \varepsilon$, and $c_2$ depends only on $d, T, a_1, \alpha_1, b, \beta$ and $C_0$.*

### 3.1.3 Convolution inequalities

In this section, we obtain some convolution inequalities for $\mathscr{G}(t,x)$ which will be used for Levi's method in Section 5. To get these inequalities we will use some estimates in [61, Section 2]. Note that by Remark 3.1.6 we already have convolution inequalities for $\tilde{\mathscr{G}}(t,x)$ (e.g. [61, Proposition 2.8]). For $a, b > 0$, let $B(a,b) = \int_0^1 s^{a-1}(1-s)^{b-1} ds = \frac{(a+b-1)!}{(a-1)!(b-1)!}$ be the beta function.

Using $L(1/2, c^{-1}, \Phi^{-1})$ and $U(1/\alpha_1, c, \Phi^{-1})$, the proof of the following lemma is same as the one in [61, Lemma 2.3]. Thus we skip the proof.

**Lemma 3.1.8.** *Assume that $\psi$ satisfies (3.1.3) and $\gamma, \delta \geq 0$, $\eta, \theta \in \mathbb{R}$ are constants satisfying $\mathbf{1}_{\gamma\geq 0}(\gamma/2)+\mathbf{1}_{\gamma<0}(\gamma/\alpha_1)+\delta/2+1-\eta > 0$. Then for every $t>0$, we have*

$$\int_0^t s^{-\eta}\Phi^{-1}(s)^{\gamma}(t-s)^{-\theta}\Phi^{-1}(t-s)^{\delta}\,ds \leq B(\frac{\delta}{2}+1-\theta, \frac{\gamma}{2}+1-\eta)t^{1-\eta-\theta}\Phi^{-1}(t)^{\gamma+\delta}. \tag{3.1.25}$$

For $0 \leq s \leq t$, let $g(s) := t^{\beta} + (2^{\beta}-1)s^{\beta} - (t+s)^{\beta}$. Then we can easily check that $g(0) = g(t) = 0$ and

$$g'(s) = \beta\left((2^{\beta}-1)s^{\beta-1} - (t+s)^{\beta-1}\right)\begin{cases} \geq 0, & s \in [0, kt], \\ \leq 0, & s \in [kt, t], \end{cases}$$

where $k := ((2^{\beta}-1)^{\frac{1}{\beta-1}}-1)^{-1} \in (0,1)$ is the constant satisfying $g'(kt) = 0$. Thus, we conclude that $g(s) \geq 0$ for any $0 \leq s \leq t$, which implies

$$t^{\beta}+s^{\beta} \geq (t+s)^{\beta}+(2-2^{\beta})(t^{\beta}\wedge s^{\beta}), \qquad \text{for all } 0<\beta<1 \text{ and } t,s>0. \tag{3.1.26}$$

Using (3.1.26) we prove the following lemma, which we need for our convolution inequalities.

**Lemma 3.1.9.** *(a) Let $0 < \beta < 1$ and $b > 0$. Then, there exists a constant $c_1 > 0$ such that for any $x \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} \exp(-b|x-z|^{\beta} - b|z|^{\beta})dz \leq c_1 \exp(-b|x|^{\beta}). \tag{3.1.27}$$

*(b) There exists a constant $c_2 > 0$ such that for any $x \in \mathbb{R}^d$ with $|x| \geq 1$,*

$$\int_{\mathbb{R}^d} (|x-z|^{-d-1}\wedge 1)(|z|^{-d-1}\wedge 1)dz \leq c_2|x|^{-d-1}. \tag{3.1.28}$$

*The constant $c_1$ depends only on $b$, $d$ and $\beta$, and $c_2$ depends only on $d$.*

**Proof.** (a) Let

$$c_1 = 2\int_{\mathbb{R}^d} \exp(-b(2-2^\beta)|z|^\beta)dz < \infty.$$

Using (3.1.26) for the second line, we arrive

$$\begin{aligned}
&\int_{\mathbb{R}^d} \exp(-b|x-z|^\beta - b|z|^\beta)dz \\
&\le \int_{\mathbb{R}^d} \exp(-b|x|^\beta)\exp\big(-b(2-2^\beta)(|z|^\beta \wedge |x-z|^\beta)\big)dz \\
&\le \exp(-b|x|^\beta)\left(\int_{\mathbb{R}^d} \exp(-b(2-2^\beta)|z|^\beta)dz + \int_{\mathbb{R}^d} \exp(-b(2-2^\beta)|x-z|^\beta)dz\right) \\
&= c_1 \exp(-b|x|^\beta).
\end{aligned}$$

This proves (3.1.27).
(b) Using $|x-z|^{-1} \wedge |z|^{-1} \le 2|x|^{-1}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} (|x-z|^{-d-1} \wedge 1)(|z|^{-d-1} \wedge 1)dz \\
\le&(\frac{2}{|x|})^{d+1}\left(\int_{|x-z|\ge|z|} (|z|^{-d-1} \wedge 1)dz + \int_{|x-z|<|z|} (|x-z|^{-d-1} \wedge 1)dz\right) \\
\le&(\frac{2}{|x|})^{d+1}\left(\int_{\mathbb{R}^d} (|z|^{-d-1} \wedge 1)dz + \int_{\mathbb{R}^d} (|x-z|^{-d-1} \wedge 1)dz\right) := c_2|x|^{-d-1}.
\end{aligned}$$

This concludes the lemma. □

For $\gamma, \delta \in \mathbb{R}$, $t > 0$ and $x \in \mathbb{R}^d$ we define

$$\mathscr{G}_\gamma^\delta(t,x) := \Phi^{-1}(t)^\gamma(|x|^\delta \wedge 1)\mathscr{G}(t,x) \text{ and } \tilde{\mathscr{G}}_\gamma^\delta(t,x) := \Phi^{-1}(t)^\gamma(|x|^\delta \wedge 1)\tilde{\mathscr{G}}(t,x).$$

Note that $\mathscr{G}_0^0(t,x) = \mathscr{G}(t,x)$, and $\tilde{\mathscr{G}}_\gamma^\delta(t,x)$ is comparable to the function $\rho_\gamma^\delta(t,x)$ in [61] by Remark 3.1.6. Also, we can easily check that for $T \ge \Phi(1)$,

$$\mathscr{G}_{\gamma_1}^\delta(t,x) \le \Phi^{-1}(T)^{\gamma_1-\gamma_2}\mathscr{G}_{\gamma_2}^\delta(t,x), \qquad (t,x) \in (0,T] \times \mathbb{R}^d, \quad \gamma_2 \le \gamma_1, \tag{3.1.29}$$

$$\mathscr{G}_\gamma^{\delta_1}(t,x) \le \mathscr{G}_\gamma^{\delta_2}(t,x), \qquad (t,x) \in (0,\infty) \times \mathbb{R}^d, \quad 0 \le \delta_2 \le \delta_1 \tag{3.1.30}$$

We record the following inequality which immediately follows from (3.1.29) and (3.1.30): for any $T \ge \Phi(1)$, $\delta \ge 0$ and $(t,x) \in (0,T] \times \mathbb{R}^d$,

$$\big(\mathscr{G}_0^\delta + \mathscr{G}_\delta^0\big)(t,x) \le \big(\Phi^{-1}(T)^\delta + 1\big)\mathscr{G}(t,x) \le 2\Phi^{-1}(T)^\delta \mathscr{G}(t,x). \tag{3.1.31}$$

Now we are ready to introduce convolution inequalities for $\mathscr{G}(t,x)$.

**Proposition 3.1.10.** *Assume that $\psi$ satisfies* (3.1.3). *Let $T \ge 1$ and $0 < \alpha < \alpha_1$.*
*(a) There exists a constant $c = c(d,T,a_1,\alpha,\alpha_1) > 0$ such that for any $0 < t \le T$, $\delta \in [0,\alpha]$ and $\gamma \in \mathbb{R}$,*

$$\int_{\mathbb{R}^d} \tilde{\mathscr{G}_\gamma^\delta}(t,x)\,dx \le ct^{-1}\Phi^{-1}(t)^{\gamma+\delta}. \tag{3.1.32}$$

*(b) There exists $C = C(\alpha,T) = C(d,T,a_1,\alpha,\alpha_1,b,\beta) > 0$ such that for all $x \in \mathbb{R}^d, \delta_1, \delta_2 \ge 0$ with $\delta_1 + \delta_2 \le \alpha$, $\gamma_1, \gamma_2 \in \mathbb{R}$ and $0 < s < t \le T$,*

$$\begin{aligned} &\int_{\mathbb{R}^d} \mathscr{G}_{\gamma_1}^{\delta_1}(t-s,x-z)\mathscr{G}_{\gamma_2}^{\delta_2}(s,z)\,dz \\ &\le C\Big((t-s)^{-1}\Phi^{-1}(t-s)^{\gamma_1+\delta_1+\delta_2}\Phi^{-1}(s)^{\gamma_2}\mathscr{G}(t,x) \\ &+ \Phi^{-1}(t-s)^{\gamma_1}s^{-1}\Phi^{-1}(s)^{\gamma_2+\delta_1+\delta_2}\mathscr{G}(t,x) \\ &+ (t-s)^{-1}\Phi^{-1}(t-s)^{\gamma_1+\delta_1}\Phi^{-1}(s)^{\gamma_2}\mathscr{G}_0^{\delta_2}(t,x) \\ &+ \Phi^{-1}(t-s)^{\gamma_1}s^{-1}\Phi^{-1}(s)^{\gamma_2+\delta_2}\mathscr{G}_0^{\delta_1}(t,x)\Big). \end{aligned} \tag{3.1.33}$$

*In particular, letting $\gamma_1 = \gamma_2 = \delta_1 = \delta_2 = 0$ in* (3.1.33) *we have*

$$\int_{\mathbb{R}^d} \mathscr{G}(t-s,x-z)\mathscr{G}(s,z)dz \le 2C\big(s^{-1} + (t-s)^{-1}\big)\mathscr{G}(t,x). \tag{3.1.34}$$

*(c) For all $x \in \mathbb{R}^d$, $0 < t \le T$, $\delta_1, \delta_2 \ge 0$ and $\theta, \eta \in [0,1]$ satisfying $\delta_1 + \delta_2 \le \alpha$, $\mathbf{1}_{\gamma_1 \ge 0}(\gamma_1/2) + \mathbf{1}_{\gamma_1 < 0}(\gamma_1/\alpha_1) + \delta_1/2 + 1 - \theta > 0$ and $\mathbf{1}_{\gamma_2 \ge 0}(\gamma_2/2) +$*

$\mathbf{1}_{\gamma_2<0}(\gamma_2/\alpha_1)+\delta_2/2+1-\eta>0$, *we have a constant* $C_2>0$ *satisfying*

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^d}(t-s)^{1-\theta}\mathscr{G}^{\delta_1}_{\gamma_1}(t-s,x-z)s^{1-\eta}\mathscr{G}^{\delta_2}_{\gamma_2}(s,z)\,dz\,ds\\ \le& C_2t^{2-\theta-\eta}\left(\mathscr{G}^{0}_{\gamma_1+\gamma_2+\delta_1+\delta_2}+\mathscr{G}^{\delta_1}_{\gamma_1+\gamma_2+\delta_2}+\mathscr{G}^{\delta_2}_{\gamma_1+\gamma_2+\delta_1}\right)(t,x)\end{aligned} \tag{3.1.35}$$

*for any* $0<t\le T$ *and* $x\in\mathbb{R}^d$*. Moreover, when* $\gamma_1,\gamma_2\ge 0$ *we further have*

$$C_2=4C\,B\left(\frac{\gamma_1+\delta_1}{2}+1-\theta,\frac{\gamma_2+\delta_2}{2}+1-\eta\right). \tag{3.1.36}$$

**Proof.** (a) See [61, Lemma 2.6(a)].
(b) By (3.1.19), it suffices to show (3.1.33) with the function $(\mathscr{G}_T)^{\delta}_{\gamma}(t,x):=\Phi^{-1}(t)^{\gamma}(|x|^{\delta}\wedge 1)\mathscr{G}_T(t,x)$. Without loss of generality we assume $T\ge\Phi(1)$ and for notational convenience we drop $T$ in the notations so we use $\mathscr{G}(t,x)$ and $\mathscr{G}^{\delta}_{\gamma}(t,x)$ instead of $\mathscr{G}_T(t,x)$ and $(\mathscr{G}_T)^{\delta}_{\gamma}(t,x)$ respectively.

First let $|x|\le\Phi^{-1}(T)$. By Remark 3.1.6 and [61, Lemma 2.6(b)], we already have that there exists $c_1>0$ satisfying (3.1.33) with $\tilde{\mathscr{G}}$. Note that $\mathscr{G}(t,x)=\tilde{\mathscr{G}}(t,x)$ since $|x|\le\Phi^{-1}(T)$. Using (3.1.20) for the left-hand side and $\mathscr{G}(t,x)=\tilde{\mathscr{G}}(t,x)$ for the right-hand side, we obtain (3.1.33) for $|x|\le\Phi^{-1}(T)$.

Now assume $|x|>\Phi^{-1}(T)$ and observe that

$$\begin{aligned}&\int_{\mathbb{R}^d}\mathscr{G}^{\delta_1}_{\gamma_1}(t-s,x-z)\mathscr{G}^{\delta_2}_{\gamma_2}(s,z)dz=\\ &\left(\int_{\substack{|z|>\Phi^{-1}(T),\\|x-z|>\Phi^{-1}(T)}}+\int_{\substack{|z|>\Phi^{-1}(T),\\|x-z|\le\Phi^{-1}(T)}}+\int_{\substack{|z|\le\Phi^{-1}(T),\\|x-z|>\Phi^{-1}(T)}}+\int_{\substack{|z|\le\Phi^{-1}(T),\\|x-z|\le\Phi^{-1}(T)}}\right)\mathscr{G}^{\delta_1}_{\gamma_1}(t-s,x-z)\mathscr{G}^{\delta_2}_{\gamma_2}(s,z)dz\\ &:=I_1+I_2+I_3+I_4.\end{aligned}$$

First we assume $0<\beta<1$ and obtain upper bounds for $I_i$, $i=1,\dots 4$. For

$I_1$, using $\Phi^{-1}(T) \geq 1$ we have

$$I_1 = \int_{|x-z|>\Phi^{-1}(T),|z|>\Phi^{-1}(T)} \mathscr{G}_{\gamma_1}^{\delta_1}(t-s, x-z)\mathscr{G}_{\gamma_2}^{\delta_2}(s,z)dz \tag{3.1.37}$$
$$= \Phi^{-1}(t-s)^{\gamma_1}\Phi^{-1}(s)^{\gamma_2} \int_{|x-z|>\Phi^{-1}(T),|z|>\Phi^{-1}(T)} \exp(-b|x-z|^\beta - b|z|^\beta)dz.$$

By (3.1.27) we obtain

$$I_1 \leq c_1\Phi^{-1}(t-s)^{\gamma_1}\Phi^{-1}(s)^{\gamma_2}\exp(-b|x|^\beta) = c_1\Phi^{-1}(t-s)^{\gamma_1}\Phi^{-1}(s)^{\gamma_2}\mathscr{G}(t,x)$$
$$\leq c_2(t-s)^{-1}\Phi^{-1}(t-s)^{\gamma_1+\delta_1+\delta_2}\Phi^{-1}(s)^{\gamma_2}\mathscr{G}(t,x),$$

where we used $\delta_1, \delta_2 \geq 0$ and $t-s \leq T$ for the last line. For the estimates of $I_2, I_3$ and $I_4$ we omit counterpart of the last line above.

For $I_2$, we have

$$I_2 = \int_{|x-z|\leq\Phi^{-1}(T),|z|>\Phi^{-1}(T)} \mathscr{G}_{\gamma_1}^{\delta_1}(t-s, x-z)\mathscr{G}_{\gamma_2}^{\delta_2}(s,z)dz$$
$$= \Phi^{-1}(t-s)^{\gamma_1}\Phi^{-1}(s)^{\gamma_2} \int_{|x-z|\leq\Phi^{-1}(T),|z|>\Phi^{-1}(T)} \tilde{\mathscr{G}}_0^{\delta_1}(t-s, x-z)\exp(-b|z|^\beta)dz.$$

Since $|x-z| \leq \Phi^{-1}(T)$, using triangular inequality we have

$$\exp(-b|z|^\beta) \leq \exp(-b|x|^\beta)\exp(b|x-z|^\beta) \leq \exp(b\Phi^{-1}(T)^\beta)\exp(-b|x|^\beta). \tag{3.1.38}$$

Thus by (3.1.32),

$$I_2 \leq \Phi^{-1}(t-s)^{\gamma_1}\Phi^{-1}(s)^{\gamma_2}\exp(-b|x|^\beta)\int_{\mathbb{R}^d} \tilde{\mathscr{G}}_0^{\delta_1}(t-s, x-z)dz$$
$$\leq c_3(t-s)^{-1}\Phi^{-1}(t-s)^{\gamma_1+\delta_1}\Phi^{-1}(s)^{\gamma_2}\mathscr{G}(t,x).$$

By the similar way, we obtain

$$I_3 \leq c_3 s^{-1}\Phi^{-1}(t-s)^{\gamma_1}\Phi^{-1}(s)^{\gamma_2+\delta_2}\mathscr{G}(t,x).$$

When $|x| \geq 2\Phi^{-1}(T)$, we have $I_4 = 0$. So we can assume $|x| < 2\Phi^{-1}(T)$ without loss of generality for the estimate of $I_4$. By (3.1.20) we have

$$I_4 \leq \int_{\mathbb{R}^d} \tilde{\mathscr{G}}^{\delta_1}_{\gamma_1}(t-s, x-z)\tilde{\mathscr{G}}^{\delta_2}_{\gamma_2}(s,z)dz \leq c_4\tilde{\mathscr{G}}(t,x).$$

Using $\tilde{\mathscr{G}}(t,x) \leq \tilde{\mathscr{G}}(t, \Phi^{-1}(T)) \leq \exp(b\Phi^{-1}(T)^\beta)\mathscr{G}(t,x)$, we can obtain desired estimates. Combining estimates for $I_1, I_2, I_3$ and $I_4$, we arrive (3.1.33) for $0 < \beta < 1$.

For the case $\beta = 1$, estimate for $I_4$ is same as above. For $I_2$ and $I_3$, instead of (3.1.38) we argue as the following: using $|x-z| \leq \Phi^{-1}(T)$ and $|x|, |z| \geq \Phi^{-1}(T)$, we have

$$\frac{1}{|z|^{d+1}}\exp(-\frac{b}{5}|z|) \leq \frac{2^{d+1}}{|x|^{d+1}}\exp(\frac{b}{5}\Phi^{-1}(T))\exp(-\frac{b}{5}|x|).$$

For $I_1$, following (3.1.37) and using (3.1.28) for the fourth line and $U(1/\alpha_1, a_1^{-1/\alpha_1}, \Phi^{-1})$ for the fifth line we have

$$\begin{aligned}
&I_1 = \Phi^{-1}(t-s)^{\gamma_1}\Phi^{-1}(s)^{\gamma_2}\int_{|x-z|>\Phi^{-1}(T), |z|>\Phi^{-1}(T)} \frac{e^{-\frac{b}{5}|x-z|-\frac{b}{5}|z|}}{|x-z|^{d+1}|z|^{d+1}}dz \\
&\leq c_1\Phi^{-1}(t-s)^{\gamma_1}\Phi^{-1}(s)^{\gamma_2}\exp(-\frac{b}{5}|x|)\int_{|x-z|>1, |z|>1}\frac{1}{|x-z|^{d+1}|z|^{d+1}}dz \\
&\leq c_1\Phi^{-1}(t-s)^{\gamma_1}\Phi^{-1}(s)^{\gamma_2}\exp(-\frac{b}{5}|x|)\int_{\mathbb{R}^d}\big(1\wedge|x-z|^{-d-1}\big)\big(1\wedge|z|^{-d-1}\big)dz \\
&\leq c_2\Phi^{-1}(t-s)^{\gamma_1}\Phi^{-1}(s)^{\gamma_2}\frac{1}{|x|^{d+1}}\exp(-\frac{b}{5}|x|) = c_2\Phi^{-1}(t-s)^{\gamma_1}\Phi^{-1}(s)^{\gamma_2}\mathscr{G}(t,x) \\
&\leq c_3(t-s)^{-1}\Phi^{-1}(t-s)^{\gamma_1+\delta_1+\delta_2}\Phi^{-1}(s)^{\gamma_2}\mathscr{G}(t,x).
\end{aligned}$$

(c) Integrating (3.1.33) with respect to $s$ from 0 to $t$. With (3.1.25), we can follow the proof of [61, Lemma 2.6(c)]. □

### 3.1.4 Heat kernel estimates for Lévy processes

Following the framework of [34, 61], we need the upper bounds of derivatives of the heat kernel for the symmetric Lévy process whose jumping kernel is $J(|y|)$ (see, for example, [61, Proposition 3.2]). To be more precise, in our case, to get the upper bound of heat kernel for non-symmetric operator of the form (3.0.1), we need correct upper bounds of the first and second order derivatives of the heat kernel for unimodal Lévy processes. In this section, we will prove that (3.1.1) and (3.1.2) are sufficient condition for the estimates of the second order derivatives in Proposition 3.1.12, which decay exponentially or subexponentially.

In this section, we fix $T \leq [1, \infty)$ and let $\nu(dy) = \nu(|y|)dy$ be an isotropic measure in $\mathbb{R}^d$ satisfying $\int_{\mathbb{R}^d} \left(1 \wedge |y|^2\right) \nu(dy) < \infty$. Throughout this section we further assume that $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ is non-increasing, differentiable function.

Here are our goals in this subsection.

**Proposition 3.1.11.** *Let $X$ be an isotropic unimodal Lévy process in $\mathbb{R}^d$ with Lévy measure $\nu(|y|)dy$ satisfying the following assumptions: $\psi$ is a non-decreasing function with $\psi(0) = 0$ satisfying* (3.1.3) *and* (3.1.4)*, and there exist constants $a > 0$ and $0 < \beta \leq 1$ such that*

$$\frac{a^{-1}}{r^d \psi(r)} \leq \nu(r) \leq \frac{a}{r^d \psi(r)}, \quad 0 < r \leq 1 \quad \text{and} \quad \nu(r) \leq a \exp(-br^{\beta}), \quad r > 1. \tag{3.1.39}$$

*Then its transition density $x \mapsto p_t(x)$ is in $C_b^\infty(\mathbb{R}^d)$ and satisfies gradient estimates*

$$|\nabla_x^k p_t(x)| \leq ct\,\mathscr{G}_{-k}^0(t,x) = \Phi^{-1}(t)^{-k} \left( \frac{1}{t\Phi^{-1}(t)^d} \wedge \theta(|x|) \right), \quad k = 0, 1$$

*for any $0 < t \leq T$ and $x \in \mathbb{R}^d$.*

With the above result, we can obtain the second gradient estimate for the isotropic unimodal Lévy process whose jumping kernel satisfies (3.1.1) and (3.1.2).

**Proposition 3.1.12.** *Suppose that $\psi$ is nondecreasing function with $\psi(0) = 0$ satisfying (3.1.3) and (3.1.4), and that Lévy measure $J(|y|)dy$ satisfies (3.1.1) and (3.1.2) with $0 < \beta \le 1$. Then, its corresponding transition density $x \mapsto p(t,x)$ is in $C_b^\infty(\mathbb{R}^d)$ and satisfies gradient estimates*

$$|\nabla_x^k p(t,x)| \le ct\,\mathscr{G}^0_{-k}(t,x) = \Phi^{-1}(t)^{-k}\left(\frac{1}{t\Phi^{-1}(t)^d} \wedge \theta(|x|)\right) \tag{3.1.40}$$

*for $k = 1,2,3$, for any $0 < t \le T$ and $x \in \mathbb{R}^d$.*

Now we will combine some results in [67, 54, 55] to prove Proposition 3.1.11. Recall that we have assumed that $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ is non-increasing differentiable function satisfying $\int_{\mathbb{R}^d}\left(1 \wedge |y|^2\right)\nu(|y|)dy < \infty$. In this subsection, instead of the function $\Phi$, we mainly use

$$\varphi(r) := \begin{cases} \dfrac{r^2}{\int_0^r s^{d+1}\nu(s)ds}, & 0 < r \le 1, \\ \varphi(1)r^2, & r > 1, \end{cases} \tag{3.1.41}$$

Note that the integral $\int_0^r s^{d+1}\nu(s)ds$ above is finite because of our assumption $\int_{\mathbb{R}^d}\left(1 \wedge |y|^2\right)\nu(|y|)dy < \infty$.

To prove Propositions 3.1.11 and 3.1.12 at once, we need to consider the following conditions on Lévy measure $\nu(|y|)dy$ which is slightly more general than (3.1.39). We assume that there exist constants $a > 0, 0 < \beta \le 1$ and $\ell \ge 0$ such that

$$\nu(r) \le ar^{-\ell}\exp(-br^\beta), \quad r > 1. \tag{3.1.42}$$

Also, we assume that there exist $a_3 > 0$ and $\alpha_3 \in (0,2]$ such that

$$a_3\Big(\frac{R}{r}\Big)^{\alpha_3} \le \frac{\varphi(R)}{\varphi(r)}, \quad 0 < r \le R < \infty. \tag{3.1.43}$$

For instance, when $X$ is an isotropic Lévy process in Proposition 3.1.11 we have $\frac{s}{a\psi(s)} \le \nu(s)s^{d+1} \le \frac{as}{\psi(s)}$, which implies $\varphi(r) \asymp \Phi(r)$. Using this and (2.1.12) we obtain (3.1.43) with $\alpha_3 = \alpha_1$. Thus, the conditions in Proposition 3.1.11 imply (3.1.42) and (3.1.43).

Under (3.1.43), we have $\varphi(r) \le cr^{\alpha_3}$ for $r \le 1$ so that

$$c^{-1}r^{-\alpha_3} \le \int_0^r \frac{s^{d+1}}{r^2}\nu(s)ds \le \int_0^r s^{d-1}\nu(s)ds \le \int_0^1 s^{d-1}\nu(s)ds, \quad r \le 1.$$

Thus, letting $r \downarrow 0$ we obtain $\int_0^1 s^{d-1}\nu(s)ds = \infty$. Also, as in (2.1.12) we obtain

$$\frac{\varphi(R)}{\varphi(r)} \le \big(\frac{R}{r}\big)^2, \quad 0 < r \le R. \tag{3.1.44}$$

In addition, since $\nu$ is non-increasing, we have for $r < 1$,

$$\varphi(r)^{-1} = r^{-2}\int_0^r s^{d+1}\nu(s)ds \ge r^{-2}\int_0^r s^{d+1}\nu(r)dr = \frac{r^d\nu(r)}{d+2}. \tag{3.1.45}$$

In this subsection except the proofs of Propositions 3.1.11 and 3.1.12 we will always assume that $\nu$ satisfies (3.1.42) and (3.1.43). Let $X$ be the Lévy process with Lévy measure $\nu(|y|)dy$, and $\xi \mapsto \phi(|\xi|)$ be the characteristic exponent of $X$. First note that $\nu(\mathbb{R}^d) = \int_{\mathbb{R}^d} \nu(|y|)dy = \infty$ because $\int_0^1 s^{d-1}\nu(s)ds = \infty$. Also, since $X$ is isotropic, characteristic exponent of $X$ is also isotropic function. Define $\Psi(r) := \sup_{|y|\le r}\phi(|y|)$ and let $\mathcal{P}(r) := \int_{\mathbb{R}^d}\big(1 \wedge \frac{|y|^2}{r^2}\big)\nu(|y|)dy$ be the Pruitt function for $X$. By [17, Lemma 1 and Proposition 2], we have that for $r > 0$,

$$\frac{2}{\pi^2 d}\mathcal{P}(r^{-1}) \le \phi(r) \le \Psi(r) \le \pi^2\phi(r) \le 2\pi^2\mathcal{P}(r^{-1}), \quad r > 0. \tag{3.1.46}$$

Using (3.1.46), we can prove the following lemma.

**Lemma 3.1.13.** *Assume that $\nu(|y|)dy$ satisfies* (3.1.42) *and* (3.1.43). *Then, $\Psi(r)$ is comparable to $\varphi(r^{-1})^{-1}$, i.e., there exists a constant $c > 0$ such that*

$$c^{-1}\varphi(r^{-1})^{-1} \le \Psi(r) \le c\varphi(r^{-1})^{-1}, \quad r > 0. \tag{3.1.47}$$

**Proof.** We claim that

$$\mathcal{P}(r) \asymp \varphi(r)^{-1} \quad \text{for} \quad r > 0. \tag{3.1.48}$$

First assume $r \le 1$ and observe that

$$\begin{aligned}
\mathcal{P}(r) &= \int_{\mathbb{R}^d} \big(1 \wedge \frac{|z|^2}{r^2}\big)\nu(z)dz \\
&= c(d)\left( r^{-2}\int_0^r s^{d+1}\nu(s)ds + \int_r^1 s^{d-1}\nu(s)ds + \int_1^\infty s^{d-1}\nu(s)ds \right) \\
&=: c(d)\big(I_1 + I_2 + I_3\big).
\end{aligned}$$

By the definition of $\varphi$ we have $I_1 = \varphi(r)^{-1}$. To estimate $I_2$, let us define $k := \lfloor \frac{\log r}{\log 2} \rfloor$, the largest integer smaller than or equal to $\frac{\log r}{\log 2}$. Then we have

$$0 \le I_2 \le \sum_{i=0}^{k} \int_{2^i r}^{2^{i+1} r} s^{d-1}\nu(s)ds =: \sum_{i=0}^{k} I_{2i}.$$

Using (3.1.43), we have

$$I_{2i} \le (2^i r)^{-2} \int_{2^i r}^{2^{i+1} r} s^{d+1}\nu(s)ds = 4\varphi(2^{i+1}r)^{-1} \le a_3 2^{2-\alpha_3(i+1)}\varphi(r)^{-1}.$$

Thus,

$$0 \le I_2 \le \sum_{i=0}^{k} I_{2i} \le \frac{2^{2-\alpha_3}}{\varphi(r)} \sum_{i=0}^{k} 2^{-\alpha_3 i} \le \frac{c_1}{\varphi(r)}. \tag{3.1.49}$$

Also, using (3.1.42) and (3.1.43) we obtain

$$0 \le I_3 \le a \int_1^\infty s^{d-\ell-1}\exp(-bs^\beta)ds = c_2 \le \frac{c_2\varphi(1)}{a_3\varphi(r)},$$

where we used $a_3 \le a_3\big(\frac{1}{r}\big)^{\alpha_3} \le \frac{\varphi(1)}{\varphi(r)}$ for the last inequality. Combining estimates of $I_1, I_2$ and $I_3$ we have proved the claim (3.1.48) for $r \le 1$.

Now assume $r > 1$. Then we have

$$\begin{aligned}\mathcal{P}(r) &= \int_{\mathbb{R}^d} \big(1 \wedge \frac{|z|^2}{r^2}\big)\nu(z)dz \\ &= c(d)\left(r^{-2}\int_0^1 s^{d+1}\nu(s)ds + \int_1^\infty \big(1 \wedge \frac{s^2}{r^2}\big)s^{d-1}\nu(s)ds\right) \\ &:= c(d)(\varphi(r)^{-1} + I_4).\end{aligned}$$

Also, using (3.1.42) we have

$$0 \le I_4 \le \int_1^\infty \frac{s^2}{r^2}s^{d-1}\nu(s)ds \le ar^{-2}\int_1^\infty s^{d-\ell+1}\exp(-bs^\beta)ds \le c_3 r^{-2}.$$

Using $\varphi(r) = \varphi(1)r^2$ for $r \ge 1$ we obtain that $\mathcal{P}(r) \asymp r^{-2} \asymp \varphi(r)^{-1}$ for $r > 1$, which implies (3.1.48) for $r > 1$. Therefore, (3.1.48) holds for any $r > 0$. Combining (3.1.48) and (3.1.46) we conclude the lemma. $\square$

Using (3.1.47), (3.1.43) and (3.1.44) we obtain the following weak scaling condition for $\Psi$: there exists a constant $c > 0$ such that

$$c^{-1}\big(\frac{R}{r}\big)^{\alpha_3} \le \frac{\Psi(R)}{\Psi(r)} \le c\big(\frac{R}{r}\big)^2, \quad 0 < r \le R < \infty. \tag{3.1.50}$$

Let $p_t(x)$ be a transition density of $X$. Since $X$ is isotropic, $x \mapsto p_t(x)$ is also isotropic function for any $t > 0$. By an abuse of notation we also denote the radial part of the heat kernel $p_t(x)$ of $X$ as $p_t(r)$, $r > 0$.

To obtain gradient estimate for $p_t(x)$, we first follow the proof of [67, Proposition 3.1] to construct a $(d+2)$-dimensional Lévy process $Y$ whose heat kernel estimate implies gradient estimate of $X$.

**Lemma 3.1.14.** *Assume that isotropic unimodal Lévy measure $\nu$ satisfies* (3.1.42) *and* (3.1.43). *Then there exists an isotropic Lévy process $Y$ in $\mathbb{R}^{d+2}$ such that its characteristic exponent is $\xi \mapsto \phi(|\xi|)$, $\xi \in \mathbb{R}^{d+2}$. Let $\nu_1(|x|)$ and $q_t(|x|)$ be the jumping kernel and heat kernel of $Y$, respectively. Then for any $r > 0$,*

$$q_t(r) = -\frac{1}{2\pi r}\frac{d}{dr}p_t(r) \tag{3.1.51}$$

*and*

$$\nu_1(r) = -\frac{1}{2\pi r}\nu'(r). \tag{3.1.52}$$

**Proof.** The existence of $Y$ and (3.1.51) are immediately followed by [67, Proposition 3.1]. Note that using (3.1.46) and (3.1.50) we have

$$\lim_{\rho\to\infty}\frac{\phi(\rho)}{\log\rho} \geq \lim_{\rho\to\infty}\frac{\Psi(\rho)}{\pi^2\log\rho} \geq \lim_{\rho\to\infty}\frac{c_1\rho^{\alpha_3}}{\log\rho} = \infty,$$

which is one of the conditions in [67, Proposition 3.1]. For (3.1.52), we just need to follow the corresponding part in the proof of [67, Theorem 1.5]. Here we provide a brief sketch for the proof for reader's convenience; As in the proof of [67, Theorem 1.5], without using the assumption that $-\nu'(r)/r$ is non-increasing, one can show that there exists an isotropic Lévy process $X^{(d+2)}$ in $\mathbb{R}^{d+2}$ with jumping kernel $\nu_1(dy)$ and that the characteristic exponent of $X^{(d+2)}$ is $\phi(r)$. Thus, $X^{(d+2)}$ and $Y$ are identical in law, which concludes the proof. To show this, only [67, (8) and (9)] are used, which follow directly from the fact that $\nu$ is isotropic, unimodal measure satisfying $\int_{\mathbb{R}^d}\left(|y|^2\wedge 1\right)\nu(dy) < \infty$. $\square$

We emphasize here that we don't impose the condition (3.1.2) on $\nu$. Thus the function $r \to \nu_1(r)$ in the above lemma may not be non-increasing.

Now we are going to establish heat kernel estimates for the process $Y$ obtained in Lemma 3.1.14, which will imply heat kernel estimate and gradient estimate of $X$ as a consequence of (3.1.51). To do this, we will check conditions **(E), (D), (P)** and **(C)** (when $\beta < 1$) in [55] for the process $X$ and $Y$, and apply [55, Theorem 4] and [54, Theorem 1].

First, we verify the condition **(E)** in [55]. Recall $\Psi(r) = \sup_{|y|\leq r}\phi(|y|)$.

**Lemma 3.1.15.** *Assume that isotropic unimodal Lévy measure $\nu$ satisfies* (3.1.42) *and* (3.1.43). *Then for any $n, m \in \mathbb{N}$, there exists a constant $c = c(n,m) > 0$ such that*

$$\int_{\mathbb{R}^n} e^{-t\phi(|z|)}|z|^m dz \leq c\Psi^{-1}(t^{-1})^{n+m}, \quad t > 0.$$

**Proof.** By (3.1.46) and (3.1.50) we have that for $0 < t$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} e^{-t\phi(|z|)}|z|^m dz \le c_1 \int_0^{\Psi^{-1}(t^{-1})} r^{n+m-1} dr + c_1 \int_{\Psi^{-1}(t^{-1})}^\infty e^{-\pi^{-2}t\Psi(r)} r^{n+m-1} dr \\
&\le c_2 \Psi^{-1}(t^{-1})^{n+m} + c_1 \int_{\Psi^{-1}(t^{-1})}^\infty e^{-c_3 t\Psi(\Psi^{-1}(t^{-1}))(r/\Psi^{-1}(t^{-1}))^{\alpha_3}} r^{n+m-1} dr \\
&= \left(c_2 + c_1 \int_1^\infty e^{-c_3 s^{\alpha_1}} s^{n+m-1} dr\right) \Psi^{-1}(t^{-1})^{n+m} = c_4 \Psi^{-1}(t^{-1})^{n+m},
\end{aligned}$$

where we have used the change of variables with $s = \frac{r}{\Psi^{-1}(t^{-1})}$ in the last line. $\square$

Note that Lemma 3.1.15 for $(n, m) = (d, 1)$ and $(n, m) = (d+2, 1)$ implies the condition **(E)** in [55] for the process $X$ and $Y$, respectively.

For $0 < \beta \le 1$ and $\ell \ge 0$, we define non-increasing functions $f$ and $\tilde{f}$ by

$$f(r) := \begin{cases} \dfrac{\varphi(1)}{r^{d+1}\varphi(r)}, & r \le 1, \\ r^{-\ell-1} e^{-br^\beta}, & r > 1 \end{cases} \quad \text{and} \quad \tilde{f}(r) := \begin{cases} \dfrac{\varphi(1)}{r^{d}\varphi(r)}, & r \le 1, \\ r^{-\ell} e^{-br^\beta}, & r > 1 \end{cases} \tag{3.1.53}$$

The functions $f$ and $\tilde{f}$ above are non-increasing since for any $0 < r \le R \le 1$,

$$\frac{1}{r^d\varphi(r)} = \int_0^1 t^{d+1}\nu(rt)dt \ge \int_0^1 t^{d+1}\nu(Rt)dt = \frac{1}{R^d\varphi(R)}.$$

Here we used that $\nu$ is nonincreasing. Note that by (3.1.42) and (3.1.45),

$$\frac{\nu(r)}{r} \le cf(r) \quad \text{and} \quad \nu(r) \le c\tilde{f}(r) \qquad \text{for} \quad r > 0 \tag{3.1.54}$$

In the next lemma we verify the condition **(D)** in [55] for both $X$ and $Y$. In fact, we are going to verify **(D)** for $X$ with the above $\tilde{f}$ and $\gamma = d$, while we use $f$ and $\gamma = d+1$ to verify **(D)** for $Y$. Let $B_d(x, r) := \{y \in \mathbb{R}^d : |x-y| < r\}$ and recall that $\mathrm{diam}(A) = \sup\{|x-y| : x, y \in A\}$ and $\nu_1(r) = -\frac{1}{2\pi r}\nu'(r)$.

**Lemma 3.1.16.** *Assume that $\nu$ satisfies* (3.1.42) *and* (3.1.43). *Then both $\nu(\mathbb{R}^d)$ and $\nu_1(\mathbb{R}^{d+2}) = \int_{\mathbb{R}^{d+2}} \nu_1(|x|)dx$ are infinite, and there exists $c > 0$ such* *that*

$$\nu(A) \le c\tilde{f}(\delta(A))[\mathrm{diam}(A)]^d, \quad A \in \mathcal{B}(\mathbb{R}^d). \tag{3.1.55}$$

*and*

$$\nu_1(A) = \int_A \nu_1(|x|)dx \le cf(\delta(A))[\mathrm{diam}(A)]^{d+1}, \quad A \in \mathcal{B}(\mathbb{R}^{d+2}). \tag{3.1.56}$$

*for some $c > 0$, where $\delta(A) := \inf\{|y| : y \in A\}$.*

**Proof.** We have already showed that $\nu(B_d(0,1)) = \nu(\mathbb{R}^d) = \infty$. For any $A \in \mathcal{B}(\mathbb{R}^d)$, using (3.1.54) we have

$$\nu(A) = \int_A \nu(|y|)dy \le \nu(\delta(A))[\mathrm{diam}(A)]^d \le c\tilde{f}(\delta(A))[\mathrm{diam}(A)]^d.$$

This concludes (3.1.55). Using $\nu'(r) \le 0$, (3.1.39), the integration by parts and the fact $\nu(B_d(0,1)) = \infty$ we have

$$\begin{aligned}
\nu_1(\mathbb{R}^{d+2}) &\ge \int_{B_{d+2}(0,1)} \nu_1(|y|)dy = c(d)\int_0^1 r^{d+1}\nu_1(r)dr \\
&= c_1 \liminf_{\varepsilon\downarrow 0} \int_\varepsilon^1 -r^d\nu'(r)dr = c_1 \liminf_{\varepsilon\downarrow 0}\left(-[r^d\nu(r)]_\varepsilon^1 + d\int_\varepsilon^1 r^{d-1}\nu(r)dr\right) \\
&= c_1 \liminf_{\varepsilon\downarrow 0}\left(\varepsilon^d\nu(\varepsilon) + d\int_\varepsilon^1 r^{d-1}\nu(r)dr - \nu(1)\right) = \infty.
\end{aligned}$$

Now it remains to prove (3.1.56). First observe that using the integration by parts, we have that for any $0 < r < R$,

$$\begin{aligned}
\int_r^R s^{d+1}\nu_1(s)ds &= -\frac{1}{2\pi}\int_r^R s^d\nu'(s)ds = \frac{1}{2\pi}\left(-[s^d\nu(s)]_r^R + d\int_r^R s^{d-1}\nu(s)ds\right) \\
&\le \frac{1}{2\pi}\left(r^d\nu(r) + \nu(r)\, d\int_r^R s^{d-1}ds\right) = \frac{1}{2\pi}\nu(r)R^d
\end{aligned} \tag{3.1.57}$$

where we used that $\nu$ is non-increasing. Denote $r := \delta(A)$ and $l :=\mathrm{diam}(A)$.

When $l \geq r/2$, using $A \subset \{y \in \mathbb{R}^{d+2} : r \leq |y| \leq r+l\}$ we obtain

$$\begin{aligned}\nu_1(A) &\leq \nu_1(\{y : r \leq |y| \leq r+l\}) = c(d) \int_r^{r+l} s^{d+1} \nu_1(s) ds \\ &\leq \frac{c(d)}{2\pi} \nu(r)(r+l)^d \leq c_1 \frac{\nu(r)}{r} l^{d+l} \leq c_3 f(r) l^{d+1},\end{aligned}$$

where we used (3.1.57) and (3.1.54) for the last line.

When $l < r/2$, choose a point $y_0 \in \bar{A}$ with $|y_0| = r$.

Since $A \subset B_{d+2}(y_0, l) \backslash B_{d+2}(0, r)$, there exists $c_4 = c_4(d) > 0$ such that

$$\int_{|y|=s} \mathbf{1}_A(y) \sigma(dy) \leq c_4 l^{d+1}$$

for any $s \in [r, r+l]$. Thus, by (3.1.57) and (3.1.54) we have

$$\begin{aligned}\nu_1(A) &\leq \nu_1(B(y_0, l) \backslash B(0, r)) \leq c_5 \int_r^{r+l} l^{d+1} \nu_1(s) ds \leq c_5 \frac{l^{d+1}}{r^{d+1}} \int_r^{r+l} s^{d+1} \nu_1(s) ds \\ &\leq \frac{c_5}{2\pi} \frac{l^{d+1}}{r^{d+1}} (r+l)^d \nu(r) \leq c_6 l^{d+1} \frac{\nu(r)}{r} \leq c_7 f(r) l^{d+1},\end{aligned}$$

which proves (3.1.56). □

Recall $\Psi(r) = \sup_{|y| \leq r} \phi(|y|)$.

**Lemma 3.1.17.** *Assume that $\nu$ satisfies* (3.1.42) *and* (3.1.43)*. For every $\kappa < 1$, there exists $c = c(\kappa) > 0$ such that*

$$\int_{\{y \in \mathbb{R}^d : |y| > r\}} \exp\left(b\kappa |y|^\beta\right) \nu(dy) \leq c \Psi(\frac{1}{r}), \quad r > 0 \tag{3.1.58}$$

*and*

$$\int_{\{y \in \mathbb{R}^{d+2} : |y| > r\}} \exp\left(b\kappa |y|^\beta\right) \nu_1(dy) \leq c \Psi(\frac{1}{r}), \quad r > 0 \tag{3.1.59}$$

**Proof.** Since (3.1.58) can be derived directly from the estimate of $I_2$ below, we only prove (3.1.59) here. Using the integration by parts, we have

$$\begin{aligned}
&\int_{|y|>r} \exp(b\kappa|y|^\beta)\nu_1(dy) = c(d)\int_r^\infty \exp(b\kappa t^\beta)t^d(-\nu'(t))dt \\
&= c(d)\left(\big[\exp(b\kappa t^\beta)t^d(-\nu(t))\big]_r^\infty + \int_r^\infty (\exp(b\kappa t^\beta)t^d)'\nu(t)dt\right). \\
&:= c(d)\big(I_1 + I_2\big).
\end{aligned}$$

For $I_1$, by (3.1.54) $\lim_{t\to\infty}\exp(b\kappa t^\beta)t^d\nu(t) \le \lim_{t\to\infty} a\exp(-b(1-\kappa)t^\beta))t^{d-\ell} = 0$ so $I_1 = \exp(b\kappa r^\beta)r^d\nu(r) \le c_1\varphi(r)^{-1}$. Now let us estimate $I_2$. First we observe that

$$\frac{d}{dt}\big(\exp(b\kappa t^\beta)t^d\big) \le c_2\begin{cases} t^{d-1}, & t\le 1\\ \exp(b\kappa t^\beta)t^{d+\beta-1}, & t>1.\end{cases}$$

Thus, for $r\ge 1$ we have

$$\int_r^\infty (\exp(b\kappa t^\beta)t^d)'\nu(t)dt \le c_2\int_r^\infty \exp(-b(1-\kappa)t^\beta)t^{d-\ell+\beta-1}dt \le \frac{c_3\varphi(1)}{\varphi(r)}.$$

For $r<1$, using above estimate, (3.1.49) and (3.1.43) we get

$$\begin{aligned}
\int_r^\infty (\exp(b\kappa t^\beta)t^d)'\nu(t)dt &= \left(\int_r^1 + \int_1^\infty\right)(\exp(b\kappa t^\beta)t^d)'\nu(t)dt \\
&\le c_2\left(\int_r^1 t^{d-1}\nu(t)dt + \int_1^\infty \exp(-b(1-\kappa)t^\beta)t^{d-\ell+\beta-1}dt\right) \\
&\le \frac{c_4}{\varphi(r)} + c_3 \le \frac{c_5}{\varphi(r)}
\end{aligned}$$

Combining above two inequalities and (3.1.47), we obtain $I_1 + I_2 \le c_6\Psi(\frac{1}{r})$. Therefore, we have proved the lemma. □

Using Lemma 3.1.17, we verify the condition **(P)** in [55] for both $X$ and $Y$. We continue to use the non-increasing functions $f$ and $\tilde f$ defined in (3.1.53).

**Lemma 3.1.18.** *Assume that isotropic unimodal Lévy measure $\nu$ satisfies*

(3.1.42) *and* (3.1.43)*. Then, there exists* $c > 0$ *such that*

$$\int_{\{y\in\mathbb{R}^d:|y|>r\}} \tilde{f}\left(s\vee|y|-\frac{|y|}{2}\right)\nu(dy) \le c\tilde{f}(s)\Psi(\frac{1}{r}), \quad r,s>0 \tag{3.1.60}$$

*and*

$$\int_{\{y\in\mathbb{R}^{d+2}:|y|>r\}} f\left(s\vee|y|-\frac{|y|}{2}\right)\nu_1(dy) \le cf(s)\Psi(\frac{1}{r}), \quad r,s>0 \tag{3.1.61}$$

**Proof.** We only prove (3.1.61) here, since (3.1.60) can be verified similarly. We claim that for any $0<\beta\le 1$, there exists $c_1>0$ such that for any $s,t>0$,

$$f(s\vee t-\frac{t}{2}) \le c_1 f(s)\exp(b\kappa t^\beta) \tag{3.1.62}$$

where $\kappa=\frac{1}{2}(2^{-\beta}+1)$. First we define

$$f_1(r) := \begin{cases} \frac{\varphi(1)}{r^{d+1}\varphi(r)}, & r\le 2 \\ r^{-\ell-1}\exp(-br^\beta), & r>2. \end{cases}$$

Then, since $f(r)=f_1(r)$ for $r\in(0,1]\cup(2,\infty)$ we have

$$c_2^{-1}f(r)\le f_1(r)\le c_2 f(r), \quad r>0. \tag{3.1.63}$$

Now assume $s\vee t>2$. Then, using $1\vee\frac{s}{2}\le s\vee t-\frac{t}{2}$ and triangular inequality,

$$\begin{aligned} f(s\vee t-\frac{t}{2}) &= (s\vee t-\frac{t}{2})^{-\ell-1}\exp\big(-b(s\vee t-\frac{t}{2})^\beta\big) \\ &\le (1\vee\frac{s}{2})^{-\ell-1}\exp(-bs^\beta)\exp(b(\frac{t}{2})^\beta) \le c_3 f(s)\exp(b(\frac{t}{2})^\beta). \end{aligned}$$

Here in the last inequality we used $\ell\ge 0$ and $\exp(-bs^\beta)\le c_3 f(s)$ for $0<s\le 2$. When $s\le 2$ and $t\le 2$, using (3.1.63), (3.1.43) and (3.1.44) with $s\vee t-\frac{t}{2}\ge\frac{s}{2}$ we obtain

$$f(s\vee t-\frac{t}{2}) \le c_2 f_1(s\vee t-\frac{t}{2}) \le \frac{c_4\varphi(1)}{s^{d+1}\varphi(s)} \le c_4 f_1(s) \le c_5 f(s).$$

Here we used $\frac{\varphi(s)}{\varphi(s\vee t-\frac{t}{2})}=\frac{\varphi(s)}{\varphi(s/2)}\frac{\varphi(s/2)}{\varphi(s\vee t-\frac{t}{2})}\leq 4a_3^{-1}$ which follows from (3.1.43) and (3.1.44). Thus, we conclude (3.1.62). Combining (3.1.62) and Lemma 3.1.17, we have proved the lemma. □

Now we obtain a priori heat kernel estimates for the process $X$ and $Y$. To state the results, we need to define *generalized inverse of* $\varphi$ by $\varphi^{-1}(t):=\inf\{s>0:\varphi(s)\geq t\}$. Using Lemmas 1.1.4 and 1.1.5, there exists $c, c_1\geq 1$ such that $L(1/2, c^{-1}, \varphi^{-1})$, $U(\alpha_3, c, \varphi^{-1})$ hold and

$$c_1^{-1}\varphi(\varphi^{-1}(r))\leq r\leq c_1\varphi(\varphi^{-1}(r)), \tag{3.1.64}$$

First we apply [54, Theorem 3] to obtain the regularity of the transition density $p_t(x)$ of $X$.

**Proposition 3.1.19.** *Let $X$ be an isotropic unimodal Lévy process in $\mathbb{R}^d$ with jumping kernel $\nu(|y|)dy$ satisfying* (3.1.42) *and* (3.1.43) *with $0<\beta\leq 1$. Then $x\to p_t(x)\in C_b^\infty(\mathbb{R}^d)$ and for any $k\in\mathbb{N}_0$ there exists $c_k>0$ such that*

$$|\nabla_x^k p_t(x)|\leq c_k\varphi^{-1}(t)^{-k}\left(\varphi^{-1}(t)^{-d}\wedge\frac{t}{|x|^d\varphi(|x|)}\right) \tag{3.1.65}$$

*for any $t>0$ and $x\in\mathbb{R}^d$.*

**Proof.** Define $h(t):=\frac{1}{\Psi^{-1}(t^{-1})}$ as in [54]. Note that by (3.1.47) and (3.1.64) we have

$$h(t)\asymp\varphi^{-1}(t),\quad t>0.$$

Applying [54, Theorem 3] for the process $X$, $p_t(x)\in C_b^\infty(\mathbb{R}^d)$ and for any $k\in\mathbb{N}$, $\gamma\in[1,d]$ and $n>\gamma$ we have constants $c_{k,n}$ satisfying

$$|\nabla_x^k p_t(x)|\leq c_{k,n}(h(t))^{-d-k}\min\left\{1, \frac{t[h(t)]^\gamma}{|x|^\gamma\varphi(|x|)}e^{-b(|x|/4)^\beta}+\left(1+\frac{|x|}{h(t)}\right)^{-n}\right\}.$$

Note that we already verified [54, (8)] at Lemma 3.1.15. Thus, using $h(t)\asymp$

$\varphi^{-1}(t)$ we obtain

$$|\nabla_x^k p_t(x)| \le \tilde{c}_{k,n} \varphi^{-1}(t)^{-d-k}$$

Also, taking $\gamma = d$, $n = d+2$ and using $h(t) \asymp \varphi^{-1}(t)$ we get

$$\begin{aligned}
|\nabla_x^k p_t(x)| &\le c_{k,n} \left( h(t)^{-k} \frac{t}{|x|^d \varphi(|x|)} e^{-b(|x|/4)^\beta} + h(t)^{-k} |x|^d \left(1 + \frac{|x|}{h(t)}\right)^{-2} \right) \\
&\le c\varphi^{-1}(t)^{-k} \left( \frac{t}{|x|^d \varphi(|x|)} + \left(\frac{\varphi^{-1}(t)}{|x|} \wedge 1\right)^2 \frac{1}{|x|^d} \right) \\
&\le \tilde{c}_{k,n} \varphi^{-1}(t)^{-k} \frac{t}{|x|^d \varphi(|x|)}.
\end{aligned}$$

The last inequality is straightforward when $|x| < \varphi^{-1}(t)$ and it follows from $L(1/2, c, \varphi^{-1})$ and (3.1.64) when $|x| \ge \varphi^{-1}(t)$. Therefore, we conclude that

$$|\nabla_x^k p_t(x)| \le c_k \varphi^{-1}(t)^{-k} \left( \varphi^{-1}(t)^{-d} \wedge \frac{t}{|x|^d \varphi(|x|)} \right).$$

□

Note that the gradient estimates in Proposition 3.1.19 is same as the ones in [61, Proposition 3.2] except that the gradient estimates in [61, Proposition 3.2] is for $t \le T$ (see Remark 3.1.6).

Combining above estimates with Lemmas 3.1.15, 3.1.16 and 3.1.18, we can apply [54, Theorem 1] for the process $X$ and $Y$. Here is the result.

**Lemma 3.1.20.** *Assume that $\nu$ satisfies* (3.1.42) *and* (3.1.43) *and $\beta = 1$. Then for any $T \ge 1$, there exists a constant $c > 0$ such that*

$$p_t(x) \le ct \exp(-\frac{b}{4}|x|) \quad \textit{and} \quad q_t(x) \le ct\varphi^{-1}(t)^{-1} \exp(-\frac{b}{4}|x|) \qquad (3.1.66)$$

*for any $0 < t \le T$ and $|x| > \varphi^{-1}(T)$.*

**Proof.** Define $h(t) := \frac{1}{\Psi^{-1}(t^{-1})}$ as in [54] and denote $q_t(|x|) = q_t(x)$. Applying Lemmas 3.1.15, 3.1.16 and 3.1.18 to [54, Thoerem 1] for the process $Y$ in

Lemma 3.1.14, we have

$$q_t(r) \le c_1 h(t)^{-1}\Big(h(t)^{-d-1} \wedge \Big[tf(\frac{r}{4}) + h(t)^{-d-1}\exp\big(-\frac{c_2 r}{h(t)}\log(1+\frac{r}{h(t)})\big)\Big]\Big)$$

for any $t, r > 0$. First observe that using $f(\frac{r}{4}) = (\frac{r}{4})^{-\ell-1}\exp(-\frac{b}{4}r)$ for $r > 4$, we obtain

$$t\varphi^{-1}(t)^{-1}f(\frac{r}{4}) \le c_6 t r^{-\ell-1}\exp(-\frac{b}{4}r) \le c_7 t\exp(-\frac{b}{4}r), \quad r > 4.$$

Let $c(T) > 4$ be a constant which is large enough to satisfy

$$\frac{c_4}{2\varphi^{-1}(T)}\log(1+c_5\frac{c(T)}{\varphi^{-1}(T)}) \ge \frac{b}{4}, \quad r > 1.$$

Then, for any $0 < t \le T$ and $r > c(T)$ we have

$$\begin{aligned}
&\varphi^{-1}(t)^{-d-1}\exp\big(-c_4\frac{r}{\varphi^{-1}(t)}\log(1+c_5\frac{r}{\varphi^{-1}(t)})\big) \\
&\le \varphi^{-1}(t)^{-d-1}\exp\big(-c_8\frac{r}{\varphi^{-1}(t)}\big)\exp\big(-\frac{c_4 r}{2\varphi^{-1}(T)}\log(1+c_5\frac{c(T)}{\varphi^{-1}(T)})\big) \\
&\le \varphi^{-1}(t)^{-d-1}\exp(-c_8\frac{r}{\varphi^{-1}(t)} - \frac{b}{4}r) \le \frac{c_9 t}{r^{d+1}\varphi(r)}\exp(-\frac{b}{4}r) \le c_9 t\exp(-\frac{b}{4}r)
\end{aligned}$$

where $c_9 = \sup_{s\ge 1} s^{d+1}\exp(-c_8 s) < \infty$. Thus,

$$q_t(r) \le c_{10} t\varphi^{-1}(t)^{-1}\exp(-\frac{b}{4}r), \quad r > c(T) = \varphi(\varphi^{-1}(c(T))).$$

Meanwhile, by (3.1.51) and (3.1.65) we have

$$q_t(r) = \frac{1}{2\pi r}|\frac{r}{dr}p_t(r)| \le \frac{c_{11}t\varphi^{-1}(t)^{-1}}{r^{d+1}\varphi(r)} \le c_{12}t\varphi^{-1}(t)^{-1}\exp(-\frac{b}{4}r + \frac{bc(T)}{4})$$

for $\varphi^{-1}(T) < r \le c(T)$. Therefore, combining above two estimates we conclude the estimate on $q$ in (3.1.66).

Note that, applying Lemmas 3.1.15, 3.1.16 and 3.1.18 to [54, Thoerem 1] for the process $X$ and using $h(t) \asymp \varphi^{-1}(t)$ we have

$$p_t(r) \le c_{10}\Big(\varphi^{-1}(t)^{-d} \wedge \Big[t\tilde{f}(r/4) + \varphi^{-1}(t)^{-d}\exp(-\frac{c_{11}r}{\varphi^{-1}(t)}\log(1+\frac{c_{12}r}{\varphi^{-1}(t)})\Big]\Big). \tag{3.1.67}$$

for any $t, r > 0$. Using (3.1.67), the estimate on $p$ in (3.1.66) can be verified similarly. □

Now we check condition **(C)** in [55] with $r_0 = 1$, $t_p = \infty$ and $\gamma = d$ for $X$ ($\gamma = d+1$ for $Y$, respectively). We need additional condition $0 < \beta < 1$ to verify the condition **(C)**.

**Lemma 3.1.21.** *Assume $\nu$ satisfies* (3.1.42) *and* (3.1.43) *with $0 < \beta < 1$. Then, there exists constant $c > 0$ such that for every $|x| \ge 2$ and $r \in (0, 1]$,*

$$\tilde{f}(r) \le cr^{-d}\Psi(\frac{1}{r}), \quad \int_{\{y\in\mathbb{R}^d:|x-y|\ge 1,|y|>r\}} \tilde{f}(|x-y|)\nu(dy) \le c\Psi(\frac{1}{r})\tilde{f}(|x|) \quad \textit{and}$$

$$f(r) \le cr^{-d-1}\Psi(\frac{1}{r}), \quad \int_{\{y\in\mathbb{R}^{d+2}:|x-y|\ge 1,|y|>r\}} f(|x-y|)\nu_1(dy) \le c\Psi(\frac{1}{r})f(|x|).$$

**Proof.** The first inequalities immediately follow from (3.1.47) and (3.1.53).

Let us show the second inequality in the first line. When $|x-y| \ge \frac{|x|}{2}$, using (3.1.26) and triangular inequality, we have $|x|^\beta \le |x-y|^\beta + (2^\beta - 1)|y|^\beta$. Thus, using this inequality and Lemma 3.1.17 we obtain

$$\begin{aligned}\int_{|x-y|\ge\frac{|x|}{2},|y|>r} f(|x-y|)\nu_1(dy) =& \int_{|x-y|\ge\frac{|x|}{2},|y|>r} |x-y|^{-\ell-1}\exp(-b|x-y|^\beta)\nu_1(dy)\\ &\le \big(\frac{|x|}{2}\big)^{-\ell-1}\int_{|y|>r}\exp(-b|x|^\beta)\exp(b(2^\beta-1)|y|^\beta)\nu_1(dy)\\ &= f(|x|)\int_{|y|>r}\exp(b(2^\beta-1)|y|^\beta)\nu_1(dy) \le c_1 f(|x|)\Psi(\frac{1}{r}).\end{aligned}$$

So, it suffices to show that there exists a constant $c_2 > 0$ such that for every

$|x| \geq 2$,

$$\int_{1\leq|x-y|\leq\frac{|x|}{2}} f(|x-y|)\nu_1(dy) \leq c_2 f(|x|). \tag{3.1.68}$$

To show this, we will divide the set $D := \{y : 1 \leq |x-y| \leq \frac{|x|}{2}\}$ into cubes with diameter 1. Let $x = (x_1, ..., x_{d+2})$. For $(a_1, ..., a_{d+2}) \in \mathbb{Z}^{d+2}$, we define $a := (\sqrt{d+2})^{-1}(a_1, ..., a_{d+2})$, and let

$$C_a := \prod_{i=1}^{d+2}[x_i + \frac{2a_i - 1}{2\sqrt{d+2}}, x_i + \frac{2a_i+1}{2\sqrt{d+2}})$$

be a cube with length $(\sqrt{d+2})^{-1}$. Since $\mathrm{diam}(C_a) = 1$ and $x+a$ is the center of cube $C_a$, for any $|a| \leq \frac{|x|+1}{2}$ we have $c_5 > 0$ independent of $a$ such that

$$\begin{aligned}
&\nu_1(C_a \cap D) \leq c_3 f(\delta(C_a \cap D)) \leq c_3 f\left(\big(|x| - |a| - \frac{1}{2}\big) \vee \frac{|x|}{2}\right) \\
&\leq c_4\big(|x| - |a|\big)^{-\ell-1}\exp\left(-b\,\big||x| - |a|\big|^{\beta}\right) \leq c_5|x|^{-\ell-1}\exp\left(-b\big(|x| - |a|\big)^{\beta}\right)
\end{aligned}$$

where we used Lemma 3.1.16 for the first inequality and triangular inequality for the second line. Thus, using $|a| - \frac{1}{2} \leq |x-y|$ on $C_a$ and

$$D \subset \bigcup_{1\leq|a|\leq\frac{|x|+1}{2}} C_a,$$

we obtain

$$\begin{aligned}
&\int_{1\leq|x-y|\leq\frac{|x|}{2}} f(|x-y|)\nu_1(dy) \leq \sum_{1\leq|a|\leq\frac{|x|+1}{2}} \int_{C_a\cap D} |x-y|^{-\ell-1}\exp(-b|x-y|^{\beta})\nu_1(dy) \\
&\leq \sum_{1\leq|a|\leq\frac{|x|+1}{2}} (|a| - \frac{1}{2})^{-\ell-1}\exp(-b(|a| - \frac{1}{2})^{\beta})\nu_1(C_a \cap D) \\
&\leq c_6|x|^{-\ell-1} \sum_{1\leq|a|\leq\frac{|x|}{2}+1} |a|^{-\ell-1}\exp(-b|a|^{\beta})\exp(-b(|x| - |a|)^{\beta}).
\end{aligned}$$

Since $|a| \le \frac{|x|+1}{2}$, by (3.1.26) we have

$$|a|^\beta + (|x| - |a|)^\beta + 1 \ge |a|^\beta + (|x| + 1 - |a|)^\beta \ge |x|^\beta + (2 - 2^\beta)|a|^\beta.$$

Thus,

$$\begin{aligned} &|x|^{-\ell-1} \sum_{1 \le |a| \le \frac{|x|+1}{2}} \exp(-b|a|^\beta) \exp(-b(|x| - |a|)^\beta) \\ &\le c_7 |x|^{-\ell-1} \exp(-b|x|^\beta) \sum_{a \in \mathbb{Z}^d \setminus \{0\}} |a|^{-\ell-1} \exp(-b(2 - 2^\beta)|a|^\beta) \le c_8 f(|x|). \end{aligned}$$

Combining above inequalities and using (3.1.47), we arrive (3.1.68). The remainder is similar. □

Now we have that conditions **(E), (D)** and **(C)** in [55] holds for the process $Y$ when $\nu$ satisfies (3.1.42) and (3.1.43) with $0 < \beta < 1$. Thus, we can apply [55, Thereoem 4] for both $X$ and $Y$.

**Lemma 3.1.22.** *Let $T \ge 1$ and assume that $\nu$ satisfies* (3.1.42) *and* (3.1.43) *with $0 < \beta < 1$. Then, there exists a constant $c > 0$ such that*

$$p_t(r) \le ctr^{-\ell} \exp(-br^\beta) \tag{3.1.69}$$

*and*

$$\left| \frac{d}{dr} p_t(r) \right| \le ct\varphi^{-1}(t)^{-1} r^{-\ell} \exp(-br^\beta) \tag{3.1.70}$$

*for any $0 < t \le T$ and $r \ge 4$.*

**Proof.** Applying [55, Theorem 4] for $Y$ and (3.1.47) we have that for $0 < t \le t_p = T$ and $r \ge 4r_0 = 4$,

$$q_t(r) \le c_1 t\varphi^{-1}(t)^{-1} f(r) = c_1 t\varphi^{-1}(t)^{-1} r^{-\ell-1} \exp(-br^\beta).$$

Combining with (3.1.51), $|\frac{d}{dr} p_t(r)| \le 2\pi r q_t(r) \le c_2 t\varphi^{-1}(t)^{-1} r^{-\ell} \exp(-br^\beta)$. This concludes (3.1.70). (3.1.69) immediately follows from applying [55, Theorem 4] for $X$. $\square$

For reader's convenience, we put the heat kernel estimates and gradient estimates in Proposition 3.1.19, and Lemmas 3.1.20 and 3.1.22 together into one proposition.

**Proposition 3.1.23.** *Let $X$ be an isotropic unimodal Lévy process in $\mathbb{R}^d$ with jumping kernel $\nu(|y|)dy$ satisfying* (3.1.42) *and* (3.1.43)*. Then, $x \mapsto p_t(x) \in C_b^\infty(\mathbb{R}^d)$ and the following holds.*
*(a) There exists a constant $c_1 > 0$ such that for $k \geq 0$,*

$$|\nabla_x^k p_t(x)| \leq c_1 \varphi^{-1}(t)^{-k} \left( \varphi^{-1}(t)^{-d} \wedge \frac{t}{r^d \varphi(r)} \right), \quad t > 0,\, x \in \mathbb{R}^d.$$

*(b) Assume $\beta = 1$. Then for any $T \geq 1$, there exists a constant $c_2 > 0$ such that for $k = 0, 1$,*

$$|\nabla_x^k p_t(x)| \leq c_2 t \varphi^{-1}(t)^{-k} \exp(-\frac{b}{4} r), \quad t \in (0, T],\, |x| > \varphi^{-1}(T).$$

*(c) Assume $0 < \beta < 1$. Then for any $T \geq 1$, there exists a constant $c_3 > 0$ such that for $k = 0, 1$,*

$$|\nabla_x^k p_t(x)| \leq c_3 t \varphi^{-1}(t)^{-k} r^{-\ell} \exp(-b r^\beta), \quad t \in (0, T],\, |x| > \varphi^{-1}(T).$$

**Proof.** (a) and (b) immediately follow from Proposition 3.1.19 and Lemma 3.1.20, respectively.
(c) Observe that for any $t \in (0, T]$, $\varphi^{-1}(T) < |x| \leq 4$ and $k = 0, 1$,

$$|\nabla_x^k p_t(x)| \leq c_1 \varphi^{-1}(t)^{-k} \frac{t}{r^d \varphi(r)} \leq c_4 t \varphi^{-1}(t)^{-k} r^{-\ell} \exp(-b r^\beta).$$

This and Lemma 3.1.22 finish the proof. $\square$

Now we are ready to prove Propositions 3.1.11 and 3.1.12.

**Proof of Proposition 3.1.11.** Now assume that $X$ is an isotropic Lévy process in Proposition 3.1.11 with Lévy measure $\nu(|y|)dy$. Recall that $\varphi(r) \asymp \Phi(r)$, and $\nu$ satisfies (3.1.42) with $\ell = 0$ and (3.1.43). Therefore, we can apply results in Proposition 3.1.23 with ths function $\Phi$ instead of $\varphi$. Using Proposition 3.1.23 and (3.1.19), we conclude that for any $t \in (0, T]$ and $x \in \mathbb{R}^d$

$$|\nabla_x^k p_t(x)| \le c_1 t \varphi^{-1}(t)^{-k} \mathscr{G}_T(t,x) \le c_2 t \mathscr{G}^0_{-k}(t,x), \quad k = 0, 1.$$

□

**Proof of Proposition 3.1.12.** (3.1.40) for $k = 0, 1$ and that $t \mapsto p(t,x)$ is in $C_b^\infty(\mathbb{R}^d)$ immediately follow from Proposition 3.1.11.

Now it suffices to prove (3.1.40) when $k = 2$. Let $X$ be an isotropic unimodal Lévy process with jumping kernel $J(|x|)dx$ satisfying (3.1.1) with $0 < \beta \le 1$ and (3.1.2), and let $\phi(|x|) = \phi(x)$ be a characteristic exponent of $X$. By Lemma 3.1.14, there exists an isotropic Lévy process $Y$ in $\mathbb{R}^{d+2}$ with characteristic exponent $\phi(r)$ satisfying (3.1.51) and (3.1.52). In particular, by (3.1.2) and (3.1.52), $Y$ is unimodal. Denote $J_1(|x|)dx$ and $q_t(|x|)$ be the Lévy density and transition density of $Y$ respectively. Using (3.1.52) we have

$$2\pi \int_s^r J_1(t)dt = -\int_s^r \frac{J'(t)}{t} dt = \frac{J(s)}{s} - \frac{J(r)}{r} - \int_s^r \frac{J(t)}{t^2} dt.$$

Since $J_1$ is non-increasing by (3.1.2), we obtain that for any $0 < s \le r$,

$$(r-s)J_1(r) \le \int_s^r J_1(t)dt \le \frac{J(s)}{2\pi s} \tag{3.1.71}$$

and

$$(r-s)J_1(s) \ge \int_s^r J_1(r)dr \ge \frac{1}{2\pi r}(J(s) - J(r)). \tag{3.1.72}$$

We claim that there exists a constant $c > 0$ such that

$$\begin{cases} \frac{c^{-1}}{r^{d+2}\psi(r)} \le J_1(r) \le \frac{c}{r^{d+2}\psi(r/2)}, & r \le 1 \\ J_1(r) \le cr^{-1}\exp(-br^\beta), & r > 1. \end{cases} \tag{3.1.73}$$

For $r \le 1$, letting $s = \frac{r}{2}$ in (3.1.71) we have $J_1(r) \le \frac{J(r/2)}{\pi r^2} \le \frac{c_1}{r^{d+2}\psi(r/2)}$ by (3.1.1). Also, taking $(r, s) = (Cr, r)$ with constant $C = \left(\frac{2}{a^2}\right)^{2/d} > 1$ in (3.1.72) we have

$$\begin{aligned} J_1(r) &\ge \frac{J(r) - J(Cr)}{2\pi C(C-1)r^2} \ge \frac{1}{2\pi C(C-1)r^2}\left(\frac{a}{r^d\psi(r)} - \frac{1}{a(Cr)^d\psi(Cr)}\right) \\ &\ge \frac{1}{C(C-1)}\left(a - \frac{1}{aC^d}\right)\frac{1}{r^{d+2}\psi(r)} = \frac{a}{2C(C-1)}\frac{1}{r^{d+2}\psi(r)}, \end{aligned}$$

where we used $\psi(Cr) \ge \psi(r)$ and $a - \frac{1}{aC^d} = \frac{a}{2}$ in the second line.

When $r > 1$, letting $s = r - 1$ in (3.1.71) we have

$$J_1(r) \le \frac{J(r-1)}{r} \le \frac{1}{r}\exp(-b(r-1)^\beta) \le e^b\frac{1}{r}\exp(-br^\beta),$$

where we used the assumptions $r > 1$ and $0 < \beta \le 1$ for the last inequality. We have proved (3.1.73).

Let $\varphi$ be the function (3.1.41) with $\nu = J_1$ and the dimension $d + 2$ (instead of $d$). Note that (3.1.73) implies that $\varphi$ satisfies that for $r < 1$,

$$c_1\Phi(r) \le 2c^{-1}\Phi(r/2) = \frac{c^{-1}r^2}{\int_0^r \frac{s}{\psi(s/2)}ds} \le \varphi(r) = \frac{r^2}{\int_0^r s^{d+3}J_1(s)ds} \le 2c\Phi(r)$$

Thus, $J_1$ satisfies (3.1.42) with $\ell = 0$ and (3.1.43) since $U(2, 1, \Phi)$ holds. Combining $\varphi(r) \asymp \Phi(r)$ and Proposition 3.1.23 for the process $Y$, we have that there is a constant $c_2 > 0$ satisfying

$$q_t(r) \le c_2 t\mathscr{G}^{(d+2)}(t, r) \quad \text{and} \quad \left|\frac{d}{dr}q_t(r)\right| \le c_2 t\Phi^{-1}(t)^{-1}\mathscr{G}^{(d+2)}(t, r)$$

for any $0 < t \le T$ and $r > 0$. From now on, assume $t \in (0, T]$ and $x \in \mathbb{R}^d$.

Also, let $r = |x|$. Combining above inequalities and (3.1.51) we have

$$
\begin{aligned}
|\nabla_x^2 p(t,x)| &= |\frac{\partial^2}{\partial r^2}p(t,r) + \frac{d-1}{r}\frac{\partial}{\partial r}p(t,r)| = 2\pi|\frac{d}{dr}\left(-rq_t(r) + (d-1)q_t(r)\right)| \\
&\le 2\pi d\left(q_t(r) + r|\frac{d}{dr}q_t(r)|\right) \le c_3 t\left(1 + r\Phi^{-1}(t)^{-1}\right)\mathscr{G}^{(d+2)}(t,r) \\
&\le c_4 t\left(1 + r\Phi^{-1}(t)^{-1}\right)\mathscr{G}_T^{(d+2)}(t,r) \qquad (3.1.74)
\end{aligned}
$$

where we used (3.1.19) for the last line. Thus, we obtain for $r \le \Phi^{-1}(t)$,

$$
|\nabla_x^2 p(t,x)| \le 2c_4 t\mathscr{G}^{(d+2)}(t,r) \le 2c_4\Phi^{-1}(t)^{-2}\mathscr{G}_T(t,r). \qquad (3.1.75)
$$

Also, for $\Phi^{-1}(t) < r \le \Phi^{-1}(T)$ we have

$$
\begin{aligned}
|\nabla_x^2 p(t,x)| &\le \frac{2c_4 tr}{\Phi^{-1}(t)}\mathscr{G}^{(d+2)}(t,r) \le \frac{2c_4 tr^2}{\Phi^{-1}(t)^2}\mathscr{G}^{(d+2)}(t,r) \\
&\le 2c_4\Phi^{-1}(t)^{-2}\frac{t}{r^d\Phi(r)} = 2c_4\Phi^{-1}(t)^{-2}\mathscr{G}_T(t,r).
\end{aligned} \qquad (3.1.76)
$$

Note that above estimates are valid for any $0 < \beta \le 1$.

Now assume $0 < \beta < 1$. Let us recall that $J_1$ satisfies (3.1.42) with $\ell = 1$ and (3.1.43). Applying Proposition 3.1.23(c) for the process $Y$ we have

$$
q_t(r) \le c_5 tr^{-1}\exp(-br^\beta) \quad \text{and} \quad |\frac{d}{dr}q_t(r)| \le c_5 t\Phi^{-1}(t)^{-1}r^{-1}\exp(-br^\beta)
$$

for $r > \Phi^{-1}(T)$. Thus, by (3.1.74)

$$
\begin{aligned}
|\nabla_x^2 p(t,x)| &\le 2\pi d\left(q_t(r) + r|\frac{d}{dr}q_t(r)|\right) \\
&\le c_6 t\big(r^{-1} + t\Phi^{-1}(t)^{-1}\big)\exp(-br^\beta) \le c_7 t\Phi^{-1}(t)^{-2}\mathscr{G}_T(t,r)
\end{aligned}
$$

for $r > \Phi^{-1}(T)$. Combining this with (3.1.75), (3.1.76) and (3.1.19) we obtain

$$
|\nabla_x^2 p_t(x)| \le c_8 t\Phi^{-1}(t)^{-2}\mathscr{G}_T(t,x) \le c_9\mathscr{G}_{-2}^0(t,x), \qquad 0 < t \le T, x \in \mathbb{R}^d.
$$

This concludes (3.1.40) for $0 < \beta < 1$.

Similarly, for $\beta = 1$ using Proposition 3.1.23(b) for the process $Y$ we have

$$q_t(r) \le c_{10} t \exp(-\frac{b}{4} r) \quad \text{and} \quad |\frac{d}{dr} q_t(r)| \le c_{10} t \Phi^{-1}(t)^{-1} \exp(-\frac{b}{4} r)$$

for $r > \Phi^{-1}(T)$. Thus, for $r > \Phi^{-1}(T)$ we have

$$\begin{aligned} |\nabla_x^2 p(t,x)| &\le 2\pi d \left( q_t(|x|) + |x| |\frac{d}{dr} q_t(|x|)| \right) \\ &\le c_{12} t \Phi^{-1}(t)^{-2} \exp(-\frac{b}{5}|x|) \le c_{13} t \Phi^{-1}(t)^{-2} \mathscr{G}_T(t,r). \end{aligned}$$

Hence, combining this with (3.1.75), (3.1.76) and (3.1.19) we obtain

$$|\nabla_x^2 p_t(x)| \le c_{14} t \Phi^{-1}(t)^{-2} \mathscr{G}_T(t,x) \le c_{15} t \mathscr{G}^0_{-2}(t,x), \quad 0 < t \le T, x \in \mathbb{R}^d,$$

which is our desired result for $\beta = 1$. □

### 3.1.5 Further properties of heat kernel for isotropic Lévy process

In this section we assume that $J$ satisfies (3.1.1) with $0 < \beta \le 1$ and (3.1.2), and that nondecreasing function $\psi$ satisfies (3.1.3) and (3.1.4). As in the previous section, let $X$ be an istropic unimodal Lévy process with jumping kernel $J(|y|)dy$ and $p(t,x)$ be the transition density of $X$. Also, let $\mathcal{L}$ be an infinitesimal generator of $X$.

Recall that $\delta_f$ is defined in (3.1.18). The following results are counterpart of [61, Proposition 3.3].

**Proposition 3.1.24.** *For every $T \ge 1$, there exists a constant $0 < c = c(d,T,a,a_1,\alpha_1,b,\beta,C_0)$ such that for every $t \in (0,T]$ and $x,y,z \in \mathbb{R}^d$,*

$$|p(t,x) - p(t,y)| \le c \left( \frac{|x-y|}{\Phi^{-1}(t)} \wedge 1 \right) t \left( \mathscr{G}(t,x) + \mathscr{G}(t,y) \right), \qquad (3.1.77)$$

*and*

$$|\delta_p(t,x;z)| \le c\left(\frac{|z|}{\Phi^{-1}(t)} \wedge 1\right)^2 t\left(\mathscr{G}(t,x\pm z)+\mathscr{G}(t,x)\right), \tag{3.1.78}$$

**Proof.** (a) Since (3.1.77) is clearly true when $\Phi^{-1}(t) \le |x-y|$ by (3.1.40), we assume that $\Phi^{-1}(t) \ge |x-y|$. Let $\alpha(\theta) = x+\theta(y-x),\ \theta \in [0,1]$ be a segment from $x$ to $y$. Then, for any $\theta \in [0,1]$ we have

$$|\alpha(\theta)| \ge |x| - |x-\alpha(\theta)| \ge |x| - |x-y| \ge |x| - \Phi^{-1}(t),$$

here we used $|x-y| \le \Phi^{-1}(t)$ for the last inequality. Thus, we obtain

$$\begin{aligned}
&|p(t,x)-p(t,y)| = |\int_0^1 \alpha'(\theta)\cdot\nabla_x p(t,\alpha(\theta))\,d\theta| \le \int_0^1 |\alpha'(\theta)||\nabla_x p(t,\alpha(\theta))|\,d\theta \\
&\le c_1\int_0^1 |\alpha'(\theta)|\Phi^{-1}(t)^{-1}t\mathscr{G}(t,\alpha(\theta))\,d\theta \le c_1|x-y|\Phi^{-1}(t)^{-1}t\mathscr{G}(t,|x|-\Phi^{-1}(t)) \\
&\le c_2\,|x-y|\Phi^{-1}(t)^{-1}t\mathscr{G}(t,x).
\end{aligned}$$

Here we used (3.1.40) with $k=1$ for the second line and (3.1.24) for the last line. This concludes (3.1.77).

Note that using (3.1.40) for $k=2$ and following the same argument as the above we can estimate $|\nabla p(t,x) - \nabla p(t,y)|$. Hence, we have a constant $c_3 > 0$ satisfying

$$|\nabla p(t,x) - \nabla p(t,y)| \le c_3|x-y|\Phi^{-1}(t)^{-2}t(\mathscr{G}(t,x)+\mathscr{G}(t,y)) \tag{3.1.79}$$

for $0 < t \le T$ and $|x-y| \le \Phi^{-1}(t)$.
(b) (3.1.78) is clearly true when $\Phi^{-1}(t) \le 2|z|$. Now assume $\Phi^{-1}(t) \ge 2|z|$. Let $\alpha(\theta) = x+\theta z,\ \theta \in [-1,1]$ be a segment from $x-z$ to $x+z$. Then, for

any $\theta \in [-1,1]$ we have $|\alpha(\theta)| \geq |x| - \Phi^{-1}(t)/2$, hence

$$\begin{aligned}
|\delta_p(t,x;z)| &= |(p(t,x) - p(t,x-z)) - (p(t,x+z) - p(t,x))| \\
&= \Big| \int_0^1 \alpha'(\theta) \cdot \nabla p(t,\alpha(\theta)) - \alpha'(-\theta) \cdot \nabla p(t,\alpha(-\theta)) d\theta \Big| \\
&= \Big| \int_0^1 z \cdot (\nabla p(t,\alpha(\theta)) - \nabla p(t,\alpha(-\theta))) d\theta \Big| \\
&\leq 4c_3\, |z| \Phi^{-1}(t)^{-2} \big( |z| t \mathscr{G}(t, |x| - \Phi^{-1}(t)) \big) \\
&\leq c_4\, \Phi^{-1}(t)^{-2} |z|^2 t \mathscr{G}(t,x).
\end{aligned}$$

Here we used $|\alpha(\theta) - \alpha(-\theta)| \leq 2|z| \leq \Phi^{-1}(t)$ and (3.1.79) for the first inequality, and (3.1.24) for the second one. □

**Proposition 3.1.25.** *For every $T \geq 1$, there exist constants $c_i > 0$, $i = 1,2$, such that for any $t \in (0,T]$ and $x \in \mathbb{R}^d$,*

$$\begin{aligned}
\int_{\mathbb{R}^d} |\delta_p(t,x;z)|\, J(|z|) dz &\leq c_1 \int_{\mathbb{R}^d} \Big( \frac{|z|}{\Phi^{-1}(t)} \wedge 1 \Big)^2 t \left( \mathscr{G}(t, x \pm z) + \mathscr{G}(t,x) \right) J(|z|) dz \\
&\leq c_2 \mathscr{G}(t,x)
\end{aligned}$$

**Proof.** By (3.1.78) we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\delta_p(t,x;z)|\, J(|z|)\, dz \\
&\quad \leq c_1 \int_{\mathbb{R}^d} \Big( \frac{|z|}{\Phi^{-1}(t)} \wedge 1 \Big)^2 t \left( \mathscr{G}(t, x \pm z) + \mathscr{G}(t,x) \right) J(|z|)\, dz \\
&\quad \leq c_2 \left( \int_{\mathbb{R}^d} \Big( \frac{|z|}{\Phi^{-1}(t)} \wedge 1 \Big)^2 t \mathscr{G}(t, x+z) J(|z|)\, dz + t \mathscr{G}(t,x) \mathcal{P}(\Phi^{-1}(t)) \right) \\
&\quad =: c_2 \left( I_1 + I_2 \right)
\end{aligned}$$

Clearly, by (3.1.48) we have

$$I_2 \leq c_3 \mathscr{G}(t,x).$$

To estimate $I_1$, we divide $I_1$ into two parts as

$$\begin{aligned}I_1 &= \int_{|z|\le\Phi^{-1}(t)} \Big(\frac{|z|}{\Phi^{-1}(t)}\Big)^2 t\mathscr{G}(t,x+z)J(|z|)\,dz + \int_{|z|>\Phi^{-1}(t)} t\mathscr{G}(t,x+z)J(|z|)\,dz\\ &=: I_{11}+I_{12}\,.\end{aligned}$$

By using (3.1.24) in the first inequality below and (3.1.48) in the third, we have

$$\begin{aligned}I_{11} &\le c_4 t\mathscr{G}(t,x)\int_{|z|\le\Phi^{-1}(t)} \Big(\frac{|z|^2}{\Phi^{-1}(t)^2}\wedge 1\Big)J(|z|)\,dz\\ &\le c_4 t\mathscr{G}(t,x)\mathcal{P}(\Phi^{-1}(t)) \le c_5\mathscr{G}(t,x)\,.\end{aligned}$$

For the estimates of $I_{12}$, we will use

$$J(|z|)\le c_6\theta(|z|) = c_6\mathscr{G}_T(t,z), \quad |z|>\Phi^{-1}(t), \tag{3.1.80}$$

which follows from (3.1.1) and (2.1.12). Using (3.1.19), (3.1.80) and (3.1.34), we arrive

$$I_{12}\le c_6 at\int_{|z|>\Phi^{-1}(t)} \mathscr{G}(t,x-z)\mathscr{G}(t,z)\,dz \le c_7\mathscr{G}(t,x).$$

Here we used (3.1.23) for the last inequality. The lemma follows from the estimates of $I_{11}, I_{12}$ and $I_2$. $\square$

Recall that $\mathcal{L}^\kappa f(x) = \lim_{\varepsilon\downarrow 0}\int_{|z|>\varepsilon}(f(x+z)-f(x))\,\kappa(x,z)J(|z|)dz$ where $J:\mathbb{R}_+\to\mathbb{R}_+$ is a non-increasing function satisfying (3.1.1) and (3.1.2) with strictly increasing function $\psi$ satisfying (3.1.3) and (3.1.4). Let $\mathfrak{K}:\mathbb{R}^d\to(0,\infty)$ be a symmetric function satisfying

$$\kappa_0\le\mathfrak{K}(z)\le\kappa_1 \quad \text{for all } z\in\mathbb{R}^d \tag{3.1.81}$$

where $\kappa_0$ and $\kappa_1$ are constants in (3.0.2). We denote $Z^{\mathfrak{K}}$ symmetric Lévy process whose jumping kernel is given by $\mathfrak{K}(z)J(|z|), z\in\mathbb{R}^d$. Then the infinitesimal generator of $Z^{\mathfrak{K}}$ is a self-adjoint operator in $L^2(\mathbb{R}^d)$ and is of the following form:

$$\begin{aligned}\mathcal{L}^{\mathfrak{K}} f(x) &= \lim_{\epsilon\downarrow 0}\int_{|z|>\epsilon}(f(x+z)-f(x))\mathfrak{K}(z)J(|z|)dz\\ &= \frac{1}{2}\lim_{\epsilon\downarrow 0}\int_{|z|>\epsilon}(f(x+z)+f(x-z)-2f(x))\mathfrak{K}(z)J(|z|)dz.\end{aligned} \tag{3.1.82}$$

(3.1.1) implies that when $f\in C_b^2(\mathbb{R}^d)$, it is not necessary to take the principal value in the last line in (3.1.82). The transition density of $Z^{\mathfrak{K}}$(i.e., the heat kerenl of $\mathcal{L}^{\mathfrak{K}}$) will be denoted by $p^{\mathfrak{K}}(t,x)$. In this section, we will observe further properties of $p^{\mathfrak{K}}(t,x)$.

**Remark 3.1.26.** The operator (3.0.1) satisfies all conditions in [61] with respect to the function $\tilde{\mathscr{G}}(t,x)$ and $\Phi(r^{-1})^{-1}$ except [61, (1.7)]: Recall from Remark 3.1.6 that $\tilde{\mathscr{G}}(t,x)$ is comparable to the function $\rho(t,x)$ in [61]. Moreover, by Lemma 3.1.13, The characteristic exponent of any symmetric Lévy process whose jumping kernel comparable to $J(|z|)$, is comparable to $\Phi(r^{-1})^{-1}$. Clearly $L(\alpha_1,a_1,\Phi)$, $U(2,1,\Phi)$ and [61, Remark 5.2] with (3.1.1) imply [61, (1.4), (1.5) and (1.9)]. Also, we obtain gradient estimates with respect to $\tilde{\mathscr{G}}(t,x)$ in Proposition 3.1.19, which are same as the gradient estimates in [61, Proposition 3.2]. Under these observations, one can follow the proofs of [61] using (3.1.1) instead of the condition [61, (1.7)] and see that [61, Theorems 1.1–1.3] hold under our setting.

Using the above Remark 3.1.26, from the remainder of this section *we use [61, Theorems 1.1–1.3] without any further remark.*

Let $\widehat{\mathfrak{K}} := \mathfrak{K} - \frac{\kappa_0}{2}$. Then, $\frac{\kappa_0}{2} \le \widehat{\mathfrak{K}}(z) \le \kappa_1$. Let $p^{\widehat{\mathfrak{K}}}$ be the heat kernel of symmetric Lévy process $Z^{\widehat{\mathfrak{K}}}$ whose jumping kernel is $\widehat{\mathfrak{K}}(z)J(|z|)dz$ and $p^{\frac{\kappa_0}{2}}(t,x)=p(\frac{\kappa_0}{2}t,x)$ be the heat kernel of symmetric Lévy process $Z^{\frac{\kappa_0}{2}}$ whose jumping kernel is $\frac{\kappa_0}{2}J(|z|)dz$. Without loss of generality, we can assume that $Z^{\widehat{\mathfrak{K}}}$ and $Z^{\frac{\kappa_0}{2}}$ are independent. By [49, Theorem 1.2], there exists a constant

$c = c(T) = c(d, T, a, a_1, \alpha_1, b, \beta, C_0, \kappa_0, \kappa_1) > 0$ such that

$$p^{\widehat{\mathfrak{K}}}(t,x) \le ct\mathscr{G}(t,x) \quad \text{for all} \quad 0 < t \le T,\ x \in \mathbb{R}^d$$

for every $\mathfrak{K}$ satisfying (3.1.81). Also, by Remark 3.1.26 we have [61, (3.21)]. We record this for the readers:

$$\frac{\partial p^{\mathfrak{K}}(t,x)}{\partial t} = \mathcal{L}^{\mathfrak{K}} p^{\mathfrak{K}}(t,x), \quad \lim_{t\downarrow 0} p^{\mathfrak{K}}(t,x) = \delta_0(x). \tag{3.1.83}$$

Since $Z^{\widehat{\mathfrak{K}}}$ and $Z^{\frac{\kappa_0}{2}}$ are independent, $Z^{\mathfrak{K}}$ and $Z^{\widehat{\mathfrak{K}}}+Z^{\frac{\kappa_0}{2}}$ have same distributions. Thus, we have

$$\begin{aligned} p^{\mathfrak{K}}(t,x) &= \int_{\mathbb{R}^d} p^{\frac{\kappa_0}{2}}(t, x-y) p^{\widehat{\mathfrak{K}}}(t,y)dy \\ &= \int_{\mathbb{R}^d} p(\frac{\kappa_0}{2}t, x-y) p^{\widehat{\mathfrak{K}}}(t,y)dy. \end{aligned}$$

Now, by the convolution inequalities in Proposition 3.1.10, we can extend the estimates in Propositions 3.1.12 and 3.1.24–3.1.25, and obtain continuity of transition density with respect to $\mathfrak{K}$. We skip the proof.

**Proposition 3.1.27.** ([60, Proposition 4.4]) *There exists a constant $c = c(d, T, a, a_1, \alpha_1, b, \beta, C_0, \kappa_0, \kappa_1) > 0$ such that for any $t \in (0, T]$ and $x, y, z \in \mathbb{R}^d$,*

$$|\nabla_x p^{\mathfrak{K}}(t,x)| \le ct\Phi^{-1}(t)^{-1}\mathscr{G}(t,x)$$

$$|p^{\mathfrak{K}}(t,x) - p^{\mathfrak{K}}(t,y)| \le ct(\Phi^{-1}(t)^{-1}|x-y| \wedge 1)(\mathscr{G}(t,x) + \mathscr{G}(t,y)) \tag{3.1.84}$$

$$|\delta_{p^{\mathfrak{K}}}(t,x;z)| \le ct((\Phi^{-1}(t)^{-1}|z|)^2 \wedge 1)(\mathscr{G}(t, x\pm z) + \mathscr{G}(t,x))$$

$$\int_{\mathbb{R}^d} |\delta_{p^{\mathfrak{K}}}(t,x;z)| J(|z|)dz \le c\mathscr{G}(t,x) \tag{3.1.85}$$

**Theorem 3.1.28.** ([60, Theorem 4.5]) *for any two symmetric functions $\mathfrak{K}_1$*

*and* $\mathfrak{K}_2$ *in* $\mathbb{R}^d$ *satisfying* (3.1.81)*, any* $t \in (0, T]$ *and* $x \in \mathbb{R}^d$*, we have*

$$\left|p^{\mathfrak{K}_1}(t,x) - p^{\mathfrak{K}_2}(t,x)\right| \le c\|\mathfrak{K}_1 - \mathfrak{K}_2\|_\infty \, t\mathscr{G}(t,x)\,,$$

$$\left|\nabla p^{\mathfrak{K}_1}(t,x) - \nabla p^{\mathfrak{K}_2}(t,x)\right| \le c\|\mathfrak{K}_1 - \mathfrak{K}_2\|_\infty \Phi^{-1}(t)^{-1} t\mathscr{G}(t,x)$$

*and*

$$\int_{\mathbb{R}^d} |\delta_{p^{\mathfrak{K}_1}}(t,x;z) - \delta_{p^{\mathfrak{K}_2}}(t,x;z)| J(|z|) dz \le c\|\mathfrak{K}_1 - \mathfrak{K}_2\|_\infty \mathscr{G}(t,x).$$

Estimates in this section are almost same with [61, Section 2 and 3] except these: First of all, the function $\mathscr{G}$ is different from [61], hence our estimates are more precise than estimates in [61]. However, we don't have estimates for third derivatives in terms of $\mathscr{G}$ of the heat kernel in Proposition 3.1.12. Thus, we do not have the improvements on [61, (3.14) and (3.18)], which are used for the gradient estimate of the function $p^\kappa(t,x,y)$ in Theorems 3.1.1-3.1.4, for instance, [61, Theorem 1.1(2) and 1.2(4)].

From now on, until the end of this section we always assume that $\kappa : \mathbb{R}^d \times \mathbb{R}^d \to (0,\infty)$ is a Borel function satisfying (3.0.2) and (3.0.3), that $J$ satisfies (3.1.1)-(3.1.2) with the function $\psi$ satisfying (3.1.3) and (3.1.4).

For a fixed $y \in \mathbb{R}^d$, let $\mathfrak{K}_y(z) = \kappa(y,z)$ and let $\mathcal{L}^{\mathfrak{K}_y}$ be the freezing operator defined by

$$\mathcal{L}^{\mathfrak{K}_y} f(x) = \lim_{\varepsilon \downarrow 0} \int_{|z|>\varepsilon} \delta_f(x;z) \kappa(y,z) J(|z|) dz. \tag{3.1.86}$$

Let $p_y(t,x) := p^{\mathfrak{K}_y}(t,x)$ be the heat kernel of the operator $\mathcal{L}^{\mathfrak{K}_y}$. Note that $\mathfrak{K}_y$ satisfies (3.1.81) so that there exists a constant $c > 0$ such that

$$p_y(t,x) \le ct\mathscr{G}(t,x) \quad \text{for all } x, y \in \mathbb{R}^d, t \in (0,T].$$

By Remark 3.1.26 and [61, Theorem 1.1], we have a continuous function

$p^\kappa(t,x,y)$ on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ solving (3.1.8) and it satisfies

$$p^\kappa(t,x,y)\le ct\widetilde{\mathscr{G}}(t,x-y),\qquad 0<t\le T\ \text{ and }\ x\in\mathbb{R}^d$$

In this part, we will investigate further estimates and regularity of $p^\kappa(t,x,y)$. We first recall the construction of $p^\kappa$ from [61, section 4]. For $t>0$ and $x,y\in\mathbb{R}^d$, define

$$\begin{aligned} q_0(t,x,y) &:= \frac{1}{2}\int_{\mathbb{R}^d}\delta_{p_y}(t,x-y;z)(\kappa(x,z)-\kappa(y,z))J(|z|)dz \\ &= \big(\mathcal{L}^{\mathfrak{K}_x}-\mathcal{L}^{\mathfrak{K}_y}\big)p_y(t,\cdot)(x-y). \end{aligned}\tag{3.1.87}$$

For $n\in\mathbb{N}$, we inductively define the function $q_n(t,x,y)$ by

$$q_n(t,x,y):=\int_0^t\int_{\mathbb{R}^d}q_0(t-s,x,z)q_{n-1}(s,z,y)dzds$$

and

$$q(t,x,y):=\sum_{n=0}^\infty q_n(t,x,y).\tag{3.1.88}$$

Finally we define

$$\psi_y(t,x):=\int_0^t\psi_y(t,x,s)ds=\int_0^t\int_{\mathbb{R}^d}p_z(t-s,x-z)q(s,z,y)dzds$$

and

$$p^\kappa(t,x,y):=p_y(t,x-y)+\psi_y(t,x)\tag{3.1.89}$$

As [61, section 4], the definitions in (3.1.87)–(3.1.89) are well-defined. In other words, each integrand in (3.1.87)–(3.1.89) is integrable and series in (3.1.88) is absolutely converge on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$.

In the next lemma, we will establish the upper bounds of $p^\kappa$.

**Theorem 3.1.29.** *For every $T\ge 1$ and $\delta_0\in(0,\delta]\cap(0,\alpha_1/2)$, there are* *constants $c_1$ and $c_2$ such that for any $t \in (0,T]$ and $x, y \in \mathbb{R}^d$,*

$$|\psi_y(t,x)| \le c_1 t\big(\mathscr{G}_0^{\delta_0} + \mathscr{G}_{\delta_0}^0\big)(t, x-y) \tag{3.1.90}$$

*and*

$$p^\kappa(t,x,y) \le c_2 t \mathscr{G}(t, x-y). \tag{3.1.91}$$

*The constants $c_1$ and $c_2$ depend on $d, T, a, a_1, \alpha_1, b, \beta, C_0, \delta_0, \delta, \kappa_0, \kappa_1$ and $\kappa_2$.*

**Proof.** We first claim that for $n \in \mathbb{N}_0$,

$$|q_n(t,x,y)| \le d_n\big(\mathscr{G}_{(n+1)\delta_0}^0 + \mathscr{G}_{n\delta_0}^{\delta_0}\big)(t, x-y) \tag{3.1.92}$$

with

$$d_n := (16C(\delta_0,T)c_2)^{n+1} \prod_{k=1}^n B(\delta_0/2, k\delta_0/2) = (16Cc_2)^{n+1} \frac{\Gamma(\delta_0/2)^{n+1}}{\Gamma((n+1)\delta_0/2)}$$

where $C = C(\delta_0, T)$ is the constant in (3.1.33). Without loss of generality, we assume that $C \ge 1/16$.

For $n = 0$, using (3.1.87), (3.0.2), (3.0.3) and (3.1.85) we have

$$\begin{aligned}
|q_0(t,x,y)| &\le \frac{1}{2}\int_{\mathbb{R}^d} |\delta_{p_y}(t, x-y; z)(\kappa(x,z) - \kappa(y,z))| J(|z|)dz \\
&\le c_1\big(|x-y|^{\delta_0} \wedge 1\big) \int_{\mathbb{R}^d} |\delta_{p_y}(t, x-y; z)| J(|z|)dz \\
&\le c_2\big(|x-y|^{\delta_0} \wedge 1\big)\mathscr{G}(t, x-y) = c_2 \mathscr{G}_0^{\delta_0}(t, x-y).
\end{aligned}$$

Suppose that (3.1.92) is valid for $n$. Then for $t \le T$,

$$\begin{aligned}
|q_{n+1}(t,x,y)| &\le \int_0^t \int_{\mathbb{R}^d} |q_0(t-s, x, z) q_n(s, z, y)| dz ds \\
&\le c_2 d_n \int_0^t \int_{\mathbb{R}^d} \mathscr{G}_0^{\delta_0}(t-s, x-z)\big(\mathscr{G}_{(n+1)\delta_0}^0 + \mathscr{G}_{n\delta_0}^{\delta_0}\big)(x, z-y) dz ds \\
&\le 16Cc_2 d_n B(\delta_0/2, (n+1)\delta_0/2)\big(\mathscr{G}_{(n+2)\delta_0}^0 + \mathscr{G}_{(n+1)\delta_0}^{\delta_0}\big)(t, x-y) \\
&= d_{n+1}\big(\mathscr{G}_{(n+2)\delta}^0 + \mathscr{G}_{(n+1)\delta_0}^{\delta_0}\big)(t, x-y)
\end{aligned}$$

here we used induction in the second line, and used (3.1.35) and (3.1.36) in the last line. For the third line, we need the following: let $\theta = \eta = 1$, $\gamma_1 = \delta_2 = 0$, $\delta_1 = \delta_0$ and $\gamma_2 = (n+1)\delta_0$ which satisfy conditions in Lemma 3.1.10(c) since $\delta_0 \in (0, \alpha_1/2)$. Then, by (3.1.35) we have

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}^d} \mathscr{G}_0^{\delta_0}(t-s, x-z)\mathscr{G}_{(n+1)\delta_0}^0(s, z-y)dzds \\ &\le 4CB(\delta_0/2, (n+1)\delta_0/2)\big(\mathscr{G}^0_{(n+2)\delta_0} + \mathscr{G}^{\delta_0}_{(n+1)\delta_0} + \mathscr{G}^0_{(n+2)\delta_0}\big)(t, x-y) \\ &\le 8CB(\delta_0/2, (n+1)\delta_0/2)\big(\mathscr{G}^0_{(n+2)\delta_0} + \mathscr{G}^{\delta_0}_{(n+1)\delta_0}\big)(t, x-y).\end{aligned}$$

Also, letting $\theta = \eta = 1$, $\gamma_1 = 0$, $\delta_1 = \delta_2 = \delta_0$ and $\gamma_2 = \delta_0$ which satisfy conditions in Lemma 3.1.10(c),

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}^d} \mathscr{G}_0^{\delta_0}(t-s, x-z)\mathscr{G}_{n\delta_0}^{\delta_0}(x, z-y)dzds \\ &\le 4CB(\delta_0/2, (n+1)\delta_0/2)\big(\mathscr{G}^0_{(n+2)\delta_0} + \mathscr{G}^{\delta_0}_{(n+1)\delta_0} + \mathscr{G}^{\delta_0}_{(n+1)\delta_0}\big)(t, x-y) \\ &\le 8CB(\delta_0/2, (n+1)\delta_0/2)\big(\mathscr{G}^0_{(n+2)\delta_0} + \mathscr{G}^{\delta_0}_{(n+1)\delta_0}\big)(t, x-y).\end{aligned}$$

Thus, (3.1.92) is valid for all $n \in \mathbb{N}_0$. Note that

$$\sum_{n=0}^{\infty} d_n \Phi^{-1}(T)^{\delta_0} := C_1(\delta_0, T) < \infty$$

since $\frac{d_{n+1}\Phi^{-1}(T)^{(n+1)\delta_0}}{d_n\Phi^{-1}(T)^{n\delta_0}} = 16Cc_2\Phi^{-1}(T)^{\delta_0}B(\delta_0/2, (n+1)\delta_0/2) \to 0$ as $n \to \infty$. So, by using (3.1.29) in the second line we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} |q_n(t,x,y)| &\le \sum_{n=0}^{\infty} d_n\big(\mathscr{G}^0_{(n+1)\delta_0} + \mathscr{G}^{\delta_0}_{n\delta_0}\big)(t, x-y) \\ &\le \sum_{n=0}^{\infty} d_n \Phi^{-1}(T)^{n\delta_0}\big(\mathscr{G}^0_{\delta_0} + \mathscr{G}^{\delta_0}_0\big)(t, x-y) = C_1\big(\mathscr{G}^0_{\delta_0} + \mathscr{G}^{\delta_0}_0\big)(t, x-y)\end{aligned}$$

for $t \le T$. Therefore, for every $t \in (0, T]$ and $x, y \in \mathbb{R}^d$,

$$|q(t,x,y)| \le C_1\big(\mathscr{G}^0_{\delta_0} + \mathscr{G}^{\delta_0}_0\big)(t, x-y). \tag{3.1.93}$$

To obtain (3.1.90) and (3.1.91), we calculate that

$$\begin{aligned}
|\psi_y(t,x)| &\le \int_0^t \int_{\mathbb{R}^d} p_z(t-s,x-z)|q(s,z,y)|\,dz\,ds \\
&\le c_3 \int_0^t \int_{\mathbb{R}^d} (t-s)\mathscr{G}(t-s,x-z)\left(\mathscr{G}^0_{\delta_0}+\mathscr{G}^{\delta_0}_0\right)(s,z-y)\,dz\,ds \\
&\le c_4 t\left(\mathscr{G}^0_{\delta_0}+\mathscr{G}^{\delta_0}_0\right)(t,x-y) \\
&\le 2c_4\Phi^{-1}(T)^{\delta_0} t\mathscr{G}(t,x-y) = c_5 t\mathscr{G}(t,x-y), \qquad \text{for all } t\in(0,T]\,.
\end{aligned}$$

Here we used (3.1.83) and (3.1.93) for the second line, (3.1.35) for the third line and (3.1.31) for the last line. Therefore, using (3.1.83) we arrive $p^\kappa(t,x,y) \le p_y(t,x-y)+|\psi_y(t,x)| \le c_6 t\mathscr{G}(t,x-y)$. □

We concludes this section with some fractional estimates on $p^\kappa(t,x,y)$.

**Lemma 3.1.30.** *For every $T\ge 1$ and $\gamma\in(0,1]\cap(0,\alpha_1)$, there exists a constant $c_3$ such that for any $t\in(0,T]$ and $x,x',y\in\mathbb{R}^d$,*

$$|p^\kappa(t,x,y)-p^\kappa(t,x',y)| \le c_3|x-x'|^\gamma\; t\Big(\mathscr{G}^0_{-\gamma}(t,x-y)+\mathscr{G}^0_{-\gamma}(t,x'-y)\Big).$$

*The constant $c_3$ depends on $d,T,a,a_1,\alpha_1,b,\beta,C_0,\gamma,\delta,\kappa_0,\kappa_1$ and $\kappa_2$.*

**Proof.** Assume that $x,x',y\in\mathbb{R}^d$ and $t\in(0,T]$. By (3.1.84) and the fact that $\gamma\le 1$, we have

$$|p_z(s,x-z)-p_z(s,x'-z)| \le c_1|x-x'|^\gamma s\big(\mathscr{G}^0_{-\gamma}(s,x-z)+\mathscr{G}^0_{-\gamma}(s,x'-z)\big)\,.$$

for any $0<s\le T$ and $z\in\mathbb{R}^d$. Thus, using above inequalities, (3.1.93) and a change of the variables, we further have that for $\delta_0 := (\delta\wedge\alpha_1/4)\in(0,\delta]\cap(0,\alpha_1/2)$,

$$\begin{aligned}
&|\psi_y(t,x)-\psi_y(t,x')| \le \int_0^t\int_{\mathbb{R}^d} |p_z(t-s,x-z)-p_z(t-s,x'-z)||q(s,z,y)|\,dz\,ds \\
&\le c_2|x-x'|^\gamma t\Big(\mathscr{G}^0_{-\gamma+\delta_0}(t,x-y)+\mathscr{G}^{\delta_0}_{-\gamma}(t,x-y)+\mathscr{G}^0_{-\gamma+\delta_0}(t,x'-y)+\mathscr{G}^{\delta_0}_{-\gamma}(t,x'-y)\Big) \\
&\le 2c_2\Phi^{-1}(T)^{\delta_0}|x-x'|^\gamma t\big(\mathscr{G}^0_{-\gamma}(t,x-y)+\mathscr{G}^0_{-\gamma}(t,x'-y)\big)
\end{aligned}$$

for all $t \le T$. Since $\gamma < \alpha_1$, the penultimate line follows from (3.1.35) (with $\theta = 0$), and the last line by (3.1.29) and (3.1.30). The lemma follows by combining above two estimates and (3.1.89). $\square$

### 3.1.6 Proof of Theorems 3.1.1-3.1.4

In this subsection we prove the main theorems in this chapter. We first prove that the function $p^\kappa(t,x,y)$ defined by (3.1.89) satisfies all statements in Theorems 3.1.1-3.1.4, then we show that $p^\kappa(t,x,y)$ is the unique solution to (3.1.8) satisfying (i)–(iii) in Theorem 3.1.1.

**Proof of Theorems 3.1.3 and 3.1.4.** It follows from Remark 3.1.26 that we can apply the results in [61, Theorem 1.1-1.4] for operator (3.0.1) with the function $\tilde{\mathscr{G}}(t,x)$. Note that the function $p^\kappa(t,x,y)$ in [61, Theorems 1.1-1.4] is constructed by the same way as (3.1.89). Therefore, Theorems 3.1.3 and 3.1.4 except (3.1.15) immediately follow from Remarks 3.1.6 and 3.1.26, and [61, Theorem 1.1(iii), 1.2 and 1.3]. Finally (3.1.15) is proved in Lemma 3.1.30. $\square$

Now we prove the lower bound estimates in Theorem 3.1.1 and Corollary 3.1.2 for the function $p^\kappa(t,x,y)$ in (3.1.89). By Theorems 3.1.3 and 3.1.4, we have that $(P_t^\kappa)_{t\ge 0}$ defined by $p^\kappa(t,x,y)$ in (3.1.89) with (3.1.16) is a Feller semigroup and there exists a Feller process $X=(X_t,\mathbb{P}_x)$ corresponding to $(P_t^\kappa)_{t\ge0}$. Moreover, by (3.1.17) for $f \in C_b^{2,\varepsilon}(\mathbb{R}^d)$,

$$f(X_t) - f(x) - \int_0^t \mathcal{L}^\kappa f(X_s)\, ds \tag{3.1.94}$$

is a martingale with respect to the filtration $\sigma(X_s, s \le t)$. Therefore, by the same argument as that in [34, Section 4.4], we have the following Lévy system formula: for every function $f : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ vanishing on the diagonal and every stopping time $S$,

$$\mathbb{E}_x \sum_{0<s\leq S} f(X_{s-}, X_s) = \mathbb{E}_x \int_0^S f(X_s, y) J_X(X_s, dy) ds\,,$$

where $J_X(x,y) := \kappa(x, y-x)J(|x-y|)$. For $A \in \mathcal{B}(\mathbb{R}^d)$ we define $\tau_A := \inf\{t \geq 0 : X_t \notin A\}$ be the exit time from $A$.

Using (3.1.94), (3.0.2) and (3.1.48), the proof of the following result is the same as the one in [61, Lemma 5.7]. We skip the proof.

**Lemma 3.1.31.** *Let $T \geq 1$. For each $\varepsilon \in (0,1)$ there exists $\lambda = \lambda(\varepsilon) > 0$ such that for every $0 < r \leq \Phi^{-1}(T)$,*

$$\sup_{x\in\mathbb{R}^d} \mathbb{P}_x \left(\tau_{B(x,r)} \leq \lambda\Phi(r)\right) \leq \varepsilon\,. \tag{3.1.95}$$

We record that by (3.1.95), for any $x \in \mathbb{R}^d$ and $0 < r \leq \Phi^{-1}(T)$ we have

$$\mathbb{E}_x[\tau_{B(x,r)}] \geq \lambda(1/2)\Phi(r)\mathbb{P}_x(\tau_{B(x,r)} > \lambda\Phi(r)) \geq \frac{\lambda}{2}\Phi(r) = c\Phi(r).$$

Now we are ready to prove the lower bound in (3.1.12).

**Lemma 3.1.32.** *The function $p^\kappa(t,x,y)$ in* (3.1.89) *satisfies* (3.1.12).

**Proof.** Fix $T \geq 1$. Let $p_y(t,x)$ be the heat kernel of the freezing operator in (3.1.86), and $J_y(z) := \kappa(y,z)J(|z|)$ and $\phi_y(z)$ be the corresponding Lévy measure and characteristic exponent, respectively. By [54, Theorem 2], there exist constants $C_1, C_2 > 0$ such that

$$p_y(t,x) \geq C_1\Phi^{-1}(t)^{-d}, \qquad t \in (0,T], y \in \mathbb{R}^d \text{ and } |x| \leq C_2\Phi^{-1}(t). \tag{3.1.96}$$

Indeed, $J_y(z)dz$ is symmetric and infinite Lévy measure by (3.0.2) and that $J(|z|)dz$ is infinite Lévy measure. To check the condition [54, (3)], we need to show that there exists a constant $c > 0$ such that

$$\int_{\mathbb{R}^d} e^{-t\phi_y(z)}|z|dz \le ch_y(t)^{-d-1}, \quad 0 < t, y \in \mathbb{R}^d$$

where $h_y(t) := \frac{1}{\Psi_y^{-1}(t^{-1})}$ and $\Psi_y(r) := \sup_{|z|\le r} \phi_y(z)$. Let $\mathcal{P}(r) := \int_{\mathbb{R}^d} \big(1 \wedge \frac{|z|^2}{r^2}\big) J(|z|)dz$ and $\mathcal{P}_y(r) := \int_{\mathbb{R}^d} \big(1 \wedge \frac{|z|^2}{r^2}\big) J_y(|z|)dz$. Then, by [54, (11)] we have

$$c_1\kappa_0\mathcal{P}(r^{-1}) \le c_1\mathcal{P}_y(r^{-1}) \le \Psi_y(r) \le 2\mathcal{P}_y(r^{-1}) \le 2\kappa_1\mathcal{P}(r^{-1}). \quad (3.1.97)$$

On the other hand, by the symmetry of $J_y$ and [54, (10)], we have

$$\phi_y(z) \ge (1-\cos 1)\int_{|\xi|\le 1/|z|} |\xi\cdot z|^2 J_y(d\xi) \ge c\int_{|\xi|\le 1/|z|} |\xi\cdot z|^2 J(|\xi|)d\xi.$$

Since by a rotation

$$\int_{|\xi|\le 1/|z|} |\xi\cdot z|^2 J(|\xi|)d\xi = |z|^2\int_{|\xi|\le 1/|z|} \xi_i^2 J(|\xi|)d\xi, \quad i = 1,\dots,d,$$

we have

$$\phi_y(z) \ge d^{-1}\kappa_0(1-\cos 1)|z|^2\int_{|\xi|\le 1/|z|} |\xi|^2 J(|\xi|)d\xi.$$

Thus, when $|z| \le 1$ we have

$$\phi_y(z) \ge d^{-1}\kappa_0(1-\cos 1)|z|^2\int_{|\xi|\le 1} |\xi|^2 J(d\xi) \ge c_2|z|^2 = c_3\Phi(|z|^{-1}),$$

whereas by (3.1.1) we have

$$\phi_y(z) \ge c|z|^2\int_{|\xi|\le 1/|z|} |\xi|^2 J(d\xi) \ge c_4|z|^2\int_0^{1/|z|} \frac{s}{\psi(s)}ds = c_4\Phi(|z|^{-1})$$

for $|z| \ge 1$. Therefore, using (3.1.48) and (3.1.46) we obtain

$$\phi_y(z) \ge c_5\Phi(|z|^{-1}) \ge c_6\mathcal{P}(|z|) \ge (c_6/2)\phi(|z|). \quad (3.1.98)$$

Moreover, (3.1.48) and (3.1.97) also imply that $h_y(t) \asymp \Phi^{-1}(t) \asymp h(t) := \frac{1}{\Psi^{-1}(t^{-1})}$. From this and (3.1.98) we can follow the proof of Lemma 3.1.15 and obtain for $t > 0$ and $y \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} e^{-t\phi_y(z)}|z|dz \le \int_{\mathbb{R}^d} e^{-c_6 t\phi(z)/2}|z|dz \le c_7 h(t)^{-d-1} \le c_8 h_y(t)^{-d-1}.$$

Note that every constant above is independent of $y$. Therefore, letting $f(r) \equiv 0$ we obtain all conditions in [54, Theorem 2] so we have (3.1.96) where $C_1 > 0$ is independent of $y$. The rest of the proof is almost identical to the one of [61, Theorem 1.4]. □

**Proof of Theorem 3.1.1.** By Remarks 3.1.6 and 3.1.26, $p^\kappa(t, x, y)$ defined in (3.1.89) satisfies (3.1.8), (3.1.10) and (3.1.11). Also, (3.1.9) and (3.1.12) follow from Theorem 3.1.29 and Lemma 3.1.32, respectively. It remains to show the uniqueness part of Theorem 3.1.1. Recall that we observe in Remark 3.1.26 that [61, (1.9)] holds. Thus all results in [61, Sections 5.1 and 5.2] hold for our case. Since properties (i)–(iii) are stronger than ones in [61, Theorem 1.1], we now see that the proof of the uniqueness part of Theorem 3.1.1 is exactly same as the one of the uniqueness part of [61, Theorem 1.1]. □

# Chapter 4

# Applications of heat kernel estimates

Heat kernel estimates are not only important themselves, but also applicable to many related topics. For instance, as a corollary of Theorem 2.1.1, we obtain a global sharp two-sided estimate of the Green function $G(x, y) = \int_0^\infty p(t, x, y)dt$.

**Corollary 4.0.1.** ([3, Corollary 1.3]) *Suppose that the assumptions in Theorem 2.1.1 hold and $d > \beta_2 \wedge 2$. Then for any $x, y \in \mathbb{R}^d$, $G(x, y) \asymp \Phi(|x-y|)|x-y|^{-d}$.*

It is well known that the Green function defined above is the fundamental solution of Poisson equation $Lu = f$ with respect to the operator

$$Lu(x) := \lim_{\varepsilon\to 0}\int_{|y-x|>\varepsilon}(u(y) - u(x))J(x, y)dy.$$

For instance, if $J(x, y) = \frac{c}{|x-y|^{d+\alpha}}$ with $\alpha \in (0, 2)$, the corresponding operator is fractional Laplacian $-(-\triangle)^{\alpha/2}$. Thus, heat kernel estimates are helpful to study partial differential equations. In Section 4.1 we will see how it works.

One of the most important consequences of heat kernel esimate is Harnack inequality. Harnack inequalities and Hölder regularities for harmonic functions are important components of the celebrated De Giorgi-Nash-Moser theory in harmonic analysis and partial differential equations.

Equivalent characterizations for parabolic Harnack inequalities (that is, Harnack inequalities for caloric functions) were obtained by [42] and [78] for Brownian motions (or equivalently, Laplace-Beltrami operators) on complete Riemannian manifolds. They showed that parablic Harnack inequalities are equivalent to the two-sided Guassian type heat kernel estimates. This result was extended to general processes in many spaces, such as graphs or metric measure spaces including fractals (see [10, 33, 39] and references therein).

For instance, if the two-sided heat kernel estimates $p(t,x,y) \asymp \frac{1}{V(x,\psi^{-1}(t))} \wedge \frac{t}{V(x,d(x,y))\psi(d(x,y))}$ in (1.0.3) holds for symmetric pure-jump process $X$ in metric measure space $(M,d,\mu)$ with volume doubling condition, then by [33, Theorem 1.17], the following parabolic Harnack inequality holds : there exist constants $0 < c_1 < c_2 < c_3 < c_4$, $0 < c_5 < 1$ and $c_6 > 0$ such that for every $x \in M$, $t_0 \geq 0$, $r > 0$ and for every non-negative function $u(t,x)$ on $[0,\infty) \times M$ that is caloric on cylinder $Q := (t_0, t_0 + c_4\psi(r)) \times B(x,r)$,

$$\operatorname*{ess\,sup}_{Q-} u \leq C_6 \operatorname*{ess\,inf}_{Q+} u,$$

where $Q_- := (t_0 + c_1\psi(r), t_0 + c_2\psi(r)) \times B(x, c_5 r)$ and $Q_+ := (t_0 + c_3\psi(r), t_0 + c_4\psi(r)) \times B(x, c_5 r)$. Moreover, parabolic Harnack inequality implies the following elliptic Harnack inequality : there exist $c > 0$ and $\delta \in (0,1)$ such that for every $x \in M$, $r > 0$ and for every nonnegative function $u$ on $M$ that is harmonic in $B(x,r)$,

$$\operatorname*{ess\,sup}_{B(x,\delta r)} h \leq c \operatorname*{ess\,inf}_{B(x,\delta r)} h.$$

In this chapter, we introduce some applications of heat kernel estimates. First one is boundary regularity for the solutions of Poisson equation based on [58]. Next one is the laws of iterated logarithms, which are properties of the sample paths of stochastic processes. We obtain various types of laws of iterated logarithms from the heat kernel estimates.

## 4.1 Boundary regularity for nonlocal operators

In this section, we study the boundary regularity of solutions of the Dirichlet problem for the nonlocal operator with a kernel of variable orders. Let $D$ be a bounded $C^{1,1}$ open set in $\mathbb{R}^d$. We consider the following Dirichlet (exterior) problem

$$\begin{cases} -Lu = f & \text{in } D, \\ u = 0 & \text{in } \mathbb{R}^d \backslash D, \end{cases} \tag{4.1.1}$$

where $L$ is symmetric operator of the form

$$Lu(x) = p.v \int_{\mathbb{R}^d \backslash \{0\}} (u(x+y) - u(x))\, J(|y|) dy, \tag{4.1.2}$$

which is infinitesimal generator for a class of isotropic Lévy processes.

The main results of this section are the existence and the uniqueness of the viscosity solution $u$ of (4.1.1), obtaining boudary decaying function $V$ of such solution $u$ and the regularity of the quotient $u/V(\delta_D)$ up to the boundary, where $\delta_D(x) := \operatorname{dist}(x, D^c)$ is distance function.

The first result is the Hölder estimates up to the boundary of solutions of the Dirichlet problem (4.1.1). Unlike the case of the fractional Laplacian, it is inappropriate to represent Hölder regularity as a single number since kernel in (4.1.2) has variable orders. Therefore it is natural to consider a generalized Hölder space.

The operators we consider in this section coincides with infinitesimal generators of isotropic unimodal Lévy processes for $C^2(\mathbb{R}^d)$ functions. Thus, we first explain the definitions and properties of Lévy processes, and some related concepts. Then we introduce some additional conditions that will be needed in this paper. With these concepts, we state our main results.

We denote by $C(D)$ the Banach space of bounded and continuous functions on $D$, equipped with the supremum norm $\|f\|_{C(D)} := \sup_{x \in D} |f(x)|$, and denote by $C^k(D), k \geq 1$, the Banach space of $k$-times continuously differentiable functions on $D$ with the norm $\|f\|_{C^k(D)} := \sum_{|\gamma|\le k} \sup_{x\in D} |D^\gamma f(x)|$. Also, denote $C_0(D) := \{u \in C(D) : u$ vanishes at the boundary of $D\}$. For $x \in \mathbb{R}^d$, define $C^1(x)$ as the collection of functions which are $C^1$ in some neighborhood of $x$. Similarly, we define $C^2(x)$, $C^{1,1}(x)$, etc. For $0 < \alpha < 1$, the Hölder space $C^\alpha(\mathbb{R}^d)$ is defined as

$$C^\alpha(\mathbb{R}^d) := \left\{ f \in C(\mathbb{R}^d) \mid \|f\|_{C^\alpha(\mathbb{R}^d)} < \infty \right\},$$

equipped with the $C^\alpha$-norm

$$\|f\|_{C^\alpha(\mathbb{R}^d)} := \|f\|_{C(\mathbb{R}^d)} + \sup_{x,y\in\mathbb{R}^d, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

Also, for given open set $D \subset \mathbb{R}^d$ we define $C^\alpha(D)$ by

$$C^\alpha(D) := \left\{ f \in C(D) \mid \|f\|_{C^\alpha(D)} < \infty \right\}$$

with the norm

$$\|f\|_{C^\alpha(D)} := \|f\|_{C(D)} + \sup_{x,y\in D, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

For given function $h : (0,\infty) \to (0,\infty)$, we define Generalized Hölder space $C^h(D)$ for bounded open set $D$ by

$$C^h(D) := \left\{ f \in C(D) \mid \|f\|_{C^h(D)} < \infty \right\},$$

equipped with the norm

$$\|f\|_{C^h(D)} := \|f\|_{C(D)} + \sup_{x,y\in D, x\neq y} \frac{|f(x)-f(y)|}{h(|x-y|)}.$$

We define seminorm $[\,\cdot\,]_{C^h(D)}$ by

$$[f]_{C^h(D)} := \sup_{x,y\in D, x\neq y} \frac{|f(x)-f(y)|}{h(|x-y|)}.$$

We denote the diameter of $D$ by $\mathrm{diam}(D)$. Note that if $h_1 \asymp h_2$ in $0 < r \leq \mathrm{diam}(D)$, $\|\cdot\|_{C^{h_1}(D)}$ and $\|\cdot\|_{C^{h_2}(D)}$ are equivalent and $C^{h_1}(D) = C^{h_2}(D)$.

We say that $D \subset \mathbb{R}^d$ (when $d \geq 2$) is a $C^{1,1}$ open set if there exist a localization radius $R_0 > 0$ and a constant $\Lambda > 0$ such that for every $z \in \partial D$ there exist a $C^{1,1}$-function $\Psi = \Psi_z : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying $\Psi(0) = 0$, $\nabla\Psi(0) = (0, \ldots, 0)$, $\|\nabla\Psi\|_\infty \leq \Lambda$, $|\nabla\Psi(x) - \nabla\Psi(w)| \leq \Lambda|x - w|$ and an orthonormal coordinate system $CS_z$ of $z = (z_1, \cdots, z_{n-1}, z_n) := (\widetilde{z}, z_n)$ with origin at $z$ such that $D \cap B(z, R_0) = \{y = (\tilde{y}, y_n) \in B(0, R_0) \text{ in } CS_z : y_n > \Psi(\widetilde{y})\}$. The pair $(R_0, \Lambda)$ will be called the $C^{1,1}$ characteristics of the open set $D$. Note that a $C^{1,1}$ open set $D$ with characteristics $(R_0, \Lambda)$ can be unbounded and disconnected, and the distance between two distinct components of $D$ is at least $R_0$. By a $C^{1,1}$ open set in $\mathbb{R}$ with a characteristic $R_0 > 0$, we mean an open set that can be written as the union of disjoint intervals so that the infimum of the lengths of all these intervals is at least $R_0$ and the infimum of the distances between these intervals is at least $R_0$.

Next we define the *viscosity solution* of $Lu = f$ in $D$. A function $u : \mathbb{R}^d \to \mathbb{R}$ which is upper (resp. lower) semicontinuous on $\overline{D}$ is said to be a *viscosity subsolution* (resp. *viscosity supersolution*) to $Lu = f$, and we write $Lu \geq f$ (resp. $Lu \leq f$) in *viscosity sense*, if for any $x \in D$ and a test function $v \in C^2(x)$ satisfying $v(x) = u(x)$ and

$$v(y) > u(y) \quad (\text{resp. } <), \quad y \in \mathbb{R}^d \setminus \{x\},$$

it holds that

$$Lv(x) \geq f(x) \quad (\text{resp. } \leq).$$

A function $u$ is said to be a *viscosity solution* if $u$ is both sub and supersolution.

### 4.1.1 Main results

Let $X = (X_t, \mathbb{P}^x, t \geq 0, x \in \mathbb{R}^d)$ be a Lévy process in $\mathbb{R}^d$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P}^x)$.For the precise definition of Lévy process, see [80, Definition 1.5]. By Lévy-Khintchine formula, the characteristic exponent of Lévy process is given by

$$\mathbb{E}^0[e^{iz\cdot X_t}] = e^{t\Phi(z)}, \quad z \in \mathbb{R}^d,$$

where

$$\Phi(z) = -\frac{1}{2} z \cdot Uz + i\gamma \cdot z + \int_{\mathbb{R}^d} \left(e^{iz\cdot x} - 1 - iz \cdot x \mathbf{1}_{\{|x|\leq 1\}}\right) J(dx)$$

with an $d \times d$ symmetric nonnegative-definite matrix $U = (U_{ij})$, $\gamma \in \mathbb{R}^d$ and a measure $J(dx)$ on $\mathbb{R}^d \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}^d \setminus \{0\}} \left(1 \wedge |x|^2\right) J(dx) < \infty.$$

Let $(P_t)_{t\geq 0}$ be a transition semigroup for $X$. Now, define *the infinitesimal generator* $A$ of $X$ by

$$Au(x) := \lim_{t \downarrow 0} \frac{P_t u(x) - u(x)}{t}$$

if the limit exists. By [85, Section 4.1], $Au$ is well-defined for $u \in C^2(\mathbb{R}^d)$ and represented by

$$\begin{aligned} Au(x) = \frac{1}{2} \sum_{i,j=1}^{n} U_{ij} \partial_{ij} u(x) + \sum_{i=1}^{n} \gamma_i \partial_i u(x) \\ + \int_{\mathbb{R}^d \setminus \{0\}} \left(u(x+y) - u(x) - \mathbf{1}_{\{|y|\leq 1\}} y \cdot \nabla u(x)\right) J(dy). \end{aligned}$$

Recall that

$$Lu(x) = p.v \int_{\mathbb{R}^d \setminus \{0\}} (u(x+y) - u(x)) \, J(|y|) dy,$$

with the function $J : \mathbb{R}_+ \to \mathbb{R}_+$. Let $X$ be an isotropic pure jump Lévy process in $\mathbb{R}^d$ with generating triplet $(0, 0, J(|y|)dy)$ and $\Phi$ be its characteristic exponent. We say that the condition **(A)** holds if the following conditions **(A1)**-**(A3)** hold :

**(A1)** $\Phi$ satisfies $L^1(2\alpha_1, a_1^{-1})$ and $U^1(2\alpha_2, a_1)$ for some $a_1 \geq 1$ and $0 < \alpha_1 \leq \alpha_2 < 1$.

**(A2)** There exists a constant $a_2 > 1$ such that

$$J(r) \leq a_2 J(r+1) \text{ for all } r > 0. \tag{4.1.3}$$

**(A3)** $r \mapsto J(r), \; -\frac{J'(r)}{r}$ is non-increasing.

Let

$$\varphi(r) := \frac{J(1)}{J(r) r^n}.$$

By [17], for any $c > 0$ we have $\Phi(r^{-1})^{-1} \asymp \varphi(r)$ in $0 < r \leq c$ with comparison constant depending only on $c$ and $d$. Thus, there exists a constant $a_3 = a_3(d, a_1) \geq 1$ such that $L(2\alpha_1, a_3^{-1}, \varphi)$ and $U(2\alpha_2, a_3, \varphi)$ hold. Note that $L(2\alpha_1, a_3^{-1}, \varphi)$ implies that $\varphi(r) \leq c r^{2\alpha_1}$ for $r \leq 1$, so by definition of $\varphi$ we see that $J(|y|)dy$ is an infinite measure.

For every open subset $D \subset \mathbb{R}^d$, the transition density $p_D(t, x, y)$ of killed process $X^D$ satisfies

$$p_D(t, x, y) = p(t, |x-y|) - \mathbb{E}^x[p(t - \tau_D, |X_{\tau_D} - y|); \tau_D < t],$$

and its transition semigroup $(P_t^D)_{t \geq 0}$ is represented by

$$P_t^D f(x) := \mathbb{E}^x[f(X_t^D)] = \int_D f(y) p_D(t, x, y) \, dy.$$

Note that, under settings above, the infinitesimal generator can be rewritten as

$$Au(x) = \frac{1}{2}\int_{\mathbb{R}^d\setminus\{0\}} (u(x+y)+u(x-y)-2u(x))\, J(|y|)dy \qquad (4.1.4)$$

for $u \in C^2(\mathbb{R}^d)$. Moreover, it is known in [4, Lemma 2.6] that (4.1.4) still holds for $u \in C^2(x) \cap C_0(\mathbb{R}^d)$. Thus, we have that $Au(x) = Lu(x)$ for any $u \in C^2(x) \cap C_0(\mathbb{R}^d)$ for the next use.

Next we will define the renewal function $V$, which will be act as a barrier. Let $Z = (Z_t)_{t\geq 0}$ be an one-dimensional Lévy process with characteristic exponent $\Phi(|z|)$ and $M_t := \sup\{Z_s : 0 \leq s \leq t\}$ be the supremum of $Z$. Let $L = (L_t)_{t\geq 0}$ be a local time of $M_t - Z_t$ at 0, which satisfies

$$L_t = \int_0^t \mathbf{1}_{\{M_t = Z_t\}}(s)ds.$$

Note that since $t \mapsto L_t$ is non-decreasing and continuous with probability 1, we can define the right-continuous inverse of $L$ by

$$L^{-1}(t) := \inf\{s > 0 : L(s) > t\}.$$

The mapping $t \mapsto L^{-1}(t)$ is non-decreasing and right-continuous a.s. The process $L^{-1} = (L_t^{-1})_{t\geq 0}$ with $L_t^{-1} = L^{-1}(t)$ is called *the ascending ladder time process* of $Z$. *The ascending ladder height process* $H = (H_t)_{t\geq 0}$ is defined as

$$H_t := \begin{cases} M_{L_t^{-1}} (= Z_{L_t^{-1}}) & \text{if} \quad L_t^{-1} < \infty, \\ \infty & \text{otherwise.} \end{cases}$$

(See [40] for details.) Define the renewal function of the ladder height process $H$ with respect to $\Phi$ by

$$V(x) = \int_0^\infty \mathbb{P}(H_s \leq x)ds, \quad x \in \mathbb{R}.$$

It is known that $V(x) = 0$ if $x \leq 0$, $V(\infty) = \infty$ and $V$ is strictly increasing, differentiable on $[0, \infty)$. So, there exists the inverse function $V^{-1} : [0, \infty) \to [0, \infty)$.

Now we are ready to introduce main assumptions in this section.

**Theorem 4.1.1.** ([58, Theorem 2.1]) *Assume that $D$ is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ and $L$ is an operator of the form (4.1.2) satisfying ($\mathbf{A}$). If $f \in C(D)$, there exists a unique viscosity solution $u$ of (4.1.1) and $u \in C^V(D)$. Moreover, we have*

$$\|u\|_{C^V(D)} \le C\|f\|_{C(D)},$$

*where $\overline{\phi}(r) := \varphi(r)^{1/2}$, for some constant $C > 0$ depending only on $d, D$, and $J$.*

**Theorem 4.1.2.** ([58, Theorem 2.2]) *Assume that $D$ is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ and $L$ is an operator of the form (4.1.2) satisfying ($\mathbf{A}$). If $f \in C(D)$ and $u$ is the viscosity solution of (4.1.1), then $u/V(\delta_D) \in C^\alpha(D)$ and*

$$\left\| \frac{u}{V(\delta_D)} \right\|_{C^\alpha(D)} \le C\|f\|_{C(D)}$$

*for some constants $\alpha > 0$ and $C > 0$ depending only on $d, D$, and $J$.*

### 4.1.2 Hölder Regularity up to the Boundary

In this section, we give the proof of Theorem 4.1.1. We start from collecting some basic properties of renewal function in [17] and [18].

**Lemma 4.1.3.** *For any $c > 0$, There exist constants $C_i(c) > 0$ for $i = 1, 2, 3$ such that*

$$C_1^{-1}\varphi(r) \le V(r)^2 \le C_1\varphi(r), \quad 0 < r \le c, \tag{4.1.5}$$

$$C_2^{-1}\left(\frac{R}{r}\right)^{\alpha_1} \le \frac{V(R)}{V(r)} \le C_2\left(\frac{R}{r}\right)^{\alpha_2}, \quad 0 < r \le R \le c \quad \text{and} \tag{4.1.6}$$

$$C_3^{-1}\left(\frac{T}{t}\right)^{1/\alpha_2} \le \frac{V^{-1}(T)}{V^{-1}(t)} \le C_3\left(\frac{T}{t}\right)^{1/\alpha_1}, \quad 0 < t \le T < V(c). \tag{4.1.7}$$

**Proof.** By [17, Corollary 3] and [18, Proposition 2.4], we have

$$(V(r))^{-2} \asymp \Phi(r^{-1}), \quad r > 0.$$

with comparison constant depending only on $d$. Combining with $\Phi(r^{-1})^{-1} \asymp \varphi(r)$ in $0 < r \le c$, we conclude (4.1.5). By (4.1.5) with $L(2\alpha_1, a_3^{-1}, \varphi)$ and $U(2\alpha_2, a_3, \varphi)$ we have (4.1.6). Using [17, Remark 4], we also obtain the weak scaling property of the inverse function in (4.1.7). □

The most important property of renewal function is the following: $w(x) := V(x_n)$ is a solution of the following Dirichlet problem :

$$\begin{cases} Lw = 0 & \text{in} \quad \mathbb{R}^d_+, \\ w = 0 & \text{in} \quad \mathbb{R}^d \backslash \mathbb{R}^d_+, \end{cases} \tag{4.1.8}$$

where $L$ is of the form (4.1.2) satisfying (**A**). (see [49, Theorem 3.3]).

The following estimates for derivatives of $V$ are in [49, Proposition 3.1] and [66, Theorem 1.2].

**Lemma 4.1.4.** *Assume $X$ is an isotropic pure jump Lévy process satisfying (4.1.3). Then $r \mapsto V(r)$ is twice-differentiable for any $r > 0$. Moreover, for any $c > 0$ there exists a constant $C(c) = C(c, n, a_1, \alpha_1, \alpha_2) > 0$ such that*

$$|V''(r)| \le C \frac{V'(r)}{r \wedge c}, \quad V'(r) \le C \frac{V(r)}{r \wedge c}. \tag{4.1.9}$$

We are going to utilize the space $C^V(D)$ in Section 4.1.2 and adopt $V(\delta_D)$ as a barrier in Section 4.1.3.

Next we introduce the following Dirichlet heat kernel estimates from [29, Corollary 1.6] and [66, Thoerem 1.1 and 1.2]. We reformulate here for the usage of our proofs.

**Theorem 4.1.5.** *Let $X$ be an isotropic unimodal Lévy process satisfying (4.1.3). Let $D \subset \mathbb{R}^d$ be a bounded $C^{1,1}$ open set satisfying* $\mathrm{diam}(D) \le 1$ *and $p_D(t, x, y)$ be the Dirichlet heat kernel for $X$ on $D$. Then $x \mapsto p_D(t, x, y)$*

*is differentiable for any $y \in D, t > 0$, and there exist constants $C_i = C_i(n, D, a_1, a_2, \alpha_1, \alpha_2, \Phi(1)) > 0$, $i = 1, \dots, 4$ satisfying the following estimates:*

*(a) For any $(t, x, y) \in (0, 1] \times D \times D$,*

$$p_D(t,x,y) \le C_1 \left(1 \wedge \frac{V(\delta_D(x))}{t^{1/2}}\right)\left(1 \wedge \frac{V(\delta_D(y))}{t^{1/2}}\right) p\left(t, |x-y|/4\right)$$

*and*

$$|\nabla_x p_D(t,x,y)| \le C_2 \left[\frac{1}{\delta_D(x) \wedge 1} \vee \frac{1}{V^{-1}(\sqrt{t})}\right] p_D(t,x,y).$$

*(b) For any $(t, x, y) \in [1, \infty) \times D \times D$,*

$$p_D(t,x,y) \le C_3 e^{-\lambda_1 t} V(\delta_D(x)) V(\delta_D(y))$$

*and*

$$|\nabla_x p_D(t,x,y)| \le C_4 \left[\frac{1}{\delta_D(x) \wedge 1} \vee \frac{1}{V^{-1}(1)}\right] p_D(t,x,y),$$

*where $-\lambda_1 = -\lambda_1(n, a_1, a_2, \alpha_1, \alpha_2, \Phi(1)) < 0$ is the largest eigenvalue of the generator of $X^{B(0,1)}$.*

In the estimates of Theorem 4.1.5, we used $\delta_D(x) \vee \delta_D(y) \le \mathrm{diam}(D) \le 1$, $V(r) \asymp \varphi(r)^{1/2}$ in $0 < r \le 1$ and $\frac{1}{V^{-1}(\sqrt{t})} \asymp \varphi^{-1}(t)$ to reformulate theorems in our references. In addition, estimates in [29, Corollary 1.6] are of the form

$$p_D(t,x,y) \le c e^{-\lambda(D) t} V(\delta_D(x)) V(\delta_D(y))$$

where $-\lambda(D) < 0$ is the largest eigenvalue of the generator of $X^D$. Using [41, (6.4.14) and Lemma 6.4.5], we have $\lambda(D) = \inf\{\int_{\mathbb{R}^d} -Lu(x)u(x)dx \,|\, \|u\|_2 = 1, \mathrm{supp}(u) \subset D\}$, thus we can obtain $\lambda_1 \le \lambda(D)$. This implies heat kernel estimates in Theorem 4.1.5(b).

Without loss of generality, we will always assume $\mathrm{diam}(D) \le 1$ in this section. We define the *Green function of* $X^D$ by

$$G^D(x,y) = \int_0^\infty p_D(t,x,y)dt$$

for $x, y \in D$ with $x \neq y$. Note that by Theorem 4.1.5(b), $G^D(x,y)$ is finite for any $x \neq y$.

We define a potential operator $R^D$ for $X^D$ as

$$R^D f(x) := \int_0^\infty \int_D p_D(t,x,y)f(y)dydt. \tag{4.1.10}$$

Using definitions of $P_t^D$ and $G^D$, we also have

$$R^D f(x) = \int_{D\setminus\{x\}} G^D(x,y)f(y)dy = \int_0^\infty P_t^D f(x)dt. \tag{4.1.11}$$

In the next subsection, we will see that $R^D$ acts as the inverse of $-A$.

First we will prove interior Hölder estimate of $R^D f$. For the next usage, we prove the following proposition for the functions in $L^\infty(D)$.

**Proposition 4.1.6.** *For any $f \in L^\infty(D)$ and any ball $B(x_0, r) \subset D$ satisfying $\delta_D(x_0) \leq 2r$, we have $R^D f \in C^V(B/2)$ and there is a constant $C = C(n, a_1, a_2, \alpha_1, \alpha_2, D, \Phi(1)) > 0$ satisfying*

$$\|R^D f\|_{C^V(B/2)} \leq C \left(\|f\|_{L^\infty(D)} + \|R^D f\|_{C(B)}\right) \tag{4.1.12}$$

*Here we have denoted $B = B(x_0, r)$ and $B/2 = B(x_0, r/2)$.*

We next capture a behavior of the function $R^D f$ near the boundary by using various estimates in [18, 29, 49]. Especially, the second assertion in the following lemma will provide the optimality of Theorem 4.1.2.

**Lemma 4.1.7.** *There exists a constant $C > 0$ such that*

$$|R^D f(x)| \leq C\|f\|_{L^\infty(D)} V(\delta_D(x))$$

*for any $f \in L^\infty(D)$ and $x \in D$. Moreover, if we further assume that $f > 0$ in $D$, then for any $\theta > 0$ there exists a constant $c > 0$ such that*

$$R^D f(x) \geq cV(\delta_D(x)) \int_{D_\theta} f dm$$

*for every $x \in D$, where $D_\theta := \{y \in D : \delta_D(y) > \theta\}$.*

**Remark 4.1.8.** *As a corollary of Lemma 4.1.7, we have*

$$\|R^D f\|_{L^\infty(D)} \leq C\|f\|_{L^\infty(D)}.$$

*Hence we can simplify* (4.1.12) *to*

$$\|R^D f\|_{C^V(B/2)} \leq \tilde{C}\|f\|_{L^\infty(D)} \tag{4.1.13}$$

*for some constant $\tilde{C} = \tilde{C}(n, a_1, a_2, \alpha_1, \alpha_2, D, \Phi(1)) > 0$.*

Now we are ready to prove Theorem 4.1.1 for the function $R^D f$.

**Proposition 4.1.9.** *Assume $f \in L^\infty(D)$. Then, $R^D f \in C^V(D)$ and there exists a constant $C > 0$ such that*

$$\|R^D f\|_{C^V(D)} \leq C\|f\|_{L^\infty(D)}.$$

*The constant $C > 0$ depends only on $n, a_1, a_2, \alpha_1, \alpha_2, D$ and* $\Phi(1)$.

**Proof.** By (4.1.13) we have

$$|R^D f(x) - R^D f(y)| \leq c_1\|f\|_{L^\infty(D)} V(|x - y|) \tag{4.1.14}$$

for all $x, y$ satisfying $|x-y| < \delta_D(x)/2$. We want to show that (4.1.14) holds, perhaps with a bigger constant, for all $x, y \in D$.

Let $(R_0, \Lambda)$ be the $C^{1,1}$ characteristics of $D$. Then $D$ can be covered by finitely many balls of the form $B(z_i, \delta_D(z_i)/2)$ with $z_i \in D$ and finitely many sets of the form $B(z_j^*, R_0) \cap D$ with $z_j^* \in \partial D$. Thus, it is enough to show that (4.1.14) holds for all $x, y \in B(z_j^*, R_0) \cap D$ possibly with a larger constant.

Fix $B(z_0^*, R_0) \cap D$ and assume that the outward normal vector at $z_0$ is $(0, \cdots, 0, -1)$. This is possible because the operator is invariant under the rotation. Now let $x = (x', x_n)$ and $y = (y', y_n)$ be two points in $B(z_0^*, R_0) \cap D$, and let $r = |x - y|$. Let us define for $k \geq 0$

$$x^k = (x', x_n + \lambda^k r) \quad \text{and} \quad y^k = (y', y_n + \lambda^k r),$$

for some $1 - 2^{-1}(1 + \Lambda^2)^{-1/2} \leq \lambda < 1$. Since $(1 + \Lambda^2)^{-1/2}(x^k)_n \leq \delta_D(x^k)$, we have

$$|x^k - x^{k+1}| = \lambda^k(1 - \lambda)r \leq \frac{1}{2\sqrt{1 + \Lambda^2}}(x^k)_n \leq \frac{1}{2}\delta_D(x^k).$$

Thus, we have from (4.1.14) that

$$\begin{aligned} |R^D f(x^k) - R^D f(x^{k+1})| &\leq c_1 \|f\|_{L^\infty(D)} V(|x^k - x^{k+1}|) \\ &= c_1 \|f\|_{L^\infty(D)} V(\lambda^k(1 - \lambda)r) \end{aligned}$$

and similarly that $|R^D f(y^k) - R^D f(y^{k+1})| \leq c_1 \|f\|_{L^\infty(D)} V(\lambda^k(1-\lambda)r)$. Moreover, note that the distance from the line segment joining $x^0$ and $y^0$ to the boundary $\partial D$ is more than $r(1-\Lambda/2)$. Thus, this line can be split into finitely many line segments of length less than $r(1 - \Lambda/2)/2$. The number of small line segments depends only on $\Lambda$. Therefore, we have $|R^D f(x^0) - R^D f(y^0)| \leq c_2 \|f\|_{L^\infty(D)} V(r)$ and hence

$$\begin{aligned} &|R^D f(x) - R^D f(y)| \leq |R^D f(x^0) - R^D f(y^0)| \\ &\qquad + \sum_{k \geq 0} \big(|R^D f(x^k) - R^D f(x^{k+1})| + |R^D f(y^k) - R^D f(y^{k+1})|\big) \\ &\leq c_3 \|f\|_{L^\infty(D)} \big(V(r) + \sum_{k \geq 0} V(\lambda^k(1 - \lambda)r)\big) \\ &\leq c_4 \|f\|_{L^\infty(D)} V(r) \left(1 + c_5 \sum_{k \geq 0} \big(\lambda^k(1 - \lambda)\big)^{\alpha_1}\right) \leq c_6 \|f\|_{L^\infty(D)} V(r). \end{aligned}$$

Recall that $r = |x - y|$. This finishes the proof. $\square$

From now on, we will prove that the function $u = -R^D f$ is the unique viscosity solution for (4.1.1) when $f \in C(D)$ by establishing the relation between viscosity solutions of (4.1.1) and solutions of the following:

$$\begin{cases} Au = f & \text{in } D, \\ u = 0 & \text{in } \mathbb{R}^d \backslash D. \end{cases} \tag{4.1.15}$$

In [4], the authors discussed the relation between operators $A$ and $L$, for instance, domain or values of the operators; see [4] for the application to heat equations.

At the beginning of this section we apply the strategies in [4] to our settings and obtain some related properties. After then, we obtain comparison principle for the viscosity solution. Combining these results, we finally obtain the existence and uniqueness for Dirichlet problems (4.1.1) and (4.1.15). Moreover, these two solutions coincide under some conditions. Also, in Section 4.1.4 we obtain Harnack inequality, which is one of the key ingredients for the standard argument of Krylov in [65]. In Section 4.1.5 we will make use of Harnack inequality and the comparison principle to prove Theorem 4.1.2. Let

$$\mathcal{D} = \mathcal{D}(D) := \{u \in C_0(D) : Au \in C(D)\}$$

be the domain of operator $A$. Recall that by [4, Lemma 2.6] we have

$$Au(x) = Lu(x) \tag{4.1.16}$$

for any $u \in C^2(x) \cap C_0(\mathbb{R}^d)$, $x \in D$. We first show that $u = -R^D f$ satisfies (4.1.15) when $f$ is continuous.

**Lemma 4.1.10.** *Let $f \in C(D)$ and define $u = -R^D f$. Then, $u$ is a solution for* (4.1.15).

**Proof.** First we claim that for any $u \in C_0(D)$ and $x \in D$,

$$Au(x) = \lim_{t \downarrow 0} \frac{P_t^D u(x) - u(x)}{t}. \tag{4.1.17}$$

To show (4.1.17), we follow the proof in [4, Theorem 2.3]. Note that our domain of operator is slightly different from it in [4, (2.8)].

We first observe that for any $u \in \mathcal{D}$ and $x \in D$,

$$\begin{aligned}
P_t^D u(x) - P_t u(x) &= \mathbb{E}^x u(X_t^D) - \mathbb{E}^x u(X_t) \\
&= \mathbb{E}^x[u(X_t^D)\mathbf{1}_{\{\tau_D \geq t\}}] - \mathbb{E}^x[u(X_t)\mathbf{1}_{\{\tau_D \geq t\}}] - \mathbb{E}^x[u(X_t)\mathbf{1}_{\{\tau_D < t\}}] \\
&= -\mathbb{E}^x[u(X_t)\mathbf{1}_{\{\tau_D < t\}}].
\end{aligned}$$

Indeed, the first and the third term in the second line cancel. Hence

$$\frac{P_t^D u(x) - P_t u(x)}{t} = -\frac{\mathbb{E}^x[u(X_t)\mathbf{1}_{\{\tau_D < t\}}]}{t} = \frac{\mathbb{E}^x[\big(u(X_{\tau_D}) - u(X_t)\big)\mathbf{1}_{\{\tau_D < t\}}]}{t}.$$

Meanwhile, by the strong Markov property we obtain

$$\left|\mathbb{E}^x\left[\big(u(X_{\tau_D}) - u(X_t)\big)\mathbf{1}_{\{\tau_D < t\}}\right]\right| \leq \mathbb{E}^x\left[\left|\mathbb{E}^{X_{\tau_D}}[u(X_0) - u(X_{t-\tau_D})]\right| \mathbf{1}_{\{\tau_D < t\}}\right].$$

Since $u \in C_0(D)$ is uniformly continuous, with stochastic continuity of Lévy process we have that for any $\varepsilon > 0$ there is $\theta = \theta(\varepsilon) > 0$ such that

$$|\mathbb{E}^z[u(X_s)] - u(z)| < \varepsilon$$

for any $z \in D$ and $0 < s \leq \theta$. Combining above two equations we conclude

$$\left|\mathbb{E}^x[\big(u(X_{\tau_D}) - u(X_t)\big)\mathbf{1}_{\{\tau_D < t\}}]\right| \leq \varepsilon \mathbb{P}^x(\tau_D < t)$$

for $0 < t \leq \theta$. Since $D$ is open, for any $x \in D$ we have a constant $r_x > 0$ such that $B(x, r_x) \subset D$. Using [20, Theroem 5.1 and Proposition 2.27(d)] there exists some $M > 0$ such that

$$\frac{\mathbb{P}^x(\tau_D < t)}{t} \leq \frac{\mathbb{P}^x(\tau_{B(x,r_x)} < t)}{t} \leq M \quad \text{for all} \quad t > 0.$$

Combining above inequalities we obtain that

$$\begin{aligned}\lim_{t\downarrow 0}\left|\frac{P_t^D u(x)-u(x)}{t}-Au(x)\right| &= \lim_{t\downarrow 0}\left|\frac{P_t^D u(x)-u(x)}{t}-\frac{P_t u(x)-u(x)}{t}\right| \\ &\le \varepsilon \lim_{t\downarrow 0}\frac{\mathbb{P}^x(\tau_D<t)}{t}\le \varepsilon M.\end{aligned}$$

Since $\varepsilon>0$ is arbitrarily, this concludes the claim.

Now we prove the lemma. Note that $u=0$ in $D^c$ immediately follows from the definition of $R^D$. Then, by (4.1.17) and (4.1.11) we have that for $x\in D$,

$$\begin{aligned}Au(x) = A(-R^D f)(x) &= -\lim_{t\downarrow 0}\frac{P_t^D(R^D f)(x)-R^D f(x)}{s} \\ &= -\lim_{t\downarrow 0}\frac{1}{t}\left[P_t^D\Big(\int_0^\infty P_s^D f(\cdot)ds\Big)(x)-\int_0^\infty P_s^D f(x)ds\right] \\ &= \lim_{t\downarrow 0}\frac{1}{t}\left(-\int_0^\infty P_{t+s}^D f(x)ds+\int_0^\infty P_s^D f(x)ds\right) \qquad (4.1.18) \\ &= \lim_{t\downarrow 0}\frac{1}{t}\left(-\int_t^\infty P_s^D f(x)ds+\int_0^\infty P_s^D f(x)ds\right) \\ &= \lim_{t\downarrow 0}\frac{\int_0^t P_s^D f(x)ds}{t}=f(x).\end{aligned}$$

Indeed, the third line follows from the semigroup property $P_s^D P_t^D = P_{s+t}^D$ and that $R^D f\in C_0(D)$ which follows from Proposition 4.1.6. This finishes the proof. $\square$

The next lemma shows that every solution of (4.1.15) is a viscosity solution of (4.1.1).

**Lemma 4.1.11.** *Assume that $f\in C(D)$ and $u\in\mathcal{D}$ satisfies $Au=f$ in $D$. Then, $u$ is a viscosity solution of $Lu=f$.*

**Proof.** For any $x_0\in D$ and test function $v\in C^2(\mathbb{R}^d)$ with $v(x_0)=u(x_0)$ and $v(y)>u(y)$ for $y\in\mathbb{R}^d\setminus\{x_0\}$, we have

$$Av(x_0)=Lv(x_0).$$

Since $v(x_0) = u(x_0)$ and $P_t^D v(x_0) \geq P_t^D u(x_0)$ for every $t > 0$, we have

$$Av(x_0) = \lim_{t \downarrow 0} \frac{P_t^D v(x_0) - v(x_0)}{t} \geq \lim_{t \downarrow 0} \frac{P_t^D u(x_0) - u(x_0)}{t} = Au(x_0).$$

Thus, we arrive

$$Lv(x_0) \geq Au(x_0),$$

which concludes that $u$ is a viscosity solution of (4.1.1). □

Now we see comparison principle in [21]. This implies the uniqueness of viscosity solution for (4.1.1).

**Theorem 4.1.12** (Comparison principle)**.** *Let $D$ be a bounded open set in $\mathbb{R}^d$. Let $u$ and $v$ be bounded functions satisfying $Lu \geq f$ and $Lv \leq f$ in $D$ in viscosity sense for some continuous function $f$, and let $u \leq v$ in $\mathbb{R}^d \setminus D$. Then $u \leq v$ in $D$.*

**Proof.** We first claim that $L$ satisfies [21, Assumption 5.1]. More precisely, there exists constant $r_0 \geq 1$ such that for every $r \geq r_0$, there exists a constant $\theta = \theta(r) > 0$ satisfying $Lw > \theta$ in $B_r$, where $w(x) = 1 \wedge \frac{|x|^2}{r^3}$.

Let $r_0 = 4$, $r \geq 4$ and $x \in B_r$. Note that by $r \geq 4$ we have

$$\frac{|y|^2}{r^3} \leq \frac{4r^2}{r^3} \leq 1, \qquad y \in B_{2r}.$$

Thus, for $y \in B_r$ we obtain

$$w(x+y) + w(x-y) - 2w(x) = \frac{|x+y|^2 + |x-y|^2 - 2|x|^2}{r^3} = \frac{2|y|^2}{r^3}.$$

On the other hand, for $y \in B_r^c$ we have

$$w(x+y) + w(x-y) - 2w(x) \geq \frac{2|y|^2}{r^3} \wedge (1 - 2w(x)) > 0.$$

Therefore, since $w \in C^2(\mathbb{R}^d)$ we have

$$
\begin{aligned}
Lw(x) &:= \frac{1}{2}\int_{\mathbb{R}^d} (w(x+y)+w(x-y)-2w(x))\, J(y)\, dy \\
&\geq \frac{1}{2}\int_{B_r} (w(x+y)+w(x-y)-2w(x))\, J(y)\, dy \\
&\geq \frac{1}{r^3}\int_{B_r} |y|^2 J(y) dy \ =: \theta(r) > 0
\end{aligned}
$$

for every $r \geq r_0 = 4$ and $x \in B_r$. Since $L$ satisfies [21, Assumption 5.1], we can apply Theorem 5.2 therein, which proves the theorem. □

The following uniqueness of viscosity solution is immediate.

**Corollary 4.1.13.** *Let $D$ be a bounded open set in $\mathbb{R}^d$ and let $f \in C(D)$. Then there is at most one viscosity solution of* (4.1.1).

Here is the main result in this section.

**Theorem 4.1.14.** *Assume that $f \in C(D)$. Then, $u = -R^D f \in \mathcal{D}$ is the unique solution of* (4.1.15). *Also, $u$ is the unique viscosity solution of* (4.1.1).

**Proof.** By Lemma 4.1.10, we have that $u = -R^D f \in \mathcal{D}$ is solution of (4.1.15). Now, Lemma 4.1.11 and Corollary 4.1.13 conclude the proof. □

**Proof of Theorem 4.1.1** By Theorem 4.1.14, the unique viscosity solution for (4.1.1) is given by $u = -R^D f$. Therefore, Proposition 4.1.9 yields the Hölder regularity of viscosity solution with respect to $C^V$-norm. By (4.1.5), we have $V \asymp \overline{\phi}$ and this concludes the proof. □

**Remark 4.1.15.** *Every viscosity supersolution $u$ to the problem*

$$
\begin{cases}
Lu \leq -1 & \text{in } D, \\
u = 0 & \text{in } \mathbb{R}^d \setminus D,
\end{cases}
$$

*satisfies $u \geq cV(\delta_D)$ for some constant $c > 0$. Indeed, letting $v = R^D 1$, we have $Lv \geq -1$ by Theorem 4.1.14. Thus, by Theorem 4.1.12 we obtain*

*$u \le v = R^D 1$. Now the conclusion follows from Lemma 4.1.7. This provides the optimality of Theorem 4.1.2.*

### 4.1.3 Boundary Regularity

Throughout this subsection, $D \subset \mathbb{R}^d$ is a bounded $C^{1,1}$ open set. Without loss of generality, we assume that $\mathrm{diam}(D) \le 1$. Since $\delta_D$ is only $C^{1,1}$ near $\partial D$, we need to use the following "regularized version" of $\delta_D$, defined in [76, Definition 2.1].

**Definition 4.1.16.** *We call $\psi : D \to (0, \infty)$ the regularized version of $\delta_D$ if $\psi \in C^{1,1}(D)$ and it satisfies*

$$\tilde{C}^{-1}\delta_D(x) \le \psi(x) \le \tilde{C}\delta_D(x),\ \|\nabla\psi(x)\| \le \tilde{C} \text{ and } \|\nabla\psi(x) - \nabla\psi(y)\| \le \tilde{C}|x-y| \tag{4.1.19}$$

*for any $x, y \in D$, where the constant $\tilde{C} > 0$ depends only on $D$.*

For $D = B(0,1)$, there exists a regularized version of $\delta_{B(0,1)}$ which is $C^2$ and isotropic. Denote this function by $\Psi$ and let $C = C(n)$ be the constant in (4.1.19) for the function $\Psi$. For any open ball $B_r := B(x_0, r)$, we will take the regularized version of $\delta_{B_r}$ which is defined by $\Psi_r(x) := \Psi(\frac{x-x_0}{r})$. Then, $\Psi_r$ satisfies

$$C^{-1}\delta_{B_r}(x) \le \Psi_r(x) \le C\delta_{B_r}(x), \quad \|\nabla\Psi_r\| \le C \quad \text{and} \quad \|\nabla^2\Psi_r(x)\| \le \frac{C}{r} \tag{4.1.20}$$

for any $x, y \in B(x_0, r)$. The last estimate follows from the fact that $\Psi \in C^2(B_r)$.

We first introduce the following three lemmas which will be used to construct a barrier for $L$.

**Lemma 4.1.17.** *Assume that $D$ is a bounded $C^{1,1}$ open set and let $\psi$ be a regularized version of $\delta_D$. Then, for every $x \in \mathbb{R}^d$ and $x_0 \in D$ we have*

$$|\psi(x) - (\psi(x_0) + \nabla\psi(x_0)\cdot(x - x_0))_+| \le \tilde{C}|x - x_0|^2 \tag{4.1.21}$$

*where* $\tilde{C}$ *is the constant in* (4.1.19)*. In addition, when* $D = B(0,r)$ *and* $\psi = \Psi_r$ *we have* (4.1.21) *with* $\tilde{C} = \frac{C}{r}$ *where* $C$ *is the constant in* (4.1.20)*.*

**Proof.** The proof for (4.1.21) is exactly the same as [76, Lemma 2.4], but we provide the proof to see the dependence of the constant $\tilde{C}$ on $r$ for the case $D = B(0,r)$.

Let $\tilde{\psi}$ be a $C^{1,1}$ extension of $\psi|_D$ satisfying $\tilde{\psi} \le 0$ in $\mathbb{R}^d \backslash D$. Then, since $\tilde{\psi} \in C^{1,1}(\mathbb{R}^d)$ we have

$$|\tilde{\psi}(x)-\psi(x_0)-\nabla\psi(x_0)\cdot(x-x_0)| = |\tilde{\psi}(x)-\tilde{\psi}(x_0)-\nabla\tilde{\psi}\cdot(x-x_0)| \le \tilde{C}|x-x_0|^2$$

for all $x \in \mathbb{R}^d$. Using $|a_+ - b_+| \le |a-b|$ and $(\tilde{\psi})_+ = \psi$, we have

$$\frac{|\psi(x) - (\psi(x_0) + \nabla\psi(x_0)\cdot(x-x_0))_+|}{|x-x_0|^2} \le \frac{|\tilde{\psi}(x) - \psi(x_0) - \nabla\psi(x_0)\cdot(x-x_0)|}{|x-x_0|^2}$$

for all $x \in \mathbb{R}^d$. If $D = B(0,r)$ and $\psi = \Psi_r$, the constant $\tilde{C}$ become $\frac{C}{r}$. Thus, the conclusion of lemma follows. □

Next lemma is a collection of inequalities which will be used for this section. Note that we can easily check these inequalities when $\varphi(r) = r^{2\alpha}$ and $V(r) = r^\alpha$ with $0 < \alpha < 1$. The inequalities (4.1.23) and (4.1.25) are in [18, Lemma 3.5]. We skip the proof.

**Lemma 4.1.18.** ([58, Lemma 4.3]) *There exists a constant* $C_1 > 0$ *such that for any* $0 < r \le 1$,

$$\int_0^r \frac{s}{\varphi(s)} ds \le \frac{C_1 r^2}{\varphi(r)}, \tag{4.1.22}$$

$$\int_r^\infty \frac{1}{s\varphi(s)} ds \le \frac{C_1}{\varphi(r)}, \tag{4.1.23}$$

$$\int_0^r \frac{1}{V(s)} ds \le \frac{C_1 r}{V(r)}, \quad \int_0^r \frac{V(s)}{s} ds \le C_1 V(r) \tag{4.1.24}$$

*and*

$$\int_r^\infty \frac{V(s)}{s\varphi(s)} ds \le \frac{C_1}{V(r)}. \tag{4.1.25}$$

Also,

The following lemma is the counterpart of [76, Lemma 2.5]

**Lemma 4.1.19.** *Let $U \subset \mathbb{R}^d$ be a $C^{1,1}$ open set, which can be unbounded. Then there exists a constant $C_2 = C_2(n, U, a_1, a_2, \alpha_1, \alpha_2) > 0$ such that for any $x \in U$ and $0 < r \leq 1$,*

$$\int_{U\cap\left(B(x,r)\setminus B(x,\delta_U(x)/2)\right)} \frac{V(\delta_U(y))}{\delta_U(y)} \frac{dy}{|x-y|^{n-2}\varphi(|x-y|)} \leq \frac{C_2 r}{V(r)}. \tag{4.1.26}$$

**Proof.** Fix $x \in U$ and denote $\rho := \delta_U(x) < 2r$, $B_r := B(x,r)$ for $r > 0$ and $B_r = \emptyset$ for $r \leq 0$. First note that there is a constant $\kappa = \kappa(U) > 0$ such that the level set $\{\delta_U \geq t\} = \{x \in U | \delta_U(x) \geq t\}$ is $C^{1,1}$ for any $t \in (0, \kappa]$ since $U$ is $C^{1,1}$. Without loss of generality we can assume $\kappa \leq r$ because $\kappa$ can be arbitrarily small.

Since $B_R \cap \{\delta_U \geq \kappa\} = \emptyset$ for every $R \leq \kappa - \rho$, we have

$$\begin{aligned}
&\int_{(B_r\setminus B_{\rho/2})\cap\{\delta_U\geq\kappa\}} \frac{V(\delta_U(y))}{\delta_U(y)} \frac{dy}{|x-y|^{n-2}\varphi(|x-y|)} \\
&= \int_{(B_r\setminus B_{\max\{\rho/2,\kappa-\rho\}})\cap\{\delta_U\geq\kappa\}} \frac{V(\delta_U(y))}{\delta_U(y)} \frac{dy}{|x-y|^{n-2}\varphi(|x-y|)} \\
&\leq \int_{(B_r\setminus B_{2\kappa/3})\cap\{\delta_U\geq\kappa\}} \frac{V(\delta_U(y))}{\delta_U(y)} \frac{dy}{|x-y|^{n-2}\varphi(|x-y|)},
\end{aligned}$$

where the last line follows from $\rho/2 \vee (\kappa - \rho) \geq \frac{2\kappa}{3}$. Using

$$\kappa \leq \delta_U(y) \leq r + \kappa \leq 2r \quad \text{and} \quad \frac{2\kappa}{3} \leq |x-y| \leq r$$

for every $y \in (B_r \backslash B_{2\kappa/3}) \cap \{\delta_U \geq \kappa\}$, we arrive that for any $x \in U$,

$$\begin{aligned}
&\int_{(B_r \backslash B_{2\kappa/3}) \cap \{\delta_U \geq \kappa\}} \frac{V(\delta_D(y))}{\delta_D(y)} \frac{dy}{|x-y|^{n-2}\varphi(|x-y|)} \\
&\leq \int_{(B_r \backslash B_{2\kappa/3}) \cap \{\delta_U \geq \kappa\}} \frac{V(2r)}{\kappa} \frac{dy}{|x-y|^{n-2}\varphi(|x-y|)} \qquad (4.1.27)\\
&\leq c_1 \frac{V(r)}{\kappa} \int_0^r \frac{s}{\varphi(s)} ds \leq c_2(\kappa) \frac{r^2}{V(r)} \leq c_2(\kappa) \frac{r}{V(r)},
\end{aligned}$$

where we used (4.1.5) and (4.1.22) for the second last inequality. Thus, it suffices to estimate the integrand (4.1.26) in the set $(B_r \backslash B_{\rho/2}) \cap \{0 < \delta_U < \kappa\}$.

We will utilize the following estimates on Hausdorff measure in [RV15], that is, there exists a constant $c_3(U) > 0$ such that that for every $x \in U$ and $t \in (0, \kappa)$,

$$\mathcal{H}^{n-1}(\{\delta_U = t\} \cap (B_{2^{-k+1}r} \backslash B_{2^{-k}r})) \leq c_3 (2^{-k} r)^{n-1} \qquad (4.1.28)$$

which follows from the fact that the level set $\{\delta_U = t\}$ is $C^{1,1}$ for $t \in (0, \kappa)$.

Let us denote $C_n := B_{r2^{-n}}$ for $n \geq 0$ and let $M \in \mathbb{N}$ be the natural number satisfying $2^{-M} r \leq \rho/2 \leq 2^{-M+1} r$. Using $|x-y| \geq 2^{-k} r$ for every $y \in C_{k-1} \backslash C_k$ and $\varphi$ is increasing for the third line, we have

$$\begin{aligned}
&\int_{(B_r \backslash B_{\rho/2}) \cap \{0 < \delta_U < \kappa\}} \frac{V(\delta_U(y))}{\delta_U(y)} \frac{dy}{|x-y|^{n-2}\varphi(|x-y|)} \\
\leq\ & \sum_{k=1}^{M} \int_{(C_{k-1} \backslash C_k) \cap \{0 < \delta_U < \kappa\}} \frac{V(\delta_U(y))}{\delta_U(y)} \frac{dy}{|x-y|^{n-2}\varphi(|x-y|)} \\
\leq\ & \sum_{k=1}^{M} \frac{1}{(2^{-k}r)^{n-2}\varphi(2^{-k}r)} \int_{(C_{k-1} \backslash C_k) \cap \{0 < \delta_U < \kappa\}} \frac{V(\delta_U(y))}{\delta_U(y)} dy \\
=\ & \sum_{k=1}^{M} \frac{1}{(2^{-k}r)^{n-2}\varphi(2^{-k}r)} \int_{(C_{k-1} \backslash C_k) \cap \{0 < \delta_U < \kappa\}} \frac{V(\delta_U(y))}{\delta_U(y)} |\nabla \delta_U(y)| dy.
\end{aligned}$$

Here we used $|\nabla \delta_U(y)| = 1$ for $y \in \{0 < \delta_U < \kappa\}$ for the last line. (See [77].) For any $1 \leq k \leq M$ and $y \in C_{k-1}$ we have $\delta_U(y) \leq 2^{-k+1} r + \rho \leq (2^{-k+1} +$

$2^{-M+2})r \le 6\cdot 2^{-k}r$, which implies $C_{k-1}\subset\{\delta_U < 6\cdot 2^{-k}r\}$. Thus, combining this with above inequality we have

$$\begin{aligned}
&\int_{(B_r\setminus B_{\rho/2})\cap\{0<\delta_U<\kappa\}} \frac{V(\delta_U(y))}{\delta_U(y)}\frac{dy}{|x-y|^{n-2}\varphi(|x-y|)} \\
\le\ & \sum_{k=1}^{M}\frac{1}{(2^{-k}r)^{n-2}\varphi(2^{-k}r)}\int_{(C_{k-1}\setminus C_k)\cap\{0<\delta_U<6\cdot 2^{-k}r\}}\frac{V(\delta_U(y))}{\delta_U(y)}|\nabla\delta_U(y)|dy.
\end{aligned} \tag{4.1.29}$$

Plugging $u(y)=\delta_U(y)$ and $g(y)=\frac{V(\delta_U(y))}{\delta_U(y)}$ into the following coarea formula

$$\int_D g(y)|\nabla u(y)|dy = \int_{-\infty}^{\infty}\left(\int_{u^{-1}(t)} g(y)d\mathcal{H}_{n-1}(y)\right)dt,$$

we obtain

$$\begin{aligned}
&\sum_{k=1}^{M}\frac{1}{(2^{-k}r)^{n-2}\varphi(2^{-k}r)}\int_{(C_{k-1}\setminus C_k)\cap\{0<\delta_U<6\cdot 2^{-k}r\}}\frac{V(\delta_U(y))}{\delta_U(y)}|\nabla\delta_U(y)|dy \\
=\ & \sum_{k=1}^{M}\frac{1}{(2^{-k}r)^{n-2}\varphi(2^{-k}r)}\int_0^{6\cdot 2^{-k}r}\int_{(C_{k-1}\setminus C_k)\cap\{d=t\}}\frac{V(t)}{t}d\mathcal{H}^{n-1}(y)dt \\
\le\ & \sum_{k=1}^{M}\frac{1}{(2^{-k}r)^{n-2}\varphi(2^{-k}r)}\int_0^{6\cdot 2^{-k}r} c_3(2^{-k}r)^{n-1}\frac{V(t)}{t}dt \\
=\ & c_3\sum_{k=1}^{M}\frac{2^{-k}r}{\varphi(2^{-k}r)}\int_0^{6\cdot 2^{-k}r}\frac{V(t)}{t}dt \le c_4\sum_{k=1}^{M}\frac{2^{-k}r}{\varphi(2^{-k}r)}V(6\cdot 2^{-k}r),
\end{aligned} \tag{4.1.30}$$

where we used (4.1.28) for the third line and (4.1.24) for the last line. Also, by (4.1.6) and (4.1.5),

$$\begin{aligned}
\sum_{k=1}^{M}\frac{2^{-k}r}{\varphi(2^{-k}r)}V(6\cdot 2^{-k}r) &\le \sum_{k=1}^{M}\frac{2^{-k}r}{V(2^{-k}r)} = \sum_{k=1}^{M}\int_{2^{-k}r}^{2^{-k+1}r}\frac{1}{V(2^{-k}r)}ds \\
&\le \int_0^r\frac{1}{V(s)}ds \le c_5\frac{r}{V(r)},
\end{aligned} \tag{4.1.31}$$

where in the last two inequalities we have used that $V$ is increasing and

(4.1.24). Using (4.1.29), (4.1.30), and (4.1.31), we conclude

$$\int_{(B_r\setminus B_{\rho/2})\cap\{d<\kappa\}} \frac{V(\delta_U(y))}{\delta_U(y)} \frac{dy}{|x-y|^{n-2}\varphi(|x-y|)} \le \frac{c_4c_5r}{V(r)}.$$

This and (4.1.27) finish the proof. $\square$

With Lemmas above, we are ready to show that $V(\psi)$ acts as a barrier of $L$ on $D$.

**Proposition 4.1.20.** *Let $L$ be given by* (4.1.2) *and $\psi$ be a regularlized version of $\delta_D$. Then there exists a constant $\tilde{C}_3 = \tilde{C}_3(n, a_1, a_2, \alpha_1, \alpha_2, D) > 0$ such that*

$$|L(V(\psi))| \le \tilde{C}_3 \quad \text{in } D. \tag{4.1.32}$$

*where $V$ is the renewal function with respect to $\Phi$. In addition, if $D = B(0, r)$ is a ball with radius $r$, there exists a constant $C_3 = C_3(n, a_1, a_2, \alpha_1, \alpha_2) > 0$ such that*

$$|L(V(\psi))| \le \frac{C_3}{V(r)} \quad \text{in } B(0, r), \tag{4.1.33}$$

*where $\psi = \Psi_r$ is a regularized version of $\delta_{B(0,r)}$ defined in* (4.1.20). *Note that $C_3$ is independent of $r$.*

**Proof.** This proof is mainly motivated by [76, Proposition 2.3]. Here we only prove (4.1.33), and the proof of (4.1.32) is similar.

Let $x_0 \in B_r := B(0, r)$ and $\rho := \delta_{B_r}(x_0)$. First we prove (4.1.33) for the case $\rho \geq \kappa r > 0$ with $\kappa = 1/(8C^2)$. In this case, we have

$$
\begin{aligned}
&|L(V(\psi))(x_0)| \\
&= \left| \int_{R^d} \left( \frac{V(\psi(x_0+y)) + V(\psi(x_0-y))}{2} - V(\psi(x_0)) \right) \frac{J(1)}{|y|^d \varphi(|y|)} dy \right| \\
&\leq \int_{B_{\kappa r/2}} \left\| \nabla^2 [V(\psi(x_*))] \right\| \frac{J(1)}{|y|^{d-2}\varphi(|y|)} dy \qquad (4.1.34) \\
&+ \int_{B^c_{\kappa r/2}} \left| \frac{V(\psi(x_0+y)) + V(\psi(x_0-y))}{2} - V(\psi(x_0)) \right| \frac{J(1)}{|y|^d \varphi(|y|)} dy, := I_1 + I_2
\end{aligned}
$$

where $x^*$ is a point on the segment between $x_0 - y$ and $x_0 + y$, so that $\delta_{B_r}(x_*) \geq \kappa r/2$ when $y \in B_{\kappa r/2}$. Using (4.1.6), (4.1.20), and Lemma 4.1.4, we have

$$
\|\nabla^2 [V(\psi(x_*))]\| \leq |V''(\psi(x))| \|\nabla \psi(x)\|^2 + |V'(\psi(x))| \|\nabla^2 \psi(x)\| \leq \frac{c_1(\kappa) V(r)}{r^2},
$$

which yields to estimate the first term of (4.1.34) by

$$
I_1 \leq c_1 \frac{V(r)}{r^2} \int_{B_{\kappa r/2}} \frac{1}{|y|^{n-2}\varphi(|y|)} dy = c_2 \frac{V(r)}{r^2} \int_0^{\kappa r/2} \frac{s}{\varphi(s)} ds \leq \frac{c_3}{V(r)}.
$$

In the last inequality above, we have used (4.1.22) and (4.1.5). For the second term, using $\psi(x) \leq C\delta_{B_r}(x) \leq Cr$ for any $x \in B_r$, we have

$$
\left| \frac{V(\psi(x_0+y)) + V(\psi(x_0-y))}{2} - V(\psi(x_0)) \right| \leq 2V(Cr) \leq c_4 V(r).
$$

Therefore,

$$
I_2 \leq c_5 V(r) \int_{\kappa r/2}^{\infty} \frac{1}{s\varphi(s)} ds \leq \frac{c_6(\kappa)}{V(r)}.
$$

In the last inequality we have used (4.1.23) and (4.1.5). Therefore, (4.1.33) for the case $\rho \geq \kappa r$ holds with $C_3 = c_3 + c_6$.

Now it suffices to consider the case $\rho < \kappa r$. Denote

$$l(x) := (\psi(x_0) + \nabla\psi(x_0) \cdot (x - x_0))_+,$$

which satisfies

$$L(V(l)) = 0 \quad \text{on} \quad \{l > 0\}$$

by (4.1.8). Note that $\psi(x_0) = l(x_0)$ and $\nabla\psi(x_0) = \nabla l(x_0)$. Moreover, by (4.1.21) we have

$$|\psi(x) - l(x)| \le \frac{C}{r}|x - x_0|^2. \tag{4.1.35}$$

For any $0 < a \le b \le C$, there exists $a_* \in [a, b]$ satisfying $|V(a) - V(b)| = |a - b|V'(a_*)$. Using Lemma 4.1.4 in the first inequality we have

$$|V(a) - V(b)| = |a - b|V'(a_*) \le c_7|a - b|\frac{V(a_*)}{a_*} \le c_8|a - b|\frac{V(a)}{a}.$$

Here we used (4.1.6) with $c = C$ for the second inequality. Therefore, for any $a, b \in (0, C]$ we have

$$|V(a) - V(b)| \le c_8|a - b|\left(\frac{V(a)}{a} + \frac{V(b)}{b}\right).$$

Also, one can easily see the following inequality

$$|V(a) - V(b)| \le c_8|a - b|\left(\frac{V(a)}{a}\mathbf{1}_{\{a>0\}} + \frac{V(b)}{b} \cdot \mathbf{1}_{\{b>0\}}\right) \tag{4.1.36}$$

for any $0 \le a, b \le C$ by using Lemma 4.1.4.

By (4.1.35) and (4.1.36) we have that for any $x \in B_r(x_0)$,

$$\begin{aligned} |V(\psi(x)) - V(\ell(x))| &\le \frac{c_8}{r}|x - x_0|^2\left(\frac{V(\psi(x))}{\psi(x)}\mathbf{1}_{\{\psi(x)>0\}} + \frac{V(\ell(x))}{\ell(x)}\mathbf{1}_{\{\ell(x)>0\}}\right) \\ &\le \frac{c_9}{r}|x - x_0|^2\left(\frac{V(\delta_{B_r}(x))}{\delta_{B_r}(x)}\mathbf{1}_{\{\delta_{B_r}(x)>0\}} + \frac{V(\ell(x))}{\ell(x)}\mathbf{1}_{\{\ell(x)>0\}}\right), \end{aligned} \tag{4.1.37}$$

where we used $\psi(x) \le C\delta_{B_r}(x) \le C$ and $\ell(x) \le C\delta_{B_r}(x_0) + Cr \le C$ for the first inequality and (4.1.6) for the second. On the other hand, for any $x \in B_{\rho/2}(x_0)$ with $\rho \le \kappa r$ we have

$$|\ell(x) - \psi(x)| \le \frac{C}{r}|x - x_0|^2 \le \frac{C}{r}\rho^2 \le C\kappa\rho$$

and

$$C^{-1}\frac{\rho}{2} \le C^{-1}\delta_{B_r}(x) \le \psi(x).$$

Thus, using $\kappa = 1/(8C^2)$ we obtain

$$\frac{1}{2}\psi(x) \le \ell(x) \le 2\psi(x) \quad \text{for any} \quad x \in B_{\rho/2}(x_0).$$

Using $\frac{\rho}{2} \le \delta_{B_r}(x) \le 2\rho$, we arrive at

$$\psi(x), \ell(x) \in [(4C)^{-1}\rho, 4C\rho].$$

Therefore, there exists $y \in ((4C)^{-1}\rho, 4C\rho)$ satisfying

$$\frac{V(\psi(x)) - V(\ell(x))}{\psi(x) - \ell(x)} = V'(y),$$

so using (4.1.35) and (4.1.9), we have

$$\begin{aligned} |V(\psi(x)) - V(\ell(x))| = |\psi(x) - \ell(x)|V'(y) &\le \frac{c_{10}}{r}|x - x_0|^2\frac{V(y)}{y} \\ &\le \frac{c_{11}}{r}|x - x_0|^2\frac{V((4C)^{-1}\rho)}{(4C)^{-1}\rho} \le \frac{c_{12}}{r}|x - x_0|^2\frac{V(\rho)}{\rho} \end{aligned} \tag{4.1.38}$$

for $x \in B_{\rho/2}(x_0)$. Here we used (4.1.9) and (4.1.6) for the second line. Also, for any $x \in B_r^c(x_0)$ we have

$$V(\ell(x)) = V(\psi(x_0) + (x - x_0)\nabla\psi(x_0)) \le V(C\rho + C|x - x_0|) \le c_{13}V(|x - x_0|)$$

and

$$V(\psi(x)) \le V(Cr) \le V(C|x - x_0|) \le c_{13}V(|x - x_0|),$$

where we have used (4.1.6) and $\rho \le r \le |x - x_0|$. Thus we obtain

$$|V(\psi) - V(\ell)|(x) \le c_{14} V(|x - x_0|) \tag{4.1.39}$$

for $x \in B_r^c(x_0)$. Therefore, by taking $x = y + x_0$ for (4.1.37), (4.1.38), and (4.1.39) we have

$$\begin{aligned}
&|V(\psi) - V(\ell)|(y + x_0) \\
&\le c \begin{cases} \frac{1}{r}\frac{V(\rho)}{\rho}|y|^2 & \text{for } y \in B_{\rho/2} \\ \frac{|y|^2}{r}\left(\frac{V(\delta_{B_r}(x_0+y))}{\delta_{B_r}(x_0+y)}\mathbf{1}_{\{\delta_{B_r}(x_0+y)>0\}} + \frac{V(l(x_0+y))}{l(x_0+y)}\mathbf{1}_{\{l(x_0+y)>0\}}\right) & \text{for } y \in B_r \backslash B_{\rho/2} \\ V(|y|) & \text{for } y \in B_r^c \end{cases}
\end{aligned}$$

where $c = c_9 \vee c_{12} \vee c_{14}$. Hence, recalling that $L(V(\ell))(x_0) = 0$ and $\psi(x_0) = \ell(x_0)$, we find that

$$\begin{aligned}
&|L(V(\psi))(x_0)| = |L\big(V(\psi(\cdot)) - V(\ell(\cdot))\big)(x_0)| \\
&= \int_{\mathbb{R}^d} |V(\psi) - V(\ell)|(x_0 + y)\frac{J(1)}{|y|^n \varphi(|y|)}dy \\
&\le \frac{c}{r}\frac{V(\rho)}{\rho}\int_{B_{\rho/2}} |y|^2 \frac{J(1)}{|y|^n\varphi(|y|)}dy + c\int_{B_r^c} V(|y|)\frac{J(1)}{|y|^n\varphi(|y|)}dy \\
&\quad + c\int_{B_r \backslash B_{\rho/2}} \frac{|y|^2}{r}\left(\frac{V(\delta_{B_r}(x_0+y))}{\delta_{B_r}(x_0+y)}\mathbf{1} + \frac{V(\ell(x_0+y))}{\ell(x_0+y)}\mathbf{1}\right)\frac{J(1)}{|y|^n\varphi(|y|)}dy \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

For I, using (4.1.22) we have

$$\begin{aligned}
\mathrm{I} &= \frac{c}{r}\frac{V(\rho)}{\rho}\int_{B_{\rho/2}} |y|^2\frac{J(1)}{|y|^n\varphi(|y|)}dy = \frac{c_{15}}{r}\frac{V(\rho)}{\rho}\int_0^{\rho/2}\frac{s}{\varphi(s)}ds \\
&\le \frac{c_{16}}{r}\frac{V(\rho)}{\rho}\frac{(\rho/2)^2}{\varphi(\rho/2)} \le \frac{c_{17}}{V(r)}\left(\frac{\rho}{r}\frac{V(r)}{V(\rho)}\right) \le \frac{c_{18}}{V(r)},
\end{aligned}$$

where we used (4.1.5) and (4.1.6) for the last two inequalities. Also, using

(4.1.25) we obtain

$$\mathrm{II} = c\int_{B_r^c} V(|y|)\frac{J(1)}{|y|^n\varphi(|y|)}dy = c_{19}\int_r^\infty \frac{V(s)}{s\varphi(s)}ds \le \frac{c_{20}}{V(r)}.$$

For the estimate of III, we first observe that for any $y \in \{\ell > 0\} := H$,

$$\left|\frac{\ell(y)}{\delta_H(y)}\right| = \|\nabla\psi(x_0)\| \le C.$$

Thus, by (4.1.6) we have

$$\frac{V(\ell(y))}{\ell(y)} \le c_{21}\frac{V(C\delta_H(y))}{\delta_H(y)} \le c_{22}\frac{V(\delta_H(y))}{\delta_H(y)}.$$

Therefore, using Lemma 4.1.19 for $B_r$ and the half plane $H := \{\ell > 0\}$ for each line, we conclude

$$\begin{aligned}
\mathrm{III} = &\frac{c}{r}\int_{B_r\cap\left(B_1(x_0)\setminus B_{\rho/2}(x_0)\right)} \frac{V(\delta_{B_r}(y))}{\delta_{B_r}(y)}\frac{J(1)}{|y-x_0|^{n-2}\varphi(|y-x_0|)}dy \\
&+ \frac{c}{r}\int_{H\cap\left(B_1(x_0)\setminus B_{\rho/2}(x_0)\right)} \frac{V(\ell(y))}{\ell(y)}\frac{J(1)}{|x-y|^{n-2}\varphi(|x_0-y|)}dy \\
\le &\frac{c_{23}}{V(r)} + \frac{c_{24}}{r}\int_{H\cap\left(B_1(x_0)\setminus B_{\rho/2}(x_0)\right)} \frac{V(\delta_H(y))}{\delta_H(y)}\frac{1}{|x-y|^{n-2}\varphi(|x_0-y|)}dy \le \frac{c_{25}}{V(r)}.
\end{aligned}$$

Combining estimates of I,II and III we arrive

$$|L(V(\psi))(x_0)| \le \mathrm{I} + \mathrm{II} + \mathrm{III} \le (c_{18} + c_{20} + c_{25})\frac{1}{V(r)}$$

and (4.1.33) follows. □

### 4.1.4 Subsolution and Harnack Inequality

In this section we construct a subsolution from the barrier we have obtained in Proposition 4.1.20. Recall that we defined the domain of infinitesimal generator $A$ by

$$\mathcal{D} = \mathcal{D}(D) = \{u \in C_0(D) : Au \in C(D)\}.$$

It is uncertain whether $V(\psi) \in \mathcal{D}(D)$ since $A(V(\psi))$ is not continuous in general. To make our barrier included in the domain of operator, we construct a new domain of generator which contains $V(\psi)$. For given $C^{1,1}$ bounded open set $D$ and open subset $U$ in $D$, define

$$\mathcal{F} = \mathcal{F}(D, U) := \{u \in C_0(D) : Au \in L^\infty(U)\}.$$

for the usage of proof. Denote $\mathcal{F}(D) = \mathcal{F}(D, D)$. Clearly for any $U_1 \subset U_2$, $\mathcal{F}(D, U_2) \subset \mathcal{F}(D, U_1)$. We first prove that $V(\psi) \in \mathcal{F}(D)$.

**Lemma 4.1.21.** *Let $\psi$ be the regularized version of $\delta_D$. Then, $A(V(\psi)) = L(V(\psi))$ in $D$. Moreover, $V(\psi) \in \mathcal{F}(D)$.*

**Proof.** Let $u \in C_0(D)$ be a twice-differentiable function in $D$. Assume that $\nabla^2 u$ is bounded in some $U \subset\subset D$. We first claim that

$$Lu(x) = Au(x) \quad \text{for any} \quad x \in U.$$

Indeed, fix $x \in U$ and let $r_x > 0$ be a constant satisfying $B = B(x, r_x) \subset U$. Without loss of generality we can assume $r_x \leq 1$. Note that there exists a constant $c_1 > 0$ such that $2|u| + r_x^2 \|\nabla^2 u\| \leq c_1$ in $U$. Then we have

$$\begin{aligned} Au(x) &= \lim_{t \downarrow 0} \frac{P_t u(x) - u(x)}{t} = \lim_{t \downarrow 0} \frac{1}{t} \left( \int_{\mathbb{R}^d} u(x+y) p(t, |y|) dy - u(x) \right) \\ &= \lim_{t \downarrow 0} \int_{\mathbb{R}^d} \left( \frac{u(x+y) + u(x-y)}{2} - u(x) \right) \frac{p(t, |y|)}{t} dy. \end{aligned} \tag{4.1.40}$$

Since there is a constant $c_2 > 0$ such that $\frac{p(t,r)}{t} \leq c_2 J(r)$ for any $t > 0$ and

$r > 0$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \left| \frac{u(x+y)+u(x-y)}{2} - u(x) \right| \frac{p(t,|y|)}{t} dy \\
&\le \left( \int_B + \int_{B^c} \right) \left| \frac{u(x+y)+u(x-y)}{2} - u(x) \right| \frac{p(t,|y|)}{t} dy \\
&\le \int_B \frac{|y|^2}{r_x^2} \frac{c_1 p(t,|y|)}{t} dy + \int_{B^c} \frac{c_1 p(t,|y|)}{t} dy \le c_1 \int_{\mathbb{R}^d} (\frac{|y|^2}{r_x^2} \wedge 1)(c_3 J(|y|)) dy < \infty
\end{aligned}
$$

for any $t > 0$ so that we can apply dominate convergence theorem in the right-handed side of (4.1.40). Thus, using $\lim_{t \downarrow 0} \frac{p(t,r)}{t} = J(r)$ we obtain

$$
\begin{aligned}
Au(x) &= \lim_{t \downarrow 0} \int_{\mathbb{R}^d} \left( \frac{u(x+y)+u(x-y)}{2} - u(x) \right) \frac{p(t,|y|)}{t} dy \\
&= \int_{\mathbb{R}^d} \left( \frac{u(x+y)+u(x-y)}{2} - u(x) \right) J(|y|) dy = Lu(x).
\end{aligned}
$$

This concludes the claim. Now, by Lemma 4.1.4 we have that $V(\psi) \in C_0(D)$ is twice-differentiable and $\nabla^2 V(\psi)$ is locally bounded on $D$. Therefore, we arrive $L(V(\psi)) = A(V(\psi))$ in $D$. It immediately follows from (4.1.32) that $V(\psi) \in \mathcal{F}(D)$. □

Now we are ready to construct a subsolution with respect to the generator $A$.

**Lemma 4.1.22** (subsolution)**.** *There exist a constant $C_4 > 0$ independent of $r$ and a radial function $w = w_r \in \mathcal{F}(B_{4r})$ satisfying*

$$
\begin{cases}
Aw \ge 0 & \text{in } B_{4r} \setminus B_r, \\
w \le V(r) & \text{in } B_r, \\
w \ge C_4 V(4r - |x|) & \text{in } B_{4r} \setminus B_r, \\
w \equiv 0 & \text{in } \mathbb{R}^d \setminus B_{4r},
\end{cases}
$$

*where $B_r := B(0, r)$.*

**Proof.** Let $\Psi = \Psi_{4r}$ be the regularized version of $\delta_{B_{4r}}$ in (4.1.20) and choose a function $\eta \in C_c^\infty(B_1)$ satisfying $\|\eta\|_{C(B_1)} = 1$ and $\eta \equiv 1$ on $B_{1/2}$. Define $\eta_r(x) := V(r)\eta(x/r) \in C_c^\infty(B_r)$. Then, we have

$$\begin{aligned}|A\eta_r(x)| = |L\eta_r(x)| &\leq \int_{\mathbb{R}^d} \left| \frac{\eta_r(x+y) + \eta_r(x-y)}{2} - \eta_r(x) \right| J(|y|)dy \\ &\leq \left( \|\nabla^2 \eta_r\|_{L^\infty(B_r)} + \|\eta_r\|_{L^\infty(B_r)} \right) \int_{\mathbb{R}^d} \left( |y|^2 \wedge 1 \right) J(|y|)dy < \infty\end{aligned}$$

for any $x \in \mathbb{R}^d$, which implies $\eta_r \in \mathcal{F}(B_{4r})$. Also, for $x \in B_{4r} \backslash B_r$,

$$\begin{aligned}A\eta_r(x) = L\eta_r(x) &= \int_{\mathbb{R}^d} \frac{\eta_r(x+y) + \eta_r(x-y)}{2} \frac{J(1)}{|y|^n \varphi(|y|)} dy \\ &= \int_{\mathbb{R}^d} \eta_r(x+y) \frac{J(1)}{|y|^n \varphi(|y|)} dy \geq \int_{B(-x, r/2)} \frac{V(r)J(1)}{|y|^n \varphi(|y|)} dy \geq \frac{c_2}{V(4r)}.\end{aligned}$$

Here we used (4.1.5) and (4.1.6) for the last inequality.

Define a function $\tilde{w}_r$ by

$$\tilde{w}_r = \frac{c_2}{C_3} V(\Psi) + \eta_r,$$

where $C_3$ is the constant in Proposition 4.1.20. We have $\tilde{w}_r \in \mathcal{F}(B_{4r})$ by Lemma 4.1.21. Also, for $x \in B_{4r} \backslash B_r$, using Proposition 4.1.20 and Lemma 4.1.21 again, we have

$$A\tilde{w}_r(x) \geq -\frac{c_2}{C_3} |LV(\Psi)(x)| + A\eta_r(x) \geq -\frac{c_2}{V(4r)} + \frac{c_2}{V(4r)} = 0$$

and

$$\tilde{w}_r(x) = \frac{c_2}{C_3} V(\Psi(x)) \geq c_3 V(\delta_D(x)) = c_3 V(4r - |x|).$$

For $x \in B_r$,

$$\tilde{w}_r(x) \leq \frac{c_2}{C_3} V(4Cr) + V(r) \leq c_4 V(r)$$

by (4.1.20) and (4.1.6). Define $w_r(x) := \frac{1}{c_4} \tilde{w}_r(x)$. Then $w_r$ satisfies all assertions in Lemma 4.1.22 with constant $C_4 = \frac{c_3}{c_4}$, which is independent of $r$. □

We end with the Harnack inequality and the maximum principle of probabilistic version. For local operators, the Harnack inequality implies Hölder regularity of solutions of differential equations. However for nonlocal operators, as Silvestre mentioned in [84], this is not true because the nonnegativity of the function $u$ is required in the whole space $\mathbb{R}^d$. The Harnack inequality, maximum principle, and the subsolution constructed in Lemma 4.1.22 will play a key role in the proof of Theorem 4.1.2. We emphasize that the following theorem is the Harnack inequality for harmonic function with respect to $A$, and it does not imply the Harnack inequality for the viscosity solution with respect to $L$. See [21] for the statement of Harnack inequality for viscosity solution.

**Theorem 4.1.23.** ([86, Theorem 2.2]) *Let $D$ be a bounded $C^{1,1}$ open set. Then, there exists a constant $C > 0$ such that for any ball $B(x_0, r) \subset D$, and any nonnegative function $u \in \mathcal{F}(D)$ satisfying $Au = 0$ a.e. in $B(x_0, r)$, we have*

$$\sup_{B(x_0,r/2)} u \leq C \inf_{B(x_0,r/2)} u.$$

Also, we have the following maximum principle.

**Lemma 4.1.24** (Maximum principle)**.** *Let $D$ be a bounded $C^{1,1}$ open set and $U$ be an open subset of $D$. If the function $u \in \mathcal{F}(D, U)$ satisfies $Au = 0$ a.e. in $U$ and $u \geq 0$ in $U^c$, then $u \geq 0$ in $\mathbb{R}^d$.*

**Proof.** Suppose that there exists $x \in U$ satisfying $u(x) < 0$. Since $u \in C_0(D)$, the set $U_- := \{x \in \mathbb{R}^d : u(x) < 0\}$ is bounded and open set with positive

Lebesgue measure. For any $t > 0$ we have

$$\int_{U_-} P_t u(x) - u(x)dx = \int_{U_-} \int_{\mathbb{R}^d} u(y)p(t, |x-y|)dydx - \int_{U_-} u(x)dx$$
$$= \int_{\mathbb{R}^d} u(y) \int_{U_-} p(t, |x-y|)dxdy - \int_{U_-} u(y)dy$$
$$= \int_{U_-^c} u(y) \int_{U_-} p(t, |x-y|)dxdy + \int_{U_-} u(y) \left( \int_{U_-} p(t, |x-y|)dx - 1 \right) dy$$
$$\geq \int_{U_-} u(y) \left( \int_{U_-} p(t, |x-y|)dx - 1 \right) dy.$$

Since $U_-$ is bounded, diam(U$_-$) =: R $< \infty$. Thus, for any $y \in U_- \subset B(y, R)$,

$$\frac{1 - \int_{U_-} p(t, |x-y|)dx}{t} \geq \frac{1 - \int_{B(y,R)} p(t, |x-y|)dx}{t} = \frac{\mathbb{P}^0(|X_t| \geq R)}{t}.$$

Using heat kernel estimates in [17, Theorem 21], $p(t, r) \asymp \left( \varphi^{-1}(t)^{-n} \wedge \frac{t}{r^n \varphi(r)} \right)$ for $(t, r) \in (0, 1] \times \mathbb{R}_+$. Note that $\frac{t}{r^n \varphi(r)} \leq \varphi^{-1}(t)^{-n}$ for $t \leq \varphi(r)$. Thus, there exists $\varepsilon = \varepsilon(R) > 0$ satisfying

$$\frac{\mathbb{P}^0(|X_t| \geq R)}{t} \geq \frac{1}{t} \int_{R \leq |z| \leq 2R} p(t, |z|)dz \geq c \int_R^{2R} \frac{1}{r\varphi(r)} dr \geq \varepsilon$$

for all $t \in (0, \varphi(R)]$.Combining above estimates we obtain

$$\int_{U_-} \frac{P_t u(x) - u(x)}{t} dx \geq -\varepsilon \int_{U_-} u(y)dy \quad \text{for all} \quad t \in (0, \varphi(R)].$$

Letting $t \to 0$, we conclude

$$0 = \int_{U_-} Au(x)dx = \lim_{t \to 0} \int_{U_-} \frac{P_t u(x) - u(x)}{t} dx \geq -\varepsilon \int_{U_-} u(y)dy > 0,$$

which is contradiction. Therefore, $u \geq 0$ in $\mathbb{R}^d$. $\square$

### 4.1.5 Proof of Theorem 4.1.2

In this subsection we will prove Theorem 4.1.2. More precisely, we prove the Hölder regularity for the function $u/V(\delta_D)$ up to the boundary of $D$. We will control the oscillation of this function using the Harnack inequality, the maximum principle and the subsolution constructed in Lemma 4.1.22.

Let us adopt notations in [75, Definition 3.3]. Let $\kappa > 0$ be a fixed small constant and let $\kappa' = 1/2 + 2\kappa$. Given $x_0 \in \partial D$ and $r > 0$, define

$$D_r = D_r(x_0) = B(x_0, r) \cap D$$

and

$$D^+_{\kappa' r} = D^+_{\kappa' r}(x_0) = B(x_0, \kappa' r) \cap \{x \in D : -x \cdot \nu(x_0) \geq 2\kappa r\},$$

where $\nu(x_0)$ is the unit outward normal at $x_0$. Since $D$ is a bounded $C^{1,1}$ open set, there exists $\rho_0 > 0$ such that for each $x_0 \in \partial D$ and $r \leq \rho_0$, there exists an orthonormal system $CS_{x_0}$ with its origin at $x_0$ and a $C^{1,1}$-function $\Psi : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying $\Psi(\tilde{0}) = 0, \nabla_{CS_{x_0}} \Psi(\tilde{0}) = 0, \|\Psi\|_{C^{1,1}} \leq \kappa$, and

$$\{y = (\tilde{y}, y_n) \text{ in } CS_{x_0} : |\tilde{y}| < 2r, \Psi(\tilde{y}) < y_n < 2r\} \subset D.$$

Then we have

$$B(y, \kappa r) \subset D_r(x_0) \text{ for all } y \in D^+_{\kappa' r}(x_0), \tag{4.1.41}$$

and we can take a $C^{1.1}$ subdomain $D^{1,1}_r$ satisfying $D_r \subset D^{1,1}_r \subset D_{2r}$ and

$$\operatorname{dist}(y, \partial D^{1,1}_r) = \delta_D(y) \tag{4.1.42}$$

for all $y \in D_r$. Since $D_r$ is not $C^{1,1}$ in general, we will use this subdomain instead of $D_r$.

Since $D$ is bounded and $C^{1,1}$ again, we can assume that for each $x_0 \in \partial D$

and $r \le \rho_0$,

$$B(y^* - 4\kappa r\nu(y^*), 4\kappa r) \subset D_r(x_0) \text{ and } B(y^* - 4\kappa r\nu(y^*), \kappa r) \subset D^+_{\kappa' r}(x_0) \tag{4.1.43}$$

for all $y \in D_{r/2}(x_0)$, where $y^* \in \partial D$ is the unique boundary point satisfying $|y - y^*| = \delta_D(y)$.

The following oscillation lemma is the key lemma to prove Theorem 4.1.2.

**Lemma 4.1.25** (Oscillation lemma)**.** *Assume $f \in C(D)$ and let $u \in \mathcal{D}$ be the viscosity solution of* (4.1.1)*. Then there exist constants $\gamma \in (0,1)$ and $C_1 > 0$, depending only on $d, a_1, a_2, \alpha_1, \alpha_2$ and $D$, such that*

$$\operatorname*{osc}_{D_r(x_0)} \frac{u}{V(\delta_D)} \le C_1 V(r)^\gamma \|f\|_{L^\infty(D)} \tag{4.1.44}$$

*for any $x_0 \in \partial D$ and $r > 0$.*

To prove the oscillation lemma, we need some preparation. Note that in the following two lemmas we aim to verify inequalities for every function $u \in \mathcal{F}$, since we want to utilize the subsolution constructed in Lemma 4.1.22. The first one is a generalized version of Harnack inequality.

**Lemma 4.1.26** (Harnack inequality)**.** *There exists a constant $C_2 > 0$ such that for any $r \le \rho_0, x_0 \in \partial D$ and nonnegative function $u \in \mathcal{F}(D, D^{1,1}_r)$,*

$$\sup_{D^+_{\kappa' r}(x_0)} \frac{u}{V(\delta_D)} \le C_2 \left( \inf_{D^+_{\kappa' r}(x_0)} \frac{u}{V(\delta_D)} + \|Au\|_{L^\infty(D^{1,1}_r)} V(r) \right). \tag{4.1.45}$$

**Proof.** We first prove that if a nonnegative function $v$ satisfies $Av = 0$ a.e. in $D^{1,1}_r$, then

$$\sup_{D^+_{\kappa' r}(x_0)} \frac{v}{V(\delta_D)} \le c \inf_{D^+_{\kappa' r}(x_0)} \frac{v}{V(\delta_D)} \tag{4.1.46}$$

for a constant $c > 0$ which is independent of $r$ and $v$. Indeed, for each $y \in D^+_{\kappa' r}$, we have $B(y, \kappa r) \subset D^{1,1}_r$ by (4.1.41) hence $Av = 0$ a.e. in $B(y, \kappa r)$.

We may cover $D^+_{\kappa' r}$ by finitely many balls $B(y_i, \kappa r/2)$. Here the number of balls is independent of $r$. By the Theorem 4.1.23, we have for each $i$,

$$\sup_{B(y_i,\kappa r/2)} v \leq c_1 \inf_{B(y_i,\kappa r/2)} v.$$

If $x \in B(y_i, \kappa r/2)$, we have $\kappa r/2 \leq \delta_D(x) \leq r/2 + 5\kappa r/2$. Thus, using (4.1.6) we obtain

$$\sup_{B(y_i,\kappa r/2)} \frac{v}{V(\delta_D)} \leq \inf_{B(y_i,\kappa r/2)} \frac{c_2 v}{V(r/2+5\kappa r/2)} \leq \inf_{B(y_i,\kappa r/2)} \frac{c_2 v}{V(\delta_D)}.$$

Now (4.1.46) follows from the standard covering argument, possibly with a larger constant. We next prove (4.1.45). Let us write $u = u_1 + u_2$, where $u_1 := u + R^{D^{1,1}_r} Au$ and $u_2 := -R^{D^{1,1}_r} Au$. We claim that $u_1 \geq 0$ in $\mathbb{R}^d$ and $Au_1 = 0$ a.e. in $D^{1,1}_r$.

Following the calculations of (4.1.17) we obtain that for any open subset $U \subset D$, $x \in U$ and $u \in \mathcal{F}(D, U)$,

$$Au(x) = \lim_{t \downarrow 0} \frac{P_t u(x) - u(x)}{t} = \lim_{t \downarrow 0} \frac{P^U_t u(x) - u(x)}{t}. \tag{4.1.47}$$

Let us emphasize that we only have used $u \in C_0(D)$ in (4.1.17) so we can repeat the same argument for $u \in \mathcal{F}(D, U)$.

Let $g \in L^\infty(U)$. Deducing $R^U g \in C_0(U)$ from Proposition 4.1.6 and (4.1.10), we obtain the following counterpart of (4.1.18): For any $x \in U$,

$$\begin{aligned} AR^U g(x) &= A\left(\int_0^\infty P^U_s g(\cdot) ds\right)(x) \\ &= \lim_{t\downarrow 0} \frac{1}{t}\left(P^U_t \left(\int_0^\infty P^U_s g(\cdot)ds\right)(x) - \int_0^\infty P^U_s g(x) ds\right) \\ &= \lim_{t\downarrow 0} \frac{1}{t}\left(\int_0^\infty P^U_{s+t} g(x) ds - \int_0^\infty P^U_s g(x) ds\right) \\ &= -\lim_{t\downarrow 0} \frac{\int_0^t P^U_s g(x) ds}{t} = -\lim_{t\downarrow 0} \frac{\int_0^t P_s g(x) ds}{t}. \end{aligned} \tag{4.1.48}$$

Here we used (4.1.47) for the first line. Let

$$U_g := \{x \in U : \lim_{r\downarrow 0} \frac{1}{r^n} \int_{B(x,r)} |g(x) - g(y)| dy = 0\}.$$

Then, we have $|U \setminus U_g| = 0$ since $g \in L^\infty(U) \subset L^1(U)$. For $x \in U_g$, we have

$$|P_t g(x) - g(x)| \le \int_{\mathbb{R}^d} p(t, |x-y|) |g(y) - g(x)| dy.$$

Let $\varepsilon > 0$. Using $p(t,r) \asymp \left(\varphi^{-1}(t)^{-n} \wedge \frac{t}{r^n \varphi(r)}\right)$ for $t \in (0,1] \times \mathbb{R}_+$ in [17, Theorem 21] again, there exist constants $c_3(\varepsilon), c_4(\varepsilon) > 0$ such that for any $t \in (0,1]$ and $r > 0$,

$$p(t,r) \le c_4 \varphi^{-1}(t)^{-n} \quad \text{and} \quad \mathbb{P}^x(|X_t| > c_3 \varphi^{-1}(t)) \le \varepsilon.$$

Indeed, using (4.1.23) and $L(2\alpha_1, a_3^{-1}, \varphi)$ we have

$$\mathbb{P}^x(|X_t| > c_3 \varphi^{-1}(t)) = \int_{|z| > c_3 \varphi^{-1}(t)} p(t, |z|) dz \le c_4 t \int_{c_3 \varphi^{-1}(t)}^\infty \frac{dr}{r \varphi(r)} \le c_5.$$

Thus, we obtain

$$\begin{aligned}
|P_t g(x) - g(x)| &\le \left( \int_{B(x, c_3 \varphi^{-1}(t))} + \int_{B(x, c_3 \varphi^{-1}(t))^c} \right) p(t, |x-y|) |g(y) - g(x)| dy \\
&\le c_4 \varphi^{-1}(t)^{-n} \int_{B(x, c_3 \varphi^{-1}(t))} |g(y) - g(x)| dy + 2\|g\|_\infty \int_{B(x, c_3 \varphi^{-1}(t))} p(t, |x-y|) dy \\
&\le c_4 \varphi^{-1}(t)^{-n} \int_{B(x, c_3 \varphi^{-1}(t))} |g(y) - g(x)| dy + 2\|g\|_\infty \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary and $x \in U_g$, we conclude

$$\lim_{t\downarrow 0} |P_t g(x) - g(x)| = 0.$$

Combining this with (4.1.48) we arrive that for any open subset $U \subset D$ and

$g \in L^\infty(D)$,

$$AR^U g = -g \quad \text{a.e. in} \quad U. \tag{4.1.49}$$

Since $u \in \mathcal{F}(D,U)$, we have $Au \in L^\infty(U)$. Thus, taking $U = D_r^{1,1}$ and $g = Au$ in (4.1.49) we conclude

$$Au_1 = Au + AR^{D_r^{1,1}} Au = 0 \quad \text{a.e. in } D_r^{1,1}.$$

Also, $u_1 \geq 0$ follows from applying Lemma 4.1.24 with above equation and $u_1 = u \geq 0$ in $\mathbb{R}^d \backslash D_r^{1,1}$.

Applying (4.1.46) to $u_1$, we get

$$\sup_{D^+_{\kappa' r}} \frac{u_1}{V(\delta_D)} \leq c_7 \inf_{D^+_{\kappa' r}} \frac{u_1}{V(\delta_D)}.$$

Meanwhile, using (4.1.42) and Lemma 4.1.7 we have

$$\begin{aligned}|u_2(x)| &\leq c_8 \|Au\|_{L^\infty(D_r^{1,1})} V(\mathrm{diam}(D_r^{1,1})) V(\mathrm{dist}(x, \partial D_r^{1,1})) \\ &\leq c_9 \|Au\|_{L^\infty(D_r^{1,1})} V(r) V(\delta_D(x))\end{aligned}$$

for all $x \in D_r^{1,1}$. Therefore, combining above two inequalities we conclude that

$$\begin{aligned}&\sup_{D^+_{\kappa' r}} \frac{u}{V(\delta_D)} \leq \sup_{D^+_{\kappa' r}} \frac{u_1}{V(\delta_D)} + \sup_{D^+_{\kappa' r}} \frac{u_2}{V(\delta_D)} \leq c_5 \inf_{D^+_{\kappa' r}} \frac{u_1}{V(\delta_D)} + \sup_{D^+_{\kappa' r}} \frac{u_2}{V(\delta_D)} \\ &\leq \inf_{D^+_{\kappa' r}} \frac{c_5 u}{V(\delta_D)} + \sup_{D^+_{\kappa' r}} \frac{|(c_5+1)u_2|}{V(\delta_D)} \leq C_2 \left( \inf_{D^+_{\kappa' r}} \frac{u}{V(\delta_D)} + \|Au\|_{L^\infty(D_r^{1,1})} V(r) \right).\end{aligned}$$

□

The next lemma gives the link between $D^+_{\kappa' r}$ and $D_{r/2}$. Here we are going to use the subsolution $w$ in Lemma 4.1.22.

**Lemma 4.1.27.** *Let $r \leq \rho_0, x_0 \in \partial D$. If $u \in \mathcal{F}(D, D_r^{1,1})$ is nonnegative,* *then there exists a constant* $C_3 = C_3(n, a_1, a_2, \alpha_1, \alpha_2, D) > 0$ *such that*

$$\inf_{D^+_{\kappa' r}(x_0)} \frac{u}{V(\delta_D)} \le C_3 \left( \inf_{D_{r/2}(x_0)} \frac{u}{V(\delta_D)} + \|Au\|_{L^\infty(D^{1,1}_r)} V(r) \right).$$

**Proof.** First assume that $Au$ is nonnegative. As in the proof of Lemma 4.1.26, we write $u = u_1 + u_2$, where $u_1 = u + R^{D^{1,1}_r} Au$ and $u_2 = -R^{D^{1,1}_r} Au$. Then $u_1$ is a nonnegative solution for

$$\begin{cases} Au_1 = 0 & \text{a.e. in } D^{1,1}_r, \\ u_1 = u & \text{in } \mathbb{R}^d \setminus D^{1,1}_r. \end{cases}$$

Let

$$m := \inf_{D^+_{\kappa' r}} \frac{u_1}{V(\delta_D)} \ge 0.$$

For $y \in D_{r/2}$, we have either $y \in D^+_{\kappa' r}$ or $\delta_D(y) < 4\kappa r$ by (4.1.43).

If $y \in D^+_{\kappa' r}$, then clearly

$$m \le \frac{u_1(y)}{V(\delta_D(y))}.$$

If $\delta_D(y) < 4\kappa r$, let $y^*$ be the closest point to $y$ on $\partial D^{1,1}_r$ and let $\tilde{y} = y^* - 4\kappa r \nu(y^*)$. By (4.1.43), we have $B_{4\kappa r}(\tilde{y}) \subset D_r$ and $B_{\kappa r}(\tilde{y}) \subset D^+_{\kappa' r}$.

Now consider $w \in \mathcal{F}(B_{4\kappa r}(\tilde{y})) \subset \mathcal{F}(D, B_{4\kappa r}(\tilde{y}) \backslash B_{\kappa r}(\tilde{y}))$ satisfying

$$\begin{cases} Aw \ge 0 & \text{in } B_{4\kappa r}(\tilde{y}) \setminus B_{\kappa r}(\tilde{y}), \\ w \le V(\kappa r) & \text{in } B_{\kappa r}(\tilde{y}), \\ w \ge c_1 V(4\kappa r - |x - \tilde{y}|) & \text{in } B_{4\kappa r}(\tilde{y}) \setminus B_{\kappa r}(\tilde{y}), \\ w \equiv 0 & \text{in } \mathbb{R}^d \setminus B_{4\kappa r}(\tilde{y}), \end{cases}$$

which can be obtained by translating the subsolution in Lemma 4.1.22. Since

$Au_1 = 0$ a.e. in $B_{4\kappa r}(\tilde{y})$, we have

$$\begin{cases} Au_1 = 0 \le A(mw) & \text{a.e. in } B_{4\kappa r}(\tilde{y}) \setminus B_{\kappa r}(\tilde{y}), \\ u_1 \ge mV(\delta_D) \ge mw & \text{in } B_{\kappa r}(\tilde{y}), \\ u_1 \ge 0 = mw & \text{in } \mathbb{R}^d \setminus B_{4\kappa r}(\tilde{y}). \end{cases}$$

Now by the maximum principle in Lemma 4.1.24 with the function $u_1 - mw$ and $U = B_{4\kappa r}(\tilde{y}) \setminus B_{\kappa r}(\tilde{y})$, we obtain $u_1 \ge mw$ in $\mathbb{R}^d$. In particular, for $y \in B_{4\kappa r}(\tilde{y}) \setminus B_{\kappa r}(\tilde{y})$,

$$u_1(y) \ge c_1 mV(4\kappa r - |y - \tilde{y}|) = c_1 mV(\delta_D(y)).$$

Therefore, we obtain

$$\inf_{D^+_{\kappa' r}} \frac{u_1}{V(\delta_D)} \le c_2 \inf_{D_{r/2}} \frac{u_1}{V(\delta_D)}.$$

On the other hand, $u_2$ satisfies

$$|u_2(x)| \le c_3 \|Au\|_{L^\infty(D^{1,1}_r)} V(r) V(\delta_D(x))$$

for all $x \in D^{1,1}_r$, which gives the desired result. $\square$

We prove the oscillation lemma (4.1.44) by using Lemmas 4.1.26 and 4.1.27.

**Proof of Lemma 4.1.25** As a consequence of Remark 4.1.8, by dividing $\|f\|_{L^\infty(D)}$ on both sides of (4.1.1) if necessary, we may assume $\|f\|_{L^\infty(D)} \le 1$ and $\|u\|_{C(D)} = \|R^D f\|_{C(D)} \le c_1$ without loss of generality. Fix $x_0 \in \partial D$. We will prove that there exist constants $c_2 > 0, \rho_1 \in (0, \rho_0/16]$, and $\gamma \in (0,1)$ and monotone sequences $(m_k)_{k\ge0}$ and $(M_k)_{k\ge0}$ such that $M_k - m_k = V(r_{k+1}/2)^\gamma$,

$$-V(\rho_1/16) \le m_k \le m_{k+1} < M_{k+1} \le M_k \le V(\rho_1/16),$$

and

$$m_k \le \frac{u}{c_2 V(\delta_D)} \le M_k \text{ in } D_{r_k} = D_{r_k}(x_0)$$

for all $k \ge 0$, where $r_k = \rho_1 8^{-k}$. If we have such constants and sequences, then for any $0 < r \le \rho_1$ we have $k \ge 0$ satisfying $r \in (r_{k+1}, r_k]$ and

$$\sup_{D_r} \frac{u}{V(\delta_D)} - \inf_{D_r} \frac{u}{V(\delta_D)} \le \sup_{D_{r_k}} \frac{u}{V(\delta_D)} - \inf_{D_{r_k}} \frac{u}{V(\delta_D)} \le M_k - m_k.$$

Also, for any $r > \rho_1$ we have

$$\sup_{D_r} \frac{u}{V(\delta_D)} - \inf_{D_r} \frac{u}{V(\delta_D)} \le c_3 \le c_4 V(\rho_1)^\gamma \le c_4 V(r)^\gamma$$

by Lemma 4.1.7. Above two inequalities conclude the lemma so it suffices to construct such constants and sequences.

Let us use the induction on $k$. The case $k = 0$ follows from Lemma 4.1.7 provided we take $c_2$ large enough. The constants $\rho_1$ and $\gamma$ will be chosen later. Assume that we have sequences up to $m_k$ and $M_k$. Let $\psi$ be the regularized version of $\delta_D$. We may assume that $\psi = \delta_D$ in $\{\delta_D(x) \le \rho_1\}$. Define

$$u_k = V(\psi)\left(\frac{u}{c_2 V(\psi)} - m_k\right) = \frac{1}{c_2} u - m_k V(\psi)$$

in $\mathbb{R}^d$. Note that $u_k \in \mathcal{F}(D)$ since $Au = f$ by the consequence of Theorem 4.1.14. Moreover, for $x \in D^{1,1}_{r_k/4}$ we have $u_k^- \in C^2(x)$ since we know that $u_k^- \equiv 0$ in $B(x_0, r_k)$ by the induction hypothesis. Thus, we have $Au_k^-(x) = Lu_k^-(x)$ by (4.1.16), which implies that $Au_k^-$ is well-defined in $D^{1,1}_{r_k/4}$, and so is $Au_k^+$. We will apply Lemmas 4.1.26 and 4.1.27 for the function $u_k^+$ and $r = r_k/4$ to find $m_{k+1}$ and $M_{k+1}$. By (4.1.32) and Lemma 4.1.21, we have

$$\begin{aligned}|Au_k^+| &\leq |Au_k| + |Au_k^-| \leq \left|\frac{1}{c_2}Au - m_k AV(\psi)\right| + |Au_k^-| \\ &\leq \left(\frac{1}{c_2}|f| + V(\rho_1/16)|L(V(\psi))|\right) + |Au_k^-| \leq c_3 + |Au_k^-|\end{aligned}$$

in $D$. Thus, we need to estimate $|Au_k^-|$ in $D^{1,1}_{r_k/4}$ for the usage of Lemmas 4.1.26 and 4.1.27.

Let $x \in D^{1,1}_{r_k/4}$. By the induction hypothesis, we have $u_k^- \equiv 0$ in $B(x_0, r_k)$, which implies that $u_k^- \in C^2(x)$. Thus, we compute the value $Au_k^-(x)$ using the operator $L$ as follows:

$$\begin{aligned}0 \leq Au_k^-(x) = Lu_k^-(x) &= \frac{1}{2}\int_{\mathbb{R}^d} \left(u_k^-(x+h) + u_k^-(x-h)\right) \frac{J(1)}{|h|^n \varphi(|h|)}\, dh \\ &= \int_{x+h \notin B_{r_k}} u_k^-(x+h) \frac{J(1)}{|h|^n \varphi(|h|)}\, dh.\end{aligned} \tag{4.1.50}$$

For any $y \in B_{r_0} \setminus B_{r_k}$, there is $0 \leq j < k$ such that $y \in B_{r_j} \setminus B_{r_{j+1}}$. Since $c_2^{-1}u \geq m_j V(\psi)$ and $\delta_D = \psi$ in $B_{r_j}$, we have

$$\begin{aligned}&u_k(y) = c_2^{-1}u(y) - m_k V(\psi(y)) \geq (m_j - m_k)V(\psi(y)) \\ &\geq (m_j - M_j + M_k - m_k)V(\delta_D(y)) \geq -(V(r_{j+1}/2)^\gamma - V(r_{k+1}/2)^\gamma)V(r_j).\end{aligned}$$

It follows from $r_{j+1} \leq |y - x_0| < r_j \leq 8|y - x_0| \leq 1$ that

$$\begin{aligned}u_k^-(y) &\leq c_4 \left(V(|y-x_0|/2)^\gamma - V(r_k/16)^\gamma\right) V(8|y-x_0|) \\ &\leq c_5 \left(V(|y-x_0|/2)^\gamma - V(r_k/16)^\gamma\right) V(|y-x_0|/2).\end{aligned} \tag{4.1.51}$$

Note that (4.1.51) possibly with a larger constant also holds for $y \in \mathbb{R}^d \setminus B_{r_0}$ because $\|u_k\|_{C(\mathbb{R}^d)} \leq c_1 c_2^{-1} + V(1/16)V(\tilde{C})$ for any $k$ and

$$\left(V(|y-x_0|/2)^\gamma - V(r_k/16)^\gamma\right) V\left(\frac{|y-x_0|}{2}\right) \geq \left(V(\rho_1/2)^\gamma - V(\rho_1/16)^\gamma\right) V(\rho_1/2)$$

for any $y \in \mathbb{R}^d \setminus B_{r_0}$. Thus, by (4.1.50) and (4.1.51), we have

$$|Au_k^-(x)| \le c_6 \int_{x+y \notin B_{r_k}} \left(V(|x+h-x_0|/2)^\gamma - V(r_k/16)^\gamma\right) \frac{V(|x+y-x_0|/2)}{|h|^n \varphi(|h|)}\, dh.$$

If $x+y \notin B_{r_k}$, then $|h| \ge |x+h-x_0| - |x-x_0| \ge r_k - r_k/2 = r_k/2$ and $|x+h-x_0| \le r_k/2 + |h| \le 2|h|$. Thus, recalling that $\mathcal{P}_1(r) = \int_r^\infty \frac{ds}{s\varphi(s)}$, we obtain

$$\begin{aligned}
&|Au_k^-(x)| \le c_6 \int_{|h| \ge r_k/2} \left(V(|h|)^\gamma - V(r_k/16)^\gamma\right) \frac{V(|h|)}{|h|^n \varphi(|h|)}\, dh \\
&\le c_7 \int_{r_k/2}^\infty \left(V(s)^\gamma - V(r_k/16)^\gamma\right) V(s) d(-\mathcal{P}_1)(s) \\
&= c_7 \Bigg( \left[- \left(V(s)^\gamma - V(r_k/16)^\gamma\right) V(s)\mathcal{P}_1(s)\right]_{r_k/2}^\infty \\
&\qquad + \int_{r_k/2}^\infty \left((1+\gamma)V(s)^\gamma - V(r_k/16)^\gamma\right) V'(s)\mathcal{P}_1(s) ds \Bigg) =: c_7 \left(\mathrm{I} + \mathrm{II}\right).
\end{aligned}$$

By (4.1.23) we have

$$\lim_{s\to\infty} \left(V(s)^\gamma - V(r_k/16)^\gamma\right) V(s)\mathcal{P}_1(s) \le c_8 \lim_{s\to\infty} \frac{V(s)^\gamma - V(r_k/16)^\gamma}{V(s)} = 0,$$

hence

$$\mathrm{I} \le c_8 \frac{V(r_k/2)^\gamma - V(r_k/16)^\gamma}{V(r_k/2)}.$$

Also, using (4.1.23) again we have

$$\begin{aligned}
\mathrm{II} &\le c_8 \int_{r_k/2}^\infty \left((1+\gamma)V(s)^\gamma - V(r_k/16)^\gamma\right) \frac{V'(s)}{V(s)^2}\, ds \\
&= c_8 \left(\frac{1+\gamma}{1-\gamma} V(r_k/2)^\gamma - V(r_k/16)^\gamma\right) \frac{1}{V(r_k/2)}.
\end{aligned}$$

Therefore, combining above two inequalities and using (4.1.6) we get

$$\begin{aligned}
|Au_k^-(x)| &\le c_9 \left( \frac{2}{1-\gamma} V(r_k/2)^\gamma - 2V(r_k/16)^\gamma \right) \frac{1}{V(r_k/2)} \\
&\le c_9 \left( \frac{2}{1-\gamma} \left(c_{10} 64^{\alpha_2}\right)^\gamma - 2\left(c_{10}^{-1} 8^{\alpha_1}\right)^\gamma \right) \frac{V(r_{k+2}/2)^\gamma}{V(r_k/4)} \\
&=: c_9 \varepsilon_\gamma \frac{V(r_{k+2}/2)^\gamma}{V(r_k/4)}
\end{aligned}$$

and hence

$$\|Au_k^+\|_{L^\infty(D^{1,1}_{r_k/4})} \le c_{11} \left( 1 + \varepsilon_\gamma \frac{V(r_{k+2}/2)^\gamma}{V(r_k/4)} \right).$$

Note that $\varepsilon_\gamma \to 0$ as $\gamma \to 0$.

Now we apply Lemma 4.1.26 and 4.1.27 for $u_k^+ \in \mathcal{F}(D, D^{1,1}_{r_k/4})$. Since $u_k = u_k^+$ and $\delta_D = \psi$ in $D_{r_k}$, we have

$$\begin{aligned}
&\sup_{D^+_{\kappa' r_k/4}} \left( \frac{u}{c_2 V(\psi)} - m_k \right) \\
&\le c_{12} \left( \inf_{D^+_{\kappa' r_k/4}} \left( \frac{u}{c_2 V(\psi)} - m_k \right) + V(r_k/4) + \varepsilon_\gamma V(r_{k+2}/2)^\gamma \right) \\
&\le c_{13} \left( \inf_{D_{r_{k+1}}} \left( \frac{u}{c_2 V(\psi)} - m_k \right) + V(r_k/4) + \varepsilon_\gamma V(r_{k+2}/2)^\gamma \right).
\end{aligned}$$

Repeating this procedure with the function $u_k = M_k V(\delta_D) - c_2^{-1} u$ instead of $u_k = c_2^{-1} u - m_k V(\delta_D)$, we also have

$$\sup_{D^+_{\kappa' r_k/4}} \left( M_k - \frac{u}{c_2 V(\psi)} \right) \le \inf_{D_{r_{k+1}}} \left( M_k - \frac{u}{c_2 V(\psi)} \right) + V(r_k/4) + \varepsilon_\gamma V(r_{k+2}/2)^\gamma.$$

Adding up these two inequalities, we obtain

$$M_k - m_k \le c_{15} \big( - \operatorname*{osc}_{D_{r_{k+1}}} \frac{u}{c_2 V(\psi)} + M_k - m_k + V(r_k/4) + \varepsilon_\gamma V(r_{k+2}/2)^\gamma \big).$$

Thus, recalling that $M_k - m_k = V(r_{k+1}/2)^\gamma$, we get

$$\begin{aligned}\operatorname*{osc}_{D_{r_{k+1}}} \frac{u}{c_2 V(\psi)} &\leq \frac{c_{15}-1}{c_{15}} V(r_{k+1}/2)^\gamma + V(r_k/4) + \varepsilon_\gamma V(r_{k+2}/2)^\gamma \\ &\leq \left(\frac{c_{15}-1}{c_{15}} c_{16}^\gamma + c_{17}^\gamma V(\rho_1)^{1-\gamma} + \varepsilon_\gamma\right) V(r_{k+2}/2)^\gamma.\end{aligned}$$

Now we choose $\gamma$ and $\rho_1$ small enough so that

$$\frac{c_{15}-1}{c_{15}} c_{16}^\gamma + c_{17}^\gamma V(\rho_1)^{1-\gamma} + \varepsilon_\gamma \leq 1,$$

and it yields that

$$\operatorname*{osc}_{D_{r_{k+1}}} \frac{u}{c_2 V(\psi)} \leq V(r_{k+2}/2)^\gamma.$$

Therefore, we are able to choose $m_{k+1}$ and $M_{k+1}$. $\square$

Finally, we prove the Theorem 4.1.2 using the Lemma 4.1.25.

**Proof of Theorem 4.1.2** By Remark 4.1.8, by dividing $\|f\|_{L^\infty(D)}$ on both sides of (4.1.1) if necessary, we may assume that $\|f\|_{L^\infty(D)} \leq 1$ and $\|u\|_{C(D)} \leq c_1$. We first show that the following holds for any $x \in D$:

$$\left[\frac{u}{V(\delta_D)}\right]_{C^\beta(B(x,r/2))} \leq \frac{C}{r^\beta V(r)}$$

for each $0 < \beta \leq \alpha_1$, where $r = \delta_D(x)$. We are going to use the inequality

$$\left[\frac{u}{V(\delta_D)}\right]_{C^\beta} \leq \|u\|_C \left[\frac{1}{V(\delta_D)}\right]_{C^\beta} + [u]_{C^\beta} \left\|\frac{1}{V(\delta_D)}\right\|_C. \tag{4.1.52}$$

By (4.1.13), $[u]_{C^V(B(x,r/2))} \leq c_2$. Thus, we have $[u]_{C^\beta(B(x,r/2))} \leq c_3$ for each $0 < \beta \leq \alpha_1$.

Since $\delta_D(y) \geq r/2$ for $y \in B(x, r/2)$, we have

$$\left\|\frac{1}{V(\delta_D)}\right\|_{C(B(x,r/2))} \leq \frac{c_4}{V(r)}$$

and

$$\begin{aligned}
\left[\frac{1}{V(\delta_D)}\right]_{C^{0,1}(B(x,r/2))} &\leq \sup_{y,z\in B(x,r/2)} \frac{|V(\delta_D(y))^{-1} - V(\delta_D(z))^{-1}|}{|y-z|} \\
&\leq \sup_{y,z\in B(x,r/2)} \frac{V'(d^*)}{V(d^*)^2} \frac{|\delta_D(y) - \delta_D(z)|}{|y-z|} \\
&\leq c_5 \left( \sup_{y,z\in B(x,r/2)} \frac{1}{d^* V(d^*)} \right) [d]_{C^{0,1}(B(x,r/2))} \\
&\leq \frac{c_6}{rV(r)},
\end{aligned}$$

where $d^*$ is a value in $[\delta_D(y), \delta_D(z)]$, so $d^* \geq r/2$. Thus, by interpolation, we obtain

$$\left[\frac{1}{V(\delta_D)}\right]_{C^{\beta}(B(x,r/2))} \leq c_7 \left\| \frac{1}{V(\delta_D)} \right\|_{C(B(x,r/2))}^{1-\beta} \left[\frac{1}{V(\delta_D)}\right]_{C^{0,1}(B(x,r/2))}^{\beta} \leq \frac{c_8}{r^\beta V(r)}$$

and it follows from (4.1.52) that

$$\left[\frac{u}{V(\delta_D)}\right]_{C^\beta} \leq \frac{c_1 c_8}{r^\beta V(r)} + \frac{c_3 c_4}{V(r)} \leq \frac{c_9}{r^\beta V(r)}. \tag{4.1.53}$$

Next, let $x, y \in D$ and let us show that

$$\left| \frac{u(x)}{V(\delta_D(x))} - \frac{u(y)}{V(\delta_D(y))} \right| \leq C|x-y|^\alpha$$

for some $\alpha > 0$. Without loss of generality, we may assume that $r := \delta_D(x) \geq \delta_D(y)$. Fix any $0 < \beta \leq \alpha_1$ and let $p > 1 + \alpha_2/\beta$. If $|x-y| \leq r^p/2$, then we have $|x-y| \leq r/2$ and $y \in B(x, r/2)$ since $r \leq 1$. Thus, by (4.1.53) we obtain

$$\left| \frac{u(x)}{V(\delta_D(x))} - \frac{u(y)}{V(\delta_D(y))} \right| \leq \frac{c_9|x-y|^\beta}{r^\beta V(r)} \leq \frac{c_{10}|x-y|^{\beta-\beta/p}}{V(|x-y|^{1/p})} \leq c_{11}|x-y|^{\beta - \frac{\beta+\alpha_2}{p}}.$$

On the other hand, if $|x-y| \geq r^p/2$, let $x_0, y_0 \in \partial D$ be boundary points satisfying $\delta_D(x) = |x-x_0|$ and $\delta_D(y) = |y-y_0|$. Then by the oscillation lemma 4.1.25 we have

$$\left|\frac{u(x)}{V(\delta_D)(x)} - \frac{u(x_0)}{V(\delta_D)(x_0)}\right| \leq c_{12} V(\delta_D(x))^\gamma, \tag{4.1.54}$$
$$\left|\frac{u(y)}{V(\delta_D)(y)} - \frac{u(y_0)}{V(\delta_D)(y_0)}\right| \leq c_{12} V(\delta_D(y))^\gamma$$

and

$$\left|\frac{u(x_0)}{V(\delta_D)(x_0))} - \frac{u(y_0)}{V(\delta_D)(y_0)}\right| \leq c_{12} V\left(\delta_D(x) + |x-y| + \delta_D(y)\right)^\gamma. \tag{4.1.55}$$

Using inequalities (4.1.54) and (4.1.55) we obtain

$$\left|\frac{u(x)}{V(\delta_D)(x))} - \frac{u(y)}{V(\delta_D)(y)}\right| \leq c_{12}\left(2V(r)^\gamma + V(2r + |x-y|)^\gamma\right) \leq c_{13}|x-y|^{\frac{\alpha_1\gamma}{p}}.$$

Therefore, taking $\alpha = \min\{\beta - (\beta+\alpha_2)/p, \alpha_1\gamma/p\}$ gives the result. □

## 4.2 Laws of iterated logarithms

Consider a random walk $(S_n)_{n\in\mathbb{N}}$ on $\mathbb{R}$ with $\mathbb{E}S_1 = 0$ and $\mathrm{var}(S_1) = 1$. [57, 56] proved that

$$\limsup_{n\to\infty} \frac{|S_n|}{(2n\log\log n)^{1/2}} = 1 \quad \text{a.s.}$$

Moreover, for Brownian motion in $\mathbb{R}$, we have

$$\limsup_{t\to 0 \text{ or } \infty} \frac{|B_t|}{(2t\log|\log t|)^{1/2}} = 1 \quad \text{a.s.}$$

Observe that compare to the scaling property $B_t \overset{d}{=} t^{1/2}B_1$, there exists $(\log\log t)^{1/2}$ difference between distribution and the limsup or liminf distance of the sample path. Thus we call these properties *the law of iterated logarithm.*

In this time, let $B_t^* := \sup\{|B_s| : 0 \leq s \leq t\}$ be the supremum of the Brownian motion in the time interval $[0,t]$. Similarly, the following *Chung-*

*type liminf law* holds :

$$\liminf_{t\to 0 \text{ or } \infty} \frac{B_t^*}{(2t/\log|\log t|)^{1/2}} = \frac{\pi}{4} \quad \text{a.s.}$$

Various types of the laws of iterated logarithm appears for may processes. For example, let $(X_t)_{t\geq 0}$ be a symmetric $\alpha$-stable process$(0<\alpha<2)$ in $\mathbb{R}^d$ without drift and $X_t^* := \sup\{|X_s| : 0 \leq s \leq t\}$. Then, by [90, 74] we have the following Chung-type liminf law : there is $c_i > 0$, $i=1,2$ such that

$$\liminf_{t\to 0 \text{ or } \infty} \frac{X_t^*}{(t/\log|\log t|)^{1/\alpha}} = c_1 \text{ (or } c_2) \quad a.s.$$

However, for any non-decreasing function $h:(0,1)\to(0,\infty)$, we have $\limsup_{t\to 0} \frac{|X_t|}{h(t)} = \infty$ a.s. if and only if $\int_0^1 \frac{dt}{h(t)^\alpha} = \infty$, and $\limsup_{t\to 0} \frac{|X_t|}{h(t)} = 0$ a.s. if and only if $\int_0^1 \frac{dt}{h(t)^\alpha} < \infty$. Thus we conclude that

$$\limsup_{t\to 0} \frac{|X_t|}{h(t)} = 0 \text{ or } \infty$$

for any non-decreasing function $h:(0,1)\to(0,\infty)$. Similarly, for any non-decreasing function $h:(1,\infty)\to(0,\infty)$, we have $\limsup_{t\to\infty} \frac{|X_t|}{h(t)} = \infty$ a.s. if and only if $\int_1^\infty \frac{dt}{h(t)^\alpha} = \infty$, and $\limsup_{t\to 0} \frac{|X_t|}{h(t)} = 0$ a.s. if and only if $\int_0^1 \frac{dt}{h(t)^\alpha} < \infty$. See [80, Proposition 47.21] for instance.

Chung-type liminf law of the $\alpha$-stable processes are similar as Brownian motion. On the other hand, totally different form of Khintchine-type law appears at the $\alpha$-stable case.

### 4.2.1 Khintchine-type laws of iterated logarithm

In this subsection, we observe when the Khintchine-type laws of iterated logarithm at the infinity holds for some class of jump processes. Furthermore, we will also see that Khintchine-type laws of iterated logarithm imply certain finite moment condition. In particular, it is finite second moment condition in Euclidean space. This is the corollary of heat kernel estimates in [3, 2] (see Section 5 of both papers). In this thesis we only introduce the Khintchine-type laws in [2] which is more general.

Throughout this subsection, we assume that $(M, d, \mu)$ satisfies $\mathrm{Ch}(A)$, $\mathrm{RVD}(d_1)$, $\mathrm{VD}(d_2)$ and $\mathrm{Diff}(F)$, where $F$ is strictly increasing and satisfies (2.2.59) with $1 < \gamma_1 \le \gamma_2$.

Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form given by (2.2.4), which satisfies $J_\psi$ with a non-decreasing function $\psi$ satisfying (2.2.17), $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$. Recall that $X = \{X_t, t \ge 0; \mathbb{P}^x, x \in M\}$ is the $\mu$-symmetric Hunt process associated with $(\mathcal{E}, \mathcal{F})$. Recall that $\Phi$ is the function defined in (2.2.18).

We first establish the zero-one law for tail events. We say that an event $A$ is *tail event* (with respect to $X$) if $A$ is $\cap_{t>0}^{\infty}\sigma(X_s : s > t)$-measurable.

**Lemma 4.2.1.** *Let $A$ be a tail event with respect to $X$. Then, either $\mathbb{P}^x(A) = 0$ for all $x \in M$ or else $\mathbb{P}^x(A) = 1$ for all $x \in M$.*

From (2.2.18) and (2.2.17) with $\mathrm{VD}(d_2)$, we easily see that the following three conditions are equivalent:

$$\sup_{x\in M} \left(\text{or } \inf_{x\in M}\right) \int_M F(d(x,y))J(x,dy) < \infty; \tag{4.2.1}$$

$$\exists\, c > 0 \quad \text{such that} \quad c^{-1}F(r) \le \Phi(r) \le cF(r), \quad \text{for all} \quad r > 1; \tag{4.2.2}$$

$$\int_1^\infty \frac{dF(s)}{\psi(s)} < \infty. \tag{4.2.3}$$

We will show that from $\mathrm{GHK}(\Phi, \psi)$, the above conditions (4.2.1)-(4.2.3) are also equivalent to the following moment condition.

**Lemma 4.2.2.** ([2, Lemma 5.2]) *Suppose that $(M, d, \mu)$ satisfies $\mathrm{RVD}(d_1)$, $\mathrm{VD}(d_2)$ and $\mathrm{Diff}(F)$ where $F$ is strictly increasing and satisfies (2.2.17), $L(\gamma_1, c_F^{-1})$ and $U(\gamma_2, c_F)$ with $1 < \gamma_1 \le \gamma_2$. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form given by (2.2.4), which satisfies $J_\psi$ with a non-decreasing function $\psi$*

*satisfying* (2.2.17)*. Then the following is also equivalent to* (4.2.1)*-*(4.2.3)*:*

$$\sup_{x\in M}\left(\text{or }\inf_{x\in M}\right)\mathbb{E}^x[F(d(x,X_t))]<\infty,\quad \forall(\text{or }\exists)\ t>0.$$

Let us define an increasing function $h(t)$ on $[16,\infty)$ by

$$h(t):=(\log\log t)F^{-1}\big(\frac{t}{\log\log t}\big).$$

**Lemma 4.2.3.** *For any $c_1>0$, $c_2\in(0,1]$ and $t\in[16,\infty)$,*

$$F_1((c_1+1)h(t),t)\geq c_1\log\log t \tag{4.2.4}$$

*and*

$$F_1(c_2h(t),t)\leq c_F^{1/(\gamma_1-1)}c_2\log\log t. \tag{4.2.5}$$

**Proof.** By the definition of $F_1$, letting $s=h(t)(\log\log t)^{-1}$ we have that for $t\geq 16$,

$$\begin{aligned}F_1\big((c_1+1)h(t),t\big)&=\sup_{s>0}\left(\frac{(c_1+1)h(t)}{s}-\frac{t}{F(s)}\right)\\ &\geq\frac{(c_1+1)h(t)}{h(t)(\log\log t)^{-1}}-\frac{t}{F(h(t)(\log\log t)^{-1})}=c_1\log\log t.\end{aligned}$$

For (4.2.5), we fix $t>0$ and let $s_0:={c_F}^{-1/(\gamma_1-1)}h(t)(\log\log t)^{-1}\leq h(t)(\log\log t)^{-1}$. If $s\leq s_0$, using $L(\gamma_1,c_F^{-1},F)$ we have

$$\frac{s}{F(s)}\geq c_F^{-1}\left(\frac{h(t)(\log\log t)^{-1}}{s}\right)^{\gamma_1-1}\frac{h(t)(\log\log t)^{-1}}{F(h(t)(\log\log t)^{-1})}\geq\frac{h(t)}{t}\geq\frac{c_2h(t)}{t}.$$

Thus, we obtain $\frac{c_2h(t)}{s}-\frac{t}{F(s)}\leq 0$ for $s\leq s_0$. Since $F_1(r,t)>0$ for all $r,t>0$, we have

$$F_1(c_2h(t),t)=\sup_{s\geq s_0}\left(\frac{c_2h(t)}{s}-\frac{t}{F(s)}\right)\leq\frac{c_2h(t)}{s_0}=c_F^{1/(\gamma_1-1)}c_2\log\log t.$$

□

Note that if $(M,d,\mu)=(\mathbb{R}^d,|\cdot|,dm)$, we have $F(r)=r^2$ and so $h(t)=(t\log\log t)^{1/2}$. Thus, the next theorem is the counterpart of [3, Theorem 5.2].

**Theorem 4.2.4.** *Suppose that* $(M,d,\mu)$ *satisfies* $\mathrm{Ch}(A)$, $\mathrm{RVD}(d_1)$, $\mathrm{VD}(d_2)$ *and* $\mathrm{Diff}(F)$ *where* $F$ *is strictly increasing and satisfies* (2.2.17), $L(\gamma_1, c_F^{-1})$ *and* $U(\gamma_2, c_F)$ *with* $1<\gamma_1\le\gamma_2$. *Let* $(\mathcal{E},\mathcal{F})$ *be a regular Dirichlet form given by* (2.2.4), *which satisfies* $J_\psi$ *with a non-decreasing function* $\psi$ *satisfying* (2.2.17). *(i) Assume that* (4.2.1) *holds. Then there exists a constant* $c\in(0,\infty)$ *such that for all* $x\in M$,

$$\limsup_{t\to\infty}\frac{d(x,X_t)}{h(t)}=c\quad \textit{for}\quad \mathbb{P}^x\text{-}a.e. \tag{4.2.6}$$

*(ii) Suppose that* (4.2.1) *does not hold. Then for all* $x\in M$, (4.2.6) *holds with* $c=\infty$.

**Proof.** Here we only provide the proof of (i). See [2, Theorem 5.4] for the proof of (ii). Fix $x\in M$. We first observe that by (2.2.59), there exist constants $a>16$ and $c_1(a)>1$ such that for any $t\ge 16$,

$$(2c_F)^{1/\gamma_1}h(t)\le h(at)\le c_1h(t). \tag{4.2.7}$$

In particular, combining (4.2.7) and $L(\gamma_1,c_F^{-1},F)$ we have

$$2F(h(t))\le F(h(at)). \tag{4.2.8}$$

Also, using $L(\gamma_1,c_F^{-1},F)$, we have for $t\ge 16$, $\frac{F(h(t))}{t/\log\log t}=\frac{F(h(t))}{F(h(t)/\log\log t)}\ge c_F^{-1}(\log\log t)^{\gamma_1}$. Thus,

$$c_F^{-1}t(\log\log t)^{\gamma_1-1}\le F(h(t)),\quad t\ge 16. \tag{4.2.9}$$

Using (4.2.8), (4.2.7) and $U(\beta_2, C_U, \psi)$ we obtain that for $n \geq 1$,

$$\int_{h(a^n)}^{h(a^{n+1})} \frac{dF(s)}{\psi(s)} \geq \big(F(h(a^{n+1})) - F(h(a^n))\big) \frac{1}{\psi(h(a^{n+1}))} \geq c_2 \frac{F(h(a^{n+1}))}{\psi(h(a^n))}$$
$$\geq c_3 a^{n+1} \frac{(\log\log a^{n+1})^{\gamma_1 - 1}}{\psi(h(a^n))} \geq c_3 \int_{a^n}^{a^{n+1}} \frac{(\log\log t)^{\gamma_1 - 1}}{\psi(h(t))} dt.$$

In particular, this and $a > 16$ imply that

$$\int_{h(a)}^{\infty} \frac{dF(s)}{\psi(s)} ds \geq c_3 \int_a^{\infty} \frac{1}{\psi(h(t))} dt. \tag{4.2.10}$$

(i) Let $k_0 \in \mathbb{N}$ be a natural number satisfying $2^{k_0} \geq a$. By (4.2.3) and (4.2.10),

$$\sum_{k=k_0}^{\infty} \frac{2^k}{\psi(h(2^k))} \leq c_4 \sum_{k=k_0}^{\infty} \int_{2^k}^{2^{k+1}} \frac{dt}{\psi(h(t))} \leq c_4 \int_a^{\infty} \frac{dt}{\psi(h(t))} < \infty. \tag{4.2.11}$$

By (4.2.2), we have $c_8^{-1} t \geq F(\Phi^{-1}(u))$ for any $u \geq 16$ and $t \leq u \leq 4t$. Thus, we have

$$F_1(d(x,y), F(\Phi^{-1}(u))) \geq F_1(d(x,y), c_8^{-1} t) = c_8^{-1} F_1(c_8 d(x,y), t). \tag{4.2.12}$$

Using GUHK($\Phi$), VD($d_2$) and (4.2.12) we have

$$\mathbb{P}^x(d(x, X_u) > Ch(t)) = \int_{\{y: d(x,y) > Ch(t)\}} p(u,x,y)\mu(dy) \tag{4.2.13}$$
$$\leq c_5 t \int_{\{d(x,y) > Ch(t)\}} \frac{\mu(dy)}{V(x, d(x,y))\psi(d(x,y))}$$
$$+ \frac{c_5}{V(x, F^{-1}(t))} \int_{\{d(x,y) > Ch(t)\}} e^{-c_7 F_1(c_8 d(x,y), t)} \mu(dy) := c_5(I + II).$$

Let us choose $C = c_8^{-1}(1 + 4c_7^{-1})$ for (4.2.13). By [32, Lemma 2.1], we have

$I \le c_{11}\frac{t}{\psi(h(t))}$. For $II$, using VD($d_2$) and (2.2.61) we have

$$\begin{aligned}
II &= \frac{1}{V(x,F^{-1}(t))}\int_{\{d(x,y)>Ch(t)\}} \exp(-c_7F_1(c_8d(x,y),t))\mu(dy)\\
&= \frac{1}{V(x,F^{-1}(t))}\sum_{i=0}^{\infty}\int_{\{C2^ih(t)<d(x,y)\le C2^{i+1}h(t)\}} \exp(-c_7F_1(c_8d(x,y),t))\mu(dy)\\
&\le \sum_{i=0}^{\infty}\frac{V(x,C2^{i+1}h(t))}{V(x,F^{-1}(t))}\exp\Big(-c_7F_1\big((1+4c_7^{-1})2^ih(t),t\big)\Big)\\
&\le c_9\exp\Big(-\frac{c_7}{2}F_1\big((1+4c_7^{-1})h(t),t\big)\Big).
\end{aligned}$$

Note that by (4.2.9), we have $h(t) \ge cF^{-1}(t)$. Using (4.2.4), we obtain

$$II \le c_{10}\exp\Big(-\frac{c_7}{2}F_1\big((1+4c_7^{-1})h(t),t\big)\Big) \le c_{13}\exp\big(-2\log\log t\big) = c_{10}(\log t)^{-2}.$$

Thus, for $C = c_8^{-1}(1+4c_7^{-1})$ and any $t \ge 16$ and $t \le u \le 4t$, we have

$$\mathbb{P}^x(d(x,X_u) > Ch(t)) \le c_{11}\big(\frac{t}{\psi(h(t))} + (\log t)^{-2}\big).$$

Using this and the strong Markov property, for $t_k = 2^k$ with $k \ge k_0+1$ we get

$$\begin{aligned}
&\mathbb{P}^x(d(x,X_s) > 2Ch(s) \text{ for some } s\in[t_{k-1},t_k]) \le \mathbb{P}^x(\tau_{B(x,Ch(t_{k-1}))} \le t_k)\\
&\le 2\sup_{s\le t_k, z\in M}\mathbb{P}^z\big(d(z,X_{t_{k+1}-s}) > Ch(t_{k-1})\big) \le c_{12}\left(\frac{1}{k^2}+\frac{2^k}{\psi(h(2^k))}\right).
\end{aligned}$$

Thus, by (4.2.11) and the Borel-Cantelli lemma, the above inequality implies that

$$\mathbb{P}^x(d(x,X_t) \le 2Ch(t) \text{ for all sufficiently large } t) = 1.$$

Thus, $\limsup_{t\to\infty}\frac{d(x,X_t)}{h(t)} \le 2C$.

On the other hand, by (4.2.2) and $L(\gamma_1, c_F^{-1}, F)$, we have $L^1(\gamma_1, c_L, \Phi)$ with some $c_L > 0$. Also, by (2.2.60) we have $U(\gamma_2, c_F, \Phi)$. Since $\gamma_1 > 1$, using

(2.2.53) we have for any $x, y \in M$ and $t \geq T$,

$$p(t, x, y) \geq c_{13} V(x, \Phi^{-1}(t))^{-1} \exp\big( - a_L \widetilde{\Phi}_1(d(x, y), t)\big), \qquad (4.2.14)$$

where $\widetilde{\Phi}(r) = r^{\gamma_2}\Phi(1)\mathbf{1}_{\{r<1\}} + \Phi(r)\mathbf{1}_{\{r\geq 1\}}$ and $\widetilde{\Phi}_1(r, t) = \mathcal{T}(\widetilde{\Phi})(r, t)$ are the functions defined in (2.2.10) and (2.2.12). Note that by $U(\gamma_2, c_F, F)$ we have $\widetilde{\Phi}(r) = r^{\gamma_2}\Phi(1) \leq c_F \frac{\Phi(1)F(r)}{F(1)}$ for $r < 1$. Using this and (4.2.2), we obtain that $\widetilde{\Phi}(r) \leq cF(r)$ for all $r > 0$. Thus, by the definitions of $\widetilde{\Phi}_1$ and $F_1$ we obtain

$$\widetilde{\Phi}_1(r, t) \leq F_1(r, \frac{t}{c}), \quad r, t > 0. \qquad (4.2.15)$$

Combining (4.2.14) and (4.2.15), we have that for all $c_0 \in (0, 1), t \geq 16$ and $t \leq u \leq 4t$,

$$\begin{aligned}\mathbb{P}^x(d(x, X_u) > c_0 h(t)) &= \int_{\{d(x,y)>c_0 h(t)\}} p(u, x, y)\mu(dy) \\ &\geq \frac{c_{16}}{V(x, \widetilde{\Phi}^{-1}(u))} \int_{\{d(x,y)>c_0 h(t)\}} e^{-a_L \widetilde{\Phi}_1(d(x,y),u)}\mu(dy) \\ &\geq \frac{c_{16}}{V(x, F^{-1}(t))} \int_{\{d(x,y)>c_0 h(t)\}} e^{-a_L F_1(d(x,y),\frac{u}{c})}\mu(dy).\end{aligned}$$

Note that by $\mathrm{RVD}(d_1)$, we have a constant $c_{17} > 0$ such that

$$V(x, c_{17} r) \geq 2V(x, r), \quad \text{for all} \quad x \in M, \ r > 0.$$

Thus, using this and (4.2.9) we have that for $u \geq t$,

$$\begin{aligned}&\frac{1}{V(x, F^{-1}(t))} \int_{\{d(x,y)>c_0 h(t)\}} e^{-a_L F_1(d(x,y),\frac{u}{c})}\mu(dy) \\ &\geq \frac{1}{V(x, F^{-1}(t))} \int_{\{c_0 h(t)<d(x,y)\leq c_0 c_{17} h(t)\}} e^{-a_L F_1(d(x,y),\frac{u}{c})}\mu(dy) \\ &\geq \frac{V(x, c_0 h(t))}{V(x, F^{-1}(t))} \exp\big( - a_L F_1(c_0 c_{17} h(t), tc^{-1})\big) \\ &\geq c_0^{d_2} C_\mu^{-1} \exp\big( - c_{18} F_1(c_0 c_{19} h(t), t)\big).\end{aligned}$$

Since the constants $c_{16}, c_{18}, c_{19}$ are independent of $c_0$, provided $c_0 > 0$ small and using (4.2.5), we have

$$\begin{aligned} \mathbb{P}^x(d(x, X_u) > c_0 h(t)) &\geq c_{16} \exp\big(-c_{18} F_1(c_0 c_{19} h(t), t)\big) \\ &\geq c_0^{d_2} c_{16} C_\mu^{-1} \exp\big(-c_{20} c_0 \log\log t\big) \geq c_0^{d_2} c_{16} C_\mu^{-1} (\log t)^{-1/2}. \end{aligned} \tag{4.2.16}$$

Thus, by the strong Markov property and (4.2.16), we have

$$\sum_{k=1}^{\infty} \mathbb{P}^x(d(X_{t_k}, X_{t_{k+1}}) \geq c_0 h(t_k) | \mathcal{F}_{t_k}) \geq \sum_{k=4}^{\infty} c_0^{d_2} c_{16} C_\mu^{-1} (\log t_k)^{-1/2} = \infty.$$

Thus, by the second Borel-Cantelli lemma,

$$\mathbb{P}^x(\limsup\{d(X_{t_k}, X_{t_{k+1}}) \geq c_0 h(t_k)\}) = 1.$$

Whence, for infinitely many $k \geq 1$, $d(x, X_{t_{k+1}}) \geq \frac{c_0 h(t_k)}{2}$ or $d(x, X_{t_k}) \geq \frac{c_0 h(t_k)}{2}$. Therefore, for all $x \in M$,

$$\limsup_{t \to \infty} \frac{d(x, X_t)}{h(t)} \geq \limsup_{k \to \infty} \frac{d(x, X_{t_k})}{h(t_k)} \geq c_{21}, \quad \mathbb{P}^x\text{-a.e.},$$

where $c_{21} > 0$ is the constant satisfying $c_{21} h(2t) \leq \frac{c_0}{2} h(t)$ for any $t \geq 16$. Since

$$\mathbb{P}^x(c_{21} \leq \limsup_{t \to \infty} \frac{d(x, X_t)}{h(t)} \leq 2C) = 1,$$

by Lemma 4.2.1 there exists a constant $c > 0$ satisfying (4.2.6).

### 4.2.2 Chung-type laws of iterated logarithm

In this subsection, we introduce my ongoing research project, that is, Chung-type liminf laws of iterated logarithm for Markov processes. We focus on such laws in general metric measure spaces under assumptions locally either near zero or near infinity. We show that, when the transition densities satisfy the near-diagonal lower bound for small time (for large time, respectively) in terms of a scale function $\phi$, then Chung-type liminf law of iterated logarithm at zero (at infinity, respectively) holds in terms of the scaling function $\phi$.

Let us now describe the main result of this paper more precisely and at the same time fix the setup and notation of the paper. As in the previous section we will assume that $(M, d)$ is a locally compact separable metric space, and $\mu$ is a positive Radon measure on $M$ with full support. We denote $B(x, r) := \{y \in M : d(x, y) < r\}$ and $V(x, r) := \mu(B(x, r))$ an open ball in $M$ and its volume, respectively. We add a cemetery point $\partial$ to $M$ and define $M_\partial := M \cup \{\partial\}$ . We first introduce local versions of volume doubling properties in Definition 1.1.6 for the metric measure space $(M, d, \mu)$.

**Definition 4.2.5.** (i) For an open set $U \subset M$, $R_0 \in (0, \infty]$ and $d_2 > 0$, we say that $(M, d, \mu)$ satisfies the *volume doubling property* $\mathrm{VD}_{R_0}(d_2, U)$ if there exists a constant $C_\mu \geq 1$ such that

$$\frac{V(x, R)}{V(x, r)} \leq C_\mu \left(\frac{R}{r}\right)^{d_2} \quad \text{for all } x \in U \text{ and } 0 < r \leq R < R_0.$$

For an open set $U \subset M$, $R_0 \in (0, \infty]$ and $d_1 > 0$, we say that $(M, d, \mu)$ satisfies the *reverse volume doubling property* $\mathrm{RVD}_{R_0}(d_1, U)$ if there exists a constant $c_\mu > 0$ such that

$$\frac{V(x, R)}{V(x, r)} \geq c_\mu \left(\frac{R}{r}\right)^{d_1} \quad \text{for all } x \in U \text{ and } 0 < r \leq R < R_0.$$

For simplicity, we write $\mathrm{VD}(d_2)$ and $\mathrm{RVD}(d_1)$ instead of $\mathrm{VD}_\infty(d_2, M)$ and $\mathrm{RVD}_\infty(d_1, M)$.

(ii) For $R_\infty > 0$ and $d_2 > 0$, we say that $(M, d, \mu)$ satisfies the *volume doubling property* $\mathrm{VD}^{R_\infty}(d_2)$ if there exists a constant $C_\mu \geq 1$ such that

$$\frac{V(x, R)}{V(x, r)} \leq C_\mu \left(\frac{R}{r}\right)^{d_2} \quad \text{for all } x \in M \text{ and } R_\infty \leq r \leq R.$$

For $R_\infty > 0$ and $d_1 > 0$, we say that $(M, d, \mu)$ satisfies the *reverse volume* *doubling property* $\mathrm{RVD}^{R_\infty}(d_1)$ if there exists a constant $c_\mu > 0$ such that

$$\frac{V(x,R)}{V(x,r)} \geq c_\mu \left(\frac{R}{r}\right)^{d_1} \quad \text{for all } x \in M \text{ and } R_\infty \leq r \leq R.$$

We assume that $X = (\Omega, \mathcal{F}_t, X_t, \theta_t, t \geq 0; \mathbb{P}^x, x \in M_\partial)$ is a Borel standard Markov process on $M$. Here $(\theta_t, t \geq 0)$ is the shift operator with respect to the process $X$ which is defined as $X_s(\theta_t \omega) = X_{s+t}(\omega)$ for all $t, s > 0$ and $\omega \in \Omega$.

A family of $[0,\infty]$-valued random variables $(A_t, t \geq 0)$ is called a (perfect) positive *continuous additive functional* (PCAF) of the process $X$, if $A$ satisfies that $A_0 = 0$, $A_t(\cdot)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$, $t \mapsto A_t$ is continuous in $[0,\infty)$ a.s. and

$$A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t \omega) \text{ for all } s,t \geq 0 \text{ and a.s. } \omega \in \Omega$$

See [41] for detail.

Since $X$ is a standard process on $M$, it is known that $X$ admits a Lévy system $(N,H)$. (see [15, 83].) Here $H$ is a PCAF of $X$ with bounded 1-potential and $N(x,dy)$ is a kernel on $M$ with $N(x,\{x\}) = 0$ for all $x \in M$.

We assume that there exists a version of Lévy system $(N,H)$ of $X$ where Revuz measure $\nu_H$ of $H$ is absolutely continuous with respect to the reference measure $\mu$. Thus, there exists a nonnegative measurable function $\nu_H(x)$ such that $\nu_H(dx) = \nu_H(x)\mu(dx)$. Let $J(x,dy) = N(x,dy)\nu(x)$. Then, we see from the Lévy system that for any nonnegative Borel function $F$ on $M \times M_\partial$ vanishing on the diagonal, $z \in M$ and $t > 0$,

$$\mathbb{E}^z\left[\sum_{s\leq t} F(X_{s-},X_s)\right] = \mathbb{E}^z\left[\int_0^t \int_{M_\partial} F(X_s,y)J(X_s,dy)ds\right].$$

The measure $J(x,dy)$ is called the *Lévy measure* of the process $X$ in the literature. See [91]. We emphasize here that $J(x,dy)$ can be identically zero and $J(x,dy)$ may not be absolutely continuous with respect to the reference measure $\mu$. Let $(P_t)_{t\ge 0}$ be the associate semigroup of $X$ in $L^2(M;\mu)$, defined by $P_t f(x) := \mathbb{E}^x[f(X_t)]$ for any $t>0$, $x\in M$ and bounded Borel measurable function $f$.

We call that a function $p:(0,\infty)\times M\times M\to[0,\infty]$ is *heat kernel* of semigroup $(P_t)_{t\ge 0}$ if the followings hold :

1. for any $t>0$, $x\in M$ and any bounded Borel measurable function $f$ vanishing at infinity,

$$P_t f(x) = \int_M p(t,x,y)f(y)\mu(dy).$$

2. For any $t,s>0$ and $x,y\in M$,

$$p(t+s,x,y) = \int_M p(t,x,z)p(t,z,y)\mu(dz).$$

For an open subset $D\subset M$, we denote the heat kernel of killed process $X^D$ by $p^D(t,x,y)$. Note that if heat kernel $p(t,x,y)$ exists, $p^D(t,x,y)$ also exists. From now on, we always assume that the function $\phi:[0,\infty)\to[0,\infty)$ is an increasing function with $\phi(0)=0$.

The following local versions of the condition NDL in Section 2.2 will be one of the main assumptions of this paper.

**Definition 4.2.6.** (i) For an open set $U\subset M$ and $R_0\in(0,\infty]$, we say that the condition $\mathrm{NDL}_{R_0}(\phi,U)$ holds if the heat kernel $p(t,x,y)$ exists and there are constants $c>0$ and $\eta\in(0,1)$ such that for any $x\in U$, $r<R_0$ and $0<t\le\phi(\eta r)$,

$$p^{B(x,r)}(t,y,z) \ge \frac{c}{V(x,\phi^{-1}(t))}, \qquad y,z\in B(x,\eta\phi^{-1}(t)).$$

For simplicity, we write $\mathrm{NDL}(\phi)$ instead of $\mathrm{NDL}_\infty(\phi,M)$.
(ii) For $R_\infty\in[0,\infty)$, we say that the condition $\mathrm{NDL}^{R_\infty}(\phi)$ holds if the heat kernel $p(t,x,y)$ exists and there are constants $\varepsilon,\eta\in(0,1)$ and $c>0$ such

that for any $x \in M$, $r > R_\infty$ and $\varepsilon\phi(\eta r) \le t \le \phi(\eta r)$,

$$p^{B(x,r)}(t,y,z) \ge \frac{c}{V(x,\phi^{-1}(t))}, \qquad y,z \in B(x,\eta\phi^{-1}(t)).$$

We will also consider local assumptions on upper bounds of tails of Lévy measure $J$.

**Definition 4.2.7.** (i) For an open set $U \subset M$ and $R_0 \in (0,\infty]$, we say that the condition $\mathrm{Tail}_{R_0}(\phi,U)$ holds if there exist $c > 0$ such that for any $x \in U$ and $r < R_0$,

$$J(x, M_\partial \setminus B(x,r)) \le \frac{c}{\phi(r)}.$$

For simplicity, we write $\mathrm{Tail}(\phi)$ instead of $\mathrm{Tail}_\infty(\phi,M)$.
(ii) For $R_\infty \in [0,\infty)$, we say that the condition $\mathrm{Tail}^{R_\infty}(\phi)$ holds if for any $x \in M$ and $r \ge R_\infty$,

$$J(x, M_\partial \setminus B(x,r)) \le \frac{c}{\phi(r)}.$$

Note that $\mathrm{Tail}(\phi)$ holds trivially if the process $X$ is a conservative diffusion process.

Here is the main results.

**Theorem 4.2.8.** *Let $\phi : (0,\infty) \to (0,\infty)$ be an increasing function.*
*(i) For $R_0 \in (0,\infty]$ and an open subset $U \subset M$, assume that the conditions* $\mathrm{VD}_{R_0}(d_2,U)$, $\mathrm{RVD}_{R_0}(d_1,U)$, $L_1(\beta_1,C_L,\phi)$, $U_1(\beta_2,C_U,\phi)$, $\mathrm{NDL}_{R_0}(\phi,U)$ *and* $\mathrm{Tail}_{R_0}(\phi,U)$ *hold. Then, there exist constants $c_1,c_2 \in (0,\infty)$ such that for all $x \in U$, there exists a constant $c_x \in [c_1,c_2]$ satisfying*

$$\liminf_{t\to 0} \frac{\sup_{0<s\le t} d(X_s,x)}{\phi^{-1}(t/\log|\log t|)} = c_x, \qquad \mathbb{P}^x\text{-}a.s.$$

*(ii) For $R_\infty \in [0,\infty)$, assume that the conditions* $\mathrm{VD}^{R_\infty}(d_2)$, $\mathrm{RVD}^{R_\infty}(d_1)$, $L^1(\beta_1,C_L,\phi)$, $U^1(\beta_2,C_U,\phi)$, $\mathrm{NDL}^{R_\infty}(\phi)$ *and* $\mathrm{Tail}^{R_\infty}(\phi)$. *Then, there exists a*

*constant* $c \in (0, \infty)$ *such that for all* $x, y \in M$,

$$\liminf_{t\to\infty} \frac{\sup_{0<s\le t} d(X_s, y)}{\phi^{-1}(t/\log\log t)} = c, \qquad \mathbb{P}^x\text{-}a.s.$$

Our results extend [59, Theorems 3.7 and 3.8] where two-sided and mixed stable-like heat kernel estimates were assumed. See [59, (3.17)].

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# 국문초록

이 학위논문에서는 다양한 마코브 확률과정의 열 커널에 대한 추정치와 그 응용에 대해 알아본다. 먼저 일반적인 거리공간에서의 대칭 헌트 확률과정에 대한 열 커널을 알아본 뒤, 유클리드 공간에서의 비대칭 점프 확률과정에 대한 열 커널을 구하는 레비의 방법론에 대해서 알아보고자 한다. 또한, 이 학위논문에서는 가장 대표적으로 알려진 열 커널 추정의 응요인 그린함수와 하낙 부등식 이외에도, 비국소 작용소의 유한한 도메인에 대한 푸아송 방정식의 해가 경계 근방에서 어떠한 속도로 감소하는지에 대해서 알아본다. 마지막으로, 브라우니안 모션에서 잘 알려진 법칙인 반복 로그 법칙을 열 커널 추정치를 이용하여 일반적인 확률과정에 대해 얻는 방법에 대해 알아보고자 한다.

**주요어휘:** 마코브 확률과정, 열커널 추정, 디리클렛 폼, 비국소 연산자, 반복 로그 법칙, 그린 함수
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