Master's Thesis of Economics

The Best-of-Five Rounds' Contest with Asymmetric Players

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February 2021

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Submitting a master's thesis of
Economics

February 2021

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Economics

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Confirming the master's thesis written by
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February 2021

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Abstract

This paper shows the solution of the fully rational model of the Best-of-Five rounds' contest between asymmetric players. The complexity of the equilibrium in a fully rational model of the best-of-N contest between asymmetric players rapidly increases as N increases, so we find the general form of the solution first and use them to get the equilibrium at each stage. We will see how the asymmetry between the players affects the probability of winning at each stage and what characteristics of equilibrium have. We find that the results are consistent with those in the previous study, which used the Best-of-Three model with asymmetric players. We suggest the reason why the Best-of-Five model is better to distinguish the effect of strategic momentum and psychological momentum. Also, We suggest testable hypotheses, interpretations of results, and theoretical frequencies calculated based on the equilibrium as a benchmark for the test using the professional tennis data like that have done in the previous study.

Keyword: Best-of-N, Best-of-Five, Asymmetric Players, Strategic Momentum, Nash Equilibrium

Student Number : 2016-20145

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Extension from the Best-of-Three Rounds model

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January 7, 2021

1 Introduction

The Best-of-N contest is the contest where contestants compete over consecutive rounds until the winner is determined by winning a majority of the N rounds. When we watch the broadcast of big, famous sports events that have best-of-N structure, like Major League Baseball World Series, Korean Series of Korean Baseball Organization, League of Legends World Championship Tournament, major tennis tournaments, or the final stage of major Go tournaments, commentators often mention the importance of the first game(or round, set) with the statistical data on the screen that shows how frequently the winner of the first game won the title. Korean Series of the KBO is a good example of it. Korean Series is the best-of-seven contest, and by 2020, the percentage of winning the series title by the team that won the first game was 73.7%(28/38). The first two consecutive wins increase this percentage to 89.5%(17/19). The team with the first three

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consecutive wins always got the title (11/11). Also, out of those eleven cases, eight Korean series ended with four consecutive wins. Though they are not a rigorous statistical analysis that shows the causal relationship, these reports easily make us believe that there exists momentum that belongs to the winner of the earlier round since the frequency that supports the comment is high enough even the advantageous situation is considered. This is one example of the common perception that momentum generated in the previous round has a powerful influence on subsequent performance. The effect of the result in one round on the players' performance in subsequent rounds is a subject of great interest to both economists and behavioral researchers. There are two distinct types of momentum that are commonly considered: strategic momentum and psychological momentum.

Psychological momentum is a commonly mentioned concept in sports broadcasting and, thus, commonly perceived as a major cause of winning streak. Psychological momentum is quite a nebulous concept even in the psychology literature but can be summed up as 'success breeds success' informally. According to the psychological momentum, even two players have the same ability, the player who won the previous round more likely to win the current round thanks to the enhanced confidence. Of course, this momentum may shift to the opponent as the player with the momentum loses. Many papers emphasize the role of psychological momentum. Iso-Ahola and Dotson (2016) is one of them and argues that psychological momentum is a key to continued success using the sports data. On the other hand, strategic momentum exists when the result of the current round affects players' incentives in the next round. Contest theory, which was formulated by Tullock (1980) and Rosen (1986), assumes that the probability of winning is determined by players' effort level and ability. Players choose their effort level to maximize their expected payoff: prize minus cost of effort in each round. In this setting, the results of the previous round affect the net incentives for winning the next round. Even without the existence of psychological momentum, the player who won the previous round faces a bigger net prize since fewer rounds are needed to win, which means less effort is needed. Thus, he exerts more effort than the opponent and more likely to win.

At first glance, those two kinds of momentum look similar in the sense that the winner of the previous round has a kind of momentum, and it improves the player's performance in the next round. However, there is a big difference. Let us think of the best-of-three contest between the homogeneous players. If the game goes to Round 3, the score is tied, 1:1, psychological momentum belongs to the winner of Round 2, thus he is more likely to win the round and win the game. However, in a fully rational model of best-of-three between homogeneous players, the player who won Round 1 has strategic momentum in Round 2, but once they go to Round 3, both players exert the same level of effort since they are tied in a score again and face the same prize no matter what happened in the previous round. Neither one of them is more likely to win in Round 3. The tendency that the winner of the first round is more likely to win the next round can be explained not only by the psychological momentum as commonly mentioned but also by the strategic momentum in a simple rational model. Malueg and Yates(2010) outlined a fully rational model that assumed the homogeneous players and tested using professional tennis data, using Shin probabilities.[1] Depken, Gandar, and Shapiro (2020) (DGS hereafter) pointed out that it is a more common case where two players have different abilities, and it becomes more difficult to distinguish those two kinds of momentum. They expanded the framework by introducing the asymmetry in players' ability in the theoretical model and tested using bigger data than the previous study. In this paper, we will solve the best-of-five model that uses the same assumption with the DGS and suggest it as a benchmark to test the data of best-of-five contests. There are some points that make this model extension worthy. First, men's tennis is a best-of-five contest, so we have big data that

[1] It is a method to calculate the implied probabilities from the betting odds using iterative approach. It allows estimating the theoretical size of insider betting. It is well known for its great accuracy, so the implements which use this method is widely used by sports bettor. This makes ex-ante winning probabilities effectively observable.

is not still used. Second, we can make an inference on what will happen in the longer formats. Lastly, as we will see later, there exist score states that make the best-of-five model is better to distinguish the effect of two momenta than the best-of-three model.

2 Literature Review

The existence of strategic and psychological momentum in the best-of-N contest is a controversial issue. Many empirical analyses have done using the data of various kinds of sports to find which momentum is more effective in the real contest, especially in sports. In the case of strategic momentum, Ferrall and Smith (1999) find little evidence of strategic effects in championship series in professional baseball, basketball, and hockey. Ozbeklik and Smith (2017) find evidence of strategic risk-taking behavior in single-elimination golf tournaments. Mago, Sheremeta, and Yates (2013) find evidence of strategic momentum rather than psychological momentum in best-of-three tennis records. Malueg and Yates (2010) also use best-of-three tennis data but consider contestants' ability using betting odds to identify equally skilled opponents and explain why these mixed results happen. Depken, Gandar, and Shapiro (2020) expand this framework by adding heterogeneity between players in best-of-three and also find evidence of strategic momentum. We apply the same theoretical model considering the heterogeneity to a best-of-five case in this paper.

The literature on psychological momentum in the best-of-N contest is mostly in the context of the idea that 'success breeds success'. When psychological momentum works in the best-of-N contest, the winner of the first round has the momentum, therefore, he is more likely to win the second round. But if the contest goes to the third round, now the momentum switches to the opponent. Like the studies on strategic momentum, empirical studies suggest divergent evidence on psychological momentum. Gilovich et al.

(1985) find that the concept 'Hot hand'[2] in basketball turns out to be a misperception. Iso-Ahola and Mobily (1980) find archival data from the racquetball tournament supports the psychological momentum hypothesis on the first game. Gayton, et al. (1993) find that scoring first or winning the first of three periods in hockey matches has a positive effect on the probability of winning the match. Page (2011) finds evidence that barely winning the first set of a tennis match through a tie-breaker has a positive effect on the probability of winning the next set. Jordan (2014) finds that in the MLB World Series, winning the first game increases the odds of winning the title. However, other studies argue that psychological momentum doesn't occur after winning the first round. Ferrall and Smith (1999) find that, after controlling the team quality using the regular-season records and past appearance in the championship series, the result of the first-round doesn't make momentum for the next round in the championship series of professional basketball, baseball, and hockey. Berger and Pope (2011) show that being slightly behind can actually increase success by increasing motivation. Using college and professional basketball records, they find that the team barely losing the first half can actually increase the odds that the team wins the second half so that they win the game. While some studies have found that the winner of the second round has psychological momentum that makes him more likely to win in the third round, other studies have shown that after two players are tied after two rounds, neither player has momentum. Richardson, Adler, and Hankes (1988) do not find psychological momentum in 163 collegiate tennis matches and see that psychological momentum is a highly individual matter. Silva, Hardy, and Crace find no evidence of psychological momentum in the third round of intercollegiate tennis records. Page (2011) finds that the winner of the second round in a tennis match has bigger odds of winning the next round and subsequent sets in best-of-three and best-of-five matches.

Vergin (2000) finds no evidence of winning streaks by a momentum based on the

[2] It means, a player who was successful in previous attempt is more likely to score.

assumption that each team has an equal probability of winning any particular game. However, Depken, Gandar, and Shapiro (2020) say this assumption is unreasonable since, in most games, the probability of winning any particular game is not the same for both teams involved emphasizing that heterogeneity of the ability of players is underestimated in many cases. Arkes and Martinez (2011) discuss the limitation of this assumption in detail. Meier et al. (2020) see strategic momentum and psychological momentum coexist and distinguish those effects by employing exogenously given interruptions, converted breakpoints, that only affect psychological momentum.

There are papers which concern the players' different abilities. Jackson and Mosurski (1997) used relative world rankings as a proxy for relative ability and find evidence of psychological momentum in Wimbledon and U.S. open matches. Malueg and Yates (2010) test best-of-three contest theory using 351 ATP single matches data from 2001 through 2007. Matches in this sample are chosen to be regarded as the competition by the players of equal ability measured by betting odds. They find that the first set winner also wins the second set in 64 percent of their matches. Moreover, when the players go to the third set, which means players are tied, they have an equal chance of winning. These accord to the predictions of contest theory.

Depken, Gandar, and Shapiro (2020) introduced the asymmetry of ability between the players in the contest theory model and tested it with expanded sample data, including the matches between the favorite and underdog. They find that both strategic momentum and psychological momentum contribute to the outcomes of the best-of-three tennis contests. We will discuss further this paper, Depken, Gandar, and Shapiro (2020), and compare the results we have in the main part.

3 Model

3.1 Elements of the Best-of-Five game

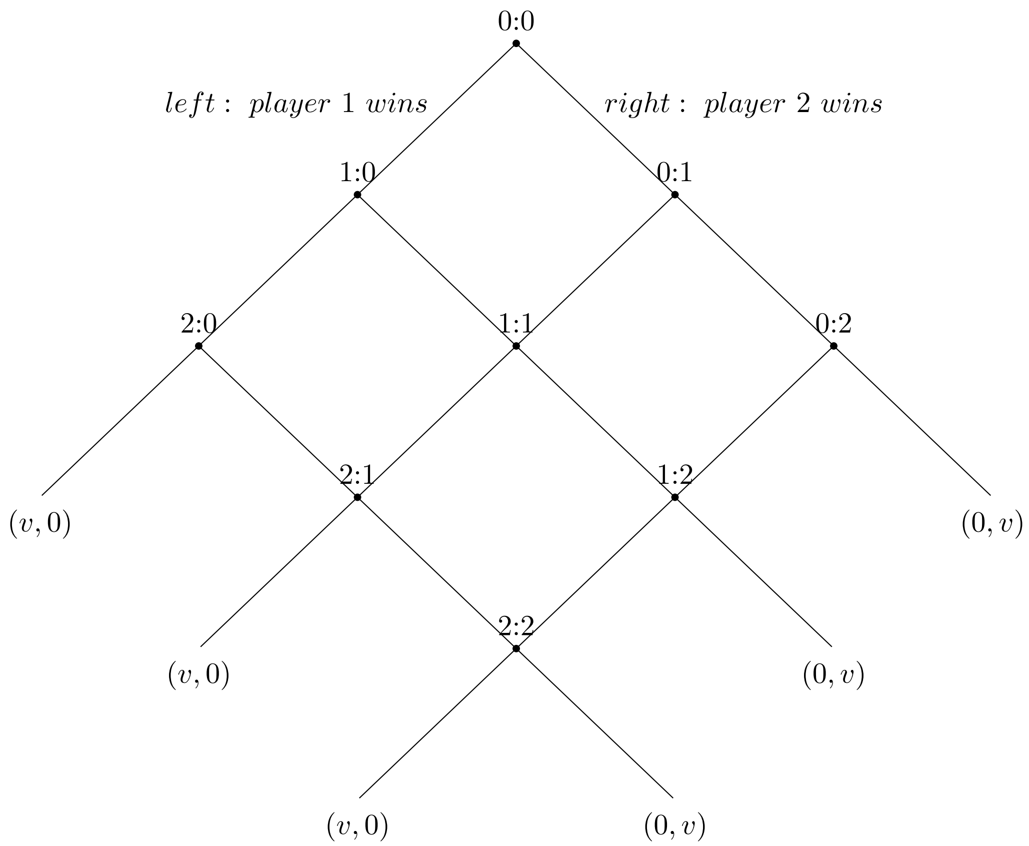
Best-of-five is common in professional men's tennis, which is the most famous section. Especially, men's tennis section in four grand slam tournaments, the Davis Cup, known as the World Cup of Tennis, and the Summer Olympics are on a best-of-five basis. Two players play the best-of-five contest, which means players compete up to 5 matches until any player wins three games first.

We model the best-of-five contest as follows. We assume that the winner of this contest gets payoff v, and the loser gets nothing. The players are heterogeneous in ability. In each round, players decide their effort level simultaneously. We denote x and y as the effort level of player 1 and 2 each. In this paper, the superscript denotes the player, and the subscript denotes the score state.[3] For a given score and efforts, the player 1's probability of winning at j:k score state, $p^1_{j:k}$, determined as

$$p^1_{j:k} = \frac{ax_{j:k}}{ax_{j:k} + y_{j:k}} \tag{3.1}$$

where a represents the degree of heterogeneity in ability. In what follows, we assume $a \geq 1$ here, which means player 1 is more potent than player 2. In this sense, we will call player 1 as *the favorite* and player 2 as *the underdog*. Analyses in this paper are done from the perspective of player 1. We can also consider the cases of $a = 1$ or $a < 1$ in analyses to consider a homogeneous case and an underdog case if needed.

[3] Notations used in this paper are listed in Appendix A.



Numbers right above the node shows the score state. (Number on the left is the score of player 1) The game tree of the best-of-five contest consists of 19 nodes; however, since the subgames below the same score states are all precisely the same and the path does not affect the result of the subgame, we can visualize the game structure like the image above, though it is not the rigorous game tree, for the sake of simplification.[4] It means nodes in the actual game tree can be categorized with the score state, so we can focus on those nine kinds of score states to solve subgame perfect equilibrium. Players adjust their effort level to maximize the expected prize minus their cost of effort. We denote the *expected continuation value of player i at score $j:k$* as $CV^i_{j:k}$ which is the equilibrium expected payoff of player i at $j:k$ score state, and it has a unique value as will be shown

[4] In section 3.2, we will see that for any given continuation values, there exists unique equilibrium.

in Section 3.2. By its definition,

$$CV^1_{j:k} = \max_{x_{j:k}} \ p^1_{j:k} \cdot CV^1_{j+1:k} + (1 - p^1_{j:k}) \cdot CV^1_{j:k+1} - x_{j:k}$$

$$CV^2_{j:k} = \max_{y_{j:k}} \ p^2_{j:k} \cdot CV^2_{j:k+1} + (1 - p^2_{j:k}) \cdot CV^2_{j+1:k} - y_{j:k}$$

where the expression of $p^1_{j:k}$ is equation (3.1). We can see that the current stage's effort level does not affect the continuation values of the next stage, so we can regard them as given constant.

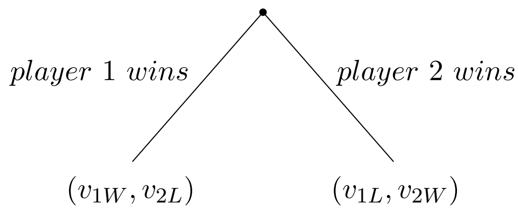
What we are most interested in here is the equilibrium winning probabilities at each score state. We will use the definition below when we compare the winning probabilities later on.

Definition 1 *For player 1, between two score states - $j : k$ and $l : m$, state $j : k$ is* ***more favorable*** *than $l : m$ when $j \geq l$, $k < m$, or $j > l$, $k \leq m$*

The definition is quite intuitive. For example, 1:0 is more favorable than 0:0 for player 1, and 0:2 is more favorable than 1:1 for player 2 according to the **Definition 1**. We can regard one's own point as *a good* and the opponent's point as *a bad*. Then, the definition of 'more favorable' is quite like the monotonicity of the preference. However, we cannot always compare the score states in this sense, for example, 1:0 and 2:1.

3.2 Equilibrium in an one-stage game

In equilibrium, the continuation value (expected payoff) for each player exists at every node. We will use simplified notations in this and the next subsection to see how the equilibrium looks like for given continuation values. We denote x and y as effort level of each player, $v_{iW}(v_{iL})$ as continuation value when player i wins (loses), p_i as the probability of winning of player i and d_i as the difference between the continuation values of player i.



Using the assumptions from the previous section, we can write the maximization problem of both players:

$$\max_x Eu_1 = p_1 v_{1W} + p_2 v_{1L} - x = \frac{ax}{ax+y} v_{1W} + \frac{y}{ax+y} v_{1L} - x$$
$$\max_y Eu_2 = p_2 v_{2W} + p_1 v_{2L} - y = \frac{y}{ax+y} v_{2W} + \frac{ax}{ax+y} v_{2L} - y$$

We solve for equilibrium of the one-stage game in Appendix A:

$$x^* = -\frac{a\left(-v_{1W}+v_{1L}\right)^2\left(v_{2L}-v_{2W}\right)}{\left(av_{1W}-av_{1L}+v_{2W}-v_{1L}\right)^2} = \frac{ad_1^2 d_2}{\left(ad_1+d_2\right)^2}$$

$$y^* = -\frac{a\left(v_{2L}-v_{2W}\right)^2\left(-v_{1W}+v_{1L}\right)}{\left(av_{1L}-av_{1W}+v_{2L}-v_{2W}\right)^2} = \frac{ad_1\, d_2^2}{\left(ad_1+d_2\right)^2}$$

where $d_i = v_{iW} - v_{iL}$. The difference between the continuation value is always positive since the player who won the previous round needs fewer rounds to win the contest than when he lost the previous round and thus needs less effort to win the game.
Given the equilibrium efforts above, we can calculate the probability of player i, p_i^*, to win the subgame.

$$p_1^* = \left(1 + \frac{d_2}{ad_1}\right)^{-1} = \left(1 + \frac{1}{a \cdot dr}\right)^{-1},$$

$$p_2^* = \left(1 + \frac{ad_1}{d_2}\right)^{-1} = (1 + a \cdot dr)^{-1},$$

where $dr = \frac{d_1}{d_2}$.

We can see here that the difference between the continuation values fully determines the equilibrium effort level and the probability of winning. Moreover, once we know the ratio of each player's difference in continuation values, we can compare the equilibrium effort level and the probability of winning between two players.

Proposition 1 *In equilibrium, the ratio of effort levels and winning probabilities are fully determined by the ratio of the difference in continuation values:*

$$\frac{x^*}{y^*} = \frac{d_1}{d_2} = dr$$

$$\frac{p_1^*}{p_2^*} = a\frac{d_1}{d_2} = a \cdot dr$$

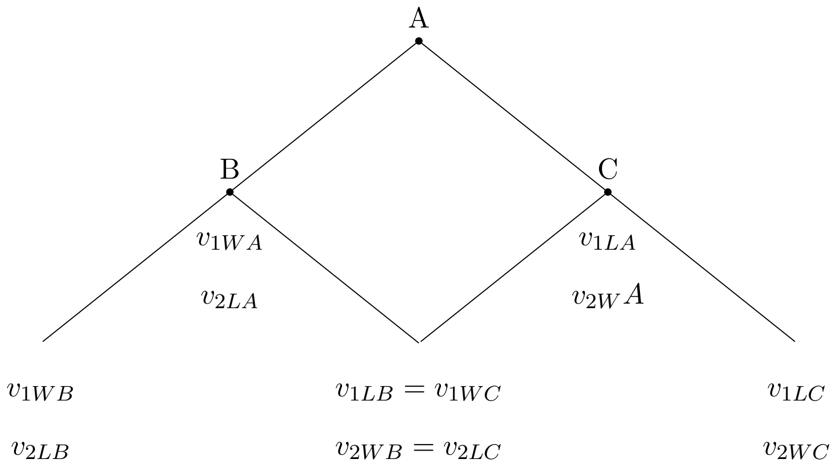
We see that given any continuation values, there exists a unique equilibrium. If there are multiple equilibria for given continuation values, equilibrium may depend on history, but we have a unique equilibrium in this model.

Corollary 1 *In the absence of psychological momentum, the previous history of each node (e.g., 1:1, 2:1, 1:2, 2:2) does not matter.*

We can think of two extreme paths to get to Round 5, score 2:2. One is, player 1 wins the first two rounds, and then player 2 wins two rounds in the streak, and vice versa. Psychological momentum will work in a totally different direction in those cases in Round 5. However, strategic momentum is generated by the incentive, regardless of history. This corollary is a testable hypothesis itself to see the existence of strategic momentum.

3.3 Equilibrium in a two-stage game

In the two-stage game, we have six continuation values. We have three kinds of the node: A, B, C, like in the picture below. We will use the same notations with the previous subsection adding the node name at the end of each subscript.



We denote p_{iB},p_{iC} as probability of winning at each node of player i and d_{iB}, d_{iC} as difference between the continuation values at node B and C for player i, that are all positive values. For example, $d_{1B} = v_{1WB} - v_{1LB}$ and $d_{2C} = v_{2WC} - v_{2LC}$. Once we know the ratio between d_{1A} and d_{2A}, we can write equilibrium effort level and probability of winning of each player at the first stage. Based on the result of one-stage game, we have the Proposition 2

Proposition 2 *Given six continuation values in a two-stage game,*

$$d_{1A} = d_{1B} \cdot p_{1B}^2 - d_{1C} \cdot p_{1C}^2 + d_{1C} = d_{1B} \cdot p_{1B}^2 + d_{1C} \cdot (1 - p_{1C}^2)$$

$$d_{2A} = -d_{2B} \cdot p_{2B}^2 + d_{2C} \cdot p_{2C}^2 + d_{2B} = d_{2C} \cdot p_{2C}^2 + d_{2B} \cdot (1 - p_{2B}^2)$$

Derivation of Proposition 2 is in Appendix C. Proposition 2 shows us the relationship between the difference values of the upper round and difference values of the lower round. We can see that the difference value of a player in the upper round consists of two parts. The first part, which consists of the values of the node where he goes after winning, and the second part, vice versa. We can see that increase in the difference values at nodes B and C increases the difference value at node A. It is because of the existence of the continuation values in the middle that two nodes share. The quadratic form of winning probability is multiplied in each part. If the winning probability increases at the node where the player goes after winning, it makes the difference value of the upper round bigger; thus, the player has more incentive to exert more effort. Thanks to Proposition 2, we can solve the equilibrium of node A without calculating the effort level and continuation value at nodes B and C. Equilibrium of the best-of-five contest is very complicated, as we will see later, and Proposition 2 we have above makes a great shortcut to get equilibrium.

4 Equilibrium of Best-of-Five Contest Game

We can use the outcome of the previous section to solve the whole Best-of-Five game. What we will focus on in this paper is the probability of winning at each node in equilibrium. We found in **Proposition** 1 that, once we have the ratio of difference between continuous values, $dr_{j:k}$, we can get equilibrium probability of winning at the $j : k$ node(s). Also, we can compare the effort level of each player since in equilibrium,

$$\frac{x^*}{y^*} = dr$$

$$\frac{p_1^*}{p_2^*} = a \cdot dr$$

This shortcut is helpful because the expression of the equilibrium effort level is more complicated than equilibrium probability, especially for Round 1.

In this section, we use backward induction from Round 5 to Round 3 to get the continuation values of Round 3 (2:0, 1:1, 0:2). Then, we substitute those continuation values into the formulas of sections 3.3 and 3.4 to have the difference ratio of Round 2 and Round 1.

4.1 Round 5

In round 5, we have only one score state: 2:2. Since this is the last round, anyone who wins this round gets v, and the other gets nothing. So, it is clear that the difference value between the payoff is v for both players; that is,

$$d^1_{2:2} = d^2_{2:2} = v$$

Therefore, the equilibrium effort level at 2:2 node is:

$$x^*_{2:2} = y^*_{2:2} = \frac{av}{(a+1)^2}$$

Two players exert same effort level in the last round. Then, equilibrium probabilities of winning are:

$$p^{1*}_{2:2} = \frac{a}{a+1}, \quad p^{2*}_{2:2} = \frac{1}{a+1}$$

We can see that for all $a > 1$, $p^1_{2:2} > \frac{1}{2} > p^2_{2:2}$.

We have continuation values from playing Round 5 for each player:

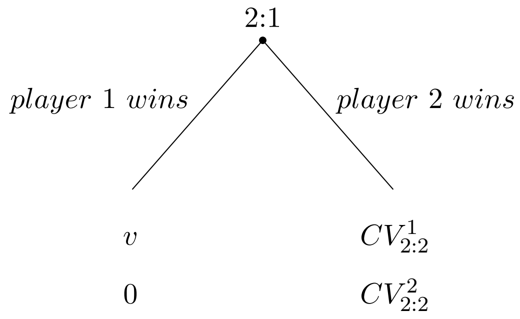
$$CV^1_{2:2} = \frac{a^2 v}{(a+1)^2}$$

$$CV^2_{2:2} = \frac{v}{(a+1)^2}$$

4.2 Round 4

Conditional on reaching Round 4, there are two possible scores: 2:1 and 1:2. We will analyze each of them in turn.

When players reach a score of 2:1, they face a situation like a figure below. If player 1 wins here, the game is over: player 1 gets payoff v, and player 2 gets nothing. However, if player 2 wins, the score becomes 2:2 and goes to the last round.



Based on the result from the previous subsection, once we get the difference between the continuation values, we can solve the equilibrium. We already solved the equilibrium of Round 5 in the previous subsection, and we have the continuation values of Round 5. The difference value between the continuation values for each player are:

$$d^1_{2:1} = v - \frac{a^2 v}{(a+1)^2}, \quad d^2_{2:1} = \frac{v}{(a+1)^2}$$

Equilibrium efforts at 2:1 are:

$$x^*_{2:1} = \frac{av\,(2\,a+1)^2}{(a+1)^2\,(2\,a^2+a+1)^2}, \quad y^*_{2:1} = \frac{av\,(2\,a+1)}{(a+1)^2\,(2\,a^2+a+1)^2}$$

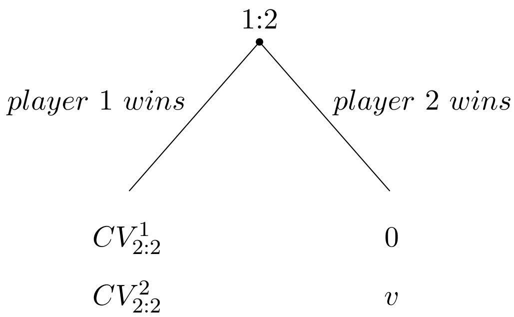
Probabilities of winning at 2:1 are:

$$p^{1*}_{2:1} = \frac{(2\,a+1)\,a}{2\,a^2+a+1}, \quad p^{2*}_{2:1} = \frac{1}{2\,a^2+a+1}$$

We have continuation values from playing at 2:1 for each player:

$$CV^1_{2:1} = \frac{a^2 v \left(4\,a^4 + 12\,a^3 + 17\,a^2 + 8\,a + 2\right)}{(a+1)^2\,(2\,a^2 + a + 1)^2}, \quad CV^2_{2:1} = \frac{v}{(a+1)^2\,(2\,a^2 + a + 1)^2}$$

On the other hand, at 1:2, player 2 needs only one more point to win the game and get payoff v, and player 1 must win this round to go last round.



Likewise,

$$d^1_{1:2} = \frac{a^2 v}{(a+1)^2}, \quad d^2_{1:2} = v - \frac{v}{(a+1)^2}$$

Equilibrium efforts at 1:2 are:

$$x^*_{1:2} = \frac{a^4 v\,(a+2)}{(a^3 + 2\,a^2 + 3\,a + 2)^2}, \quad y^*_{1:2} = \frac{a^3 v\,(a+2)^2}{(a^3 + 2\,a^2 + 3\,a + 2)^2}$$

Probabilities of winning at 1:2 are:

$$p_{1:2}^{1*} = \frac{a^2}{a^2+a+2}, \quad p_{1:2}^{2*} = \frac{a+2}{a^2+a+2}$$

We have continuation values from playing at 2:1 node for each player:

$$CV_{1:2}^{1} = \frac{va^6}{(a^3+2\,a^2+3\,a+2)^2}, \quad CV_{1:2}^{2} = \frac{v\left(2\,a^4+8\,a^3+17\,a^2+12\,a+4\right)}{(a^3+2\,a^2+3\,a+2)^2}$$

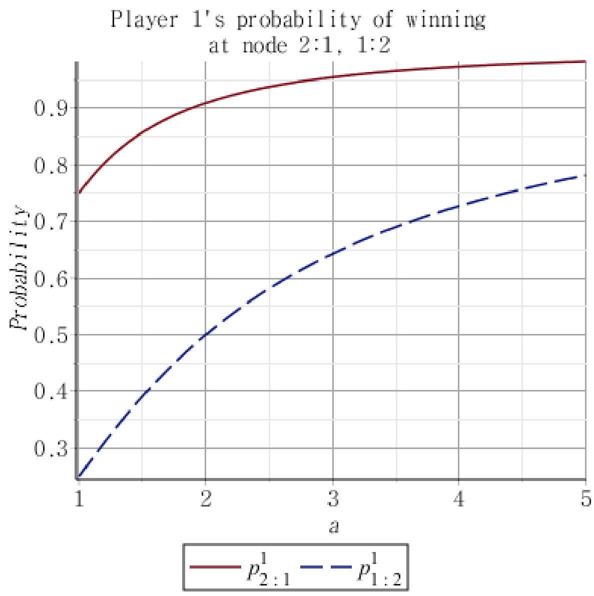


Figure 1: Equilibrium Winning Probabilities in Round 4

At the node 2:1, $dr_{2:1} = 2a+1 > 1$ for all positive a, not only for $a > 1$. It means whoever ahead in the score in Round 4 exerts more effort than the opponent. We can also see this in the figure on the left above. Moreover, a stronger player who is ahead in Round 4 exerts effort more than three times than the opponent.

Result 1 *At the score state 2:1, the player 1 is more likely to win round 4, i.e.,* $p^1_{2:1} > 1/2$, $\forall a > 1$

The proof is clear from the expression of $p^1_{2:1}$. In other words, if player 1 is in a favorable state in Round 4, he is more likely to win.

Result 2 *At the score state 1:2, player 1 is more likely to win round 4 if* $a > 2$, *The player 2 is more likely to win Round 4 if* $a < 2$

When $a < 2$, that is, when the difference in abilities between two players is small enough, player 1 is more likely to lose. However, when $a > 2$, player 1 is relatively stronger than the opponent, the player 1 is more likely to win despite player 1 is behind in Round 4.

Results here are exactly the same with the results of Round 2 of Best-of-Three analysis in 'Strategic and Psychological Momentum in Professional Tennis' (2020) because they both solved the second round from behind. However, there is a big difference in interpretation. In Round 2 of Best-of-Three, there are two score states, 1:0 and 0:1, and each of them has only one history. So it is possible to compare the expectation between psychological momentum and strategic momentum. But each state of Round 4 of Best-of-Five, 2:1 and 1:2, has three histories to get each state, which means, the favorite may have won or lost in the previous round. We can not say in which way psychological momentum work when the score state is all information we have.

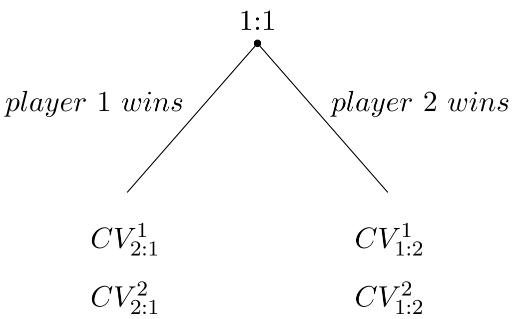
This is the reason why those two states are remarkable. Except for the tied states (e.g., 1:1, 2:2), they are the only states with multiple paths to get each of them in the Best-of-Five game, which do not exist in Best-of Three model. [5] We can see that, at 2:1 and 1:2, strategic momentum works for the one who is ahead according to our model and

[5] In the Best-of-Three model, there are four score states: 0:0, 1:1, 1:0, and 0:1, and 1:1 is the only score state that has multiple paths before.

psychological momentum works for the winner of Round 3 according to its concept. It means, at those two score states, two kinds of momentum are much less correlated than in Round 2 of the Best-of-Three case. So we can expect that the test using the Best-of-Five model and data is better to find the evidence of each momentum and distinguish its effect. We can also infer that the longer format than Best-of-Five will be better since we will have more score states with this characteristic.

4.3 Round 3

Conditional on reaching Round 3, there are three possible scores: 2:0, 1:1, and 0:2.



Difference values at 1:1 are:

$$d^1_{1:1} = CV^1_{2:1} - CV^1_{1:2} = 2\,\frac{\left(8\,a^7 + 28\,a^6 + 58\,a^5 + 83\,a^4 + 80\,a^3 + 55\,a^2 + 20\,a + 4\right) a^2 v}{\left(a^3 + 2\,a^2 + 3\,a + 2\right)^2 \left(2\,a^2 + a + 1\right)^2}$$

$$d^2_{1:1} = CV^2_{1:2} - CV^2_{2:1} = 2\,\frac{\left(4\,a^7 + 20\,a^6 + 55\,a^5 + 80\,a^4 + 83\,a^3 + 58\,a^2 + 28\,a + 8\right) va}{\left(a^3 + 2\,a^2 + 3\,a + 2\right)^2 \left(2\,a^2 + a + 1\right)^2}$$

Equilibrium efforts at 1:1 are:

$$x^*_{1:1} = 2\,\frac{a^4 v\left(4\,a^7+20\,a^6+55\,a^5+80\,a^4+83\,a^3+58\,a^2+28\,a+8\right)\left(8\,a^7+28\,a^6+58\,a^5+83\,a^4+80\,a^3+55\,a^2+20\,a+4\right)^2}{\left(2\,a^2+3\,a+2\right)^2\left(a^2+a+2\right)^2\left(2\,a^2+a+1\right)^2\left(4\,a^6+4\,a^5+11\,a^4+10\,a^3+11\,a^2+4\,a+4\right)^2(a+1)^4}$$

$$y^*_{1:1} = 2\,\frac{\left(4\,a^7+20\,a^6+55\,a^5+80\,a^4+83\,a^3+58\,a^2+28\,a+8\right)^2\left(8\,a^7+28\,a^6+58\,a^5+83\,a^4+80\,a^3+55\,a^2+20\,a+4\right)a^3 v}{\left(2\,a^2+3\,a+2\right)^2\left(a^2+a+2\right)^2\left(2\,a^2+a+1\right)^2\left(4\,a^6+4\,a^5+11\,a^4+10\,a^3+11\,a^2+4\,a+4\right)^2}$$

Probabilities of winning at 1:1 are:

$$p^{1*}_{1:1} = \frac{a^2\left(8\,a^7+28\,a^6+58\,a^5+83\,a^4+80\,a^3+55\,a^2+20\,a+4\right)}{(a+1)\left(2\,a^2+3\,a+2\right)\left(4\,a^6+4\,a^5+11\,a^4+10\,a^3+11\,a^2+4\,a+4\right)}$$

$$p^{2*}_{1:1} = \frac{4\,a^7+20\,a^6+55\,a^5+80\,a^4+83\,a^3+58\,a^2+28\,a+8}{(a+1)\left(2\,a^2+3\,a+2\right)\left(4\,a^6+4\,a^5+11\,a^4+10\,a^3+11\,a^2+4\,a+4\right)}$$

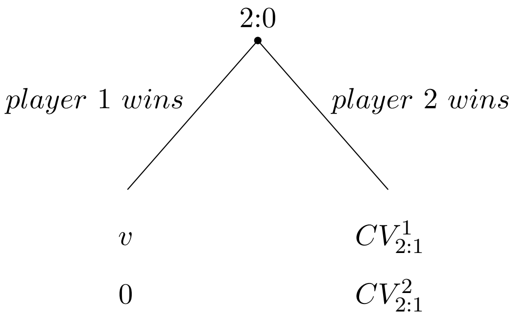
We have continuation values from playing at 1:1 for each player:

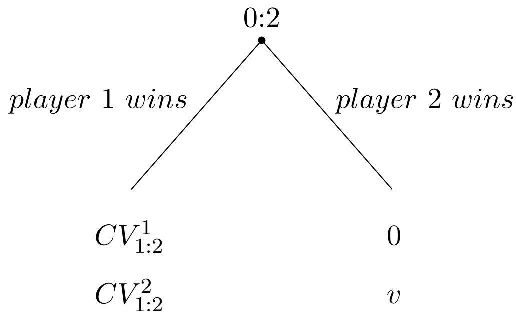
$$CV^1_{1:1} = \frac{nCV^1_{1:1}}{dCV^1_{1:1}}$$

$$CV^2_{1:1} = \frac{nCV^2_{1:1}}{dCV^2_{1:1}}$$

where the prefix n and d means numerator and denominator each. Expressions of numerator and denominator are in Appendix D.1.

Similarly, we can also solve for the 2:0 and 0:2.





Probabilities of winning at 2:0 are:

$$p^{1*}_{2:0} = \frac{a\,(2\,a+1)\left(4\,a^2+2\,a+1\right)}{8\,a^4+8\,a^3+4\,a^2+a+1}, \quad p^{2*}_{2:0} = \frac{1}{8\,a^4+8\,a^3+4\,a^2+a+1}$$

Continuation values from playing at 2:0 for each player:

$$CV^1_{2:0} = \frac{\left(256\,a^{12}+1280\,a^{11}+3136\,a^{10}+5056\,a^9+6016\,a^8+5616\,a^7+4276\,a^6+2604\,a^5+1237\,a^4+446\,a^3+123\,a^2+24\,a+3\right)va^2}{(a+1)^2\left(2\,a^2+a+1\right)^2\left(8\,a^4+8\,a^3+4\,a^2+a+1\right)^2}$$

$$CV^2_{2:0} = \frac{v}{(a+1)^2\left(2\,a^2+a+1\right)^2\left(8\,a^4+8\,a^3+4\,a^2+a+1\right)^2}$$

Probabilities of winning at 0:2 are:

$$p^{1*}_{0:2} = \frac{a^4}{a^4 + a^3 + 4\,a^2 + 8\,a + 8}. \quad p^{2*}_{0:2} = \frac{(a+2)\left(a^2 + 2\,a + 4\right)}{a^4 + a^3 + 4\,a^2 + 8\,a + 8}$$

Continuation values from playing at 0:2 node for each player are:

$$CV^1_{0:2} = \frac{v a^{14}}{(a+1)^2\,(a^2 + a + 2)^2\,(a^4 + a^3 + 4\,a^2 + 8\,a + 8)^2}$$

$$CV^2_{0:2} = \frac{v\left(3\,a^{12} + 24\,a^{11} + 123\,a^{10} + 446\,a^9 + 1237\,a^8 + 2604\,a^7 + 4276\,a^6 + 5616\,a^5 + 6016\,a^4 + 5056\,a^3 + 3136\,a^2 + 1280\,a + 256\right)}{(a+1)^2\,(a^2 + a + 2)^2\,(a^4 + a^3 + 4\,a^2 + 8\,a + 8)^2}$$

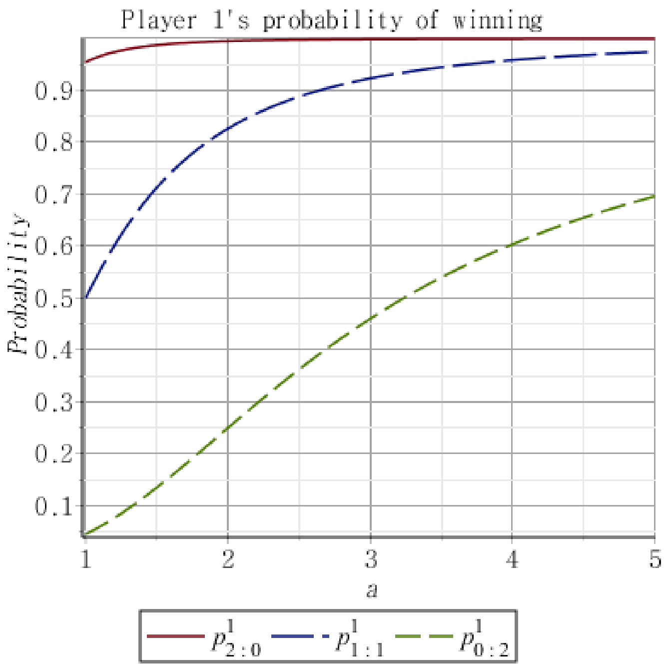


Figure 2: Winning Probabilities of Round 3

Result 3 *In Round 3, player 1's probability of winning is always higher in more favorable state.i.e.,* $p^1_{2:0} > p^1_{1:1} > p^1_{0:2}$, $\forall a > 1$

Proof. This is clear from the expressions of difference in probabilities since they are consist of terms with positive coefficient only.

$p^1_{2:0} - p^1_{1:1} =$

$$\frac{2a\left(4\,a^6+20\,a^5+51\,a^4+64\,a^3+47\,a^2+20\,a+4\right)\left(2\,a^2+a+1\right)^2}{(8\,a^4+8\,a^3+4\,a^2+a+1)\,(a+1)\,(2\,a^2+3\,a+2)\,(4\,a^6+4\,a^5+11\,a^4+10\,a^3+11\,a^2+4\,a+4)} > 0,$$

$\forall a > 1$

$p^1_{1:1} - p^1_{0:2} =$

$$\frac{2a^2\left(4\,a^6+20\,a^5+47\,a^4+64\,a^3+51\,a^2+20\,a+4\right)\left(a^2+a+2\right)^2}{(a+1)\,(2\,a^2+3\,a+2)\,(4\,a^6+4\,a^5+11\,a^4+10\,a^3+11\,a^2+4\,a+4)\,(a^4+a^3+4\,a^2+8\,a+8)} > 0,$$

$\forall a > 1$ ■

Figure 2 also illustrates the result above.

Result 4 *At the score state 2:0 & 1:1, the player 1 is more likely to win Round 3, i.e.,* $p^1_{2:0} > 1/2$, $p^1_{1:1} > 1/2$, $\forall a > 1$. *At the score state 0:2, the player 1 is more likely to win Round 3 if* $a > 3.236$, *The player 2 is more likely to win Round 3 if* $a < 3.236$

Proof.

$$p^1_{2:0} - \frac{1}{2} = \frac{1}{2}\,\frac{\left(4\,a^2+2\,a-1\right)\left(2\,a^2+a+1\right)}{8\,a^4+8\,a^3+4\,a^2+a+1} > 0, \quad \forall a > 1$$

$$p^1_{1:1} - \frac{1}{2} = \frac{1}{2}\,\frac{(a-1)\left(a^2+a+2\right)\left(2\,a^2+a+1\right)\left(4\,a^4+12\,a^3+15\,a^2+12\,a+4\right)}{(a+1)\,(2\,a^2+3\,a+2)\,(4\,a^6+4\,a^5+11\,a^4+10\,a^3+11\,a^2+4\,a+4)} > 0,$$

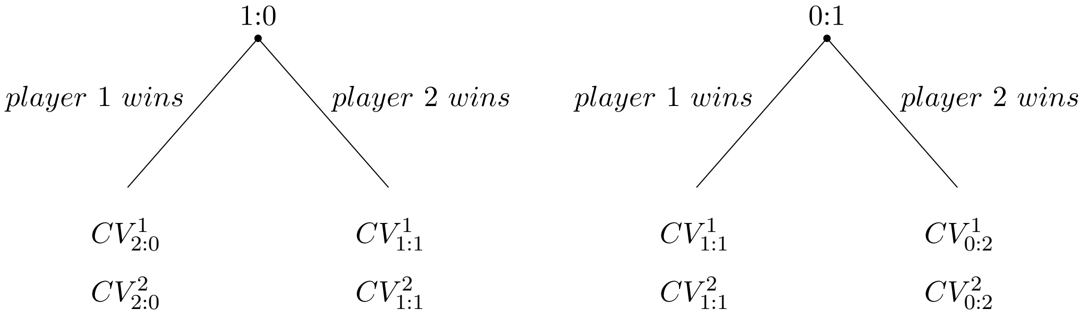
$\forall a > 1$ ■

This result is consistent with Round 5 and Round 4. When the score is tied, like in Round 5, 2:2, the favorite is more likely to win, and when the favorite is ahead, like in Round 4, 2:1, more likely to win for any value of $a > 1$.

Unlike at 2:1 and 1:2 in Round 4, there is only one history to get to 2:0 and 0:2 each. At 2:0, strategic momentum and psychological momentum expect in the same direction. In the case of 0:2, when the value of a is smaller than 3.236, expectations of both momentum coincide, but when a is bigger than that, the predictions of both momentum are opposite. Psychological momentum suggests that, at 0:2, player 2 has the momentum, so he is more likely to win. However, strategic momentum suggests that if player 1 is stronger enough than player 2, player 1 is more likely to win though he is in a disadvantageous state. Compared to 1:2, the critical point that makes the favorite more likely to win is higher than at 0:2. 1:2 is a better situation for player 1 than 0:2, and player 1 has more incentive to exert effort to win the round for any given value of a. It makes a critical point to be higher in 0:2 than 1:2.

4.4 Round 2

Conditional on reaching Round 4, there are two possible scores: 2:1 and 1:2. Since we have every continuous value of round 3, we do not need to solve the equilibrium effort level to get the difference values of upper rounds. As stated before, the difference ratio includes all information we are interested in, e.g., the probability of winning and equilibrium effort level ratio.



$$dr_{1:0} = \frac{d^1_{1:0}}{d^2_{1:0}} = \frac{ndr_{1:0}}{ddr_{1:0}},$$

$$dr_{0:1} = \frac{d^1_{0:1}}{d^2_{0:1}} = \frac{ndr_{0:1}}{ddr_{0:1}},$$

where the prefix n and d means numerator and denominator each. Expressions of numerator and denominator are in Appendix D.2. We can plot Figure 3 with those two expressions. Likewise, in Round 3, we can see that probability of winning is higher in a more favorable state , *i.e.*, $p^1_{1:0} > p^1_{0:1}$. Figure 3 illustrates it.

Result 5 *If the player 1 is ahead in Round 2, player 1 is more likely to win, i.e.,* $p^1_{1:0} > 1/2$*,* $\forall a > 1$.

Proof.

$$p^1_{1:0} - \frac{1}{2} = \frac{1}{2} \cdot \frac{A}{B}$$

Expressions of A and B are in Appendix D. There is only one factor that includes terms with the negative coefficient in A, and it has a structure where higher degree coefficients are positive while lower degree coefficients are negative. When $a = 1$, the value of the term is 9257472, which is positive. So A over B is also positive according to the **Lemma 1** below ■

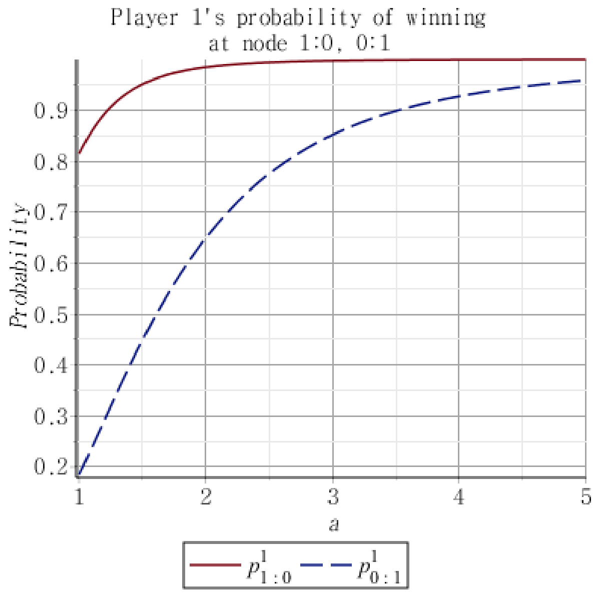


Figure 3: Winning probabilities in Round 2

This result is consistent with Round 3 (2:0) and Round 4 (2:1).

Lemma 1 *For polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + ...a_0$ such that coefficients at above degree k are positive, and coefficients at below degree k are negative, if $P(w)$ is positive for $w \geq 1$, then $a_n x^n + a_{n-1}x^{n-1} + ...a_0 > 0 \quad \forall x > w$*

Proof.

$P(w) = \sum_{i=0}^{n} a_i w^i$ and this is positive for $w \geq 1$ as given. Let's think of y such that $y > w$. $P(y) - P(w) = \sum_{i=0}^{n} a_i y^i - \sum_{i=0}^{n} a_i w^i = \sum_{i=0}^{n} a_i (y^i - w^i)$. Since $(y^m - w^m)$ is increasing function of m given $y > w \geq 1$ and coefficients of higher order terms are positive and $P(w)$ is positive as given, $P(y) - P(w) > 0 \; \forall y > m \geq 1$. Therefore, $P(y)$ is also positive for all $y > w \geq 1$ ■

Result 6 *At the score state 0:1, player 1 is more likely to win round 2 if $a > 1.609$. Player 2 is more likely to win if $a < 1.609$.*

At 1:0, both momenta expect in the same way: player 1 is more likely to win for all $a > 1$. However, at 0:1, when the value of a is smaller than 1.609, expectations of both momentum coincide, but when a is greater than that, strategic momentum goes against the psychological momentum. At 0:1, the critical point that makes player 1 more likely to win is lower than 0:2 and 1:2. 0:1 is more favorable than 0:2. We cannot compare 0:1 and 1:2 in this sense, so we will compare the winning probabilities in the next subsection.

4.5 Round 1

With the six continuation values of Round 3, we can directly get the difference ratio of Round 1 using the **Proposition** 2, and it includes essential information we are interested in: the probability of winning and effort ratio in equilibrium.

$$
\begin{aligned}
dr_{0:0} &= \frac{d_{1:0}^1 {p_{1:0}^1}^2 + d_{0:1}^1(1 - {p_{0:1}^1}^2)}{d_{0:1}^2 {p_{0:1}^2}^2 + d_{1:0}^2(1 - {p_{1:0}^2}^2)} \\
&= \frac{ndr_{0:0}}{ddr_{0:0}}
\end{aligned}
$$

Expressions of numerator and denominator are in Appendix D.3. Now we have equilibrium probabilities of winning at every score state.

4.6 Comparison of winning probabilities

The results we have so far have been the outcome within each round. We will see what happens in a whole game from now on. In a previous study of DGS (2020)[6], the winning probability of the favorite in Round 1 is higher than in Round 3 for any given value of a, though two players are tied in both round. There are three tied states in the Best-of-Five contest: 0:0, 1:1, and 2:2, and we will see whether it is consistent with the previous study.

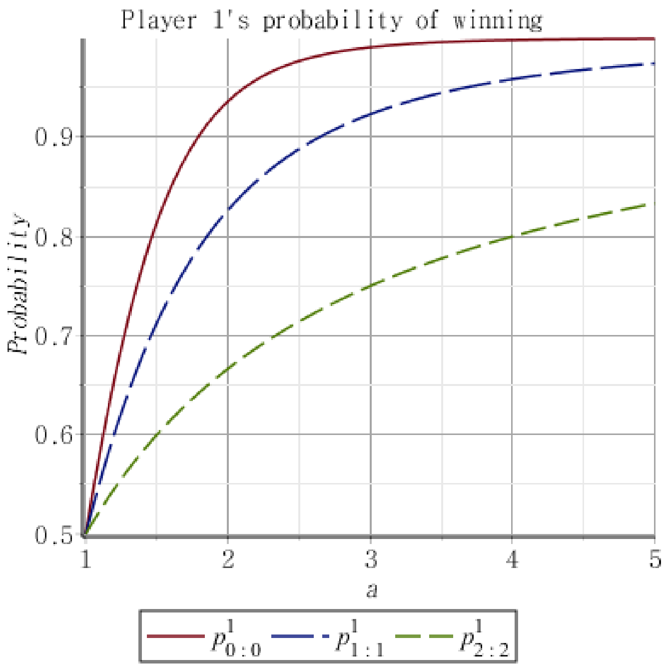


Figure 4: Comparison of winning probabilities of three tied states

Result 7 $p^1_{0:0} > p^1_{1:1} > p^1_{2:2} \quad , \forall a > 1$

Proof.

$$p^1_{1:1} - p^1_{2:2} = 2\,\frac{(a-1)\left(4\,a^6+12\,a^5+23\,a^4+26\,a^3+23\,a^2+12\,a+4\right)a}{(a+1)\left(2\,a^2+3\,a+2\right)\left(4\,a^6+4\,a^5+11\,a^4+10\,a^3+11\,a^2+4\,a+4\right)} > 0 \quad \forall a > 1$$

[6] 'Strategic and Psychological Momentum in Professional Tennis'

Similarly, $p^1_{0:0} - p^1_{1:1}$, is also consist of $(a-1)$ and positive terms[7] therefore, it is positive for all $a > 1$. ■

Those three states are symmetric in the sense that both players are tied, but player 2's incentive to win the earlier round is not as high as the later round. For the weaker player, though those three states are all tied in score, a later round is always better because it is not likely to get that score state from the earlier round because of the asymmetry of ability. However, once the underdog gets to Round 5, he has more incentive to exert effort than previous tied rounds. Figure 4 depicts this result.

Result 7 can be interpreted as the longer the contest, the more likely stronger player will win, therefore, fewer surprises. There are general beliefs that the favorite is more likely to win in the best-of-five than in the best-of-three in the tennis match. Two figures below are from the article titled 'Tennis Grand Slam trading strategy' from Smarket website.[8]

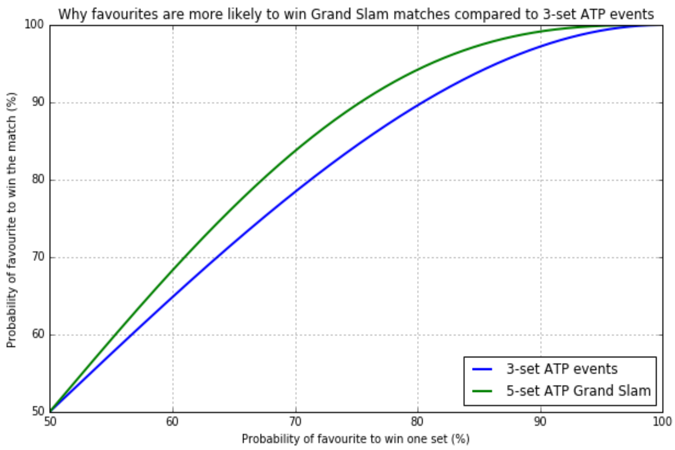


Figure 5: Probability of favorite to win the match given the probability of favorite to win one set

[7] Expression is omitted since it's too long

[8] `https://help.smarkets.com/hc/en-gb/articles/115000821689-Tennis-Grand-Slam-trading-strategy`

Figure 5 shows how the probability of winning the entire match(vertical axis) is calculated given the probability of winning a set(horizontal axis) in each of the best-of-three and best-of-five format. We can see that the probability of favorite to win the match is always higher in the longer format for any given probability of favorite to win one set.

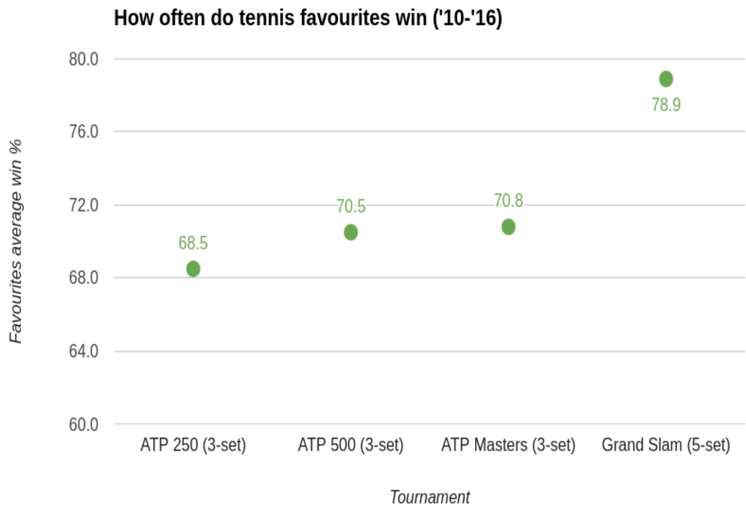


Figure 6: Tennis match data from '10 '16

Figure 6 shows us how often the pre-game favorite wins the match in each section on the horizontal axis. We can see that there is less upset in Best-of-Five tennis matches than Best-of-Three tennis matches. Based on this data, the writer advises the bettor that there is lower risk in the longer format, which means a lower variance of profit. A veteran bettor who has made a consistent profit in tennis betting for more than 16 years points out that bettors should focus on the best-of-five matches and not to bet on the women's tournament match which is the best-of-three format as an answer to the question, 'Is tennis worth betting on seriously?'.[9] Result 7 implies that strategic momentum can well explain the data suggested above since in our model, the probability of winning the first set by the favorite increases as the number of total rounds in best-of-N increases, thus

[9] https://www.olbg.com/forum/viewtopic.php?start=60&t=25183

increases the probability of winning the match. Also, Result 7 is one of the things that are hardly inferred when we assume psychological momentum.

Result 8 *If the player 1 is ahead or tied in score, player 1 is more likely to win, i.e.,* $p^1_{j:k} > 1/2$*,* $\forall a > 1, j \geq k$

The result above is a summary of **Result 1, 4, 5, 7**. In a Best-of-Five game, if the favorite is not in an inferior position in terms of score, he is more likely to win the round in the absence of psychological momentum.

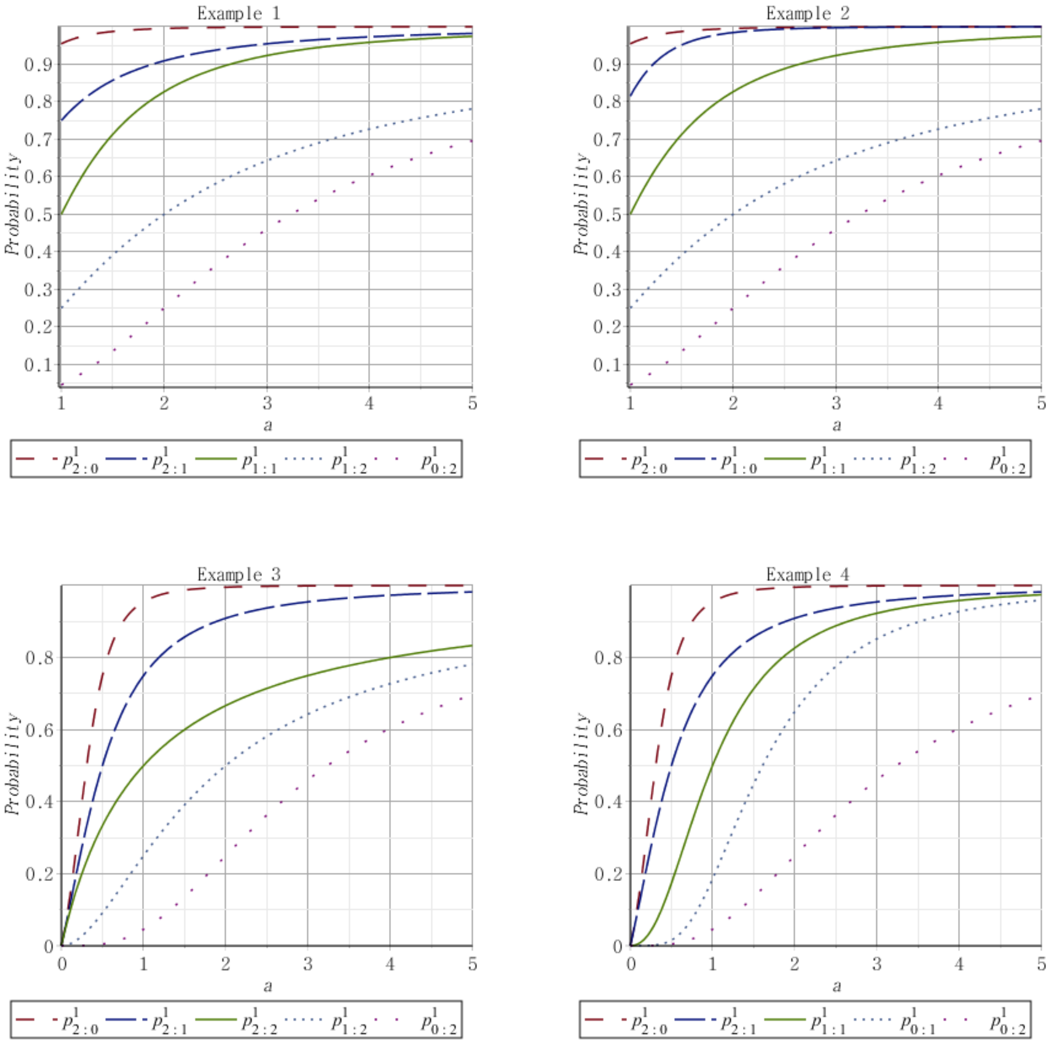


Figure 7: Examples of higher winning probability in more favorable situation

In the analysis of Round 3, we found that player 1's probability of winning is always higher in **more favorable** situation for all $a > 1$. We can also find that this also happens when it comes to the whole game.

Figure 7 above shows us that

$p^1_{2:0} > p^1_{2:1} > p^1_{1:1} > p^1_{1:2} > p^1_{0:2}$, $\forall a > 1$ (Example 1),

$p^1_{2:0} > p^1_{1:0} > p^1_{1:1} > p^1_{1:2} > p^1_{0:2}$, $\forall a > 1$ (Example 2),

$p^1_{2:0} > p^1_{2:1} > p^1_{2:2} > p^1_{1:2} > p^1_{0:2}$, $\forall a > 1$ (Example 3),

$p^1_{2:0} > p^1_{2:1} > p^1_{1:1} > p^1_{0:1} > p^1_{0:2}$, $\forall a > 1$ (Example 4).

Those four examples above are part of combinations of states that can be ordered according to the definition of 'more favorable'. We can see here that in a more favorable situation, the higher the probability of winning for any given value of a.

Result 9 *For both players, the probability of winning is always higher in a more favorable situation in terms of score regardless of the size of a.*

Proof. To say that player 1's probability of winning is always higher in more favorable situation, we need to show that $p^1_{2:0} > p^1_{1:0}$, $p^1_{2:0} > p^1_{2:1}$, $p^1_{1:0} > p^1_{0:0}$, $p^1_{1:0} > p^1_{1:1}$, $p^1_{2:1} > p^1_{1:1}(> p^1_{2:2})$, $p^1_{0:1} > p^1_{0:2}$, $p^1_{0:0} > p^1_{0:1}$, $(p^1_{1:1} >)p^1_{2:2} > p^1_{1:2}$, $p^1_{1:2} > p^1_{0:2}$ $\forall a > 1$. So we need to check eleven cases in total, but inequalities in parentheses are already checked in the proof of **Result 8**. Those nine inequalities hold for all $a > 1$ since all the difference between the two possibilities consists of only terms with positive coefficients. We omit the expressions of differences between possibilities since expressions of all winning possibilities are included in the paper.[10] ■

According to the definition of '**more favorable**' in this paper, it's not applicable to all combinations of score state. We will see the comparison of the winning probability between the score states that fall on this point.

[10]Those inequalities hold not only for $a > 1$ but also $a \leq 1$. It is intuitive since if the state is more favorable by a player, it must be less favorable by the opponent according to the definition of 'more favorable'

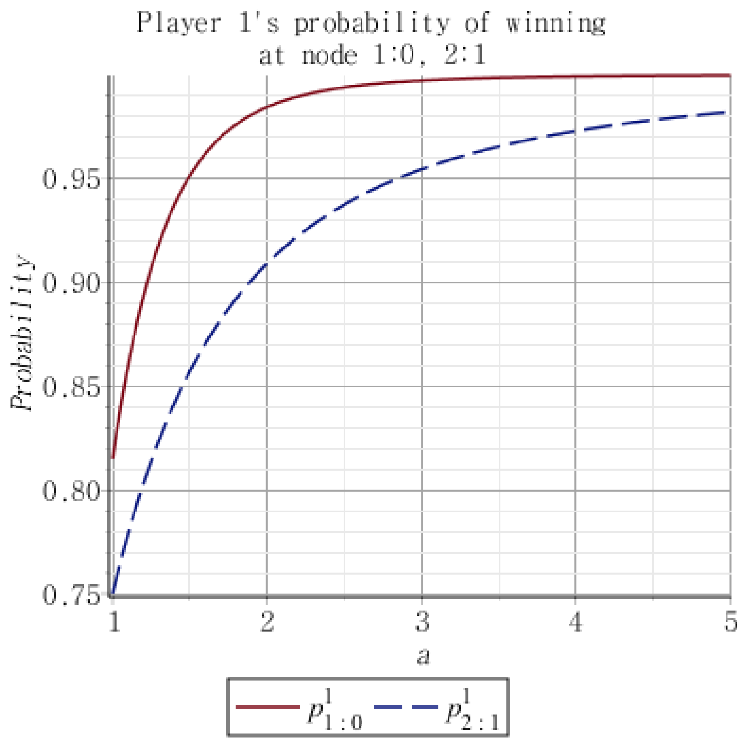


Figure 8: Winning probabilities at 1:0 and 2:1

Result 10 $p^1_{1:0} > p^1_{2:1} \quad \forall a > 1.$

Proof.

$$p^1_{1:0} - p^1_{2:1} = \frac{A}{B}$$

Expressions of A and B are in Appendix D. When $a = 1$, the value of longest term in numerator that only includes terms with negative coefficients is 3527774208, and is positive. According to **Lemma 1**, $p^1_{1:0} - p^1_{2:1} > 0 \quad \forall a > 1$ ■

The same logic can be applied here with **Result 9**. Player 1 is ahead one point in both score state, but it is less likely to get 2:1 from 1:0 for player 2, the underdog. Then player 2 has less incentive to exert effort in 1:0 than 2:1. However, once he gets there, now he has more incentive to exert effort.

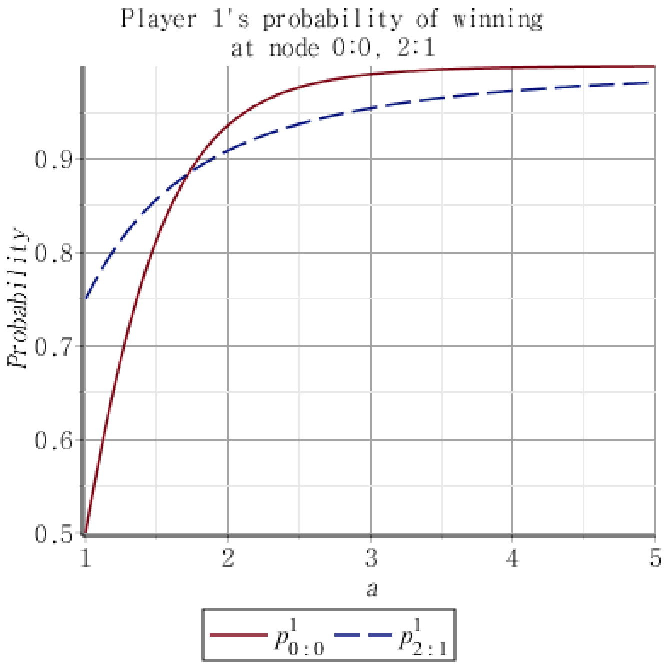


Figure 9: Intersection of $p^1_{0:0}$ and $p^1_{2:1}$

Result 11 $p^1_{0:0} < p^1_{2:1}$ *when the value of* a *is relatively small, but as the value of* a *increases, there is a breakpoint that turns to* $p^1_{0:0} > p^1_{1:2}$*. The value of the breakpoint is 1.721.*

Proof. There is only one polynomial factor, which includes the terms with minus coefficient in the factorized expression of $p^1_{0:0} - p^1_{2:1}$. From degree 170 to 117, coefficients are positive and negative below 117. According to the **Lemma 1**, $p^1_{0:0} - p^1_{2:1} > 0 \;\; \forall a > 1.721$ and vice versa. ■

Result 12 *$p^1_{0:1} < p^1_{1:2}$ when the value of a is relatively small, but as the value of a increases, there is a breakpoint that turns to $p^1_{0:1} > p^1_{1:2}$. The value of the breakpoint is 1.2693.*

$p^1_{0:1} < p^1_{2:2}$ when the value of a is relatively small, but as the value of a increases, there is a breakpoint that turns to $p^1_{0:1} > p^1_{2:2}$ The value of the breakpoint is 2.0876.

Proof. It is clear that there exists an intersection in each case. We can confirm that this is unique by checking the first-order derivative.

$$\frac{d(p^1_{0:1} - p^1_{1:2})}{da} = \frac{A}{B} > 0 \quad \forall a > 1$$

$$\frac{d(p^1_{0:1} - p^1_{2:2})}{da} = \frac{C}{D} > 0 \quad \forall a > 1$$

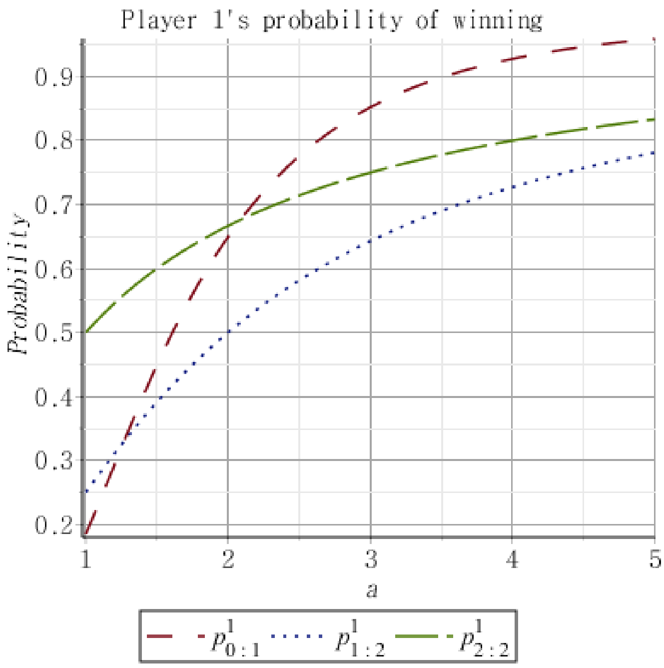


Figure 10: Two intersections of winning probabilities

Since coefficients of every term in A, B, C, D are strictly positive, the difference between two probabilities increases as the value of a increases. Expressions of A, B, C, D are in Appendix C.4. ■

After losing round 1, player 1 has less incentive to exert effort than in score state 1:2 or 2:2 when a is relatively small. Nevertheless, when the value of a is relatively large, losing round 1 affects less to the incentive of player 1. We can see here that when the favorite is stronger enough than the opponent, he exerts much more effort than his opponent. Figure 10 illustrates these results.

5 Theoretical Frequencies

In this section, we will calculate the theoretical frequencies to compare with the observed data. DGS (2020) use betting odds data to calculate the Shin (1992 and 1993) subjective probabilities of the favorite to win the match as the observed frequency. Shin probability is a method to calculate the implied probabilities from the betting odds using an iterative approach. It allows for estimating the theoretical size of insider betting.[11] It is well known for its incredible accuracy, so the Python implementation that uses this method is widely used by sports bettors. Shin probabilities make ex-ante winning probabilities effectively observable, and this helps us link the asymmetry of ability in our model and the data. Observed frequencies are counted in the following way. For each match record, implied probabilities are calculated using Shin's method. Match records can be categorized according to the winning probabilities. [12] Then, we can count the observed frequencies for each score state and ex-ante winning probability.

Theoretical frequencies are calculated in the following way. We denote p^F as the theoretical winning probability of the favorite. Theoretical probabilities are calculated based on the model we used in the previous section. Thus, the theoretical probability that the favorite wins the match is

[11]This is a theoretical model to explain a bias called 'favorite-longshot bias'(FLB), which is the tendency for high odds/low probability betting propositions(i.e., long shots) to have subjective win probabilities above observed win proportions and low odds/high probability propositions (i.e., favorites) to have subjective win probabilities below observed win proportions. Early explanations of FLB stressed demand-side explanations such as risk-loving behavior by bettors. Instead of viewing the FLB as stemming from the inability of books to evaluate the actual win probabilities, Shin sees the FLB as a response to an asymmetric and adverse selection problem stemming from the presence of insiders who possess superior information than the book on some horses in the race. As a result of insiders' presence, books engineer an FLB to pass the costs of losses arising from insider betting on to less-informed bettors.

[12]For observed frequency, DGS (2020) uses 24 intervals between 0.5 and 1.0 in the sense that the player is the favorite. We also use those 24 intervals when we calculate theoretical frequencies.

$p^F =$

$$p^1_{0:0} \cdot p^1_{1:0} \cdot p^1_{2:0} + p^2_{0:0} \cdot p^1_{0:1} \cdot p^1_{1:1} \cdot p^1_{2:1} + p^1_{0:0} \cdot p^2_{1:0} \cdot p^1_{1:1} \cdot p^1_{2:1} + p^1_{0:0} \cdot p^1_{1:0} \cdot p^2_{2:0} \cdot p^1_{2:1} + p^1_{0:0} \cdot p^1_{1:0} \cdot p^2_{2:0} \cdot p^2_{2:1} \cdot p^1_{2:2} + p^1_{0:0} \cdot p^2_{1:0} \cdot p^1_{1:1} \cdot p^2_{2:1} \cdot p^1_{2:2} + p^1_{0:0} \cdot p^2_{1:0} \cdot p^2_{1:1} \cdot p^1_{1:2} \cdot p^1_{2:2} + p^2_{0:0} \cdot p^2_{0:1} \cdot p^1_{0:2} \cdot p^1_{1:2} \cdot p^1_{2:2} + p^2_{0:0} \cdot p^1_{0:1} \cdot p^1_{1:1} \cdot p^2_{2:1} \cdot p^1_{2:2} + p^2_{0:0} \cdot p^1_{0:1} \cdot p^2_{1:1} \cdot p^1_{1:2} \cdot p^1_{2:2}$$

Let a_r denote the value of a such that $p^F(a_r) = r$. That is, a_r is the value of player 1's relative ability such that player 1 wins the match with probability r. We can calculate the array of a_r for each r in $\{0.5, 0.52 \ldots ,0.96\}$ using the vpasolve function(numerical solve command) in Matlab software. By substituting a with a_r in the equilibrium probability of winning at each node, we can have the predicted probabilities that the favorite wins at each node.

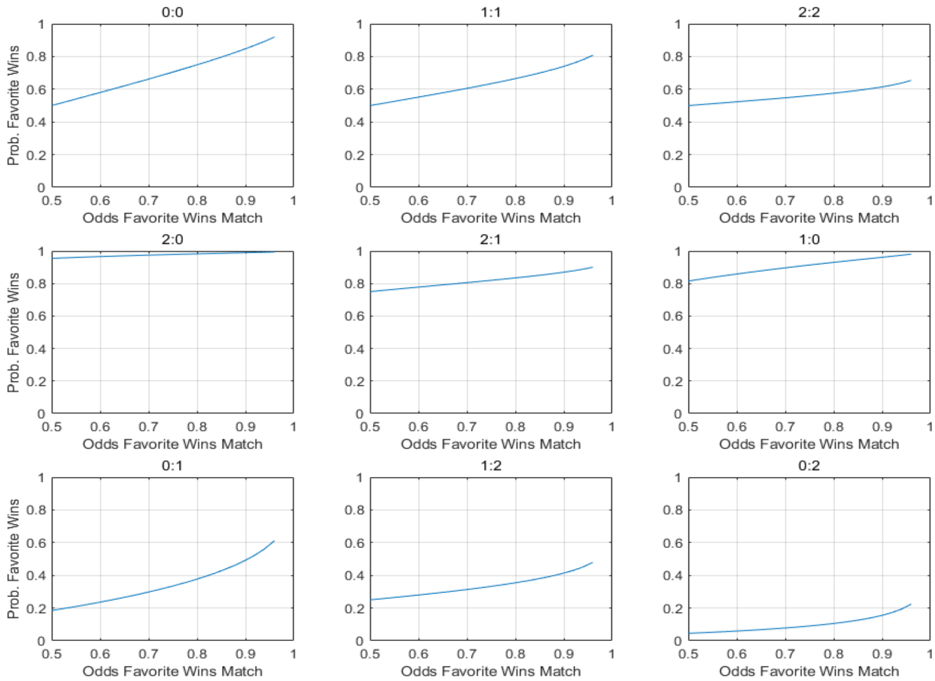


Figure 11: Theoretical Frequencies

Figure 11 illustrates the theoretically predicted frequencies at each node. We can see that the relative position of graphs in Figure 11 are the same as the graphs we have studied so far. Of course, it is because the probability of winning the match strictly increases as the value of a increases.

Graphs on the first line are the tied states, graphs on the second line are for the states that the favorite is ahead, and graphs on the third-line are for the states that the favorite is behind in terms of score. When $a = 1$(Odds favorite wins the match is 0.5) and the favorite wins the first round, the winning probability is around 80%. This may seem exaggerated; however, this fits well with the KBO Korean Series example mentioned in the introduction.

6 Conclusions

We solved the fully rational model of the best-of-five contest with asymmetric players and suggested that strategic momentum also may well explain the winning streak that seems caused by the psychological momentum.

Malueg and Yates (2010) outlined the fully rational model of best-of-three between the symmetric players and tested using professional tennis match data. Depken, Gandar, and Shapiro(2020) expanded this study introducing the asymmetry between the players, and they use the bigger data as restriction of equal ability removes. They show that the first set outcome, which is free from the psychological momentum, can well be matched with the theoretical model, and both momenta affect the outcome of the best-of-three contest. We expand the previous study by extending the model to the best-of-five. Since men's tennis is mostly in the longer format, the results we looked at so far can be a good benchmark for the empirical research that assumes strategic momentum.

Results found in the model of best-of-five contests are consistent with those of the model of best-of-three contests. Among the tied score states, the favorite's winning probability is higher in upper rounds. Moreover, as the number of the score states increases, we could have more combinations to compare the equilibrium winning probabilities. We find that the winning probability is always higher in a more favorable situation, and we can infer that this will also happen for the contests that are longer than the best-of-five.

Also, there is a new finding in the best-of-five model. There are two distinctive score states that may have multiple histories in Round 4, that is, 2:1 and 1:2. There also exists a node that has multiple histories in the best-of-three model, that is, 1:1, but this is the only one, and it is tied in terms of score. In those score states, two kinds of momentum are much less correlated than in other score states since the strategic momentum definitely works for the one ahead, no matter what happened in the previous round. That is why those two can be the right places to distinguish the effect of two kinds of momentum using the professional tennis match data. Thus, we can say that the Best-of-Five model would be better to find evidence of strategic momentum. Also, we can infer that the test which uses the model and data of a longer format will be better than our model because there will be more score states that have this characteristic.

The momentum generated by the result of the previous round may be a psychological one or a strategic one. We expect both to exist, but strategic momentum is usually undervalued since it is less intuitive than psychological momentum.[13] That is why it is crucial to understand the existence of strategic incentives that players have in the contest-type setting. The Best-of-N setting is prevalent, not only in the tennis tournament. Think of when we play a bet with friends on billiards, video games, or rock-scissor-paper. Most of the significant, important sports events like those mentioned in the introduction have the Best-of-N setting. Moreover, betting data that was used to test the model in previous studies are also widely available in various sports. We expect that we will see the evidence of strategic momentum in studies using various sports data later on.

[13]Or, maybe incentive factor is also considered as a part of psychological momentum, though they work in totally different way.

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Appendix

A Notations

- $p^i_{j:k}$: Probability of player i win at $j:k$ score nodes.(j: player 1's score. k: player 2's score.)
- $x_{j:k}$: Effort level of player 1 at $j:k$ score nodes.
- $y_{j:k}$: Effort level of player 2 at $j:k$ score nodes.
- $CV^i_{j:k}$: Continuation value(Expected payoff in equilibrium) of player i at $j:k$ score nodes.
- $d^i_{j:k}$: Difference between continuation values at $j:k$ score node(s). (Positive value only. e.g. for player 1 at $j:k$ nodes, $d^1_{j:k} = CV^1_{j+1:k}$-$CV^1_{j:k+1}$)
- $dr_{j:k}$: Ratio of difference between continuation values at $j:k$ score node(s).

$$dr_{j:k} = \frac{d^1_{j:k}}{d^2_{j:k}}$$

B Derivation of 3.3.1

F.O.C:

$$\frac{d}{dx}Eu_1 = \frac{av_{1W}}{ax+y} - \frac{a^2xv_{1W}}{(ax+y)^2} - \frac{yv_{1L}\,a}{(ax+y)^2} - 1 = 0$$

$$\frac{d}{dy}Eu_2 = -\frac{axv_{2L}}{(ax+y)^2} + \frac{v_{2W}}{ax+y} - \frac{yv_{2W}}{(ax+y)^2} - 1 = 0$$

Since $x > 0, y > 0$

$$x = \frac{-y + \sqrt{-yv_{1L}\,a + av_{1W}\,y}}{a} = \frac{-y + \sqrt{ayd_1}}{a}$$

$$y = -ax + \sqrt{-axv_{2L} + av_{2W}\,x} = -ax + \sqrt{axd_2}$$

We have:

$$\frac{x}{y} = \frac{d_1}{d_2}$$

Then,

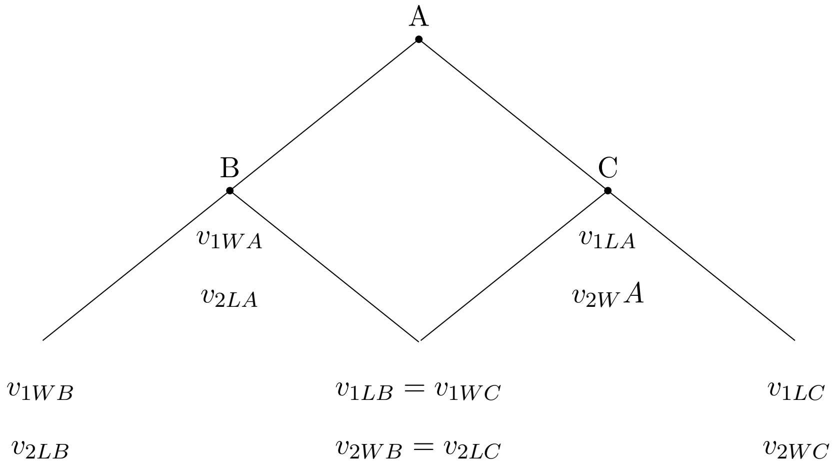
$$x^* = \frac{a{d_1}^2 d_2}{\left(ad_1 + d_2\right)^2}$$

$$y^* = \frac{ad_1\,{d_2}^2}{\left(ad_1 + d_2\right)^2}$$

$$p_1^* = \left(1 + \frac{d_2}{ad_1}\right)^{-1}$$

$$p_2^* = \left(1 + \frac{ad_1}{d_2}\right)^{-1}$$

C Derivation of 3.3.2



Node B and C share the continuation values in the middle. This helps us to write down the difference ratio of the first stage only with the difference values of the second stage. We can see this below.

C.1 Continuation value at B node

Using the result of the one-stage structure-that equilibrium effort level and the probability of winning is the function of difference ratio-, we can write the continuation value at the B node(v_{1W}, v_{2L}) in terms of continuation values.

$$v_{1WA} = p_{1B}v_{1WB} + (1 - p_{1B})v_{1LB} - x_B = p_{1B}d_{1B} + v_{1LB} - x_B$$

$$= \frac{{d_{1B}}^2 a}{ad_{1B} + d_{2B}} + v_{1LB} - \frac{d_{2B}\, a{d_{1B}}^2}{(ad_{1B} + d_{2B})^2}$$

$$v_{2LA} = p_{2B}v_{2WB} + (1-p_{2B})v_{2LB} - y_B = -p_{1B}d_{2B} + v_{2WB} - y_B$$

$$= -\frac{d_{2B}\,d_{1B}\,a}{ad_{1B}+d_{2B}} + v_{2WB} - \frac{ad_{1B}\,{d_{2B}}^2}{(ad_{1B}+d_{2B})^2}$$

C.2 Continuation value at C node

Likewise:

$$v_{1LA} = -\frac{d_{1C}\,d_{2C}}{ad_{1C}+d_{2C}} + v_{1WC} - \frac{d_{2C}\,a{d_{1C}}^2}{(ad_{1C}+d_{2C})^2}$$

$$v_{2WA} = \frac{{d_{2C}}^2}{ad_{1C}+d_{2C}} + v_{2LC} - \frac{ad_{1C}\,{d_{2C}}^2}{(ad_{1C}+d_{2C})^2}$$

C.3 Difference values of first stage

Now we can get d_{1A}, d_{2A}:

$$d_{1A} = \frac{a^4{d_{1B}}^3{d_{1C}}^2 + 2\,a^3{d_{1B}}^3 d_{1C}\,d_{2C} + 2\,a^3{d_{1B}}^2{d_{1C}}^2 d_{2C} + a^2{d_{1B}}^3{d_{2C}}^2 + a^2{d_{1B}}^2 d_{1C}\,{d_{2C}}^2}{(ad_{1B}+d_{2B})^2\,(ad_{1C}+d_{2C})^2}$$

$$+\frac{4\,a^2 d_{1B}\,d_{2B}\,{d_{1C}}^2 d_{2C} + 2\,ad_{1B}\,d_{2B}\,d_{1C}\,{d_{2C}}^2 + 2\,a{d_{2B}}^2{d_{1C}}^2 d_{2C} + {d_{2B}}^2 d_{1C}\,{d_{2C}}^2}{(ad_{1B}+d_{2B})^2\,(ad_{1C}+d_{2C})^2}$$

$$d_{2A} = \frac{a^4{d_{1B}}^2 d_{2B}\,{d_{1C}}^2 + 2\,a^3{d_{1B}}^2 d_{2B}\,d_{1C}\,d_{2C} + 2\,a^3 d_{1B}\,{d_{2B}}^2{d_{1C}}^2 + a^2{d_{1B}}^2 d_{2B}\,{d_{2C}}^2 + a^2{d_{1B}}^2{d_{2C}}^3}{\left(ad_{1B} + d_{2B}\right)^2\left(ad_{1C} + d_{2C}\right)^2}$$
$$+\frac{4\,a^2 d_{1B}\,{d_{2B}}^2 d_{1C}\,d_{2C} + 2\,ad_{1B}\,{d_{2B}}^2{d_{2C}}^2 + 2\,ad_{1B}\,d_{2B}\,{d_{2C}}^3 + {d_{2B}}^2{d_{2C}}^3}{\left(ad_{1B} + d_{2B}\right)^2\left(ad_{1C} + d_{2C}\right)^2}$$

$$\frac{d_{1A}}{d_{2A}} = \frac{nd_{1A}}{nd_{2A}}$$

where nd_{iA} means numerator of d_{iA}:

$$nd_{1A} = a^4{d_{1B}}^3{d_{1C}}^2 + 2\,a^3{d_{1B}}^3 d_{1C}\,d_{2C} + 2\,a^3{d_{1B}}^2{d_{1C}}^2 d_{2C} + a^2{d_{1B}}^3{d_{2C}}^2 + a^2{d_{1B}}^2 d_{1C}\,{d_{2C}}^2 + 4\,a^2 d_{1B}\,d_{2B}\,{d_{1C}}^2 d_{2C} + 2\,ad_{1B}\,d_{2B}\,d_{1C}\,{d_{2C}}^2 + 2\,ad_{2B}{}^2{d_{1C}}^2 d_{2C} + {d_{2B}}^2 d_{1C}\,{d_{2C}}^2$$

$$nd_{2A} = a^4{d_{1B}}^2 d_{2B}\,{d_{1C}}^2 + 2\,a^3{d_{1B}}^2 d_{2B}\,d_{1C}\,d_{2C} + 2\,a^3 d_{1B}\,{d_{2B}}^2{d_{1C}}^2 + a^2{d_{1B}}^2 d_{2B}\,{d_{2C}}^2 + a^2{d_{1B}}^2{d_{2C}}^3 + 4\,a^2 d_{1B}\,{d_{2B}}^2 d_{1C}\,d_{2C} + 2\,ad_{1B}\,{d_{2B}}^2{d_{2C}}^2 + 2\,ad_{1B}\,d_{2B}\,{d_{2C}}^3 + {d_{2B}}^2{d_{2C}}^3$$

C.4 Simplification of d_{1A}, d_{2A}

We substitute $d_{1B}, d_{2B}, d_{1C}, d_{2B}$ with A, B, C, D to make the difference values we got in previous section easier to handle.

$d_{1A} =$

$$\frac{A^3C^2a^4 + 2\,A^3CDa^3 + 2\,A^2C^2Da^3 + A^3D^2a^2 + A^2CD^2a^2 + 4\,ABC^2Da^2 + 2\,ABCD^2a + 2\,B^2C^2Da + B^2CD^2}{(Aa + B)^2\,(Ca + D)^2}$$

$d_{2A} =$

$$\frac{A^2BC^2a^4 + 2\,A^2BCDa^3 + 2\,AB^2C^2a^3 + A^2BD^2a^2 + A^2D^3a^2 + 4\,AB^2CDa^2 + 2\,AB^2D^2a + 2\,ABD^3a + B^2D^3}{(Aa + B)^2\,(Ca + D)^2}$$

C.4.1 Numerator of d_{1A}

Let $nd_{1A} = numerator\ of\ d_{1A}$. Then,

$$\frac{nd_{1A}}{A^2C^2} = Aa^4+2\,\frac{ADa^3}{C}+2\,Da^3+\frac{AD^2a^2}{C^2}+\frac{D^2a^2}{C}+4\,\frac{BDa^2}{A}+2\,\frac{BD^2a}{AC}+2\,\frac{B^2Da}{A^2}+\frac{B^2D^2}{A^2C}$$

Substitute $B/A = i$, $D/C = j$.

$$\frac{nd_{1A}}{A^2C^2} = Aa^4 + 2\,Aa^3j + Aa^2j^2 + 2\,Da^3 + 4\,Da^2i + Da^2j + 2\,Dai^2 + 2\,Daij + Di^2j$$

$$= (2\,Da + Dj)\,i^2 + \left(4\,Da^2 + 2\,Daj\right)i + Aa^4 + 2\,Aa^3j + Aa^2j^2 + 2\,Da^3 + Da^2j$$

$$= \left\{Aa^4 + 2\,Aja^3 + Aj^2a^2\right\} + \left\{(2\,Da + Dj)\,i^2 + \left(4\,Da^2 + 2\,Daj\right)i + 2\,Da^3 + Da^2j\right\}$$

$$= \left\{Aa^2\,(a+j)^2\right\} + \left\{D\,(a+i)^2\,(j+2\,a)\right\}$$

C.4.2 Numerator of d_{2A}

Similarly,

$$\frac{nd_{2A}}{A^2C^2} = Ba^4+2\,\frac{BDa^3}{C}+2\,\frac{B^2a^3}{A}+\frac{BD^2a^2}{C^2}+\frac{D^3a^2}{C^2}+4\,\frac{B^2Da^2}{AC}+2\,\frac{B^2D^2a}{AC^2}+2\,\frac{BD^3a}{AC^2}+\frac{B^2D^3}{A^2C^2}$$

$$= Ba^4 + 2\,Ba^3i + 2\,Ba^3j + 4\,Ba^2ij + Ba^2j^2 + 2\,Baij^2 + Da^2j^2 + 2\,Daij^2 + Di^2j^2$$

$$= a\,(a+2\,i)\left(Ba^2 + 2\,Baj + Bj^2 + Dj^2\right) + Di^2j^2$$

$$= \left\{(a+j)^2\,Ba\,(a+2\,i)\right\} + \left\{a\,(a+2\,i)\,Dj^2 + Di^2j^2\right\}$$

$$= Dj^2 (a+i)^2 + (a+j)^2 Ba (a+2i)$$

C.4.3 Simple expression of d_{1A} , d_{2A}

Denominator of both difference value(d_{1A} , d_{2A}) is $(Aa+B)^2 (Ca+D)^2$. Divide this by A^2C^2:

$$\frac{(Aa+B)^2 (Ca+D)^2}{A^2C^2} = (a+i)^2 (a+j)^2$$

Now we can rewrite d_{1A} , d_{2A} as:

$$d_{1A} = \frac{D(a+i)^2 (a+j) + (a+i)^2 Da + Aa^2 (a+j)^2}{(a+i)^2 (a+j)^2} = \frac{D}{a+j} + \frac{Da}{(a+j)^2} + \frac{Aa^2}{(a+i)^2}$$

$$d_{2A} = \frac{Dj^2 (a+i)^2 + (a+j)^2 Ba (a+i) + (a+j)^2 iBa}{(a+i)^2 (a+j)^2} = \frac{Dj^2}{(a+j)^2} + \frac{Ba}{a+i} + \frac{Bai}{(a+i)^2}$$

Since $\frac{a}{a+i} = p_{1B}$, $\frac{i}{a+i} = p_{2B}$, $\frac{a}{a+j} = p_{1C}$, $\frac{j}{a+j} = p_{2C}$, $\frac{p_{1C}}{p_{2C}} = \frac{Ca}{D}$ in equilibrium,

$$d_{1A} = \frac{Dp_{1C}}{a} + \frac{D{p_{1C}}^2}{a} + A{p_{1B}}^2 = A{p_{1B}}^2 + Cp_{1C}\,p_{2C} + Cp_{2C}$$

$$d_{2A} = Bp_{1B}\,p_{2B} + D{p_{2C}}^2 + Bp_{1B}$$

By its definition, $p_{2C} = 1 - p_{1C},\ p_{1B} = 1 - p_{2B}$

$$d_{1A} = A{p_{1B}}^2 - C{p_{1C}}^2 + C = d_{1B}{p_{1B}}^2 - d_{1C}{p_{1C}}^2 + d_{1C}$$

$$d_{2A} = -B{p_{2B}}^2 + D{p_{2C}}^2 + B = -d_{2B}{p_{2B}}^2 + d_{2C}{p_{2C}}^2 + d_{2B}$$

D Expressions omitted in the text

D.1 Numerator and denominator of continuation values at 1:1

$nCV_{1:1}^1 = (256\,a^{22} + 3072\,a^{21} + 19968\,a^{20} + 89856\,a^{19} + 309088\,a^{18} + 854784\,a^{17} + 1960960\,a^{16} + 3808272\,a^{15} + 6345049\,a^{14} + 9147876\,a^{13} + 11468710\,a^{12} + 12527796\,a^{11} + 11914672\,a^{10} + 9833144\,a^9 + 6999694\,a^8 + 4258740\,a^7 + 2187023\,a^6 + 931980\,a^5 + 322308\,a^4 + 87648\,a^3 + 17872\,a^2 + 2496\,a + 192)va^6$

$dCV_{1:1}^1 = (a^3 + 2\,a^2 + 3\,a + 2)^2(8\,a^9 + 28\,a^8 + 62\,a^7 + 103\,a^6 + 135\,a^5 + 135\,a^4 + 103\,a^3 + 62\,a^2 + 28\,a + 8)(a + 1)4\,a^6 + 4\,a^5 + 11\,a^4 + 10\,a^3 + 11\,a^2 + 4\,a + 4)t(2\,a^2 + a + 1)^2(2\,a^2 + 3\,a + 2)$

$nCV_{1:1}^2 = (192\,a^{22} + 2496\,a^{21} + 17872\,a^{20} + 87648\,a^{19} + 322308\,a^{18} + 931980\,a^{17} + 2187023\,a^{16} + 4258740\,a^{15} + 6999694\,a^{14} + 9833144\,a^{13} + 11914672\,a^{12} + 12527796\,a^{11} + 11468710\,a^{10} + 9147876\,a^9 + 6345049\,a^8 + 3808272\,a^7 + 1960960\,a^6 + 854784\,a^5 + 309088\,a^4 + 89856\,a^3 + 19968\,a^2 + 3072\,a + 256)v$

$dCV_{1:1}^2 = dCV_{1:1}^1$

D.2 Numerator and denominator of difference ratio at Round 2

D.2.1 $dr_{1:0}$

$ndr_{1:0} = (98304\,a^{31} + 1310720\,a^{30} + 9232384\,a^{29} + 44658688\,a^{28} + 164980736\,a^{27} + 493030400\,a^{26} + 1235051520\,a^{25} + 2656024832\,a^{24} + 4986960512\,a^{23} + 8276995200\,a^{22} + 12257672928\,a^{21} + 16314984208\,a^{20} + 19628182680\,a^{19} + 21440884716\,a^{18} + 21340500800\,a^{17} + 19405098927\,a^{16} + 16149329084\,a^{15} + 12311419852\,a^{14} + 8596876436\,a^{13} + 5492255490\,a^{12} + 3202771324\,a^{11} + 1698582960\,a^{10} + 815094572\,a^9 + 351478119\,a^8 + 134963728\,a^7 + 45603984\,a^6 + 13349312\,a^5 + 3314528\,a^4 + 677120\,a^3 + 108288\,a^2 + 12288\,a + 768)a$

$ddr_{1:0} = 12288\,a^{29} + 184320\,a^{28} + 1487872\,a^{27} + 8231936\,a^{26} + 34343168\,a^{25} + 113675264\,a^{24} + 308449664\,a^{23} + 702235264\,a^{22} + 1365336784\,a^{21} + 2298659536\,a^{20} + 3388277044\,a^{19} + 4411491900\,a^{18} + 5109316471\,a^{17} + 5293728314\,a^{16} + 4928663300\,a^{15} + 4138058916\,a^{14} + 3141534634\,a^{13} + 2160846280\,a^{12} + 1348296720\,a^{11} + 763509360\,a^{10} + 392148087\,a^9 + 182314942\,a^8 + 76419568\,a^7 + 28685280\,a^6 + 9538784\,a^5 + 2762176\,a^4 + 677120\,a^3 + 133632\,a^2 + 19200\,a + 1536$

D.2.2 $dr_{0:1}$

$ndr_{0:1} = (1536\,a^{29} + 19200\,a^{28} + 133632\,a^{27} + 677120\,a^{26} + 2762176\,a^{25} + 9538784\,a^{24} + 28685280\,a^{23} + 76419568\,a^{22} + 182314942\,a^{21} + 392148087\,a^{20} + 763509360\,a^{19} + 1348296720\,a^{18} + 2160846280\,a^{17} + 3141534634\,a^{16} + 4138058916\,a^{15} + 4928663300\,a^{14} + 5293728314\,a^{13} + 5109316471\,a^{12} + 4411491900\,a^{11} +$

$3388277044\,a^{10}+2298659536\,a^{9}+1365336784\,a^{8}+702235264\,a^{7}+308449664\,a^{6}+113675264\,a^{5}+34343168\,a^{4}+8231936\,a^{3}+1487872\,a^{2}+184320\,a+12288)a^{3}$

$ddr_{0:1} = 768\,a^{31}+12288\,a^{30}+108288\,a^{29}+677120\,a^{28}+3314528\,a^{27}+13349312\,a^{26}+45603984\,a^{25}+134963728\,a^{24}+351478119\,a^{23}+815094572\,a^{22}+1698582960\,a^{21}+3202771324\,a^{20}+5492255490\,a^{19}+8596876436\,a^{18}+12311419852\,a^{17}+16149329084\,a^{16}+19405098927\,a^{15}+21340500800\,a^{14}+21440884716\,a^{13}+19628182680\,a^{12}+16314984208\,a^{11}+12257672928\,a^{10}+8276995200\,a^{9}+4986960512\,a^{8}+2656024832\,a^{7}+1235051520\,a^{6}+493030400\,a^{5}+164980736\,a^{4}+44658688\,a^{3}+9232384\,a^{2}+1310720\,a+98304$

D.3 Expression omitted in proof of Result 5

$A=(a+1)(2\,a^{2}+3\,a+2)(4\,a^{6}+4\,a^{5}+11\,a^{4}+10\,a^{3}+11\,a^{2}+4\,a+4)(8\,a^{4}+8\,a^{3}+4\,a^{2}+a+1)(1536\,a^{20}+13568\,a^{19}+65152\,a^{18}+210624\,a^{17}+506496\,a^{16}+952992\,a^{15}+1441256\,a^{14}+1778212\,a^{13}+1794754\,a^{12}+1467965\,a^{11}+939539\,a^{10}+420838\,a^{9}+63244\,a^{8}-100671\,a^{7}-126677\,a^{6}-92024\,a^{5}-49796\,a^{4}-21040\,a^{3}-6768\,a^{2}-1536\,a-192)$

$B=(2\,a^{2}+a+1)(49152\,a^{31}+630784\,a^{30}+4276224\,a^{29}+19875840\,a^{28}+70420480\,a^{27}+201459200\,a^{26}+482329856\,a^{25}+990233856\,a^{24}+1774369984\,a^{23}+2813033312\,a^{22}+3989359648\,a^{21}+5107413256\,a^{20}+5948373280\,a^{19}+6341878858\,a^{18}+6219262853\,a^{17}+5627724558\,a^{16}+4705829072\,a^{15}+3635797268\,a^{14}+2591956698\,a^{13}+1701280220\,a^{12}+1025534520\,a^{11}+566307250\,a^{10}+285774761\,a^{9}+131452734\,a^{8}+54942160\,a^{7}+20762016\,a^{6}+7032352\,a^{5}+2102720\,a^{4}+540416\,a^{3}+113664\,a^{2}+17664\,a+1536)$

D.4 Expression omitted in proof of Result 10

$A=2\,a(a+1)(24576\,a^{29}+290816\,a^{28}+1841152\,a^{27}+8004608\,a^{26}+26461696\,a^{25}+70151936\,a^{24}+153841408\,a^{23}+284437888\,a^{22}+448522272\,a^{21}+606876752\,a^{20}+705134680\,a^{19}+699242180\,a^{18}+580805938\,a^{17}+384387541\,a^{16}+170585650\,a^{15}-3587528\,a^{14}-107829586\,a^{13}-143301238\,a^{12}-131487730\,a^{11}-98295300\,a^{10}-63085800\,a^{9}-35515255\,a^{8}-17671408\,a^{7}-7759696\,a^{6}-2979008\,a^{5}-982624\,a^{4}-270592\,a^{3}-59136\,a^{2}-9216\,a-768)$

$B=(2\,a^{2}+a+1)(49152\,a^{31}+630784\,a^{30}+4276224\,a^{29}+19875840\,a^{28}+70420480\,a^{27}+201459200\,a^{26}+482329856\,a^{25}+990233856\,a^{24}+1774369984\,a^{23}+2813033312\,a^{22}+3989359648\,a^{21}+5107413256\,a^{20}+5948373280\,a^{19}+6341878858\,a^{18}+6219262853\,a^{17}+5627724558\,a^{16}+4705829072\,a^{15}+3635797268\,a^{14}+2591956698\,a^{13}+1701280220\,a^{12}+1025534520\,a^{11}+566307250\,a^{10}+285774761\,a^{9}+131452734\,a^{8}+54942160\,a^{7}+20762016\,a^{6}+7032352\,a^{5}+2102720\,a^{4}+540416\,a^{3}+113664\,a^{2}+17664\,a+1536)$

D.5 Numerator and denominator of $dr_{0:0}$

$ndr_{0:0} =$

$a^3(22412794049557105213 44\,a^{167} + 1531540926719735522918 40\,a^{166} +$
$53265250213997435934474 24\,a^{165} + 125578335116110763044896768\,a^{164}$
$+ 2255685866473116150609739776\,a^{163} + 32899595501158155314169643008\,a^{162} + 405544684145904623458927509504\,a^{161} +$
$4342545929120227838402739830784\,a^{160} + 41207143850651654752196603412480\,a^{159} +$
$351797016251777998065262913388544\,a^{158} + 2734293968582820997323832663474176\,a^{157} +$
$19532521370441690978932553272000512\,a^{156} + 129244087778931657382845729631895552\,a^{155} +$
$797288906689738630961506086071304192\,a^{154} + 4610503763210984080520101429889728512\,a^{153} +$
$25109451578509119371950546683974123520\,a^{152} + 129310118163011093830933371582303698944\,a^{151} +$
$631913443944250912236672739879298269184\,a^{150} + 2939331377850529829300648441224460500992\,a^{149} +$
$13049235320315177928542602279006754570240\,a^{148} + 55426181513457572429317416909149021667328\,a^{147} +$
$225722002056047184083194858054234281082880\,a^{146} + 883082592160994058880850496298476158058496\,a^{145} +$
$3324708915641950760730682124195490858795008\,a^{144} + 12064656537156800661850757036705710189051904\,a^{143} +$
$42257746654167144253973010706458030471380992\,a^{142} + 143051772442018872007662539622221458670878720\,a^{141} +$
$468589072611085251284064615235465250730934272\,a^{140} + 1486880454382901929184776108403168587269799936\,a^{139} +$
$4574874362806023064676039985656514601056993280\,a^{138} + 13661586391585606672683235703215704308229079040\,a^{137} +$
$39628762008379569240082827698661414866526404608\,a^{136} + 111750103042441128230039074184408182442736746496\,a^{135} +$
$306569446024907787603929950163822020827222638592\,a^{134} + 818739978356557137767350823422286192497498521600\,a^{133} +$
$2129959561079975587665078630330285919409226645504\,a^{132} + 5400801912000568046885909860822825526215680458752\,a^{131} +$
$13354969711418710506609879243950889250461096869888\,a^{130} + 32221519729844305479601968402701036471848798257152\,a^{129} +$
$75888167918472364758252207047265960046828057526272\,a^{128} + 174550275174852208146299544710879029030724692869120\,a^{127} +$
$392253660612998004201442198016352745418923272306688\,a^{126} + 861557198257752656874382813296139080188561582981120\,a^{125} +$
$1850258728145905902376543623569264704130814184521728\,a^{124} + 3886542135258053395114090673674123456117517615890432\,a^{123} +$
$7987661772654516321593018660141053329111075616980992\,a^{122} + 16067052808209984069984794286936271561110525378560000\,a^{121} +$
$31640302711717790579898298181140215425006882384510976\,a^{120} + 61017399162839048701045947350015134017889440217890816\,a^{119} +$
$115262992557991166294980320804651316469988285191290880\,a^{118} + 213332890680004190030294479885199429139862776903237632\,a^{117} +$
$386955082328994187533778038991881095097773391930187776\,a^{116} + 688012001085917087866233125021654316808138791874666496\,a^{115} +$
$1199381209778710524385595810289235889421938136251387904\,a^{114} + 2050372120433812904071281982521306064108267482852440064\,a^{113} +$
$3438004603023944363489437458793366090542451863917077504\,a^{112} + 5655337062518394125532050685213991617706290913213622656\,a^{111} +$
$9127772572103326353866554093670920214490894688856067968\,a^{110} + 14457646863184134235230295591290797596940992570302979968\,a^{109} +$
$22476333201412361239869885107958810871852392054187918256\,a^{108} + 34301522333641843525908102646590473582852198092858661720\,a^{107} +$
$51395255828070672268000508391504781152418141964253644880\,a^{106} + 75615820500722324881277747694082305092014292267747563588\,a^{105} +$
$109254002340239670567052124696639917951843763857247186667\,a^{104} + 155041858264717426302501587337163407589333617663875989692\,a^{103} +$
$216120643279316295622673705722645458889424680929965411774\,a^{102} + 295954468021420847004225673630813366069261855298094314224\,a^{101} +$
$398178286681734016228837474614657383374765638159534280957\,a^{100} + 526373410054103429640044233142060194888242989715792041368\,a^{99} +$
$683771078553618479153507249590793961146064556816106255708\,a^{98} + 872894433972000768232198115471817183353246365793204565268\,a^{97} +$
$1095160656263009170818679770627538223971806639969411480609\,a^{96} + 1350476671849875774364223384061958719640324537453390306140\,a^{95} +$
$1636871821004031292621635758814025808421800462832028518102\,a^{94} + 1950217103486736120928002641097155689389257746011067050280\,a^{93} +$
$2284081151262790113661526096437029354822565703586820548423\,a^{92} + 2629766507206274294206652647421395209443749568288903124024\,a^{91} +$
$2976555675207601518936926247804972090679155477539007683528\,a^{90} + 3312175543572018621781746479676927593219507100239343673704\,a^{89} +$
$3623463284452178446632836572483398852722592482534630279246\,a^{88} + 3897189957767810300821683668145558118041228716326799564920\,a^{87} +$
$4120973767657043055250017752275396854716231821871129118492\,a^{86} + 4284197265985083705299545480495385960810293936500543006720\,a^{85} +$
$4378835128634232583431511407633497705589541709612905014834\,a^{84} + 4400103443628278317443056124460554580113466560185498444752\,a^{83} +$
$4346857903731691588876721672319127325112473057000828467720\,a^{82} + 4221695055777117044010805251003698182307177761142659755144\,a^{81} +$
$4030744225383110190973743691343351524098121124062928650034\,a^{80} + 3783173124671108140716854391244162085697703526829275307800\,a^{79} +$
$3490462317328022593226658982560101947916446038273734275836\,a^{78} + 3165528058140518651899148326742312689670378089008109849776\,a^{77} +$
$2821786281313486866106653679670254130782092558721203760846\,a^{76} + 2472251290982619851497922093536722073049370476229434021 52\,a^{75} +$
$2128751645370108080542659744410391391345049911116519662896\,a^{74} + 1801325263813065036763551698620810628227952359763083045556\,a^{73} +$
$1497829622531187437299857829510843022712404504847298250951\,a^{72} + 1223775263575199676629199213442935845380402418994622959948\,a^{71} +$
$982365706827969486190529923998638557381692686599559959206\,a^{70} + 774707336590042025644296467021479593091537347156479196752\,a^{69} +$
$600140763877396871933588644547941137921001532785510190241\,a^{68} + 456640978624686395736357933093838526976461361941485584 24\,a^{67} +$
$341236490124636204968443524552228541890280112289583321020\,a^{66} + 250405894442164410267168807235516742281093949327259851876\,a^{65} +$
$180421730913199909743842852554804766165161572286938680445\,a^{64} + 127623904679230719005197492955944278425606331838374735500\,a^{63} +$
$88616503245956349322923133931291919290765112776119688750\,a^{62} + 60391223061966590325306796412049568787690343749508496744\,a^{61} +$
$40387177760451381434128144059978187257564963235337371243\,a^{60} + 26500491692161173280579500912942611625479015832519901192\,a^{59} +$
$17058140240015245265593182666361277255906879078821854104\,a^{58} + 10769597154636183494159193384429836652109267864877116832\,a^{57} +$
$6667698004317279347421915615621081656625641435454239792\,a^{56} + 4047393960730024087739048966899156597656564260969841408\,a^{55} +$
$2408286144932631564220061259094462322718116745324678400\,a^{54} + 1404361837313064200727401284318041909031022894801751040\,a^{53} +$
$802395764306897210583333658606826046859787128282022912\,a^{52} + 449088651684483635679960619051013960247917143410909184\,a^{51} +$
$246149572391186064609040894089864530891268846361350144\,a^{50} + 132091108835145633070793117967307232491412506787659776\,a^{49} +$
$69379863117571657973286464316004082364239827766304768\,a^{48} + 35657432343123129683246805995909517925648684640763904\,a^{47} +$
$17926193630817764671144723031608677456771705833455616\,a^{46} + 8812616776503707878812375598469552609467963192115200\,a^{45} +$
$4234968798554607951305091495618146567358864285761536\,a^{44} + 1988683572602901632964792920949019021558136677859328\,a^{43} +$
$912187810567769220348023988481369798320173019561984\,a^{42} + 408535409645037934309981589898821327125986632794112\,a^{41} +$
$178573058040262175236500826467506411832335610675200\,a^{40} + 76145875721951083935514210969990909946861558169600\,a^{39} +$
$31660169743126227760869962786620085587256878825472\,a^{38} + 12829112673093385577128313950404623004787753877504\,a^{37} +$
$5063650043668554267735536179186623172288308051968\,a^{36} + 1945662968544620219397456158198433924934061260 8\,a^{35} +$
$727355110164527269512359803067769418307696852992\,a^{34} + 264376018815477047319304058368751582922183540736\,a^{33} +$
$93368045512764600066251659725559782274932670464\,a^{32} + 32015458058638230698880965257368239042413461504\,a^{31} +$
$10650496021756458970254695949568718067789201408\,a^{30} + 3434541665628500012970927463173157444250501120\,a^{29} +$
$1072682400059089119120714086651556916688322560\,a^{28} + 324162799167509253781168163295260574534860800\,a^{27} +$
$94689017674028350642891025506700488463089664\,a^{26} + 26705461835090441478774596404090556771205120\,a^{25} +$
$7263481292137722761154200801031669269135360\,a^{24} + 1902696582204389961355030577241481818406912\,a^{23} +$
$479357520830961518865939409315612894167040\,a^{22} + 115969184139900275118005485495358412816384\,a^{21} +$
$26895546623060499981690883645316381278208\,a^{20} + 5968402696986771433102044841289543843840\,a^{19} +$
$1264655030099963593240817552584739913728\,a^{18} + 255278363700759595697195424934814285824\,a^{17} +$
$48961190393447588946212269319641890816\,a^{16} + 8896234210087273470214580490614079488\,a^{15} +$
$1526225093867325951913745851529297920\,a^{14} + 246271004484852372434988352399212544\,a^{13} +$
$37208783953975025532201208050089984\,a^{12} + 5236406426606629236226799901343744\,a^{11} +$
$682128098320921718632941722009600\,a^{10} + 81634839108006596704482329886720\,a^{9} +$
$8893153358224938967862096166912\,a^{8} + 871752919634909283467390877696\,a^{7} + 75762767717845860707525984256\,a^{6} +$
$5724299993062850063889334272\,a^{5} + 365951392349455629372358656\,a^{4} + 19025240308929358492336128\,a^{3} + 755015435104142789443584$
$a^{2} + 20358287928347703902208\,a + 2801599256194638151 68)$

$ddr_{0:0} =$

$$
\begin{aligned}
&280159925619463815168\,a^{167} + 2035828792834770390208\,a^{166} + 755015435104142789443584\,a^{165} + 19025240308929358492336128\,a^{164}\\
&+ 365951392349455629372358656\,a^{163} + 57242999930628500638893334272\,a^{162} + 75762767717845860707525984256\,a^{161}\\
&+ 871752919634909283467390877696\,a^{160} + 8893153358224938967862096166912\,a^{159} + 81634839108006596704482329886720\,a^{158} +\\
&682128098320921718632941722009600\,a^{157} + 5236406426606629236226799901343744\,a^{156} + 37208783953975025532201208050089984\,a^{155} +\\
&246271004484852372434988352399212544\,a^{154} + 1526225093867325951913745851529297920\,a^{153} +\\
&8896234210087273470214580490614079488\,a^{152} + 48961190393447588946212269319641890816\,a^{151} +\\
&255278363700759595697195424934814285824\,a^{150} + 1264655030099963593240817552584739913728\,a^{149} +\\
&5968402696986771433102044841289543843840\,a^{148} + 26895546623060499981690883645316381278208\,a^{147} +\\
&115969184139900275118005485495358412816384\,a^{146} + 479357520830961518865939409315612894167040\,a^{145} +\\
&1902696582204389961355030577241481818406912\,a^{144} + 7263481292137722761154200801031669269135360\,a^{143} +\\
&26705461835090441478774596404090556771205120\,a^{142} + 94689017674028350642891025506700488463089664\,a^{141} +\\
&324162799167509253781168163295260574534860800\,a^{140} + 1072682400059089119120714086651556916688322560\,a^{139} +\\
&34345416665628500012970927463173157444250501120\,a^{138} + 10650496021756458970254695949568718067789201408\,a^{137} +\\
&32015458058638230698880965257368239042413461504\,a^{136} + 93368045512764600066251659725559782274932670464\,a^{135} +\\
&264376018815477047319304058368751582922183540736\,a^{134} + 727355110164527269512359803067769418307696852992\,a^{133} +\\
&1945662968544620219193974561581984339249340612608\,a^{132} + 5063650043668554267735536179186623172288308051968\,a^{131} +\\
&12829112673093385577128313950404623004787753877504\,a^{130} + 31660169743126227760869962786620085587256878825472\,a^{129} +\\
&76145875721951083935514210969990909946861558169600\,a^{128} + 178573058040262175236500826467506411832335610675200\,a^{127} +\\
&408535409645037934309981589898821327125986632794112\,a^{126} + 912187810567769220348023988481369798320173019561984\,a^{125} +\\
&1988683572602901632964792920949019021558136677859328\,a^{124} + 4234968798554607951305091495618146567358864285761536\,a^{123} +\\
&8812616776503707878812375598469552609467963192115200\,a^{122} + 17926193630817764671144723031608677456771705833455616\,a^{121} +\\
&35657432343123129683246805995909517925648684640763904\,a^{120} + 69379863117571657973286464316004082364239827766304768\,a^{119} +\\
&132091108835145633070793117967307232491412506787659776\,a^{118} + 246149572391186064609040894089864530891268846361350144\,a^{117} +\\
&449088651684483635679960619051013960247917143410909184\,a^{116} + 802395764306897210583333658606826046859787128282022912\,a^{115} +\\
&1404361837313064200727401284318041909031022894801751040\,a^{114} + 2408286144932631564220061259094462322718116745324678400\,a^{113} +\\
&4047393960730024087739048966899156597656564260969841408\,a^{112} + 6667698004317279347421915615621081656625641435454239792\,a^{111} +\\
&10769597154636183494159193384429836652109267864877116832\,a^{110} + 17058140240015245265593182666361277255906879078821854104\,a^{109} +\\
&26500491692161173280579500912942611625479015832519901192\,a^{108} + 40387177760451381434128144059978187257564963235337371243\,a^{107} +\\
&60391223061966590325306796412049568787690343749508496744\,a^{106} + 88616503245956349322923133931291919290765112776119688750\,a^{105} +\\
&127623904679230719005197492955944278425606331838374735500\,a^{104} + 180421730913199909743842852554804766165161572286938680445\,a^{103} +\\
&250405894442164410267168807235516742281093949327259851876\,a^{102} + 341236490124636204968443524552228541890280112289583321020\,a^{101} +\\
&456640978624686395736357933093838526976946136194148558424\,a^{100} + 600140763877396871933588644547941137921001532785510190241\,a^{99} +\\
&774707336590042025644296467021479593091537347156479196752\,a^{98} + 982365706827969486190529923998638557381692686599559959206\,a^{97} +\\
&1223775263575199676629199213442935845380402418994622959948\,a^{96} + 1497829622531187437299857829510843022712404504847298250951\,a^{95} +\\
&1801325263813065036763551698620810628227952359763083045556\,a^{94} + 2128751645370108080542659744410391391345049911116519662896\,a^{93} +\\
&2472251290982619851497922309353672207304937047622943402152\,a^{92} + 2821786281313486866106653679670254130782092558721203760846\,a^{91} +\\
&3165528058140518651899148326742312689670378089008109849776\,a^{90} + 3490462317328022593226658982560101947916446038273734275836\,a^{89} +\\
&3783173124671108140716854391244162085697703526829275307800\,a^{88} + 4030744225383110190973743691343351524098121124062928650034\,a^{87} +\\
&4221695055777117044010805251003698182307177761142659755144\,a^{86} + 4346857903731691588876721672319127325112473057000828467720\,a^{85} +\\
&4400103443628278317443056124460554580113466560185498444752\,a^{84} + 4378835128634232583431511407633497705589541709612905014834\,a^{83} +\\
&4284197265985083705299545480495385960810293936500543006720\,a^{82} + 4120973767657043055250017752275396854716231821871129118492\,a^{81} +\\
&3897189957767810300821683668145558118041228716326799564920\,a^{80} + 3623463284452178446632836572483398852722592482534630279246\,a^{79} +\\
&3312175543572018621781746479676927593219507100239343673704\,a^{78} + 2976555675207601518936926247804972090679155477539007683528\,a^{77} +\\
&2629766507206274294206652647421395209443749568288903124024\,a^{76} + 2284081151262790113661526096437029354822565703586820548423\,a^{75} +\\
&1950217103486736120928002641097155689389257746011067050280\,a^{74} + 1636871821004031292621635758814025808421800462832028518102\,a^{73} +\\
&1350476671849875774364223384061958719640324537453390306140\,a^{72} + 1095160656263009170818679770627538223971806639969411480609\,a^{71} +\\
&872894433972000768232198115471817183353246365793204565268\,a^{70} + 683771078553618479153507249590793961146064556816106255708\,a^{69} +\\
&526373410054103429640044233142060194888242989715792041368\,a^{68} + 398178286681734016228837474614657383374765638159534280957\,a^{67} +\\
&295954468021420847004225673630813366069261855298094314224\,a^{66} + 216120643279316295622673705722645458889424680929965411774\,a^{65} +\\
&155041858264717426302501587337163407589333617663875989692\,a^{64} + 109254002340239670567052124696639917951843763857247186667\,a^{63} +\\
&75615820500722324881277747694082305092014292267747563588\,a^{62} + 51395255828070672268000508391504781152418141964253644880\,a^{61} +\\
&34301522333641843525908102646590473582852198092858661720\,a^{60} + 22476333201412361239869885107958810871852392054187918256\,a^{59} +\\
&14457646863184134235230295591290797596940992570302979968\,a^{58} + 9127772572103326353866554093670920214490894688856067968\,a^{57} +\\
&5655337062518394125532050685213991617706290913213622656\,a^{56} + 3438004603023944363489437458793366090542451863917077504\,a^{55} +\\
&2050372120433812904071281982521306064108267482852440064\,a^{54} + 1199381209778710524385595810289235889421938136251387904\,a^{53} +\\
&688012001085917087866233125021654316808138791874666496\,a^{52} + 386955082328994187533778038991881095097773391930187776\,a^{51} +\\
&213332890680004190030294479885199429139862776903237632\,a^{50} + 115262992557991166294980320804651316469988285191290880\,a^{49} +\\
&61017399162839048701045947350015134017889440217890816\,a^{48} + 31640302711717790579898298181140215425006882384510976\,a^{47} +\\
&16067052808209984069984794286936271561110525378560000\,a^{46} + 7987661772654516321593018660141053329111075616980992\,a^{45} +\\
&3886542135258053395114090673674123456117517615890432\,a^{44} + 1850258728145905902376543623569264704130814184521728\,a^{43} +\\
&861557198257752656874382813296139080188561582981120\,a^{42} + 392253660612998004201442198016352745418923272306688\,a^{41} +\\
&174550275174852208146299544710879029030724692869120\,a^{40} + 75888167918472364758252207047265960046828057526272\,a^{39} +\\
&32221519729844305479601968402701036471848798257152\,a^{38} + 13354969711418710506609879243950889250461096869888\,a^{37} +\\
&5400801912000568046885909860822825526215680458752\,a^{36} + 2129959561079975587665078630330285919409226645504\,a^{35} +\\
&818739978356557137767350823422286192497498521600\,a^{34} + 306569446024907787603929950163822020827222638592\,a^{33} +\\
&111750103042441128230039074184408182442736746496\,a^{32} + 39628762008379569240082827698661414866526404608\,a^{31} +\\
&13661586391585606672683235703215704308229079040\,a^{30} + 4574874362806023064676039985656514601056993280\,a^{29} +\\
&1486880454382901929184776108403168587269799936\,a^{28} + 468589072611085251284064615235465250730934272\,a^{27} +\\
&143051772442018872007662539622221458670878720\,a^{26} + 42257746654167144253973010706458030471380992\,a^{25} +\\
&12064656537156800661850757036705710189051904\,a^{24} + 3324708915641950760730682124195490858795008\,a^{23} +\\
&883082592160994058880850496298476158058496\,a^{22} + 225722002056047184083194858054234281082880\,a^{21} +\\
&55426181513457572429317416909149021667328\,a^{20} + 13049235320315177928542602279006754570240\,a^{19} +\\
&2939331377850529829300648441224460500992\,a^{18} + 631913443944250912236672739879298269184\,a^{17} +\\
&129310118163011093830933371582303698944\,a^{16} + 25109451578509119371950546683974123520\,a^{15} +\\
&4610503763210984080520101429889728512\,a^{14} + 797288906689738630961506086071304192\,a^{13} +\\
&129244087778931657382845729631895552\,a^{12} + 19532521370441690978932553272000512\,a^{11} +\\
&2734293968582820997323832663474176\,a^{10} + 351797016251777998065262913388544\,a^{9} + 41207143850651654752196603412480\,a^{8} +\\
&4342545929120227838402739830784\,a^{7} + 405544684145904623458927509504\,a^{6} + 32899595501158155314169643008\,a^{5} +\\
&2255685866473116150609739776\,a^{4} + 125578335116110763044896768\,a^{3} + 5326525021399743593447424\,a^{2} + 153154092671973552291840\,a\\
&+ 224127940495571052134 4
\end{aligned}
$$

D.6 Numerator and Denominator of first order derivatives

$$A = a^3(2359296\,a^{60} + 71368704\,a^{59} + 1137180672\,a^{58} + 12559712256\,a^{57} + 107248287744\,a^{56} + 750586183680\,a^{55} + 4464367173632\,a^{54} + 23133873397760\,a^{53} + 106352180959232\,a^{52} + 439829468220928\,a^{51} + 1654349582626816\,a^{50} + 5709904243425792\,a^{49} + 18216248214654720\,a^{48} + 54045408982697280\,a^{47} + 149882006837803264\,a^{46} + 390220318507612448\,a^{45} + 957267139311978212\,a^{44} + 2219566201570644441\,a^{43} + 4876963271482244840\,a^{42} + 10177026125091004008\,a^{41} + 20204718189439321480\,a^{40} + 38217110272931621564\,a^{39} + 68945320636355510696\,a^{38} + 118722251666798593016\,a^{37} + 195236777143916215632\,a^{36} + 306698762661169543782\,a^{35} + 460269503409187581880\,a^{34} + 659802117748166863768\,a^{33} + 903237438490428884568\,a^{32} + 1180335928374435332364\,a^{31} + 1471649740287704462072\,a^{30} + 1749582645129725269096\,a^{29} + 1981944123504784750412\,a^{28} + 2137627945954130731657\,a^{27} + 2193178224882984915520\,a^{26} + 2138416781149886272768\,a^{25} + 1979325582539300181920\,a^{24} + 1737110069763071658928\,a^{23} + 1443588893693079745408\,a^{22} + 1134280771228282275072\,a^{21} + 841273662570269974528\,a^{20} + 587875200062244730624\,a^{19} + 386237663657653970944\,a^{18} + 238023215186046441472\,a^{17} + 137219786180870922240\,a^{16} + 73777707384269578240\,a^{15} + 36866772607679135744\,a^{14} + 17053513270781804544\,a^{13} + 7268753489876746240\,a^{12} + 2839514991396847616\,a^{11} + 1010254312350154752\,a^{10} + 324914077762584576\,a^{9} + 93613087543787520\,a^{8} + 23895380436451328\,a^{7} + 5328703437406208\,a^{6} + 1019364635574272\,a^{5} + 163156546551808\,a^{4} + 21069448609792\,a^{3} + 2070845325312\,a^{2} + 138915348480\,a + 4831838208)$$

$$B = (a^2 + a + 2)^2(1536\,a^{31} + 17664\,a^{30} + 113664\,a^{29} + 540416\,a^{28} + 2102720\,a^{27} + 7032352\,a^{26} + 20762016\,a^{25} + 54942160\,a^{24} + 131452734\,a^{23} + 285774761\,a^{22} + 566307250\,a^{21} + 1025534520\,a^{20} + 1701280220\,a^{19} + 2591956698\,a^{18} + 3635797268\,a^{17} + 4705829072\,a^{16} + 5627724558\,a^{15} + 6219262853\,a^{14} + 6341878858\,a^{13} + 5948373280\,a^{12} + 5107413256\,a^{11} + 3989359648\,a^{10} + 2813033312\,a^{9} + 1774369984\,a^{8} + 990233856\,a^{7} + 482329856\,a^{6} + 201459200\,a^{5} + 70420480\,a^{4} + 19875840\,a^{3} + 4276224\,a^{2} + 630784\,a + 49152)^2$$

$$C = a^3(2359296\,a^{60} + 71368704\,a^{59} + 1137180672\,a^{58} + 12559712256\,a^{57} + 107248287744\,a^{56} + 750586183680\,a^{55} + 4464367173632\,a^{54} + 23133873397760\,a^{53} + 106352180959232\,a^{52} + 439829468220928\,a^{51} + 1654349582626816\,a^{50} + 5709904243425792\,a^{49} + 18216248214654720\,a^{48} + 54045408982697280\,a^{47} + 149882006837803264\,a^{46} + 390220318507612448\,a^{45} + 957267139311978212\,a^{44} + 2219566201570644441\,a^{43} + 4876963271482244840\,a^{42} + 10177026125091004008\,a^{41} + 20204718189439321480\,a^{40} + 38217110272931621564\,a^{39} + 68945320636355510696\,a^{38} + 118722251666798593016\,a^{37} + 195236777143916215632\,a^{36} + 306698762661169543782\,a^{35} + 460269503409187581880\,a^{34} + 659802117748166863768\,a^{33} + 903237438490428884568\,a^{32} + 1180335928374435332364\,a^{31} + 1471649740287704462072\,a^{30} + 1749582645129725269096\,a^{29} + 1981944123504784750412\,a^{28} + 2137627945954130731657\,a^{27} + 2193178224882984915520\,a^{26} + 2138416781149886272768\,a^{25} + 1979325582539300181920\,a^{24} + 1737110069763071658928\,a^{23} + 1443588893693079745408\,a^{22} + 1134280771228282275072\,a^{21} + 841273662570269974528\,a^{20} + 587875200062244730624\,a^{19} + 386237663657653970944\,a^{18} + 238023215186046441472\,a^{17} + 137219786180870922240\,a^{16} + 73777707384269578240\,a^{15} + 36866772607679135744\,a^{14} + 17053513270781804544\,a^{13} + 7268753489876746240\,a^{12} + 2839514991396847616\,a^{11} + 1010254312350154752\,a^{10} + 324914077762584576\,a^{9} + 93613087543787520\,a^{8} + 23895380436451328\,a^{7} + 5328703437406208\,a^{6} + 1019364635574272\,a^{5} + 163156546551808\,a^{4} + 21069448609792\,a^{3} + 2070845325312\,a^{2} + 138915348480\,a + 4831838208)$$

$$D = (a^2 + a + 2)^2(1536\,a^{31} + 17664\,a^{30} + 113664\,a^{29} + 540416\,a^{28} + 2102720\,a^{27} + 7032352\,a^{26} + 20762016\,a^{25} + 54942160\,a^{24} + 131452734\,a^{23} + 285774761\,a^{22} + 566307250\,a^{21} + 1025534520\,a^{20} + 1701280220\,a^{19} + 2591956698\,a^{18} + 3635797268\,a^{17} + 4705829072\,a^{16} + 5627724558\,a^{15} + 6219262853\,a^{14} + 6341878858\,a^{13} + 5948373280\,a^{12} + 5107413256\,a^{11} + 3989359648\,a^{10} + 2813033312\,a^{9} + 1774369984\,a^{8} + 990233856\,a^{7} + 482329856\,a^{6} + 201459200\,a^{5} + 70420480\,a^{4} + 19875840\,a^{3} + 4276224\,a^{2} + 630784\,a + 49152)^2$$

국문초록

김진욱
사회과학대학 경제학 전공
서울대학교 대학원

삼세판이라는 말이 누구에게도 낯설게 느껴지지 않을 만큼 다전제 형식의 경쟁은 일상적인 가위바위보 게임부터, 프로 스포츠 경기에 이르기까지 광범위하게 이루어진다. 특히 여러 종목에서 정규시즌 이후의 플레이오프나 토너먼트의 상위 라운드는 거의 대부분 다전제 형식으로 진행된다.

이 논문에서 관심을 두고 있는 것은 다전제에서 존재하는 모멘텀이다. 다전제의 매 경기 결과가 각 플레이어 또는 팀의 역량에 의해 독립적으로 결정되는 것이 아니라, 이전 경기의 결과 역시 이후 경기의 결과에 영향을 준다는 것이다. 이런 현상은 행동연구학이나 경제학 공통의 관심 주제이며, 대표적으로 심리적 모멘텀(Psychological Momentum)과 전략적 모멘텀(Strategic Momentum)이 다전제에서 연승을 야기하는 대표적인 요인으로 거론된다.

초기의 연구에서는 동질적인 참가자 간의 3전 2선승제 경쟁에서 각 라운드의 결과가 두 참가자의 선택된 노력 수준에 의해 확률적으로 결정된다는 가정 하에서 모델을 분석하고, 프로테니스경기의 데이터를 이용해 검정을 하였다. 이후 여기에 참가자 간의 비대칭성을 도입한 이론적 모델을 바탕으로 기존 연구보다 더 폭넓은 데이터를 이용하여 검정한 연구에서 1) 심리적 모멘텀이 부재하는 첫 번째 라운드는 전략적 모멘텀으로 잘 설명이 가능하고 2) 전체적으로 전략적 모멘텀과 심리적 모멘텀이 함께 작용하고 있음을 보였다.

이 논문은 기존의 연구에서 사용된 비대칭적 참가자들에 의한 3전 2선승제의 이론적 모델을 확장하여 5전 3선승제의 설정에서의 균형을 도출하고, 결과를 해석하여 더 폭넓은 데이터를 이용한 검정의 준거로 제시하고자 한다. 테니스 종목 중 가장 인기가 많은 메이저 토너먼트 남자 부문은 대부분 3선승제로 진행된다.

비교적 간단한 모델의 설정에도 불구하고, 역진귀납의 과정에서 비대칭의 정도를 나타내는 변수의 존재로 인해 다전제의 라운드 수가 많아질수록 논문의 부록에서 볼 수 있듯이 균형의 복잡성이 급격하게 증가하게 된다. 이에, 우선 이 논문의 모델에서 일반적인 균형의 형태를 분석한 뒤, 역진 귀납의 순서를 따라 각 라운드에서의 균형을 도출하고, 그 결과를 해석한다. 이 논문의 전신인 2선승제 비대칭 참가자 모델에서 얻을 수 있는 결론들이 3선승제 모델에서도 일관되게 확인되는 것을 알 수 있었을 뿐만 아니라, 모델의 균형을 해석하는 과정에서 2선승제의 모델에서는 존재하지 않았던, 다중 경로를 가짐과 동시에 전략적 모멘텀이 점수가 앞서고 있는 플레이어에게 존재하는 노드가 새롭게 등장하는 것을 확인함으로써, 3선승제 모델로 확장하여 분석하는 의의를 제시함과 동시에 더 긴 다전제의 모델과 데이터를 이용하면 두가지 모멘텀이 작용하는 정도를 검정을 통해 더욱 효과적으로 구분할 수 있고, 이를 전략적 모멘텀의 존재에 대한 강력한 근거로 제시할 수 있을 것이라는 추론을 할 수 있다.

또한, 기존의 연구에서 이론적 모델과 실제 경기의 데이터 간의 비교를 위해 스포츠 베팅에서의 배당률 기록을 이용하였다. 이 논문에서 실제 데이터와의 비교까지 이루어지는 데에는 한계가 있어, 본문 마지막 부분에 이론적으로 도출된 전체 균형 승률과 각 라운드의 승률 간 관계를 나타낸 수식과 그래프를 제시하고 해석하는 것으로 이 논문을 마치게 된다.

주요어: 다전제, 5전 3선승제, 비대칭 참가자, 전략적 모멘텀, 내쉬 균형
학 번: 2016-20145

Acknowledgement

Most of all, I would like to thank my advisor, Professor Dmitry A. Shapiro for the patient guidance, encouragement, detailed feedback on every update of the draft, a lot of proper pressures that kept me standing at the front line, and advice he has provided throughout the whole process of this paper. Moreover, I really appreciate his patience with my poor English communication skills and consideration of my circumstances during the whole time. Beers and dinners we had together sometimes made my experience in SNU richer than ever. Also, other committee members, Professor Yves Gueron and Professor Yena Park are greatly appreciated for their meticulous review and helpful feedback.

I would like to acknowledge my colleagues in the Laboratory of Liberal Studies of Economics and HGU friends for giving me great help in many ways. Especially, I want to say sorry for bothering you guys so often.

I wish to show my gratitude to my parents and brother for giving me a chance to study in graduate school. Without their dedication, I would not have been able to finish the course. Also, I am deeply grateful to my mother-in-law and my sister-in-law for always being warm and comfortable for me.

Finally, I wish to express my deepest gratitude to my beloved wife, who has been greatly supportive and has been considerate throughout the whole process. Most of all, thank you for trusting me and encouraging me whenever I was stuck at the crossroads. I owe you everything.