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Combinatorics of highest weight crystals of type D
(D형 최고 무게 결정의 조합론)

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Combinatorics of highest weight crystals of type D

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Abstract

In this thesis, we study the crystals of type D from a combinatorial viewpoint. We focus on especially the crystals $B(\lambda)$ and $B(\infty)$, where $B(\infty)$ is the crystal of the negative half of the quantum group and $B(\lambda)$ is the crystal of an integrable highest weight irreducible module with highest weight λ.

As a main result, we obtain a simple description of the crystal structure of $B(\infty)$ in terms of Lusztig's parametrization using the PBW basis associated with a certain reduced expression of the longest element of the Weyl group. Also, we develop a combinatorial algorithm on $B(\lambda)$, which is compatible with the crystal structure of $B(\infty)$. These results establish an explicit combinatorial description of the crystal embedding from $B(\lambda)$ into $B(\infty)$.

Our study of the crystal structure of $B(\lambda)$ and $B(\infty)$ has several interesting applications such as an affine crystal theoretic interpretation of Robinson-Schensted-Knuth type correspondence of type D, a new formula for the branching multiplicity from GL_n to O_n, and a new combinatorial model of Kirillov-Reshetikhin crystals of type $\mathrm{D}_n^{(1)}$ associated with the spin node.

Key words: Quantum groups, Crystal bases, Kirillov-Reshetikhin crystals, Robinson-Schensted-Knuth correspondence, Branching rules, Generalized exponents
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Contents

Chapter 1

Introduction

The notion of *quantum group* was introduced independently by Drinfeld [16] and Jimbo [40] around 1985 in their study to construct the solution of quantum Yang-Baxter equation. The quantum group is a q-deformation of the universal enveloping algebra of a symmetrizable Kac-Moody algebra. Over the past 35 years, it turns out that the quantum group is a fundamental algebraic structure to shed light on many branches of mathematics and mathematical physics.

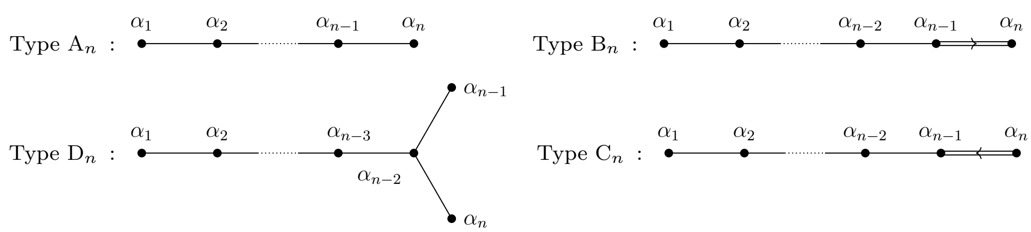
The *crystal base* introduced by Kashiwara [43] has been developed intensively by many authors since it has very nice properties reflecting an internal combinatorial structure of the representations of the quantum group. Moreover, it has interesting applications and connections in many branches of mathematics. We refer the reader to [7, 30, 45, 48, 49, 56] for more details.

The crystal base has a colored oriented graph structure, so-called *crystal graph* (or *crystal* for short). This enables us to reduce several problems in representation theory of the quantum group to combinatorial ones. For example, the crystals provide us an elegant answer to the problem of decomposing tensor product of modules over the quantum groups [73, 84].

There are various combinatorial descriptions of the crystals (of classical types) [47, 50, 73, 83], which enable us to understand the structure of the crystals more deeply and reveal interesting connections with several areas such as the theory of symmetric functions, the representations of symmetric groups and mathematical physics.

1.1 Main results

The main goal of this thesis is to understand the structure of the crystals over the classical quantum groups from a combinatorial viewpoint, where the classical quantum groups are the q-deformation of the universal enveloping algebras of Lie algebras whose corresponding Dynkin diagrams are given as follows.



In particular, we focus on the following objects:

(1) the crystal $B(\infty)$ of the negative half of the quantum group (see Section 2.2.1),

(2) the crystal $B(\lambda)$ of an integrable highest weight irreducible module with highest weight λ (see Section 2.1.2).

The structure of the crystals of types A is already well-understood in several viewpoints, and the crystals of types BC may be understood sometimes by using the ones of type A through Kashiwara's similarity method [46]. On the other hand, the structure of the crystals of type D is largely independent of the one of types ABC. In addition, the well-known combinatorial descriptions of the crystals of type D are often complicated rather than other types (see [50] or Section 4.1.2). In order to overcome these difficulties, we consider the crystals of type D based on the recent works [63, 64, 103].

First, we describe the crystal structure of $B(\infty)$ in terms of Lusztig's parametrization [76, 77] using PBW basis associated with a certain reduced expression of the longest element in Weyl group by using the result in [103]. More precisely, the crystal $B(\infty)$ is equal to the PBW basis at $q = 0$ [77, 98]. It is independent of the choice of a reduced expression, and then the actions of the crystal operators on the crystal of PBW basis can be described by a simple way with respect to reduced expressions satisfying a certain condition which is called simply braided [103].

We find a simply braided reduced expression $\mathbf{i}_0$ associated with the maximal Levi subalgebra of type A_{n-1} and we show that the actions of the crystal operators associated with $\mathbf{i}_0$ are described by a simple rule similar to the one obtained from the tensor product rule of crystals. This is the first main result in this thesis. We should remark that our reduced expression is different from the one in [103].

As an application, it enables us to establish a crystal theoretic interpretation of an analog of the RSK correspondence for type D (see Section 1.2.1), and then we obtain a new combinatorial model of Kirillov-Reshetikhin crystals of type $\mathrm{D}_n^{(1)}$ related to spin node (see Section 1.2.3).

Second, we study the crystal $B(\lambda)$ using a combinatorial model in [63, 64]. In particular, we develop a combinatorial algorithm on that model, so-called *separation* (see Section 4.3), which is consisting of a non-trivial sequence of Schützenberger's jeu de taquin slidings. We show that the algorithm is compatible with the crystal structure of PBW basis associated with $\mathbf{i}_0$. This is the second main result in this thesis. The compatibility allows us to obtain an explicit combinatorial description of the embedding of crystals of type D (up to a weight shift)

$$B(\lambda) \hookrightarrow B(\infty),$$

following the approach for types BC [66]. Also, as an application, we apply the separation algorithm to study the branching rule from GL_n to O_n (see Section 1.2.2).

1.2 Applications

Let us explain the applications of the main results more precisely.

1.2.1 RSK correspondence of type D

The *Robinson-Schensted-Knuth correspondence* (RSK correspondence or RSK for short) is a weight-preserving bijection between the set $\mathcal{M}_{m\times n}$ of $m \times n$ matrices with nonnegative integers and the set $\mathcal{T}_{m,n}$ of pairs of semistandard tableaux of same shape with letters in $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$ [55] (see also [7, 23] or Section 3.1.2). The RSK correspondence has nice relationships with the theory of symmetric functions, the representations of symmetric groups and classical groups, see [23, 81, 96, 102, 105] for more details.

The RSK correspondence has also an interpretation in terms of the crystals. More precisely, the sets $\mathcal{M}_{m\times n}$ and $\mathcal{T}_{m,n}$ have type $\mathrm{A}_{m-1} \times \mathrm{A}_{n-1}$-crystal structures, respectively, and then the RSK is an isomorphism of the crystals [68]. Furthermore, it is shown in [62] that there exist affine crystal structures of type $\mathrm{A}^{(1)}_{m+n-1}$ on both $\mathcal{M}_{m\times n}$ and $\mathcal{T}_{m,n}$, respectively, and the RSK can be extended to an isomorphism of affine crystals of type $\mathrm{A}^{(1)}_{m+n-1}$ [62]. This can be viewed as an affine crystal theoretic interpretation of Cauchy identity.

Recently, it is shown in [65] that the type A_{m+n-1}-crystal structure of $\mathcal{M}_{m\times n}$ coincides with the one of the *quantum nilpotent subalgebra* [24, 51] associated with the maximal Levi subalgebra of type $\mathrm{A}_{m-1}\times\mathrm{A}_{n-1}$.

Motivated by this result, we consider an analog of the RSK correspondence for type D_n due to Burge [6], which is a weight-preserving bijection between the set of strictly upper triangular $n\times n$ matrices with nonnegative integers and the set of semistandard tableaux with columns of even length. Then we introduce affine crystal structures on both sides of the map. Note that the affine crystal structure of type $\mathrm{D}^{(1)}_n$ on the set of the semistandard tableaux with columns of even length is already known in [62].

To define an affine crystal structure of type $\mathrm{D}^{(1)}_n$ on the set of strictly triangular $n\times n$ matrices, we consider the crystal of the PBW basis associated with $\mathbf{i}_0$ (recall Section 1.1). Then a certain subcrystal of the crystal of the PBW basis associated with $\mathbf{i}_0$ can be identified with the set of strictly triangular $n\times n$ matrices and it is the crystal of the quantum nilpotent subalgebra associated with the maximal Levi subalgebra of type A_{n-1}. Furthermore, it is extended to an affine crystal of type $\mathrm{D}^{(1)}_n$ in a natural way.

Finally, we show that the Burge correspondence is an isomorphism of the affine crystals of type $\mathrm{D}^{(1)}_n$. This is a new affine crystal theoretic interpretation of the Littlewood identity for type D (see [75, 80] for the identity). We remark that an interpretation of the Littlewood identity for types BC is obtained from the result of type A above by using Kashiwara's similarity method and the symmetric property of the RSK correspondence [66].

1.2.2 Branching rules for $(\mathrm{GL}_n,\,\mathrm{O}_n)$

Given a pair $(\mathbf{A},\mathbf{B})$ of groups or algebras with $\mathbf{B}\subset\mathbf{A}$, and an irreducible representation π of $\mathbf{A}$, it is often useful to know the decomposition of π into irreducible representations of $\mathbf{B}$ (for example, see [12, 56, 92] for applications). A *branching rule* (or *branching law*) from $\mathbf{A}$ to $\mathbf{B}$ is to describe an irreducible representation of $\mathbf{B}$ or its multiplicity in the restriction of π to $\mathbf{B}$. In particular, if η is an irreducible representation of $\mathbf{B}$, then the later one is often called *branching multiplicity* of η in the restriction of π to $\mathbf{B}$. If the representations π and η are obvious, then we call it simply branching multiplicity from $\mathbf{A}$ to $\mathbf{B}$.

Let GL_n, Sp_n and O_n be the general linear group, symplectic group and orthogonal group of rank n over $\mathbb{C}$, respectively. These groups are called *classical groups*. It is well-known that a finite-dimensional irreducible representation of classical groups is parametrized by a partition of n.

In [74, 75], Littlewood proved that if a finite-dimensional irreducible representation of GL_n parametrized by a partition λ satisfies the condition that the length of λ is less than or equal to $\frac{n}{2}$, then the branching multiplicity from GL_n to Sp_n or O_n is equal to a subtraction-free sum in terms of Littlewood-Richardson coefficients (LR coefficient for short). The range of λ is often called the stable range [33]. It is natural to ask whether the Littlewood's formulas can be generalized to *arbitrary* finite-dimensional irreducible representations of GL_n outside the stable range.

A subtraction-free formula of branching multiplicity from GL_n to Sp_n generalizing the Littlewood's formula for the case of Sp_n is known due to Sundaram [104, 105]. Recently, Lecouvey-Lenart also obtain another combinatorial formula generalizing Littlewood's one for the case of Sp_n [71] by extending the approach in [67].

Motivated by the recent works [67, 71], we also extend the approach in [67] for the case of O_n by using the separation algorithm presented here. Then it is enable us to obtain a new subtraction-free formula of branching multiplicity from GL_n to O_n generalizing the Littlewood's formula for the case of O_n. More precisely, the separation algorithm induces an embedding which associates the branching multiplicity from GL_n to O_n with LR coefficients satisfying a certain condition. In particular, the condition on LR coefficients is vanished if λ is in the stable range. Consequently, we obtain a subtraction-free formula of branching multiplicity from GL_n to O_n outside the stable range, which generalizes the Littlewood's formula for the case of O.

We should remark that there are already numerous works to extend the Littlewood's formula for O_n (see [18, 33] and references therein), but most of which are obtained in an algebraic way and they do not give a subtraction-free formula in many cases. To the best of our knowledge, there seems to be no a subtraction-free formula of the branching multiplicity from GL_n to O_n in full generality.

1.2.3 Kirillov-Reshetikhin crystals of type $\mathrm{D}_n^{(1)}$ associated with spin node

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, and let $\widehat{\mathfrak{g}}$ be the affine Kac-Moody algebra of untwisted type corresponding to $\mathfrak{g}$ [41]. Let $U_q'(\widehat{\mathfrak{g}})$ be the *quantum affine algebra* of $\widehat{\mathfrak{g}}$ [16, 40]. The finite-dimensional representations of $U_q'(\widehat{\mathfrak{g}})$ have been studied intensively since they have important connections to various areas in mathematics and mathematical physics. For example, see [10, 19] and references therein.

By Chari-Pressley's classification [10, 11], each isomorphism class of finite-dimensional irreducible representations (of type 1) is parametrized by an n-tuple $\mathbf{P} = (P_i(u))_{1\leq i\leq n}$ of polynomials with constant term 1, where n is the rank of $\mathfrak{g}$. The polynomial $\mathbf{P}$ is often called the Drinfeld's polynomial [17].

The Kirillov-Reshetikhin (KR for short) module $W_{s,a}^{(r)}$ is the finite-dimensional irreducible $U_q'(\widehat{\mathfrak{g}})$-module associated with the Drinfeld polynomial $\mathbf{P} = (P_i(u))_{1\leq i\leq n}$

$$P_i(u) := \begin{cases} \prod_{j=1}^{s}(1 - aq^{s-2j+1}u) & \text{if } i = r, \\ 1 & \text{otherwise,} \end{cases}$$

where $1 \leq r \leq n$, $s \in \mathbb{Z}_+$ and $a \in \mathbb{C}^\times$ [54]. It is now well-known that the family $\left\{ W_{s,a}^{(r)} \right\}$ plays an important role in the category of the finite-dimensional representations of $U_q'(\widehat{\mathfrak{g}})$ (cf. [9, 69]).

It was conjectured by Hatayama et al.[28] that for $1 \leq r \leq n$ and $s \in \mathbb{Z}_+$, there exists $a_{r,s} \in \mathbb{C}^\times$ such that $W_{s,a_{r,s}}^{(r)}$ has a crystal base. The conjecture has been proved for all nonexceptional types [91] (see also [42] for type $\mathrm{A}_n^{(1)}$, [89] for type $\mathrm{D}_n^{(1)}$ with $1 \leq r \leq n-2$) and some exceptional types (with certain r) [86, 87]. Let $B^{r,s}$ denote the crystal associated with $W_{s,a_{r,s}}^{(r)}$, which is called *KR crystal* for short.

It is an important problem to describe the structure of $B^{r,s}$. A description of the crystal structure of $B^{r,s}$ is known for nonexceptional types and some exceptional nodes on a case-by-case approach [21] (see also references therein).

Another combinatorial model for $B^{r,s}$ of type $\mathrm{A}_n^{(1)}$ and types $\mathrm{D}_{n+1}^{(2)}$, $\mathrm{C}_n^{(1)}$, $\mathrm{D}_n^{(1)}$ with exceptional nodes is introduced in [62] by using the RSK correspondence as an isomorphism of affine crystals of type $\mathrm{A}_n^{(1)}$. The advantage of the approach in [62] is that the description of the action of the 0th crystal operators on $B^{r,s}$ is given uniformly for the cases considered above, and very simple compared to the ones in previous works.

In this thesis, we give a new polytope realization of KR crystals $B^{n,s}$ ($s \in \mathbb{Z}_+$) of type $\mathrm{D}_n^{(1)}$ associated with the spin node, which is isomorphic to the one using tableaux in [62] through Burge correspondence. To do this, we consider the crystal of PBW basis associated to $\mathbf{i}_0$ (recall Section 1.1). In particular, we use the affine crystal structure for the quantum nilpotent subalgebra associated to the maximal Levi subalgebra of type A_{n-1}. This approach allows us to use the formula of ε_n^*-statistic with respect to $*$-crystal structure on $B(\infty)$ due to Berenstein-Zelevinsky [4]. Then we obtain a new formula of the ε_n^*-statistic in terms of non-intersecting double paths defined on the positive roots of the quantum nilpotent subalgebra, which gives the polytope realization of $B^{n,s}$.

1.3 Organization

This thesis is organized as follows.

- In Chapter 2, we review necessary background on quantum groups, representations of quantum groups and crystal bases based on [7, 10, 30, 43, 45].

- In Chapter 3, we describe explicitly the crystal structure of $B(\infty)$ in terms of Lustig's parametrization using PBW basis associated with a certain reduced expression of the longest element of the Weyl group. Then we apply the result to obtain a crystal theoretic interpretation of Burge correspondence (Theorem 3.3.3), that is, we show that Burge correspondence is an isomorphism of crystals of type D_n. Also, we give a combinatorial formula for the shape of a tableau obtained from Burge correspondence (Theorem 3.3.6), which is indeed a byproduct of the realization of ε_n^*-statistic (Theorem 6.2.4).

- In Chapter 4, we briefly review a combinatorial model of $B(\lambda)$ in [63, 64] (Section 4.1.3), and then we develop a combinatorial algorithm called separation on this model. We show that the algorithm is compatible with the structure of the crystal of the parabolic Verma module associated with the maximal Levi subalgebra of type A_{n-1} (Theorem 4.4.3). By combining this result with our description of $B(\infty)$, we give a combinatorial description of the crystal embedding from $B(\lambda)$ into $B(\infty)$ in type D_n (Theorem 4.5.3).

- In Chapter 5, we give a new subtraction-free formula of the branching multiplicity from GL_n to O_n (Theorem 5.4.14) generalizing the Littlewood's formula for the case of O_n. As a byproduct, we also obtain a new formula of generalized exponents of types BD (Theorem 5.5.6) following the idea in [71] for type C.

- In Chapter 6, we obtain a new polytope realization of the KR crystal $B^{n,s}$ of type $\mathrm{D}_n^{(1)}$ associated with the spin node (Theorem 6.2.2) by using the crystal of the quantum nilpotent subalgebra (Section 3.2.4) and an explicit formula of the ε_n^*-statistic (Theorem 6.2.4), and we extend the Burge correspondence to an isomorphism of affine crystals of type $\mathrm{D}_n^{(1)}$.

- In Chapter 7, we give the detailed proofs for some results in this thesis.

Chapter 2

Crystal bases

In this chapter, we review necessary background on quantum groups, representations of quantum groups and crystal bases based on [7, 10, 30, 43, 45].

This chapter is organized as follows. In Section 2.1, we introduce the definition of the quantum group over a symmetrizable Kac-Moody algebra and its basic properties. In Section 2.1.1, we review the well-known results on the integrable representations of the quantum group. In Section 2.1.2, we introduce the crystal base for the integrable representations of the quantum group, which is a central notion in this thesis, and review the fundamental results for the crystal bases in [43].

2.1 Quantum groups

Let $\mathbb{Z}_+$ denote the set of non-negative integers. Let $\mathfrak{g}$ be the Kac-Moody algebra associated with a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$ indexed by a set I. We denote by $D = \mathrm{diag}(s_i \in \mathbb{Z}_{>0} \mid i \in I)$ a diagonal matrix such that DA is symmetric. If A is symmetric, we often say $\mathfrak{g}$ is the symmetric Kac-Moody algebra. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$.

Let $P^\vee$ be the dual weight lattice, $P = \mathrm{Hom}_{\mathbb{Z}}(P^\vee, \mathbb{Z})$ the weight lattice, $\Pi^\vee = \{ h_i \mid i \in I \} \subset P^\vee$ the set of simple coroots, and $\Pi = \{ \alpha_i \mid i \in I \} \subset P$ the set of simple roots of $\mathfrak{g}$ such that $\langle h_i, \alpha_j \rangle = a_{ij}$ for $i, j \in I$. Let P^+ be the set of integral dominant weights. We denote by ϖ_i the i-th fundamental weight for $i \in I$. Let Q be the root lattice and $Q_+ = \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$. There is a partial ordering on $\mathfrak{h}^*$ defined by $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q_+$ for $\lambda, \mu \in \mathfrak{h}^*$. Let $(\cdot\, , \cdot)$ be the standard nondegenerate symmetric bilinear form on $\mathfrak{h}$ in [41]. Then it induces the nondegenerate symmetric bilinear form on $\mathfrak{h}^*$. We denote it by the same notation $(\ \cdot\ ,\ \cdot\ )$. Note that $\langle h_i, \alpha_j\rangle = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ and $(\alpha_i, \varpi_j) = \delta_{ij}$ for $i, j \in I$.

Given $n \in \mathbb{Z}$ and any symbol x, we define the notation

$$[n]_x = \frac{x^n - x^{-1}}{x - x^{-1}}\,.$$

Put $[0]_x! = 1$ and $[n]_x! = [n]_x[n-1]_x \dots [1]_x$ for $n \in \mathbb{Z}_{>0}$. For $m, n \in \mathbb{Z}_{\geq 0}$ $m \geq n \geq 0$, we define

$$\begin{bmatrix} m \\ n \end{bmatrix}_x = \frac{[m]_x!}{[n]_x![m-n]_x!}$$

Then, $[n]_q$ and $\begin{bmatrix} m \\ n \end{bmatrix}_q$ are called q-integers and q-binomial coefficients, respectively.

Definition 2.1.1. [16, 40] The *quantum group* $U_q(\mathfrak{g})$ associated with a Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is the associative algebra over $\mathbb{Q}(q)$ with 1 generated by the elements e_i, f_i $(i \in I)$ and q^h $(q \in P^\vee)$ with the following defining relations:

$$\begin{aligned}
&q^0 = 1, \quad q^h q^{h'} = q^{h+h'} \quad \text{for } h, h' \in P^\vee\,,\\
&q^h e_i q^{-h} = q^{\alpha_i(h)} e_i \quad \text{for } h \in P^\vee\,,\\
&q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \quad \text{for } h \in P^\vee\,,\\
&e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \quad \text{for } i, j \in I,\\
&\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0 \quad \text{for } i \neq j,\\
&\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0 \quad \text{for } i \neq j,
\end{aligned} \tag{2.1.1}$$

where $q_i = q^{s_i}$ and $K_i = q^{s_i h_i}$.

Let us review the basic properties of the quantum group $U_q(\mathfrak{g})$. Set $\deg f_i = -\alpha_i$, $\deg e_i = \alpha_i$ and $\deg q^h = 0$. The quantum group has the root space decomposition

$$U_q(\mathfrak{g}) = \bigoplus_{\alpha \in Q} U_q(\mathfrak{g})_\alpha,$$

where $U_q(\mathfrak{g})_\alpha = \{\, u \in U_q(\mathfrak{g}) \mid q^h u q^{-h} = q^{\alpha(h)} u$ for all $h \in P^\vee \,\}$. For $x \in U_q(\mathfrak{g})_\alpha$ $(\alpha \in Q)$, we denote by $\mathrm{wt}(x) = \alpha$.

It is well known that $U_q(\mathfrak{g})$ has Hopf algebra structure (see [30, Section 1.5] for definition) with the comultiplication Δ, the counit ε, and the antipode S defined by

$$\begin{aligned}
&\Delta(q^h) = q^h \otimes q^h, \\
&\Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i\,, \\
&\varepsilon(q^h) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0\,, \\
&S(q^h) = q^{-h}, \quad S(e_i) = -e_i K_i, \quad S(f_i) = -K_i^{-1} f_i
\end{aligned}$$

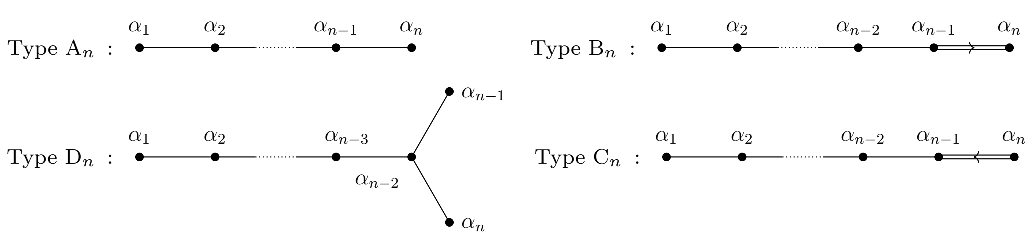
for $h \in P^\vee$ and $i \in I$.

Let $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) be the subalgebra of $U_q(\mathfrak{g})$ generated by the elements e_i (resp. f_i) for all $i \in I$, and let $U_q^0(\mathfrak{g})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by q^h $(h \in P^\vee)$. Then the quantum group $U_q(\mathfrak{g})$ has the triangular decomposition given by

$$U_q(\mathfrak{g}) \simeq U_q^-(\mathfrak{g}) \otimes U_q^0(\mathfrak{g}) \otimes U_q^+(\mathfrak{g}).$$

Here $\simeq$ means the isomorphism of vector spaces.

Remark 2.1.2. In this thesis, we consider mainly the Kac-Moody algebras of finite types associated with the following Dynkin diagrams:



(see Kac's classification [41, Chapter 4]). To emphasize the types, we often use the notation $U_q(X_n)$ instead of $U_q(\mathfrak{g})$, where $X = \mathrm{A}, \mathrm{B}, \mathrm{C}$ or D and $\mathfrak{g}$ is the Kac-Moody algebra of type X_n.

2.1.1 Representations of quantum groups

In this section, let us briefly review the integrable representations of the quantum group.

Definition 2.1.3. Let V be a vector space over $\mathbb{Q}(q)$.

(1) A *representation* of $U_q(\mathfrak{g})$ on V is an $\mathbb{Q}(q)$-algebra homomorphism

$$\phi : U_q(\mathfrak{g}) \longrightarrow \mathrm{End}_{\mathbb{Q}(q)}(V).$$

(2) A vector space V is called a $U_q(\mathfrak{g})$-*module* if there is a bilinear map $U_q(\mathfrak{g}) \times V \longrightarrow V$, denoted by $(x, v) \mapsto xv$, satisfying

$$(xy)v = x(yv), \qquad 1v = v$$

for $x, y \in U_q(\mathfrak{g})$ and $v \in V$.

Let ϕ be a representation of $U_q(\mathfrak{g})$ on V. Then it defines a $U_q(\mathfrak{g})$-module structure on V by

$$xv := \phi(x)(v) \text{ for } x \in U_q(\mathfrak{g}),\ v \in V.$$

Conversely, if V is a $U_q(\mathfrak{g})$-module, then it gives a representation of $U_q(\mathfrak{g})$ on V by

$$\begin{array}{rcl} \phi :\ U_q(\mathfrak{g}) & \longrightarrow & \mathrm{End}_{\mathbb{Q}(q)}(V)\ , \\ x & \longmapsto & (x,\ \cdot\ ) \end{array}$$

where $(x,\ \cdot\ )$ is the endomorphism of V induced from the bilinear map in Definition 2.1.3(2). Hence, we often say that V is a *representation of* $U_q(\mathfrak{g})$ if V is a $U_q(\mathfrak{g})$-module.

Definition 2.1.4. Let V be a $U_q(\mathfrak{g})$-module be given.

(1) V is called a *weight module* if it admits a weight space decomposition

$$V = \bigoplus_{\lambda \in P} V_\lambda,$$

where $V_\lambda = \left\{ v \in V \mid q^h v = q^{\lambda(h)} v \text{ for all } h \in P^\vee \right\}$. A vector $v \in V_\lambda$ is called a *weight vector* of weight λ.

(2) If $e_i v = 0$ for all $i \in I$, then it is called a *maximal weight vector*.

(3) If $V_\lambda \neq 0$, then λ is called a *weight* of V and V_λ is the weight space of weight $\lambda \in P$. The dimension of V_λ is called the *weight multiplicity* of λ. We denote by $\mathrm{wt}(V)$ the set of weights of V.

(4) If $\dim V_\lambda < \infty$ for all $\lambda \in \mathrm{wt}(V)$, then we define the *character* of V by

$$\mathrm{ch}V = \sum_{\lambda} \dim V_\lambda e^\lambda,$$

where $\{ e^\lambda \mid \lambda \in P \}$ is formal basis of the group algebra $\mathbb{Z}[P]$ with multiplication given by $e^\lambda e^\mu = e^{\lambda+\mu}$.

(5) A $\mathbb{Q}(q)$-subspace $W \subset V$ is called $U_q(\mathfrak{g})$-*submodule* of V if $xW \subset W$ for all $x \in U_q(\mathfrak{g})$. The $U_q(\mathfrak{g})$-module V is called *irreducible* (or *simple*) if it has no submodule other than 0 and V.

(6) A weight module V is called a *highest weight module* of *highest weight* $\lambda \in \mathfrak{h}^*$ if there exists a non-zero vector $v_\lambda \in V$, called *highest weight vector*, such that

$$e_i v_\lambda = 0 \quad (i \in I), \quad q^h v_\lambda = q^{\lambda(h)} v_\lambda \quad (h \in \mathfrak{h}), \quad V = U_q(\mathfrak{g}) v_\lambda.$$

Example 2.1.5. Let $U_q(\mathfrak{sl}_2)$ be the quantum group generated by $e, f, K^{\pm}$ with the defining relations:

$$KeK^{-1} = q^2 e, \quad KfK^{-1} = q^{-2} f, \quad ef - fe = \frac{K - K^{-1}}{q - q^{-1}}.$$

Let $V(n) := \bigoplus_{i=1}^n \mathbb{Q}(q) v_i$ be the $(n+1)$-dimensional vector space over $\mathbb{Q}(q)$ and we define the $U_q(\mathfrak{sl}_2)$-actions on $V(n)$ by

$$kv_i = q^{n-2i} v_i, \quad ev_i = [n-i+1]_q v_{i-1}, \quad fv_i = [i+1]_q v_{i+1}, \tag{2.1.2}$$

where we assume that $v_{-1} = v_{n+1} = 0$. It is enough to show that the above actions satisfies the relations (2.1.1). For example, for $i \neq 0, n$,

$$\begin{aligned} (ef - fe)v_i &= \Big([i+1]_q [n-i]_q - [n-i+1]_q [i]_q \Big) v_i \\ &= \frac{q^{n-2i} - q^{-n+2i}}{q - q^{-1}} v_i = \frac{K - K^{-1}}{q - q^{-1}} v_i \end{aligned}$$

We can check the other relations similarly. Therefore, it is a straightforward calculation to check that $V(n)$ is a representation of $U_q(\mathfrak{sl}_2)$. Note that $V(n)$ is an irreducible highest weight module with highest weight $n\varpi_1$.

Furthermore, $V(n)$ is irreducible highest wight module and any finite-dimensional irreducible representation of $U_q(\mathfrak{sl}_2)$ is of this form [39, Theorem 2.6] (cf. [34, Section 7.2]). $\square$

We say that $x \in U_q(\mathfrak{g})$ is *locally nilpotent* on V if for any $v \in V$, there exists a positive integer N such that $x^N v = 0$. Then a weight module V is *integrable* if the operators e_i and f_i are locally nilpotent on V for all $i \in I$. For $\lambda \in \mathfrak{h}^*$, set $D(\lambda) = \{\, \mu \in \mathfrak{h}^* \mid \mu \leq \lambda \,\}$.

Definition 2.1.6. (cf. §1.2 in [43]) The category $\mathcal{O}^q_{\mathrm{int}}$ consists of $U_q(\mathfrak{g})$-module V satisfying the following conditions:

(1) V has a weight space decomposition $V = \bigoplus_{\lambda \in P} V_\lambda$ with $\dim V_\lambda < \infty$ for all $\lambda \in P$,

(2) there exists a finite number of elements $\lambda_1, \ldots, \lambda_s \in P$ such that

$$\mathrm{wt}(V) \subset D(\lambda_1) \bigcup \cdots \bigcup D(\lambda_s),$$

(3) the operators e_i and f_i are locally nilpotent on V for all $i \in I$.

The morphisms are taken to be the usual $U_q(\mathfrak{g})$-module homomorphisms.

Note that the category $\mathcal{O}^q_{\mathrm{int}}$ is closed under taking direct sums or tensor product of finitely many $U_q(\mathfrak{g})$-modules.

The following results are well known due to Lusztig. We refer to [30, Chapter 3] for more details (cf. [34, 41]).

Theorem 2.1.7. *Let $V(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight $\lambda \in P$.*

(1) *$V(\lambda)$ belongs to the category $\mathcal{O}^q_{int}$ if and only if $\lambda \in P^+$.*

(2) *If V is a highest weight $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{int}$ with highest weight $\lambda \in P$, then $\lambda \in P^+$ and $V \simeq V(\lambda)$.*

(3) *Every irreducible $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{int}$ is isomorphic to $V(\lambda)$ for some $\lambda \in P^+$.*

(4) *Every $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{int}$ is isomorphic to a direct sum of irreducible highest weight modules $V(\lambda)$ with $\lambda \in P^+$.*

2.1.2 Crystal bases

In this section, we review the notion of crystal base for $U_q(\mathfrak{g})$-modules which are objects of the category $\mathcal{O}^q_{\mathrm{int}}$.

Let V be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{\mathrm{int}}$ and let $V = \bigoplus_{\lambda \in P} V_\lambda$ be the weight space decomposition. For $m \in \mathbb{Z}_{\geq 0}$, we denote by $x^{(m)}$ the m-th divided power of x given by

$$x^{(m)} = \frac{1}{[m]_q!} x^m ,$$

where $x = e_i$ or f_i for $i \in I$. For each $i \in I$, any weight vector $v \in V_\lambda$ ($\lambda \in \mathrm{wt}(V)$) can be written in the form

$$v = v_0 + f_i v_1 + \cdots + f_i^{(N)} v_N, \tag{2.1.3}$$

where $N \in \mathbb{Z}_{\geq 0}$ and $v_k \in V_{\lambda + k\alpha_i} \cap \ker e_i$ for all $k = 1, \ldots, N$ (cf. [30, Lemma 4.1.1]).

Definition 2.1.8. [43, Section 2.2] The *Kashiwara operators* $\tilde{e}_i$ and $\tilde{f}_i$ ($i \in I$) on V are defined by

$$\tilde{e}_i v = \sum_{k=1}^{N} f_i^{(k-1)} v_k, \qquad \tilde{f}_i v = \sum_{k=0}^{N} f_i^{(k+1)} v_k \,.$$

Example 2.1.9. Consider Example 2.1.5. By (2.1.2),

$$v_i = f^{(i)} v_0,$$

where $v_0 \in \ker e$. By definition,

$$\tilde{e}\left(f^{(i)} v_0\right) = f^{(i-1)} v_0, \qquad \tilde{f}\left(f^{(i)} v_0\right) = f^{(i+1)} v_0 \,.$$

Then we may express the operators $\tilde{e}$ and $\tilde{f}$ by

$$v_0 \longrightarrow f\, v_0 \longrightarrow \cdots \longrightarrow f^{(n-1)}\, v_0 \longrightarrow f^{(n)}\, v_0.$$

Let $\mathbb{A}_0$ be the subring of $\mathbb{Q}(q)$ consisting of rational functions regular at $q = 0$, that is,

$$\mathbb{A}_0 = \left\{ \frac{f}{g} \;\middle|\; f, g \in \mathbb{Q}[q],\ g(0) \neq 0 \right\}.$$

Definition 2.1.10. Let V be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{\mathrm{int}}$. A free $\mathbb{A}_0$-submodule L of V is called a *crystal lattice* if

(1) L generates V as a vector space over $\mathbb{Q}(q)$,

(2) $L = \bigoplus_{\lambda \in P} L_\lambda$, where $L_\lambda = L \cap V_\lambda$ for all $\lambda \in P$,

(3) $\tilde{e}_i L \subset L$ and $\tilde{f}_i L \subset L$ for all $i \in I$.

Since the operators $\tilde{e}_i$ and $\tilde{f}_i$ preserve the lattice L, they also define operators on L/qL and we use the same symbols. We denote by $\mathbf{0}$ a formal symbol.

Definition 2.1.11. A *crystal base* of a $U_q(\mathfrak{g})$-module V in the category $\mathcal{O}^q_{\mathrm{int}}$ is a pair (L, B) satisfying the following conditions:

(1) L is a crystal lattice of M,

(2) B is a $\mathbb{Q}$-basis of L/qL,

(3) $B = \bigsqcup_{\lambda \in P} B_\lambda$, wehre $B_\lambda = B \cap (L_\lambda / qL_\lambda)$,

(4) $\tilde{e}_i B \subset B \cup \{\mathbf{0}\}$, $\tilde{f}_i B \subset B \cup \{\mathbf{0}\}$ for all $i \in I$,

(5) for any $b, b' \in B$ and $i \in I$, we have $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$.

Take B as the set of vertices and define the I-colored arrows on B by

$$b \xrightarrow{\ i\ } b' \ \text{ if and only if } \ \tilde{f}_i b = b' \quad (i \in I).$$

Then B is often identified with the above I-colored oriented graph called the *crystal graph* of V. For $i \in I$ and $b \in B_\lambda$ ($\lambda \in P^+$), we define the maps $\varepsilon_i, \varphi_i : B \longrightarrow \mathbb{Z}$ by

$$\begin{aligned} \varepsilon_i(b) &= \max\{\, k \geq 0 \mid \tilde{e}_i^k b \in B \,\}, \\ \varphi_i(b) &= \max\{\, k \geq 0 \mid \tilde{f}_i^k b \in B \,\}. \end{aligned} \tag{2.1.4}$$

Then the map satisfies the following:

$$\begin{aligned} &\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b) \rangle, \\ &\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1, \qquad \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1, \quad \text{ if } \tilde{e}_i b \in B, \\ &\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \qquad \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1, \quad \text{ if } \tilde{f}_i b \in B. \end{aligned} \tag{2.1.5}$$

We refer to §2.4 [43] for the proof of the first one. The other ones are obtained from Definition 2.1.11.

Example 2.1.12. Let us recall Example 2.1.9. Then we may set

$$L(n) := \bigoplus_{i=0}^{n} \mathbb{A}_0 \left(f^{(i)} v_0\right) , \qquad B(n) := \left\{ b_i := f^{(i)} v_0 + qL(n) \mid 0 \le i \le n \right\} .$$

Then the pair $(L(n), B(n))$ is a crystal base of $V(n)$ and by definition of ε and φ, we see

$$b_0 \underbrace{\longrightarrow \cdots \longrightarrow}_{\varepsilon(b_i)} b_i \underbrace{\longrightarrow \cdots \longrightarrow}_{\varphi(b_i)} b_n ,$$

where $\varepsilon(b_i)$ (resp. $\varphi(b_i)$) is equal to the number of the arrows between b_i and b_0 (resp. b_n).

Definition 2.1.13. Let V be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{\mathrm{int}}$ with crystal bases (L_j, B_j) for $j = 1, 2$. We say that two crystal bases (L_1, B_1) and (L_2, B_2) are *isomorphic* if there is an $\mathbb{A}_0$-linear isomorphism $\psi : L_1 \longrightarrow L_2$ such that

(1) ψ commutes with all $\tilde{e}_i$ and $\tilde{f}_i$ for $i \in I$,

(2) the induced $\mathbb{Q}$-linear isomorphism $\overline{\psi} : L_1/qL_1 \longrightarrow L_2/qL_2$ defines a bijection $\overline{\psi} : B_1 \cup \{0\} \longrightarrow B_2 \cup \{0\}$ that commutes with all $\tilde{e}_i$ and $\tilde{f}_i$ for $i \in I$.

Let $\lambda \in P^+$ and let $V(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$-module of highest weight λ. Put v_λ to be the highest weight vector of $V(\lambda)$. We define $L(\lambda)$ to be the free $\mathbb{A}_0$-submodule $V(\lambda)$ spanned by the vectors of the form $\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} v_\lambda$ for $r \ge 0$ and $i_k \in I$, and set

$$B(\lambda) = \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} v_\lambda + qL(\lambda) \in L(\lambda)/qL(\lambda) \;\middle|\; r \ge 0,\, i_k \in I \right\} \setminus \left\{ 0 \right\}.$$

Theorem 2.1.14. [43] *Let M be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{int}$ and let $V(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$-module of highest weight $\lambda \in P^+$.*

(1) *The pair $(L(\lambda), B(\lambda))$ is a crystal base of $V(\lambda)$.*

(2) *There exists a unique crystal base (L, B) of M. If M is isomorphic to $\bigoplus_{\lambda \in P^+} V(\lambda)^{\oplus m_\lambda}$, then*

$$(L, B) \xrightarrow{\simeq} \left(\bigoplus_{\lambda \in P^+} L(\lambda)^{\oplus m_\lambda},\ \bigsqcup_{\lambda \in P^+} B(\lambda)^{\oplus m_\lambda} \right) ,$$

where $m_\lambda \in \mathbb{Z}_{\ge 0}$ is the multiplicity of $V(\lambda)$ in M. $\square$

The stability of tensor product on crystal bases is very nice and important feature of crystal bases, which is called *tensor product rule*. It also plays crucial role in the proof of the results in [43].

Theorem 2.1.15 (Tensor product rule)**.** *Let V_j be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{int}$ with crystal base (L_j, B_j) for $j = 1, 2$. Set $L = L_1 \otimes_{\mathbb{A}_0} L_2$ and $B = B_1 \times B_2$. Then (L, B) is a crystal base of $V_1 \otimes_{\mathbb{Q}(q)} V_2$, where the action of Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ on B for $i \in I$ are given by*

$$\begin{aligned}
\tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\
\tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \epsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \epsilon_i(b_2). \end{cases}
\end{aligned} \tag{2.1.6}$$

with

$$\begin{aligned}
&\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2), \\
&\varepsilon_i(b_1 \otimes b_2) = \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \mathrm{wt}(b_1), h_i \rangle\}, \\
&\varphi_i(b_1 \otimes b_2) = \max\{\varphi_i(b_1) + \langle \mathrm{wt}(b_2), h_i \rangle, \varphi_i(b_2)\},
\end{aligned}$$

where we assume that $\mathbf{0}$ is a formal symbol and $\mathbf{0} \otimes b_2 = b_1 \otimes \mathbf{0} = \mathbf{0}$. □

Corollary 2.1.16.

(1) *The vector $b_1 \otimes b_2 \in B_1 \otimes B_2$ is a maximal vector if and only if $\tilde{e}_i b_1 = 0$ and $\langle h_i, wt(b_1) \rangle \geq \varepsilon_i(b_2)$ for all $i \in I$.*

(2) *Let V_j be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{int}$ with crystal base (L_j, B_j) for $j = 1, \ldots N$. Then the vector $b_1 \otimes \cdots \otimes b_N \in B_1 \otimes \cdots \otimes B_N$ is a maximal vector if and only if $b_1 \otimes \cdots \otimes b_k$ is a maximal vector for all $k = 1, \ldots, N$.* □

2.2 Crystals

The structure of crystal graph is characterized by the following maps:

(1) $\mathrm{wt} : B \longrightarrow P$ defined by $b \in B_\lambda \mapsto \mathrm{wt}(b) = \lambda$,

(2) Kashiwara operators $\tilde{e}_i, \tilde{f}_i : B \longrightarrow B \cup \{0\}$ (Definition 2.1.8),

(3) the maps $\varphi_i, \varepsilon_i : B \longrightarrow \mathbb{Z}$ given in (2.1.4).

In particular, these maps satisfies the properties (2.1.5). The abstract notion of crystals is defined by the above maps with the properties as follows:

Definition 2.2.1. A *crystal* associated with the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is a set B together with the maps

$$\begin{aligned} &\mathrm{wt} : B \longrightarrow P, \qquad \tilde{e}_i, \tilde{f}_i : B \longrightarrow B \cup \{0\}, \\ &\varepsilon_i, \varphi_i : B \longrightarrow \mathbb{Z} \cup \{-\infty\}, \end{aligned}$$

where $i \in I$, satisfying the following conditions:

(1) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b) \rangle$ for all $i \in I$,

(2) $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$ if $\tilde{e}_i b \in B$,

(3) $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$ if $\tilde{f}_i b \in B$,

(4) $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1, \qquad \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1, \qquad$ if $\tilde{e}_i b \in B$,

(5) $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \qquad \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1, \qquad$ if $\tilde{f}_i b \in B$,

(6) $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$ for $b, b' \in B$ and $i \in I$,

(7) if $\varphi_i(b) = -\infty$ for $b \in B$, then $\tilde{e}_i b = \tilde{f}_i b = \mathbf{0}$,

where $\mathbf{0}$ is a formal symbol.

We often say that B is a $U_q(\mathfrak{g})$-*crystal* (or $\mathfrak{g}$-*crystal* for short), where $U_q(\mathfrak{g})$ is the quantum group associated with the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$. For $v \in B$, if $\tilde{e}_i v = \mathbf{0}$ for all $i \in I$, then v is called $\mathfrak{g}$-*highest weight vector*.

Example 2.2.2.

(1) The crystal graph B of a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{\mathrm{int}}$ is a $\mathfrak{g}$-crystal.

(2) For $\lambda \in P$, let $T_\lambda = \{t_\lambda\}$ and for all $i \in I$, we define

$$\mathrm{wt}(t_\lambda) = \lambda, \quad \tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = \mathbf{0}, \quad \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty.$$

Then T_λ is a $\mathfrak{g}$-crystal.

We define the *tensor product rule of crystals* as follows:

Definition 2.2.3. The *tensor product rule* $B_1 \otimes B_2$ of crystals B_1 and B_2 is defined to be the set $B_1 \times B_2$ with the crystal structure given by

$$\begin{aligned}
&\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2), \\
&\varepsilon_i(b_1 \otimes b_2) = \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \mathrm{wt}(b_1), h_i\rangle\}, \\
&\varphi_i(b_1 \otimes b_2) = \max\{\varphi_i(b_1) + \langle \mathrm{wt}(b_2), h_i\rangle, \varphi_i(b_2)\}, \\
&\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\
&\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \epsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \epsilon_i(b_2), \end{cases}
\end{aligned} \tag{2.2.1}$$

for $i \in I$. Here, we assume that $\mathbf{0} \otimes b_2 = b_1 \otimes \mathbf{0} = \mathbf{0}$.

Definition 2.2.4. Let B_1 and B_2 be crystals associated with Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$. A *crystal morphism* (or *morphism of crystals*) $\Psi : B_1 \longrightarrow B_2$ is a map

$$\Psi : \ B_1 \cup \{\mathbf{0}\} \longrightarrow B_2 \cup \{\mathbf{0}\}$$

such that

(1) $\Psi(\mathbf{0}) = \mathbf{0}$,

(2) if $b \in B_1$ and $\Psi(b) \in B_2$, then

$$\mathrm{wt}(\Psi(b)) = \mathrm{wt}(b), \quad \varepsilon_i(\Psi(b)) = \varepsilon_i(b), \quad \varphi_i(\Psi(b)) = \varphi_i(b)$$

for all $i \in I$,

(3) if $b, b' \in B_1$, $\Psi(b), \Psi(b') \in B_2$ and $\tilde{f}_i b = b'$, then

$$\tilde{f}_i \Psi(b) = \Psi(b'), \quad \Psi(b) = \tilde{e}_i \Psi(b'),$$

for all $i \in I$.

The category of crystals is a tensor category, see [45, Section 7].

Definition 2.2.5.

(1) A crystal morphism is called *strict* if it commutes with all $\tilde{e}_i$ and $\tilde{f}_i$ for $i \in I$.

(2) A crystal morphism $\Psi : B_1 \longrightarrow B_2$ is called an *embedding* if Ψ induces an injective map from $B_1 \cup \{\mathbf{0}\}$ to $B_2 \cup \{\mathbf{0}\}$.

(3) A crystal morphism $\Psi : B_1 \longrightarrow B_2$ is called an *isomorphism* if it is a bijection from $B_1 \cup \{\mathbf{0}\}$ to $B_2 \cup \{\mathbf{0}\}$.

(4) We say that B_1 is a *subcrystal* of B_2 if there exists an embedding from B_1 to B_2.

(5) We say that B_1 is *isomorphic* to B_2 if there exists an isomorphism between B_1 and B_2, and write $B_1 \equiv_{\mathfrak{g}} B_2$ or simply $B_1 \equiv B_2$ if there is no confusion.

(6) For $b_1 \in B_1$ and $b_2 \in B_2$, we say that b_1 *is equivalent to* b_2 if there is an isomorphism of crystals $\psi : C(b_1) \longrightarrow C(b_2)$ such that $\psi(b_1) = b_2$, where $C(b_i)$ is the connected component of b_i in B_i for $i = 1, 2$, and write $b_1 \equiv_{\mathfrak{g}} b_2$ or simply $b_1 \equiv b_2$ if there is no confusion.

(7) We say that B is a *regular* $\mathfrak{g}$*-crystal* if for each pair $i, j \in I$ with $i \neq j$, B is isomorphic as a $\mathfrak{g}_{\{i,j\}}$-crystal to an union of integrable highest weight $U_q(\mathfrak{g}_{\{i,j\}})$-crystals, where $\mathfrak{g}_{\{i,j\}}$ is the Lie algebra associated with Dynkin diagram containing i and j, and all edges between them.

2.2.1 Crystal base of $U_q^-(\mathfrak{g})$

Let us recall that $U_q^-(\mathfrak{g})$ is the $\mathbb{Q}(q)$-subalgebra of $U_q(\mathfrak{g})$ generated by f_i for all $i \in I$. For $\lambda \in P^+$, let $V(\lambda)$ be an integrable irreducible $U_q(\mathfrak{g})$-module with highest weight λ and highest weight vector v_λ. Then there is a natural $U_q^-(\mathfrak{g})$-linear surjective map

$$\begin{aligned} \pi_\lambda :\ U_q^-(\mathfrak{g}) &\longrightarrow V(\lambda)\ . \\ P &\longmapsto Pv_\lambda \end{aligned} \tag{2.2.2}$$

It is known that the surjective map induces the $U_q^-(\mathfrak{g})$-linear isomorphism as follows:

$$U_q^-(\mathfrak{g}) \Big/ \Big(\textstyle\sum_{i \in I} U_q^-(\mathfrak{g}) f_i^{\langle h_i, \lambda \rangle + 1} \Big) \longrightarrow V(\lambda)$$

(see [48, Lemma 3.2.7] for more detail).

By taking $\langle h_i, \lambda\rangle \to \infty$, we regard $U_q^-(\mathfrak{g})$ as the inverse limit of $V(\lambda)$ and the operators $K_i e_i$ may converge an operator of $U_q^-(\mathfrak{g})$, denoted by e_i', which plays a similar role as the action of e_i on an integrable $U_q(\mathfrak{g})$-module. We regard f_i as an operator on $U_q^-(\mathfrak{g})$ by the left multiplication. Then the operators e_i' and f_i $(i \in I)$ induce modified root operators on $U_q^-(\mathfrak{g})$ such as the Kashiwara operators (recall Definition (2.1.8)). By using these operators, we define the crystal base of $U_q^-(\mathfrak{g})$ [43, Section 3] (see also [45, Section 8]).

Remark 2.2.6. Let us denote by $B(\infty)$ the crystal of $U_q^-(\mathfrak{g})$. In [48, Chapter 7], Kashiwara explains the crystal $B(\infty)$ as the (direct) limit of the crystal $B(\lambda)$ when $\lambda \to \infty$. On the other hand, in [7, Chapter 12], the authors explain $B(\infty)$ as the crystal of Verma module with highest weight 0.

Let us explain in more detail following [43]. Let $\mathscr{B}_q(\mathfrak{g})$ be the algebra generated by e_i' and f_i $(i \in I)$ with the relations:

$$e_i' f_j = q_i^{-\langle h_i,\alpha_j\rangle} f_j e_i' + \delta_{ij},$$
$$\sum_{n=0}^{1-\langle h_i,\alpha_j\rangle} (-1)^n \begin{bmatrix} 1-\langle h_i,\alpha_j\rangle \\ n \end{bmatrix}_{q_i} e_i'^n e_j' e_i'^{1-\langle h_i,\alpha_j\rangle-n} = 0,$$
$$\sum_{n=0}^{1-\langle h_i,\alpha_j\rangle} (-1)^n \begin{bmatrix} 1-\langle h_i,\alpha_j\rangle \\ n \end{bmatrix}_{q_i} f_i'^n f_j' f_i'^{1-\langle h_i,\alpha_j\rangle-n} = 0.$$

The algebra $\mathscr{B}_q(\mathfrak{g})$ is called the *reduced q-analogue*.

Lemma 2.2.7. [43, Lemma 3.4.1] *For any $P \in U_q^-(\mathfrak{g})$, there exists unique $Q, R \in U_q^-(\mathfrak{g})$ such that*

$$[e_i, P] = \frac{K_i Q - K_i^{-1} R}{q - q^{-1}}$$

We define the endomorphisms $e_i', e_i'' : U_q^-(\mathfrak{g}) \longrightarrow U_q^-(\mathfrak{g})$ by

$$e_i'(P) = R, \qquad e_i''(P) = Q.$$

Then these endomorphisms satisfy

$$e_i'' f_j = q_i^{\langle h_i,\alpha_j\rangle} f_j e_i'' + \delta_{ij}, \quad e_i' f_j = q_i^{-\langle h_i,\alpha_j\rangle} f_j e_i' + \delta_{ij},$$

where f_j is understood as an endomorphism of $U_q^-(\mathfrak{g})$ by the left multiplication. Consequently, we obtain $\mathscr{B}_q(\mathfrak{g})$-module structure on $U_q^-(\mathfrak{g})$ by using the endomorphisms e_i' and f_j for $i, j \in I$ [43, Lemma 3.4.2].

The $\mathscr{B}_q(\mathfrak{g})$-module structure on $U_q^-(\mathfrak{g})$ induces the following decomposition (cf. (2.1.3))

$$U_q^-(\mathfrak{g}) = \bigoplus_{n=0} f_i^{(n)} \ker e_i'$$

(see [43, Proposition 3.2.1]). Now we define the modified root operators $\tilde{e}_i$ and $\tilde{f}_i$ on $U_q^-(\mathfrak{g})$ by

$$\tilde{e}_i \left(f_i^{(n)} u \right) = f_i^{(n-1)} u, \quad \tilde{f}_i \left(f_i^{(n)} u \right) = f_i^{(n+1)} u,$$

where $u \in \ker e_i'$ and $i \in I$.

Definition 2.2.8. A *crystal base* of a $\mathscr{B}_q(\mathfrak{g})$-module M is a pair (L, B) satisfying the following conditions:

(1) L is a crystal lattice of M,

(2) B is a $\mathbb{Q}$-basis of L/qL,

(3) $B = \bigsqcup_{\lambda \in P} B_\lambda$, wehre $B_\lambda = B \cap (L_\lambda / qL_\lambda)$,

(4) $\tilde{e}_i B \subset B \cup \{\mathbf{0}\}$, $\tilde{f}_i B \subset B \cup \{\mathbf{0}\}$ for all $i \in I$,

(5) for any $b, b' \in B$ and $i \in I$, we have $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$.

Let $L(\infty)$ be the free $\mathbb{A}_0$-submodule of $U_q^-(\mathfrak{g})$ generated by $\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \cdot 1$, and let $B(\infty)$ be the subset of $L(\infty)$ given by

$$B(\infty) = \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \cdot 1 + qL(\infty) \in L(\infty)/qL(\infty) \;\middle|\; r \geq 0,\, i_k \in I \right\} \setminus \{ \mathbf{0} \}.$$

Theorem 2.2.9. [43] *The pair $(L(\infty), B(\infty))$ is a crystal base of $U_q^-(\mathfrak{g})$.* □

The crystal base $(L(\infty), B(\infty))$ has a nice compatibility with the crystal base $(L(\lambda), B(\lambda))$ as follows.

Theorem 2.2.10. [43] *Let π_λ be the surjective $U_q^-(\mathfrak{g})$-linear homomorphism given in (2.2.2).*

(1) $\pi_\lambda(L(\infty)) = L(\lambda)$ *and it induces the surjective homomorphism*

$$\overline{\pi}_\lambda : L(\infty)/qL(\infty) \longrightarrow L(\lambda)/qL(\lambda).$$

(2) $\tilde{f}_i \circ \overline{\pi}_\lambda = \overline{\pi}_\lambda \circ \tilde{f}_i$ *for all* $i \in I$.

(3) *If* $b \in B(\infty)$ *satisfies* $\overline{\pi}_\lambda(b) \neq \mathbf{0}$, *then* $\tilde{e}_i \overline{\pi}_\lambda(b) = \overline{\pi}_\lambda(\tilde{e}_i b)$.

(4) $B(\lambda)$ *is isomorphic to* $\{\, b \in B(\infty) \mid \overline{\pi}_\lambda(b) \neq \mathbf{0} \,\}$. □

Note that by Theorem 2.2.10(4) we have an embedding of $B(\lambda)$ into $B(\infty)$

$$\Xi_\lambda : B(\lambda) \hookrightarrow B(\infty) \otimes T_\lambda \,. \tag{2.2.3}$$

Let us consider the image of the embedding Ξ_λ. To do this, we consider the anti-automorphism $*$ of $U_q(\mathfrak{g})$ as $\mathbb{Q}(q)$-algebra given by

$$e_i^* = e_i, \quad f_i^* = f_i, \quad (q^h)^* = q^{-h} \,.$$

Theorem 2.2.11. [43, Proposition 5.2.4, Proposition 6.1.1], [44, Theorem 2.1.1] *We have*

$$L(\infty)^* = L(\infty), \quad B(\infty)^* = B(\infty).$$

Then we define $*$-crystal operators $\tilde{e}_i^*$ and $\tilde{f}_i^*$ by

$$\tilde{e}_i^* = *\tilde{e}_i*, \qquad \tilde{f}_i^* = *\tilde{f}_i*,$$

and we also define $\varepsilon_i^* : B(\infty) \longrightarrow \mathbb{Z} \cup \{-\infty\}$ by

$$\varepsilon_i^*(b) := \max\{n \in \mathbb{Z} \mid (\tilde{e}_i^*)^n(b) \neq \mathbf{0}\}.$$

Proposition 2.2.12. [45, Proposition 8.2] (cf. [56, Lemma 10.2.2]) *For any* $\lambda \in P^+$, *the image of* Ξ_λ *is given by*

$$\{\, b \otimes t_\lambda \in B(\infty) \otimes T_\lambda \mid \varepsilon_i^*(b) \leq \langle h_i, \lambda \rangle \ \text{for all } i \in I \} \,.$$

Proof. It follows from Theorem 2.2.10(4), [43, Lemma 7.3.2] and [98, Lemma 3.4.1]. □

2.2.2 PBW basis and crystals

Let us review another formulation of $B(\infty)$ [77, 98] for finite types using Poincaré-Birkhoff-Witt type bases that was considered in the study of the *canonical bases* of $U_q^-(\mathfrak{g})$ for types ADE by Lusztig [76, 79].

Let W be Weyl group of $\mathfrak{g}$ generated by the simple reflections s_i $(i \in I)$ given by

$$s_i(\beta) = \beta - \frac{2(\beta, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i \tag{2.2.4}$$

for $\beta \in \mathfrak{h}^*$. For $w \in W$, $R(w)$ be the set of reduced expressions of w, that is,

$$R(w) = \{\, \mathbf{i} = (i_1, \dots, i_m) \mid w = s_{i_1} \dots s_{i_m} \text{ and } m \text{ is minimal } .\}$$

We denote by $\ell(w)$ the length m, and we call ℓ the length function of W. Let w_0 be the longest element of W of length N. Then it is known that for $\mathbf{i} \in R(w_0)$,

$$\Phi^+ = \left\{ \beta_1 := \alpha_{i_1}\, ,\, \beta_2 := s_{i_1}(\alpha_{i_2})\, , \cdots ,\, \beta_N := s_{i_1} \dots s_{i_{N-1}}(\alpha_{i_N})\, \right\} \tag{2.2.5}$$

is equal to the set of positive roots of $\mathfrak{g}$ (see [93] and reference therein).

For each $i \in I$, there is an $\mathbb{Q}(q)$-algebra automorphism T_i of $U_q(\mathfrak{g})$ due to Lusztig [78, 79]

$$T_i : U_q(\mathfrak{g}) \longrightarrow U_q(\mathfrak{g}) \tag{2.2.6}$$

given by

$$\begin{aligned}
&T_i(q^h) = q^{s_i(h)}, \quad T_i(e_i) = -f_i K_i, \quad T_i(f_i) = -K_i^{-1} e_i\, , \\
&T_i(e_j) = \sum_{r+s=-\langle h_i, \alpha_j \rangle} (-1)^r q_i^{-r} e_i^{(s)} e_j e_i^{(r)} \qquad \text{if } i \neq j, \\
&T_i(f_j) = \sum_{r+s=-\langle h_i, \alpha_j \rangle} (-1)^r q_i^{r} f_i^{(r)} f_j f_i^{(s)} \qquad \text{if } i \neq j.
\end{aligned}$$

Here the automorphism T_i is equal to $T''_{i,1}$ in [79]. Let us take $\mathbf{i} = (i_1, \dots, i_N) \in R(w_0)$ and put

$$f_{\beta_k} = T_{i_1} \dots T_{i_{k-1}}(f_{i_k}) \quad (1 \le k \le N). \tag{2.2.7}$$

For $\mathbf{c} = (c_{\beta_1}, \ldots, c_{\beta_N}) \in \mathbb{Z}_+^N$,

$$b_{\mathbf{i}}(\mathbf{c}) = f_{\beta_1}^{(c_{\beta_1})} f_{\beta_2}^{(c_{\beta_2})} \cdots f_{\beta_N}^{(c_{\beta_N})}, \tag{2.2.8}$$

where $f_{\beta_k}^{(c_{\beta_k})}$ is the divided power of f_{β_k} for $1 \leq k \leq N$. Then the set

$$B_{\mathbf{i}} := \left\{ b_{\mathbf{i}}(\mathbf{c}) \mid \mathbf{c} \in \mathbb{Z}_+^N \right\}$$

is a $\mathbb{Q}(q)$-basis of $U_q^-(\mathfrak{g})$, which is called a *PBW basis* associated to $\mathbf{i}$ [78, 98] (see also [79]). Furthermore, we have the following.

Theorem 2.2.13. [77, 98] *Let $\mathbf{i}$ be a reduced expression of w_0.*

(1) *$L(\infty)$ is generated by $B_{\mathbf{i}}$, which is independent of choice of $\mathbf{i} \in R(w_0)$.*

(2) *Let $\pi : L(\infty) \longrightarrow L(\infty)/qL(\infty)$ be the canonical projection. Then the image of $B_{\mathbf{i}}$ is equal to $B(\infty)$.* □

We identify $\mathbf{B}_{\mathbf{i}} := \mathbb{Z}_+^N$ with a crystal $\pi(B_{\mathbf{i}})$ under the map $\mathbf{c} \mapsto b_{\mathbf{i}}(\mathbf{c})$, and call $\mathbf{c} \in \mathbf{B}_{\mathbf{i}}$ an $\mathbf{i}$-*Lusztig data*. Then $\mathbf{B}_{\mathbf{i}}$ is called the crystal of $\mathbf{i}$-Lusztig datum. We often call it *PBW crystal* for short if there is no confusion for $\mathbf{i}$.

2.2.3 Quantum nilpotent subalgebras

In this section, we assume that $\mathfrak{g}$ is a symmetrizable Kac-Moody algebra. Note that the automorphism T_i ($i \in I$) (2.2.6) is available in this setting [79, Chapter 37] (see also [98, Proposition 1.3.1]).

Let $w \in W$ and $\mathbf{i} = (i_1, \ldots, i_m) \in R(w)$ be given, where $m \in \mathbb{Z}_+$. In this case, we also define the root vectors f_{β_k} and $b_{\mathbf{i}}(\mathbf{c})$ as in (2.2.7) and (2.2.8), respectively, where $1 \leq k \leq m$ and $\mathbf{c} \in \mathbb{Z}_+^m$.

The following commutation relation on root vectors is known as *Levendorskii–Soibelman formula*, see [72, Section 5.5.2, Proposition], [1, Proposition 7] and [51, Theorem 4.10] for more details.

Theorem 2.2.14. *For $j < k$,*

$$f_{\beta_k}^{(c_k)} f_{\beta_j}^{(c_j)} - q^{-(c_k\beta_k, c_j\beta_j)} f_{\beta_j}^{(c_j)} f_{\beta_k}^{(c_k)} = \sum_{\mathbf{c}'} Q_{\mathbf{c}'} b_{\mathbf{i}}(\mathbf{c}'), \tag{2.2.9}$$

where $Q_{\mathbf{c}'} \in \mathbb{Q}(q)$ *and* $\mathbf{c}' = (c'_i) \in \mathbb{Z}_+^m$. *If* $Q_{\mathbf{c}'} \neq 0$, *then* $c'_j < c_j$ *and* $c'_k < c_k$ *with* $\sum_{j \le m \le k} c'_m \beta_m = c_j \beta_j + c_k \beta_k$. □

Example 2.2.15. Let us consider the case $\mathfrak{g} = A_3$ with $I = \{1,2,3\}$. We choose a reduced expression $\mathbf{i} = 213231 \in R(w_0)$. Then the ordering is given by

$$\begin{aligned}&\beta_1 \prec \beta_2 \prec \beta_3 \prec \beta_4 \prec \beta_5 \prec \beta_6 \\ &= \alpha_2 \prec \alpha_1 + \alpha_2 \prec \alpha_2 + \alpha_3 \prec \alpha_1 + \alpha_2 + \alpha_3 \prec \alpha_3 \prec \alpha_1.\end{aligned}$$

By Theorem 2.2.14, we have

$$f_{\beta_4} f_{\beta_1} - f_{\beta_1} f_{\beta_4} = g(q) f_{\beta_2} f_{\beta_3}\,,$$

for some $g(q) \in \mathbb{Q}(q)$. On the other hand, we can check the commutation relation for f_{β_1} and f_{β_4} directly using the defining relations (2.1.1), which is given by

$$f_{\beta_4} f_{\beta_1} - f_{\beta_1} f_{\beta_4} = (q^{-1} - q) f_{\beta_2} f_{\beta_3}.$$

In general, it is difficult to describe the coefficient $Q_{\mathbf{c}'}$ in (2.2.9).

Let $U_q^-(w)$ be the $\mathbb{Q}(q)$-subspace of $U_q^-(\mathfrak{g})$ generated by $\{\, b_{\mathbf{i}}(\mathbf{c}) \mid \mathbf{c} \in \mathbb{Z}_+^m \,\}$ [79, Section 40.2]. It is known in [79, Proposition 40.2.1] that it does not depend on $\mathbf{i} \in R(w)$. Note that when $\mathfrak{g}$ is of finite type and $w = w_0$, we have $U_q^-(w_0) = U_q^-(\mathfrak{g})$. By Theorem 2.2.14, the subspace $U_q^-(w)$ is the $\mathbb{Q}(q)$-subalgebra of $U_q^-(\mathfrak{g})$ generated by $\{\, f_{\beta_k} \mid 1 \le k \le m \,\}$. The $\mathbb{Q}(q)$-subalgebra $U_q^-(w)$ is called the *quantum nilpotent subalgebra* associated with $w \in W$ [24, 51].

Chapter 3

PBW crystal and RSK correspondence of type D

In this chapter, we give a crystal theoretic interpretation of Burge correspondence which can be viewed as an analog of Robinson-Schensted-Knuth correspondence of type D_n.

As a byproduct, we give a combinatorial formula for the shape of the semistandard tableaux obtained from Burge correspondence. This is a non-trivial analog of Greene's result in type A_n [25].

The results in this chapter are based on [36].

3.1 Robinson-Schensted-Knuth correspondence

In this section, let us review the crystal theoretic interpretation of Robinson-Schensted-Knuth (RSK for short) correspondence following [60].

3.1.1 Notations

The notations in this section are used throughout this thesis (cf. [23]). Let $\mathbb{N}$ be the set of positive integers with the usual linear ordering and let $\overline{\mathbb{N}}$ be the set consisting of $\overline{i}$ ($i \in \mathbb{N}$) with the linear ordering $\overline{i} > \overline{j}$ for $i < j \in \mathbb{N}$. For $n \in \mathbb{N}$, we put $[n] = \{\, 1, \ldots, n \,\}$ and $[\overline{n}] = \{\, \overline{1}, \ldots, \overline{n} \,\}$. Let $\mathbb{Z}_+$ denote the set of non-negative integers. Let $\mathscr{P}$ be the set of partitions or Young diagrams. We let $\mathscr{P}_n = \{\, \lambda \in \mathscr{P} \,|\, \ell(\lambda) \leq n \,\}$ for $n \geq 1$, where $\ell(\lambda)$ is the length of λ. Let λ^π be the skew Young diagram obtained by 180°-rotation of λ.

Example 3.1.1. Let $\lambda = (2, 2, 1)$ be given. Then we have

$$\lambda \quad \longleftrightarrow \quad \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \multicolumn{1}{|c|}{} \\ \cline{1-1}\end{array}\ , \qquad \lambda^{\pi} = (1, 2, 2) \quad \longleftrightarrow \quad \begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{} \\ \hline \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} \\ \hline \end{array}$$

For a skew Young diagram λ/μ, we denote by $SST_{\mathcal{A}}(\lambda/\mu)$ the set of semistandard tableaux of shape λ/μ with entries in a subset $\mathcal{A}$ of $\mathbb{N}$ or $\overline{\mathbb{N}}$. We put $SST(\lambda/\mu) = SST_{\mathbb{N}}(\lambda/\mu)$ for short. For $T \in SST_{\mathcal{A}}(\lambda/\mu)$, let $w(T)$ be the word given by reading the entries of T column by column from right to left and from top to bottom in each column, and let $\mathrm{sh}(T)$ denote the shape of T. Let H_{λ} and $H_{\lambda^{\pi}}$ be the tableaux in $SST(\lambda)$ and $SST(\lambda^{\pi})$, respectively, where the i-th entry from the top in each column is filled with i for $i \geq 1$. We denote by $\mathcal{W}_n$ (resp. $\mathcal{W}_n^{\vee}$) the set of words $w = w_1 \ldots w_r$ with $w_k \in [\overline{n}]$ (resp. $w_k \in [n]$) for all $1 \leq k \leq r$. Put $\mathcal{W} = \sqcup_{n \geq 1} \mathcal{W}_n$ and $\mathcal{W}^{\vee} = \sqcup_{n \geq 1} \mathcal{W}_n^{\vee}$.

Example 3.1.2. When $\lambda = (2, 2, 1)$, the tableaux H_{λ} and $H_{\lambda^{\pi}}$ are given by

$$H_{\lambda} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1}\end{array}\ , \qquad H_{\lambda^{\pi}} = \begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{1} \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} \\ \hline \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

Then $w(H_{\lambda}) = 12123 \in \mathcal{W}_3$ and $\mathrm{sh}(H_{\lambda}) = (2, 2, 1) \in \mathscr{P}_3$.

For $a \in \mathcal{A}$ and $T \in SST_{\mathcal{A}}(\lambda)$ with $\lambda \in \mathscr{P}_n$ and $\mathcal{A} = [n]$, $[\overline{n}]$, $\mathbb{N}$ or $\overline{\mathbb{N}}$, we denote by $T \leftarrow a$ the tableau obtained by applying the Schensted's column insertion of a into T in the usual way, see [23]. Then for a word $w = w_1 \ldots w_r \in \mathcal{W} \sqcup \mathcal{W}^{\vee}$, we define $(T \leftarrow w) = (((T \leftarrow w_1) \leftarrow w_2) \leftarrow \cdots \leftarrow w_r)$. For a semistandard tableau S, we define $(T \leftarrow S) = (T \leftarrow w(S))$.

Example 3.1.3. Suppose that the tableau T and the word w are given by

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 4 \\ \cline{1-2} \multicolumn{1}{|c|}{5} \\ \cline{1-1}\end{array}\ , \qquad w = 231.$$

Then we have

$$(T \leftarrow 2) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 4 \\ \cline{1-3} \multicolumn{1}{|c|}{5} \\ \cline{1-1}\end{array} \rightsquigarrow ((T \leftarrow 2) \leftarrow 3) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 4 \\ \cline{1-3} 3 & 5 \\ \cline{1-2}\end{array} \rightsquigarrow (T \leftarrow w) = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 4 \\ \cline{1-3} 3 & 5 \\ \cline{1-2}\end{array}$$

We also define the reverse column insertion as follows. For $a \in \mathcal{A}$ and $T \in SST_{\mathcal{A}}(\lambda^{\pi})$

with $\lambda \in \mathscr{P}_n$ and $\mathcal{A} = [n]$, $[\overline{n}]$, $\mathbb{N}$ or $\overline{\mathbb{N}}$, let $a \to T$ be the tableau obtained by applying the Schensted's column insertion of a into T in a *reverse way* starting from the rightmost column. For a word $w = w_1 \dots w_r \in \mathcal{W} \sqcup \mathcal{W}^\vee$, we define $(w \to V) = (w_r \to (\cdots \to (w_1 \to T)))$. For a semistandard tableau S, we define $(S \to V) = (w(S) \to V)$.

Example 3.1.4. For $\lambda = (2, 2, 1)$,

$$T = \begin{ytableau} \none & 2 \\ 2 & 3 \\ 4 & 4 \end{ytableau} \quad \rightsquigarrow \quad 2 \to T = \begin{ytableau} \none & \none & 2 \\ \none & 2 & 3 \\ 2 & 4 & 4 \end{ytableau}\,.$$

For $T \in SST(\lambda/\mu)$, let $T^{\nwarrow}$ be the unique semistandard tableau such that $\mathrm{sh}(T^{\nwarrow}) \in \mathscr{P}$ and $w(T^{\nwarrow})$ is Knuth equivalent to $w(T)$. We define $T^{\searrow}$ in a similar way such that $\mathrm{sh}(T^{\searrow}) \in \mathscr{P}^\pi$. Note that if $\mathrm{sh}(T^{\nwarrow}) = \nu$, then $\mathrm{sh}(T^{\searrow}) = \nu^\pi$. For $w = w_1 \dots w_r \in \mathcal{W} \sqcup \mathcal{W}^\vee$, we define $P(w)^{\nwarrow} = ((w_r \leftarrow w_{r-1}) \leftarrow \cdots \leftarrow w_1)$ and $P(w)^{\searrow} = (w_r \to (\cdots \to w_2 \to w_1))$.

Example 3.1.5. Let $\lambda = (3, 2)$ and $\mu = (1)$. Then

$$T = \begin{ytableau} \none & 1 & 2 \\ 1 & 2 \end{ytableau} \in SST(\lambda/\mu) \quad \rightsquigarrow \quad T^{\nwarrow} = \begin{ytableau} 1 & 1 & 2 \\ 2 \end{ytableau}\,, \quad T^{\searrow} = \begin{ytableau} \none & \none & 1 \\ 1 & 2 & 2 \end{ytableau}\,.$$

3.1.2 Crystals and RSK correspondence

In this section, we review the RSK correspondence and its crystal interpretation following [60].

For $m, n \in \mathbb{Z}_{\geq 1}$, put

$$\mathcal{T}_{m,n} := \bigsqcup_{\substack{\lambda \in \mathscr{P} \\ \ell(\lambda) \leq \min\{m,n\}}} SST_{[m]}(\lambda) \times SST_{[n]}(\lambda)\,.$$

Let $\mathcal{M}_{m,n}$ be the set of all $m \times n$ matrices with nonnegative integers. Then, for each $M = (m_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} \in \mathcal{M}_{m \times n}$, we define a biword $(\mathbf{a}, \mathbf{b}) \in \mathcal{W}_m^\vee \times \mathcal{W}_n^\vee$ by reading the entries of M from bottom to top and from left to right such that there are m_{ij} biletters (a_k, b_k) with $a_k = i$ and $b_k = j$. Similarly, we define a biword $(\mathbf{a}, \mathbf{b}) \in \mathcal{W}_m^\vee \times \mathcal{W}_n^\vee$ by reading the entries of M from top to bottom and from right to left such that there are m_{ij} biletters (a'_k, b'_k) with $a'_k = i$ and $b'_k = j$. We often write $M = M(\mathbf{a}, \mathbf{b}) = M(\mathbf{a}', \mathbf{b}')$.

Example 3.1.6. For example, when $m = n = 3$ and M is given by

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we have the biwords $(\mathbf{a}, \mathbf{b})$ and $(\mathbf{a}', \mathbf{b}')$ of M given by

$$\begin{array}{rcccccc} \mathbf{a} = & 2 & 1 & 2 & 2 & 3 & 1 \\ \mathbf{b} = & 1 & 1 & 2 & 2 & 3 & 3 \end{array}, \qquad \begin{array}{rcccccc} \mathbf{a}' = & 1 & 1 & 2 & 2 & 2 & 3 \\ \mathbf{b}' = & 3 & 1 & 2 & 2 & 1 & 3 \end{array}.$$

Note that each letter of $\mathbf{a}$ and $\mathbf{a}'$ is the row index of M and each letter of $\mathbf{b}$ and $\mathbf{b}'$ is the column index of M.

Let us recall that RSK correspondence κ is a weight-preserving bijection given by

$$\begin{aligned} \kappa \,:\, \mathcal{M}_{m\times n} &\longrightarrow \mathcal{T}_{m,n} \\ M &\longmapsto (P(\mathbf{a})^{\nwarrow},\, Q(\mathbf{b})^{\nwarrow}) \end{aligned}, \tag{3.1.1}$$

where $P(\mathbf{a})^{\nwarrow}$ is the tableau obtained from the word $\mathbf{a}$ by the Schensted's column insertion and $Q(\mathbf{b})^{\nwarrow}$ is the recording tableau associated with the word $\mathbf{b}$.

Example 3.1.7. Consider Example 3.1.6. The pair $\kappa(M) = (P(\mathbf{a})^{\nwarrow}, Q(\mathbf{b})^{\nwarrow})$ is obtained as follows:

$$\begin{aligned} &\left(\begin{ytableau} 2 \end{ytableau}\,,\begin{ytableau} 1 \end{ytableau}\right) \rightsquigarrow \left(\begin{ytableau} 1 & 2 \end{ytableau}\,,\begin{ytableau} 1 & 1 \end{ytableau}\right) \rightsquigarrow \left(\begin{ytableau} 1 & 2 \\ 2 \end{ytableau}\,,\begin{ytableau} 1 & 1 \\ 2 \end{ytableau}\right) \rightsquigarrow \left(\begin{ytableau} 1 & 2 & 2 \\ 2 \end{ytableau}\,,\begin{ytableau} 1 & 1 & 2 \\ 2 \end{ytableau}\right) \\ &\rightsquigarrow \left(\begin{ytableau} 1 & 2 & 2 \\ 2 \\ 3 \end{ytableau}\,,\begin{ytableau} 1 & 1 & 2 \\ 2 \\ 3 \end{ytableau}\right) \rightsquigarrow \left(\begin{ytableau} 1 & 1 & 2 & 2 \\ 2 \\ 3 \end{ytableau}\,,\begin{ytableau} 1 & 1 & 2 & 3 \\ 2 \\ 3 \end{ytableau}\right) = (P(\mathbf{a})^{\nwarrow}, Q(\mathbf{b})^{\nwarrow}) \end{aligned}$$

Remark 3.1.8. By *Symmetry theorem* for RSK correspondence [23, Theorem, p.40], one can check that

$$Q(\mathbf{b})^{\nwarrow} = P(\mathbf{b}')^{\nwarrow}.$$

When we interpret the RSK correspondence in terms of crystals, we use the recording tableau of κ as $P(\mathbf{b}')^{\nwarrow}$ instead of $Q(\mathbf{b})^{\nwarrow}$.

Now, let us explain the crystal interpretation of κ. To do this, we introduce the notion of the *bicrystal* over the pair of general linear Lie algebras $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$ for $m, n \in \mathbb{Z}_{\geq 1}$. Put $I_k = \{ 1, \dots, k \}$ for $k \in \mathbb{Z}_{\geq 1}$.

Definition 3.1.9. We say that $\mathbf{B}$ is a $(\mathfrak{gl}_m, \mathfrak{gl}_n)$-*bicrystal* if it is a $\mathfrak{gl}_m$-crystal under $\tilde{e}_i$ and $\tilde{f}_i$ for $i \in I_m$ and also a $\mathfrak{gl}_n$-crystal under $\tilde{e}'_j$ and $\tilde{f}'_j$ for $j \in I_n$ such that $\tilde{e}_i$ and $\tilde{f}_i$ *commute* with $\tilde{e}'_j$ and $\tilde{f}'_j$ for all i, j.

We may identify a word $w = w_1 \cdots w_r \in \mathcal{W}^\vee$ with $w_1 \otimes \cdots \otimes w_r$. Then $\mathcal{W}^\vee$ has the crystal structure induced from the crystal $[n]$ by tensor product rule (2.2.1), respectively, where the crystal $[n]$ is given by

$$[n] \ : \ 1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n\,.$$

Indeed, the crystal $[n]$ is the crystal graph of the natural representation of $U_q(\mathfrak{gl}_n)$ [50] (see also [30, Example 4.2.7]).

Let us consider a crystal of biwords. We define $\Omega_{m,n}$ to be the set of biwords $(\mathbf{a}, \mathbf{b}) \in \mathcal{W}_m^\vee \times \mathcal{W}_n^\vee$ such that

(1) $\mathbf{a} = a_1 \dots a_r$ and $\mathbf{b} = b_1 \dots b_r$ for some $r \geq 0$,

(2) $(a_1, b_1) \preceq \cdots \preceq (a_r, b_r)$, where the ordering $\preceq$ is given by

$$(a, b) \preceq (c, d) \quad \Longleftrightarrow \quad (b < d) \ \text{or} \ (b = d \ \text{and} \ a > c)$$

for (a, b), $(c, d) \in [n] \times [n]$.

Then we define crystal operators $\tilde{e}_i$, $\tilde{f}_i$ on $\Omega_{m,n}$ for $i \in I_m$ as follows. For $i \in I_m$ and $(\mathbf{a}, \mathbf{b}) \in \Omega_{m,n}$, we define

$$\tilde{e}_i(\mathbf{a}, \mathbf{b}) = (\tilde{e}_i \mathbf{a}, \mathbf{b}), \qquad \tilde{f}_i(\mathbf{a}, \mathbf{b}) = (\tilde{f}_i \mathbf{a}, \mathbf{b})\,,$$

where we assume that $x_i(\mathbf{a}, \mathbf{b}) = \mathbf{0}$ if $x_i \mathbf{a} = \mathbf{0}$ ($x = e,\ f$), and set

$$\mathrm{wt}(\mathbf{a}, \mathbf{b}) = \mathrm{wt}(\mathbf{a}), \quad \varepsilon_i(\mathbf{a}, \mathbf{b}) = \varepsilon_i(\mathbf{a}), \quad \varphi_i(\mathbf{a}, \mathbf{b}) = \varphi_i(\mathbf{a}).$$

Then $\Omega_{m,n}$ is a $\mathfrak{gl}_m$-crystal with respect to $(\tilde{e}_i, \tilde{f}_i, \mathrm{wt}, \varepsilon_i, \varphi_i)_{i \in I_m}$.

Next, set

$$\Omega'_{m,n} = \{ (\mathbf{c}, \mathbf{d}) \in \mathcal{W}_m \times \mathcal{W}_n \ | \ (\mathbf{d}, \mathbf{c}) \in \Omega_{n,m} \}\ .$$

By the similar way as in $\Omega_{m,n}$, we define a $\mathfrak{gl}_n$-crystal structure on $\Omega'_{m,n}$ as follows. For $j \in I_n$ and $(\mathbf{c}, \mathbf{d}) \in \Omega'_{m,n}$, we define

$$\tilde{e}'_j(\mathbf{c}, \mathbf{d}) = (\mathbf{c}, \tilde{e}'_j \mathbf{d}), \qquad \tilde{f}'_j(\mathbf{c}, \mathbf{d}) = (\mathbf{c}, \tilde{f}'_j \mathbf{d}),$$

and set

$$\mathrm{wt}(\mathbf{c}, \mathbf{d}) = \mathrm{wt}(\mathbf{d}), \quad \varepsilon'_j(\mathbf{c}, \mathbf{d}) = \varepsilon_j(\mathbf{d}), \quad \varphi'_j(\mathbf{c}, \mathbf{d}) = \varphi_j(\mathbf{d}).$$

Then $\Omega'_{m,n}$ is a $\mathfrak{gl}_n$-crystal with respect to $(\tilde{e}'_i, \tilde{f}'_i, \mathrm{wt}, \varepsilon'_i, \varphi'_i)_{i \in I_m}$.

By definition, for $M \in \mathcal{M}_{m,n}$, the biwords $(\mathbf{a}, \mathbf{b})$ and $(\mathbf{a}', \mathbf{b}')$ are contained in $\Omega_{m,n}$ and $\Omega'_{m,n}$, respectively. Then, the maps

$$(\mathbf{a}, \mathbf{b}) \longmapsto M = M(\mathbf{a}, \mathbf{b}) \ , \qquad (\mathbf{a}', \mathbf{b}') \longmapsto M = M(\mathbf{a}', \mathbf{b}')$$

are bijective and induce the $\mathfrak{gl}_m$-crystal structure and the $\mathfrak{gl}_n$-crystal structure on $\mathcal{M}_{m,n}$, respectively. Then one can check that $\mathcal{M}_{m,n}$ becomes a $(\mathfrak{gl}_m, \mathfrak{gl}_n)$-bicrystal. Note that $\mathcal{T}_{m,n}$ has the natural $(\mathfrak{gl}_m, \mathfrak{gl}_n)$-bicrystal structure.

Theorem 3.1.10. [15] *The map κ (3.1.1) is an isomorphism of $(\mathfrak{gl}_m, \mathfrak{gl}_n)$-bicrystals.* $\square$

3.2 PBW crystals

3.2.1 Description of $\tilde{f}_i$

Suppose that $\mathfrak{g}$ is of finite type. Let $\mathbf{i} \in R(w_0)$ be given. For $\beta \in \Phi^+$, we denote by $\mathbf{1}_\beta$ the element in $\mathbf{B_i}$ where $c_\beta = 1$ and $c_\gamma = 0$ for $\gamma \in \Phi^+ \setminus \{\beta\}$. The Kashiwara operators $\tilde{f}_i$ or $\tilde{f}_i^*$ on $\mathbf{B_i}$ for $i \in I$ is not easy to describe in general except

$$\begin{aligned} \tilde{f}_i \mathbf{c} &= (c_1 + 1, c_2, \ldots, c_N) = \mathbf{c} + \mathbf{1}_{\alpha_i}, \qquad \text{when } \beta_1 = \alpha_i, \\ \tilde{f}_i^* \mathbf{c} &= (c_1, \ldots, c_{N-1}, c_N + 1) = \mathbf{c} + \mathbf{1}_{\alpha_i}, \quad \text{when } \beta_N = \alpha_i, \end{aligned} \tag{3.2.1}$$

for $\mathbf{c} \in \mathbf{B_i}$ [79].

Let us review the results in [103], where it is shown that $\tilde{f}_i$ can be described more explicitly in terms of so-called *signature rule* under certain conditions on $\mathbf{i}$ with respect to $i \in I$ (in [103], the authors call it *bracketing rule*). For simplicity, let us assume that $\mathfrak{g}$ is of types A, D or E from now on.

Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_s)$ be a sequence with $\sigma_u \in \{+, -, \cdot\}$. We replace a pair $(\sigma_u, \sigma_{u'}) = (+, -)$, where $u < u'$ and $\sigma_{u''} = \cdot$ for $u < u'' < u'$, with $(\cdot, \cdot)$, and repeat this process as far as possible until we get a sequence with no $-$ placed to the right of $+$. We denote the resulting sequence by σ^{red}. For another sequence $\tau = (\tau_1, \ldots, \tau_t)$, we denote by $\sigma \cdot \tau$ the concatenation of σ and τ.

Recall that a total order $\prec$ on the set Φ^+ of positive roots is called *convex* if either $\gamma \prec \gamma' \prec \gamma''$ or $\gamma'' \prec \gamma' \prec \gamma$ whenever $\gamma' = \gamma + \gamma''$ for $\gamma, \gamma', \gamma'' \in \Phi^+$. It is well-known that there exists a one-to-one correspondence between $R(w_0)$ and the set of convex orders on Φ^+, where the convex order $\prec$ associated with $\mathbf{i} = (i_1, \ldots, i_N) \in R(w_0)$ is given by

$$\beta_1 \prec \beta_2 \prec \ \ldots \prec \beta_N, \tag{3.2.2}$$

where β_k is as in (2.2.5) [93].

There exists a reduced expression $\mathbf{i}'$ obtained from $\mathbf{i}$ by a 3-term braid move $(i_k, i_{k+1}, i_{k+2}) \rightarrow (i_{k+1}, i_k, i_{k+1})$ with $i_k = i_{k+2}$ if and only if

$$\{\, \beta_k, \beta_{k+1}, \beta_{k+2} \,\}$$

forms the positive roots of type A_2, where the corresponding convex order $\prec'$ is given by replacing $\beta_k \prec \beta_{k+1} \prec \beta_{k+2}$ with $\beta_{k+2} \prec' \beta_{k+1} \prec' \beta_k$. Also there exists a reduced expression $\mathbf{i}'$ obtained from $\mathbf{i}$ by a 2-term braid move $(i_k, i_{k+1}) \rightarrow (i_{k+1}, i_k)$ if and only if β_k and β_{k+1} are orthogonal, where the associated convex ordering $\prec'$ is given by replacing $\beta_k \prec \beta_{k+1}$ with $\beta_{k+1} \prec' \beta_k$.

Given $i \in I$, suppose that $\mathbf{i}$ is *simply braided for* $i \in I$, that is, if one can obtain $\mathbf{i}' = (i'_1, \ldots, i'_N) \in R(w_0)$ with $i'_1 = i$ by applying a sequence of braid moves consisting of either a 2-term move or 3-term braid move $(\gamma, \gamma', \gamma'') \rightarrow (\gamma'', \gamma', \gamma)$ with $\gamma'' = \alpha_i$, see [103, Definition 4.1].

Suppose that

$$\Pi_s = \{\gamma_s, \gamma'_s, \gamma''_s\} \tag{3.2.3}$$

is the triple of positive roots of type A_2 with $\gamma'_s = \gamma_s + \gamma''_s$ and $\gamma''_s = \alpha_i$ corresponding to the s-th 3-term braid move for $1 \leq s \leq t$.

For $\mathbf{c} \in \mathbf{B_i}$, let

$$\sigma_i(\mathbf{c}) = (\underbrace{- \cdots -}_{c_{\gamma'_1}} \underbrace{+ \cdots +}_{c_{\gamma_1}} \cdots \underbrace{- \cdots -}_{c_{\gamma'_t}} \underbrace{+ \cdots +}_{c_{\gamma_t}}). \tag{3.2.4}$$

Then we have the following description of $\tilde{f}_i$ on $\mathbf{B_i}$.

Theorem 3.2.1. [103, Theorem 4.6] *Let $\mathbf{i} \in R(w_0)$ and $i \in I$. Suppose that $\mathbf{i}$ is simply braided for i. Let $\mathbf{c} \in \mathbf{B_i}$ be given.*

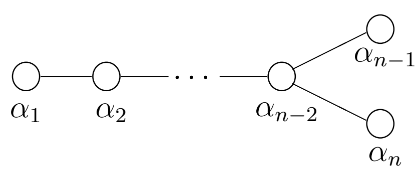
(1) *If there exists $+$ in $\sigma_i(\mathbf{c})^{\mathrm{red}}$ and the leftmost $+$ appears in c_{γ_s}, then*

$$\tilde{f}_i\mathbf{c} = \mathbf{c} - \mathbf{1}_{\gamma_s} + \mathbf{1}_{\gamma'_s}.$$

(2) *If there exists no $+$ in $\sigma_i(\mathbf{c})^{\mathrm{red}}$, then $\tilde{f}_i\mathbf{c} = \mathbf{c} + \mathbf{1}_{\alpha_i}$.* □

3.2.2 Kac-Moody algebra of type D_n

From now on, we assume that $\mathfrak{g}$ is the Kac-Moody algebra associated to the Dynkin diagram



where $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq n-1$, and $\alpha_n = \epsilon_{n-1} + \epsilon_n$. Also, we assume that the weight lattice is $P = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$, where $\{\, \epsilon_i \,|\, 1 \leq i \leq n \,\}$ is an orthonormal basis with respect to the symmetric bilinear form $(\ ,\ )$, that is, $(\epsilon_i, \epsilon_j) = \delta_{ij}$ for $1 \leq i, j \leq n$. The set of positive roots is $\Phi^+ = \{\, \epsilon_i \pm \epsilon_j \,|\, 1 \leq i < j \leq n \,\}$. Recall that W acts faithfully on P by $s_i(\epsilon_i) = \epsilon_{i+1}$, $s_i(\epsilon_k) = \epsilon_k$ for $1 \leq i \leq n-1$ and $k \neq i, i+1$, and $s_n(\epsilon_{n-1}) = -\epsilon_n$ and $s_n(\epsilon_k) = \epsilon_k$ for $k \neq n-1, n$ (recall (2.2.4)). The fundamental weights are $\varpi_i = \sum_{k=1}^i \epsilon_k$ for $1 \leq i \leq n-2$, $\varpi_{n-1} = (\epsilon_1 + \cdots + \epsilon_{n-1} - \epsilon_n)/2$ and $\varpi_n = (\epsilon_1 + \cdots + \epsilon_{n-1} + \epsilon_n)/2$.

Let

$$\mathcal{P}_n = \left\{\, (\lambda_1, \ldots, \lambda_n) \,\middle|\, \lambda_i \in \tfrac{1}{2}\mathbb{Z},\ \lambda_i - \lambda_{i+1} \in \mathbb{Z}_+,\ \lambda_{n-1} \geq |\lambda_n| \,\right\}. \tag{3.2.5}$$

For $\lambda \in \mathcal{P}_n$, we put

$$\omega_\lambda = \sum_{i=1}^{n} \lambda_i \epsilon_i \,. \tag{3.2.6}$$

Then $P^+ = \{\omega_\lambda \,|\, \lambda \in \mathcal{P}_n\}$ is the set of dominant integral weights. We put $\mathtt{sp}^+ = \left((\frac{1}{2})^n\right)$ and $\mathtt{sp}^- = ((\frac{1}{2})^{n-1}, -\frac{1}{2}))$ for simplicity. We also identify $\lambda \in \mathcal{P}_n$ with a (generalized) Young diagram, which may have a half-width box on the leftmost column, see [50, Section 6.7].

3.2.3 PBW crystal of type D_n

In this subsection, we give an explicit description of the signature rule in type D_n associated with the reduced expression $\mathbf{i}_0 \in R(w_0)$ whose convex order on Φ^+ is given by

$$\begin{aligned} &\epsilon_i + \epsilon_j \prec \epsilon_k - \epsilon_l, \\ \epsilon_i + \epsilon_j \prec \epsilon_k + \epsilon_l \quad &\Longleftrightarrow \quad (j > l) \text{ or } (j = l,\, i > k), \\ \epsilon_i - \epsilon_j \prec \epsilon_k - \epsilon_l \quad &\Longleftrightarrow \quad (i < k) \text{ or } (i = k,\, j < l), \end{aligned} \tag{3.2.7}$$

for $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$. An explicit form of $\mathbf{i}_0$ is as follows. For $1 \leq k \leq n-1$, put

$$\mathbf{i}_k = \begin{cases} (n, n-2, \ldots, k+1, k), & \text{if } k \text{ is odd}, \\ (n-1, n-2, \ldots, k+1, k), & \text{if } k \text{ is even}, \\ (n), & \text{if } n \text{ is even and } k = n-1, \end{cases}$$

$$\mathbf{i}'_k = \begin{cases} (n-1, n-2, \ldots, k+1, k), & \text{if } n \text{ is even and } 1 \leq k \leq n-1, \\ (n, n-2, \ldots, k+1, k), & \text{if } n \text{ is odd and } 1 \leq k \leq n-2, \\ (n), & \text{if } n \text{ is odd and } k = n-1. \end{cases}$$

Let $\mathbf{i}^J = \mathbf{i}_1 \cdot \mathbf{i}_2 \cdot \cdots \cdot \mathbf{i}_{n-1}$ and $\mathbf{i}_J = \mathbf{i}'_1 \cdot \mathbf{i}'_2 \cdot \cdots \cdot \mathbf{i}'_{n-1}$. Then

$$\mathbf{i}_0 = \mathbf{i}^J \cdot \mathbf{i}_J, \tag{3.2.8}$$

where $\mathbf{i} \cdot \mathbf{j}$ denotes the concatenation of $\mathbf{i} \in I^{\times r}$ and $\mathbf{j} \in I^{\times s}$. We write $\mathbf{i}_0 = (i_1, \ldots, i_N)$, where $i_1 = n$, and put $\mathbf{i}^J = (i_1, \ldots, i_M)$, and $\mathbf{i}_J = (i_{M+1}, \ldots, i_N)$ with $N = n^2 - n$ and

$M = N/2$.

Example 3.2.2. We have

$$\begin{aligned} &\mathbf{i}^J = (4,2,1,3,2,4), && \mathbf{i}_J = (3,2,1,3,2,3), && \text{when } n = 4, \\ &\mathbf{i}^J = (5,3,2,1,4,3,2,5,3,4), && \mathbf{i}_J = (5,3,2,1,5,3,2,5,3,5), && \text{when } n = 5. \end{aligned}$$

The associated convex order when $n = 4$ is

$$\begin{aligned} &\epsilon_3 + \epsilon_4 \prec \epsilon_2 + \epsilon_4 \prec \epsilon_1 + \epsilon_4 \prec \epsilon_2 + \epsilon_3 \prec \epsilon_1 + \epsilon_3 \prec \epsilon_1 + \epsilon_2 \\ &\prec \epsilon_1 - \epsilon_2 \prec \epsilon_1 - \epsilon_3 \prec \epsilon_1 - \epsilon_4 \prec \epsilon_2 - \epsilon_3 \prec \epsilon_2 - \epsilon_4 \prec \epsilon_3 - \epsilon_4. \end{aligned}$$

For $\mathbf{c} = (c_\beta) \in \mathbf{B}_{\mathbf{i}_0}$, we write

$$c_{\beta_k} = \begin{cases} c_{\overline{j}\,\overline{i}}, & \text{if } \beta_k = \epsilon_i + \epsilon_j \text{ for } 1 \le i < j \le n, \\ c_{j\overline{i}}, & \text{if } \beta_k = \epsilon_i - \epsilon_j \text{ for } 1 \le i < j \le n. \end{cases} \tag{3.2.9}$$

Proposition 3.2.3. *For $i \in I \setminus \{n\}$, there exists a reduced expression $\mathbf{i} \in R(w_0)$, which is equal to $\mathbf{i}_0$ up to 2-term braid moves, such that $\mathbf{i}$ is simply braided for $i \in I$ and the signature $\sigma_i(\mathbf{c})$ for $\mathbf{c} \in \mathbf{B}_{\mathbf{i}}$ (recall (3.2.4)) is given by*

$$\sigma_i(\mathbf{c}) = \sigma_{i,1}(\mathbf{c}) \cdot \sigma_{i,2}(\mathbf{c}) \cdot \sigma_{i,3}(\mathbf{c}),$$

where

$$\begin{aligned} \sigma_{i,1}(\mathbf{c}) &= (\underbrace{-\cdots-}_{c_{\overline{n}\,\overline{i}}}\ \underbrace{+\cdots+}_{c_{\overline{n}\,\overline{i+1}}}\underbrace{-\cdots-}_{c_{\overline{n-1}\,\overline{i}}}\ \underbrace{+\cdots+}_{c_{\overline{n-1}\,\overline{i+1}}}\ \cdots\ \underbrace{-\cdots-}_{c_{\overline{i+2}\,\overline{i}}}\ \underbrace{+\cdots+}_{c_{\overline{i+2}\,\overline{i}}}), \\ \sigma_{i,2}(\mathbf{c}) &= (\underbrace{-\cdots-}_{c_{\overline{i}\,\overline{i-1}}}\ \underbrace{+\cdots+}_{c_{\overline{i+1}\,\overline{i-1}}}\underbrace{-\cdots-}_{c_{\overline{i}\,\overline{i-2}}}\ \underbrace{+\cdots+}_{c_{\overline{i+1}\,\overline{i-2}}}\ \cdots\ \underbrace{-\cdots-}_{c_{\overline{i}\,\overline{1}}}\ \underbrace{+\cdots+}_{c_{\overline{i+1}\,\overline{1}}}), \\ \sigma_{i,3}(\mathbf{c}) &= (\underbrace{-\cdots-}_{c_{i+1\,\overline{1}}}\ \underbrace{+\cdots+}_{c_{i\,\overline{1}}}\underbrace{-\cdots-}_{c_{i+1\,\overline{2}}}\ \underbrace{+\cdots+}_{c_{i\,\overline{2}}}\ \cdots\ \underbrace{-\cdots-}_{c_{i+1\,\overline{i-1}}}\ \underbrace{+\cdots+}_{c_{i\,\overline{i-1}}}\underbrace{-\cdots-}_{c_{i+1\,\overline{i}}}). \end{aligned} \tag{3.2.10}$$

Here we assume that c_{ab} is zero when it is not defined and a 2-term braid move means that $ij = ji$ for $i, j \in I$ such that $a_{ij} = 0$.

Proof. We assume that n is even since the proof for n odd is almost identical (see Example 3.2.5). Let us fix $i \in I \setminus \{n\}$.

Step 1. We first observe that if the first letter $n-1$ in $\mathbf{i}'_i$ corresponds to i_k in $\mathbf{i}_0$ for some k, then $\beta_k = \alpha_i$.

Step 2. Let $i_k = n-1$ be as in *Step 1*. Suppose that $i \neq 1$. Then we can apply 2-term move or 3-term braid move $(\gamma, \gamma', \gamma'') \to (\gamma'', \gamma', \gamma)$ with $\gamma'' = \alpha_i$ to $\mathbf{i}_0$ (indeed to the subword $\mathbf{i}'_1 \cdot \mathbf{i}'_2 \cdots \mathbf{i}'_{i-1} \cdot i_k$) to get $\mathbf{i}_0^{(3)} = (i_1, \ldots, i_M, j, \ldots)$ with $j = n-i$. We can check that 3-term braid move occurs once in each $\mathbf{i}'_s$ for $s = 1, \ldots, i-1$, and the positive roots of the corresponding root system of type A_2 is

$$\Pi_s^{(3)} = \{\epsilon_s - \epsilon_i, \epsilon_s - \epsilon_{i+1}, \alpha_i\} \tag{3.2.11}$$

for $s = 1, \cdots, i-1$. We assume that $\Pi_s^{(3)}$ is empty when $i = 1$.

Step 3. We consider the reduced word $\mathbf{i}_0^{(3)}$. Suppose that $i \neq 1$ or $j \neq n$. First, we apply 2-moves only to $\mathbf{i}_j \cdot \mathbf{i}_{j+1}$ so that the last $i-2$ letters in $\mathbf{i}_j$ and the first $i-2$ letters are shuffled by a permutation of length $(i-2)(i-1)/2$ and hence appear in an alternative way. We denote this subword by $\bar{\mathbf{i}}_j \cdot \bar{\mathbf{i}}_{j+1}$.

Then we apply 2-term move or 3-term braid move $(\gamma, \gamma', \gamma'') \to (\gamma'', \gamma', \gamma)$ with $\gamma'' = \alpha_i$ to the subword $\bar{\mathbf{i}}_j \cdot \bar{\mathbf{i}}_{j+1} \cdot \mathbf{i}_{j+2} \cdots \mathbf{i}_{n-1} \cdot j$ to obtain a word starting with j'' where j'' is n (resp. $n-1$) when j is odd (resp. even). We denote the resulting whole word by $\mathbf{i}_0^{(2)}$.

Here we have $i-1$ 3-term braid moves only in $\bar{\mathbf{i}}_j \cdot \bar{\mathbf{i}}_{j+1}$ and the positive roots of the corresponding root system of type A_2 is

$$\Pi_s^{(2)} = \{\epsilon_{i+1} + \epsilon_s, \epsilon_i + \epsilon_s, \alpha_i\} \tag{3.2.12}$$

for $1 \leq s \leq i-1$, and the order of occurrence of 3-term braid move is when s ranges from 1 to $i-1$. If $i = 1$, then we assume that $\Pi_s^{(2)}$ is empty, and $\mathbf{i}_0^{(2)} = \mathbf{i}_0^{(3)}$.

Step 4. Finally, we apply 2-term move or 3-term braid move $(\gamma, \gamma', \gamma'') \to (\gamma'', \gamma', \gamma)$ with $\gamma'' = \alpha_i$ to the subword $\mathbf{i}_1 \cdots \mathbf{i}_j \cdot j''$ of $\mathbf{i}_0^{(2)}$ to obtain a word starting with i, and denote the resulting whole word by $\mathbf{i}_0^{(1)}$. In this case, 3-term braid move occurs once in each $\mathbf{i}_s$ for $s = 1, \ldots, j-1$, and the positive roots of the corresponding root system of type A_2 is

$$\Pi_s^{(1)} = \{\epsilon_{n-s+1} + \epsilon_{i+1}, \epsilon_{n-s+1} + \epsilon_i, \alpha_i\} \tag{3.2.13}$$

for $s = 1, \cdots, j-1$.

By the above steps, we conclude that $\mathbf{i}_0^{(1)} \in R(w_0)$ is obtained from $\mathbf{i}_0$ by applying

2-term move or 3-term braid move $(\gamma, \gamma', \gamma'') \to (\gamma'', \gamma', \gamma)$ with $\gamma'' = \alpha_i$. We define

$$\mathbf{i} := \bar{\mathbf{i}}^J \cdot \mathbf{i}_J,$$

where $\bar{\mathbf{i}}^J = \mathbf{i}_1 \cdot \cdots \cdot \bar{\mathbf{i}}_j \cdot \bar{\mathbf{i}}_{j+1} \cdot \cdots \cdot \mathbf{i}_{n-1}$ (recall (3.2.8)) and $\bar{\mathbf{i}}_j \cdot \bar{\mathbf{i}}_{j+1}$ is obtained from $\mathbf{i}_j \cdot \mathbf{i}_{j+1}$ in *Step 3*. By *Step 1*–*Step 4*, the reduced expression $\mathbf{i}$ is simply braided for i. It follows from Theorem 3.2.1 that the sequence $\sigma_i(\mathbf{c})$ in (3.2.4) for $\mathbf{c} \in \mathbf{B_i}$ is given by (3.2.10), where the positive roots of the root systems of type A_2 associated with $\sigma_{i,j}(\mathbf{c})$ are given by $\Pi_s^{(j)}$ for $j = 1, 2, 3$ in (3.2.13), (3.2.12), and (3.2.11). □

Remark 3.2.4.

(1) For $i \in I \setminus \{n\}$, the crystal operator $\tilde{f}_i$ on $\mathbf{B}_{\mathbf{i}_0}$ may be understood by

$$\mathbf{B}_{\mathbf{i}_0} \xrightarrow{R_{\mathbf{i}_0}^{\mathbf{i}}} \mathbf{B}_{\mathbf{i}} \xrightarrow{\tilde{f}_i} \mathbf{B}_{\mathbf{i}} \xrightarrow{R_{\mathbf{i}}^{\mathbf{i}_0}} \mathbf{B}_{\mathbf{i}_0},$$

where $R_{\mathbf{i}_0}^{\mathbf{i}}$ (resp. $R_{\mathbf{i}}^{\mathbf{i}_0}$) is the transition map from $\mathbf{B}_{\mathbf{i}_0}$ to $\mathbf{B}_{\mathbf{i}}$ (resp. from $\mathbf{B}_{\mathbf{i}}$ to $\mathbf{B}_{\mathbf{i}_0}$) (cf. [76]). The map $R_{\mathbf{i}_0}^{\mathbf{i}}$ corresponds to the 2-term braid moves from $\mathbf{i}_j \cdot \mathbf{i}_{j+1}$ to $\bar{\mathbf{i}}_j \cdot \bar{\mathbf{i}}_{j+1}$ (see *Step 3* in the proof of Proposition 3.2.3), which is simply given by exchanging the multiplicities related to them, and the map $R_{\mathbf{i}}^{\mathbf{i}_0}$ is the inverse of it. Therefore, the crystal operator $\tilde{f}_i$ on $\mathbf{B}_{\mathbf{i}_0}$ can be described in the same way as in Theorem 3.2.1 with $\sigma_i(\mathbf{c})$ in Proposition 3.2.3.

(2) In type A_n, for $r \in I$, we take $\mathbf{i}_0 = \mathbf{i}^J \cdot \mathbf{i}_J \in R(w_0)$ such that $\mathbf{i}^J$ and $\mathbf{i}_J$ are given by

$$\mathbf{i}^J = \mathbf{i}_1 \cdot \mathbf{i}_2 \cdot \cdots \cdot \mathbf{i}_{n-r+1}, \qquad \mathbf{i}_J = \mathbf{i}_{n-r+2} \cdot \mathbf{i}_{n-r+3} \cdot \cdots \cdot \mathbf{i}_{2n-r}\,. \tag{3.2.14}$$

where $\mathbf{i}_s$ is defined by

$$\mathbf{i}_s = \begin{cases} (r+s-1,\, r+s-2, \ldots,\, s+1,\, s) & \text{if } 1 \le s \le n-r+1, \\ (r+1,\, r,\, \cdots,\, r-s+1) & \text{if } n-r+1 < s \le n, \\ (1,\, 2,\, \cdots,\, n-r+s-1) & \text{if } n < s \le 2n-r\,. \end{cases}$$

By a similar argument as in the proof of Proposition 3.2.3, $\mathbf{i}_0$ is simply braided as in the sense of Proposition 3.2.3 and we obtain the sequence $\sigma_i(\mathbf{c})$ associated with the reduced expression $\mathbf{i}_0 = \mathbf{i}^J \cdot \mathbf{i}_J$ (3.2.14). Then it is straightforward to check that

$\mathbf{B}_{\mathbf{i}_0}$ coincides with the crystal $\mathcal{B}_\Omega = \mathbf{B}_{\mathbf{i}}$ in [65], where Ω is the Dynkin quiver of type A_n and $\mathbf{i}$ is adapted to Ω (see also [95, 99]). Indeed, the reduced expression $\mathbf{i}_0$ is also adapted to the Dynkin quiver A_n.

Example 3.2.5. Let us illustrate $\sigma_i(\mathbf{c})$ for $\mathbf{c} \in \mathbf{B}_{\mathbf{i}_0}$ when $n = 5$ and $i = 3$. Consider $\mathbf{i}_0 = \mathbf{i}^J \cdot \mathbf{i}_J = (i_1, \ldots, i_{20})$ (see Example 3.2.2). Note that $i_{18} = 5$ is the first letter in $\mathbf{i}'_3$, and $\beta_{18} = \alpha_3$.

For convenience, let $\rightsquigarrow$ (resp. $\longrightarrow$) mean the 3-term (resp. 2-term) braid move. Then

$$\begin{aligned}
\mathbf{i}_J = (5,3,2,1,5,3,2,\mathbf{5},3,5) &\longrightarrow (5,3,2,1,5,3,\mathbf{5},2,3,5) \\
&\rightsquigarrow (5,3,2,1,\mathbf{3},5,3,2,3,5) \quad \cdots \quad \Pi_2^{(3)} \\
&\longrightarrow (5,3,2,\mathbf{3},1,5,3,2,3,5) \\
&\rightsquigarrow (5,\mathbf{2},3,2,1,5,3,2,3,5) \quad \cdots \quad \Pi_1^{(3)} \\
&\longrightarrow (\mathbf{2},5,3,2,1,5,3,2,3,5),
\end{aligned}$$

where $\Pi_2^{(3)} = \{\epsilon_2 - \epsilon_3, \epsilon_2 - \epsilon_4, \alpha_3\}$ and $\Pi_1^{(3)} = \{\epsilon_1 - \epsilon_3, \epsilon_1 - \epsilon_4, \alpha_3\}$. Here the bold letter denotes the one corresponding to α_3 in the associated convex order on Φ^+.

Next, we have $\mathbf{i}_2 \cdot \mathbf{i}_3 = (4,3,\mathbf{2},\mathbf{5},3) \longrightarrow \bar{\mathbf{i}}_2 \cdot \bar{\mathbf{i}}_3 = (4,3,\mathbf{5},\mathbf{2},3)$, and hence

$$\begin{aligned}
\bar{\mathbf{i}}_2 \cdot \bar{\mathbf{i}}_3 \cdot \mathbf{2} = (4,3,5,2,3,\mathbf{2}) &\rightsquigarrow (4,3,5,\mathbf{3},2,3) \quad \cdots \quad \Pi_2^{(2)} \\
&\rightsquigarrow (4,\mathbf{5},3,5,2,3) \quad \cdots \quad \Pi_1^{(2)} \\
&\longrightarrow (\mathbf{5},4,3,5,2,3),
\end{aligned}$$

where $\Pi_2^{(2)} = \{\epsilon_1 + \epsilon_4, \epsilon_1 + \epsilon_3, \alpha_3\}$ and $\Pi_1^{(2)} = \{\epsilon_2 + \epsilon_4, \epsilon_2 + \epsilon_3, \alpha_3\}$. Finally,

$$\begin{aligned}
\mathbf{i}_1 \cdot \mathbf{5} = (5,3,2,1,\mathbf{5}) &\longrightarrow (5,3,2,\mathbf{5},1) \\
&\longrightarrow (5,3,\mathbf{5},2,1) \\
&\rightsquigarrow (\mathbf{3},5,3,2,1) \quad \cdots \quad \Pi_1^{(1)},
\end{aligned}$$

where $\Pi_1^{(3)} = \{\epsilon_4 + \epsilon_5, \epsilon_3 + \epsilon_5, \alpha_3\}$. Thus $\mathbf{i}_0$ is simply braided for $i = 3$. Hence, for $\mathbf{c} \in \mathbf{B}_{\mathbf{i}_0}$,

$$\begin{aligned}
&\sigma_{3,1}(\mathbf{c}) = (\underbrace{-\cdots-}_{c_{\overline{5}\overline{3}}}\ \underbrace{+\cdots+}_{c_{\overline{5}\overline{4}}}), \quad \sigma_{3,2}(\mathbf{c}) = (\underbrace{-\cdots-}_{c_{\overline{3}\overline{2}}}\ \underbrace{+\cdots+}_{c_{\overline{4}\overline{2}}}\underbrace{-\cdots-}_{c_{\overline{3}\overline{1}}}\ \underbrace{+\cdots+}_{c_{\overline{4}\overline{1}}}), \\
&\sigma_{3,3}(\mathbf{c}) = (\underbrace{-\cdots-}_{c_{4\overline{1}}}\ \underbrace{+\cdots+}_{c_{3\overline{1}}}\underbrace{-\cdots-}_{c_{4\overline{2}}}\ \underbrace{+\cdots+}_{c_{3\overline{2}}}\underbrace{-\cdots-}_{c_{4\overline{3}}}).
\end{aligned}$$

3.2.4 Crystal $\mathbf{B}^J$ of quantum nilpotent subalgebra

Put $J = I \setminus \{n\}$. Let $\mathfrak{l}$ be the Levi subalgebra of $\mathfrak{g}$ associated with $\{\, \alpha_i \,|\, i \in J \,\}$ of type A_{n-1}. Then

$$\Phi^+ = \Phi^+(J) \cup \Phi_J^+,$$

where $\Phi_J^+ = \{\, \epsilon_i - \epsilon_j \,|\, 1 \le i < j \le n \,\}$ is the set of positive roots of $\mathfrak{l}$ and $\Phi^+(J) = \{\, \epsilon_i + \epsilon_j \,|\, 1 \le i < j \le n \,\}$ is the set of roots of the nilradical $\mathfrak{u}$ of the parabolic subalgebra of $\mathfrak{g}$ associated with $\mathfrak{l}$.

Set

$$\begin{aligned} \mathbf{B}^J &= \left\{\, \mathbf{c} = (c_\beta) \in \mathbf{B}_{\mathbf{i}_0} \,\middle|\, c_\beta = 0 \text{ unless } \beta \in \Phi^+(J) \,\right\}, \\ \mathbf{B}_J &= \left\{\, \mathbf{c} = (c_\beta) \in \mathbf{B}_{\mathbf{i}_0} \,\middle|\, c_\beta = 0 \text{ unless } \beta \in \Phi_J \,\right\}. \end{aligned} \tag{3.2.15}$$

which we regard them as subcrystals of $\mathbf{B}_{\mathbf{i}_0}$, where we assume that $\widetilde{e}_n\mathbf{c} = \widetilde{f}_n\mathbf{c} = \mathbf{0}$ with $\varepsilon_n(\mathbf{c}) = \varphi_n(\mathbf{c}) = -\infty$ for $\mathbf{c} \in \mathbf{B}_J$. The subcrystal $\mathbf{B}^J$ is the *crystal of the quantum nilpotent subalgebra* $U_q^-(w^J)$, where $w^J = s_{i_1} \cdots s_{i_M}$ with $\mathbf{i}^J = (i_1, \ldots, i_M)$, which can be viewed as a *q-deformation* of $U(\mathfrak{u}^-)$.

Proposition 3.2.6.

(1) *The crystal $\mathbf{B}_J$ is isomorphic to the crystal of $U_q^-(\mathfrak{l})$ as an $\mathfrak{l}$-crystal.*

(2) *The map*

$$\begin{aligned} \mathbf{B}_{\mathbf{i}_0} &\longrightarrow \mathbf{B}^J \otimes \mathbf{B}_J \\ \mathbf{c} &\longmapsto \mathbf{c}^J \otimes \mathbf{c}_J \end{aligned} \tag{3.2.16}$$

is an isomorphism of $\mathfrak{g}$-crystals.

Proof. (1) It follows directly from comparing the crystal structure of $U_q^-(\mathfrak{l})$ given in [99, Section 4.1] (see also [65, Section 4.2]).

(2) It follows from Theorem 3.2.1, Proposition 3.2.3, and the tensor product rule of crystals. □

We have the characterization of the crystal $\mathbf{B}^J$ as follows.

Proposition 3.2.7. *We have* $\mathbf{B}^J = \{\, \mathbf{c} \,|\, \varepsilon_i^*(\mathbf{c}) = 0 \ (i \in J) \,\}$.

Proof. It follows from [76, Section 2.1] and Proposition 3.2.3(2). □

From Proposition 3.2.3, we also obtain the decomposition of $\mathbf{B}^J$ as a $\mathfrak{l}$-crystal.

Proposition 3.2.8.

(1) $\boldsymbol{c} = (c_{\overline{j}\overline{i}}) \in \mathbf{B}^J$ *is an* $\mathfrak{l}$*-highest weight vector if and only if*

$$c_{\overline{n}\,\overline{n-1}} \geq c_{\overline{n-2}\,\overline{n-3}} \geq \cdots, \quad c_{\overline{j}\overline{i}} = 0 \text{ elsewhere.} \tag{3.2.17}$$

(2) *As an* $\mathfrak{l}$*-crystal, we have*

$$\mathbf{B}^J \cong \bigsqcup_{\lambda} B_J(\lambda),$$

where the union is over $\lambda = \sum_{i=1}^{n} \lambda_i \epsilon_i \in P$ *such that* $0 \geq \lambda_1 = \lambda_2 \geq \lambda_3 = \lambda_4 \geq \cdots$, *and* $B_J(\lambda)$ *is the crystal of an integrable highest weight* $U_q(\mathfrak{l})$*-module for* λ.

Proof. It is enough to prove (1) because (1) implies (2) by Propositions 2.2.12 and 3.2.7 (see (6.2.3)). It is immediate from Proposition 3.2.3 that if $\mathbf{c}$ satisfies (3.2.17), then $\widetilde{e}_i\mathbf{c} = \mathbf{0}$ for $i \in I \setminus \{n\}$. Conversely, suppose that $\mathbf{c} = (c_{\overline{j}\overline{i}}) \in \mathbf{B}^J$ is an $\mathfrak{l}$-highest weight vector. If $c_{\overline{j}\overline{i}} \neq 0$ for some $(i,j) \notin \{\,(n-1,n),(n-3,n-2),\dots\}$, then choose $c_{\overline{j}\overline{i}} \neq 0$ whose corresponding root $\epsilon_i + \epsilon_j$ is minimal with respect to (3.2.7). If $j - i > 1$, then $\widetilde{e}_i\mathbf{c} \neq \mathbf{0}$, and if $j - i = 1$, then $\widetilde{e}_j\mathbf{c} \neq \mathbf{0}$. This is a contradiction. Next, if $c_{\overline{i+2}\,\overline{i+1}} < c_{\overline{i}\,\overline{i-1}}$ for some $i \geq 2$, then we have $\widetilde{e}_i\mathbf{c} \neq \mathbf{0}$, which is also a contradiction. Hence $\mathbf{c}$ satisfies (3.2.17). □

Remark 3.2.9. Recall that there is a one-to-one correspondence between the reduced expressions of w_0 and the convex orderings on Φ^+ [93]. Then the subexpression $(i_1,\dots,i_M)$ corresponding to the roots of $\mathfrak{u}$ always appears as the first M entries (up to 2-term braid moves) in any reduced expression of w_0 such that the positive roots of $\mathfrak{u}$ precede those of $\mathfrak{l}$ with respect to the corresponding convex ordering. Here a 2-term braid move means $ij = ji$ for $i,j \in I$ such that $|i-j| > 1$.

3.2.5 Notation for $\mathbf{B}^J$

Let Δ_n be the arrangements of dots in the plane to represent the $(n-1)$-th triangular number. We often identify Δ_n with $\Phi^+(J)$ in such a way that $\epsilon_{k+1}+\epsilon_{l+1}$, $\epsilon_{k+1}+\epsilon_l$ and $\epsilon_k+\epsilon_l$ for $1 \leq k,l \leq n-1$ are the vertices of a triangle of minimal shape in Δ_n as follows:

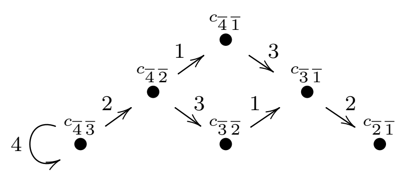
$$\begin{array}{ccc} & \overset{\epsilon_k+\epsilon_{l+1}}{\bullet} & \\ \overset{\epsilon_{k+1}+\epsilon_{l+1}}{\bullet} & & \overset{\epsilon_k+\epsilon_l}{\bullet} \end{array} \tag{3.2.18}$$

We also identify $\mathbf{c} \in \mathbf{B}^J$ with an array of c_β's in $\mathbf{c}$ with c_β at the corresponding dot in Δ_n.

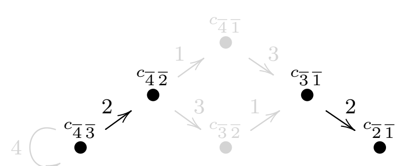
Example 3.2.10. For $n = 5$ and $\mathbf{c} \in \mathbf{B}^J$, we have

$$\Delta_5 = \begin{array}{ccccccc} & & & \overset{\epsilon_1+\epsilon_5}{\bullet} & & & \\ & & \overset{\epsilon_2+\epsilon_5}{\bullet} & & \overset{\epsilon_1+\epsilon_4}{\bullet} & & \\ & \overset{\epsilon_3+\epsilon_5}{\bullet} & & \overset{\epsilon_2+\epsilon_4}{\bullet} & & \overset{\epsilon_1+\epsilon_3}{\bullet} & \\ \overset{\epsilon_4+\epsilon_5}{\bullet} & & \overset{\epsilon_3+\epsilon_4}{\bullet} & & \overset{\epsilon_2+\epsilon_3}{\bullet} & & \overset{\epsilon_1+\epsilon_2}{\bullet} \end{array} \qquad \mathbf{c} = \begin{array}{ccccccc} & & & c_{\overline{51}} & & & \\ & & c_{\overline{52}} & & c_{\overline{41}} & & \\ & c_{\overline{53}} & & c_{\overline{42}} & & c_{\overline{31}} & \\ c_{\overline{54}} & & c_{\overline{43}} & & c_{\overline{32}} & & c_{\overline{21}} \end{array}$$

Example 3.2.11. Under the above convention, for $n = 4$, the description of $\tilde{f}_i$ $(i \in I)$ is as follows (recall Theorem 3.2.1 and Proposition 3.2.3(2)):



For example, when $i = 2$, the signature $\sigma_2(\mathbf{c})$ is given by Lusztig datum associated with the arrows labeled 2 as follows:



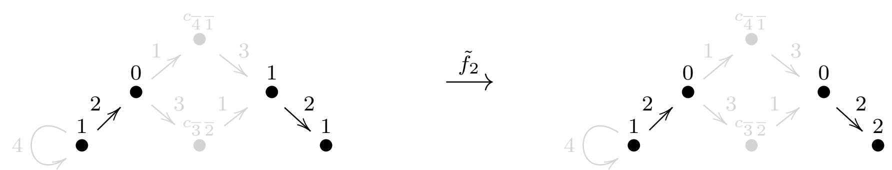
Hence we have

$$\sigma_2(\mathbf{c}) = \big(\underbrace{-\cdots-}_{c_{\overline{42}}}\underbrace{+\cdots+}_{c_{\overline{43}}}\underbrace{-\cdots-}_{c_{\overline{21}}}\underbrace{+\cdots+}_{c_{\overline{31}}}\big).$$

Then the reduced signature $\sigma_2^{\mathrm{red}}(\mathbf{c})$ determines an arrow labeled 2 at which we apply $\tilde{f}_2$. For instance, if $c_{\overline{42}} = 0$, $c_{\overline{43}} = 1$, $c_{\overline{21}} = 1$ and $c_{\overline{31}} = 1$, then

$$\sigma_2(\mathbf{c}) = (\cdot\ +\ -+) \longrightarrow \sigma_2^{\mathrm{red}}(\mathbf{c}) = (\cdot\ \cdot\ \cdot +)$$

and we have



3.3 Burge correspondence

3.3.1 RSK of type $\mathbf{D}_n$

Let us recall a variation of RSK correspondence for type D due to Burge [6]. Put

$$\mathcal{T}^{\searrow} := \bigsqcup_{\substack{\lambda \in \mathscr{P}_n \\ \lambda' : \text{even}}} SST_{[\overline{n}]}(\lambda^{\pi}), \tag{3.3.1}$$

where we say that λ' is even if each part of λ' is even. Let Ω be the set of biwords $(\mathbf{a}, \mathbf{b}) \in \mathcal{W} \times \mathcal{W}$ such that

(1) $\mathbf{a} = a_1 \cdots a_r$ and $\mathbf{b} = b_1 \cdots b_r$ for some $r \geq 0$,

(2) $a_i < b_i$ for $1 \leq i \leq r$,

(3) $(a_1, b_1) \leq \cdots \leq (a_r, b_r)$,

where $(a, b) < (c, d)$ if and only if $(a < c)$ or $(a = c$ and $b > d)$ for (a, b), $(c, d) \in \mathcal{W} \times \mathcal{W}$.

We denote by $\mathbf{c}(\mathbf{a}, \mathbf{b})$ the unique element in $\mathbf{B}^J$ corresponding to $(\mathbf{a}, \mathbf{b})$ such that $c_{ab} = |\{\, k \mid (a_k, b_k) = (a, b) \,\}|$.

Example 3.3.1. Suppose that $n = 5$. Let $\mathbf{c} \in \mathbf{B}^J$ be given by

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 0 & & \\ & 1 & & 2 & & 1 & \\ 2 & & 1 & & 0 & & 1 \end{array}$$

Then the corresponding biword $\mathbf{c} = \mathbf{c}(\mathbf{a}, \mathbf{b})$ for $(\mathbf{a}, \mathbf{b}) \in \Omega$ is given by

$$\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} \overline{5} & \overline{5} & \overline{5} & \overline{5} & \overline{5} & \overline{4} & \overline{4} & \overline{4} & \overline{3} & \overline{2} \\ \overline{1} & \overline{2} & \overline{3} & \overline{4} & \overline{4} & \overline{2} & \overline{2} & \overline{3} & \overline{1} & \overline{1} \end{pmatrix}.$$

For $(\mathbf{a}, \mathbf{b}) \in \Omega$ with $\mathbf{a} = a_1 \cdots a_r$ and $\mathbf{b} = b_1 \cdots b_r$, we define a sequence of tableaux $P_r, P_{r-1}, \ldots, P_1$ inductively as follows:

(1) let P_1 be a vertical domino $\begin{array}{|c|}\hline a_r \\ \hline b_r \\ \hline \end{array}$,

(2) if P_{k+1} is given for $1 \leq k \leq r-1$, then define P_k to be the tableau obtained by first applying the column insertion to get $b_k \to P_{k+1}$, and then adding $\boxed{a_k}$ at the conner of $b_k \to P_{k+1}$ located above the box $\mathrm{sh}(b_k \to P_{k+1})/\mathrm{sh}(P_{k+1})$.

We put $P^{\searrow}(\mathbf{a}, \mathbf{b}) := P_1$. It is not difficult to see from the definition that $P^{\searrow}(\mathbf{a}, \mathbf{b}) \in SST(\lambda^\pi)$ for some $\lambda \in \mathscr{P}$ such that λ' is even.

For $\mathbf{c} \in \mathbf{B}^J$, let $P^{\searrow}(\mathbf{c}) = P^{\searrow}(\mathbf{a}, \mathbf{b})$ where $\mathbf{c} = \mathbf{c}(\mathbf{a}, \mathbf{b})$. Since the map $(\mathbf{a}, \mathbf{b}) \mapsto P^{\searrow}(\mathbf{a}, \mathbf{b})$ is a bijection from Ω to $\mathcal{T}^{\searrow}$ [6], we have a bijection

$$\begin{aligned} \kappa^{\searrow} : \mathbf{B}^J &\longrightarrow \mathcal{T}^{\searrow} \\ \mathbf{c} &\longmapsto P^{\searrow}(\mathbf{c}) \end{aligned} \quad . \tag{3.3.2}$$

Example 3.3.2. Let us consider Example 3.3.1. The following is the sequence of tableaux $P_r, P_{r-1}, \ldots, P_1 =: P^{\searrow}(\mathbf{a}, \mathbf{b})$ given in the definition of $\kappa^{\searrow}$ (3.3.2):

$$\begin{array}{c}\bar{2}\\ \bar{1}\end{array} \xrightarrow{\substack{\bar{3}\\ \bar{1}}} \begin{array}{cc}\bar{3}&\bar{2}\\ \bar{1}&\bar{1}\end{array} \xrightarrow{\substack{\bar{4}\\ \bar{3}}} \begin{array}{cc}&\bar{4}\\ &\bar{3}\\ \bar{3}&\bar{2}\\ \bar{1}&\bar{1}\end{array} \xrightarrow{\substack{\bar{4}\\ \bar{2}}} \begin{array}{ccc}&&\bar{4}\\ &&\bar{3}\\ \bar{4}&\bar{2}&\bar{2}\\ \bar{3}&\bar{1}&\bar{1}\end{array} \xrightarrow{\substack{\bar{4}\\ \bar{2}}} \begin{array}{cccc}&&&\bar{4}\\ &&&\bar{3}\\ \bar{4}&\bar{4}&\bar{2}&\bar{2}\\ \bar{3}&\bar{2}&\bar{1}&\bar{1}\end{array}$$

$$\xrightarrow{\substack{\bar{5}\\ \bar{4}}} \begin{array}{cccc}&&\bar{5}&\bar{4}\\ &&\bar{4}&\bar{3}\\ \bar{4}&\bar{4}&\bar{2}&\bar{2}\\ \bar{3}&\bar{2}&\bar{1}&\bar{1}\end{array} \xrightarrow{\substack{\bar{5}\\ \bar{4}}} \begin{array}{ccccc}&&&\bar{5}&\bar{4}\\ &&&\bar{4}&\bar{3}\\ \bar{5}&\bar{4}&\bar{4}&\bar{2}&\bar{2}\\ \bar{4}&\bar{3}&\bar{2}&\bar{1}&\bar{1}\end{array} \xrightarrow{\substack{\bar{5}\\ \bar{3}}} \begin{array}{cccccc}&&&&\bar{5}&\bar{4}\\ &&&&\bar{3}&\bar{3}\\ \bar{5}&\bar{5}&\bar{4}&\bar{4}&\bar{2}&\bar{2}\\ \bar{4}&\bar{4}&\bar{3}&\bar{2}&\bar{1}&\bar{1}\end{array}$$

$$\xrightarrow{\substack{\bar{5}\\ \bar{2}}} \begin{array}{ccccccc}&&&&&\bar{5}&\bar{4}\\ &&&&&\bar{3}&\bar{3}\\ \bar{5}&\bar{5}&\bar{5}&\bar{4}&\bar{4}&\bar{2}&\bar{2}\\ \bar{4}&\bar{4}&\bar{3}&\bar{2}&\bar{2}&\bar{1}&\bar{1}\end{array} \xrightarrow{\substack{\bar{5}\\ \bar{1}}} \begin{array}{cccccccc}&&&&&&\bar{5}&\bar{4}\\ &&&&&&\bar{3}&\bar{3}\\ \bar{5}&\bar{5}&\bar{5}&\bar{5}&\bar{4}&\bar{4}&\bar{2}&\bar{2}\\ \bar{4}&\bar{4}&\bar{3}&\bar{2}&\bar{2}&\bar{1}&\bar{1}&\bar{1}\end{array}$$

Here we use the notation $T \xrightarrow{\substack{\bar{j}\\ \bar{i}}} T'$ when $T = P_{k+1}$, $T' = P_k$ and $(a_k, b_k) = (\bar{j}, \bar{i})$. Hence, we have

$$\kappa^{\searrow}(\mathbf{c}) = \begin{array}{cccccccc} & & & & & & \overline{5} & \overline{4} \\ & & & & & & \overline{3} & \overline{3} \\ \overline{5} & \overline{5} & \overline{5} & \overline{5} & \overline{4} & \overline{4} & \overline{2} & \overline{2} \\ \overline{4} & \overline{4} & \overline{3} & \overline{2} & \overline{2} & \overline{1} & \overline{1} & \overline{1} \end{array}$$

Let us recall the $\mathfrak{g}$-crystal structure on $\mathcal{T}^{\searrow}$ [62, Section 5.2]. We regard $[\overline{n}] = \{\, \overline{n} < \cdots < \overline{1} \,\}$ as the crystal of dual natural representation of $\mathfrak{l}$ with $\mathrm{wt}(\overline{k}) = -\epsilon_k$. Then $\mathcal{W}$ is a regular $\mathfrak{l}$-crystal, where $w = w_1 \dots w_r$ is identified with $w_1 \otimes \cdots \otimes w_r$. For $\lambda \in \mathscr{P}_n$, $SST(\lambda)$ is a regular $\mathfrak{l}$-crystal with lowest weight $-\sum_{i=1}^{n} \lambda_i \epsilon_i$, where T is identified with $w(T)$ [50]. In particular $\mathcal{T}^{\searrow}$ is a regular $\mathfrak{l}$-crystal.

Let $T \in \mathcal{T}^{\searrow}$ be given. For $k \geq 1$, let t_k be the entry in the top of the k-th column of T (enumerated from the right). Consider $\sigma = (\sigma_1, \sigma_2, \dots)$, where

$$\sigma_k = \begin{cases} + \,, & \text{if } t_k > \overline{n-1} \text{ or the } k\text{-th column is empty,} \\ - \,, & \text{if the } k\text{-th column has both } \overline{n-1} \text{ and } \overline{n} \text{ as its entries,} \\ \cdot \,, & \text{otherwise.} \end{cases}$$

Then $\widetilde{e}_n T$ is obtained from T by removing $\begin{array}{|c|}\hline \overline{n} \\ \hline \overline{n-1} \\ \hline \end{array}$ in the column corresponding to the rightmost $-$ in σ^{red} (recall Section 3.2.1 for σ^{red}). If there is no such $-$ sign, then we define $\widetilde{e}_n T = \mathbf{0}$, and $\widetilde{f}_n T$ is obtained from T by adding $\begin{array}{|c|}\hline \overline{n} \\ \hline \overline{n-1} \\ \hline \end{array}$ column corresponding to the leftmost $+$ in σ^{red}. Hence $\mathcal{T}^{\searrow}$ is a $\mathfrak{g}$-crystal with respect to wt, ε_i, φ_i, $\widetilde{e}_i$, $\widetilde{f}_i$ ($i \in I$), where $\varepsilon_n(T) = \max\{\, k \,|\, \widetilde{e}_n^{\,k} T \neq 0 \,\}$ and $\varphi_n(T) = \varepsilon_n(T) + \langle \mathrm{wt}(T), h_n \rangle$.

Now we are in position to state the crystal theoretic interpretation of Burge correspondence.

Theorem 3.3.3. *The bijection $\kappa^{\searrow}$ in* (3.3.2) *is an isomorphism of $\mathfrak{g}$-crystals.*

Proof. The key observation is that Burge correspondence can be described by an inductive algorithm using RSK correspondence for skew tableaux. Here the skew RSK correspondence is introduced by Sagan-Stanley [97]. Recall that it is known that RSK correspondence is an isomorphism of crystals of type A, see Section 1.2.1 and references therein. Then we compare the crystal structure of $\mathbf{B}^J$ in Proposition 3.2.3 with the one of $\mathcal{T}^{\searrow}$. The detailed proof is given in Section 7.1.2. $\square$

3.3.2 Shape formula

For $\mathbf{c} \in \mathbf{B}^J$, let

$$\lambda(\mathbf{c}) = (\lambda_1(\mathbf{c}) \geq \ldots \geq \lambda_\ell(\mathbf{c})) \tag{3.3.3}$$

be the partition corresponding to the regular $\mathfrak{l}$-subcrystal of $\mathbf{B}^J$ including $\mathbf{c}$, that is, $\lambda(\mathbf{c})^\pi = \mathrm{sh}(\kappa^{\searrow}(\mathbf{c}))$ by Theorem 3.3.3. Note that $\ell = 2[\frac{n}{2}]$ and $\lambda_{2i-1}(\mathbf{c}) = \lambda_{2i}(\mathbf{c})$ for $1 \leq i \leq [\frac{n}{2}]$.

We can characterize the whole partition $\lambda(\mathbf{c})$ in terms of *double paths* on Δ_n as follows.

Definition 3.3.4. A *path in* Δ_n is a sequence $p = (\gamma_1, \ldots, \gamma_s)$ in $\Phi^+(J)$ for some $s \geq 1$ such that

(1) $\gamma_1, \ldots, \gamma_s \in \Phi^+(J)$,

(2) if $\gamma_i = \epsilon_k + \epsilon_{l+1}$ for some $k < l$, then $\gamma_{i+1} = \epsilon_{k+1} + \epsilon_{l+1}$ or $\epsilon_k + \epsilon_l$ (see (3.2.18)),

(3) $\gamma_s = \epsilon_k + \epsilon_{k+1}$ for some k.

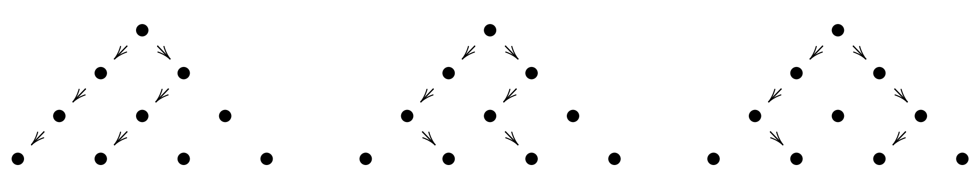
For $\beta \in \Phi^+(J)$, a *double path at* β *in* Δ_n is a pair of paths $\mathbf{p} = (p_1, p_2)$ in Δ_n of the same length with $p_1 = (\gamma_1, \ldots, \gamma_s)$ and $p_2 = (\delta_1, \ldots, \delta_s)$ such that

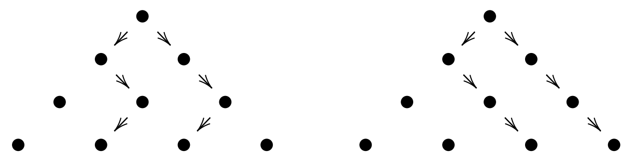
(1) $\gamma_1 = \delta_1 = \beta$,

(2) γ_i is located to the strictly left of δ_i for $2 \leq i \leq s$,

(3) $\gamma_s = \epsilon_{k+1} + \epsilon_{k+2}$, $\delta_s = \epsilon_k + \epsilon_{k+1}$ for some $k \geq 1$.

Example 3.3.5. For a double path $\mathbf{p} = (p_1, p_2)$ at β, if we draw an arrow from γ_i to γ_{i+1} in p_1 and from δ_i to δ_{i+1} in p_2, then p_1 and p_2 form a pair of non-intersecting paths starting from β going downward to the bottom row in Δ_n with p_1 on the left, and p_2 on the right. The following is the list of double paths $\mathbf{p}$ at $\epsilon_1 + \epsilon_5$ in Δ_5.





For $\mathbf{c} \in \mathbf{B}^J$ and a double path $\mathbf{p}$, let

$$||\mathbf{c}||_{\mathbf{p}} = \sum_{\beta \text{ lying on } \mathbf{p}} c_\beta. \tag{3.3.4}$$

Theorem 3.3.6. *For $\mathbf{c} \in \mathbf{B}^J$ and $1 \le l \le [\frac{n}{2}]$, we have*

$$\lambda_1(\mathbf{c}) + \lambda_3(\mathbf{c}) + \cdots + \lambda_{2l-1}(\mathbf{c}) = \max_{\mathbf{p}_1,\ldots,\mathbf{p}_l} \{\, ||\mathbf{c}||_{\mathbf{p}_1} + \cdots + ||\mathbf{c}||_{\mathbf{p}_l} \,\},$$

where $\mathbf{p}_1, \ldots, \mathbf{p}_l$ are mutually non-intersecting double paths in Δ_n and each $\mathbf{p}_i$ starts at the $(2i-1)$-th row of Δ_n for $1 \le i \le l$.

Proof. The proof is given in Section 7.1.3 (see also Remark 6.2.6(2)). $\square$

Example 3.3.7. Let $n = 6$ and let $\mathbf{c} \in \mathbf{B}^J$ be given by

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 2 & & 3 & & & \\
 & & 2 & & 1 & & 1 & & \\
 & 1 & & 3 & & 2 & & 1 & \\
2 & & 3 & & 2 & & 0 & & 3
\end{array}$$

Then we have

$$\kappa^{\searrow}(\mathbf{c}) = \begin{array}{ccccccccccccccccccc}
 & & & & & & & & & & & & & & & & & \overline{6} & \overline{6} \\
 & & & & & & & & & & & & & & & & & \overline{5} & \overline{5} \\
 & & & & & & & & & & & & & \overline{6} & \overline{6} & \overline{5} & \overline{4} & \overline{4} & \overline{4} \\
 & & & & & & & & & & & & & \overline{5} & \overline{3} & \overline{3} & \overline{3} & \overline{3} & \overline{3} \\
\overline{6} & \overline{6} & \overline{6} & \overline{6} & \overline{5} & \overline{5} & \overline{5} & \overline{5} & \overline{5} & \overline{4} & \overline{3} & \overline{3} & \overline{2} & \overline{2} & \overline{2} & \overline{2} & \overline{2} & \overline{2} & \overline{2} \\
\overline{5} & \overline{5} & \overline{5} & \overline{4} & \overline{4} & \overline{4} & \overline{4} & \overline{4} & \overline{3} & \overline{2} & \overline{1} & \overline{1} & \overline{1} & \overline{1} & \overline{1} & \overline{1} & \overline{1} & \overline{1} & \overline{1}
\end{array}$$

where $\lambda(\mathbf{c}) = (19, 19, 6, 6, 2, 2)$.

On the other hand, the double path $\mathbf{p}$ at $\epsilon_1 + \epsilon_6$ given by

$$
\begin{array}{ccccccccccccccccc}
&&&&&&&&\mathbf{1}&&&&&&&& \\
&&&&&&&\swarrow&&\searrow&&&&&&& \\
&&&&&&\mathbf{2}&&&&\mathbf{3}&&&&&& \\
&&&&&\swarrow&&&&&&\searrow&&&&& \\
&&&&\mathbf{2}&&&&1&&&&\mathbf{1}&&&& \\
&&&&&\searrow&&&&&&\swarrow&&&&& \\
&&1&&&&\mathbf{3}&&&&\mathbf{2}&&&&1&& \\
&&&&&\swarrow&&&&\swarrow&&&&&&& \\
2&&&&\mathbf{3}&&&&\mathbf{2}&&&&0&&&&3
\end{array}
$$

has maximal value $||\mathbf{c}||_{\mathbf{p}} = 19$, and the pair of double paths $\mathbf{p}_1$ and $\mathbf{p}_2$ at $\epsilon_1 + \epsilon_6$ and $\epsilon_3 + \epsilon_6$, respectively, given by

$$
\begin{array}{ccccccccccccccccc}
&&&&&&&&\mathbf{1}&&&&&&&& \\
&&&&&&&\swarrow&&\searrow&&&&&&& \\
&&&&&&\mathbf{2}&&&&\mathbf{3}&&&&&& \\
&&&&&&&\searrow&&&&\searrow&&&&& \\
&&&&\mathbf{2}&&&&\mathbf{1}&&&&\mathbf{1}&&&& \\
&&&\swarrow&&\searrow&&&&\searrow&&&&\searrow&&& \\
&&\mathbf{1}&&&&\mathbf{3}&&&&\mathbf{2}&&&&\mathbf{1}&& \\
&&&\searrow&&&&\searrow&&&&\searrow&&&&\searrow& \\
2&&&&\mathbf{3}&&&&\mathbf{2}&&&&\mathbf{0}&&&&\mathbf{3}
\end{array}
$$

has maximal value $||\mathbf{c}||_{\mathbf{p}_1} + ||\mathbf{c}||_{\mathbf{p}_2} = 25$. By Theorem 3.3.6, we have

$$
\lambda_1(\mathbf{c}) = 19, \quad \lambda_1(\mathbf{c}) + \lambda_3(\mathbf{c}) = 25, \quad \lambda_1(\mathbf{c}) + \lambda_3(\mathbf{c}) + \lambda_5(\mathbf{c}) = 27,
$$

which implies $\lambda_3(\mathbf{c}) = 6$, $\lambda_5(\mathbf{c}) = 2$, and hence $\lambda(\mathbf{c}) = (19, 19, 6, 6, 2, 2)$.

Remark 3.3.8. Suppose that $\mathfrak{g}$ is of type A_n and $\mathfrak{l}$ is of type $\mathrm{A}_r \times \mathrm{A}_s$ with $r + s = n - 1$. The associated crystal $B(U_q(\mathfrak{u}^-))$ can be realized as the set of $(r+1) \times (s+1)$ non-negative integral matrices (see [65, Section 4.3]). For $M \in B(U_q(\mathfrak{u}^-))$, let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be the shape of the tableaux corresponding to M under RSK. It is a well-known result due to Greene [25] (cf. [23]) that $\lambda_1 + \cdots + \lambda_l$ is a maximal sum of entries in M lying on mutually non-intersecting l lattice paths on $(r+1) \times (s+1)$ array of points from northeast to southwest. A similar result when $\mathfrak{g}$ is of types BC is obtained by folding crystals of type A with $r = s$. Hence, Theorem 3.3.6 is a non-trivial generalization of [25] to the case of type D. We can also recover the result in [25] by using the same argument as in Section 7.1.3.

Chapter 4

Crystal embedding from $B(\lambda)$ into $B(\infty)$

In this chapter, we describe the crystal embedding (2.2.3) for type D_n in a combinatorial way. More precisely, the embedding is obtained as follows:

$$\mathbf{KN}_\lambda \xrightarrow{\cong} \mathbf{T}_\lambda \xrightarrow{(a)} \mathbf{V}_\lambda \xrightarrow{(b)} \mathbf{B}_{\mathbf{i}_0} \otimes T_\lambda,$$

where the crystals are given by

$\mathbf{KN}_\lambda$: the crystal of Kashiwara-Nakashima tableaux for type D_n of shape λ [50], which is isomorphic to $B(\lambda)$ (Section 4.1.2),

$\mathbf{T}_\lambda$: the spinor model of type D_n associated with λ, which is isomorphic to $\mathbf{KN}_\lambda$ [63, 64] (cf. [66]) (Section 4.1.3),

$\mathbf{V}_\lambda$: the crystal of parabolic Verma module associated with the maximal Levi subalgebra of type A_{n-1} (Section 4.4.1),

$\mathbf{B}_{\mathbf{i}_0}$: the crystal of $\mathbf{i}_0$-Lusztig datum in Section 3.2.3, which is isomorphic to the crystal $B(\infty)$,

and the crystal embedding (a) is obtained from a combinatorial algorithm on $\mathbf{T}_\lambda$ compatible with the crystal structure of $\mathbf{V}_\lambda$, which is developed in Section 4.3, and the crystal embedding (b) is obtained by using Burge correspondence (recall Section 3.3.1) and a well-known result for type A.

This chapter is based on [38].

4.1 Highest weight crystals for type D_n

4.1.1 Tableaux with two columns

For $a, b, c \in \mathbb{Z}_+$, let $\lambda(a, b, c)$ be a skew Young diagram with at most two columns given by $(2^{b+c}, 1^a)/(1^b)$. Let T be a tableau of shape $\lambda(a, b, c)$. We denote the left and right columns of T by $T^{\mathtt{L}}$ and $T^{\mathtt{R}}$, respectively.

Let T be a tableau. If necessary, we assume that it is placed on the plane with a horizontal line L, say $\mathbb{P}_L$, such that any box in T is either below or above L, and at least one edge of a box in T meets L. We denote by $T^{\mathtt{body}}$ and $T^{\mathtt{tail}}$ the subtableaux of T above and below L, respectively. For example,

$$T = \begin{array}{ccccc} & & \boxed{\overline{5}} & & \\ & & \boxed{\overline{4}} & & \\ & \boxed{\overline{5}} & \boxed{\overline{3}} & & \\ & \boxed{\overline{4}} & \boxed{\overline{2}} & & \\ \cdots\cdots & & & \cdots\cdots & L \\ & \boxed{\overline{3}} & & & \\ & \boxed{\overline{1}} & & & \end{array} \qquad T^{\mathtt{body}} = \begin{array}{cc} & \boxed{\overline{5}} \\ & \boxed{\overline{4}} \\ \boxed{\overline{5}} & \boxed{\overline{3}} \\ \boxed{\overline{4}} & \boxed{\overline{2}} \end{array} \qquad T^{\mathtt{tail}} = \begin{array}{c} \boxed{\overline{3}} \\ \boxed{\overline{1}} \end{array}$$

where the dotted line denotes L.

For a tableau U with the shape of a single column, let $\mathrm{ht}(U)$ denote the height of U and we put $U(i)$ (resp. $U[i]$) to be i-th entry of U from bottom (resp. top). We also write

$$U = (U(\ell), \ldots, U(1)) = (U[1], \ldots, U[\ell]),$$

where $\ell = \mathrm{ht}(U)$. Suppose that U is a tableau in $\mathbb{P}_L$. To emphasize gluing and cutting tableaux with respect to L, we also write

$$U^{\mathtt{body}} \boxplus U^{\mathtt{tail}} = U, \quad U \boxminus U^{\mathtt{tail}} = U^{\mathtt{body}}.$$

For a sequence of tableaux $U_1, U_2, \ldots, U_m$ in $\mathbb{P}_L$, whose shapes are single columns, let us say that $(U_1, U_2, \ldots, U_m)$ is *semistandard along* L if they form a semistandard tableau T of a skew shape with U_i the i-th column of T from the left.

4.1.2 Kashiwara-Nakashima tableaux of type D_n

Let us recall Section 3.1 and 3.2.2 for the notations. Suppose that $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{P}_n$ is given. The notion of Kashiwara-Nakashima tableaux (KN tableaux, for short) of type D [50] is a combinatorial model of $B(\omega_\lambda)$. In this thesis, we need an analogue, which is obtained from the one in [50] by applying 180° rotation and replacing i and $\overline{i}$ (resp. $\overline{i}$ with i). For the reader's convenience, let us give its definition and crystal structure.

In this section, we assume that $[n] \cup [\overline{n}]$ has the ordering given by

$$1 < 2 < \cdots < n-1 < \begin{array}{c} n \\ \overline{n} \end{array} < \overline{n-1} < \cdots < \overline{2} < \overline{1}.$$

Definition 4.1.1. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{P}_n$, let T be a tableau of shape λ^π with entries in $[n] \cup [\overline{n}]$ such that

(1) $T(i,j) \not\geq T(i+1,j)$ and $T(i,j) \leq T(i,j+1)$ for each i and j,

(2) n and $\overline{n}$ can appear successively in T other than half-width boxes,

(3) i and $\overline{i}$ do not appear simultaneously in the half-width boxes,

where $T(i,\,j)$ denotes the entry in T located in the i-th row from the bottom and the j-th column from the right. Then T is called a *KN tableau of type D_n* if it satisfies the following conditions:

($\mathfrak{d}$-1) If $T(p,j) = \overline{i}$ and $T(q,j) = i$ for some $i \in [n]$ with $p < q$, then $(q-p) + i > \lambda'_j$.

($\mathfrak{d}$-2) Suppose $\lambda_n \geq 0$ and $\lambda'_j = n$. If $T(k,j) = n$ (resp. $\overline{n}$), then k is odd (resp. even).

($\mathfrak{d}$-3) Suppose $\lambda_n < 0$ and $\lambda'_j = n$. If $T(k,j) = n$ (resp. $\overline{n}$), then k is even (resp. odd).

($\mathfrak{d}$-4) If either $T(p,j) = \overline{a}$, $T(q,j) = \overline{b}$, $T(r,j) = b$ and $T(s,j+1) = a$ or $T(p,j) = \overline{a}$, $T(q,j+1) = \overline{b}$, $T(r,j+1) = b$ and $T(s,j+1) = a$ with $p \leq q < r \leq s$ and $a \leq b < n$, then $(q-p) + (s-r) < b - a$.

($\mathfrak{d}$-5) Suppose $T(p,j) = \overline{a}$, $T(s,j+1) = a$ with $p < s$. If there exists $p \leq q < s$ such that either $T(q,j)$, $T(q+1,j) \in \{\,n, \overline{n}\,\}$ with $T(q,j) \neq T(q+1,j)$ or $T(q,j+1)$, $T(q+1,j+1) \in \{\,n, \overline{n}\,\}$ with $T(q,j+1) \neq T(q+1,j+1)$, then $s - p \leq n - a$.

($\mathfrak{d}$-6) It is not possible that $T(p,j) \in \{\,n, \overline{n}\,\}$ and $T(s,j+1) \in \{\,n, \overline{n}\,\}$ with $p < s$.

($\mathfrak{d}$-7) Suppose $T(p,j) = \overline{a}$, $T(s,j+1) = a$ with $p < s$. If $T(q,j+1) \in \{n, \overline{n}\}$, $T(r,j) \in \{n, \overline{n}\}$ and $s-q+1$ is either odd or even with $p \leq q < r \leq s$ and $a < n$, then $s-p < n-a$.

We denote by $\mathbf{KN}_\lambda$ the set of KN tableaux of shape λ^π.

Recall that $\mathbf{KN}_{(1)}$ has the following crystal structure isomorphic to that of $B(\varpi_1)$.

$$
\begin{tikzcd}
 & & & & \boxed{n} \arrow[dr, "n"] & & & & \\
\boxed{1} \arrow[r, "1"] & \boxed{2} \arrow[r, "2"] & \cdots \arrow[r, "n-2"] & \boxed{n-1} \arrow[ur, "n-1"] \arrow[dr, "n"'] & & \boxed{\overline{n-1}} \arrow[r, "n-2"] & \cdots \arrow[r, "2"] & \boxed{\overline{2}} \arrow[r, "1"] & \boxed{\overline{1}} \\
 & & & & \boxed{\overline{n}} \arrow[ur, "n-1"'] & & & &
\end{tikzcd}
$$

where $\boxed{a} \stackrel{i}{\longrightarrow} \boxed{b}$ means $\widetilde{f}_i\,\boxed{a} = \boxed{b}$ with $\widetilde{f}_i$ the Kashiwara operator for $i \in I$, and $\mathrm{wt}(\boxed{i}) = \epsilon_i$, $\mathrm{wt}(\boxed{\overline{i}}) = -\epsilon_i$. for $i = 1, \ldots, n$. On the other hand, $\mathbf{KN}_{\mathtt{sp}+}$ and $\mathbf{KN}_{\mathtt{sp}-}$ have crystal structures isomorphic to those of $B(\varpi_n)$ and $B(\varpi_{n-1})$ which are the crystals of spin representations with highest weights ϖ_n and ϖ_{n-1}, respectively. For $i \in I$, $\widetilde{f}_i$ on $\mathbf{KN}_{\mathtt{sp}\pm}$ is given by

$$
\begin{array}{|c|}
\vdots \\ \hline i \\ \hline \vdots \\ \hline i+1 \\ \hline \vdots
\end{array}
\xrightarrow{\ \widetilde{f}_i\ (i \neq n)\ }
\begin{array}{|c|}
\vdots \\ \hline i+1 \\ \hline \vdots \\ \hline \overline{i} \\ \hline \vdots
\end{array}
\qquad\qquad
\begin{array}{|c|}
\vdots \\ \hline n-1 \\ \hline n \\ \hline \vdots
\end{array}
\xrightarrow{\ \widetilde{f}_n\ }
\begin{array}{|c|}
\vdots \\ \hline \overline{n} \\ \hline \overline{n-1} \\ \hline \vdots
\end{array}. \tag{4.1.1}
$$

Let $\lambda \in \mathcal{P}_n$ be given. Let us identify $T \in \mathbf{KN}_\lambda$ with its word $w(T)$ so that we may regard

$$
\mathbf{KN}_\lambda \subset \begin{cases} \left(\mathbf{KN}_{(1)}\right)^{\otimes \mathrm{N}}, & \text{if } \lambda_n \in \mathbb{Z}, \\ \mathbf{KN}_{\mathtt{sp}^\pm} \otimes \left(\mathbf{KN}_{(1)}\right)^{\otimes \mathrm{N}}, & \text{if } \lambda_n \notin \mathbb{Z}, \end{cases}
$$

where N is the number of letters in $w(T)$ except for the one in half-width boxes. Then $\mathbf{KN}_\lambda$ is invariant under $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in I$, and

$$
\mathbf{KN}_\lambda \cong B(\omega_\lambda),
$$

(see [50, Theorem 6.7.1]).

4.1.3 Spinor model for type D_n

The spinor model is originated from the combinatorial description of the crystal of an integrable highest weight module in a parabolic BGG category over quantum ortho-symplectic superalgebra [63, 64], where the category is established via *Super duality* developed by Cheng-Lam-Wang (see [14, Chapter 6] and references therein). As a byproduct, this should induce a combinatorial model of the crystal $B(\lambda)$ for an integrable highest weight irreducible module $V(\lambda)$ over the classical Lie algebras of types BCD. In [63, Section 7.2], the author explains briefly the connection with Kashiwara-Nakashima tableaux.

In this section, we review the spinor model for type D_n and we give an explicit crystal isomorphism between spinor model and Kashiwara-Nakashima tableaux of type D in Section 4.2. Note that the isomorphism for types BC is given in [66, Section 3.3].

Definition 4.1.2. For $T \in SST_{[\overline{n}]}(\lambda(a,b,c))$ and $0 \le k \le \min\{a,b\}$, we slide down $T^{\mathtt{R}}$ by k positions to have a tableau T' of shape $\lambda(a-k, b-k, c+k)$. We define

$$\mathfrak{r}_T = \max\{\, k \mid T' \text{ is semistandard} \,\}.$$

Definition 4.1.3. For $T \in SST_{[\overline{n}]}(\lambda(a,b,c))$ with $\mathfrak{r}_T = 0$, we define $\mathcal{E}T$ and $\mathcal{F}T$ as follows:

(1) $\mathcal{E}T$ is tableau in $SST_{[\overline{n}]}(\lambda(a-1, b+1, c))$ obtained from T by applying Schütenberger's jeu de taquin sliding to the position below the bottom of $T^{\mathtt{R}}$, when $a > 0$,

(2) $\mathcal{F}T$ is tableau in $SST_{[\overline{n}]}(\lambda(a+1, b-1, c))$ obtained from T by applying jeu de taquin sliding to the position above the top of $T^{\mathtt{L}}$, when $b > 0$.

Here we assume that $\mathcal{E}T = \mathbf{0}$ and $\mathcal{F}T = \mathbf{0}$ when $a = 0$ and $b = 0$, respectively, where $\mathbf{0}$ is a formal symbol. In general, if $\mathfrak{r}_T = k$, then we define $\mathcal{E}T = \mathcal{E}T'$ and $\mathcal{F}T = \mathcal{F}T'$, where T' is obtained from T by sliding down $T^{\mathtt{R}}$ by k positions and hence $\mathfrak{r}_{T'} = 0$.

Definition 4.1.4. We define

$$\begin{aligned}
&\mathbf{T}(a) = \left\{\, T \,|\, T \in SST_{[\overline{n}]}(\lambda(a,b,c)),\ b, c \in 2\mathbb{Z}_+,\ \mathfrak{r}_T \le 1 \,\right\} \quad (0 \le a \le n-1), \\
&\overline{\mathbf{T}}(0) = \bigsqcup_{b,c\,\in 2\mathbb{Z}_+} SST_{[\overline{n}]}(\lambda(0,b,c+1)), \qquad \mathbf{T}^{\mathrm{sp}} = \bigsqcup_{a \in \mathbb{Z}_+} SST_{[\overline{n}]}((1^a)), \\
&\mathbf{T}^{\mathrm{sp}+} = \{\, T \,|\, T \in \mathbf{T}^{\mathrm{sp}},\, \mathfrak{r}_T = 0 \,\}, \quad \mathbf{T}^{\mathrm{sp}-} = \{\, T \,|\, T \in \mathbf{T}^{\mathrm{sp}},\, \mathfrak{r}_T = 1 \,\},
\end{aligned}$$

where $\mathfrak{r}_T$ of $T \in \mathbf{T}^{\mathrm{sp}}$ is defined to be the residue of $\mathrm{ht}(T)$ modulo 2.

For $T \in \mathbf{T}(a)$, we define $(T^{\mathtt{L}*}, T^{\mathtt{R}*})$ when $\mathfrak{r}_T = 1$, and $({}^{\mathtt{L}}T, {}^{\mathtt{R}}T)$ by

$$
\begin{aligned}
(T^{\mathtt{L}*}, T^{\mathtt{R}*}) &= \ ((\mathcal{F}T)^{\mathtt{L}}, (\mathcal{F}T)^{\mathtt{R}}), \\
({}^{\mathtt{L}}T, {}^{\mathtt{R}}T) &= ((\mathcal{E}^{a^*}T)^{\mathtt{L}}, (\mathcal{E}^{a^*}T)^{\mathtt{R}}) \quad (a^* = a - \mathfrak{r}_T)
\end{aligned}
\tag{4.1.2}
$$

The following admissibility corresponds to the conditions on Kashiwara-Nakashima tableaux [50].

Definition 4.1.5. Let a, a' be given with $0 \leq a' \leq a \leq n-1$. We say a pair (T, S) is *admissible*, and write $T \prec S$ if it is one of the following cases:

(1) $(T, S) \in \mathbf{T}(a) \times \mathbf{T}(a')$ or $\mathbf{T}(a) \times \mathbf{T}^{\mathrm{sp}}$ with

$$
\begin{aligned}
&\text{(i)} \quad \mathrm{ht}(T^{\mathtt{R}}) \leq \mathrm{ht}(S^{\mathtt{L}}) - a' + 2\mathfrak{r}_T\mathfrak{r}_S, \\
&\text{(ii)} \quad \begin{cases} T^{\mathtt{R}}(i) \leq {}^{\mathtt{L}}S(i), & \text{if } \mathfrak{r}_T\mathfrak{r}_S = 0, \\ T^{\mathtt{R}*}(i) \leq {}^{\mathtt{L}}S(i), & \text{if } \mathfrak{r}_T\mathfrak{r}_S = 1, \end{cases} \\
&\text{(iii)} \quad \begin{cases} {}^{\mathtt{R}}T(i + a - a') \leq S^{\mathtt{L}}(i), & \text{if } \mathfrak{r}_T\mathfrak{r}_S = 0, \\ {}^{\mathtt{R}}T(i + a - a' + \varepsilon) \leq S^{\mathtt{L}*}(i), & \text{if } \mathfrak{r}_T\mathfrak{r}_S = 1, \end{cases}
\end{aligned}
$$

for $i \geq 1$. Here $\varepsilon = 1$ if $S \in \mathbf{T}^{\mathrm{sp}-}$ and 0 otherwise, and we assume that $a' = \mathfrak{r}_S$, $S = S^{\mathtt{L}} = {}^{\mathtt{L}}S = S^{\mathtt{L}*}$ when $S \in \mathbf{T}^{\mathrm{sp}}$.

(2) $(T, S) \in \mathbf{T}(a) \times \overline{\mathbf{T}}(0)$ with $T \prec S^{\mathtt{L}}$ in the sense of (1), regarding $S^{\mathtt{L}} \in \mathbf{T}^{\mathrm{sp}-}$.

(3) $(T, S) \in \overline{\mathbf{T}}(0) \times \overline{\mathbf{T}}(0)$ or $\overline{\mathbf{T}}(0) \times \mathbf{T}^{\mathrm{sp}-}$ with $(T^{\mathtt{R}}, S^{\mathtt{L}}) \in \overline{\mathbf{T}}(0)$.

Remark 4.1.6.

(1) For $T \in \mathbf{T}(a)$, we assume that $T \in \mathbb{P}_L$ such that the subtableau of single column with height a is below L and hence equal to $T^{\mathtt{tail}}$.

$$
T = \quad
\begin{array}{|c|c|}
\cline{2-2}
\multicolumn{1}{c|}{} & \overline{5} \\
\cline{2-2}
\multicolumn{1}{c|}{} & \overline{4} \\
\hline
\overline{5} & \overline{3} \\
\hline
\overline{4} & \overline{2} \\
\hline
\overline{3} & \multicolumn{1}{c}{} \\
\cline{1-1}
\overline{1} & \multicolumn{1}{c}{} \\
\cline{1-1}
\end{array}
\ \ {\scriptstyle L} \ \in \mathbf{T}(2)
$$

(2) Let $S \in \mathbf{T}^{\mathrm{sp}}$ with $\varepsilon = \mathfrak{r}_S$. We may assume that $S = U^{\mathtt{L}}$ for some $U \in \mathbf{T}(\varepsilon)$, where $U^{\mathtt{R}}(i)$ $(i \geq 1)$ are sufficiently large so that $S = U^{\mathtt{L}} = {}^{\mathtt{L}}U$. Then we may understand

the condition Definition 4.1.5(1) for $(T,S) \in \mathbf{T}(a) \times \mathbf{T}^{\mathrm{sp}}$ as induced from the one for $(T,U) \in \mathbf{T}(a) \times \mathbf{T}(\varepsilon)$.

(3) Let $T \in \overline{\mathbf{T}}(0)$ be given. We assume that $T^{\mathtt{L}}, T^{\mathtt{R}} \in \mathbf{T}^{\mathrm{sp}-}$ so that $T^{\mathtt{tail}}$ is non-empty. This means that $(T^{\mathtt{L}})^{\mathtt{tail}}$ and $(T^{\mathtt{R}})^{\mathtt{tail}}$ are non-empty in the sense of (2).

Let $\mathbf{B}$ be one of $\mathbf{T}(a)$ $(0 \leq a \leq n-1)$, $\mathbf{T}^{\mathrm{sp}}$, and $\overline{\mathbf{T}}(0)$. The $\mathfrak{g}$-crystal structure on $\mathbf{B}$ [64] is given as follows. Let $T \in \mathbf{B}$ given. For $i \in I \setminus \{ n \}$, we define $\widetilde{e}_i, \widetilde{f}_i$ by regarding $\mathbf{B}$ as an $\mathfrak{l}$-subcrystal of $\bigsqcup_{\lambda \in \mathscr{P}_n} SST_{[\overline{n}]}(\lambda)$ [50], where we consider the set $[\overline{n}]$ as the dual crystal of $[n]$ that is the crystal of vector representation of $\mathfrak{l}$. For $i = n$ and $T \in \mathbf{B}$, we define $\widetilde{e}_n T$ and $\widetilde{f}_n T$ as follows:

(1) if $\mathbf{B} = \mathbf{T}^{\mathrm{sp}}$, then $\widetilde{e}_n T$ is the tableau obtained by removing a domino $\begin{array}{|c|}\hline \overline{n} \\ \hline n-1 \\ \hline \end{array}$ from T if it is possible, and $\mathbf{0}$ otherwise, and $\widetilde{f}_n T$ is given in a similar way by adding $\begin{array}{|c|}\hline \overline{n} \\ \hline n-1 \\ \hline \end{array}$,

(2) if $\mathbf{B} = \mathbf{T}(a)$ or $\overline{\mathbf{T}}(0)$, then $\widetilde{e}_n T = \widetilde{e}_n\left(T^{\mathtt{R}} \otimes T^{\mathtt{L}}\right)$ and $\widetilde{f}_n T = \widetilde{f}_n\left(T^{\mathtt{R}} \otimes T^{\mathtt{L}}\right)$ regarding $\mathbf{B} \subset (\mathbf{T}^{\mathrm{sp}})^{\otimes 2}$.

The weight of $T \in \mathbf{B}$ is given by

$$\mathrm{wt}(T) = \begin{cases} 2\varpi_n + \sum_{i \geq 1} m_i \epsilon_i, & \text{if } T \in \mathbf{T}(a) \text{ or } \overline{\mathbf{T}}(0), \\ \varpi_n + \sum_{i \geq 1} m_i \epsilon_i, & \text{if } T \in \mathbf{T}^{\mathrm{sp}}. \end{cases},$$

where m_i is the number of occurrences of $\overline{i}$ in T. Then $\mathbf{B}$ is a regular $\mathfrak{g}$-crystal with respect to $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in I$, and

$$\begin{aligned} &\mathbf{T}(a) \cong \mathbf{B}(\varpi_{n-a}) \quad (2 \leq a \leq n-1), \\ &\mathbf{T}(0) \cong \mathbf{B}(2\varpi_n), \quad \overline{\mathbf{T}}(0) \cong \mathbf{B}(2\varpi_{n-1}), \quad \mathbf{T}(1) \cong \mathbf{B}(\varpi_{n-1} + \varpi_n), \\ &\mathbf{T}^{\mathrm{sp}-} \cong \mathbf{B}(\varpi_{n-1}), \quad \mathbf{T}^{\mathrm{sp}+} \cong \mathbf{B}(\varpi_n). \end{aligned}$$

([64, Proposition 4.2]). Note that the highest weight element H of $\mathbf{B}$ is of the following form:

$$H = \begin{cases} \emptyset \boxplus H_{(1^a)} & \text{if } \mathbf{B} = \mathbf{T}(a) \text{ with } 2 \leq a \leq n-1, \\ \emptyset \boxplus \boxed{\overline{n}} & \text{if } \mathbf{B} = \mathbf{T}(1), \\ \emptyset & \text{if } \mathbf{B} = \mathbf{T}(0) \text{ or } \mathbf{B} = \mathbf{T}^{\mathrm{sp}+}, \\ \boxed{\overline{n}} & \text{if } \mathbf{B} = \mathbf{T}^{\mathrm{sp}-}, \end{cases} \tag{4.1.3}$$

where $\emptyset$ is the empty tableau and $H_{(1)} \in SST_{[n]}((1^a))$ $(2 \leq a \leq n-1)$ such that $H_{(1^a)}[k] = \overline{n-k+1}$ $(1 \leq k \leq a)$, that is,

$$H_{(1^a)} = \begin{array}{|c|} \hline \overline{n} \\ \hline \overline{n-1} \\ \hline \vdots \\ \hline \overline{n-a+1} \\ \hline \end{array}\,. \tag{4.1.4}$$

Note that the empty tableau $\emptyset$ is an element of $SST_{[\overline{n}]}((0))$.

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{P}_n$ be given. Let us recall ω_λ (3.2.6). Then, ω_λ is written by

$$\omega_\lambda = \begin{cases} \sum_{i=1}^{\ell} \varpi_{n-a_i} + p\varpi_{n-1} + q(2\varpi_n) + r\varpi_n, & \text{if } \lambda_n \geq 0, \\ \sum_{i=1}^{\ell} \varpi_{n-a_i} + p\varpi_n + \overline{q}(2\varpi_{n-1}) + \overline{r}\varpi_{n-1}, & \text{if } \lambda_n < 0, \end{cases} \tag{4.1.5}$$

where $a_\ell \geq \cdots \geq a_1 \geq 1$, p is the number of i such that $a_i = 1$ and (q, r) (resp. $(\overline{q}, \overline{r})$) is given by $2\lambda_n = 2q + r$ with $r \in \{0, 1\}$ (resp. $-2\lambda_n = 2\overline{q} + \overline{r}$ with $\overline{r} \in \{0, 1\}$).

Let

$$\widehat{\mathbf{T}}_\lambda = \begin{cases} \mathbf{T}(a_\ell) \times \cdots \times \mathbf{T}(a_1) \times \mathbf{T}(0)^{\times q} \times (\mathbf{T}^{\mathrm{sp}+})^r, & \text{if } \lambda_n \geq 0, \\ \mathbf{T}(a_\ell) \times \cdots \times \mathbf{T}(a_1) \times \overline{\mathbf{T}}(0)^{\times \overline{q}} \times (\mathbf{T}^{\mathrm{sp}-})^{\overline{r}}, & \text{if } \lambda_n < 0, \end{cases} \tag{4.1.6}$$

and regard it as a crystal by identifying

$$\mathbf{T} = (\ldots, T_2, T_1) \in \widehat{\mathbf{T}}_\lambda \longleftrightarrow T_1 \otimes T_2 \otimes \ldots .$$

We define

$$\mathbf{T}_\lambda = \{ \mathbf{T} = (\ldots, T_2, T_1) \in \widehat{\mathbf{T}}_\lambda \mid T_{i+1} \prec T_i \text{ for all } i \},$$

where $\prec$ is given in Definition 4.1.5. Then $\mathbf{T}_\lambda \subset \widehat{\mathbf{T}}_\lambda$ is invariant under $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in I$, and then

$$\mathbf{T}_\lambda \cong \mathbf{B}(\omega_\lambda), \tag{4.1.7}$$

(see [64, Theorem 4.3–4.4]). We call $\mathbf{T}_\lambda$ the *spinor model for* $\mathbf{B}(\omega_\lambda)$. We note that the highest weight element $\mathbf{H}_\lambda$ of $\mathbf{T}_\lambda$ is of the form:

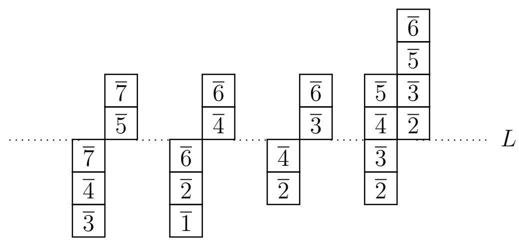
$$\mathbf{H}_\lambda = \begin{cases} H_\ell \otimes \cdots \otimes H_1 \otimes H_0^{\otimes q} \otimes H_+^{\otimes r}, & \text{if } \lambda_n \geq 0, \\ H_\ell \otimes \cdots \otimes H_1 \otimes H_0^{\otimes \overline{q}} \otimes H_-^{\otimes \overline{r}}, & \text{if } \lambda_n < 0, \end{cases} \tag{4.1.8}$$

where H_i and $H_\pm$ are the highest weight element of $\mathbf{T}(a_i)$ and $\mathbf{T}^{\mathrm{sp}\pm}$ given in (4.1.3), respectively.

Example 4.1.7. Let $n = 8$ and $\lambda = (4,4,4,4,4,2) \in \mathcal{P}_8$ be given. By (4.1.5), we have

$$\omega_\lambda = 2\varpi_{8-3} + 2\varpi_{8-2} = 2\varpi_5 + 2\varpi_6,$$

with $\ell = 4$ and $(a_4, a_3, a_2, a_1) = (3,3,2,2)$. Let $\mathbf{T} = (T_4, T_3, T_2, T_1)$ given by



T_4 T_3 T_2 T_1

where the dotted line is the common horizontal line L. Then $T_4 \prec T_3 \prec T_2 \prec T_1$, and hence $\mathbf{T} \in \mathbf{T}_\lambda$.

Definition 4.1.8. For $\lambda \in \mathcal{P}_n$, let

$$\mathbf{H}(\lambda) = \{\, \mathbf{T} \,|\, \mathbf{T} \in \mathbf{T}_\lambda,\ \widetilde{e}_i \mathbf{T} = 0 \ (i \neq n) \,\}.$$

and call $\mathbf{T} \in \mathbf{H}_\lambda$ an $\mathfrak{l}$-*highest weight vector* in $\mathbf{T}_\lambda$ for simplicity.

Note that for $\mathbf{T} \in \mathbf{T}_\lambda$, we have $\mathbf{T} \in \mathbf{H}(\lambda)$ if and only if $\mathbf{T} \equiv_{\mathfrak{l}} H_\mu$ for some $\mu \in \mathscr{P}$.

We also need the following partial order $\lhd$ in Section 4.3.

Definition 4.1.9. Let $\mathbf{B}$ be one of $\mathbf{T}(b)$ $(0 \leq b < n)$, $\mathbf{T}^{sp}$, and $\overline{\mathbf{T}}(0)$. For $(T, S) \in \mathbf{T}(a) \times \mathbf{B}$ with $a \in \mathbb{Z}_+$, we write $T \lhd S$ if the pair $({}^{R}T, S^{L})$ forms a semistandard tableau of a skew shape, where we assume that ${}^{R}T$ and S^{L} are arranged along L as follows:

$$\begin{aligned}
{}^{R}T &= (\ldots, {}^{R}T(a+1)) \boxplus ({}^{R}T(a), \ldots, {}^{R}T(1)), \\
S^{L} &= (\ldots, S^{L}(b+1)) \boxplus (S^{L}(b), \ldots, S^{L}(1)).
\end{aligned}$$

Here we understand S in the sense of Remark 4.1.6 and put $b = \mathrm{ht}(S^{\mathtt{tail}})$ when $S \in \mathbf{T}^{\mathrm{sp}-}$ or $\overline{\mathbf{T}}(0)$.

4.2 Isomorphism from $\mathbf{KN}_\lambda$ to $\mathbf{T}_\lambda$

Let us give an explicit description of the isomorphisms between $\mathbf{KN}_\lambda$ and $\mathbf{T}_\lambda$ for $\lambda \in \mathcal{P}_n$ (cf. [66, Section 3.3] for type B_n and C_n).

Let $\mathbf{B}$ be one of $\mathbf{T}(a)$ $(0 \le a \le n-1)$, $\mathbf{T}^{\mathrm{sp}\pm}$, and $\overline{\mathbf{T}}(0)$. For $T \in \mathbf{B}$, we define a tableau $\widetilde{T}$ as follows:

(1) Suppose that $\mathbf{B} = \mathbf{T}^{\mathrm{sp}\pm}$. Let $\widetilde{T}$ be the unique tableau in $SST_{[\overline{n}]}((1^n))$ such that $\bar{i}$ appears in T if and only if $\bar{i}$ appears in $\widetilde{T}$ for $1 \le i \le n$.

(2) Suppose that $\mathbf{B} = \mathbf{T}(a)$ $(0 \le a < n-1)$ or $\overline{\mathbf{T}}(0)$.

   (i) First, let $\widetilde{{}^{\mathtt{R}}T}$ be the unique tableau in $SST_{[n]}(1^m)$ with $m = n - \mathrm{ht}({}^{\mathtt{R}}T)$ such that i appears in $\widetilde{{}^{\mathtt{R}}T}$ if and only if $\bar{i}$ does not appear in ${}^{\mathtt{R}}T$ for $i \in [n]$.

   (ii) We define $\widetilde{T}$ to be the tableau of single column obtained by putting the single-column tableau consisting of $\begin{array}{|c|}\hline \overline{n} \\ \hline n \\ \hline \end{array}$ with height $b - 2\mathfrak{r}_T$ between ${}^{\mathtt{L}}T$ and $\widetilde{{}^{\mathtt{R}}T}$, where ${}^{\mathtt{L}}T$ is located below $\widetilde{{}^{\mathtt{R}}T}$.

Example 4.2.1. Let $n = 8$ and let $T \in \mathbf{T}(2)$ be T_1 in Example 4.1.7 with $\mathrm{sh}(T) = \lambda(2,2,2)$ and $\mathfrak{r}_T = 1$, where we have

$$T^{\mathtt{L}} = \begin{array}{|c|}\hline \bar{5} \\ \hline \bar{4} \\ \hline \bar{3} \\ \hline \bar{2} \\ \hline \end{array}\ , \quad T^{\mathtt{R}} = \begin{array}{|c|}\hline \bar{6} \\ \hline \bar{5} \\ \hline \bar{3} \\ \hline \bar{2} \\ \hline \end{array}\ , \quad {}^{\mathtt{L}}T = \begin{array}{|c|}\hline \bar{5} \\ \hline \bar{3} \\ \hline \bar{2} \\ \hline \end{array}\ , \quad {}^{\mathtt{R}}T = \begin{array}{|c|}\hline \bar{6} \\ \hline \bar{5} \\ \hline \bar{4} \\ \hline \bar{3} \\ \hline \bar{2} \\ \hline \end{array}\ .$$

Then $\widetilde{{}^{\mathtt{R}}T}$ is given by

$${}^{\mathtt{R}}T = \begin{array}{|c|}\hline \bar{6} \\ \hline \bar{5} \\ \hline \bar{4} \\ \hline \bar{3} \\ \hline \bar{2} \\ \hline \end{array} \longrightarrow \begin{array}{|c|}\hline 1 \\ \hline 7 \\ \hline 8 \\ \hline \end{array} = \widetilde{{}^{\mathtt{R}}T}.$$

and hence

$$T = \begin{array}{cc} & \multicolumn{1}{|c|}{\bar{6}} \\ \cline{2-2} & \multicolumn{1}{|c|}{\bar{5}} \\ \hline \multicolumn{1}{|c|}{\bar{5}} & \multicolumn{1}{c|}{\bar{3}} \\ \hline \multicolumn{1}{|c|}{\bar{4}} & \multicolumn{1}{c|}{\bar{2}} \\ \hline \multicolumn{1}{|c|}{\bar{3}} & \\ \cline{1-1} \multicolumn{1}{|c|}{\bar{2}} & \\ \cline{1-1} \end{array} \longrightarrow \begin{array}{|c|}\hline 1 \\ \hline 7 \\ \hline 8 \\ \hline \bar{5} \\ \hline \bar{3} \\ \hline \bar{2} \\ \hline \end{array} = \widetilde{T}.$$

Note that since $b = 2$ and $\mathrm{r}_T = 1$, there is no domino $\begin{array}{|c|}\hline \overline{n} \\ \hline n \\ \hline\end{array}$ in $\widetilde{T}$. On the other hand, if $T \in \mathbf{T}^{\mathrm{sp}-}$ is given as follows, then we have

$$T = \begin{array}{|c|}\hline \overline{5} \\ \hline \overline{4} \\ \hline \overline{1} \\ \hline\end{array} \longrightarrow \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 6 \\ \hline 7 \\ \hline 8 \\ \hline \overline{5} \\ \hline \overline{4} \\ \hline \overline{1} \\ \hline\end{array} = \widetilde{T}.$$

Lemma 4.2.2. *The map sending T to $\widetilde{T}$ gives an isomorphisms of crystals*

$$\Phi : \mathbf{B} \longrightarrow \begin{cases} \mathbf{KN}_{\mathtt{sp}\pm}, & \textit{if } \mathbf{B} = \mathbf{T}^{\mathrm{sp}\pm}, \\ \mathbf{KN}_{(1^{n-a})}, & \textit{if } \mathbf{B} = \mathbf{T}(a) \textit{ or } \overline{\mathbf{T}}(0). \end{cases}$$

Proof. ***Case 1.*** $\mathbf{B} = \mathbf{T}^{\mathrm{sp}\pm}$. It is straightforward to see that Φ is a weight preserving bijection and by (4.1.1), it is a morphism of crystals. Hence Φ is an isomorphism.

Case 2. Suppose that $\mathbf{B} = \mathbf{T}(a)$ $(0 \leq a < n-1)$ or $\overline{\mathbf{T}}(0)$. Let $T \in \mathbf{B}$ given. First we claim $\widetilde{T} \in \mathbf{KN}_{(1^{n-a})}$. Suppose that $\widetilde{T} \notin \mathbf{KN}_{(1^{n-a})}$. Then by the condition ($\mathfrak{d}$-1) there exists $i \in [n]$ such that

$$(q-p) + i \leq n - a \quad (p < q). \tag{4.2.1}$$

Put $x = n - a - q$ and $y = p$. We note that x is the number of entries in $[n]$ smaller than i in $\widetilde{T}$, and y is the number of entries in $[\overline{n}]$ equal to or larger than $\overline{i}$ in $\widetilde{T}$. Take k such that ${}^{\mathtt{L}}T(k) = \overline{i}$. Then we have ${}^{\mathtt{L}}T(k) > {}^{\mathtt{R}}T(k)$ by (4.2.1). This contradicts to the fact that the pair $({}^{\mathtt{L}}T, {}^{\mathtt{R}}T)$ forms a semistandard tableau when the two columns are placed on the common bottom line. Hence $\widetilde{T} \in \mathbf{KN}_{(1^{n-a})}$.

Second we show that $T \equiv_{\mathfrak{l}} \widetilde{T}$, where $\equiv_{\mathfrak{l}}$ denotes the crystal equivalence as elements of $\mathfrak{l}$-crystals. By the construction of $\widetilde{{}^{\mathtt{R}}T}$, it is not difficult to check that $\widetilde{{}^{\mathtt{R}}T} \equiv_{\mathfrak{l}} {}^{\mathtt{R}}T$ (more precisely as elements of $\mathfrak{sl}_n$-crystals). Put $\mathtt{D}_0$ to be the single-column tableau consisting of the domino $\begin{array}{|c|}\hline \overline{n} \\ \hline n \\ \hline\end{array}$ with height $b - 2\mathfrak{r}_T$. By the tensor product rule of crystals, we see that $\{\mathtt{D}_0\}$ is the crystal of the trivial representation of $\mathfrak{l}$. This implies that $\widetilde{{}^{\mathtt{R}}T} \equiv_{\mathfrak{l}} \widetilde{{}^{\mathtt{R}}T} \otimes \mathtt{D}_0$ and thus

$$T \equiv_{\mathfrak{l}} {}^{\mathtt{R}}T \otimes {}^{\mathtt{L}}T \equiv_{\mathfrak{l}} \widetilde{{}^{\mathtt{R}}T} \otimes {}^{\mathtt{L}}T \equiv_{\mathfrak{l}} \widetilde{{}^{\mathtt{R}}T} \otimes \mathtt{D}_0 \otimes {}^{\mathtt{L}}T \equiv_{\mathfrak{l}} \widetilde{T}$$

Next we claim that $\widetilde{T'} = \widetilde{f}_n \widetilde{T}$, where $T' = \widetilde{f}_n T$. Let $T \in SST_{[\overline{n}]}(\lambda(a,b,c))$ and

$T' \in SST_{[\overline{n}]}(\lambda(a,b',c'))$. Let us consider the case when $\widetilde{f}_n(T^{\mathtt{R}} \otimes T^{\mathtt{L}}) \neq \mathbf{0}$ and $\widetilde{f}_n(T^{\mathtt{R}} \otimes T^{\mathtt{L}}) = T^{\mathtt{R}} \otimes (\widetilde{f}_n T^{\mathtt{L}})$. The proof of the other cases is similar. In this case, we have $b' = b-2$, $c' = c+2$ by definition of $\widetilde{f}_n$, and that $T^{\mathtt{L}}[1]$ and $T^{\mathtt{R}}[1]$ must satisfy that

$$\overline{n-2} \leq T^{\mathtt{L}}[1], \quad T^{\mathtt{R}}[1] \leq \overline{n-1}.$$

Then we have

$$\widetilde{f}_n \widetilde{T} = \begin{cases} {}^{\mathtt{R}}\widetilde{T} \otimes \widetilde{f}_n(\mathtt{D}_0) \otimes {}^{\mathtt{L}}T, & \text{if } b > 2, \\ \widetilde{f}_n({}^{\mathtt{R}}\widetilde{T}) \otimes {}^{\mathtt{L}}T, & \text{if } b = 2, \end{cases}$$

where $\widetilde{f}_n(\mathtt{D}_0)$ is obtained from $\mathtt{D}_0$ by replacing $\begin{array}{|c|}\hline \overline{n} \\ \hline n \\ \hline\end{array}$ by $\begin{array}{|c|}\hline \overline{n} \\ \hline \overline{n-1} \\ \hline\end{array}$ at the bottom of $\mathtt{D}_0$. Note that the bottom entry of $\widetilde{f}_n({}^{\mathtt{R}}\widetilde{T})$ is given by

$$\begin{cases} \boxed{\overline{n}}, & \text{if } T^{\mathtt{R}}[1] = \overline{n}, \\ \boxed{\overline{n-1}}, & \text{if } T^{\mathtt{R}}[1] = \overline{n-1}. \end{cases}$$

On the other hand, we can check that ${}^{\mathtt{L}}T'$ is obtained from ${}^{\mathtt{L}}T$ by putting the domino $\begin{array}{|c|}\hline \overline{n} \\ \hline \overline{n-1} \\ \hline\end{array}$ (resp. $\boxed{\overline{n}}$ if $T^{\mathtt{R}}[1] = \overline{n}$, and $\boxed{\overline{n-1}}$ if $T^{\mathtt{R}}[1] = \overline{n-1}$) on the top of ${}^{\mathtt{L}}T$ when $b > 2$ (resp. $b = 2$). Now it is easy to see that $\widetilde{f}_n \widetilde{T}$ is equal to $\widetilde{T}'$.

Consequently, Φ is a morphism of crystals. Since Φ is injective and sends the highest weight elements of $\mathbf{T}(a)$ to that of $\mathbf{KN}_{(1^{n-a})}$, Φ is an isomorphism. $\square$

Next let us describe the inverse map of Φ. Let $T \in \mathbf{KN}_{(1^a)} \cup \mathbf{KN}_{\mathtt{sp}^{\pm}}$ $(0 < a \leq n)$ be given. Then we define $\Psi_a(T)$ and $\Psi_{\mathrm{sp}\pm}(T)$ if $T \in \mathbf{KN}_{(1^a)}$ and $T \in \mathbf{KN}_{\mathtt{sp}^{\pm}}$ respectively as follows:

(1) Let T_+ (resp. T_-) be the subtableau in T with entries in $[n]$ (resp. $[\overline{n}]$) except for dominos $\begin{array}{|c|}\hline \overline{n} \\ \hline n \\ \hline\end{array}$

(2) Let $\widetilde{T_+}$ be the single-column tableau with height $n - \mathrm{ht}(T_+)$ such that i appears in T_+ if and only if $\overline{i}$ does not appear in $\widetilde{T_+}$.

(3) We define $\Psi_a(T)$ and $\Psi_{\mathrm{sp}\pm}(T)$ by

$$\Psi_{\mathrm{sp}\pm}(T) = T_-, \quad \Psi_a(T) = \mathcal{F}^{n-a-\epsilon}(T_-, \widetilde{T_+}), \tag{4.2.2}$$

where

$$\epsilon = \begin{cases} 0, & \text{if } \mathrm{ht}(\widetilde{T_+}) - a \text{ is even,} \\ 1, & \text{if } \mathrm{ht}(\widetilde{T_+}) - a \text{ is odd.} \end{cases}$$

We note that $\Psi_a(T)$ has residue 1 if $\mathrm{ht}(\widetilde{T_+}) - a$ is odd, otherwise 0.

It is not difficult to check that the map Ψ_a (resp. $\Psi_{\mathrm{sp}\pm}$) is the inverse of Φ. Hence by Lemma 4.2.2, we have the following.

Lemma 4.2.3. *The maps Ψ_a and $\Psi_{\mathrm{sp}\pm}$ are isomorphisms of crystals*

$$\Psi_{\mathrm{sp}\pm} : \mathbf{KN}_{\mathbf{sp}\pm} \longrightarrow \mathbf{T}^{\mathrm{sp}\pm}, \quad \Psi_a : \mathbf{KN}_{(1^a)} \longrightarrow \mathbf{T}(n-a) \quad (0 < a \leq n),$$

Now we consider the isomorphism for any $\lambda \in \mathcal{P}_n$. Let $\mu' = (a_\ell, \ldots, a_1)$, where $a_1, \ldots, a_\ell$ are given in (4.1.5), with $\ell = \mu_1$. For $T \in \mathbf{KN}_\lambda$, let

$$\begin{cases} (T_\ell, \ldots, T_1), & \text{if } \lambda_n \in \mathbb{Z}, \\ (T_\ell, \ldots, T_1, T_0), & \text{if } \lambda_n \notin \mathbb{Z}, \end{cases}$$

denote the sequence of columns of T, where T_0 is the column of T with half-width boxes, and $T_1, T_2, \ldots$ are the other columns enumerated from right to left.

Theorem 4.2.4. *For $\lambda \in \mathcal{P}_n$, the map*

$$\Psi_\lambda : \mathbf{KN}_\lambda \longrightarrow \mathbf{T}_\lambda \tag{4.2.3}$$

defined by

$$\Psi_\lambda(T) = \begin{cases} (\Psi_{\mu'_\ell}(T_\ell), \ldots, \Psi_{\mu'_1}(T_1)), & \text{if } \lambda_n \in \mathbb{Z}, \\ (\Psi_{\mu'_\ell}(T_\ell), \ldots, \Psi_{\mu'_1}(T_1), \Psi_{\mathrm{sp}\pm}(T_0)), & \text{if } \lambda_n \notin \mathbb{Z}, \end{cases}$$

is an isomorphism of crystals from $\mathbf{KN}_\lambda$ to $\mathbf{T}_\lambda$, where we take $\Psi_{\mathrm{sp}+}$ and $\Psi_{\mathrm{sp}-}$ if $\lambda_n \geq 0$ and $\lambda_n < 0$, respectively.

Proof. By Lemma 4.2.3, the map Ψ_λ is an embedding of crystals into $\widehat{\mathbf{T}}_\lambda$. Also the map Ψ_λ sends the highest weight element of $\mathbf{KN}_\lambda$ to the one of $\mathbf{T}_\lambda$ (cf. (4.1.8)). Then by (5.3.1), the image of Ψ_λ is isomorphic to $\mathbf{T}_\lambda$. $\square$

Example 4.2.5. Let $\lambda = (4^5, 2) \in \mathcal{P}_8$ be given. Consider

$$T = \begin{array}{|c|c|c|c|} \cline{3-4} \multicolumn{2}{c|}{} & 1 & 1 \\ \hline 1 & 3 & 4 & 7 \\ \hline 2 & 5 & 5 & 8 \\ \hline 6 & 7 & 7 & \overline{5} \\ \hline 8 & 8 & 8 & \overline{3} \\ \hline \overline{7} & \overline{6} & \overline{4} & \overline{2} \\ \hline \end{array} \in \mathbf{KN}_\lambda \,.$$

Put

$$T_4 = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 6 \\ \hline 8 \\ \hline \overline{7} \\ \hline \end{array}\,, \quad T_3 = \begin{array}{|c|} \hline 3 \\ \hline 5 \\ \hline 7 \\ \hline 8 \\ \hline \overline{6} \\ \hline \end{array}\,, \quad T_2 = \begin{array}{|c|} \hline 1 \\ \hline 4 \\ \hline 5 \\ \hline 7 \\ \hline 8 \\ \hline \overline{4} \\ \hline \end{array}\,, \quad T_1 = \begin{array}{|c|} \hline 1 \\ \hline 7 \\ \hline 8 \\ \hline \overline{5} \\ \hline \overline{3} \\ \hline \overline{2} \\ \hline \end{array}\,.$$

By definition of $((T_i)_-\,, \widetilde{(T_i)_+})$,

$$\big((T_4)_-\,, \widetilde{(T_4)_+}\big) = \left(\boxed{\overline{7}}\,, \begin{array}{|c|} \hline \overline{7} \\ \hline \overline{5} \\ \hline \overline{4} \\ \hline \overline{3} \\ \hline \end{array} \right), \quad \big((T_3)_-, \widetilde{(T_3)_+}\big) = \left(\boxed{\overline{6}}\,, \begin{array}{|c|} \hline \overline{6} \\ \hline \overline{4} \\ \hline \overline{2} \\ \hline \overline{1} \\ \hline \end{array} \right),$$

$$\big((T_2)_-\,, \widetilde{(T_2)_+}\big) = \left(\boxed{\overline{4}}\,, \begin{array}{|c|} \hline \overline{6} \\ \hline \overline{3} \\ \hline \overline{2} \\ \hline \end{array} \right), \quad \big((T_1)_-\,, \widetilde{(T_1)_+}\big) = \left(\begin{array}{|c|} \hline \overline{5} \\ \hline \overline{3} \\ \hline \overline{2} \\ \hline \end{array}\,, \begin{array}{|c|} \hline \overline{6} \\ \hline \overline{5} \\ \hline \overline{4} \\ \hline \overline{3} \\ \hline \overline{2} \\ \hline \end{array} \right).$$

Since $\mathrm{ht}(\widetilde{(T_i)_+}) - \mathrm{ht}(T_i)$ is odd for $i = 1, 2, 3, 4$, we have by (4.2.2)

$$\Psi_5(T_4) = \begin{array}{cc} & \boxed{\overline{7}} \\ & \boxed{\overline{5}} \\ \boxed{\overline{7}} & \\ \boxed{\overline{4}} & \\ \boxed{\overline{3}} & \end{array}, \quad \Psi_5(T_3) = \begin{array}{cc} & \boxed{\overline{6}} \\ & \boxed{\overline{4}} \\ \boxed{\overline{6}} & \\ \boxed{\overline{2}} & \\ \boxed{\overline{1}} & \end{array}, \quad \Psi_6(T_2) = \begin{array}{cc} & \boxed{\overline{6}} \\ & \boxed{\overline{3}} \\ \boxed{\overline{4}} & \\ \boxed{\overline{2}} & \end{array}, \quad \Psi_6(T_1) = \begin{array}{cc} & \boxed{\overline{6}} \\ & \boxed{\overline{5}} \\ \boxed{\overline{5}} & \boxed{\overline{3}} \\ \boxed{\overline{4}} & \boxed{\overline{2}} \\ \boxed{\overline{3}} & \\ \boxed{\overline{2}} & \end{array}.$$

Hence, we obtain

$$\Psi_\lambda(T) = \cdots \begin{array}{cccccccc} & & & & & & & \boxed{\overline{6}} \\ & & & & & & & \boxed{\overline{5}} \\ & \boxed{\overline{7}} & & \boxed{\overline{6}} & & \boxed{\overline{6}} & \boxed{\overline{5}} & \boxed{\overline{3}} \\ & \boxed{\overline{5}} & & \boxed{\overline{4}} & & \boxed{\overline{3}} & \boxed{\overline{4}} & \boxed{\overline{2}} \\ \boxed{\overline{7}} & & \boxed{\overline{6}} & & \boxed{\overline{4}} & & \boxed{\overline{3}} & \\ \boxed{\overline{4}} & & \boxed{\overline{2}} & & \boxed{\overline{2}} & & \boxed{\overline{2}} & \\ \boxed{\overline{3}} & & \boxed{\overline{1}} & & & & & \end{array} \cdots \in \mathbf{T}_\lambda.$$

4.3 Separation algorithm

We introduce a combinatorial algorithm on $\mathbf{T}_\lambda$, so-called *separation*, which plays a crucial role in Chapters 4 and 5, and this section is one of main parts in this chapter.

Roughly speaking, the separation is given by sliding horizontally the tails of $\mathbf{T}$ (using the jeu de taquin sliding) to the leftmost one as far as possible so that the resulting tableau gives a semistandard tableau U of normal shape and the remaining one in the body gives a semistandard tableau of anti-normal shape. For example, see Example 4.3.2.

We remark that the separation algorithm of types BC is already present in [66]. However, the separation algorithm of types BC does not work well on the spinor model of type D due to more involved conditions for admissibility in $\mathbf{T}_\lambda$ (Recall Definition 4.1.5).

To overcome this difficulty, we introduce an operator *sliding* which is given by a non-trivial sequence of jeu de taquin slidings, and also moves a tail in $\mathbf{T}$ by one position to the left horizontally.

A key property is that our sliding is compatible with the type A crystal structure on $\mathbf{T}_\lambda$ so that we obtain another element $\widetilde{\mathbf{T}} \in \mathbf{T}_{\widetilde{\lambda}}$ and $\mathbf{T} = \widetilde{\mathbf{T}} \otimes U$ as an element in a crystal of type A, where U is the leftmost column in $\mathbf{T}$ and $\widetilde{\lambda}$ is a partition smaller than λ. Hence this enables us to define the separation algorithm by applying the sliding successively.

Moreover, in Section 4.4, we see that the separation induces an $\mathfrak{g}$-crystal embedding from spinor model into the crystal of parabolic Verma module associated with the maximal Levi subalgebra of type A. This is the second main part in this chapter.

4.3.1 Sliding

Let us recall that $\mathbf{T}_\lambda$ is a subcrystal of $(\mathbf{T}^{\mathrm{sp}})^{\otimes N}$ for some N. We may identify $(\mathbf{T}^{\mathrm{sp}})^{\otimes N}$ with

$$\mathbf{E}^N := \bigsqcup_{(u_N,\ldots,u_1)\in\mathbb{Z}_+^n} SST_{[\overline{n}]}(1^{u_N}) \times \cdots \times SST_{[\overline{n}]}(1^{u_1}).$$

We use an $(\mathfrak{l}, \mathfrak{sl}_N)$-bicrystal structure on $\mathbf{E}^N$ in [66, Lemma 5.1]. The $\mathfrak{l}$-crystal structure on $\mathbf{E}^N$ with respect to $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in I$ is naturally induced from that of $(\mathbf{T}^{\mathrm{sp}})^{\otimes N}$. On the other hand, the $\mathfrak{sl}_N$-crystal structure is defined as follows (recall Definition 4.1.3).

Let $(U_N, \dots, U_1) \in \mathbf{E}^N$ given. For $1 \le j \le N-1$ and $\mathcal{X} = \mathcal{E}, \mathcal{F}$, we define

$$\mathcal{X}_j(U_N, \dots, U_1) = \begin{cases} (U_r, \dots, \mathcal{X}(U_{j+1}, U_j), \dots, U_1), & \text{if } \mathcal{X}(U_{j+1}, U_j) \neq \mathbf{0}, \\ \mathbf{0}, & \text{if } \mathcal{X}(U_{j+1}, U_j) = \mathbf{0}. \end{cases}$$

where $\mathcal{X}(U_{j+1}, U_j)$ is understood to be $\mathcal{X}U$ for some $U \in SST_{[\overline{n}]}(\lambda(a,b,c))$ with $\mathfrak{r}_U = 0$, $U^L = U_{j+1}$ and $U^R = U_j$.

Let l be the number of components in $\widehat{\mathbf{T}}_\lambda$ in (4.1.6) except $\mathbf{T}^{\mathrm{sp}\pm}$. Consider an embedding of sets

$$\begin{array}{ccc} \mathbf{T}_\lambda & \longrightarrow & \mathbf{E}^{2l+1} \\ \mathbf{T} = (T_l, \dots, T_1, T_0) & \longmapsto & (T_l^{\mathtt{L}}, T_l^{\mathtt{R}}, \dots, T_1^{\mathtt{L}}, T_1^{\mathtt{R}}, T_0) \end{array}, \tag{4.3.1}$$

where T_0 is regarded as

$$T_0 \in \begin{cases} \{\emptyset\}, & \text{if } \lambda_n \in \mathbb{Z}, \\ \mathbf{T}^{\mathrm{sp}\pm}, & \text{if } \lambda_n \notin \mathbb{Z}. \end{cases}$$

Here $\{\emptyset\}$ is the crystal of trivial module. We identify $\mathbf{T} = (T_l, \dots, T_1, T_0) \in \mathbf{T}_\lambda$ with its image $\mathbf{U} = (U_{2l}, \dots, U_1, U_0)$ under (4.3.1) so that $T_0 = U_0$ and (T_{i+1}, T_i) is given by

$$(T_{i+1}, T_i) = (U_{j+2}, U_{j+1}, U_j, U_{j-1}) = (T_{i+1}^{\mathtt{L}}, T_{i+1}^{\mathtt{R}}, T_i^{\mathtt{L}}, T_i^{\mathtt{R}}), \tag{4.3.2}$$

with $j = 2i$ for $1 \le i \le l-1$.

Now we define an operator $\mathcal{S}_j$ on $\mathbf{T}$ for $j = 2i$ for $1 \le i \le l-1$ by

$$\mathcal{S}_j = \begin{cases} \mathcal{F}_j^{a_i}, & \text{if } T_{i+1} \triangleleft\, T_i, \\ \mathcal{E}_j \mathcal{E}_{j-1} \mathcal{F}_j^{a_i - 1} \mathcal{F}_{j-1}, & \text{if } T_{i+1} \not\triangleleft\, T_i, \end{cases} \tag{4.3.3}$$

where $\mathcal{S}_j$ is understood as the identity operator when $a_i = 0$, and $\triangleleft$ is given in Definition 4.1.9.

The following lemma is crucial in Section 4.3.2.

Lemma 4.3.1. *Let $\mathbf{T} = (\dots, T_{i+1}, T_i, \dots) \in \mathbf{T}_\lambda$ be given.*

1. *We have $\mathcal{S}_j \mathbf{T} = (\dots, U_{j+2}, \widetilde{U}_{j+1}, \widetilde{U}_j, U_{j-1}, \dots)$ for some $\widetilde{U}_{j+1}$ and $\widetilde{U}_j$, and $(U_{j+2}, \widetilde{U}_{j+1}, \widetilde{U}_j, U_{j-1})$ is semistandard along L.*
2. *Suppose that $\widetilde{e}_k \mathbf{T} \neq 0$ for some $k \in J$ and put $\mathbf{S} = \widetilde{e}_k \mathbf{T} = (\dots, S_{i+1}, S_i, \dots)$. Then $T_{i+1} \triangleleft T_i$ if and only if $S_{i+1} \triangleleft S_i$.*

Proof. The proof is given in Section 7.2.1. □

4.3.2 Separation when $\lambda_n \geq 0$

Let us assume $\lambda \in \mathcal{P}_n$ with $\lambda_n \geq 0$. The case when $\lambda_n < 0$ is considered in Section 4.3.3.

Let $\mathbf{T} = (T_l, \ldots, T_1, T_0) \in \mathbf{T}_\lambda$ be given. Since $T_i \in \mathbb{P}_L$ for $0 \leq i \leq l$, we may consider the $(l+1)$-tuples

$$(T_\ell^{\texttt{body}}, \ldots, T_1^{\texttt{body}}, T_0^{\texttt{body}}), \quad (T_\ell^{\texttt{tail}}, \ldots, T_1^{\texttt{tail}}, T_0^{\texttt{tail}}) \tag{4.3.4}$$

to form tableaux in $\mathbb{P}_L$. But in general, they are not necessarily semistandard along L, and $(T_l^{\texttt{body}}, \ldots, T_1^{\texttt{body}}, T_0^{\texttt{body}})$ may not be of a partition shape along L. So instead of cutting $\mathbf{T}$ with respect to L directly as in (4.3.4), we introduce an algorithm to separate $\mathbf{T}$ into two semistandard tableaux, which preserves $\mathfrak{l}$-crystal equivalence.

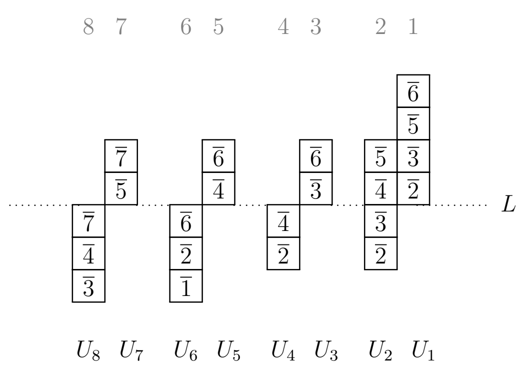
More precisely, we introduce an algorithm to get a semistandard tableau $\overline{\mathbf{T}}$ in $\mathbb{P}_L$ such that

(S1) $\overline{\mathbf{T}}$ is Knuth equivalent to $\mathbf{T}$, that is, $\overline{\mathbf{T}} \equiv_{\mathfrak{l}} \mathbf{T}$,

(S2) $\overline{\mathbf{T}}^{\texttt{tail}} \in SST_{[\overline{n}]}(\mu)$ and $\overline{\mathbf{T}}^{\texttt{body}} \in SST_{[\overline{n}]}(\delta^\pi)$ for some $\delta \in \mathscr{P}_n^{(1,1)}$, where $\mu \in \mathscr{P}_n$ is given by
$$\mu' = (a_\ell, \ldots, a_1) \tag{4.3.5}$$
with a_i as in (4.1.5).

We call this algorithm *separation* (see [66] for types BC). Let us explain this with an example before we deal it in general.

Example 4.3.2. Let $\mathbf{T} = (T_4, T_3, T_2, T_1) \in \mathbf{T}_{(4,4,4,4,4,2)}$ be given in Example 4.1.7.

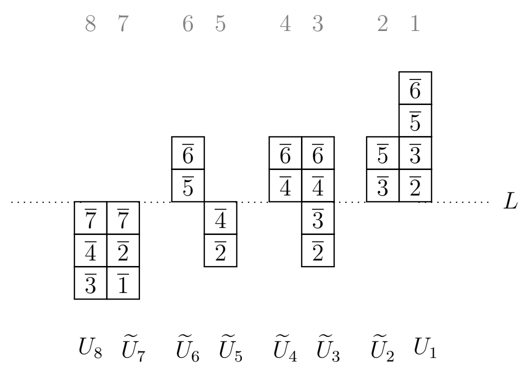


where $(U_1, \ldots, U_8)$ denotes the image of (T_4, T_3, T_2, T_1) under (4.3.1).

First we consider $\triangleleft$ in Definition 4.1.9 on (T_{i+1}, T_i) for $1 \leq i \leq 3$. Then we can check that $T_4 \not\triangleleft T_3$, $T_3 \not\triangleleft T_2$ and $T_2 \triangleleft T_1$. By (4.3.3), we have

$$\mathcal{S}_6 = \mathcal{E}_6\mathcal{E}_5\mathcal{F}_6^2\mathcal{F}_5, \quad \mathcal{S}_4 = \mathcal{E}_4\mathcal{E}_3\mathcal{F}_4^2\mathcal{F}_3, \quad \mathcal{S}_2 = \mathcal{F}_2^2.$$

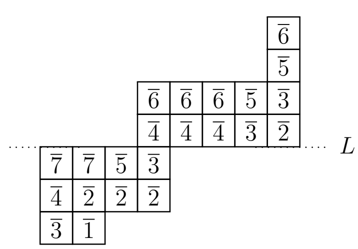
Now, we apply these operators $\mathcal{S}_6$, $\mathcal{S}_4$, and then $\mathcal{S}_2$ to $\mathbf{T}$ to have



We observe that (recall Definition 4.1.5)

$$(\widetilde{U}_7, \widetilde{U}_6) \prec (\widetilde{U}_5, \widetilde{U}_4) \prec (\widetilde{U}_3, \widetilde{U}_2), \tag{4.3.6}$$

So we can apply the above process to (4.3.6), and repeat it until there is no tail to move to the left horizontally. Consequently we have



Hence we obtain two semistandard tableaux of shape δ^π and μ, where $\delta = (4, 4, 1, 1)$ and $\mu = (4, 4, 2)$.

Now let $\mathbf{T} = (T_l, \ldots, T_1, T_0) \in \mathbf{T}_\lambda$ be given and let $\mathbf{U} = (U_{2l}, \ldots, U_1, U_0)$ be its image under (4.3.1). We use the induction on the number of columns in $\mathbf{T}$ to define $\overline{\mathbf{T}}$. If $n \leq 3$, then let $\overline{\mathbf{T}}$ is given by putting together the columns in $\mathbf{U}$ horizontally along L.

Suppose that $n \geq 4$. First, we consider

$$\mathcal{S}_2 \dots \mathcal{S}_{2l-2}\mathbf{T} = \mathcal{S}_2 \dots \mathcal{S}_{2l-2}\mathbf{U} = (U_{2l}, \widetilde{U}_{2l-1}, \dots, \widetilde{U}_2, U_1, U_0) \in \mathbf{E}^{2l+1},$$

and let

$$\widetilde{\mathbf{U}} = (\widetilde{U}_{2l-1}, \dots, \widetilde{U}_2, U_1, U_0) \in \mathbf{E}^{2l}. \tag{4.3.7}$$

Note that applying $\mathcal{S}_{2i}$ to $\mathcal{S}_{2i+2} \dots \mathcal{S}_{2l-2}\mathbf{T}$ for $1 \leq i \leq l-1$ is well-defined by Lemma 4.3.1(1).

Let $\widetilde{\lambda} \in \mathcal{P}_n$ be such that $\omega_\lambda - \omega_{\widetilde{\lambda}} = \omega_{n-a}$ with $a = \mathrm{ht}(U_{2l}^{\mathtt{tail}})$.

The following lemma is the crucial initial step for the sliding algorithm.

Lemma 4.3.3. *Suppose that* $\mathbf{T} \in \mathbf{H}(\lambda)$ (*recall Definition 4.1.8*). *Then there exits a unique* $\widetilde{\mathbf{T}} \in \mathbf{H}(\widetilde{\lambda})$ *such that the image of* $\widetilde{\mathbf{T}}$ *under* (4.3.1) *is equal to* $\widetilde{\mathbf{U}}$.

Proof. To prove this lemma, we observe how the sliding and separation work on $\mathfrak{l}$-highest weight vectors. We give the description explicitly in Sections 5.3.3 and 5.3.4 and give the proof of this lemma in Section 7.2.2. □

We generalize the above lemma for arbitrary $\mathbf{T} \in \mathbf{T}_\lambda$ as follows.

Lemma 4.3.4. *For* $\mathbf{T} \in \mathbf{T}_\lambda$, *there exits a unique* $\widetilde{\mathbf{T}} \in \mathbf{T}_{\widetilde{\lambda}}$ *such that the image of* $\widetilde{\mathbf{T}}$ *under* (4.3.1) *is equal to* $\widetilde{\mathbf{U}}$.

Proof. There exists an $\mathfrak{l}$-highest weight vector $\mathbf{H} \in \mathbf{T}_\lambda$ such that $\mathbf{H} = \widetilde{e}_{i_1} \dots \widetilde{e}_{i_r}\mathbf{U}$ for some $i_1, \dots, i_r \in I$. Put $\mathbf{U}^\sharp = (U_{2l-1}, \dots, U_1)$ and let $\mathbf{H}^\sharp$ be obtained from $\mathbf{H}$ by removing its leftmost column, say U'_{2l}. By Corollary 2.1.16(2), $\mathbf{H}^\sharp$ is also an $\mathfrak{l}$-highest weight vector. Identifying $\mathbf{U} = \mathbf{U}^\sharp \otimes U_{2l}$, we observe that by tensor product rule (2.2.3),

$$\begin{aligned}\mathbf{H}^\sharp \otimes U'_{2l} &= \widetilde{e}_{i_1} \dots \widetilde{e}_{i_r}\mathbf{U} \\ &= \widetilde{e}_{i_1} \dots \widetilde{e}_{i_r}\left(\mathbf{U}^\sharp \otimes U_{2l}\right) \\ &= \left(\widetilde{e}_{j_1} \dots \widetilde{e}_{j_s}\mathbf{U}^\sharp\right) \otimes \left(\widetilde{e}_{k_1} \dots \widetilde{e}_{k_t}U_{2l}\right)\end{aligned}$$

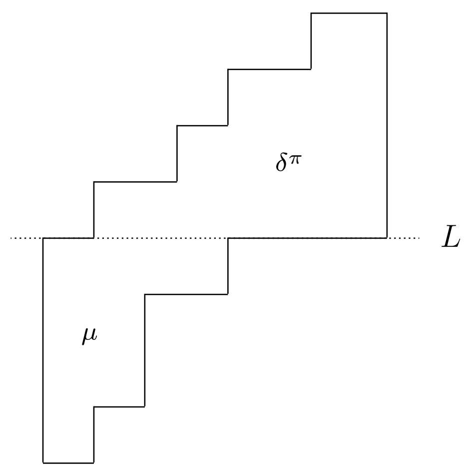
where $\{i_1, \dots, i_r\} = \{j_1, \dots, j_s\} \cup \{k_1, \dots, k_l\}$. Hence $\mathbf{H}^\sharp = \widetilde{e}_{j_1} \dots \widetilde{e}_{j_s}\mathbf{U}^\sharp$, and

$$\widetilde{\mathbf{U}} = \mathcal{S}_2 \dots \mathcal{S}_{2l-2}\mathbf{U}^\sharp = \mathcal{S}_2 \dots \mathcal{S}_{2l-2}\left(\widetilde{f}_{j_s} \dots \widetilde{f}_{j_1}\mathbf{H}^\sharp\right) = \widetilde{f}_{j_s} \dots \widetilde{f}_{j_1}\left(\mathcal{S}_2 \dots \mathcal{S}_{2l-2}\mathbf{H}^\sharp\right),$$

since $\mathbf{E}^{2l}$ is an $(\mathfrak{l}, \mathfrak{sl}_{2l})$-bicrystal. Note that the operator $\mathcal{S}_2 \ldots \mathcal{S}_{2l-2}$ (4.3.3) is well-defined on $\mathbf{H}^\sharp$ by Lemma 4.3.1(2). Then $\mathcal{S}_2 \ldots \mathcal{S}_{2l-2}\mathbf{H}^\sharp \in \mathbf{T}_{\widetilde{\lambda}}$ by Lemma 4.3.3, which implies that $\widetilde{\mathbf{U}} \in \mathbf{T}_{\widetilde{\lambda}}$. □

Put $\mathbb{T} = \widetilde{\mathbf{T}}$, which is given in Lemma 7.3.7. By induction hypothesis, there exists a tableau $\overline{\mathbb{T}}$ satisfying (S1) and (S2) associated with $\mathbb{T}$. Then we define $\overline{\mathbf{T}}$ to be the tableau in $\mathbb{P}_L$ obtained by putting together the leftmost column of $\mathbf{T}$, that is, U_{2l}, and $\overline{\mathbb{T}}$ along L. By definition, $\mathrm{sh}(\overline{\mathbf{T}}) = \eta$ is of the following form:

$\eta =$



(4.3.8)

for some $\delta \in \mathscr{P}_n^{(1,1)}$.

Proposition 4.3.5. *Under the above hypothesis, $\overline{\mathbf{T}}$ satisfies (S1) and (S2).*

Proof. By definition, it is clear that $\overline{\mathbf{T}} \equiv_{\mathfrak{l}} \overline{T}$, which implies (S1). By Lemma 4.3.1, $(U_{2l}^{\mathtt{body}}, \overline{\mathbb{T}}^{\mathtt{body}})$ and $(U_{2l}^{\mathtt{tail}}, \overline{\mathbb{T}}^{\mathtt{tail}})$ are semistandard along L, which implies (S2). □

4.3.3 Separation when $\lambda_n < 0$

Now, we consider the algorithm for separation when $\lambda_n < 0$. The algorithm in this case is almost identical with the case $\lambda_n \geq 0$ except for the spin columns of odd height (recall Definition 4.1.4). We deal with these columns in the sense of Remark 4.1.6(2).

Let us assume that $\lambda_n < 0$. Recall that $-2\lambda_n = 2\overline{q} + \overline{r}$ with $\overline{r} \in \{0, 1\}$. For $\mathbf{T} \in \mathbf{T}_\lambda$, we may write

$$\mathbf{T} = (T_l, \ldots, T_{m+1}, T_m, \ldots, T_1, T_0),$$

for some $m \geq 1$ such that $T_i \in \mathbf{T}(a_i)$ for some a_i $(m+1 \leq i \leq l)$, $T_i \in \overline{\mathbf{T}}(0)$ $(1 \leq i \leq m)$

and $T_0 \in \mathbf{T}^{\mathrm{sp}-}$ (resp. $T_0 = \emptyset$) if $\overline{r} = 1$ (resp. $\overline{r} = 0$). We identify $\mathbf{T}$ with

$$\mathbf{U} = (U_{2l}, \ldots, U_{2m}, U_{2m-1}, \ldots, U_1, U_0)$$

under (4.3.1).

Remark 4.3.6. For the spin column U_{2m}, we apply the sliding algorithm in Section 4.3.1 as follows. Let $U = H_{(1^n)}$ in (4.1.4). Consider the pair

$$(U_{2m+2},\, U_{2m+1},\, U_{2m},\, U),$$

where we regard $U = U \boxplus \emptyset \in \mathbb{P}_L$ and $(U_{2m}, U) \in \mathbf{T}(\varepsilon)$ as in Remark 4.1.6(2). By Lemma 4.3.1, we have

$$(U_{2m+2},\, U_{2m+1},\, U_{2m},\, U) \xrightarrow{\mathcal{S}_{2m}} (U_{2m+2},\, \widetilde{U}_{2m+1},\, \widetilde{U}_{2m},\, U),$$

for some $\widetilde{U}_{2m+1}$ and $\widetilde{U}_{2m}$, and by our choice of U, we have

$$(U_{2m+2},\, U_{2m+1},\, U_{2m}) \equiv_{\mathfrak{l}} (U_{2m+2},\, \widetilde{U}_{2m+1},\, \widetilde{U}_{2m}).$$

Now, we use the induction on the number of columns in $\mathbf{T}$ to define $\overline{\mathbf{T}}$ satisfying (S1) and (S2) where $\mu \in \mathscr{P}_n$ in this is given by

$$\mu' = (a_\ell, \ldots, a_1, \underbrace{1, \ldots, 1}_{-2\lambda_n}). \tag{4.3.9}$$

If $n \leq 3$, then let $\overline{\mathbf{T}}$ be given by putting together the columns in $\mathbf{U}$ along L.

Suppose that $n \geq 4$. First, we consider $\mathcal{S}_{2l-2}\mathcal{S}_{2l-4} \ldots \mathcal{S}_{2m}\mathbf{T}$, where $\mathcal{S}_{2m}$ is understood as in Remark 4.3.6.. Then we have

$$\mathcal{S}_{2l-2}\mathcal{S}_{2l-4} \ldots \mathcal{S}_{2m}\mathbf{T} = (U_{2l}, \widetilde{U}_{2l-1}, \ldots, \widetilde{U}_{2m+1}, \widetilde{U}_{2m}, U_{2m-1}, \ldots, U_1, U_0), \tag{4.3.10}$$

for some $\widetilde{U}_i$ for $2m \leq i \leq 2l-1$. Let

$$\widetilde{\mathbf{U}} = (\widetilde{U}_{2l-1}, \ldots, \widetilde{U}_{2m+1}, \widetilde{U}_{2m}, U_{2m-1}, \ldots, U_1, U_0).$$

The following is an analogue of Lemma 7.3.7 for $\lambda_n < 0$.

Lemma 4.3.7. *Let $\widetilde{\lambda}$ be such that $\omega_\lambda - \omega_{\widetilde{\lambda}} = \omega_{n-a}$ with $a = \mathrm{ht}(U_{2l}^{\mathtt{tail}})$. Then there exists a unique $\widetilde{\mathbf{T}} \in \mathbf{T}_{\widetilde{\lambda}}$ such that the image of $\widetilde{\mathbf{T}}$ is equal to $\widetilde{\mathbf{U}}$ under* (4.3.1).

Proof. Let $\mathbf{U}' = (U_{2l}, U_{2l-1}, \ldots, U_{2m+1}, U_{2m}, U)$. Let $\nu = \mathrm{wt}(\mathbf{U}')$ and let $\widetilde{\nu}$ be given by $\nu - \widetilde{\nu} = \omega_{n-a}$ with $a = \mathrm{ht}(U_{2l}^{\mathtt{tail}})$. By Lemma 7.3.7 and Remark 4.3.6, there exists a unique $\widetilde{\mathbf{S}} \in \mathbf{T}_{\widetilde{\nu}}$ whose image under (4.3.1) is $\widetilde{\mathbf{U}'} = (\widetilde{U}_{2l-1}, \ldots, \widetilde{U}_{2m}, U)$. By semistandardness of (U_{2m}, U_{2m-1}) (cf. Remark 4.3.6), we have

$$(\widetilde{U}_{2m+1}, \widetilde{U}_{2m}) \prec (U_{2m-1}, U_{2m-2}).$$

Note that we may regard $(U_{2j-1}, U_{2j-2}) \in \overline{\mathbf{T}}(0)$ $(2 \leq j \leq m)$, and $(U_1, U_0) \in \overline{\mathbf{T}}(0)$ if $U_0 \neq \emptyset$, $(U_1, U_0) \in \mathbf{T}^{\mathrm{sp}-}$ otherwise. Therefore there exists a unique $\widetilde{\mathbf{T}} \in \mathbf{T}_{\widetilde{\lambda}}$ which is equal to $\widetilde{\mathbf{U}}$ under (4.3.1). □

Put $\mathbb{T} = \widetilde{\mathbf{T}}$ given in Lemma 4.3.7. By induction hypothesis, there exists a tableau $\overline{\mathbb{T}}$ satisfying (S1) and (S2) associated with $\mathbb{T}$. Then we define $\overline{\mathbf{T}}$ to be the tableau in $\mathbb{P}_L$ obtained by putting together the leftmost column of $\mathbf{T}$, that is, U_{2l}, and $\overline{\mathbb{T}}$ along L. By definition, $\mathrm{sh}(\overline{\mathbf{T}}) = \eta$ is of the form (4.3.8) with μ in (4.3.9).

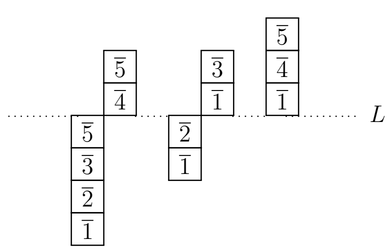
Proposition 4.3.8. *Under the above hypothesis, $\overline{\mathbf{T}}$ satisfies (S1) and (S2).*

Proof. It follows from the same argument as in the proof of Proposition 4.3.5 with Lemma 4.3.7. □

Example 4.3.9. Let $n = 5$ and $\lambda = \left(\frac{5}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}\right)$. Then we have

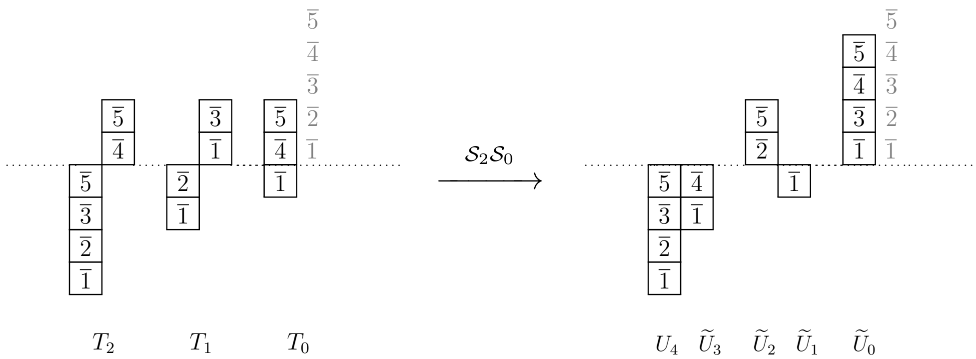
$$\omega_\lambda = \varpi_1 + \varpi_3 + \varpi_4, \quad \mathbf{T}_\lambda \subset \mathbf{T}(4) \times \mathbf{T}(2) \times \mathbf{T}^{\mathrm{sp}-}.$$

Let us consider $\mathbf{T} = (T_2, T_1, T_0) \in \mathbf{T}_\lambda$ given by



T_2 T_1 T_0

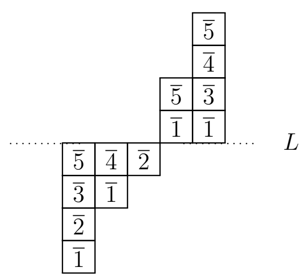
We regard T_0 as in Remark 4.1.6(1). Then we have $T_2 \not\trianglelefteq T_1$ and $T_1 \not\trianglelefteq T_0$. By applying (4.3.10) (cf. Remark 4.3.6) we obtain



where $U = H_{(1^5)}$ is the single-column tableau consisting of the numbers in gray.

Finally, we apply $\mathcal{S}_1$ to $(U_4, \widetilde{U}_3, \widetilde{U}_2, \widetilde{U}_1, \widetilde{U}_0)$ and then we have $\overline{\mathbf{T}}$ given by

$\overline{\mathbf{T}} =$



where $\overline{\mathbf{T}}^{\texttt{body}}$ (resp. $\overline{\mathbf{T}}^{\texttt{tail}}$) is the semistandard tableau located above L (resp. below L) whose shape is $(2,2,1,1)^\pi$ (resp. $(3,2,1,1)$).

4.4 Embedding from $\mathbf{T}_\lambda$ into $\mathbf{V}_\lambda$

4.4.1 Crystal of parabolic Verma module

For $\lambda \in \mathcal{P}_n$, let

$$\mathbf{V}_\lambda := \left(\bigsqcup_{\delta \in \mathscr{P}_n^{(1,1)}} SST_{[\overline{n}]}(\delta^\pi) \right) \times SST_{[\overline{n}]}(\mu)\,,$$

where $\mu \in \mathscr{P}_n$ is given by (4.3.5) if $\lambda_n \geq 0$, and by (4.3.9) otherwise. Recall that

$$\mathbf{V} := \bigsqcup_{\delta \in \mathscr{P}_n^{(1,1)}} SST_{[\overline{n}]}(\delta^\pi),$$

has a $\mathfrak{g}$-crystal structure (see [62, Section 5.2] for details). On the other hand, we regard the $\mathfrak{l}$-crystal

$$\mathbf{S}_\mu := SST_{[\overline{n}]}(\mu),$$

as a $\mathfrak{g}$-crystal, by defining $\widetilde{e}_n T = \widetilde{f}_n T = \mathbf{0}$ with $\varphi_n(T) = \varepsilon_n(T) = -\infty$ for $T \in SST_{[\overline{n}]}(\mu)$. Then we may regard $\mathbf{V}_\lambda$ as a $\mathfrak{g}$-crystal by letting

$$\mathbf{V}_\lambda = \mathbf{V} \otimes \mathbf{S}_\mu, \tag{4.4.1}$$

which can be viewed as the crystal of a parabolic Verma module induced from a highest weight $\mathfrak{l}$-module with highest weight λ (cf. [61, Section 3] and [66, Theorem 4.3] for types BC).

Remark 4.4.1. In [61], Kwon interpret Littlewood identities for types B_∞ and C_∞ by using the crystal base theory. In particular, this induces a new combinatorial model for the crystal graph of an integrable highest weight irreducible module by characterizing ε_0^* (see [61, Theorem 3.11, Remark 3.12]).

It would be interesting to develop an analog of the above result for type D_∞ by using the results in this thesis. Indeed, the formula in Theorem 3.3.6 may be viewed as a characterization of ε_0^* on $\mathbf{V}_0$.

Let us recall the actions of $\widetilde{e}_n$ and $\widetilde{f}_n$ on $\mathbf{T}_\lambda$ and $\mathbf{V}_\lambda$ in more details. We let

$$\mathtt{vd} = \begin{array}{|c|}\hline \overline{n} \\ \hline \overline{n-1} \\ \hline \end{array}$$

be the vertical domino with entries $\overline{n}$ and $\overline{n-1}$.

Suppose that $\mathbf{T} = (T_l, \ldots, T_1, T_0) \in \mathbf{T}_\lambda$ is given, where

$$\mathbf{T} = (U_{2\ell}, \ldots, U_i, \ldots, U_1, U_0)$$

under the identification (4.3.1). We define a sequence $\sigma = (\sigma_0, \sigma_1, \dots, \sigma_{2l})$ by

$$\sigma_i = \begin{cases} + & \text{if } U_i = \emptyset \text{ or } U_i[1] \geq \overline{n-2}, \\ - & \text{if } \texttt{vd} \text{ is located in the top of } U_i, \\ \cdot & \text{otherwise,} \end{cases} \tag{4.4.2}$$

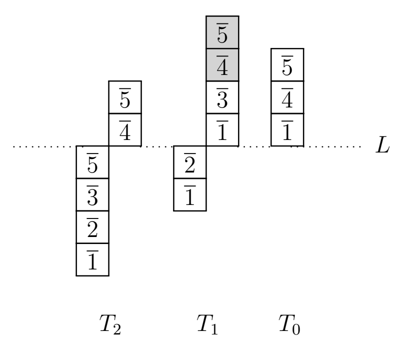
where we put $\sigma_0 = \cdot$ if $U_0 = \emptyset$. Let σ^{red} be the sequence obtained from σ by replacing the pairs of neighboring signs $(+,-)$ (ignoring $\cdot$) with $(\,\cdot\,,\,\cdot\,)$ as far as possible. If there exists a $-$ in σ^{red}, then we define $\widetilde{e}_n\mathbf{T}$ to be the one obtained by removing `vd` in U_i corresponding to the rightmost i such that $\sigma_i = -$ in σ^{red}. We define $\widetilde{e}_n\mathbf{T} = \mathbf{0}$, otherwise. Similarly, we define $\widetilde{f}_n\mathbf{T}$ by adding `vd` in U_i corresponding to the leftmost i such that $\sigma_i = +$ in σ^{red}.

Next, suppose that $T \in \mathbf{V}_\lambda$ is given. We define a sequence $\tau = (\tau_0, \tau_1, \dots)$ by (4.4.2). Note that τ is an infinite sequence where $\tau_i = +$ for all sufficiently large i. Then we define τ^{red} and hence $\widetilde{e}_nT$ and $\widetilde{f}_nT$ in the same way as in $\mathbf{T}_\lambda$.

Example 4.4.2. Let us consider $\mathbf{T} \in \mathbf{T}_\lambda$ in Example 4.3.9. Then the sequences σ is obtained from $\mathbf{T}$ by reading component σ_i from right to left and the sequence σ^{red} is reduced from σ by replacing the pair $(\,+\,,\,-\,)$ to $(\,\cdot\,,\,\cdot\,)$. Consequently, we have

$$\sigma = (\,-\,,\,+\,,\,+\,,\,-\,,\cdot\,) \quad \longrightarrow \quad \sigma^{\mathrm{red}} = (\,-\,,\,+\,,\,\cdot\,,\,\cdot\,,\cdot\,)\,.$$

Therefore, $\tilde{f}_5\mathbf{T}$ is given by



On the other hand, we consider $T \in \mathbf{V}_\lambda$ given by

$$T = \begin{array}{cc} & \boxed{\overline{5}} \\ & \boxed{\overline{4}} \\ \boxed{\overline{5}} & \boxed{\overline{3}} \\ \boxed{\overline{1}} & \boxed{\overline{1}} \end{array} \otimes \begin{array}{ccc} \boxed{\overline{5}} & \boxed{\overline{4}} & \boxed{\overline{2}} \\ \boxed{\overline{3}} & \boxed{\overline{1}} & \\ \boxed{\overline{2}} & & \\ \boxed{\overline{1}} & & \end{array} \in \mathbf{V}_\lambda = \mathbf{V} \otimes \mathbf{S}_\mu\,.$$

Then the sequences τ and τ^{red} are obtained by the similar way as in the sequences σ and σ^{red}, so we have

$$\tau = (\,-\,,\,\cdot\,,+\,,\,+\,,\,\cdots\,) \quad \longrightarrow \quad \tau^{\mathrm{red}} = (\,-\,,\,\cdot\,,+\,,\,+\,,\,\cdots\,)\,.$$

Hence $\tilde{f}_5\, T$ is given by

$$T \;=\; \begin{array}{|c|c|c|}\cline{3-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & \overline{5} \\ \cline{3-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & \overline{4} \\ \hline \overline{5} & 5 & \overline{3} \\ \hline \overline{4} & \overline{1} & \overline{1} \\ \hline \end{array} \;\otimes\; \begin{array}{|c|c|c|}\hline \overline{5} & \overline{4} & \overline{2} \\ \hline \overline{3} & \overline{1} & \multicolumn{1}{c}{} \\ \cline{1-2} \overline{2} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \overline{1} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad .$$

Note that the subsequence of τ^{red} consisting of first 3 components is equal to the sequence σ^{red} by ignoring the dot $\cdot$. This holds in general, see Lemma 4.4.5.

4.4.2 Embedding as $\mathfrak{g}$-crystals

The following is the main result in this section.

Theorem 4.4.3. *The map*

$$\begin{array}{rccc} \chi_\lambda : & \mathbf{T}_\lambda & \longrightarrow & \mathbf{V}_\lambda \otimes T_{r\omega_n} \\ & \mathbf{T} & \longmapsto & \overline{\mathbf{T}}^{\mathtt{body}} \otimes \overline{\mathbf{T}}^{\mathtt{tail}} \otimes t_{r\omega_n} \end{array} \tag{4.4.3}$$

is an embedding of $\mathfrak{g}$-crystals, where $r = (\lambda, \omega_n)$.

First, we consider the injectivity of χ_λ (4.4.3) on $\mathfrak{l}$-highest weight vectors.

Lemma 4.4.4. *Suppose that $\mathbf{T}$ and $\mathbf{S}$ are contained in $\mathbf{H}(\lambda)$. If $\chi_\lambda(\mathbf{T}) = \chi_\lambda(\mathbf{S})$, then we have $\mathbf{T} = \mathbf{S}$.*

Proof. Suppose that $\chi_\lambda(\mathbf{T}) = \chi_\lambda(\mathbf{S})$. Since $\mathbf{T}$, $\mathbf{S}$ are $\mathfrak{l}$-highest weight vectors, by Corollary 2.1.16, we have

$$\overline{\mathbf{T}}^{\mathtt{body}} = H_{\delta^\pi}, \qquad \overline{\mathbf{S}}^{\mathtt{body}} = H_{\gamma^\pi}$$

for some $\delta, \gamma \in \mathscr{P}^{(1,1)}$. Since χ_λ is weight-preserving and $\mathbf{T}$, $\mathbf{S}$ are $\mathfrak{l}$-highest weight vectors, we have

$$(H_\delta \leftarrow \overline{\mathbf{T}}^{\mathtt{tail}}) = (H_\gamma \leftarrow \overline{\mathbf{S}}^{\mathtt{tail}}) = H_\zeta \tag{4.4.4}$$

for some $\zeta \in \mathscr{P}$. Thus, we have $\delta = \gamma$ and $\overline{\mathbf{T}}^{\mathtt{body}} = \overline{\mathbf{S}}^{\mathtt{body}}$. By [23, Proposition 1, p.19] and (4.4.4), we have $\overline{\mathbf{T}}^{\mathtt{tail}} = \overline{\mathbf{S}}^{\mathtt{tail}}$. Then one can check that the sliding and separation algorithms are reversible on $\mathfrak{l}$-highest weight vectors, see Sections 5.3.3–5.3.4 for more details. We complete the proof. $\square$

Next, we investigate the signatures (4.4.2) for $\mathbf{T}$ and $\overline{\mathbf{T}}^{\mathsf{body}}$.

Lemma 4.4.5. *Let $\overline{\sigma} = (\overline{\sigma}_0, \ldots, \overline{\sigma}_{2l})$ be the subsequence of τ^{red} consisting of its first $2l+1$ components for $\overline{\mathbf{T}}^{\mathsf{body}}$. If we ignore the dot $\cdot$ in* (4.4.2), *then*

$$\sigma^{\mathrm{red}} = \overline{\sigma}^{\mathrm{red}}.$$

Proof. We first consider a pair (T_{i+1}, T_i) in $\mathbf{T}$. Let $(\sigma_{j-1},\ \sigma_j,\ \sigma_{j+1},\ \sigma_{j+2})$ be the subsequence of σ corresponding to (T_{i+1}, T_i) with $j = 2i$. Let

$$\mathcal{S}_j\mathbf{T} = (\ldots, U_{j+2}, \widetilde{U}_{j+1}, \widetilde{U}_j, U_{j-1}, \ldots)$$

(see Lemma 4.3.1). We denote by $(\sigma_{j-1}, \widetilde{\sigma}_j, \widetilde{\sigma}_{j+1}, \sigma_{j+2})$ the sequence defined by (4.4.2) corresponding to $(U_{j+2},\ \widetilde{U}_{j+1},\ \widetilde{U}_j,\ U_{j-1})$.

Case 1. $\mathfrak{r}_{i+1}\,\mathfrak{r}_i = 0$. Note that $\mathcal{S}_j = \mathcal{F}_j^{a_i}$ by Lemma 7.2.1 (1). Then we have $(\sigma_{j+1}, \sigma_j) = (\widetilde{\sigma}_{j+1}, \widetilde{\sigma}_j)$ by the similar argument in the proof of [66, Theorem 5.7].

Case 2. $\mathfrak{r}_{i+1}\,\mathfrak{r}_i = 1$. The relation between the two pairs (σ_{j+1}, σ_j) and $(\widetilde{\sigma}_{j+1}, \widetilde{\sigma}_j)$ is given in Table A.2.

Let $\mathbf{U} := \mathcal{S}_2 \ldots \mathcal{S}_{2l-2}\mathbf{T} := (U_{2l}, \widetilde{U}_{2l-1}, \ldots, \widetilde{U}_2, U_1, U_0)$ and let $\dot{\sigma}$ be the sequence given by (4.4.2) corresponding to $\mathbf{U}$. Then we have

$$\dot{\sigma} = (\sigma_{2l}, \widetilde{\sigma}_{2l-1}, \ldots, \widetilde{\sigma}_2, \sigma_1, \sigma_0).$$

It is straightforward to check that $\sigma^{\mathrm{red}} = \dot{\sigma}^{\mathrm{red}}$. Note that $\sigma_{2l} = \overline{\sigma}_{2l}$ by definition. By Lemma 7.3.7 and 4.3.7, we may use an inductive argument for $(\widetilde{U}_{2\ell-1}, \ldots, \widetilde{U}_2, U_1, U_0)$ to have $\sigma^{\mathrm{red}} = \dot{\sigma}^{\mathrm{red}} = \overline{\sigma}^{\mathrm{red}}$. This completes the proof. $\square$

Now we prove Theorem 4.4.3 as follows.

Proof of Theorem 4.4.3. We use the following notations under the identification (4.3.1) in this proof.

(1) $\widetilde{\mathbf{T}} = (\widetilde{U}_{2l}, \widetilde{U}_{2l-1}, \ldots)$: the sequence of tableaux with a column shape obtained from $\mathbf{T}$ by shifting the tails one position to the left as in Sections 4.3.2 and 4.3.3.

(2) $\mathbb{T} = (\widetilde{U}_{2l-1}, \widetilde{U}_{2l-2}, \dots)$: the sequence of tableaux obtained from $\widetilde{\mathbf{T}}$ by removing the column $\widetilde{U}_{2l}$.

(3) $\mathtt{T} = (\mathtt{U}_{2l}, \mathtt{U}_{2l-1}, \dots)$: the sequence obtained from $\mathbf{T}$ by applying $\widetilde{f}_n$, that is,

$$\mathtt{T} = \widetilde{f}_n \, \mathbf{T}.$$

(4) $\widetilde{\mathtt{T}}$: the sequence obtained from $\mathtt{T}$ by shifting the tails one position to the left as in Sections 4.3.2 and 4.3.3 and then removing the left-most column of $\mathtt{T}$.

(5) $\widetilde{\sigma} = (\widetilde{\sigma}_0, \widetilde{\sigma}_1, \dots, \widetilde{\sigma}_{2l})$: the sequence of (4.4.2) associated with $\widetilde{\mathbf{T}}$.

Recall that $\mathbb{T} \in \mathbf{T}_{\widetilde{\lambda}}$ by Lemmas 7.3.7 and 4.3.7, where $\widetilde{\lambda}$ satisfies $\omega_\lambda - \omega_{\widetilde{\lambda}} = \omega_{n-a}$ with $a = \mathrm{ht}(U_{2l}^{\mathtt{tail}})$.

First we show that χ_λ is injective. Suppose that $\chi_\lambda(\mathbf{T}) = \chi_\lambda(\mathbf{T}')$ for some $\mathbf{T}, \mathbf{T}' \in \mathbf{T}_\lambda$. There exists $i_1, \dots, i_r \in J$ such that $\widetilde{e}_{i_1} \dots \widetilde{e}_{i_r} \chi_\lambda(\mathbf{T}) = \widetilde{e}_{i_1} \dots \widetilde{e}_{i_r} \chi_\lambda(\mathbf{T}')$ is an $\mathfrak{l}$-highest weight vector. Note that χ_λ is a morphism of $\mathfrak{l}$-crystals by Proposition 4.3.5. Therefore, we have

$$\chi_\lambda(\widetilde{e}_{i_1} \dots \widetilde{e}_{i_r} \mathbf{T}) = \chi_\lambda(\widetilde{e}_{i_1} \dots \widetilde{e}_{i_r} \mathbf{T}').$$

By Lemma 4.4.4, we have

$$\widetilde{e}_{i_1} \dots \widetilde{e}_{i_r} \mathbf{T} = \widetilde{e}_{i_1} \dots \widetilde{e}_{i_r} \mathbf{T}'.$$

Hence $\mathbf{T} = \mathbf{T}'$, and χ_λ is injective.

Now it remains to show that

$$\widetilde{f}_n \mathbf{T} \neq \mathbf{0} \text{ and } \chi_\lambda(\mathbf{T}) \neq \mathbf{0} \text{ for } \mathbf{T} \in \mathbf{T}_\lambda \quad \Longrightarrow \quad \chi_\lambda(\widetilde{f}_n \mathbf{T}) = \widetilde{f}_n \chi_\lambda(\mathbf{T}). \tag{4.4.5}$$

If $\widetilde{f}_n$ acts on U_{2l} or U_k, where $k = 0$ if $U_0 \neq \emptyset$, and $k = 1$ if $U_0 = \emptyset$, then the claim (4.4.5) follows from Lemmas 4.3.1 and 4.4.5.

Suppose that $\widetilde{f}_n$ acts on U_k for $k < 2l$. To prove (4.4.5), it is enough to show that

$$\widetilde{\mathtt{T}} = \widetilde{f}_n \mathbb{T}. \tag{4.4.6}$$

Indeed, if (4.4.6) holds, then by induction on the number of columns in $\mathbf{T}$, we have

$$
\begin{aligned}
\overline{\widetilde{f}_n \mathbf{T}} &= \overline{(U_{2l},\ \widetilde{\mathtt{T}})} \\
&= \overline{\left(U_{2l},\ \widetilde{f}_n \mathbb{T}\right)} && \text{by (4.4.6)} \\
&= \left(U_{2l},\ \overline{\widetilde{f}_n \mathbb{T}}\right) \\
&= \left(U_{2l},\ \widetilde{f}_n \overline{\mathbb{T}}\right) && \text{by induction hypothesis} \\
&= \widetilde{f}_n \left(U_{2l},\ \overline{\mathbb{T}}\right) && \text{by Lemma 4.4.5,}
\end{aligned}
$$

Hence we have $\overline{\widetilde{f}_n \mathbf{T}} = \widetilde{f}_n \overline{\mathbf{T}}$ which implies (4.4.5).

Now we verify (4.4.6). Let us recall Table A.2 and then we have

$$\sigma^{\mathrm{red}} = \widetilde{\sigma}^{\mathrm{red}}. \tag{4.4.7}$$

We prove (4.4.6) for the case $U_k^{\texttt{tail}} \neq \emptyset$ with non-trivial sub-cases. The proof for other cases or the case $U_k^{\texttt{tail}} = \emptyset$ is almost identical.

Let us recall the action of $\widetilde{f}_n$ in Section 4.4.1. Then for the case $U_k^{\texttt{tail}} \neq \emptyset$, the signature σ_{k+1} (4.4.2) associated with U_{k+1} must be $\cdot$ or $+$. We consider four cases as follows.

Case 1. If $\mathrm{ht}(U_{k+1}^{\texttt{body}}) \leq \mathrm{ht}(U_k^{\texttt{body}}) < \mathrm{ht}(U_{k-1}^{\texttt{body}}) = \mathrm{ht}(U_k^{\texttt{body}}) + 2$, then by definition of $\mathcal{S}_k$ the top entry of U_{k+1} can not be moved to the right in $\widetilde{\mathbf{T}}$. Thus we obtain $(\sigma_k, \sigma_{k+1}) = (\widetilde{\sigma}_k, \widetilde{\sigma}_{k+1})$, which implies (4.4.6).

Case 2. If $\mathrm{ht}(U_{k+1}^{\texttt{body}}) = \mathrm{ht}(U_k^{\texttt{body}}) + 2 = \mathrm{ht}(U_{k-1}^{\texttt{body}})$, then the signature σ_{k+1} (4.4.2) can not be $+$. Otherwise $(\mathtt{U}_{k+2}, \mathtt{U}_{k+1}) \not\prec (\mathtt{U}_k, \mathtt{U}_{k-1})$ for Definition 4.1.5 (ii), which is a contradiction. Thus $\sigma_{k+1} = \cdot$. We observe that

$$
\begin{aligned}
&(\sigma_k, \sigma_{k+1}) = (+, \cdot\,) \longrightarrow (\widetilde{\sigma}_k, \widetilde{\sigma}_{k+1}) = (\,\cdot\,, +), \\
&U_{k+1}[1] = \boxed{\overline{n}} \text{ or } \boxed{\overline{n-1}}, \quad U_{k+1}[2] \geq \overline{n-2}.
\end{aligned} \tag{4.4.8}
$$

It is straightforward to check from the definition of $\mathcal{S}_k$ that when we apply $\mathcal{S}_k$ to $\mathtt{T}$, the domino $\mathtt{vd}$ in $\mathtt{U}_k$ is changed as follows:

$$\begin{cases} \boxed{\overline{n-1}} \text{ in } \mathtt{U}_k \text{ is moved to } \boxed{\overline{n}} \text{ in } \mathtt{U}_{k+1} \text{ below} & \text{if } \mathtt{U}_{k+1}[1] = \boxed{\overline{n}}, \\ \boxed{\overline{n}} \text{ in } \mathtt{U}_k \text{ is moved to } \boxed{\overline{n-1}} \text{ in } \mathtt{U}_{k+1} \text{ above} & \text{if } \mathtt{U}_{k+1}[1] = \boxed{\overline{n-1}}. \end{cases} \tag{4.4.9}$$

Combining (4.4.7), (4.4.8) and (4.4.9), we conclude that (4.4.6) holds in this case.

Case 3. If $\mathrm{ht}(U_{k+1}^{\mathtt{body}}) \leq \mathrm{ht}(U_k^{\mathtt{body}})$ and $\mathrm{ht}(U_k^{\mathtt{body}}) + 2 < \mathrm{ht}(U_{k-1}^{\mathtt{body}})$, then we have

$$(U_{k+2}, U_{k+1}) \triangleleft (U_k, U_{k-1}) \longrightarrow (\mathtt{U}_{k+2}, \mathtt{U}_{k+1}) \triangleleft (\mathtt{U}_k, \mathtt{U}_{k-1}).$$

By (4.4.7), this implies that (4.4.6) holds in this case.

Case 4. Suppose $\mathrm{ht}(U_{k+1}^{\mathtt{body}}) = \mathrm{ht}(U_k^{\mathtt{body}}) + 2$ and $\mathrm{ht}(U_k^{\mathtt{body}}) + 2 < \mathrm{ht}(U_{k-1}^{\mathtt{body}})$. First we consider the case $\sigma_{k+1} = \cdot$. If there exists i such that $U_{k+1}(i) > U_k(i + a_k - 1)$, where $a_k = \mathrm{ht}(U_k^{\mathtt{ail}})$, then by definition

$$(U_{k+2}, U_{k+1}) \not\triangleleft (U_k, U_{k-1}) \longrightarrow (\mathtt{U}_{k+2}, \mathtt{U}_{k+1}) \not\triangleleft (\mathtt{U}_k, \mathtt{U}_{k-1}).$$

Form this we observe that the domino $\mathtt{vd}$ in $\mathtt{U}_k$ is not moved when we apply the sliding to $\mathtt{T}$. Also we note that

$$(\sigma_k, \sigma_{k+1}) = (+, \cdot\,) \longrightarrow (\widetilde{\sigma}_k, \widetilde{\sigma}_{k+1}) = (+, \cdot\,).$$

Combining these observations with (4.4.7), we obtain (4.4.6) in this case. If there is no such i, we have

$$(U_{k+2}, U_{k+1}) \not\triangleleft (U_k, U_{k-1}) \longrightarrow (\mathtt{U}_{k+2}, \mathtt{U}_{k+1}) \triangleleft (\mathtt{U}_k, \mathtt{U}_{k-1}),$$
$$(\sigma_k, \sigma_{k+1}) = (+, \cdot\,) \longrightarrow (\widetilde{\sigma}_k, \widetilde{\sigma}_{k+1}) = (\,\cdot\,, +).$$

Note that $\widetilde{U}_k[1] = U_{k+1}[1] \leq \overline{n-1}$ and we observe that (4.4.9) also holds in this case. Hence we have (4.4.6).

Second we consider the case $\sigma_{k+1} = +$. In this case, we obtain

$$(U_{k+2}, U_{k+1}) \not\triangleleft (U_k, U_{k-1}) \longrightarrow (\mathtt{U}_{k+2}, \mathtt{U}_{k+1}) \not\triangleleft (\mathtt{U}_k, \mathtt{U}_{k-1}),$$

since $U_{k+1}[1] > \overline{n-1}$. Also it is clear that $(\widetilde{\sigma}_k, \widetilde{\sigma}_{k+1}) = (+, +)$ by definition of $\mathcal{S}_k$. On the other hand, the domino $\mathtt{vd}$ in $\mathtt{U}_k$ can not be moved to the left when we apply the sliding to $\mathtt{T}$. Consequently, we have (4.4.6). We complete the proof of Theorem 4.4.3. $\square$

4.5 Lusztig data of Kashiwara-Nakashima tableaux of type D

Let $\mu \in \mathscr{P}_n$ be given and let $\epsilon_\mu = \sum_{i=1}^{n} \mu_i \epsilon_{n-i}$. Consider the map

$$\begin{array}{rcl} SST_{[\overline{n}]}(\mu) & \longrightarrow & \mathbf{B}_J \otimes T_{\epsilon_\mu} \\ T & \longmapsto & \mathbf{c}_J(T) \otimes t_{\epsilon_\mu} \end{array}, \tag{4.5.1}$$

where $\mathbf{c}_J(T)$ is the one such that the multiplicity $c_{\epsilon_i - \epsilon_j}$ is equal to the number of $\overline{i}$'s appearing in the $(n-j+1)$th row of T for $1 \leq i < j \leq n$.

Proposition 4.5.1. *The map* (4.5.1) *is an embedding of* $\mathfrak{g}$*-crystals.*

Proof. It is a well-known fact that the map is an embedding of $\mathfrak{l}$-crystals (cf. [65]). By definition of $\widetilde{e}_n$ and $\widetilde{f}_n$ on $\mathbf{S}_\mu$ and $\mathbf{B}_J$, it becomes a morphism of $\mathfrak{g}$-crystals. $\square$

Let us recall Burge correspondence (3.3.2). We denote by $\mathbf{c}^J(\,\cdot\,)$ the inverse of (3.3.2). For the reader's convenience, we briefly review the isomorphism $\mathbf{c}^J(\,\cdot\,)$ as follows.

Now, let $T \in SST(\delta^\pi) \subset \mathbf{V}$ be given. Then $\mathbf{c}^J(T)$ is given by the following steps:

(1) Let $\overline{x}_1$ be the smallest entry in T such that it is located at the leftmost among such entries and let T' be the tableau obtained from T by removing $\overline{x}_1$.

(2) Take $\overline{y}_1$ which is the entry in T below $\overline{x}_1$. Let T'' be the tableau obtained from T' by applying the inverse of the Schensted's column insertion to $\overline{y}_1$. Then we obtain an entry $\overline{z}_1$ such that $(T'' \leftarrow \overline{z}_1) = T'$.

(3) We apply the above steps to $T_1 := T''$ instead of T. Then we denote by $\overline{x}_2$ and $\overline{z}_2$ the entries obtained from T_1 and let $T_2 := T_1''$.

(4) In general, repeat this process for $T_{i+1} := T_i''$ $(i = 1, 2, \dots)$ until there is no entries in T_{i+1}, and let $\overline{x}_{i+1}$ and $\overline{z}_{i+1}$ be the entries from T_{i+1} by this process.

(5) Finally we obtain a biword $(\mathbf{a},\mathbf{b}) \in \Omega$ given by

$$\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} \overline{x}_1 & \overline{x}_2 & \dots & \overline{x}_\ell \\ \overline{z}_1 & \overline{z}_2 & \dots & \overline{z}_\ell \end{pmatrix},$$

and define $\mathbf{c}^J(T) \in \mathbf{B}^J$ to be the one corresponding to $(\mathbf{a},\mathbf{b})$ (see Example 4.5.4).

Then we have the following.

Corollary 4.5.2. *For $\lambda \in \mathcal{P}_n$, the map*

$$\begin{array}{rcl} \mathbf{V}_\lambda \otimes T_{r\omega_n} & \longrightarrow & \mathbf{B}^J \otimes \mathbf{B}_J \otimes T_{\omega_\lambda} \\ S \otimes T \otimes t_{r\omega_n} & \longmapsto & \mathbf{c}^J(S) \otimes \mathbf{c}_J(T) \otimes t_{\omega_\lambda} \end{array}, \tag{4.5.2}$$

is an embedding of $\mathfrak{g}$-crystals, where $r = (\lambda, \omega_n)$. □

We are now in a position to state the main result in this chapter.

Theorem 4.5.3. *For $\lambda \in \mathcal{P}_n$, we have an embedding Ξ_λ of $\mathfrak{g}$-crystals given by*

$$\begin{tikzcd} \mathbf{KN}_\lambda \arrow[rr, "\Xi_\lambda"] \arrow[d, "(4.2.3)"'] & & \mathbf{B}_{\mathbf{i}_0} \otimes T_{\omega_\lambda} \\ \mathbf{T}_\lambda \arrow[r, "(4.4.3)"'] & \mathbf{V}_\lambda \otimes T_{r\omega_n} \arrow[r, "(4.5.2)"'] & \mathbf{B}^J \otimes \mathbf{B}_J \otimes T_{\omega_\lambda} \arrow[u, "(3.2.16)"'] \end{tikzcd}$$

Proof. It follows from Theorems 4.2.4, 4.4.3, Proposition 3.2.6(2) and Corollary 4.5.2. □

Example 4.5.4. Set $n = 5$ and $\lambda = \left(\frac{5}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}\right)$. Let $T \in \mathbf{KN}_\lambda$ be given by

$$T = \begin{array}{ccc} & & \boxed{2} \\ & & \boxed{3} \\ & \boxed{4} & \boxed{\overline{5}} \\ & \boxed{5} & \boxed{\overline{4}} \\ \boxed{\overline{5}} & \boxed{\overline{1}} & \boxed{\overline{1}} \end{array}.$$

By (4.2.3) and (4.4.3) (see Example 4.3.9), we have

$$\Psi_\lambda(T) = \begin{array}{ccccc} & & & & \overline{5} \\ & \overline{5} & & \overline{3} & \overline{4} \\ & \overline{4} & & \overline{1} & \overline{1} \\ \overline{5} & & \overline{2} & & \\ \overline{3} & & \overline{1} & & \\ \overline{2} & & & & \\ \overline{1} & & & & \end{array} \xrightarrow{(4.4.3)} \begin{array}{ccccc} & & & & \overline{5} \\ & & & & \overline{4} \\ & & & \overline{5} & \overline{3} \\ & & & \overline{1} & \overline{1} \\ \overline{5} & \overline{4} & \overline{2} & & \\ \overline{3} & \overline{1} & & & \\ \overline{2} & & & & \\ \overline{1} & & & & \end{array}$$

Note that $\sigma = (\,-,\,+,\,+,\,-,\,\cdot\,)$ and $\overline{\sigma} = (\,-,\,\cdot\,,+,\,\cdot\,,\cdot\,)$. Also

$$\sigma^{\mathrm{red}} = (\,-,\,+,\,\cdot\,,\cdot\,,\cdot\,), \quad \overline{\sigma}^{\mathrm{red}} = (\,-,\,\cdot\,,+,\,\cdot\,,\cdot\,)$$

(cf. Lemma 4.4.5).

Put $\mathbf{T} = \Psi_\lambda(T)$. Then

$$\overline{\mathbf{T}}^{\mathtt{body}} = \begin{array}{cc} & \overline{5} \\ & \overline{4} \\ \overline{5} & \overline{3} \\ \overline{1} & \overline{1} \end{array}, \qquad \overline{\mathbf{T}}^{\mathtt{tail}} = \begin{array}{ccc} \overline{5} & \overline{4} & \overline{2} \\ \overline{3} & \overline{1} & \\ \overline{2} & & \\ \overline{1} & & \end{array}.$$

Let us recall the convention in Section 3.2.5 for $\mathbf{c}^J(\overline{\mathbf{T}}^{\mathtt{body}})$ associated with $\Phi^+(J)$ with the convex order (3.2.7), that is, we identify $(c_{\beta_1}, \ldots, c_{\beta_{10}}) \in \mathbf{B}^J$ with

$$\begin{array}{ccccccc} & & & c_{\beta_4} & & & \\ & & c_{\beta_3} & & c_{\beta_7} & & \\ & c_{\beta_2} & & c_{\beta_6} & & c_{\beta_9} & \\ c_{\beta_1} & & c_{\beta_5} & & c_{\beta_8} & & c_{\beta_{10}} \end{array}.$$

Here $\beta_1 = \alpha_5$. Similarly, we use the above notation for $\mathbf{c}_J(\mathbf{T}^{\mathtt{tail}})$ with respect to Φ_J and the convex order (3.2.7), that is, we identify $(\beta_{11}, \dots, \beta_{20}) \in \mathbf{B}_J$ with

$$
\begin{array}{ccccccc}
 & & & c_{\beta_{14}} & & & \\
 & & c_{\beta_{13}} & & c_{\beta_{17}} & & \\
 & c_{\beta_{12}} & & c_{\beta_{16}} & & c_{\beta_{19}} & \\
c_{\beta_{11}} & & c_{\beta_{15}} & & c_{\beta_{18}} & & c_{\beta_{20}}
\end{array} .
$$

Here $\beta_{11} = \alpha_1$, $\beta_{15} = \alpha_2$, $\beta_{18} = \alpha_3$ and $\beta_{20} = \alpha_4$.

Now we find $\mathbf{c}^J\,(\overline{\mathbf{T}}^{\mathsf{body}})$ by the steps (1)–(5), see Proposition 4.5.1 below.

$$
\begin{array}{cc}
\cline{2-2} & \multicolumn{1}{|c|}{\overline{5}} \\
\cline{2-2} & \multicolumn{1}{|c|}{\overline{4}} \\
\hline
\multicolumn{1}{|c|}{\overline{\mathbf{5}}} & \multicolumn{1}{c|}{\overline{3}} \\
\hline
\multicolumn{1}{|c|}{\overline{\mathbf{1}}} & \multicolumn{1}{c|}{\overline{1}} \\
\hline
\end{array}
\longrightarrow
\begin{array}{|c|}
\hline \overline{\mathbf{5}} \\ \hline \overline{\mathbf{4}} \\ \hline \overline{3} \\ \hline \overline{1} \\ \hline
\end{array}\,, \quad
\begin{pmatrix} \overline{5} \\ \overline{1} \end{pmatrix}
\longrightarrow
\begin{array}{|c|}
\hline \overline{\mathbf{3}} \\ \hline \overline{\mathbf{1}} \\ \hline
\end{array}\,, \quad
\begin{pmatrix} \overline{5} & \overline{5} \\ \overline{4} & \overline{1} \end{pmatrix}
\longrightarrow
\emptyset\,, \quad
\begin{pmatrix} \overline{5} & \overline{5} & \overline{3} \\ \overline{4} & \overline{1} & \overline{1} \end{pmatrix} .
$$

Thus we have

$$
\begin{array}{cc}
\cline{2-2} & \multicolumn{1}{|c|}{\overline{5}} \\
\cline{2-2} & \multicolumn{1}{|c|}{\overline{4}} \\
\hline
\multicolumn{1}{|c|}{\overline{5}} & \multicolumn{1}{c|}{\overline{3}} \\
\hline
\multicolumn{1}{|c|}{\overline{1}} & \multicolumn{1}{c|}{\overline{1}} \\
\hline
\end{array}
\longrightarrow
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 0 & & 1 & \\
1 & & 0 & & 0 & & 0
\end{array}
\in \mathbf{B}^J .
$$

Next by definition (4.5.1), we have $\mathbf{c}_J(\mathbf{T}^{\mathtt{tail}})$ as follows.

$$
\begin{array}{ccc}
\hline
\multicolumn{1}{|c|}{\overline{5}} & \multicolumn{1}{c|}{\overline{4}} & \multicolumn{1}{c|}{\overline{2}} \\
\hline
\multicolumn{1}{|c|}{\overline{3}} & \multicolumn{1}{c|}{\overline{1}} & \\
\cline{1-2}
\multicolumn{1}{|c|}{\overline{2}} & & \\
\cline{1-1}
\multicolumn{1}{|c|}{\overline{1}} & & \\
\cline{1-1}
\end{array}
\longrightarrow
\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 1 & & 1 & & \\
 & 0 & & 0 & & 0 & \\
1 & & 1 & & 1 & & 1
\end{array}
\in \mathbf{B}_J .
$$

Hence we obtain the Lusztig data for the KN tableau T associated with $\mathbf{i}_0$, that is,

$$
\Xi_\lambda(T) = (1,\,0,\,0,\,1,\,0,\,0,\,0,\,0,\,1,\,0,\,1,\,0,\,1,\,0,\,1,\,0,\,1,\,1,\,0,\,1) \otimes t_{\omega_\lambda}.
$$

Remark 4.5.5. It would be interesting to characterize explicitly the image of the embedding Ξ_λ (see [66] for types BC) in a combinatorial way.

Chapter 5

Branching rules for classical groups

Let GL_n, Sp_n and O_n be the general linear group, symplectic group and orthogonal group of rank n over $\mathbb{C}$, respectively. Let $V_{\mathrm{G}_n}^{\lambda}$ be a finite-dimensional irreducible G_n-module parametrized by a partition λ, where $\mathrm{G}_n = \mathrm{GL}_n, \mathrm{Sp}_n$ or O_n [26]. Then we define

$$\left[\, V_{\mathrm{GL}_n}^{\lambda},\, V_{\mathrm{G}_n}^{\mu} \,\right] := \dim \mathrm{Hom}_{\mathrm{G}_n}\left(V_{\mathrm{G}_n}^{\mu} \,:\, V_{\mathrm{GL}_n}^{\lambda} \right), \tag{5.0.1}$$

where $\mathrm{G}_n = \mathrm{Sp}_n$ or O_n, which is called *branching multiplicity* from GL_n to G_n associated with λ and μ. In [74, 75], Littlewood proved that if $\ell(\lambda) \le \frac{n}{2}$, then

$$\left[V_{\mathrm{GL}_n}^{\lambda} : V_{\mathrm{Sp}_n}^{\mu} \right] = \sum_{\delta \in \mathscr{P}^{(2)}} c_{\delta' \mu}^{\lambda}, \qquad \left[V_{\mathrm{GL}_n}^{\lambda} : V_{\mathrm{O}_n}^{\mu} \right] = \sum_{\delta \in \mathscr{P}^{(2)}} c_{\delta \mu}^{\lambda}, \tag{5.0.2}$$

where $c_{\beta\gamma}^{\alpha}$ is the Littlewood-Richardson coefficient corresponding to partitions α, β, γ, and $\mathscr{P}^{(2)}$ denotes the set of partition with even parts.

In this chapter, we shall give a new combinatorial formula for $\left[\, V_{\mathrm{GL}_n}^{\lambda},\, V_{\mathrm{O}_n}^{\mu} \,\right]$ generalizing the Littlewood's formula above for O_n in full generality. Let us explain briefly the proof idea.

The separation algorithm induces the following embedding:

$$\mathtt{LR}_{\lambda}^{\mu}(\mathfrak{d}) \hookrightarrow \bigsqcup_{\substack{\delta \in \mathscr{P}_n \\ \delta\,:\,\mathrm{even}}} \mathtt{LR}_{\delta \mu^{\pi}}^{\lambda}\ ,$$

where $\mathtt{LR}_{\lambda}^{\mu}(\mathfrak{d})$ is the set of the $\mathfrak{l}$-highest weight vectors with weight λ' in the spinor model of type D_∞, and $\mathtt{LR}_{\beta\gamma}^{\alpha}$ is the set of the companion tableaux of Littlewood-Richardson tableaux associated with α, β, γ. Note that the above embedding is not surjective in general. We characterize the image of the embedding completely. Then, by using the dual pair $(\mathrm{O}_n, \mathfrak{d}_\infty)$ on Fock space [107], $\left[\, V^\lambda_{\mathrm{GL}_n},\, V^\mu_{\mathrm{O}_n} \,\right]$ is equal to the branching multiplicity $\mathfrak{c}^\mu_\lambda(\mathfrak{d})$ from $\mathfrak{d}_\infty$ to $\mathfrak{l}$ associated with μ' and λ'. Here $\mathfrak{d}_\infty$ and $\mathfrak{l}$ are the Kac-Moody algebras of types D_∞ and $A_{+\infty}$ (see [41, §7.11]). Since $\mathfrak{c}^\mu_\lambda(\mathfrak{d}) = |\mathtt{LR}^\mu_\lambda(\mathfrak{d})|$, this gives us a combinatorial formula for $\left[\, V^\lambda_{\mathrm{GL}_n},\, V^\mu_{\mathrm{O}_n} \,\right]$. As an application, we obtain a combinatorial formula of the generalized exponent for types BD following the idea in [71] for type C.

The results in this chapter are based on [37].

5.1 Littlewood-Richardson tableaux

Let us recall the notations in Section 3.1.1. In this subsection, we review some combinatorics related to the Littlewood-Richardson tableaux (LR tableaux, for short) (see [23]).

Let $\mathscr{P}^{(2)} = \{\, \lambda \in \mathscr{P} \,|\, \lambda = (\lambda_i)_{i\geq 1}, \lambda_i \in 2\mathbb{Z}_+ \ (i \geq 1) \,\}$, and let $\mathscr{P}^{(1,1)} = \{\, \lambda' \,|\, \lambda \in \mathscr{P}^{(2)} \,\}$, where λ' is the conjugate of λ. Put $\mathscr{P}^{(2,2)} = \mathscr{P}^{(1,1)} \cap \mathscr{P}^{(2)}$. For $\diamond \in \{(1,1), (2), (2,2)\}$ and $\ell \geq 1$, we put $\mathscr{P}^\diamond_\ell = \mathscr{P}^\diamond \cap \mathscr{P}_\ell$.

For $\lambda, \mu, \nu \in \mathscr{P}$, let $\mathtt{LR}^\lambda_{\mu\nu}$ be the set of Littlewood-Richardson tableaux S of shape λ/μ with content ν. There is a natural bijection from $\mathtt{LR}^\lambda_{\mu\nu}$ to the set of $T \in SST(\nu)$ such that $(H_\mu \leftarrow T) = H_\lambda$, where each i in the jth row of $S \in \mathtt{LR}^\lambda_{\mu\nu}$ corresponds to j in the ith row of T. We call such T a *companion tableau* of $S \in \mathtt{LR}^\lambda_{\mu\nu}$.

We also need the following anti-version of LR tableaux which is used frequently in this chapter.

Definition 5.1.1. We define $\mathtt{LR}^\lambda_{\mu\nu^\pi}$ to be the set of $S \in SST(\lambda/\mu)$ with content ν^π such that $w(T) = w_1 \ldots w_r$ is an *anti-lattice word*, that is, the number of i in $w_k \ldots w_r$ is greater than or equal to that of $i-1$ for each $k \geq 1$ and $1 < i \leq \ell(\nu)$.

Let us call $S \in \mathtt{LR}^\lambda_{\mu\nu^\pi}$ a Littlewood-Richardson tableau of shape λ/μ with content ν^π. As in case of $\mathtt{LR}^\lambda_{\mu\nu}$, the map from $S \in \mathtt{LR}^\lambda_{\mu\nu^\pi}$ to its companion tableau gives a natural bijection from $\mathtt{LR}^\lambda_{\mu\nu^\pi}$ to the set of $T \in SST(\nu^\pi)$ such that $(H_\mu \leftarrow T) = H_\lambda$.

From now on, all the LR tableaux are assumed to be the corresponding companion tableaux unless otherwise specified.

Finally, let us recall a bijection

$$\psi : \mathtt{LR}^{\lambda'}_{\mu'\nu'} \longrightarrow \mathtt{LR}^\lambda_{\mu\nu^\pi}, \tag{5.1.1}$$

which may be viewed as an analogue of Halon-Sundaram's bijection [27] for an anti-dominant content (cf. [23, Appendix A.3], [71, Remark 6.3] and references therein).

Let $S \in \mathtt{LR}^{\lambda'}_{\mu'\nu'}$ be given, that is, $(H_{\mu'} \leftarrow S) = H_{\lambda'}$. Let $S^1, \ldots, S^p$ denote the columns of S enumerated from the right. For $1 \leq i \leq p$, let $H^i = (H^{i-1} \leftarrow S^i)$ with $H^0 = H_{\mu'}$ so that $H^p = H_{\lambda'}$. Define $Q(H_{\mu'} \leftarrow S) \in SST(\lambda/\mu)$ to be the tableau such that the horizontal strip $\mathrm{sh}(H^i)'/\mathrm{sh}(H^{i-1})'$ is filled with $1 \leq i \leq p$.

On the other hand, let $U \in \mathtt{LR}^{\lambda}_{\mu\nu^\pi}$ be given, that is, $\mathrm{sh}(H_\mu \leftarrow U) = H_\lambda$. Let U_i denote the i-th row of U from the top, and let $H_i = (H_{i-1} \leftarrow U_i)$ with $H_0 = H_\mu$ for $1 \leq i \leq p$. Define $Q(H_\mu \leftarrow U)$ to be tableau such that the horizontal strip $\mathrm{sh}(H_i)/\mathrm{sh}(H_{i-1})$ is filled with $1 \leq i \leq p$.

Then for each $S \in \mathtt{LR}^{\lambda'}_{\mu'\nu'}$, there exists a unique $U \in SST(\nu^\pi)$ such that $(H_\mu \leftarrow U) = H_\lambda$ and $Q(H_\mu \leftarrow U) = Q(H_{\mu'} \leftarrow S)$. We define $\psi(S) = U$. Since the correspondence from S to U is reversible, ψ is a bijection from $\mathtt{LR}^{\lambda'}_{\mu'\nu'}$ to $\mathtt{LR}^{\lambda}_{\mu\nu^\pi}$.

Example 5.1.2. Let $\lambda = (7, 6, 4, 3, 2)$, $\mu = (6, 4, 2, 2)$, and $\nu = (2, 2, 2, 1, 1)$. Let $S \in \mathtt{LR}^{\lambda'}_{\mu'\nu'}$ be given by

$$S = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 3 & 5 & 7 \\ \hline 2 & 4 & 6 & \multicolumn{2}{c}{} \\ \cline{1-3} \end{array} \quad .$$

The recording tableau $Q(H_{\mu'} \leftarrow S)$ is given by

$$Q(H_{\mu'} \leftarrow S) = \begin{array}{ccccccc} & & & & & & \boxed{1} \\ & & & & \boxed{2} & \boxed{3} & \\ & & \boxed{3} & \boxed{4} & & & \\ & & \boxed{4} & & & & \\ \boxed{5} & \boxed{5} & & & & & \end{array} \quad .$$

Then the corresponding $U = \psi(S) \in \mathtt{LR}^{\lambda}_{\mu\nu^\pi}$ with $Q(H_\mu \leftarrow U) = Q(H_{\mu'} \leftarrow S)$ is given by

$$U = \begin{array}{cc} & \boxed{1} \\ & \boxed{2} \\ \boxed{2} & \boxed{3} \\ \boxed{3} & \boxed{4} \\ \boxed{5} & \boxed{5} \end{array}$$

5.2 Howe duality on Fock space

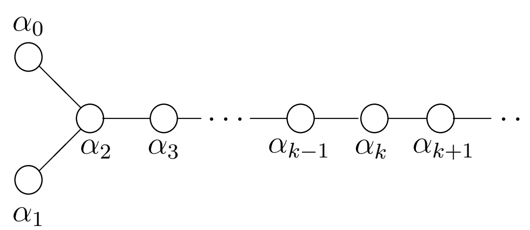
It has been known that a duality result obtained from commuting actions of two algebraic objects on a space is a powerful tool to study their irreducible representations. For example, Schur-Weyl duality for the pair (GL_n, S_k) on the tensor space $\otimes_{i=1}^k \mathbb{C}^n$ provides us a characterization of irreducible representations of GL_n or S_k, their characters, reciprocity laws, and several useful formulas. We refer to [26, Chapter 9] for more details.

There are numerous dualities in the spirit of Schur-Weyl duality. In particular, Howe provided a uniform formulation for several dualities involving Lie groups, Lie (sup)algebras or Weyl-Clifford (super)algebras [32] (see also [14, Chapter 5], [26, Section 5.6] and references therein). In particular, Howe duality on Fock space between classical groups and infinite-dimensional Lie algebras was developed systematically due to Wang [107].

In this section, we review Howe duality for the dual pair $(\mathrm{O}_n, \mathfrak{d}_\infty)$ on Fock space.

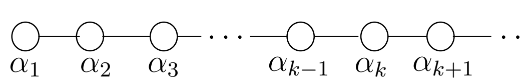
5.2.1 Kac-Moody algebra of type D_∞

In this chapter, we assume that $\mathfrak{g}$ is the Kac-Moody Lie algebra of type D_∞ whose Dynkin diagram, set of simple roots $\Pi = \{\, \alpha_i \,|\, i \in I \,\}$, and fundamental weight Λ_i $(i \in I)$ are given by



$$\Pi = \{\, \alpha_0 = -\epsilon_1 - \epsilon_2,\ \alpha_i = \epsilon_i - \epsilon_{i+1}\ (i \geq 1) \,\},$$
$$\Lambda_i = \begin{cases} \Lambda_0 + \epsilon_1, & \text{if } i = 1, \\ 2\Lambda_0 + \epsilon_1 + \cdots + \epsilon_i, & \text{if } i > 1. \end{cases}$$

Here we assume that the index set for simple roots is $I = \mathbb{Z}_+$, and the weight lattice is $P = \mathbb{Z}\Lambda_0 \oplus \left(\bigoplus_{i \geq 1} \mathbb{Z}\epsilon_i\right)$. In this chapter, we often use the notations $\mathfrak{d}_\infty$ and $\mathfrak{l}_{+\infty}$ when we refer to the corresponding Kac-Moody algebras for type D_∞ and $A_{+\infty}$, respectively or we denote by $\mathfrak{d}$ and $\mathfrak{l}$ simply. Note that $\mathfrak{l}$ is the subalgebra of $\mathfrak{d}$ associated with $\Pi \setminus \{\alpha_0\}$, which is of type $A_{+\infty}$ whose Dynkin diagram is given by



We refer to [41, §7.11] and [14, Section 5.4.1] for more details (see also [82, Section 2]).

Put $\mathfrak{k} = \mathfrak{g}$ or $\mathfrak{l}$. For a dominant integral weight Λ for $\mathfrak{k}$, let $\mathbf{B}(\mathfrak{k}, \Lambda)$ be the $\mathfrak{k}$-crystal associated with an irreducible highest weight $U_q(\mathfrak{k})$-module with highest weight Λ. We denote by $\mathfrak{g}_m$ (resp. $\mathfrak{l}_m$) the subalgebra of $\mathfrak{g}$ whose Dynkin diagram corresponds to $\alpha_0, \cdots, \alpha_{m-1}$ (resp. $\alpha_1, \cdots, \alpha_{m-1}$).

We denote by $\pi_m(\Lambda)$ the dominant integral weight of $\mathfrak{k}_m$ under the canonical projection π_m. Then, the crystal $\mathbf{B}(\mathfrak{k}_m, \pi_m(\Lambda))$ can be obtained from $\mathbf{B}(\mathfrak{k}, \Lambda)$ by restricting the index set I associated with $\mathfrak{k}_m$, which is the crystal associated with a finite-dimensional irreducible highest weight $U_q(\mathfrak{k}_m)$-module with highest weight $\pi_m(\Lambda)$.

Remark 5.2.1. It is parallel with [43] (see also [30, Chapter 5]) to prove the existence and uniqueness of the crystal base for an irreducible highest weight $U_q(\mathfrak{g})$-module in an infinite rank analog of $\mathcal{O}^q_{\rm int}$ (recall Definition 2.1.6), where $\mathfrak{g}$ is of types X_∞ for $X = \mathrm{A}, \mathrm{B}, \mathrm{C}$ or D. Furthermore, following [30, Sections 3.4–3.5], the category is also semisimple tensor category in which the classical limit of irreducible modules is isomorphic to the canonical counterpart over $U(\mathfrak{g})$. We refer to [30, 70] for more details.

5.2.2 Dual pair $(\mathrm{O}_n, \mathfrak{d}_\infty)$ on Fock space

Let us review the duality theorem for dual pair $(\mathrm{O}_n, \mathfrak{d}_\infty)$ following [13, Section 2.3.3]. We refer to [41, Chapter 14], [14, Section 5.4.2, Appendix A.4] and references therein for the exposition on (fermionic) Fock spaces (see also [107, Sections 3–4]).

The Clifford algebra $\widehat{\mathcal{C}}^\ell$ is an algebra generated by $\Psi^{\pm,p}_r$, where $1 \le p \le \ell$ and $r \in \frac{1}{2}+\mathbb{Z}$ with anti-commutation relations are given by

$$\left[\Psi^{+,p}_r,\, \Psi^{-,q}_s\right]_+ = \delta_{p,q}\delta_{r,-s} I\,, \quad \left[\Psi^{+,p}_r,\, \Psi^{+,q}_s\right]_+ = \left[\Psi^{-,p}_r,\, \Psi^{-,q}_s\right]_+ = 0\,,$$

for all $r, s \in \frac{1}{2} + \mathbb{Z}$, $1 \le p,\, q \le \ell$, and let $\mathcal{F}^\ell$ be the fermionic Fock space of ℓ pairs of fermions

$$\Psi^{\pm,p}(z) = \sum_{r \in \frac{1}{2}+\mathbb{Z}} \Psi^{\pm,p}_r z^{-r-\frac{1}{2}}, \quad 1 \le p \le \ell\,.$$

Then $\mathcal{F}^\ell$ is the simple $\widehat{\mathcal{C}}^\ell$ generated by the vacuum vector $|0\rangle$ which satisfies $\Psi^{\pm,p}_r|0\rangle = 0$ for all $r > 0$ and $1 \le p \le \ell$.

Let

$$\phi(z) = \sum_{r\in\frac{1}{2}+\mathbb{Z}} \phi_r z^{-r-\frac{1}{2}}$$

be a neutral fermionic field whose components satisfy the anti-commutation relations:

$$[\phi_r,\, \phi_s]_+ = \delta_{r,-s} I, \quad r, s \in \frac{1}{2} + \mathbb{Z}\,.$$

We denote by $\mathcal{F}^{\frac{1}{2}}$ the Fock space of $\phi(z)$ generated by the vacuum vector $|0\rangle$ that is annihilated by ϕ_r for $r > 0$. Then $\mathcal{F}^{\ell+\frac{1}{2}}$ is the tensor product of $\mathcal{F}^{\ell}$ and $\mathcal{F}^{\frac{1}{2}}$.

Let us consider the dual pair $(\mathrm{O}_n, \mathfrak{d}_\infty)$ for $n \geq 1$. We define

$$\mathcal{P}(\mathrm{O}_n) = \{\, \mu = (\mu_1, \cdots, \mu_n) \,|\, \mu_i \in \mathbb{Z}_+,\ \mu_1 \geq \ldots \geq \mu_n,\ \mu_1' + \mu_2' \leq n \,\},$$

where $\mu' = (\mu_1', \mu_2', \cdots)$ is the conjugate partition of μ. Recall that $\mathcal{P}(\mathrm{O}_n)$ parameterizes the complex finite-dimensional representations of the orthogonal group O_n, see [13, Section 2.3.3] (see also [26, Section 5.5.5]).

We may also use $\mathcal{P}(\mathrm{O}_n)$ to parametrize P_+ for $\mathfrak{g}$. More precisely, for $\mu \in \mathcal{P}(\mathrm{O}_n)$, if we put

$$\Lambda(\mu) = n\Lambda_0 + \mu_1'\epsilon_1 + \mu_2'\epsilon_2 + \cdots\,, \tag{5.2.1}$$

where $\mu' = (\mu_1', \mu_2', \ldots)$ is the conjugate partition of μ, then we have $P_+ = \{\, \Lambda(\mu) \,|\, \mu \in \bigsqcup_n \mathcal{P}(\mathrm{O}_n) \,\}$ the set of dominant integral weights for $\mathfrak{g}$. For $\mu \in \mathcal{P}(\mathrm{O}_n)$, we denote by $V_{\mathrm{O}_n}^{\mu}$ the finite-dimensional irreducible O_n-module. Let $L(\mathfrak{d}_\infty, \Lambda(\mu))$ be the irreducible highest weight $\mathfrak{d}_\infty$-module.

It is known that there exists a commuting action of $\mathrm{O}_n \times \mathfrak{d}_\infty$ on $\mathcal{F}^{\frac{n}{2}}$, see [14, Lemmas 5.48–5.49] for more details. Thus we have the following duality theorem.

Theorem 5.2.2. [107, Theorems 3.2, 4.1] *As an* $(\mathrm{O}_n, \mathfrak{d}_\infty)$*-module, we have*

$$\mathcal{F}^{\frac{n}{2}} \cong \bigoplus_{\mu\in\mathcal{P}(\mathrm{O}_n)} L(\mathfrak{d}_\infty, \Lambda(\mu)) \otimes V_{\mathrm{O}_n}^{\mu}\,.$$

5.3 Separation on $\mathfrak{l}$-highest weight vectors

In this section, we revisit the spinor model over $U_q(\mathfrak{d}_\infty)$ and describe explicitly the behavior of the separation algorithm developed in Section 4.3 on $\mathfrak{l}$-highest weight vectors.

5.3.1 Revisit of spinor model over $U_q(\mathbf{D}_\infty)$

We review the spinor model over the infinite rank Lie algebra of type D_∞. This is almost identical with the one in Section 4.1.3, but we should not confuse the notations and the set of letters. Here we use $\mathbb{N}$ as the set of letters. For example, see Example 5.3.2.

By abuse of notation, we also use the same notations

$$\begin{aligned}
&\mathbf{T}(a) = \{\, T \,|\, T \in SST(\lambda(a,b,c)),\ b,c \in 2\mathbb{Z}_+,\ \mathfrak{r}_T \leq 1 \,\} \quad (a \in \mathbb{Z}_+),\\
&\overline{\mathbf{T}}(0) = \bigsqcup_{b,c\,\in 2\mathbb{Z}_+} SST(\lambda(0,b,c+1)), \qquad \mathbf{T}^{\mathrm{sp}} = \bigsqcup_{a\in\mathbb{Z}_+} SST((1^a)),\\
&\mathbf{T}^{\mathrm{sp}+} = \{\, T \,|\, T \in \mathbf{T}^{\mathrm{sp}},\ \mathfrak{r}_T = 0 \,\}, \quad \mathbf{T}^{\mathrm{sp}-} = \{\, T \,|\, T \in \mathbf{T}^{\mathrm{sp}},\ \mathfrak{r}_T = 1 \,\}.
\end{aligned}$$

Let $\mathbf{B}$ be one of $\mathbf{T}(a)$ $(a \in \mathbb{Z}_+)$, $\mathbf{T}^{\mathrm{sp}}$, and $\overline{\mathbf{T}}(0)$. Let us describe the $\mathfrak{g}$-crystal structure on $\mathbf{B}$. Let $T \in \mathbf{B}$ given. Recall that $SST(\lambda)$ $(\lambda \in \mathscr{P})$ has an $\mathfrak{l}$-crystal structure [50]. So we may regard $\mathbf{B}$ as a subcrystal of an $\mathfrak{l}$-crystal $\bigsqcup_{\lambda\in\mathscr{P}} SST(\lambda)$ and hence define $\widetilde{e}_i T$ and $\widetilde{f}_i T$ for $i \in I \setminus \{0\}$. Let $\mathrm{wt}_{\mathfrak{l}}(T) = \sum_{i\geq 1} m_i \epsilon_i$ be the $\mathfrak{l}$-weight of T, where m_i is the number of occurrences of i in T. Next, we define $\widetilde{e}_0 T$ and $\widetilde{f}_0 T$ as follows:

(1) When $\mathbf{B} = \mathbf{T}^{\mathrm{sp}}$, we define $\widetilde{e}_0$ to be the tableau obtained from T by removing a domino $\begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix}$ if T has $\begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix}$ on its top, and $\mathbf{0}$ otherwise. We define $\widetilde{f}_0 T$ in a similar way by adding $\begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix}$.

(2) When $\mathbf{B} = \mathbf{T}(a)$ or $\overline{\mathbf{T}}(0)$, we define $\widetilde{e}_0 T = \widetilde{e}_0 \left(T^{\mathtt{R}} \otimes T^{\mathtt{L}}\right)$ regarding $\mathbf{B} \subset \left(\mathbf{T}^{\mathrm{sp}}\right)^{\otimes 2}$ by tentor product rule (2.2.1). We define $\widetilde{f}_0 T$ similarly.

Put

$$\begin{aligned}
&\mathrm{wt}(T) = \begin{cases} 2\Lambda_0 + \mathrm{wt}_{\mathfrak{l}}(T), & \text{if } T \in \mathbf{T}(a) \text{ or } \overline{\mathbf{T}}(0), \\ \Lambda_0 + \mathrm{wt}_{\mathfrak{l}}(T), & \text{if } T \in \mathbf{T}^{\mathrm{sp}}. \end{cases},\\
&\varepsilon_i(T) = \max\{\, k \,|\, \widetilde{e}_i^{\,k} T \neq \mathbf{0} \,\} \quad \varphi_i(T) = \max\{\, k \,|\, \widetilde{f}_i^{\,k}(T) \neq \mathbf{0} \,\}.
\end{aligned}$$

Then $\mathbf{B}$ is a $\mathfrak{g}$-crystal with respect to $\widetilde{e}_i$ and $\widetilde{f}_i$, ε_i, and φ_i for $i \in I$. By [64, Proposition

4.2], we have

$$
\begin{aligned}
&\mathbf{T}(a) \cong \mathbf{B}(\Lambda_a) \quad (a \geq 2),\\
&\mathbf{T}(0) \cong \mathbf{B}(2\Lambda_0), \quad \overline{\mathbf{T}}(0) \cong \mathbf{B}(2\Lambda_1), \quad \mathbf{T}(1) \cong \mathbf{B}(\Lambda_0 + \Lambda_1),\\
&\mathbf{T}^{\mathrm{sp}-} \cong \mathbf{B}(\Lambda_1), \quad \mathbf{T}^{\mathrm{sp}+} \cong \mathbf{B}(\Lambda_0).
\end{aligned}
$$

For $\mu \in \mathcal{P}(\mathrm{O}_n)$, let $q_\pm$ and $r_\pm$ be non-negative integers such that

$$
\begin{cases}
n - 2\mu'_1 = 2q_+ + r_+, & \text{if } n - 2\mu'_1 \geq 0,\\
2\mu'_1 - n = 2q_- + r_-, & \text{if } n - 2\mu'_1 < 0,
\end{cases}
$$

where $r_\pm = 0, 1$. Let $\overline{\mu} = (\overline{\mu}_i) \in \mathscr{P}$ be such that $\overline{\mu}'_1 = n - \mu'_1$ and $\overline{\mu}'_i = \mu'_i$ for $i \geq 2$ and let $M_+ = \mu'_1$ and $M_- = \overline{\mu}'_1$. Put

$$
\widehat{\mathbf{T}}(\mu, n) = \begin{cases}
\mathbf{T}(\mu_1) \times \cdots \times \mathbf{T}(\mu_{M_+}) \times \mathbf{T}(0)^{\times q_+} \times (\mathbf{T}^{\mathrm{sp}+})^{\times r_+}, & \text{if } n - 2\mu'_1 \geq 0,\\
\mathbf{T}(\overline{\mu}_1) \times \cdots \times \mathbf{T}(\overline{\mu}_{M_-}) \times \overline{\mathbf{T}}(0)^{\times q_-} \times (\mathbf{T}^{\mathrm{sp}-})^{\times r_-}, & \text{if } n - 2\mu'_1 < 0.
\end{cases} \tag{5.3.1}
$$

Define

$$
\mathbf{T}(\mu, n) = \{\, \mathbf{T} = (\ldots, T_2, T_1) \in \widehat{\mathbf{T}}(\mu, n) \,|\, T_{i+1} \prec T_i \ (i \geq 1) \,\}.
$$

When considering $\mathbf{T} = (\ldots, T_2, T_1) \in \widehat{\mathbf{T}}(\mu, n)$, it is often helpful to imagine that $T_1, T_2, \ldots$ are arranged from right to left on a plane, where the horizontal line L separates $T_i^{\mathtt{body}}$ and $T_i^{\mathtt{tail}}$ simultaneously.

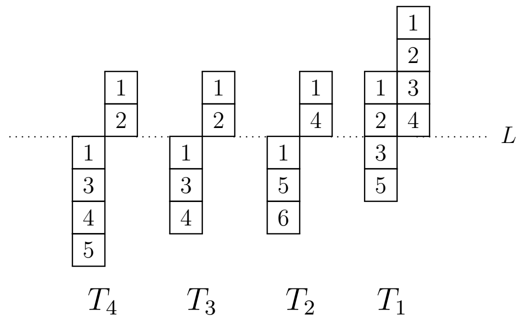
We regard $\widehat{\mathbf{T}}(\mu, n)$ as a $\mathfrak{g}$-crystal by identifying $\mathbf{T} = (\ldots, T_2, T_1) \in \widehat{\mathbf{T}}(\mu, n)$ with $T_1 \otimes T_2 \otimes \ldots$, and regard $\mathbf{T}(\mu, n)$ as its subcrystal. Then we have the following.

Theorem 5.3.1. [64, Theorem 4.3–4.4] *For $\mu \in \mathcal{P}(\mathrm{O}_n)$, $\mathbf{T}(\mu, n)$ is a connected crystal with highest weight $\Lambda(\mu)$. Furthermore, we have*

$$
\mathbf{T}(\mu, n) \cong \mathbf{B}(\Lambda(\mu)).
$$

We call $\mathbf{T}(\mu, n)$ the *spinor model for* $\mathbf{B}(\Lambda(\mu))$ in type D_∞.

Example 5.3.2. Let $n = 8$ and $\mu = (4, 3, 3, 2) \in \mathcal{P}(\mathrm{O}_8)$ and let $\mathbf{T} = (T_4, T_3, T_2, T_1)$ given by



Then we can check that $T_4 \prec T_3 \prec T_2 \prec T_1$. Thus, $\mathbf{T} \in \mathbf{T}(\mu, 8)$. where the dotted line denotes the common horizontal line L.

From now on, we fix $\mu \in \mathcal{P}(\mathrm{O}_n)$ throughout this chapter.

Definition 5.3.3. Let

$$\mathbf{H}(\mu, n) = \{\, \mathbf{T} \mid \mathbf{T} \in \mathbf{T}(\mu, n),\ \widetilde{e}_i \mathbf{T} = \mathbf{0}\ (i \neq 0)\,\},$$

and call $\mathbf{T} \in \mathbf{H}(\mu, n)$ an $\mathfrak{l}$-*highest weight vector*. Let $\lambda \in \mathscr{P}_n$ and $\mu \in \mathcal{P}(\mathrm{O}_n)$ are given. We define

$$\mathtt{LR}^{\mu}_{\lambda}(\mathfrak{d}) = \{\, \mathbf{T}\ \mid \mathbf{T} \in \mathbf{H}(\mu, n),\ \mathbf{T} \equiv_{\mathfrak{l}} H_{\lambda'}\,\}\,, \tag{5.3.2}$$

and write $c^{\mu}_{\lambda}(\mathfrak{d}) = |\mathtt{LR}^{\mu}_{\lambda}(\mathfrak{d})|$.

Note that $c^{\mu}_{\lambda}(\mathfrak{d})$ is equal to the multiplicity of irreducible highest weight $\mathfrak{l}$-module with highest weight $\sum_{i\geq 1} \lambda'_i \epsilon_i$ in the irreducible highest weight $\mathfrak{g}$-module with highest weight $\Lambda(\mu)$ (recall Remark 5.2.1).

Theorem 5.3.4. [67, Theorem 5.3] *For $\lambda \in \mathscr{P}_n$ and $\mu \in \mathcal{P}(\mathrm{O}_n)$, we have*

$$\left[\, V^{\lambda}_{\mathrm{GL}_n}, V^{\mu}_{\mathrm{G}_n} \,\right] = c^{\mu}_{\lambda}(\mathfrak{d})\,.$$

Sketch of proof. We outline the proof of [67, Theorem 5.3].

In the proof of [67, Theorem 5.3], the author constructs an explicit actions of GL_n on $\mathcal{F}^{\frac{n}{2}}$ so that its restriction to O_n coincides with the action of O_n in Theorem 5.2.2.

Following [20, 107], there exists a commuting action of $\mathfrak{l} \times \mathrm{GL}_n$ on $\mathcal{F}^{\frac{n}{2}}$ so that

$$\mathcal{F}^{\frac{n}{2}} \cong \bigoplus_{\lambda \in \mathscr{P}} L(\mathfrak{l}, \lambda') \otimes V^{\lambda}_{\mathrm{GL}_n}. \tag{5.3.3}$$

By combining Theorem 5.2.2 and (5.3.3), we complete the proof (cf. [107, Section 7.1] for reciprocity laws). □

Remark 5.3.5. Let $\mathfrak{a}_\infty$ be the Lie algebra of type A_∞ consisting of all matrices $(c_{ij})_{i,j\mathbb{Z}}$ of infinite size with finitely many non-zero entries. In [20, 107], the authors consider the dual pair $(\mathrm{GL}_n, \mathfrak{a}_\infty)$ on $\mathcal{F}^n$ and obtain the decomposition of $\mathcal{F}^n$ as $(\mathrm{GL}_n, \mathfrak{a}_\infty)$-module by computing the joint $(\mathrm{GL}_n, \mathfrak{a}_\infty)$-highest weight vectors explicitly. Since $\mathfrak{l} = \mathfrak{l}_{+\infty}$ is a subalgebra of $\mathfrak{a}_\infty$, one can check that the arguments for the pair $(\mathrm{GL}_n, \mathfrak{a}_\infty)$ also hold for the pair $(\mathrm{GL}_n, \mathfrak{l}_{+\infty})$ on $\mathcal{F}^{\frac{n}{2}}$ (see [14, Section 5.4.3] or [107, Section 3]).

5.3.2 $\mathfrak{l}$-highest weight vectors

The goal of this subsection is to give some necessary conditions for $\mathbf{T} \in \mathbf{T}(\mu, n)$ to be in $\mathbf{H}(\mu, n)$. Note that for $\mathbf{T} \in \mathbf{T}(\mu, n)$, we have $\mathbf{T} \in \mathbf{H}(\mu, n)$ if and only if $\mathbf{T} \equiv_{\mathfrak{l}} H_\lambda$ for some $\lambda \in \mathscr{P}$. Hence $\mathbf{H}(\mu, n)$ parametrizes the connected $\mathfrak{l}$-crystals in $\mathbf{T}(\mu, n)$.

First, we consider the case $n - 2\mu'_1 \geq 0$. Suppose that $n = 2l + r$, where $l \geq 1$ and $r = 0, 1$. Let $\mathbf{T} \in \mathbf{T}(\mu, n)$ be given and write

$$\mathbf{T} = (T_l, \ldots, T_1, T_0), \tag{5.3.4}$$

where $T_i \in \mathbf{T}(a_i)$ for some $a_i \in \mathbb{Z}_+$ $(1 \leq i \leq l)$, and $T_0 \in \mathbf{T}^{\mathrm{sp}+}$ (resp. $T_0 = \emptyset$) when $r = 1$ (resp. $r = 0$). Let $\mathrm{sh}(T_i) = \lambda(a_i, b_i, c_i)$ and $\mathfrak{r}_{T_i} = \mathfrak{r}_i$ for $1 \leq i \leq l$.

The lemma below follows directly from the tensor product rule in Definition 2.2.3 (cf. Corollary 2.1.16).

Lemma 5.3.6. *Put $U_0 = T_0$, $U_{2k-1} = T^{\mathtt{R}}_k$ and $U_{2k} = T^{\mathtt{L}}_k$ for $1 \leq k \leq l$. Then $\mathbf{T}$ is an $\mathfrak{l}$-highest weight element if and only if $(U_i, \ldots, U_0)$ is a $\mathfrak{l}$-highest weight element for $i \geq 0$, where we understand $(U_i, \ldots, U_0) = U_0 \otimes \cdots \otimes U_i$ as an element of an $\mathfrak{l}$-crystal.* □

Definition 5.3.7. Let $\mathbf{H}^\circ(\mu, n)$ be a subset of $\mathbf{T}(\mu, n)$ consisting of $\mathbf{T} = (T_i)$ satisfying the following conditions: for each $i \geq 1$,

(H0) $T_0[k] = k$ for $k \geq 1$,

(H1) $T_i^{\mathtt{L}}$ and $T_i^{\mathtt{R}}$ are of the form

$$\begin{aligned}
T_i^{\mathtt{R}} &= \big(1,2,\ldots,b_i+c_i-1,T_i^{\mathtt{R}}(1)\big) \boxplus \emptyset,\\
T_i^{\mathtt{L}} &= (1,2,\ldots,c_i-1,c_i) \boxplus \big(T_i^{\mathtt{L}}(a_i),\ldots,T_i^{\mathtt{L}}(1)\big)\,,
\end{aligned}$$

(H2) the entries $T_i^{\mathtt{R}}(1)$ and $T_i^{\mathtt{L}}(a_i)$ satisfy

$$\begin{aligned}
&\text{(i)} \ \ \text{if } \mathfrak{r}_i = 0, \text{then } T_i^R(1) = b_i + c_i,\\
&\text{(ii)} \ \ \text{if } \mathfrak{r}_i = 1, \text{ then} \begin{cases} T_i^{\mathtt{R}}(1) = b_i + c_i \text{ or } T_i^{\mathtt{R}}(1) \geq c_{i-1} + 1 + \mathfrak{r}_{i-1},\\ T_i^{\mathtt{L}}(a_i) = c_i + 1.\end{cases}
\end{aligned}$$

Here we assume that $c_0 = \infty$ and $\mathfrak{r}_0 = 0$.

Let us recall that $0 \leq \mathfrak{r}_T \leq 1$ for $T \in \mathbf{T}(\mu, n)$ (see Definition 4.1.2 for definition of $\mathfrak{r}_T$).

Lemma 5.3.8. *For $\mathbf{T} \in \mathbf{H}^{\circ}(\mu, n)$, we have either $T_{i+1}^{R}(1) < T_i^{L}(a_i)$ or $T_{i+1}^{R}(1) > T_i^{L}(a_i)$ for each i. Furthermore, $T_{i+1}^{R}(1) > T_i^{L}(a_i)$ implies $\mathfrak{r}_i\mathfrak{r}_{i+1} = 1$, and $\mathfrak{r}_i\mathfrak{r}_{i+1} = 0$ implies $T_{i+1}^{R}(1) < T_i^{L}(a_i)$.*

Proof. Let $i = 1, \ldots, l-1$ be given. If $\mathfrak{r}_i\mathfrak{r}_{i+1} = 0$, then by Definition 4.1.5(1)-(i) and (H1), we have $T_{i+1}^{\mathtt{R}}(1) = b_{i+1} + c_{i+1} \leq c_i < T_i^{\mathtt{L}}(a_i)$.

Suppose that $\mathfrak{r}_i\mathfrak{r}_{i+1} = 1$. By (H2)(ii), we have $T_i^{\mathtt{L}}(a_i) = c_i + 1$, and $T_{i+1}^{\mathtt{R}}(1) = b_{i+1} + c_{i+1}$ or $\geq c_i + 2$. If $T_{i+1}^{\mathtt{R}}(1) \geq c_i + 2$, then it is clear that $T_{i+1}^{\mathtt{R}}(1) > T_i^{\mathtt{L}}(a_i)$. So we assume that $T_{i+1}^{\mathtt{R}}(1) = b_{i+1} + c_{i+1}$. Note that $T_{i+1}^{\mathtt{R}}(1) = b_{i+1} + c_{i+1} \leq c_i + 2$ by Definition 4.1.5(1)-(i). If $b_{i+1} + c_{i+1} = c_i + 2$, then $T_{i+1}^{\mathtt{R}}(1) > T_i^{\mathtt{L}}(a_i)$. If $b_{i+1} + c_{i+1} < c_i + 2$, then $b_{i+1} + c_{i+1} \leq c_i$ since both $b_{i+1} + c_{i+1}$ and c_i are even. So $T_{i+1}^{\mathtt{R}}(1) < T_i^{\mathtt{L}}(a_i)$.

Finally, suppose that $T_{i+1}^{\mathtt{R}}(1) > T_i^{\mathtt{L}}(a_i)$. If $\mathfrak{r}_i\mathfrak{r}_{i+1} = 0$, then by Definition 4.1.5(1)-(ii) we have $T_{i+1}^{\mathtt{R}}(1) \leq T_i^{\mathtt{L}}(a_i + 1) < T_i^{\mathtt{L}}(a_i)$ which is a contradiction. This proves the lemma. $\square$

Now we verify that the $\mathfrak{l}$-highest weight elements satisfy Definition 5.3.7(H0)–(H2). In particular, the admissibility in Definition 4.1.5 implies the condition (H2).

Proposition 5.3.9. *We have*

$$\mathbf{H}(\mu, n) \subset \mathbf{H}^{\circ}(\mu, n).$$

Proof. Suppose that $\mathbf{T} \in \mathbf{H}(\mu, n)$. By Lemma 5.3.6, it is clear that T_0 satisfies (H0). Let $w(T_0)w(T_1)\dots w(T_l) = w_1 w_2 \dots w_\ell$, and

$$P_l = ((((w_1 \leftarrow w_2) \leftarrow w_3) \leftarrow \cdots) \leftarrow w_\ell). \tag{5.3.5}$$

By definition of $\mathbf{H}(\mu, n)$, we have $P_l = H_\nu$ for some $\nu \in \mathscr{P}$. Let h_l be the height of the rightmost column of ν.

Let us use induction on l to show that $\mathbf{T} \in \mathbf{H}^\circ(\mu, n)$. We also claim that

$$h_l = c_l + \mathfrak{r}_l. \tag{5.3.6}$$

Suppose that $l = 1$. Since $\mathbf{T}$ is an $\mathfrak{l}$-highest weight element and $\mathbf{T} \equiv_{\mathfrak{l}} \mathbf{T}^{\mathtt{R}} \otimes \mathbf{T}^{\mathtt{L}}$ by Lemma 5.3.6, it is straightforward to check that $\mathbf{T}$ satisfies (H1) and (H2). It is clear that $c_1 = h_1 + \mathfrak{r}_1$.

Suppose that $l > 1$. Let $\mathbf{T}_{l-1} = (T_{l-1}, \dots, T_1, T_0)$ and let P_{l-1} be the tableau in (5.3.5) corresponding to $\mathbf{T}_{l-1}$. By induction hypothesis, $\mathbf{T}_{l-1}$ satisfies (H1) and (H2). Put

$$P_l^\flat = (P_{l-1} \leftarrow w(T_l^{\mathtt{R}})).$$

Then $P_l^\flat = H_\eta$ for some $\eta \in \mathscr{P}$ by Lemma 5.3.6, and $P_l = (P_l^\flat \leftarrow w(T_l^{\mathtt{L}}))$. We consider two cases as follows.

Case 1. Suppose that $\mathfrak{r}_l = 0$. Note that by Definition 4.1.5(1)-(i), we have $b_l + c_l \leq c_{l-1}$. Also, by Definition 4.1.5(1)-(ii), $T_l^{\mathtt{R}}(i) \leq {}^{\mathtt{L}}T_{l-1}(i)$ for $1 \leq i \leq b_l + c_l$. By (H1) on $\mathbf{T}_{l-1}$, we have ${}^{\mathtt{L}}T_{l-1}[k] = k$ for $1 \leq k \leq \overline{c}_{l-1}$, where $\overline{c}_{l-1} = c_{l-1} + \mathfrak{r}_{l-1}$. Hence

$$T_l^{\mathtt{R}}(i) \leq c_{l-1} - i + \mathfrak{r}_{l-1} + 1, \tag{5.3.7}$$

for $1 \leq i \leq b_l + c_l$. Then (5.3.7) implies that each letter of $w(T_l^{\mathtt{R}})$ is inserted to create a box to the right of the leftmost column of P_{l-1} when we consider the insertion $(P_{l-1} \leftarrow w(T_l^{\mathtt{R}}))$. Since $P_l^\flat = H_\eta$, we have $T_l^{\mathtt{R}}[k] = k$ for $1 \leq k \leq b_l + c_l$.

By semistandardness of $T_l^{\mathtt{body}}$, we have

$$(T_l^{\mathtt{L}})^{\mathtt{body}}(i) \leq T_l^{\mathtt{R}}(i),$$

for $i \geq 1$. This implies that each letter of $w((T_l^{\mathtt{L}})^{\mathtt{body}})$ is inserted to create a box to the right of the leftmost column of $P_l^\flat$ when we consider the insertion $(P_l^\flat \leftarrow w((T_l^{\mathtt{L}})^{\mathtt{body}}))$, and $T_l^L[k] = k$ for $1 \leq k \leq c_l$. Hence $\mathbf{T}$ satisfies (H1), (H2), and (5.3.6).

Case 2. Suppose that $\mathfrak{r}_l = 1$. When $\mathfrak{r}_{l-1} = 0$, we see that T_l satisfies the conditions (H1) and (H2) by the same argument in the previous case. In particular, (5.3.7) implies that $T_l^{\mathtt{R}}(1) = b_l + c_l$. Since $T_l^{\mathtt{L}}(a_l) \leq T_l^{\mathtt{R}}(1)$ and $P_l^\flat = H_\eta$, we also have $T_l^{\mathtt{L}}(a_l) = c_l + 1$ and (5.3.6).

Now assume that $\mathfrak{r}_{l-1} = 1$. When $\mathfrak{r}_l\mathfrak{r}_{l-1} = 1$, we need to consider the $*$-pair $(T_l^{\mathtt{L}*}, T_l^{\mathtt{R}*})$ of T_l in (4.1.2) (recall Definition 4.1.5). Then, by Definition 4.1.5(1)-(ii) and the condition (H1) on $\mathbf{T}_{l-1}$, we have

$$T_l^{\mathtt{R}*}(i) \leq {}^{\mathtt{L}}T_{l-1}(i) = c_{l-1} - i + 2.$$

We claim that $T_l^{\mathtt{R}}[k] = k$ for $1 \leq k \leq b_l + c_l - 1$. Let k be such that $T_l^{\mathtt{R}*}(i) = T_l^{\mathtt{R}}(i)$ for $1 \leq i \leq k-1$, and $T_l^{\mathtt{R}*}(i) = T_l^{\mathtt{R}}(i+1)$ for $i \geq k$. Since $(\mathcal{F}T_l, T_{l-1}, \ldots, T_1) \equiv_{\mathfrak{l}} (T_l, T_{l-1}, \ldots, T_1)$, which is an $\mathfrak{l}$-highest weight element, we see from Lemma 5.3.6 that each letter of $w(T_l^{\mathtt{R}*})$ is inserted to create a box to the right of the leftmost column of P_{l-1} when we consider the insertion $(P_{l-1} \leftarrow w(T_l^{\mathtt{R}*}))$, and $T_l^{R*}[i] = i$ for $1 \leq i \leq b_l + c_l - 1$. This implies that $T_l^R(k)$ is between m and $m+1$ for some $m \in \mathbb{Z}_+$, and hence $k = b_l + c_l$ since $(P_{l-1} \leftarrow w(T_l^{\mathtt{R}*}))$ is an $\mathfrak{l}$-highest weight element. This proves the claim, and $T_l^{\mathtt{R}}$ satisfies (H1). Furthermore, the claim implies that $T_l^{\mathtt{R}}(1)$ satisfies (H2)(ii) because $P_l^\flat$ is an $\mathfrak{l}$-highest weight element.

We consider $T_l^{\mathtt{L}}$. By the same argument as in *Case 1*, we have $T_l^{\mathtt{L}}[j] = j$ for $1 \leq j \leq c_l$, and $k = 1$ (in the previous argument) implies that $T_l^{\mathtt{L}}(a_l) \leq b_l + c_l - 1$. Therefore, we have $T_l^{\mathtt{L}}(a_l) = c_l + 1$ since the tableau obtained by $((P_l^\flat \leftarrow w(T_l^{\mathtt{body}})) \leftarrow T_l^{\mathtt{L}}(a_l))$ is an $\mathfrak{l}$-highest weight element. Finally, we can check easily that (5.3.6) holds. $\square$

Now, we consider the case $n - 2\mu_1' < 0$. Recall that $\overline{\mu} = (\overline{\mu}_i) \in \mathscr{P}$ be such that $(\overline{\mu}')_1 = n - \mu_1'$ and $(\overline{\mu}')_i = \mu_i'$ for $i \geq 2$.

Let $\mathbf{T} \in \mathbf{T}(\mu, n)$ be given. Suppose that $n = 2l + r$, where $l \geq 1$ and $r = 0, 1$. By (5.3.1), we have

$$\mathbf{T} = (T_l, \ldots, T_{m+1}, T_m, \ldots, T_1, T_0), \tag{5.3.8}$$

where $T_i \in \mathbf{T}(a_i)$ for some $a_i \in \mathbb{Z}_+$ $(m+1 \leq i \leq l)$, $T_i \in \overline{\mathbf{T}}(0)$ $(1 \leq i \leq m)$, and $T_0 \in \mathbf{T}^{\mathrm{sp}-}$ (resp. $T_0 = \emptyset$) when $r = 1$ (resp. $r = 0$). Here $m = q_-$ in (5.3.1). Under (4.3.1), we identify $\mathbf{T}$ with

$$\mathbf{U} = (U_{2l}, \ldots, U_{2m+1}, U_{2m}, \ldots, U_1, U_0).$$

We may also assume that $U_i \in \mathbf{T}^{\mathrm{sp}-}$ for $0 \leq i \leq 2m$. The following is an analogue of

Definition 5.3.7 when $n - 2\mu_1' < 0$.

Definition 5.3.10. Let $\mathbf{H}^\circ(\mu, n)$ be the set of $\mathbf{T} \in \mathbf{T}(\mu, n)$ such that

(1) $U_i[k] = k$ $(k \geq 1)$ for $0 \leq i \leq 2m$,

(2) T_i satisfies (H1) and (H2) in Definition 5.3.7 for $m + 1 \leq i \leq l$.

Proposition 5.3.11. *We have* $\mathbf{H}(\mu, n) \subset \mathbf{H}^\circ(\mu, n)$.

Proof. Let $\mathbf{T} \in \mathbf{H}(\mu, n)$ be given. By Lemma 5.3.6 and the admissibility of $T_{i+1} \prec T_i$ for $0 \leq i \leq m - 1$, we have $U_i[k] = k$ $(k \geq 1)$ for $0 \leq i \leq 2m$. Hence $\mathbf{T}$ satisfies (1). The condition (2) can be verified by almost the same argument as in Proposition 5.3.9. $\square$

Example 5.3.12. Let $\mathbf{T} = (T_2, T_1) \in \mathbf{T}(2, 2)$ with $\mathfrak{r}_1 = \mathfrak{r}_2 = 1$ given by

$$
\begin{array}{cccccc}
 & & & & & \boxed{1} \\
 & & & & & \boxed{2} \\
 & \boxed{1} & & & \boxed{1} & \boxed{3} \\
 & \boxed{4} & & & \boxed{2} & \boxed{4} \\
\boxed{1} & & & & \boxed{3} & \\
\boxed{5} & & & & \boxed{5} & \\
 & & & & & \\
T_2^{\mathtt{L}} & T_2^{\mathtt{R}} & & & T_1^{\mathtt{L}} & T_1^{\mathtt{R}}
\end{array}
$$

We have $w(T_1)w(T_2) = (12341235)(1415)$ and the corresponding tableau (5.3.5) is

$$
\begin{array}{|c|c|c|c|}
\hline
1 & 1 & 1 & 1 \\
\hline
2 & 2 \\
\cline{1-2}
3 & 3 \\
\cline{1-2}
4 & 4 \\
\cline{1-2}
5 & 5 \\
\cline{1-2}
\end{array}
$$

Thus $\mathbf{T}$ is an $\mathfrak{l}$-highest weight vector.

Remark 5.3.13. In [35], when n is odd, the author characterizes completely the $\mathfrak{l}$-highest weight vectors, see [35, Theorem 3.11]. On the other hand, in this thesis, it is enough to consider the necessary conditions for $\mathbf{T} \in \mathbf{T}(\mu, n)$ to be in $\mathbf{H}(\mu, n)$ (without a condition on n).

Let us recall the pairing $\triangleleft$ in Definition 4.1.9. The description of $\triangleleft$ is simple on $\mathbf{H}^\circ(\mu, n)$ as follows.

Proposition 5.3.14. *Let $\mathbf{T} \in \mathbf{H}^{\circ}(\mu, n)$ be given and write*

$$\mathbf{T} = (\,\dots, T_{i+1},\, T_i,\, \dots).$$

Then $T_{i+1} \triangleleft T_i$ if and only if $T^{R}_{i+1}(1) < T^{L}_{i}(a_i)$. Furthermore, $T_{i+1} \not\triangleleft T_i$ if and only if $T^{R}_{i+1}(1) > T^{L}_{i}(a_i)$.

Proof. It follows directly from Lemma 5.3.8 and Proposition 5.3.9 (with Definition 4.1.5). □

5.3.3 Sliding on $\mathfrak{l}$-highest weight vectors

Let us recall Sections 4.1.1 and 4.3.1. Let $\mathbf{T} \in \mathbf{H}^{\circ}(\mu, n)$ be given. If we regard the operator $\mathcal{S}_j$ on $\mathbf{H}^{\circ}(\mu, n)$, then by Proposition 5.3.14, we can write $\mathcal{S}_j$ by

$$\mathcal{S}_j = \begin{cases} \mathcal{F}_j^{a_i} & \text{if } U_{j+1}(1) < U_j(a_i), \\ \mathcal{E}_j \mathcal{E}_{j-1} \mathcal{F}_j^{a_i - 1} \mathcal{F}_{j-1} & \text{if } U_{j+1}(1) > U_j(a_i). \end{cases}$$

Here we use the identification as in Section 4.3.1. Then it works simply on $\mathbf{H}^{\circ}(\mu, n)$ as follows (see Example 5.3.18).

Lemma 5.3.15. *Under the above hypothesis, we have*

$$\mathcal{S}_j \mathbf{U} = \Big(\dots, U_{j+2}, \widetilde{U}_{j+1}, \widetilde{U}_j, U_{j-1}, \dots\Big),$$

where

(i) *if $U_{j+1}(1) < U_j(a_i)$, then*

$$\widetilde{U}_{j+1} = U_{j+1} \boxplus U_j^{\texttt{tail}}, \qquad \widetilde{U}_j = U_j \boxminus U_j^{\texttt{tail}},$$

(ii) *if $U_{j+1}(1) > U_j(a_i)$, then*

$$\begin{aligned} \widetilde{U}_{j+1} &= (U_{j+1}(b_i + c_i), \dots, U_{j+1}(3)) \boxplus (U_{j+1}(2),\, U_j(a_i - 1), \dots, U_j(1)), \\ \widetilde{U}_j &= (U_j(a_i + c_i), \dots, U_j(a_i), U_{j+1}(1)) \boxplus \emptyset. \end{aligned}$$

Proof. **(i)** : Suppose that $U_{j+1}(1) < U_j(a_i)$. Then we have

$$\begin{aligned}\mathcal{S}_j\mathbf{U} &= (\dots, \mathcal{F}^{a_i}(U_{j+1}, U_i), \dots) \\ &= (\dots, U_{j+1} \boxplus U_j^{\mathtt{tail}}, U_j \boxminus U_j^{\mathtt{tail}}, \dots),\end{aligned}$$

which is given by cutting $U_j^{\mathtt{tail}}$ and then putting it below U_{j+1}.

(ii) : Suppose that $U_{j+1}(1) > U_j(a_i)$. By Lemma 5.3.8, we have $\mathfrak{r}_i\mathfrak{r}_{i+1} = 1$. Then we have $\mathcal{S}_j\mathbf{U} = \mathcal{E}_j\mathcal{E}_{j-1}\mathcal{F}_j^{a_i-1}\mathcal{F}_{j-1}\mathbf{U}$. Ignoring the components other than (T_{i+1}, T_i), we have

$$\begin{aligned}&\mathcal{E}_j\mathcal{E}_{j-1}\mathcal{F}_j^{a_i-1}\mathcal{F}_{j-1}\,(T_{i+1}, T_i) \\ &\qquad=\mathcal{E}_j\mathcal{E}_{j-1}\mathcal{F}_j^{a_i-1}\,(U_{j+2}, U_{j+1}, \mathcal{F}(U_j, U_{j-1})) \\ &\qquad=\mathcal{E}_j\mathcal{E}_{j-1}\mathcal{F}_j^{a_i-1}\left(U_{j+2}, U_{j+1}, U_j^*, U_{j-1}^*\right) \\ &\qquad=\mathcal{E}_j\mathcal{E}_{j-1}\left(U_{j+2}, \mathcal{F}^{a_i-1}(U_{j+1}, U_j^*), U_{j-1}^*\right) \\ &\qquad=\mathcal{E}_j\mathcal{E}_{j-1}\left(U_{j+2}, U_{j+1} \boxplus U_j^{*\mathtt{tail}}, U_j^* \boxminus U_j^{*\mathtt{tail}}, U_{j-1}^*\right) \\ &\qquad=\mathcal{E}_j\left(U_{j+2}, U_{j+1} \boxplus U_j^{*\mathtt{tail}}, \mathcal{E}(U_j^* \boxminus U_j^{*\mathtt{tail}}, U_{j-1}^*)\right) \\ &\qquad=\mathcal{E}_j\left(U_{j+2}, U_{j+1} \boxplus U_j^{*\mathtt{tail}}, U_j^{\uparrow} \boxminus U_j^{*\mathtt{tail}}, U_{j-1}\right) \\ &\qquad=\left(U_{j+2}, \mathcal{E}(U_{j+1} \boxplus U_j^{*\mathtt{tail}}, U_j^{\uparrow} \boxminus U_j^{*\mathtt{tail}}), U_{j-1}\right) \\ &\qquad=\left(U_{j+2}, \widetilde{U}_{j+1}, \widetilde{U}_j, U_{j-1}\right),\end{aligned}$$

where

$$\begin{aligned}U_{j-1}^* &= T_i^{\mathtt{R}*} = (U_{j-1}(b_i + c_i), \dots, U_{j-1}(2)) \boxplus \emptyset, \\ U_j^* &= T_i^{\mathtt{L}*} = (U_j(a_i + c_i), \dots, U_j(a_i), U_{j-1}(1)) \boxplus (U_j(a_i - 1), \dots, U_j(1)), \\ U_j^{\uparrow} &= (U_j(a_i + c_i), \dots, U_j(a_i)) \boxplus (U_j(a_i - 1), \dots, U_j(1)).\end{aligned}$$

This proves the lemma. □

Corollary 5.3.16. *Under the above hypothesis, we have the following.*

(1) *For $j = 2$, there exists unique $T, S \in \mathbf{T}(0)$ such that $(T^{\mathtt{L}}, T^{\mathtt{R}}) = (\widetilde{U}_2, U_1)$ and $(S^{\mathtt{L}}, S^{\mathtt{R}}) = (U_1, U_0)$ if U_0 is non-empty.*

(2) *For $j = 2i$ with $1 \le i \le l - 1$, there exists a unique $T \in \mathbf{T}(a_i)$ such that $(T^{\mathtt{L}}, T^{\mathtt{R}}) = (\widetilde{U}_{j+1}, \widetilde{U}_j)$ and the residue of T is 0 if $U_{j+1}(1) < U_j(a_i)$ and 1 if $U_{j+1}(1) > U_j(a_i)$.*

(3) *The pair* $(U_{2l}, \widetilde{U}_{2l-1})$ *forms a semistandard tableau when the columns are put together horizontally along* L.

Proof. (1) and (3) follow directly from Definition 4.1.5 and the description of $(\widetilde{U}_{j+1}, \widetilde{U}_j)$ in Lemma 5.3.15.

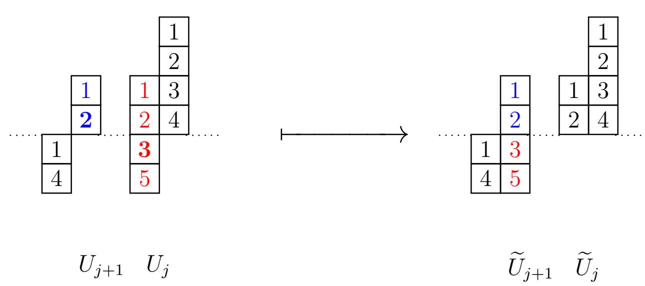
By definition of $\mathcal{S}_j$, it is not difficult to see that $(\widetilde{U}_{j+1}, \widetilde{U}_j)$ forms a semistandard tableau, say T of shape $\lambda(a_i, b'_i, c'_i)$ for some $b'_i, c'_i \in \mathbb{Z}_+$ such that $(T^{\mathtt{L}}, T^{\mathtt{R}}) = (\widetilde{U}_{j+1}, \widetilde{U}_j)$. The residue of T follows immediately from the description of $(\widetilde{U}_{j+1}, \widetilde{U}_j)$ in Lemma 5.3.15. This proves (2). $\square$

Corollary 5.3.17. *We have* $\mathcal{S}_j\mathcal{S}_k = \mathcal{S}_k\mathcal{S}_j$ *for* $j \neq k$, *and* $\mathcal{S}_2\mathcal{S}_4 \dots \mathcal{S}_{2l-2}\mathbf{U} \equiv_{\mathfrak{l}} \mathbf{U}$.

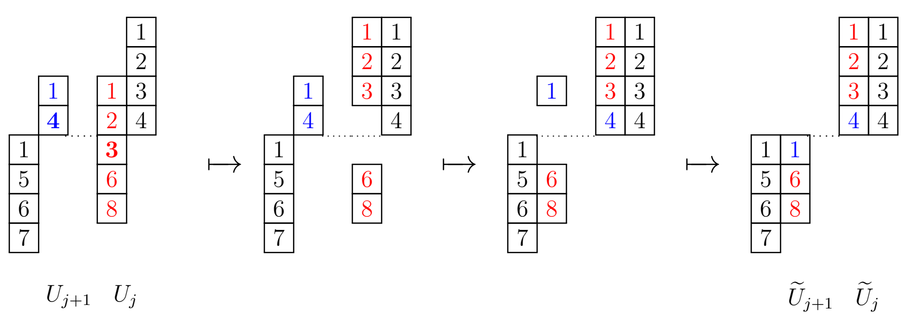
Proof. Since $\mathcal{S}_j$ changes only U_j, U_{j+1} by Lemma 5.3.15, it is clear that $\mathcal{S}_j\mathcal{S}_k = \mathcal{S}_k\mathcal{S}_j$ for $j \neq k$. The $\mathfrak{l}$-crystal equivalence follows from the fact that $\mathbf{E}^n$ is a $(\mathfrak{l}, \mathfrak{sl}_n)$-bicrystal. $\square$

Example 5.3.18.

(1) The following is an illustration of $\mathcal{S}_j$ when $U_{j+1}(1) < U_j(a_i)$.



(2) The following is an illustration of $\mathcal{S}_j$ when $U_{j+1}(1) > U_j(a_i)$.



5.3.4 Separation on $\mathfrak{l}$-highest weight vectors

We apply the separation algorithm developed in Sections 4.3.2–4.3.3 to $\mathbf{T} \in \mathbf{H}(\mu, n)$. We should remark that the process in Sections 4.3.2–4.3.3 also holds under the setting in Section 5.3.1. Then we have the following.

Proposition 5.3.19. *There exists $\overline{\mathbf{T}} \in SST(\eta)$, where η is given as in* (4.3.8), *such that*

(1) *$\overline{\mathbf{T}}$ is Knuth equivalent to $\mathbf{T}$, that is, $\overline{\mathbf{T}} \equiv_{\mathfrak{l}} \mathbf{T} \equiv_{\mathfrak{l}} H_\lambda$ for some $\lambda \in \mathscr{P}$,*

(2) *$\overline{\mathbf{T}}^{\texttt{tail}} \in \mathtt{LR}^{\lambda'}_{\delta' \mu'}$ and $\overline{\mathbf{T}}^{\texttt{body}} = H_{(\delta')^\pi}$ for some $\delta \in \mathscr{P}^{(2)}$.*

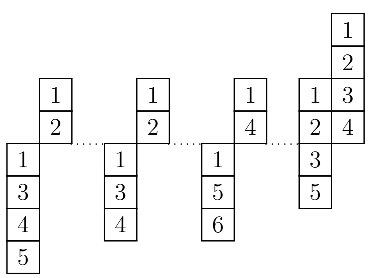
Proof.

(1) : By Propositions 4.3.5 and 4.3.8, we have $\mathbf{T} \equiv_{\mathfrak{l}} \overline{\mathbf{T}}$, and $\overline{\mathbf{T}} \equiv_{\mathfrak{l}} \overline{\mathbf{T}}^{\texttt{body}} \otimes \overline{\mathbf{T}}^{\texttt{tail}}$. Since $\mathrm{sh}(\overline{\mathbf{T}}^{\texttt{body}}) = (\delta')^\pi$ for some $\delta \in \mathscr{P}^{(2)}$, we should have $\overline{\mathbf{T}}^{\texttt{body}} = H_{(\delta')^\pi}$.

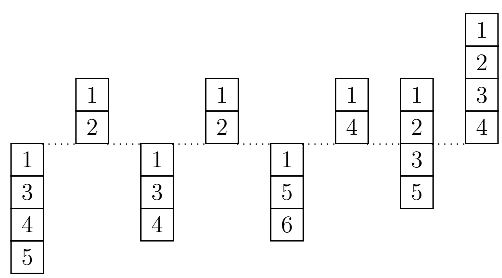
(2) : It follows from the fact that $\overline{\mathbf{T}} \equiv_{\mathfrak{l}} H_{(\delta')^\pi} \otimes \overline{\mathbf{T}}^{\texttt{tail}} \equiv_{\mathfrak{l}} \left(\overline{\mathbf{T}}^{\texttt{tail}} \rightarrow H_{(\delta')^\pi}\right) = H_{\lambda'}$. $\square$

Let us illustrate the separation on $\mathbf{H}(\mu, n)$.

Example 5.3.20. Let us consider the case $n - 2\mu_1' \geq 0$. Let $n = 8$ and $\mu = (4, 3, 3, 2) \in \mathcal{P}(\mathrm{O}_8)$. Let $\mathbf{T} = (T_4, T_3, T_2, T_1) \in \mathbf{H}(\mu, 8)$ and $\mathbf{U} = (U_8, \ldots, U_1)$ be given by

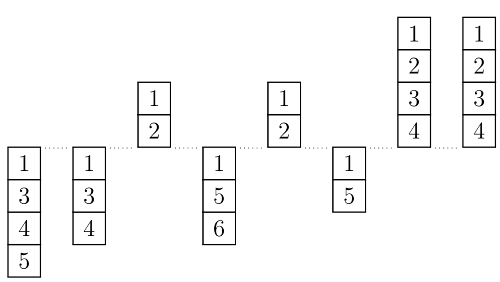


T_4 T_3 T_2 T_1



U_8 U_7 U_6 U_5 U_4 U_3 U_2 U_1

Applying $\mathcal{S}_6 \mathcal{S}_4 \mathcal{S}_2$ to $\mathbf{U}$, we get



The sequence of columns except the leftmost one (in gray) corresponds to $\mathbf{S} \in \mathbf{H}(\widetilde{\mu}, 7)$ with $\widetilde{\mu} = (3, 3, 2)$.

$$
\begin{array}{cccccccc}
 & & & & & & 1 & 1 \\
 & & & & & & 2 & 2 \\
 & & & 1 & & 1 & 3 & 3 \\
 & & & 2 & & 2 & 4 & 4 \\
\color{gray}{1} & 1 & 1 & & 1 & & & \\
\color{gray}{3} & 3 & 5 & & 5 & & & \\
\color{gray}{4} & 4 & 6 & & & & & \\
\color{gray}{5} & & & & & & &
\end{array}
$$

Applying this process again to $\mathbf{S}$, we get

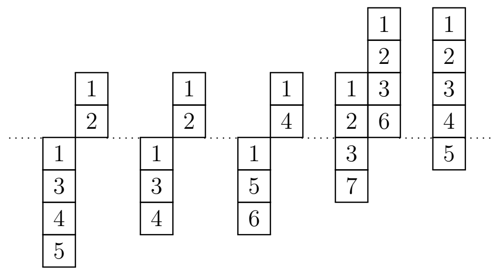
$$
\begin{array}{cccccccc}
 & & & & & & 1 & 1 \\
 & & & & & & 2 & 2 \\
 & & & & 1 & 1 & 3 & 3 \\
 & & & & 2 & 2 & 4 & 4 \\
\color{gray}{1} & 1 & 1 & 1 & & & & \\
\color{gray}{3} & 3 & 5 & 5 & & & & \\
\color{gray}{4} & 4 & 6 & & & & & \\
\color{gray}{5} & & & & & & &
\end{array}
$$

Therefore, $\overline{\mathbf{T}}$ is given by

$$
\begin{array}{cccccccc}
 & & & & & & 1 & 1 \\
 & & & & & & 2 & 2 \\
 & & & & 1 & 1 & 3 & 3 \\
 & & & & 2 & 2 & 4 & 4 \\
1 & 1 & 1 & 1 & & & & \\
3 & 3 & 5 & 5 & & & & \\
4 & 4 & 6 & & & & & \\
5 & & & & & & &
\end{array}
$$

Example 5.3.21. Let us consider the case $n - 2\mu'_1 < 0$. Let $n = 9$ and $\mu = (4, 3, 3, 2, 1) \in \mathcal{P}(\mathrm{O}_9)$. We have $n - 2\mu'_1 < 0$ and $\overline{\mu} = (4, 3, 3, 2)$. Let $\mathbf{T} \in \mathbf{T}(\mu, 9)$ be given by

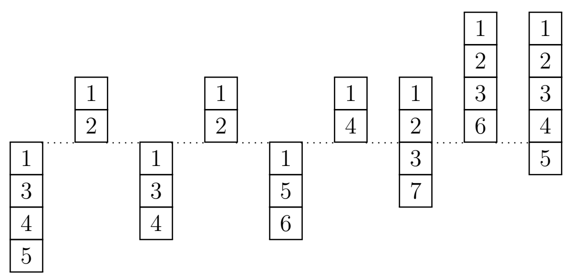
$\mathbf{T} =$



$T_4 \quad T_3 \quad T_2 \quad T_1 \quad T_0$

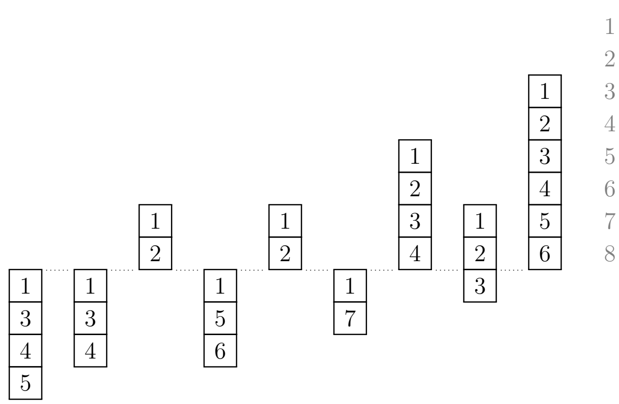
It corresponds to

$\mathbf{U} =$



$U_8 \quad U_7 \quad U_6 \quad U_5 \quad U_4 \quad U_3 \quad U_2 \quad U_1 \quad U_0$

Putting $U = H_{(1^8)}$ at the rightmost column and applying the sliding algorithm, we get



$\widetilde{U}_8 \quad \widetilde{U}_7 \quad \widetilde{U}_6 \quad \widetilde{U}_5 \quad \widetilde{U}_4 \quad \widetilde{U}_3 \quad \widetilde{U}_2 \quad \widetilde{U}_1 \quad \widetilde{U}_0 \quad U$

Then $\widetilde{\mathbf{U}} = (\widetilde{U}_7, \widetilde{U}_6, \widetilde{U}_5, \widetilde{U}_4, \widetilde{U}_3, \widetilde{U}_2, \widetilde{U}_1, \widetilde{U}_0)$ corresponds to $\widetilde{\mathbf{T}} \in \mathbf{H}(\widetilde{\mu}, 8)$, with $\widetilde{\mu} = (3, 3, 2, 1)$.

Repeating this process to $\widetilde{\mathbf{U}}$ as in Example 5.3.20 (recall subsection 4.3.2), we get $\overline{\mathbf{T}}$

$$\overline{\mathbf{T}} = \begin{ytableau} \none & \none & \none & \none & \none & \none & \none & \none & 1 \\ \none & \none & \none & \none & \none & \none & \none & \none & 2 \\ \none & \none & \none & \none & \none & \none & \none & 1 & 3 \\ \none & \none & \none & \none & \none & \none & \none & 2 & 4 \\ \none & \none & \none & \none & 1 & 1 & 1 & 3 & 5 \\ \none & \none & \none & \none & 2 & 2 & 2 & 4 & 6 \\ 1 & 1 & 1 & 1 & 3 \\ 3 & 3 & 5 & 7 \\ 4 & 4 & 6 \\ 5 \end{ytableau}$$

Hence

$$\overline{\mathbf{T}}^{\mathtt{tail}} = \begin{ytableau} 1 & 1 & 1 & 1 & 3 \\ 3 & 3 & 5 & 7 \\ 4 & 4 & 6 \\ 5 \end{ytableau} \qquad \overline{\mathbf{T}}^{\mathtt{body}} = \begin{ytableau} \none & \none & \none & \none & 1 \\ \none & \none & \none & \none & 2 \\ \none & \none & \none & 1 & 3 \\ \none & \none & \none & 2 & 4 \\ 1 & 1 & 1 & 3 & 5 \\ 2 & 2 & 2 & 4 & 6 \end{ytableau}$$

5.4 Branching rules from $\mathbf{D}_\infty$ to $\mathbf{A}_{+\infty}$

5.4.1 Branching multiplicity formulas from $\mathbf{D}_\infty$ to $\mathbf{A}_{+\infty}$

Let $\delta^{\mathtt{rev}} = (\delta^{\mathtt{rev}}_1, \dots, \delta^{\mathtt{rev}}_n)$ be the reverse sequence of $\delta = (\delta_1, \dots, \delta_n)$, that is, $\delta^{\mathtt{rev}}_i = \delta_{n-i+1}$, for $1 \le i \le n$. We put $p = \mu'_1$, $q = \mu'_2$, and $r = (\overline{\mu})'_1$ if $n - 2\mu'_1 < 0$.

Definition 5.4.1. For $S \in \mathtt{LR}^{\lambda'}_{\delta'\mu'}$, let $s_1 \le \cdots \le s_p$ denote the entries in the first row, and $t_1 \le \cdots \le t_q$ the entries in the second row of S. Let $1 \le m_1 < \cdots < m_p < n$ be the sequence defined inductively from p to 1 as follows:

$$m_i = \max\{\, k \mid \delta^{\mathtt{rev}}_k \in X_i,\ \delta^{\mathtt{rev}}_k < s_i \,\},$$

where

$$X_i = \begin{cases} \{\, \delta^{\mathtt{rev}}_i, \dots, \delta^{\mathtt{rev}}_{2i-1} \,\} \setminus \{\delta^{\mathtt{rev}}_{m_{i+1}}, \dots, \delta^{\mathtt{rev}}_{m_p}\}, & \text{if } 1 \le i \le r, \\ \{\, \delta^{\mathtt{rev}}_i, \dots, \delta^{\mathtt{rev}}_{n-p+i} \,\} \setminus \{\delta^{\mathtt{rev}}_{m_{i+1}}, \dots, \delta^{\mathtt{rev}}_{m_p}\}, & \text{if } r < i \le p, \end{cases}$$

(we assume that $r = p$ when $n - 2\mu'_1 \geq 0$). Let $n_1 < \cdots < n_q$ be the sequence such that n_j is the j-th smallest integer in $\{j+1, \cdots, n\} \setminus \{m_{j+1}, \cdots, m_p\}$ for $1 \leq j \leq q$. Then we define $\overline{\mathtt{LR}}^{\lambda'}_{\delta'\mu'}$ to be a subset of $\mathtt{LR}^{\lambda'}_{\delta'\mu'}$ consisting of S satisfying

$$t_j > \delta^{\mathtt{rev}}_{n_j}, \tag{5.4.1}$$

for $1 \leq j \leq q$. We put $\overline{c}^{\lambda}_{\delta\mu} = |\overline{\mathtt{LR}}^{\lambda'}_{\delta'\mu'}|$.

Remark 5.4.2. Let $S \in \mathtt{LR}^{\lambda'}_{\delta'\mu'}$ be given. Let us briefly explain the well-definedness of the sequence $(m_i)_{1\leq i\leq p}$ in Definition 5.4.1. We may assume that $n - 2\mu'_1 \geq 0$ since the arguments for $n - 2\mu'_1 < 0$ are similar.

It is enough to verify that $\delta^{\mathtt{rev}}_i < s_i$ for $1 \leq i \leq p$. Let $H' = (s_1 \to (s_2 \to \ldots (s_p \to H_{\delta'})))$. Then $\mathrm{sh}(H')/\mathrm{sh}(H_{\delta'})$ is a horizontal strip of length p. If there exists s_i such that $\delta^{\mathtt{rev}}_i \geq s_i$, then we should have $\ell(\lambda) > n$, which is a contradiction to $\lambda \in \mathscr{P}_n$. By definition of m_i, we also note that

$$\begin{cases} i \leq m_i \leq 2i - 1 & \text{for } 1 \leq i \leq r, \\ i \leq m_i \leq n - p + i & \text{for } r < i \leq p, \end{cases}$$

where $r = p$ when $n - 2\mu'_1 \geq 0$.

Example 5.4.3. Let $n = 8$, $\mu = (2, 2, 2, 1, 1) \in \mathcal{P}(\mathrm{O}_8)$, $\lambda = (5, 4, 4, 3, 2, 2) \in \mathscr{P}_8$, and $\delta = (4, 2, 2, 2, 2) \in \mathscr{P}^{(2)}_8$. Note that $n - 2\mu'_1 = -2 < 0$ and $r = (\overline{\mu})'_1 = 3$.

Let us consider the Littlewood-Richardson tableau $S \in \mathtt{LR}^{\lambda'}_{\delta'\mu'}$ given by

$$S = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 3 & 3 & 5 \\ \hline 2 & 4 & 4 \\ \cline{1-3} \end{array} \quad .$$

Then the sequences $(m_i)_{1\leq i\leq 5}$ and $(n_j)_{1\leq j\leq 3}$ are $(1, 3, 5, 7, 8)$ and $(2, 4, 6)$, respectively, and S satisfies the condition (5.4.1):

$$t_1 = 2 > 0 = \delta^{\mathtt{rev}}_{n_1}, \quad t_2 = 4 > 2 = \delta^{\mathtt{rev}}_{n_2}, \quad t_3 = 4 > 2 = \delta^{\mathtt{rev}}_{n_3}.$$

Hence $S \in \overline{\mathtt{LR}}^{\lambda'}_{\delta'\mu'}$.

Now we are in a position to state the main result in this chapter.

Theorem 5.4.4. *For $\mu \in \mathcal{P}(\mathrm{O}_n)$ and $\lambda \in \mathscr{P}_n$, we have a bijection*

$$\begin{array}{ccc} \mathtt{LR}^{\mu}_{\lambda}(\mathfrak{d}) & \longrightarrow & \bigsqcup_{\delta \in \mathscr{P}_n^{(2)}} \overline{\mathtt{LR}}^{\lambda'}_{\delta'\mu'} \ . \\ \mathbf{T} & \longmapsto & \overline{\mathbf{T}}^{\mathtt{tail}} \end{array}$$

Proof. We give the proof in Sections 7.3.2 and 7.3.3. □

Corollary 5.4.5. *Under the above hypothesis, we have*

$$c^{\mu}_{\lambda}(\mathfrak{d}) = \sum_{\delta \in \mathscr{P}_n^{(2)}} \overline{c}^{\lambda}_{\delta\mu}.$$

Let us give another description of $c^{\mu}_{\lambda}(\mathfrak{d})$ which is simpler than $\overline{\mathtt{LR}}^{\lambda'}_{\delta'\mu'}$, and also plays an important role in Section 5.5.

Definition 5.4.6. For $U \in \mathtt{LR}^{\lambda}_{\delta\mu^{\pi}}$ (see subsection 3.1.1), let $\sigma_1 > \cdots > \sigma_p$ denote the entries in the rightmost column and $\tau_1 > \cdots > \tau_q$ the second rightmost column of U, respectively. Let $m_1 < \cdots < m_p$ be the sequence defined by

$$m_i = \begin{cases} \min\{n - \sigma_i + 1, 2i - 1\}, & \text{if } 1 \leq i \leq r, \\ \min\{n - \sigma_i + 1, n - p + i\}, & \text{if } r < i \leq p. \end{cases}$$

and let $n_1 < \cdots < n_q$ be the sequence such that n_j is the j-th smallest number in $\{\, j+1, \ldots, n \,\} \setminus \{\, m_{j+1}, \ldots, m_p \,\}$. Then we define $\underline{\mathtt{LR}}^{\lambda}_{\delta\mu}$ to be the subset of $\mathtt{LR}^{\lambda}_{\delta\mu^{\pi}}$ consisting of U such that

$$\tau_j + n_j \leq n + 1, \tag{5.4.2}$$

for $1 \leq j \leq q$. We put $\underline{c}^{\lambda}_{\delta\mu} = |\underline{\mathtt{LR}}^{\lambda}_{\delta\mu}|$.

Example 5.4.7. We keep the assumption in Example 5.4.3 and consider the Littlewood-Richardson tableau $U \in \mathtt{LR}^{\lambda}_{\delta\mu^{\pi}}$ given by

$$U = \begin{array}{|c|c|} \hline \multicolumn{1}{c|}{} & 1 \\ \cline{2-2} \multicolumn{1}{c|}{} & 2 \\ \hline 2 & 3 \\ \hline 3 & 4 \\ \hline 6 & 6 \\ \hline \end{array}$$

The sequences $(m_i)_{1\leq i\leq 5}$ and $(n_j)_{1\leq j\leq 3}$ are $(1,3,5,7,8)$ and $(2,4,6)$, respectively. Then U satisfies the condition (5.4.2):

$$\begin{aligned}
\tau_1 + n_1 &= 6 + 2 = 8 \leq 8 + 1 = n + 1,\\
\tau_2 + n_2 &= 3 + 4 = 7 \leq 8 + 1 = n + 1,\\
\tau_3 + n_3 &= 2 + 6 = 8 \leq 8 + 1 = n + 1.
\end{aligned}$$

Hence $U \in \underline{\mathtt{LR}}^{\lambda}_{\mu\delta}$.

Now, one can show that $\overline{c}^{\lambda}_{\delta\mu} = \underline{c}^{\lambda}_{\delta\mu}$ by using the bijection ψ (5.1.1).

Lemma 5.4.8. *The sequences $(m_i)_{1\leq i\leq p}$ and $(n_j)_{1\leq j\leq q}$ for S in Definition 5.4.1 are equal to the ones for $U = \psi(S)$ in Definition 5.4.6.*

Proof. We assume that $n - 2\mu'_1 \geq 0$. The proof for the case $n - 2\mu'_1 < 0$ is similar. Suppose that $S \in \mathtt{LR}^{\lambda'}_{\delta'\mu'}$ is given. Let $s_1 \leq \cdots \leq s_p$ denote the entries in the first row of S. Let $(m'_i)_{1\leq i\leq p}$ and $(n'_j)_{1\leq j\leq q}$ be the sequences for S in Definition 5.4.1. Put $U = \psi(S)$. Let $\sigma_1 > \cdots > \sigma_p$ be the rightmost column of U and let $(m_i)_{1\leq i\leq p}$ and $(n_j)_{1\leq j\leq q}$ be the sequences for U as in Definition 5.4.6.

It is enough to show that $m'_i = m_i$ for $1 \leq i \leq p$, which clearly implies $n'_j = n_j$ for $1 \leq j \leq q$. Let us enumerate the column of δ' by $n, n-1, \ldots, 1$ from left to right. Consider the vertical strip $V^i := \mathrm{sh}(H^i)/\mathrm{sh}(H^{i-1})$ filled with i for $1 \leq i \leq p$ (recall (5.1.1) below). By definition of ψ (5.1.1), we see that the upper most box in V^i is located in the $(n - \sigma_i + 1)$-th column in δ'.

Let $i \in \{1, \ldots, p\}$ be given. First, we have $m'_i \leq n - \sigma_i + 1$ by definition of m'_i. Since $m'_i \leq 2i - 1$, we have $m'_i \leq m_i = \min\{n - \sigma_i + 1, 2i - 1\}$. Next, we claim that $m_i \leq m'_i$. If $n - \sigma_i + 1 \leq 2i - 1$, then we have $\delta^{\mathtt{rev}}_{n-\sigma_i+1} < s_i$, and hence $m_i \leq n - \sigma_i + 1 \leq m'_i$ by definition of m'_i. If $n - \sigma_i + 1 > 2i - 1$, then we have $m_i = 2i - 1 = m'_i$. This proves that $m_i = m'_i$. □

Theorem 5.4.9. *For $\mu \in \mathcal{P}(\mathrm{O}_n)$, $\lambda \in \mathscr{P}_n$ and $\delta \in \mathscr{P}^{(2)}_n$, the bijection $\psi : \mathtt{LR}^{\lambda'}_{\mu'\nu'} \longrightarrow \mathtt{LR}^{\lambda}_{\mu\nu^\pi}$ in (5.1.1) induces a bijection from $\overline{\mathtt{LR}}^{\lambda'}_{\delta'\mu'}$ to $\underline{\mathtt{LR}}^{\lambda}_{\delta\mu}$.*

Proof. Let $S \in \mathtt{LR}^{\lambda'}_{\delta'\mu'}$ given and put $U = \psi(S)$. We keep the conventions in the proof of Lemma 5.4.8. By definition of ψ, the second upper most box in V^j is located at the $(n - \tau_j + 1)$-th column in δ'. By Lemma 5.4.8, we see that $\delta^{\mathtt{rev}}_{n_j} < t_j$ if and only if $n - \tau_j + 1 \geq n_j$ or $\tau_j + n_j \leq n + 1$. Therefore, $S \in \overline{\mathtt{LR}}^{\lambda'}_{\delta'\mu'}$ if and only if $U \in \underline{\mathtt{LR}}^{\lambda}_{\delta\mu}$. □

Example 5.4.10. Let $n = 8$, $\mu = (2,2,2,1,1) \in \mathcal{P}(\mathrm{O}_8)$, $\lambda = (5,4,4,3,2,2) \in \mathscr{P}_8$, and $\delta = (4,2,2,2,2) \in \mathscr{P}_8^{(2)}$. Let S be the Littlewood-Richardson tableau in Example 5.4.3. We enumerate the columns of S as follows:

$$\begin{array}{ccccc} \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} & \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline \end{array} & \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline \end{array} & \begin{array}{|c|}\hline 3 \\ \hline \end{array} & \begin{array}{|c|}\hline 5 \\ \hline \end{array} \\ \\ S^5 & S^4 & S^3 & S^2 & S^1 \end{array}$$

Then the insertion and recording tableaux are given by

$$S \to H_{\delta'} = \begin{array}{cccccccc} 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 & \boxed{1} & & \\ 2 & 2 & 2 & 2 & 2 & \boxed{2} & & \\ 3 & \boxed{3} & \boxed{3} & \boxed{3} & & & & \\ 4 & \boxed{4} & \boxed{4} & & & & & \\ \boxed{5} & & & & & & & \end{array}, \qquad Q(S \to H_{\delta'}) = \begin{array}{ccccc} & & & & \boxed{1} \\ & & \boxed{2} & \boxed{3} & \\ & & \boxed{3} & \boxed{4} & \\ & & \boxed{4} & & \\ \\ \boxed{5} & \boxed{5} & & & \end{array}.$$

Then $\psi(S)$ is obtained by

$$\left(\begin{array}{c} \begin{array}{lccccc} 8\cdots & 1 & 1 & 1 & 1 & \boxed{1} \\ 7\cdots & 2 & 2 & \boxed{2} & \boxed{2} & \\ 6\cdots & 3 & 3 & \boxed{3} & \boxed{3} & \\ 5\cdots & 4 & 4 & \boxed{4} & & \\ 4\cdots & 5 & 5 & & & \\ 3\cdots & \boxed{6} & \boxed{6} & & & \\ 2\cdots & & & & & \\ 1\cdots & & & & & \end{array} \\ \psi(S) \to H_\delta \end{array}, \begin{array}{c} \begin{array}{ccccc} & & & & \boxed{1} \\ & & \boxed{2} & \boxed{3} & \\ & & \boxed{3} & \boxed{4} & \\ & & \boxed{4} & & \\ \\ \boxed{5} & \boxed{5} & & & \end{array} \\ \\ Q(\psi(S) \to H_\delta) \end{array} \right) \longrightarrow \begin{array}{cc} & \boxed{1} \\ & \boxed{2} \\ \boxed{2} & \boxed{3} \\ \boxed{3} & \boxed{4} \\ \boxed{6} & \boxed{6} \end{array} = \psi(S).$$

(Here the numbers in gray denote the enumeration of columns of δ'.) Thus we have

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) = (6,4,3,2,1), \qquad (\tau_1, \tau_2, \tau_3) = (6,3,2).$$

Note that the enumeration of the rows of $U := \psi(S)$ is given by

$$\begin{array}{ll} U_1 & \phantom{\boxed{2}}\boxed{1} \\ U_2 & \phantom{\boxed{2}}\boxed{2} \\ U_3 & \boxed{2}\boxed{3} \\ U_4 & \boxed{3}\boxed{4} \\ U_5 & \boxed{6}\boxed{6} \end{array} \quad .$$

Under the above correspondence, we observe that σ_i $(1 \leq i \leq 5)$ and τ_j $(1 \leq j \leq 3)$ record the positions of s_i and t_j in δ', respectively, and vice versa. For example, the entry τ_2 in U^4 is located at $(n - \tau_2 + 1)$-th row in δ, which implies that t_2 is located at $(n - \tau_2 + 1)$-th column in δ'. This correspondence implies $\psi(S) \in \underline{\mathtt{LR}}^{\lambda}_{\delta\mu}$ (cf. Example 5.4.7).

Corollary 5.4.11. *Under the above hypothesis, we have*

$$c^{\mu}_{\lambda}(\mathfrak{d}) = \sum_{\delta \in \mathscr{P}^{(2)}_n} \underline{c}^{\lambda}_{\delta\mu}.$$

Proof. This follows from $\overline{c}^{\lambda}_{\delta\mu} = \underline{c}^{\lambda}_{\delta\mu}$. □

We have another characterization of $\underline{c}^{\lambda}_{\delta\mu}$ in terms of usual LR tableaux (*not* companion tableaux) by considering the bijection between LR tableaux and their companion ones (recall Section 5.1).

Corollary 5.4.12. *Let U be an LR tableau of shape λ/δ with content μ^{π} and let σ_i be the row index of the leftmost $\mu'_1 - i + 1$ in U for $1 \leq i \leq \mu'_1$, and τ_j the row index of the second leftmost $\mu'_2 - j + 1$ in U for $1 \leq j \leq \mu'_2$. Let $m_1 < \cdots < m_{\mu'_1}$ be the sequence given by $m_i = \min\{n - \sigma_i + 1, 2i - 1\}$, and let $n_1 \leq \cdots \leq n_{\mu'_2}$ be the sequence such that n_j is the j-th smallest number in $\{\, j+1, \ldots, n \,\} \setminus \{\, m_{j+1}, \ldots, m_{\mu'_1} \,\}$. Then, $\underline{c}^{\lambda}_{\delta\mu}$ is equal to the number of U such that*

$$\tau_j + n_j \leq n + 1,$$

for $1 \leq j \leq \mu'_2$. □

We recover the Littlewood's formula (5.0.2) from Corollary 5.4.11.

Corollary 5.4.13. *Under the above hypothesis, if $\ell(\lambda) \leq \frac{n}{2}$, then*

$$c_\lambda^\mu(\mathfrak{d}) = \sum_{\delta \in \mathscr{P}_n^{(2)}} c_{\delta\mu}^\lambda.$$

Proof. We claim that $\mathtt{LR}_{\delta\mu^\pi}^\lambda = \underline{\mathtt{LR}}_{\delta\mu}^\lambda$. Let $U \in \mathtt{LR}_{\delta\mu^\pi}^\lambda$ be given. Let $H' = (\sigma_1 \to (\cdots \to (\sigma_p \to H_\delta)))$. Note that $\sigma_i + i - 1 \leq \ell(\mathrm{sh}(H')) = \ell(\lambda) \leq \frac{n}{2}$ for $1 \leq i \leq p$. So we have

$$n - \sigma_i + 1 \geq 2i \quad (1 \leq i \leq p). \tag{5.4.3}$$

Otherwise, we have $n - i < \sigma_i + i - 1 \leq \frac{n}{2}$ and hence $\frac{n}{2} < i \leq p = \mu_1' \leq \ell(\lambda)$, which is a contradiction. By definition m_i and n_j, we have

$$m_i = 2i - 1, \quad n_j = 2j, \tag{5.4.4}$$

for $1 \leq i \leq p$ and $1 \leq j \leq q$. By (5.4.3) and (5.4.4) we have

$$\tau_j \leq \sigma_j \leq n - 2j + 1 \quad (1 \leq j \leq q),$$

which implies that U satisfies (5.4.2), that is, $U \in \underline{\mathtt{LR}}_{\delta\mu}^\lambda$. This proves the claim. By Theorem 5.4.9, we have $c_{\delta\mu}^\lambda = \underline{c}_{\delta\mu}^\lambda$. □

5.4.2 Branching multiplicity formulas from GL_n to O_n

We assume that the base field is $\mathbb{C}$. Let $V_{\mathrm{GL}_n}^\lambda$ denote the finite-dimensional irreducible GL_n-module corresponding to $\lambda \in \mathscr{P}_n$, and $V_{\mathrm{O}_n}^\mu$ the finite-dimensional irreducible module O_n-module corresponding to $\mu \in \mathcal{P}(\mathrm{O}_n)$.

Then we have the following new combinatorial description of $\left[V_{\mathrm{GL_n}}^\lambda : V_{\mathrm{O}_n}^\mu\right]$.

Theorem 5.4.14. *For $\lambda \in \mathscr{P}_n$ and $\mu \in \mathcal{P}(\mathrm{O}_n)$, we have*

$$\left[V_{\mathrm{GL_n}}^\lambda : V_{\mathrm{O_n}}^\mu\right] = \sum_{\delta \in \mathscr{P}_n^{(2)}} \overline{c}_{\delta\mu}^\lambda = \sum_{\delta \in \mathscr{P}_n^{(2)}} \underline{c}_{\delta\mu}^\lambda.$$

Proof. It follows from Theorem 5.3.4, Corollaries 5.4.5 and 5.4.11. □

Example 5.4.15. Let us compare the formula in Theorem 5.4.14 with the one by Enright and Willenbring in [18, Theorem 4].

Let $\mu, \nu \in \mathcal{P}(\mathrm{O}_n)$ be given by

$$\mu = (\underbrace{d, \overbrace{2, \ldots, 2}^{a}, \overbrace{1, \ldots, 1}^{b}, \overbrace{0, \ldots, 0}^{c}}_{n}),$$
$$\nu = (\underbrace{d, \overbrace{2, \ldots, 2}^{c}, \overbrace{1, \ldots, 1}^{b}, \overbrace{0, \ldots, 0}^{a}}_{n}).$$

where a, b, c, d are positive integers with $d \geq 2$. Then we have for $\lambda \in \mathscr{P}_n$

$$\left[V^{\lambda}_{\mathrm{GL_n}} : V^{\mu}_{\mathrm{O_n}}\right] = \sum_{\xi \in \mathscr{P}^{(2)}_n} c^{\lambda'}_{\xi'\mu'} - \sum_{\upsilon \in \mathscr{P}^{(2)}_n} c^{\lambda'}_{\upsilon'\nu'},$$

(see [18, Section 7 (7.11)]). Suppose that $n = 8$, $a = b = d = 2, c = 3$ and $\lambda = (5, 4, 4, 3, 2, 2, 0, 0) \in \mathscr{P}_8$. Then it is straightforward to check that for $\xi, \upsilon \in \mathscr{P}^{(2)}_8$

$$c^{\lambda'}_{\xi'\mu'} = \begin{cases} 1, & \text{if } \xi = (4, 2, 2, 2, 2) \text{ or } (4, 4, 2, 2), \\ 0, & \text{otherwise,} \end{cases}$$
$$c^{\lambda'}_{\upsilon'\nu'} = \begin{cases} 1, & \text{if } \upsilon = (4, 2, 2, 2), \\ 0, & \text{otherwise.} \end{cases}$$

Hence we have

$$\left[V^{\lambda}_{\mathrm{GL_8}} : V^{\mu}_{\mathrm{O_8}}\right] = \sum_{\xi \in \mathscr{P}^{(2)}_8} c^{\lambda'}_{\xi'\mu'} - \sum_{\upsilon \in \mathscr{P}^{(2)}_8} c^{\lambda'}_{\upsilon'\nu'} = 2 - 1 = 1.$$

On the other hand, the following tableaux S_α and S_β are the unique tableaux in $\mathtt{LR}^{\lambda'}_{\alpha'\mu'}$ and $\mathtt{LR}^{\lambda'}_{\beta'\mu'}$, respectively, where $\alpha = (4, 2, 2, 2, 2)$ and $\beta = (4, 4, 2, 2)$:

$$S_\alpha = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 3 & 3 & 5 \\ \hline 2 & 4 & 4 \\ \cline{1-3} \end{array} \qquad S_\beta = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 3 & 3 & 5 \\ \hline 2 & 2 & 4 \\ \cline{1-3} \end{array}$$

We see that $S_\alpha \in \overline{\mathtt{LR}}^{\lambda'}_{\alpha'\mu'}$ and $\psi(S_\alpha) \in \underline{\mathtt{LR}}^{\lambda}_{\alpha\mu}$ (see Examples 5.4.3 and 5.4.10). On the other hand, for S_β, the sequence $(m_i)_{1 \leq i \leq 5}$ and $(n_j)_{1 \leq j \leq 3}$ are given by $(1, 3, 5, 6, 8)$ and $(2, 4, 7)$, respectively. Then $S_\beta \notin \overline{\mathtt{LR}}^{\lambda'}_{\beta'\mu'}$ since $t_3 = 4 = \delta^{\mathtt{rev}}_{n_3}$. We can also check that $\psi(S_\beta) \notin \underline{\mathtt{LR}}^{\lambda}_{\beta\mu}$ (cf. Example 5.4.10). By Theorem 5.4.14, we have

$$\left[V^\lambda_{\mathrm{GL}_8} : V^\mu_{\mathrm{O}_8}\right] = \sum_{\delta\in\mathscr{P}^{(2)}_8} \overline{c}^\lambda_{\delta\mu} = \sum_{\delta\in\mathscr{P}^{(2)}_n} \underline{c}^\lambda_{\delta\mu} = 1.$$

5.4.3 Comparing other works

Let us compare the results in this chapter with [67, 71, 104]. Let us briefly recall Sundaram's formula for (5.0.1) when $\mathrm{G}_n = \mathrm{Sp}_n$ [104]. She constructs a bijection between the set of oscillating tableaux appearing in Berele's correspondence for Sp_n [2] and the set of pairs of the standard tableaux and LR tableaux with the symplectically fitting lattice word. Then it is shown that these LR tableaux count the branching multiplicity (5.0.1). In fact, Sundaram's formula also can be described in terms of similar flag condition with

(5.4.2)
$$\left[\, V^\lambda_{\mathrm{GL}_n}, V^\mu_{\mathrm{Sp}_n} \right] = \sum_{\delta\in\mathscr{P}^{(1,1)}_n} \left| \left\{ T \in \mathtt{LR}^\lambda_{\mu\delta} \,\middle|\, \sigma_{2i+1} \le \frac{n}{2} + 1 \right\} \right|,$$

where σ_j is the rightmost entry in the jth row of T from top. Recall that T is the companion tableau of LR tableau. We remark that Lecouvey-Lenart provide a conjectural bijection between the Sundaram's LR tableaux and the flagged LR tableaux for type C_n in [71].

On the other hand, Theorem 5.4.4 recovers [67, Theorem 4.8] as follows. For $\mathbf{T} = (T_l, \ldots, T_0) \in \mathtt{LR}^\mu_\lambda(\mathfrak{d})$, let $\mathbf{T}^{\mathtt{tail}} = (T^{\mathtt{tail}}_l, \ldots, T^{\mathtt{tail}}_0)$. We may regard $\mathbf{T}^{\mathtt{tail}}$ as a column-semistandard tableau of shape μ' by putting together $T^{\mathtt{tail}}_i$'s horizontally. It is shown in [67, Theorem 4.8] that if $\ell(\lambda) \le n/2$, then the map sending $\mathbf{T}$ to $\mathbf{T}^{\mathtt{tail}}$ gives a bijection

$$\mathtt{LR}^\mu_\lambda(\mathfrak{d}) \longrightarrow \bigsqcup_{\delta\in\mathscr{P}^{(2)}_n} \mathtt{LR}^{\lambda'}_{\delta'\mu'} .$$

By Lemma 5.4.8 and (5.4.4), we have $\overline{\mathbf{T}}^{\mathtt{tail}} = \mathbf{T}^{\mathtt{tail}}$ if $\ell(\lambda) \le n/2$, and hence Theorem 5.4.4 recovers [67, Theorem 4.8].

Also, we may have an analogue of Theorem 5.4.4 for types BC, that is, a multiplicity formula with respect to the branching from B_∞ and C_∞ to $\mathrm{A}_{+\infty}$, respectively. More precisely, let $\mathbf{T}^{\mathfrak{g}}(\mu, n)$ be the spinor model for the integrable highest weight module over the Kac-Moody algebra of type B_∞ and C_∞ when $\mathfrak{g} = \mathfrak{b}$ and $\mathfrak{c}$, respectively, corresponding to $\mu \in \mathscr{P}(\mathrm{G}_n)$ via Howe duality. Here $\mathscr{P}(\mathrm{G}_n)$ denotes the set of partitions parametrizing the finite-dimensional irreducible representations of an algebraic group G_n (see [67, Section 2] for more details).

For $\lambda \in \mathscr{P}_n$, let $\mathtt{LR}^{\mu}_{\lambda}(\mathfrak{g})$ be the set of $\mathbf{T} \in \mathbf{T}^{\mathfrak{g}}(\mu, n)$ which is an $\mathfrak{l}$-highest weight element with highest weight λ' (cf. (5.3.2)). Let $\delta \in \mathscr{P}^{\diamond}_n$ be given, where $\diamond = (1)$ for $\mathfrak{g} = \mathfrak{b}$ and $\diamond = (1,1)$ for $\mathfrak{g} = \mathfrak{c}$ (here we understand $\mathscr{P}^{(1)} = \mathscr{P}$). Put

$$\overline{\mathtt{LR}}^{\lambda'}_{\delta'\mu'} = \left\{ S \in \mathtt{LR}^{\lambda'}_{\delta'\mu'} \,|\, s_i > \delta^{\mathtt{rev}}_{2i} \ (1 \le i \le \mu'_1) \right\},$$

where $s_1 \le \cdots \le s_{\mu'_1}$ are the entries in the first row of S.

We may apply the same arguments in Section 4.3 to $\mathbf{T}^{\mathfrak{g}}(\mu, n)$. Then by Propositions 5.3.9 and 5.3.19, we have for $\mathbf{T} = (T_l, \ldots, T_1) \in \mathtt{LR}^{\mu}_{\lambda}(\mathfrak{g})$ that

$$\mathbf{T}^{\mathtt{tail}} = (T^{\mathtt{tail}}_l, \ldots, T^{\mathtt{tail}}_0) \in \overline{\mathtt{LR}}^{\lambda'}_{\delta'\mu'},$$

for some $\delta \in \mathscr{P}^{\diamond}_n$. Furthermore, the map

$$\begin{array}{ccc} \mathtt{LR}^{\mu}_{\lambda}(\mathfrak{g}) & \longrightarrow & \bigsqcup_{\delta \in \mathscr{P}^{\diamond}_n} \overline{\mathtt{LR}}^{\lambda'}_{\delta'\mu'} \\ \mathbf{T} & \longmapsto & \mathbf{T}^{\mathtt{tail}} \end{array}$$

is a bijection. The map ψ in (5.1.1) induces a bijection from $\overline{\mathtt{LR}}^{\lambda'}_{\delta'\mu'}$ to $\underline{\mathtt{LR}}^{\lambda}_{\delta\mu}$, where

$$\underline{\mathtt{LR}}^{\lambda}_{\delta\mu} = \left\{ U \in \mathtt{LR}^{\lambda}_{\delta\mu^{\pi}} \,|\, \sigma_i + 2i \le n+1 \ (1 \le i \le \mu'_1) \right\}, \tag{5.4.5}$$

where $\sigma_1 > \cdots > \sigma_{\mu'_1}$ are the entries in the rightmost column of U. Therefore,

$$c^{\mu}_{\lambda}(\mathfrak{g}) = \sum_{\delta \in \mathscr{P}^{\diamond}_n} \overline{c}^{\lambda}_{\delta\mu} = \sum_{\delta \in \mathscr{P}^{\diamond}_n} \underline{c}^{\lambda}_{\delta\mu},$$

where $c^{\mu}_{\lambda}(\mathfrak{g}) = |\mathtt{LR}^{\mu}_{\lambda}(\mathfrak{g})|$, $\overline{c}^{\lambda}_{\delta\mu} = |\overline{\mathtt{LR}}^{\lambda'}_{\delta'\mu'}|$, and $\underline{c}^{\lambda}_{\delta\mu} = |\underline{\mathtt{LR}}^{\lambda}_{\delta\mu}|$. This is a generalization of [67, Theorem 4.8] for types BC, which also recovers [71, Theorem 6.8] for type C.

We remark that the flag condition in (5.4.5) is different from the one in [71, Section 6.3] because we use the bijection (5.1.1) whose image is the set of LR tableaux with anti-lattice word (cf. [71, Theorem 6.2]).

Remark 5.4.16. Recently, an orthogonal analogue of Sundaram type bijection [105] is given for SO_{2n+1} [40], where oscillating tableaux are replaced by vacillating tableaux, and the Sundaram's LR tableaux are replaced by so-called alternative orthogonal LR tableaux, which are in (highly non-trivial) bijection with $\mathtt{LR}^{\mu}_{\lambda}(\mathfrak{d})$.

5.5 Generalized exponents

5.5.1 Generalized exponents

Let $\mathfrak{g}$ be a simple Lie algebra of rank n over $\mathbb{C}$, and G the adjoint group of $\mathfrak{g}$. Let $S(\mathfrak{g})$ be the symmetric algebra generated by $\mathfrak{g}$, and $S(\mathfrak{g})^G$ the space of G-invariants with respect to the adjoint action. Let $\mathcal{H}(\mathfrak{g})$ be the space of polynomials annihilated by G-invariant differential operators with constant coefficients and no constant term. It is shown by Kostant [59] that $S(\mathfrak{g})$ is a free $S(\mathfrak{g})^G$-module generated by $\mathcal{H}(\mathfrak{g})$, that is,

$$S(\mathfrak{g}) = S(\mathfrak{g})^G \otimes \mathcal{H}(\mathfrak{g}).$$

Let t be an indeterminate. Let Φ^+ denote the set of positive roots and $\Phi = \Phi^+ \cup -\Phi^+$ the set of roots of $\mathfrak{g}$. We define the graded character of $S(\mathfrak{g})$ by

$$\mathrm{ch}_t S(\mathfrak{g}) = \frac{1}{(1-t)^n \prod_{\alpha\in\Phi}(1-te^{\alpha})}. \tag{5.5.1}$$

Then it is also shown in [59] that the graded character of $\mathcal{H}(\mathfrak{g})$ is determined by

$$\mathrm{ch}_t S(\mathfrak{g}) = \frac{\mathrm{ch}_t \mathcal{H}(\mathfrak{g})}{\prod_{i=1}^{n}(1-t^{d_i})}, \tag{5.5.2}$$

where $d_i = m_i + 1$ for $i = 1, \dots, n$ and m_i are the classical exponents of $\mathfrak{g}$.

For $\mu \in P_+$, let $V_{\mathfrak{g}}^{\mu}$ be the irreducible representation of $\mathfrak{g}$ with highest weight μ. The generalized exponent associated with $\mu \in P_+$ is a graded multiplicity of $V_{\mathfrak{g}}^{\mu}$ in $\mathcal{H}(\mathfrak{g})$, that is,

$$E_t(V_{\mathfrak{g}}^{\mu}) = \sum_{k\geq 0} \dim \mathrm{Hom}_{\mathfrak{g}}(V_{\mathfrak{g}}^{\mu}, \mathcal{H}^k(\mathfrak{g}))t^k,$$

where $\mathcal{H}^k(\mathfrak{g})$ is the k-th homogeneous space of degree k. It is shown in [29] that

$$E_t(V_{\mathfrak{g}}^{\mu}) = K^{\mathfrak{g}}_{\mu 0}(t),$$

where $K^{\mathfrak{g}}_{\mu 0}(t)$ is the Lustig t-weight multiplicity for $V^{\mu}_{\mathfrak{g}}$ at weight 0. In other words, we have

$$\mathrm{ch}_t\mathcal{H}(\mathfrak{g}) = \sum_{\mu\in P_+} K^{\mathfrak{g}}_{\mu 0}(t)\mathrm{ch}V^{\mu}_{\mathfrak{g}}.$$

In [71], a new combinatorial realization of $K^{\mathfrak{sp}_n}_{\mu 0}(t)$ is given in terms of the LR tableaux which give a branching formula for (5.0.2) for $\mathrm{G}_n = \mathrm{Sp}_n$. The goal of this section is to give combinatorial formulas for $K^{\mathfrak{so}_n}_{\mu 0}(t)$ following the idea in [71] as a main application of Theorem 5.4.4.

5.5.2 Combinatorial formula of generalized exponents

Suppose that $\mathfrak{g} = \mathfrak{so}_n$ for $n \geq 3$, that is, $\mathfrak{g} = \mathfrak{so}_{2m+1}, \mathfrak{so}_{2m}$ for some m. We assume that the weight lattice for $\mathfrak{g}$ is $P = \bigoplus_{i=1}^m \mathbb{Z}\epsilon_i$ so that Φ^+ is $\{\, \epsilon_i \pm \epsilon_j, \epsilon_k \,|\, 1 \leq i < j \leq m, 1 \leq k \leq m \,\}$, and $\{\, \epsilon_i \pm \epsilon_j \,|\, 1 \leq i < j \leq m \,\}$ when $\mathfrak{g} = \mathfrak{so}_{2m+1}$, and $\mathfrak{g} = \mathfrak{so}_{2m}$, respectively. Let

$$\Delta^{\mathfrak{g}}_t = \frac{1}{\prod_{1\leq i<j\leq n}(1 - tx_ix_j)}.$$

By using the Littlewood identity (when $t = 1$), we have

$$\Delta^{\mathfrak{g}}_t = \sum_{\lambda\in\mathscr{P}^{(1,1)}_n} t^{|\lambda|/2}\mathrm{ch}V^{\lambda}_{\mathrm{GL}_n}, \tag{5.5.3}$$

where $|\lambda| = \sum_{i\geq 1}\lambda_i$ for $\lambda = (\lambda_i)_{i\geq 1}$. Note that (5.5.1) can be obtained from $\Delta^{\mathfrak{g}}_t$ by specializing it with respect to the torus of SO_n (see for example [71, Section 2.2]).

For $\mu \in \mathcal{P}(\mathrm{O}_n)$, put

$$[\![V^{\lambda}_{\mathrm{GL}_n} : V^{\mu}_{\mathrm{O}_n}]\!] = \begin{cases} \left[V^{\lambda}_{\mathrm{GL}_n} : V^{\mu}_{\mathrm{O}_n}\right] + \left[V^{\lambda}_{\mathrm{GL}_n} : V^{\overline{\mu}}_{\mathrm{O}_n}\right], & \text{if } \mu \neq \overline{\mu}, \\ \left[V^{\lambda}_{\mathrm{GL}_n} : V^{\mu}_{\mathrm{O}_n}\right], & \text{if } \mu = \overline{\mu}. \end{cases}$$

Proposition 5.5.1. *For $\mu \in \mathscr{P}_m$, we have*

$$\frac{K^{\mathfrak{so}_{2m+1}}_{\mu 0}(t)}{\prod_{i=1}^m(1 - t^{2i})} = \sum_{\lambda\in\mathscr{P}^{(1,1)}_{2m+1}} [\![V^{\lambda}_{\mathrm{GL}_{2m+1}} : V^{\mu}_{\mathrm{O}_{2m+1}}]\!]\, t^{|\lambda|/2},$$

$$\frac{K^{\mathfrak{so}_{2m}}_{\mu 0}(t)}{(1 - t^m)\prod_{i=1}^{m-1}(1 - t^{2i})} = \sum_{\lambda\in\mathscr{P}^{(1,1)}_{2m}} [\![V^{\lambda}_{\mathrm{GL}_{2m}} : V^{\mu}_{\mathrm{O}_{2m}}]\!]\, t^{|\lambda|/2},$$

where we regard μ in $K^{\mathfrak{g}}_{\mu 0}(t)$ as a dominant integral weight $\mu_1\epsilon_1 + \cdots + \mu_m\epsilon_m \in P_+$.

Proof. Suppose that $\mathfrak{g} = \mathfrak{so}_{2m+1}$. For $\mu \in \mathcal{P}(\mathrm{O}_{2m+1})$ with $\ell(\mu) \leq m$, we have $\mathrm{ch}V^{\mu}_{\mathrm{O}_{2m+1}} = \mathrm{ch}V^{\overline{\mu}}_{\mathrm{O}_{2m+1}} = \mathrm{ch}V^{\mu}_{\mathfrak{so}_{2m+1}}$. By taking restriction of (5.5.3) with respect to O_{2m+1}, we have

$$\mathrm{ch}_t S(\mathfrak{g}) = \sum_{\substack{\mu\in\mathcal{P}(\mathrm{O}_{2m+1})\\ \ell(\lambda)\leq m}} \left(\sum_{\lambda \in \mathscr{P}^{(1,1)}_{2m+1}} t^{|\lambda|/2} \left[\!\!\left[V^{\lambda}_{\mathrm{GL}_{2m+1}} : V^{\mu}_{\mathrm{O}_{2m+1}} \right]\!\!\right] \right) \mathrm{ch}V^{\mu}_{\mathfrak{so}_{2m+1}}. \tag{5.5.4}$$

Next, suppose that $\mathfrak{g} = \mathfrak{so}_{2m}$. For $\mu \in \mathcal{P}(\mathrm{O}_{2m})$ with $\ell(\mu) < m$, we have $\mathrm{ch}V^{\mu}_{\mathrm{O}_{2m}} = \mathrm{ch}V^{\overline{\mu}}_{\mathrm{O}_{2m}} = \mathrm{ch}V^{\mu}_{\mathfrak{so}_{2m}}$. For $\mu \in \mathcal{P}(\mathrm{O}_{2m})$ with $\ell(\mu) = m$, we have $\mathrm{ch}V^{\mu}_{\mathrm{O}_{2m}} = \mathrm{ch}V^{\mu}_{\mathfrak{so}_{2m}} + \mathrm{ch}V^{\mu^\sigma}_{\mathfrak{so}_{2m}}$, where $\mu^\sigma = \mu_1\epsilon_1 + \cdots + \mu_{m-1}\epsilon_{m-1} - \mu_m\epsilon_m$. Similarly, we have

$$\begin{aligned} \mathrm{ch}_t S(\mathfrak{g}) = & \sum_{\substack{\mu\in\mathcal{P}(\mathrm{O}_{2m})\\ \ell(\mu)< m}} \left(\sum_{\lambda \in \mathscr{P}^{(1,1)}_{2m}} t^{|\lambda|/2} \left[\!\left[V^{\lambda}_{\mathrm{GL}_{2m}} : V^{\mu}_{\mathrm{O}_{2m}} \right]\!\right] \right) \mathrm{ch}V^{\mu}_{\mathfrak{so}_{2m}} \\ & + \sum_{\substack{\mu\in\mathcal{P}(\mathrm{O}_{2m})\\ \ell(\mu)= m}} \left(\sum_{\nu \in \mathscr{P}^{(1,1)}_{2m}} t^{|\nu|/2} \left[\!\left[V^{\lambda}_{\mathrm{GL}_{2m}} : V^{\mu}_{\mathrm{O}_{2m}} \right]\!\right] \right) \left(\mathrm{ch}V^{\mu}_{\mathfrak{so}_{2m}} + \mathrm{ch}V^{\mu^\sigma}_{\mathfrak{so}_{2m}}\right). \end{aligned} \tag{5.5.5}$$

Now, combining (5.5.2) and (5.5.4), (5.5.5), we obtained the identities. □

Suppose that $P = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$ is the weight lattice of $\mathfrak{gl}_n$. For $1 \leq i \leq n-1$, let $\varpi_i = \epsilon_1 + \cdots + \epsilon_i$ be the ith fundamental weight.

Let $\mu \in \mathscr{P}_n$ be given. We identify μ with $\mu_1\epsilon_1 + \cdots + \mu_n\epsilon_n$. Let $SST_n(\mu)$ (resp. $SST_n(\mu^\pi)$) be the subset of $SST(\mu)$ (resp. $SST(\mu^\pi)$) consisting of T with entries in $\{1, \ldots, n\}$, which is a $\mathfrak{gl}_n$-crystal with highest weight μ. For $T \in SST_n(\mu)$ or $SST_n(\mu^\pi)$, put

$$\varphi(T) = \sum_{i=1}^{n-1} \varphi_i(T)\varpi_i, \quad \varepsilon(T) = \sum_{i=1}^{n-1} \varepsilon_i(T)\varpi_i.$$

Definition 5.5.2. [1] For $\rho \in \mathscr{P}_n$, we say that T is *ρ-distinguished* if

$$\varphi(T) = \lambda - \rho, \quad \varepsilon(T) = \delta - \rho,$$

[1] In this thesis, we use the notation ρ as a partition, not the half sum of positive roots.

for some $(\lambda, \delta) \in \mathscr{P}_n^{(1,1)} \times \mathscr{P}_n^{(2)}$.

We put

$$\begin{aligned} D_n(\mu) &= \{\, T \in SST_n(\mu^\pi) \,|\, T \text{ is } \rho\text{-distinguished for some } \rho \in \mathscr{P}_n \,\}, \\ \mathscr{P}_T &= \{\, \rho \in \mathscr{P}_n \,|\, T \text{ is } \rho\text{-distinguished} \,\} \quad (T \in D_n(\mu)). \end{aligned} \tag{5.5.6}$$

Lemma 5.5.3. *For $T \in D_n(\mu)$, there exists a unique $\rho_T \in \mathscr{P}$ such that $\mathscr{P}_T = \rho_T + \mathscr{P}_n^{(2,2)}$, where ρ_T is determined by*

$$\rho_T = \sum_{\substack{1 \le i \le n-1 \\ i \equiv 0 \mod 2}} (\varepsilon_i(T) \ mod\ 2)\varpi_i.$$

Proof. It follows from the same argument as in [71, Lemma 4.4, Proposition 4.5] with Definition 5.5.2. □

Definition 5.5.4. *We define $\underline{D}_n(\mu)$ to be the subset of $D_n(\mu)$ consisting of T satisfying the condition* (5.4.2).

Proposition 5.5.5. *For $\mu \in \mathscr{P}_m$, we have*

$$\sum_{\lambda \in \mathscr{P}_n^{(1,1)}} [\![V_{\mathrm{GL}_n}^\lambda : V_{\mathrm{O}_n}^\mu]\!] \, t^{|\lambda|/2} = \frac{1}{\prod_{i=1}^m (1 - t^{2i})} \sum_{T \in \mathbb{D}_n(\mu)} t^{|\varphi(T) + \rho_T|/2},$$

where

$$\mathbb{D}_n(\mu) = \begin{cases} \underline{D}_n(\mu) \sqcup \underline{D}_n(\overline{\mu}), & \text{if } \mu \neq \overline{\mu}, \\ \underline{D}_n(\mu), & \text{if } \mu = \overline{\mu}. \end{cases} \tag{5.5.7}$$

Proof. Recall that we have bijections for $\mu \in \mathcal{P}(\mathrm{O}_n)$

$$\bigsqcup_{\lambda \in \mathscr{P}_n^{(1,1)}} \mathtt{LR}_\lambda^\mu(\mathfrak{d}) \longrightarrow \bigsqcup_{\lambda \in \mathscr{P}_n^{(1,1)}} \bigsqcup_{\delta \in \mathscr{P}_n^{(2)}} \overline{\mathtt{LR}}_{\delta'\mu'}^{\lambda'} \longrightarrow \bigsqcup_{\lambda \in \mathscr{P}_n^{(1,1)}} \bigsqcup_{\delta \in \mathscr{P}_n^{(2)}} \underline{\mathtt{LR}}_{\delta\mu}^\lambda,$$

where the first one is given in Theorem 5.4.4 and the second one in Theorem 5.4.9. By definition of $\underline{D}_n(\mu)$, we have a bijection

$$\begin{array}{ccc} \bigsqcup_{\lambda \in \mathscr{P}_n^{(1,1)}} \bigsqcup_{\delta \in \mathscr{P}_n^{(2)}} \underline{\mathtt{LR}}_{\delta\mu}^\lambda & \longrightarrow & \bigsqcup_{T \in \underline{D}_n(\mu)} \{\, T \,\} \times \mathscr{P}_T \\ T & \longmapsto & (T, \lambda - \varphi(T)) = (T, \delta - \varepsilon(T)) \end{array} . \tag{5.5.8}$$

By Lemma 5.5.3, we have a bijection

$$\begin{array}{ccc} \bigsqcup_{T\in \underline{D}_n(\mu)}\{\,T\,\}\times \mathscr{P}_T & \longrightarrow & \bigsqcup_{T\in \underline{D}_n(\mu)}\{\,T\,\}\times \mathscr{P}_n^{(2,2)} \,. \\ (T,\rho) & \longmapsto & (T,\rho-\rho_T) \end{array} \tag{5.5.9}$$

Therefore, we have from (5.5.8) and (5.5.9)

$$\begin{aligned} \sum_{\lambda\in\mathscr{P}_n^{(1,1)}} \left[V_{\mathrm{GL}_n}^{\lambda} : V_{\mathrm{O}_n}^{\mu}\right] t^{|\lambda|/2} &= \sum_{\lambda\in\mathscr{P}_n^{(1,1)}} \sum_{\delta\in\mathscr{P}_n^{(2)}} \underline{c}_{\delta\mu}^{\lambda} t^{|\lambda|/2} = \sum_{T\in\underline{D}_n(\mu)} \sum_{\rho\in\mathscr{P}_T} t^{|\varphi(T)+\rho|/2} \\ &= \sum_{T\in\underline{D}_n(\mu)} t^{|\varphi(T)+\rho_T|/2} \sum_{\kappa\in\mathscr{P}_n^{(2,2)}} t^{|\kappa|/2} \\ &= \sum_{T\in\underline{D}_n(\mu)} t^{|\varphi(T)+\rho_T|/2} \frac{1}{\prod_{i=1}^{m}(1-t^{2i})}, \end{aligned}$$

which implies the identity. □

We have the following new combinatorial formulas for $K_{\mu 0}^{\mathfrak{so}_n}(t)$.

Theorem 5.5.6. *For $\mu\in\mathscr{P}_m$, we have*

$$\begin{aligned} K_{\mu 0}^{\mathfrak{so}_{2m+1}}(t) &= \sum_{T\in\mathbb{D}_{2m+1}(\mu)} t^{|\varphi(T)+\rho_T|/2}, \\ K_{\mu 0}^{\mathfrak{so}_{2m}}(t) &= \frac{1}{1+t^m} \sum_{T\in\mathbb{D}_{2m}(\mu)} t^{|\varphi(T)+\rho_T|/2}, \end{aligned}$$

where $\mathbb{D}_n(\mu)$ is given in (5.5.7).

Proof. It follows from Propositions 5.5.1 and 5.5.5. □

Remark 5.5.7. Since $K_{\mu 0}^{\mathfrak{so}_{2m}}(t)$ is a polynomial in t, the polynomial

$$\sum_{T\in\mathbb{D}_{2m}(\mu)} t^{|\varphi(T)+\rho_T|/2}$$

is divisible by $1+t^m$. From the positivity of Kostka-Foulkes polynomial $K_{\mu 0}^{\mathfrak{so}_{2m}}(t)$, one may expect a decomposition of $\mathbb{D}_{2m}(\mu)=X_1\sqcup X_2$ together with a bijection $\tau: X_1\longrightarrow X_2$ such that

$$|\varphi(\tau(T))+\rho_{\tau(T)}| = 2m + |\varphi(T)+\rho_T|.$$

Remark 5.5.8. In [71], Lecouvey-Lenart provide a bijection between the distinguished tableaux for type C_n and the symplectic King tableaux with weight 0 (see [71, Section 6.4] for more details). We do not know yet whether there is an analogue of the above bijection which maps an orthogonal distinguished tableau (Definition 5.5.4) to an orthogonal tableau (with weight 0), which is from already known models (for example, [52, 53, 57, 88, 94, 106]) or a new one).

Chapter 6

Affine crystals

In this chapter, we extend the crystal $\mathbf{B}^J$ to an affine crystal[1] and then we give a new combinatorial model of KR crystal $B^{n,s}$ ($s \in \mathbb{Z}_{\geq 1}$) obtained from the crystal $\mathbf{B}^J$ by computing the ε_n^*-statistic explicitly. Also, we prove that Burge correspondence is an isomorphism of affine crystals of type $D_n^{(1)}$.

The results in this chapter are based on [36].

6.1 Quantum affine algebras and crystals

In this section, let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra with the index set $I = \{1, \ldots, n\}$. We denote by $\widehat{\mathfrak{g}}$ the corresponding affine Kac-Moody algebra of untwisted type with index set $\widehat{I} = \{0, 1, \ldots, n\}$ [41]. Let $\widehat{P}^\vee$ be the dual weight lattice given by

$$\widehat{P}^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n \oplus \mathbb{Z}d.$$

Then $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} \widehat{P}^\vee$ is the Cartan subalgebra. Let $\widehat{P} = \{\lambda \in \mathfrak{h}^* \mid \lambda(\widehat{P}^\vee) \subset \mathbb{Z}\}$ be the weight lattice of $\widehat{\mathfrak{g}}$. We denote by Λ_i the ith fundamental weight of $\widehat{\mathfrak{g}}$. Put $\widehat{\Pi} = \{\alpha_i \mid i \in \widehat{I}\}$ and $\widehat{\Pi}^\vee = \{h_i \mid i \in \widehat{I}\}$ to be the sets of simple roots and simple coroots, respectively. Let δ be the null root (see [41, Chapter 5]). Note that

$$\widehat{P} = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_n \oplus \mathbb{Z}\delta.$$

[1] In this thesis, an *affine crystal* means a $\widehat{\mathfrak{g}}$-crystal or the crystal graph of the crystal base of a certain finite-dimensional irreducible $U_q'(\widehat{\mathfrak{g}})$-module.

We denote by $\widehat{P}^{+}$ the set of dominant integral weights of $\widehat{\mathfrak{g}}$. Let $U_q(\widehat{\mathfrak{g}})$ be the quantum group of $\widehat{\mathfrak{g}}$ over $\mathbb{C}(q)$, which is called *quantum affine algebra* (recall Definition 2.1.1). Then the subalgebra of $U_q(\widehat{\mathfrak{g}})$ generated by $e_i, f_i, K_i^{\pm}$ for $i \in \widehat{I}$, denoted by $U_q'(\widehat{\mathfrak{g}})$, is also called the *quantum affine algebra*. We remark that all non-trivial irreducible representations of $U_q(\widehat{\mathfrak{g}})$ are infinite-dimensional (cf. [8, Section 2.6]). On the other hand, there exist finite-dimensional irreducible representations of $U_q'(\widehat{\mathfrak{g}})$.

By Chari-Pressley's classification [10,11], each isomorphism class of finite-dimensional irreducible representations (of type 1) is parametrized by an n-tuple $\mathbf{P} = (P_i(u))_{1\le i\le n}$ of polynomials with constant term 1, where n is the rank of $\mathfrak{g}$. The polynomial $\mathbf{P}$ is called Drinfeld's polynomials due to an analog result for Yangian earlier by Drinfeld [17].

The Kirillov-Reshetikhin (KR for short) module $W_{s,a}^{(r)}$ is the finite-dimensional irreducible $U_q'(\widehat{\mathfrak{g}})$-module with the Drinfeld polynomial $\mathbf{P} = (P_i(u))_{1\le i\le n}$

$$P_i(u) := \begin{cases} \prod_{j=1}^{s} (1 - aq^{s-2j+1}u) & \text{if } i = r, \\ 1 & \text{otherwise,} \end{cases}$$

where $1 \le r \le n$, $s \in \mathbb{Z}_+$ and $a \in \mathbb{C}^\times$ [54].

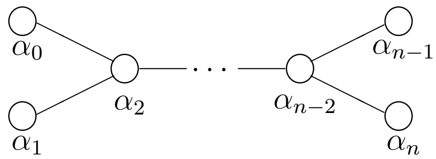
It was conjectured by Hatayama et al.[28] that for $1 \le r \le n$ and $s \in \mathbb{Z}_+$, there exists $a_{r,s} \in \mathbb{C}^\times$ such that $W_{s,a_{r,s}}^{(r)}$ has a crystal base. The conjecture has been proved for all nonexceptional types [91] (see also [42] for type $A_n^{(1)}$, [89] for type $D_n^{(1)}$ with $1 \le r \le n-2$) and some exceptional types (with certain r) [86, 87]. Let $B^{r,s}$ be the crystal of $W_{s,a_{r,s}}^{(r)}$, which is often called KR crystal for short.

The following lemma is useful to describe the KR crystals $B^{r,s}$ (recall Definition 2.2.5(7)).

Lemma 6.1.1. [101, Lemma 2.6] *Let $\widehat{\mathfrak{g}}$ be an affine Kac-Moody algebra of non-exceptional type with index set $\widehat{I} = \{0, 1, \ldots, n\}$. For $r \in \widehat{I} \setminus \{0\}$ and $s > 0$, any regular $\widehat{\mathfrak{g}}$-crystal B which is isomorphic to $B^{r,s}$ as a $\mathfrak{g}$-crystal is also isomorphic to $B^{r,s}$ as a $\widehat{\mathfrak{g}}$-crystal.* □

6.2 Kirillov-Reshetikhin crystals $B^{n,s}$ of type $\mathrm{D}_n^{(1)}$

Let $\widehat{\mathfrak{g}}$ be an affine Kac-Moody algebra of type $\mathrm{D}_n^{(1)}$ with $\widehat{I} = \{\,0, 1, \ldots, n\,\}$ the index set for the simple roots.



For $r \in \{0, n\}$, let $\widehat{\mathfrak{g}}_r$ be the subalgebra of $\widehat{\mathfrak{g}}$ corresponding to $\{\, \alpha_i \,|\, i \in \widehat{I} \setminus \{r\} \,\}$. Then $\widehat{\mathfrak{g}}_0 = \mathfrak{g}$, and $\widehat{\mathfrak{g}}_0 \cap \widehat{\mathfrak{g}}_n = \mathfrak{l}$, where $\mathfrak{l}$ is the subalgebra of type A_{n-1}. We regard $P = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$ as a sublattice of $\widehat{P}/\mathbb{Z}\delta$ by putting $\epsilon_1 = \Lambda_1 - \Lambda_0$, $\epsilon_2 = \Lambda_2 - \Lambda_1 - \Lambda_0$, $\epsilon_k = \Lambda_k - \Lambda_{k-1}$ for $k = 3, \ldots, n-2$, $\epsilon_{n-1} = \Lambda_{n-1} + \Lambda_n - \Lambda_{n-2}$ and $\epsilon_n = \Lambda_n - \Lambda_{n-1}$. In particular, we have $\alpha_0 = -\epsilon_1 - \epsilon_2$ in P. If ϖ'_i are the fundamental weights for $\widehat{\mathfrak{g}}_n$ for $i \in \widehat{I} \setminus \{n\}$, then $\varpi'_i = \varpi_i$ for $i \in \widehat{I} \setminus \{0, n\}$ and $\varpi'_0 = -\varpi_n$.

Let us recall Sections 3.2.4–3.2.5 for the crystal $\mathbf{B}^J$. For $\mathbf{c} \in \mathbf{B}^J$, we define

$$\begin{aligned} &\widetilde{e}_0\mathbf{c} = \mathbf{c} + \mathbf{1}_{\epsilon_1+\epsilon_2}, \quad \widetilde{f}_0\mathbf{c} = \begin{cases} \mathbf{c} - \mathbf{1}_{\epsilon_1+\epsilon_2}, & \text{if } c_{\epsilon_1+\epsilon_2} > 0, \\ \mathbf{0}, & \text{otherwise}, \end{cases} \\ &\varphi_0(\mathbf{c}) = \max\{\, k \,|\, \widetilde{f}_0^k \mathbf{c} \neq \mathbf{0} \,\}, \quad \varepsilon_0(\mathbf{c}) = \varphi_0(\mathbf{c}) - \langle \mathrm{wt}(\mathbf{c}), h_0 \rangle. \end{aligned} \tag{6.2.1}$$

Lemma 6.2.1. *The set $\mathbf{B}^J$ is a $\widehat{\mathfrak{g}}$-crystal with respect to* wt, ε_i, φ_i, $\widetilde{e}_i$, $\widetilde{f}_i$ *for $i \in \widehat{I}$, where* wt *is the restriction of* $\mathrm{wt} : \mathbf{B} \longrightarrow P$ *to* $\mathbf{B}^J$.

Proof. It follows directly from (6.2.1). □

Next, we consider the subcrystal $\mathbf{B}^{J,s}$ of $\mathbf{B}^J$ given by

$$\mathbf{B}^{J,s} := \{\, \mathbf{c} \in \mathbf{B}^J \,|\, \varepsilon_n^*(\mathbf{c}) \leq s \,\}, \tag{6.2.2}$$

where $s \geq 1$. By Propositions 2.2.12 and 3.2.7 (cf. [44]), we have

$$B(s\varpi_n) \cong \mathbf{B}^{J,s} \otimes T_{s\varpi_n}, \qquad \bigcup_{s \geq 1} \mathbf{B}^{J,s} = \mathbf{B}^J, \tag{6.2.3}$$

as $\mathfrak{g}$-crystals.

The following theorem is one of main results in this chapter.

Theorem 6.2.2. *For $s \geq 1$, $\mathbf{B}^{J,s} \otimes T_{s\varpi_n}$ is a regular $\widehat{\mathfrak{g}}$-crystal and*

$$\mathbf{B}^{J,s} \otimes T_{s\varpi_n} \cong B^{n,s},$$

where $B^{n,s}$ is the Kirillov-Reshetikhin crystal of type $D_n^{(1)}$ associated with $s\varpi_n$.

Proof. By (6.2.3), $\mathbf{B}^{J,s} \otimes T_{s\varpi_n}$ is a regular $\widehat{\mathfrak{g}}_0$-crystal. By Proposition 3.2.3, we see that the $\widehat{\mathfrak{g}}_n$-crystal $\mathbf{B}^{J,s} \otimes T_{s\varpi_n}$ is isomorphic to the dual of the $\widehat{\mathfrak{g}}_0$-crystal $\mathbf{B}^{J,s} \otimes T_{s\varpi_n}$ assuming that $\widehat{\mathfrak{g}}_n \cong \widehat{\mathfrak{g}}_0$ under the correspondence $\alpha_i \leftrightarrow -\alpha_{n-i}$ for $0 \le i \le n-1$. This implies that $\mathbf{B}^{J,s} \otimes T_{s\varpi_n}$ is a regular $\widehat{\mathfrak{g}}_n$-crystal, and hence a regular $\widehat{\mathfrak{g}}$-crystal. It is known that $B^{n,s}$ is classically irreducible, that is, $B^{n,s} \cong B(s\varpi_n)$ as a $\widehat{\mathfrak{g}}_0$-crystal (see [21]). Therefore, it follows from Lemma 6.1.1 that $\mathbf{B}^{J,s} \otimes T_{s\varpi_n} \cong B^{n,s}$. □

Remark 6.2.3. The inclusions $\mathbf{B}^{J,s} \hookrightarrow \mathbf{B}^{J,t} \hookrightarrow \mathbf{B}^{J}$ for $s \le t$ are embeddings of $\widehat{\mathfrak{g}}$-crystals (recall (6.2.3)), and hence $\mathbf{B}^{J}$ is a direct limit of $\left\{ \mathbf{B}^{J,s} \,|\, s \in \mathbb{Z}_+ \right\}$.

Let us recall the result in Section 3.3.2. Indeed, we can characterize ε_n^* in terms of the double paths on Δ_n.

Theorem 6.2.4. *For $\mathbf{c} \in \mathbf{B}^J$,*

$$\varepsilon_n^*(\mathbf{c}) = \max\{\, ||\mathbf{c}||_{\mathbf{p}} \,|\, \mathbf{p} \textit{ is a double path in } \Delta_n \,\}.$$

Proof. This formula is obtained from an explicit computation of the formula in [4, Theorem 3.7] for the transition matrix between Lusztig's parametrization and string parametrization of $B(\infty)$. We give the detailed proof in Section 7.1.3 (see also Section 7.1.1). □

Example 6.2.5. We refer the reader to Appendix A.1.5 for the crystal graph for $\mathbf{B}^{J,2}$ with $n = 4$ (cf. [62, Figure 3]).

Remark 6.2.6.

(1) Let $\theta = \epsilon_1 + \epsilon_n$ be the longest root in Φ^+. Since θ is located at the top of Δ_n, the formula in Theorem 6.2.4 is equivalent to

$$\varepsilon_n^*(\mathbf{c}) = \max\{\, ||\mathbf{c}||_{\mathbf{p}} \,|\, \mathbf{p} \text{ is a double path at } \theta \text{ in } \Delta_n \,\}.$$

(2) For $\mathbf{c} \in \mathbf{B}^J$, we have $\lambda_1(\mathbf{c}) = \varepsilon_n^*(\mathbf{c})$. Let us explain it in more detail. We define

$$\mathcal{T}_s^{\searrow} := \bigsqcup_{\substack{\lambda \in \mathscr{P}_n \\ \lambda' : \text{even},\ \lambda \subset (s^n)}} SST(\lambda^\pi)$$

and we regard it as a subcrystal of $\mathcal{T}^{\searrow}$ (recall Section 3.3.1). It is known [62, Lemma 5.1] that $\mathcal{T}_s^{\searrow} \otimes T_{s\varpi_n}$ is isomorphic to $B(s\varpi_n)$. Then the crystal isomorphism $\kappa^{\searrow}$ (3.3.2) induces an isomorphism of $\mathfrak{g}$-crystals between $\mathbf{B}^{J,s}$ and $\mathcal{T}_s^{\searrow}$. Therefore, $\varepsilon_n^*(\mathbf{c})$ is equal to the number of columns of $\kappa^{\searrow}(\mathbf{c})$ (cf. [30, Proposition 4.5.8]).

(3) By Theorem 6.2.4, we have

$$\mathbf{B}^{J,s} = \bigcap_{\mathbf{p}} \left\{ \mathbf{c} \in \mathbf{B}^J \mid ||\mathbf{c}||_{\mathbf{p}} \leq s \right\},$$

where $\mathbf{p}$ runs over the double paths in Δ_n. This gives a polytope realization of the KR crystal $B^{n,s}$. Moreover, $\{B^{J,s}\}$ is a family of perfect KR crystals by [22, Theorem 1.2].

6.3 Burge correspondence of type $\mathbf{D}_n^{(1)}$

In this section, we extend Burge correspondence to an isomorphism of affine crystals (recall Section 3.3). We keep the notations in Section 3.1.1. Let us recall that the affine crystal structure on $\mathbf{B}^J$ is given in (6.2.1). Following [62], we consider the set

$$\mathcal{T} = \left\{ [T] \mid T \in \mathcal{T}^{\searrow} \right\},$$

where $[T]$ is the Knuth equivalence class of T. We give an affine crystal structure on $\mathcal{T}$ following [62]. Consequently, it is isomorphic to the affine crystal $\mathbf{B}^J$ via Burge correspondence.

Let us explain it in more detail. We define

$$\mathcal{T}^{\nwarrow} := \bigsqcup_{\substack{\lambda \in \mathscr{P}_n \\ \lambda' : \text{even}}} SST_{[\overline{n}]}(\lambda).$$

As in the case $\mathcal{T}^{\searrow}$ (3.3.1), we define the $\widehat{\mathfrak{g}}_n$-crystal structure as follows.

We define the $\mathfrak{l}$-crystal structure on $\mathcal{T}^{\nwarrow}$ in the same way as in $\mathcal{T}^{\searrow}$. Let $T \in \mathcal{T}^{\nwarrow}$ be given. For $k \geq 1$, let t_k be the entry in the bottom of the k-th column of T (enumerated from the left). Consider $\sigma = (\dots, \sigma_2, \sigma_1)$, where

$$\sigma_k = \begin{cases} - \ , & \text{if } t_k < \overline{2} \text{ or the } k\text{-th column is empty,} \\ + \ , & \text{if the } k\text{-th column has both } \overline{1} \text{ and } \overline{2} \text{ as its entries,} \\ \cdot \ , & \text{otherwise.} \end{cases}$$

Then $\widetilde{e}_0 T$ is given by adding $\begin{array}{|c|}\hline \overline{2} \\ \hline \overline{1} \\ \hline \end{array}$ to the bottom of the column corresponding to the right-most $-$ in σ^{red}, and $\widetilde{f}_0 T$ is obtained from T by removing $\begin{array}{|c|}\hline \overline{2} \\ \hline \overline{1} \\ \hline \end{array}$ in the column corresponding to the left-most $+$ in σ^{red}. If there is no such $+$ sign, then we define $\widetilde{f}_0 T = \mathbf{0}$. Hence $\mathcal{T}^{\nwarrow}$ is a $\widehat{\mathfrak{g}}_n$-crystal with respect to wt, ε_i, φ_i, $\widetilde{e}_i$, $\widetilde{f}_i$ $(i \in \widehat{I} \setminus \{n\})$, where $\varphi_0(T) = \max\{\, k \mid \widetilde{f}_0^k T \neq \mathbf{0}\,\}$ and $\varepsilon_0(T) = \varphi_0(T) - \langle \mathrm{wt}(T), h_0 \rangle$.

Next, we define an analog of $\kappa^{\searrow}$ from $\mathbf{B}^J$ to $\mathcal{T}^{\nwarrow}$ as follows. Let Ω' be the set of biwords $(\mathbf{a}, \mathbf{b}) \in \mathcal{W} \times \mathcal{W}$ satisfying the same conditions as in Ω except that $<$ is replaced by $<'$, where $(a, b) <' (c, d)$ if and only if $(b < d)$ or $(b = d$ and $a < c)$ for (a, b) and $(c, d) \in \mathcal{W} \times \mathcal{W}$. We define $\mathbf{c}'(\mathbf{a}, \mathbf{b})$ in the same way as in $\mathbf{c}(\mathbf{a}, \mathbf{b})$. Given $(\mathbf{a}, \mathbf{b}) \in \Omega'$ with $\mathbf{a} = a_1 \cdots a_r$ and $\mathbf{b} = b_1 \cdots b_r$, define a sequence of tableaux $P_1, P_2, \dots, P_r$ inductively as follows:

(1) let P_1 be a vertical domino $\begin{array}{|c|}\hline a_1 \\ \hline b_1 \\ \hline \end{array}$,

(2) if P_{k-1} is given for $2 \leq k \leq r$, then define P_k to be the tableau obtained by first applying the column insertion to get $a_k \to P_{k-1}$, and then adding $\boxed{b_k}$ at the conner of $a_k \to P_{k-1}$ located below the box $\mathrm{sh}(a_k \to P_{k-1})/\mathrm{sh}(P_{k-1})$,

and put $P^{\nwarrow}(\mathbf{a}, \mathbf{b}) := P_r$. For $\mathbf{c} \in \mathbf{B}^J$, let $P^{\nwarrow}(\mathbf{c}) = P^{\nwarrow}(\mathbf{a}, \mathbf{b})$ where $\mathbf{c} = \mathbf{c}'(\mathbf{a}, \mathbf{b})$. Then we also have a bijection

$$\begin{array}{rccc} \kappa^{\nwarrow} : & \mathbf{B}^J & \longrightarrow & \mathcal{T}^{\nwarrow} \\ & \mathbf{c} & \longmapsto & P^{\nwarrow}(\mathbf{c}) \end{array} . \tag{6.3.1}$$

Theorem 6.3.1. *The bijection* $\kappa^{\nwarrow}$ *in* (6.3.1) *is an isomorphism of* $\widehat{\mathfrak{g}}_n$*-crystals.*

Proof. The proof is identical with the one of Theorem 3.3.3 (see Section 7.1.2). □

For a semistandard tableau T of skew shape, let $[T]$ denote the equivalence class of T

with respect to Knuth equivalence (see [23] for definition). If we define

$$\widetilde{x}_i[T] = \begin{cases} [\widetilde{x}_0 T^{\nwarrow}], & \text{if } i = 0, \\ [\widetilde{x}_n T^{\searrow}], & \text{if } i = n, \\ [\widetilde{x}_i T], & \text{otherwise,} \end{cases}$$

for $i \in \widehat{I}$ and $x = e, f$ (we assume that $[\mathbf{0}] = \mathbf{0}$), then the set

$$\mathcal{T} = \left\{ [T] \,|\, T \in \mathcal{T}^{\searrow} \right\} = \left\{ [T] \,|\, T \in \mathcal{T}^{\nwarrow} \right\} \tag{6.3.2}$$

is a $\widehat{\mathfrak{g}}$-crystal with respect to $\widetilde{e}_i$, $\widetilde{f}_i$ ($i \in I$), where wt, ε_i, and φ_i are well-defined on $[T]$ [62, Section 5.3]. Therefore,

Corollary 6.3.2. *The map*

$$\begin{array}{rcl} \kappa^{\mathfrak{d}} : \mathbf{B}^J & \longrightarrow & \mathcal{T} \\ \mathbf{c} & \longmapsto & [P^{\nwarrow}(\mathbf{c})] = [P^{\searrow}(\mathbf{c})] \end{array},$$

is an isomorphism of $\widehat{\mathfrak{g}}$-crystals. □

For $s \geq 1$, let $\mathcal{T}^s = \{ [T] \,|\, \ell(\mathrm{sh}(T)') \leq s \} \subset \mathcal{T}$. It is shown in [62, Theorem 5.4] that $\mathcal{T}^s \otimes T_{s\varpi_n} \cong B^{n,s}$. Therefore, we have the following.

Corollary 6.3.3. *The map $\kappa^{\mathfrak{d}}$ when restricted to $\mathbf{B}^{J,s}$ gives an isomorphism of $\widehat{\mathfrak{g}}$-crystals*

$$\kappa^{\mathfrak{d}} : \mathbf{B}^{J,s} \longrightarrow \mathcal{T}^s .$$

Remark 6.3.4. It is already known in [62] that the matrix realization of $\mathbf{B}^{r,s}$ for type $A_{n-1}^{(1)}$ is obtained from RSK correspondence. Furthermore, by the folding technique [46], the approach is available for types $\mathrm{D}_{n+1}^{(2)}$ and $\mathrm{C}_n^{(1)}$. Although the tableau description of $B^{n,s}$ for type $\mathrm{D}_n^{(1)}$ is known in [62, Section 5.3], RSK correspondence does not seem to be extended to an isomorphism of $\mathrm{D}_n^{(1)}$-crystals. Therefore, it does not give a matrix realization of $B^{n,s}$ for type $\mathrm{D}_n^{(1)}$ by the approach in [62].

The crystal $\mathbf{B}^{J,s}$ can be viewed as the matrix realization of $B^{n,s}$ mentioned in [62, Remark 5.5] and Burge correspondence is an analog of RSK for type $\mathrm{D}_n^{(1)}$ in this viewpoint.

Remark 6.3.5. The combinatorics on $B^{r,s}$ for $r \neq 1, n-1, n$ is more complicated than the cases $r = 1, n-1$ and n. For example, see [100] for the KR crystal $B^{2,s}$ for type $\mathrm{D}_n^{(1)}$. Note that the KR crystal $B^{2,s}$ is not classically irreducible and it is decomposed as D_n-crystal into a direct sum of classical D_n-crystals as follows:

$$B^{2,s} \cong \bigoplus_{k=1}^{n} B(k\varpi_2),$$

where $B(k\varpi_2)$ is the D_n-crystal of an integral highest weight irreducible module with highest weight $k\varpi_2$. This yields that the description of 0th crystal operators $\widetilde{e}_0$ and $\widetilde{f}_0$ are more complicated than (6.2.1).

Chapter 7

Proofs

7.1 In Chapters 3 and 6

7.1.1 Formula of Berenstein-Zelevinsky

Let us recall a result on Lusztig's parametrization and string parametrization of $B(\infty)$ due to Berenstein-Zelevinsky [4], which plays a crucial role in proving Theorems 3.3.6 and 6.2.4.

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra. We keep the notations in Chapter 2. For $i \in I$, let $B_i = \{ (x)_i \,|\, x \in \mathbb{Z} \}$ be the abstract crystal given by $\mathrm{wt}((x)_i) = x\alpha_i$, $\varepsilon_i((x)_i) = -x$, $\varphi_i((x)_i) = x$, $\varepsilon_j((x)_i) = -\infty$, $\varphi_j((x)_i) = -\infty$ for $j \neq i$ and $\tilde{e}_i(x)_i = (x+1)_i$, $\tilde{f}_i(x)_i = (x-1)_i$, $\tilde{e}_j(x)_i = \tilde{f}_j(x)_i = 0$ for $j \neq i$. It is well-known that for any $i \in I$, there is a unique embedding of crystals [44]

$$\Psi_i \,:\, B(\infty) \hookrightarrow B(\infty) \otimes B_i$$

sending $b_\infty \mapsto b_\infty \otimes (0)_i$, where b_∞ is the highest weight element in $B(\infty)$. This embedding satisfies that for $b \in B(\infty)$, $\Psi_i(b) = b' \otimes (-a)_i$, where $a = \varepsilon_i(b^*)$ and $b' = (\tilde{e}_i^a(b^*))^*$. Given $b \in B(\infty)$ and a sequence of indices $\mathbf{i} = (i_1, \cdots, i_l)$ in I, consider the sequence $b_k \in B(\infty)$ and $a_k \in \mathbb{Z}_+$ for $1 \leq k \leq l-1$ defined inductively by

$$b_0 = b, \quad \Psi_{i_k}(b_{k-1}) = b_k \otimes (-a_k)_{i_k}.$$

The sequence $t_{\mathbf{i}}(b) = (a_l, \cdots, a_1)$ is called the *string of b in direction* $\mathbf{i}$. By construction, it can be reformulated by

$$a_k = \varepsilon_{i_k}(\tilde{e}_{i_{k-1}}^{a_{k-1}} \cdots \tilde{e}_{i_1}^{a_1} b^*), \tag{7.1.1}$$

for $2 \leq k \leq l$, where $a_1 = \varepsilon_{i_1}(b^*)$.

Suppose that $\mathfrak{g}$ is of finite type. Let V be a finite-dimensional $\mathfrak{g}$-module and V_λ denote the weight space of V for $\lambda \in \mathrm{wt}(V)$, where $\mathrm{wt}(V)$ is the set of weights of V.

Definition 7.1.1. For $\lambda, \mu \in \mathrm{wt}(V)$, an $\mathbf{i}$-*trail from* λ *to* μ *in* V is a sequence of weights $\pi = (\lambda = \nu_0, \nu_1, \ldots, \nu_l = \mu)$ in $\mathrm{wt}(V)$ satisfying the following conditions:

(1) for $1 \leq k \leq l$, $\nu_{k-1} - \nu_k = d_k(\pi)\alpha_{i_k}$ for some $d_k(\pi) \in \mathbb{Z}_+$,

(2) $e_{i_1}^{d_1(\pi)} \cdots e_{i_l}^{d_l(\pi)}$ is a non-zero linear map from V_μ to V_λ.

Remark 7.1.2. When V is a module with a *minuscule*[1] highest weight, then the condition (1) implies (2). Furthermore, if B is a crystal of V, then we have $\widetilde{e}_{i_1}^{d_1(\pi)} \cdots \widetilde{e}_{i_l}^{d_l(\pi)} B_\mu = B_\lambda$ or $\widetilde{f}_{i_l}^{d_l(\pi)} \cdots \widetilde{f}_{i_1}^{d_1(\pi)} B_\lambda = B_\mu$.

Let $\mathbf{i} = (i_1, \cdots, i_N) \in R(w_0)$ given. Let $\mathbf{i}^* := (i_1^*, \cdots, i_N^*)$ and $\mathbf{i}^{\mathrm{op}} := (i_N, \cdots, i_1)$, where $i \mapsto i^*$ is the involution on I given by $w_0(\alpha_i) = -\alpha_{i^*}$. For $\mathbf{c} \in \mathbf{B}_\mathbf{i}$, we have by [4, Proposition 3.3]

$$b_\mathbf{i}(\mathbf{c})^* = b_{\mathbf{i}^{*\mathrm{op}}}(\mathbf{c}^{\mathrm{op}}), \tag{7.1.2}$$

where $\mathbf{c}^{\mathrm{op}} = (c_k^{\mathrm{op}})$ is given by $c_k^{\mathrm{op}} = c_{N-k}$ for $\mathbf{c} = (c_k)$.

Theorem 7.1.3 ([4], Theorem 3.7)**.** *For* $\mathbf{i}, \mathbf{i}' \in R(w_0)$ *and* $\mathbf{c} \in \mathbf{B}_\mathbf{i}$, *let* $\boldsymbol{t} = t_\mathbf{i}(b_{\mathbf{i}'}(\boldsymbol{c})^*)$. *Then* $\boldsymbol{t} = (t_k)$ *and* $\boldsymbol{c} = (c_m)$ *are related as follows : for any* $k = 1, \cdots, N$

$$t_k = \min_{\pi_1} \left\{ \sum_{m=1}^{N} d_m(\pi_1) c_m \right\} - \min_{\pi_2} \left\{ \sum_{m=1}^{N} d_m(\pi_2) c_m \right\}, \tag{7.1.3}$$

where π_1 *(resp.* π_2*) runs over* $\mathbf{i}'$*-trails from* $s_{i_1} \cdots s_{i_{k-1}} \varpi_{i_k}$ *(resp. from* $s_{i_1} \cdots s_{i_k} \varpi_{i_k}$*) to* $w_0 \varpi_{i_k}$ *in the fundamental representation* $V(\varpi_{i_k})$.

Remark 7.1.4. The string parametrization of $b \in B(\infty)$ given by (7.1.1) is the string parametrization of b^* in [5] (see also [85, Remark in Section 2]).

[1] This means that Weyl group acts transitively on the weights.

7.1.2 Proof of Theorem 3.3.3

We keep the notations in Chapter 3. We assume that $\Delta_i \subset \Delta_n$ for $1 \leq i \leq n$, where both of Δ_i and Δ_n share the same southeast corner. For $\mathbf{c} = (c_k) \in \mathbf{B}^J$, let $\mathbf{c}_{\Delta_i} \in \mathbf{B}^J$ (resp. $\mathbf{c}_{\Delta_i^c} \in \mathbf{B}^J$) whose component in Δ_i (resp. $\Delta_i^c := \Delta_n \setminus \Delta_i$) is c_k and 0 elsewhere. Let

$$\mathbf{B}^J_{\Delta_i} = \{\, \mathbf{c} \in \mathbf{B}^J \,|\, \mathbf{c}_{\Delta_i^c} = \mathbf{0} \,\}.$$

Fix $i \in I \setminus \{n\}$. Let $\mathbf{c} \in \mathbf{B}^J_{\Delta_{i+1}}$ given with $\mathbf{c} = \mathbf{c}(\mathbf{a}, \mathbf{b})$ for a unique $(\mathbf{a}, \mathbf{b}) = (a_1 \dots a_r, b_1 \dots b_r) \in \Omega$. We divide $(\mathbf{a}, \mathbf{b})$ into two biwords $(\mathbf{a}', \mathbf{b}') = (a_1 \dots a_s, b_1 \dots b_s)$ and $(\mathbf{a}'', \mathbf{b}'') = (a_{s+1} \dots a_r, b_{s+1} \dots b_r)$ where $a_s \leq \overline{i}$ and $a_{s+1} > \overline{i}$ so that

$$\begin{pmatrix} \mathbf{a}' \\ \mathbf{b}' \end{pmatrix} = \begin{pmatrix} \overline{i+1} \\ \overline{1} \end{pmatrix}^{c_{\overline{i+1}\overline{1}}} \cdots \begin{pmatrix} \overline{i+1} \\ \overline{i} \end{pmatrix}^{c_{\overline{i+1}\overline{i}}} \begin{pmatrix} \overline{i} \\ \overline{1} \end{pmatrix}^{c_{\overline{i}\overline{1}}} \cdots \begin{pmatrix} \overline{i} \\ \overline{i-1} \end{pmatrix}^{c_{\overline{i}\overline{i-1}}}. \tag{7.1.4}$$

Here the superscript means the multiplicity of each biletter.

Let $\mathbf{c}' = \mathbf{c}(\mathbf{a}', \mathbf{b}')$ and $\mathbf{c}'' = \mathbf{c}(\mathbf{a}'', \mathbf{b}'') \in \mathbf{B}^J_{\Delta_{i-1}}$ be the corresponding Lusztig data. Put $T = \kappa^{\searrow}(\mathbf{c}'')$. Then $\mathrm{sh}(T) = \mu^\pi$ for some $\mu \in \mathscr{P}_{i-1}$ such that μ' is even. We define $(\mathtt{P}(\mathbf{c}), \mathtt{Q}(\mathbf{c}))$ by

(1) $\mathtt{P}(\mathbf{c}) = ((T \leftarrow b_s) \leftarrow \cdots \leftarrow b_1)$,

(2) $\mathtt{Q}(\mathbf{c}) \in SST((\lambda/\mu)^\pi)$, where $\mathrm{sh}(\mathtt{P}_k)/\mathrm{sh}(\mathtt{P}_{k-1})$ is filled with a_k for $1 \leq k \leq s$.

Here $\lambda = \mathrm{sh}(\mathtt{P}(\mathbf{c}))^\pi$, $\mathtt{P}_k = ((T \leftarrow b_k) \leftarrow \cdots \leftarrow b_1)$, and $\mathtt{P}_0 = T$. The pair $(\mathtt{P}(\mathbf{c}), \mathtt{Q}(\mathbf{c}))$ can be viewed as a skew-analogue of RSK correspondence applying insertion of $(\mathbf{a}', \mathbf{b}')$ into T (cf. [23, Proposition 1 in Section 5.1] and [97]).

Lemma 7.1.5. *Under the above hypothesis, we have*

$$\mathtt{Q}(\widetilde{f}_i \mathbf{c}) = \widetilde{f}_i \mathtt{Q}(\mathbf{c}).$$

Proof. Considering the action of $\widetilde{f}_i$ on the subcrystal $\mathbf{B}^J_{\Delta_{i+1}}$ of $\mathbf{B}^J$ described in Proposition 3.2.3, we may apply [60, Proposition 4.6 and Remark 4.8(1)] to have $\mathtt{Q}(\widetilde{f}_i \mathbf{c}) = \widetilde{f}_i \mathtt{Q}(\mathbf{c})$ (because we can naturally identify each element of $\mathbf{B}^J_{\Delta_{k+1}}$ with an element of the crystal $\mathcal{M}$ in [60, Section 3]). □

Let $\ell(\lambda) = 2m$ for some $m \geq 1$. For $1 \leq l \leq m$, let V_l be the subtableau of $\mathtt{Q}(\mathbf{c})$ lying in the $(2l-1)$-th and $2l$-th rows from the bottom, and let U_l be the subtableau of

$\mathtt{P}(\mathbf{c})$ corresponding to V_l. Note that U_l and V_l are of anti-normal shapes, and $V_l^{\nwarrow}$ is the tableau of normal shape obtained from V_l by jeu de taquin to the northwest corner. We also let $\mathtt{P}(\mathbf{c})_l$ be the subtableau of $\mathtt{P}(\mathbf{c})$ lying above the $(2l-2)$-th row from the bottom, where $\mathtt{P}(\mathbf{c})_1 = \mathtt{P}(\mathbf{c})$.

Now we *glue* each $V_l^{\nwarrow}$ to $\mathtt{P}(\mathbf{c})$ to define a tableau $\mathtt{T}(\mathbf{c})$ by the following inductive algorithm;

($\mathbf{g}$-1) Let $\mathtt{T}(\mathbf{c})_m$ be the tableau obtained from $\mathtt{P}(\mathbf{c})_m$ by gluing $V_m^{\nwarrow}$ to U_m (so that $V_m^{\nwarrow}$ and U_m form a two-rowed rectangle W_m).

($\mathbf{g}$-2) Consider a tableau obtained from $\mathtt{P}(\mathbf{c})_{m-1}$ by gluing $V_{m-1}^{\nwarrow}$ to U_{m-1} and replacing $\mathtt{P}(\mathbf{c})_m$ by $\mathtt{T}(\mathbf{c})_m$. If the number of columns in W_m is greater than $\mu_{2m-3} - \mu_{2m-1}$ for $m \geq 2$, then we move dominos $\begin{array}{|c|}\hline \overline{i+1} \\ \hline \overline{i} \\ \hline\end{array}$ down to the next two rows as many as the difference, and denote the resulting tableau by $\mathtt{T}(\mathbf{c})_{m-1}$.

($\mathbf{g}$-3) Repeat ($\mathbf{g}$-2) to have $\mathtt{T}(\mathbf{c})_{m-2}, \ldots, \mathtt{T}(\mathbf{c})_1$, and let $\mathtt{T}(\mathbf{c}) = \mathtt{T}(\mathbf{c})_1$.

Example 7.1.6. Suppose that $n = 6$ and $i = 4$. Let $\mathbf{c} \in \mathbf{B}^J_{\Delta_5}$ be given by

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 0 & & \mathbf{3} & & & \\
 & & 0 & & \mathbf{1} & & \mathbf{1} & & \\
 & 0 & & \mathbf{3} & & \mathbf{2} & & 1 & \\
0 & & \mathbf{3} & & \mathbf{2} & & 0 & & 3
\end{array}$$

where $\mathbf{c}'$ is given by the entries in bold letters. Then

$$T = \kappa^{\searrow}(\mathbf{c}'') = \begin{array}{|c|c|c|c|}\hline \overline{3} & \overline{2} & \overline{2} & \overline{2} \\ \hline \overline{1} & \overline{1} & \overline{1} & \overline{1} \\ \hline\end{array}\,, \qquad \begin{pmatrix} \mathbf{a}' \\ \mathbf{b}' \end{pmatrix} = \begin{pmatrix} \overline{5}^{\,3} & \overline{5}^{\,1} & \overline{5}^{\,3} & \overline{5}^{\,3} & \overline{4}^{\,1} & \overline{4}^{\,2} & \overline{4}^{\,2} \\ \overline{1} & \overline{2} & \overline{3} & \overline{4} & \overline{1} & \overline{2} & \overline{3} \end{pmatrix}.$$

By definition, the pair $(\mathtt{P}(\mathbf{c}), \mathtt{Q}(\mathbf{c}))$ is given by

$$\mathtt{P}(\mathbf{c}) = \begin{array}{cccccccccc}
 & & & & & & & & \overline{4} & \overline{4} \\
 & & & & & & \overline{3} & \overline{3} & \overline{3} & \overline{3} \\
 & & & \overline{4} & \overline{3} & \overline{2} & \overline{2} & \overline{2} & \overline{2} & \overline{2} \\
\overline{3} & \overline{2} & \overline{1} & \overline{1} & \overline{1} & \overline{1} & \overline{1} & \overline{1} & \overline{1} & \overline{1}
\end{array}\,, \qquad \mathtt{Q}(\mathbf{c}) = \begin{array}{cccccccccc}
 & & & & & & & & \overline{5} & \overline{5} \\
 & & & & & & \overline{5} & \overline{5} & \overline{4} & \overline{4} \\
 & & & \overline{5} & \overline{5} & \overline{5} & & & & \\
\overline{5} & \overline{5} & \overline{5} & \overline{4} & \overline{4} & \overline{4} & & & &
\end{array}$$

where U_1 and V_1 (resp. U_2 and V_2) are given in blue (resp. in red). Since

$$V_1^{\nwarrow} = \begin{array}{|c|c|c|c|c|c|}\hline \overline{5} & \overline{5} & \overline{5} & \overline{5} & \overline{5} & \overline{5} \\ \hline \overline{4} & \overline{4} & \overline{4} \\ \cline{1-3} \end{array}\ , \quad V_2^{\nwarrow} = \begin{array}{|c|c|c|c|}\hline \overline{5} & \overline{5} & \overline{5} & \overline{5} \\ \hline \overline{4} & \overline{4} \\ \cline{1-2} \end{array}$$

we have by algorithm (**g**-1)–(**g**-3),

$$\mathtt{T}(\mathbf{c}) = \begin{array}{ccccccccccc|c|c|c|c|} \cline{12-15} &&&&&&&&&&& \overline{5} & \overline{5} & \overline{4} & \overline{4} \\ \cline{12-15} &&&&&&&&&&& \overline{3} & \overline{3} & \overline{3} & \overline{3} \\ \hline \multicolumn{1}{|c|}{\overline{5}} & \multicolumn{1}{c|}{\overline{5}} & \multicolumn{1}{c|}{\overline{5}} & \multicolumn{1}{c|}{\overline{5}} & \multicolumn{1}{c|}{\overline{5}} & \multicolumn{1}{c|}{\overline{5}} & \multicolumn{1}{c|}{\overline{5}} & \multicolumn{1}{c|}{\overline{5}} & \multicolumn{1}{c|}{\overline{4}} & \multicolumn{1}{c|}{\overline{3}} & \multicolumn{1}{c|}{\overline{2}} & \overline{2} & \overline{2} & \overline{2} & \overline{2} \\ \hline \multicolumn{1}{|c|}{\overline{4}} & \multicolumn{1}{c|}{\overline{4}} & \multicolumn{1}{c|}{\overline{4}} & \multicolumn{1}{c|}{\overline{4}} & \multicolumn{1}{c|}{\overline{4}} & \multicolumn{1}{c|}{\overline{3}} & \multicolumn{1}{c|}{\overline{2}} & \multicolumn{1}{c|}{\overline{1}} & \multicolumn{1}{c|}{\overline{1}} & \multicolumn{1}{c|}{\overline{1}} & \multicolumn{1}{c|}{\overline{1}} & \overline{1} & \overline{1} & \overline{1} & \overline{1} \\ \hline \end{array}$$

Lemma 7.1.7. *Under the above hypothesis, we have*

(1) $\mathtt{T}(\mathbf{c}) = \kappa^{\searrow}(\mathbf{c})$,

(2) $\kappa^{\searrow}(\widetilde{f}_i\mathbf{c}) = \widetilde{f}_i\kappa^{\searrow}(\mathbf{c})$.

Proof.

(1) : Let $C_{i+1} = \sum_{1\leq j\leq i} c_{\overline{i+1\,j}}$. We use induction on C_{i+1}. Note that when $C_{i+1} = 0$, we clearly have $\mathtt{T}(\mathbf{c}) = \kappa^{\searrow}(\mathbf{c})$ by definition of $\mathtt{T}(\mathbf{c})$.

First, assume that $C_{i+1} = 1$. Then $c_{\overline{i+1\,j}} = 1$ for some j. Suppose that the box in $\mathtt{P}(\mathbf{c})$, which appears after insertion of the corresponding $\overline{j}$, belongs to U_l for some $1 \leq l \leq m$. Recall that $\mu^\pi = \mathrm{sh}(T)$. Let $d = \mu_{2l-3} - \mu_{2l-1}$ and let u be the length of the bottom row of U_l.

Case 1. Suppose that $\ell(\mathrm{sh}(U_l)) = 2$ and $d > u$. Then we have

$$U_l = \begin{array}{|c|c|c|c|c|} \cline{5-5} \multicolumn{4}{c|}{} & \blacksquare \\ \hline \ & \ & \cdots & \ & \ \\ \hline \end{array}\ , \quad W_l = \begin{array}{|c|c|c|c|c|c|} \hline \overline{i+1} & \overline{i} & \overline{i} & \cdots & \overline{i} & \blacksquare \\ \hline \overline{i} & & & \cdots & & \\ \hline \end{array}\ .$$

where the gray box denotes the one created after the insertion of $\overline{j}$. In this case, the domino in the leftmost column of W_l does not move to lower rows. Hence it is clear that $\mathtt{T}(\mathbf{c})$ coincides with $\kappa^{\searrow}(\mathbf{c})$.

Case 2. Suppose that $\ell(\mathrm{sh}(U_l)) = 2$ and $d = u$. Then we have

$$U_l = \begin{array}{|c|c|c|c|c|} \cline{5-5} \multicolumn{4}{c|}{} & \blacksquare \\ \hline \ & \ & \cdots & \ & \ \\ \hline \end{array}\ , \quad W_l = \begin{array}{|c|c|c|c|c|c|} \hline \overline{i+1} & \overline{i} & \overline{i} & \cdots & \overline{i} & \blacksquare \\ \hline \overline{i} & & & \cdots & & \\ \hline \end{array}\ .$$

In this case, the leftmost domino in W_l moves down to a lower row by ($\mathbf{g}$-2), and it is easy to see that $\mathtt{T}(\mathbf{c}) = \kappa^{\searrow}(\mathbf{c})$.

Case 3. Finally suppose that $\ell(\mathrm{sh}(U_l)) = 1$. Then

$$U_l = \begin{array}{|c|c|c|c|c|c|} \hline \blacksquare & & & \cdots & & \\ \hline \end{array}\ , \quad W_l = \begin{array}{|c|c|c|c|c|c|} \hline \overline{i+1} & \bar{i} & \bar{i} & \cdots & \bar{i} & \bar{i} \\ \hline \bar{i} & & & \cdots & & \\ \hline \end{array}\ .$$

As in Case 1, the domino in the leftmost column of W_l does not move to lower rows, and hence $\mathtt{T}(\mathbf{c}) = \kappa^{\searrow}(\mathbf{c})$.

Next, we assume that $C_{i+1} > 1$. Let $(\underline{\mathbf{a}}', \underline{\mathbf{b}}')$ be the biword removing (a_1, b_1) in $(\mathbf{a}', \mathbf{b}')$ in (7.1.4), and let $\underline{\mathbf{c}} = \mathbf{c}(\underline{\mathbf{a}}', \underline{\mathbf{b}}')$. Note that $(a_1, b_1) = (\overline{i+1}, \bar{j})$ for some j.

By induction hypothesis, we have $\mathtt{T}(\underline{\mathbf{c}}) = \kappa^{\searrow}(\underline{\mathbf{c}})$. On the other hand, when we apply the insertion of $(\overline{i+1}, \bar{j})$ into $\mathtt{T}(\underline{\mathbf{c}})$, the possible cases are given similarly as above. Then it is straightforward to check that the tableau obtained by insertion of $(\overline{i+1}, \bar{j})$ into $\mathtt{T}(\underline{\mathbf{c}})$ (see the step (2) in the definition of $P^{\searrow}(\mathbf{c}) = P^{\searrow}(\mathbf{a}, \mathbf{b})$ (3.3.2)) is equal to $\mathtt{T}(\mathbf{c})$. Therefore, we have $\mathtt{T}(\mathbf{c}) = \kappa^{\searrow}(\mathbf{c})$. This completes the induction.

(2) : By definition of $\mathtt{T}(\mathbf{c})$ and Lemma 7.1.5, we have $\mathtt{T}(\widetilde{f}_i\mathbf{c}) = \widetilde{f}_i\mathtt{T}(\mathbf{c})$. Then we have $\kappa^{\searrow}(\widetilde{f}_i\mathbf{c}) = \widetilde{f}_i\kappa^{\searrow}(\mathbf{c})$ by (1). □

Proof of Theorem 3.3.3. It suffices prove that for $i \in I$ and $\mathbf{c} \in \mathbf{B}^J$

$$\kappa^{\diamond}(\widetilde{f}_i\mathbf{c}) = \widetilde{f}_i\kappa^{\diamond}(\mathbf{c}) \quad (\diamond = \searrow, \nwarrow)$$

We prove only the case when $\diamond = \searrow$ since the proof for the other case is identical.

Suppose first that $i \in I \setminus \{n\}$. By Proposition 3.2.3 ($\sigma_{k,3}(\mathbf{c})$ is trivial in this case), we have

$$\mathbf{c} = \mathbf{c}_{\Delta_{i+1}^c} \otimes \mathbf{c}_{\Delta_{i+1}},$$

as elements of $\mathfrak{gl}_2$-crystals with respect to $\widetilde{e}_i, \widetilde{f}_i$.

Let us denote by $\xrightarrow{B}$ the insertion of a biword into a tableau following the algorithm given in (3.3.2). If we ignore the entries smaller than $\overline{i+1}$, then $\mathbf{c}_{\Delta_{i+1}^c} \xrightarrow{B} \kappa^{\searrow}(\mathbf{c}_{\Delta_{i+1}})$ is equal to a usual Schensted's column insertion. Hence

$$\left(\mathbf{c}_{\Delta_{i+1}^c} \xrightarrow{B} \kappa^{\searrow}(\mathbf{c}_{\Delta_{i+1}})\right) = \mathbf{c}_{\Delta_{i+1}^c} \otimes \kappa^{\searrow}(\mathbf{c}_{\Delta_{i+1}}), \tag{7.1.5}$$

as elements of $\mathfrak{gl}_{i+1}$-crystals with respect to $\widetilde{e}_j, \widetilde{f}_j$ for $1 \le j \le i$. Moreover, the subtableau of $\kappa^{\searrow}(\mathbf{c})$ consisting of entries $\overline{n}, \ldots, \overline{i+2}$ is invariant under the action of $\widetilde{f}_j$ on $\kappa^{\searrow}(\mathbf{c})$ for $1 \le j \le i$ since it depends only on the Knuth equivalence class of the subtableau with entries $\overline{i+1}, \ldots, \overline{1}$ by definition of $\kappa^{\searrow}$.

Case 1. Suppose that $\tilde{f}_i\mathbf{c} = \mathbf{c}_{\Delta_{i+1}^c} \otimes \widetilde{f}_i\mathbf{c}_{\Delta_{i+1}}$. Then we have

$$\begin{aligned}
\kappa^{\searrow}(\tilde{f}_i\mathbf{c}) &= \left(\mathbf{c}_{\Delta_{i+1}^c} \xrightarrow{B} \kappa^{\searrow}(\tilde{f}_i\mathbf{c}_{\Delta_{i+1}})\right) \\
&= \left(\mathbf{c}_{\Delta_{i+1}^c} \xrightarrow{B} \tilde{f}_i\kappa^{\searrow}(\mathbf{c}_{\Delta_{i+1}})\right) \quad \text{by Lemma 7.1.7(2)} \\
&= \tilde{f}_i\left(\mathbf{c}_{\Delta_{i+1}^c} \xrightarrow{B} \kappa^{\searrow}(\mathbf{c}_{\Delta_{i+1}})\right) \quad \text{by (7.1.5)} \\
&= \tilde{f}_i\kappa^{\searrow}(\mathbf{c}).
\end{aligned} \tag{7.1.6}$$

Case 2. Suppose that $\tilde{f}_i\mathbf{c} = \widetilde{f}_i\mathbf{c}_{\Delta_{i+1}^c} \otimes \mathbf{c}_{\Delta_{i+1}}$. By the same argument as in (7.1.6), we have $\kappa^{\searrow}(\widetilde{f}_i\mathbf{c}) = \widetilde{f}_i\kappa^{\searrow}(\mathbf{c})$.

Next, suppose that $i = n$. We may identify $\mathbf{c}$ with the pair $(\mathbf{c}_{\Delta_{n-1}^c}, \mathbf{c}_{\Delta_{n-1}})$. Regarding $\mathbf{c}$ as an element of $\mathbf{B}^J_{\Delta_{n+1}}$, the crystal of type D_{n+1}, we have by Lemma 7.1.7(1) the following commuting diagram;

$$\begin{tikzcd}
\mathbf{c} \arrow[r, mapsto] \arrow[d, mapsto] & (\mathbf{c}_{\Delta_{n-1}^c}, \mathbf{c}_{\Delta_{n-1}}) \arrow[r, mapsto] & (\mathbf{c}_{\Delta_{n-1}^c}, \kappa^{\searrow}(\mathbf{c}_{\Delta_{n-1}})) \arrow[d, mapsto, "\text{(i)}"] \\
\kappa^{\searrow}(\mathbf{c}) = \mathtt{T}(\mathbf{c}) & & (\mathtt{P}(\mathbf{c}), \mathtt{Q}(\mathbf{c})) \arrow[ll, mapsto, "\text{(ii)}"']
\end{tikzcd}$$

Now, we can apply the same argument for the proof of [60, Theorem 3.6] to see that the composition of (i) and (ii) commutes with $\tilde{f}_n$. Therefore, we have $\kappa^{\searrow}(\widetilde{f}_n\mathbf{c}) = \widetilde{f}_n\kappa^{\searrow}(\mathbf{c})$. □

7.1.3 Proofs of Theorems 3.3.6 and 6.2.4

From now on we assume that $\mathfrak{g}$ is of type D_n $(n \ge 4)$ and let $\mathbf{i}_0 = (i_1, \ldots, i_N) \in R(w_0)$ given in (3.2.8) with $\mathbf{i}^J = (i_1, \ldots, i_M)$ and $\mathbf{i}_J = (i_{M+1}, \ldots, i_N)$.

We first prove Theorem 6.2.4 using Berenstein-Zelevinsky formula (7.1.3). Note that we generalize the argument in the proof of Theorem 6.2.4 to verify Theorem 3.3.6. In order to describe (7.1.3) explicitly, we characterize the certain trails related to $\mathbf{i}_0$ in the fundamental representation $V(\varpi_n)$ (recall Definition 7.1.1).

We have $n^* = n-1$ (resp. $(n-1)^* = n$) when n is odd, and $i^* = i$ otherwise. Put

$$\mathbf{j}_0 = (j_1, \ldots, j_N) := \mathbf{i}_0^{*\mathrm{op}} = (i_N^*, \ldots, i_1^*). \tag{7.1.7}$$

Recall that the crystal $B(\varpi_n)$ of $V(\varpi_n)$ can be realized as

$$B(\varpi_n) = \{\, \tau = (\tau_1, \ldots, \tau_n) \,|\, \tau_k = \pm \ (1 \le k \le n) \,\},$$

where $\mathrm{wt}(\tau) = \frac{1}{2}\sum_{k=1}^n \tau_k \epsilon_k$ and

$$(\ldots\ldots, \underbrace{+,+}_{\tau_{n-1},\tau_n}) \xrightarrow{\widetilde{f}_n} (\ldots\ldots,-,-), \quad (\ldots, \underbrace{+,-}_{\tau_i,\tau_{i+1}}, \ldots) \xrightarrow{\widetilde{f}_i} (\ldots,-,+,\ldots), \tag{7.1.8}$$

$(1 \le i \le n-1)$ with the highest weight element $(+,\ldots,+)$ [50]. Since the spin representation $V(\varpi_n)$ is minuscule (recall Remark 7.1.2), any $\mathbf{i}_0$-trail $\pi = (\nu_0, \ldots, \nu_N)$ in $V(\varpi_n)$ can be identified with a sequence $b_0, \ldots, b_N$ in $B(\varpi_n)$ such that $\mathrm{wt}(b_k) = \nu_k$ and $\widetilde{f}_{i_k}^{d_k(\pi)} b_{k-1} = b_k$ with $d_k(\pi) = 0, 1$ for $1 \le k \le N$.

Lemma 7.1.8. *There exists a unique $(i_N^*, \ldots, i_{M+1}^*) = (j_1, \ldots, j_M)$-trail from $\varpi_n - \alpha_n = \mathrm{wt}(+,\ldots,+,-,-)$ to $\varpi_n + \alpha_0 = \mathrm{wt}(-,-,+\ldots,+)$. We denote this trail by $(\widetilde{\nu}_0, \ldots, \widetilde{\nu}_M)$.*

Proof. Considering the crystal structure on $B(\varpi_n)$ (7.1.8), we see that up to 2-term braid move $(j_1', j_2', \ldots, j_{2n-4}') = (n-2, n-1, n-3, n-2, \ldots, 1, 2)$ is the unique sequence of indices in I such that

$$\widetilde{f}_{j_{2n-4}'} \cdots \widetilde{f}_{j_2'} \widetilde{f}_{j_1'} (+,\ldots,+,-,-) = (-,-,+\ldots,+).$$

On the other hand, there exists a subsequence $(j_1', j_2', \ldots, j_{2n-4}')$ of $(j_1, \ldots, j_N)$. Since no other subsequence gives $(n-2, n-1, n-3, n-2, \ldots, 1, 2)$ up to 2-braid move by definition of $\mathbf{i}_J$, $(j_1', j_2', \ldots, j_{2n-4}')$ determines a unique such trail. $\square$

Let $\mathcal{T}$ be the set of $\mathbf{j}_0$-trails π from $s_n\varpi_n$ to $w_0\varpi_n$ in $V(\varpi_n)$. For $\mathbf{c} = (c_k) \in \mathbf{B}^J$ and $\pi \in \mathcal{T}$, let

$$||\mathbf{c}||_\pi = (1 - d_N(\pi))c_1 + \cdots + (1 - d_{M+1}(\pi))c_M = \sum_{k=1}^M (1 - d_{N-k+1}(\pi))\, c_k. \tag{7.1.9}$$

Recall that $c_k = c_{\beta_k}$ for $\beta_k \in \Phi^+(J)$ $(1 \leq k \leq M)$ with respect to the order (3.2.2) and hence (3.2.7). Let us simply write $\mathbf{c} = (c_1, \dots, c_M)$. The following lemma plays a crucial role in the proof of Theorem 6.2.4.

Lemma 7.1.9. *For* $\mathbf{c} \in \mathbf{B}^J$, *we have* $\varepsilon_n^*(\mathbf{c}) = \max\{\, ||\mathbf{c}||_\pi \,|\, \pi \in \mathcal{T}\,\}$.

Proof. Let $\mathbf{c} = (c_k) \in \mathbf{B}^J$ given. Since $\varepsilon_n^*(\mathbf{c}) = \varepsilon_n^*(b_{\mathbf{i}_0}(\mathbf{c}))$, we have $\varepsilon_n^*(\mathbf{c}) = t_1$, where $(t_k) = \mathbf{t}_{\mathbf{i}_0}(b_{\mathbf{i}_0}(\mathbf{c})) = \mathbf{t}_{\mathbf{i}_0}(b_{\mathbf{i}_0}(\mathbf{c})^{**}) = \mathbf{t}_{\mathbf{i}_0}(b_{\mathbf{i}_0^{*\mathrm{op}}}(\mathbf{c}^{\mathrm{op}})^*)$ by (7.1.2).

One can check that applying $\widetilde{f}_{j_N} \cdots \widetilde{f}_{j_{M+1}}$ to $(+, \dots, +)$ gives a unique $\mathbf{j}_0$-trail from ϖ_n and $w_0\varpi_n$. Hence by (7.1.3), we have

$$t_1 = \sum_{1 \leq k \leq M} c_k - \min_\pi \left\{ \sum_{1 \leq k \leq M} d_{N-k+1}(\pi) c_k \right\},$$

where π is a $\mathbf{j}_0$-trail from $s_n\varpi_n$ to $w_0\varpi_n$. Hence $t_1 = \max\{\, ||\mathbf{c}||_\pi \,|\, \pi \in \mathcal{T}\,\}$. $\square$

For $\pi = (\nu_0, \dots, \nu_N) \in \mathcal{T}$, let $\pi_J = (\nu_0, \dots, \nu_M)$, $\pi^J = (\nu_{M+1}, \dots, \nu_N)$, and

$$\mathcal{T}' = \{\, \pi \,|\, \pi_J = (\widetilde{\nu}_0, \dots, \widetilde{\nu}_M)\,\} \subset \mathcal{T},$$

where $(\widetilde{\nu}_0, \dots, \widetilde{\nu}_M)$ as in Lemma 7.1.8.

Lemma 7.1.10. *For* $\mathbf{c} \in \mathbf{B}^J$, *we have* $\varepsilon_n^*(\mathbf{c}) = \max\{\, ||\mathbf{c}||_\pi \,|\, \pi \in \mathcal{T}'\,\}$.

Proof. For simplicity, we assume that n is even so that $w_0\varpi_n = -\varpi_n$. The proof for odd n is almost identical. Let $\pi = (\nu_0, \dots, \nu_N) \in \mathcal{T}$ given. It suffices to show that there exists $\pi' \in \mathcal{T}'$ such that $||\mathbf{c}||_\pi \leq ||\mathbf{c}||_{\pi'}$.

If $\pi \in \mathcal{T}'$, then $\nu_M = \mathrm{wt}(-, -, + \dots, +)$ by Lemma 7.1.8. Suppose that $\pi \notin \mathcal{T}'$. Since $j_k \neq n$ for $1 \leq k \leq M$, we have $\nu_M = \mathrm{wt}(\tau)$ where $\tau = (\tau_1, \dots, \tau_n)$ with $\tau_p = \tau_q = -$ for some $(p, q) \neq (1, 2)$ and $\tau_i = +$ otherwise.

Since $\pi \in \mathcal{T}$, there exists a subsequence $(j'_1, \dots, j'_{N'})$ of $(j_{M+1}, \dots, j_N)$ such that

$$\widetilde{f}_{j'_{N'}} \cdots \widetilde{f}_{j'_2} \widetilde{f}_{j'_1} \tau = (-, \dots, -),$$

the lowest weight element. Ignoring j'_k such that $-$'s in τ is moved to the left by $\widetilde{f}_{j'_k}$ (7.1.8), we obtain a subsequence $(j''_1, \dots, j''_{N''})$ of $(j'_1, \dots, j'_{N'})$ such that

$$\widetilde{f}_{j''_{N''}} \cdots \widetilde{f}_{j''_2} \widetilde{f}_{j''_1} (+, \dots, +) = (+, +, -, \dots, -).$$

This implies that there exists a unique $\pi' \in \mathcal{T}'$ such that for $M+1 \leq k \leq N$

$$d_k(\pi') = \begin{cases} 1, & \text{if } k = j''_l \text{ for some } 1 \leq l \leq N'', \\ 0, & \text{otherwise.} \end{cases}$$

Hence we have $||\mathbf{c}||_\pi \leq ||\mathbf{c}||_{\pi'}$ by construction of π'. $\square$

Recall that $\mathbf{c} = (c_k) \in \mathbf{B}^J$ is by convention identified with the array, where c_k is placed at the position of β_k in Δ_n for $1 \leq k \leq M$ (see Example 3.2.10).

We note that if we consider the array (j_k) for $M+1 \leq k \leq N$, where j_k is placed at the position of β_{N-k+1} in Δ_n, then the r-th row from the top is filled with r for $1 \leq r \leq n-2$ and the bottom row is filled with $\ldots, n-1, n, n-1, n$ from right to left.

Let $\mathcal{D}$ be the set of arrays, where either 0 or 1 is placed in each r-th row of Δ_n from the top $(1 \leq r \leq n-1)$ satisfying the following conditions;

(1) the three entries in the first two rows are 0,

(2) the number of 1's in each r-th row is $r-2$ for $3 \leq r \leq n-1$,

(3) if $r > 3$ (resp. $r < n-1$) and there are two 1's in the r-th row such that the entries in the same row between them are zero, then there is exactly one 1 in the $(r-1)$-th row (resp. $(r+1)$-th row) between them,

(4) the j_k's corresponding to 1's in the $(n-1)$-th row are $n, n-1, n, \ldots$ from right to left.

We write $\mathbf{d} = (d_k) \in \mathcal{D}$, where d_k denotes the entry at the position of β_{N-k+1} in Δ_n for $M+1 \leq k \leq N$.

Example 7.1.11. When $n = 6$, we have

$$\mathbf{c} = \begin{array}{ccccccccc} &&&& c_5 &&&& \\ &&& c_4 && c_9 &&& \\ && c_3 && c_8 && c_{12} && \\ & c_2 && c_7 && c_{11} && c_{14} & \\ c_1 && c_6 && c_{10} && c_{13} && c_{15} \end{array} \qquad \mathbf{d} = \begin{array}{ccccccccc} &&&& d_{26} &&&& \\ &&& d_{27} && d_{22} &&& \\ && d_{28} && d_{23} && d_{19} && \\ & d_{29} && d_{24} && d_{20} && d_{17} & \\ d_{30} && d_{25} && d_{21} && d_{18} && d_{16} \end{array}$$

$$
\begin{array}{ccccccccc}
 & & & & j_{26} & & & & \\
 & & & j_{27} & & j_{22} & & & \\
 & & j_{28} & & j_{23} & & j_{19} & & \\
 & j_{29} & & j_{24} & & j_{20} & & j_{17} & \\
j_{30} & & j_{25} & & j_{21} & & j_{18} & & j_{16}
\end{array}
\quad = \quad
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 2 & & 2 & & & \\
 & & 3 & & 3 & & 3 & & \\
 & 4 & & 4 & & 4 & & 4 & \\
6 & & 5 & & 6 & & 5 & & 6
\end{array}
$$

□

For $\pi \in \mathcal{T}'$, let $\mathbf{d}(\pi)$ denote the array $(d_k(\pi))$, where $d_k(\pi)$ is placed in the position of β_{N-k+1} in Δ_n for $M+1 \leq k \leq N$.

Lemma 7.1.12. *The map sending π to $\mathbf{d}(\pi)$ is a bijection from $\mathcal{T}'$ to $\mathcal{D}$.*

Proof. Let us assume that n is even since the proof for odd n is the same. We first show that the map is well-defined. Let $\pi = (\nu_0, \ldots, \nu_N) \in \mathcal{T}'$ given, where $\nu_M = \mathrm{wt}(-,-,+\ldots,+)$. Let $(j'_1, \ldots, j'_L)$ be the subsequence of $(j_{M+1}, \ldots, j_N)$ such that $d_{j'_k}(\pi) = 1$. Then

$$\widetilde{f}_{j'_L} \cdots \widetilde{f}_{j'_2} \widetilde{f}_{j'_1}(-,-,+\ldots,+) = (-,-,-,\ldots,-),$$

equivalently,

$$\widetilde{f}_{j'_L} \cdots \widetilde{f}_{j'_2} \widetilde{f}_{j'_1}(+,+,+\ldots,+) = (+,+,-,\ldots,-). \tag{7.1.10}$$

From (7.1.8), (7.1.10) and the array (j_k) on Δ_n, one can check that (i) $L = (n-3)(n-2)/2$, (ii) $3 \leq j'_k \leq n$, (iii) the array $\mathbf{d}(\pi)$ satisfies the conditions (1) and (2) for $\mathcal{D}$. To verify the condition (3), let us enumerate $-$'s appearing in (7.1.10) from left to right by $-_1, -_2, \ldots$.

For $3 \leq r \leq n-1$, let $1_{(r-2,r)}, \ldots, 1_{(2,r)}, 1_{(1,r)}$ denote the entries 1 of $\mathbf{d}(\pi)$ in the r-th row, which are enumerated from the right.

For $1 \leq k \leq L$, suppose that j'_k corresponds to $1_{(s,r)}$ in $\mathbf{d}(\pi)$ for some s with $r = j'_k$. It is not difficult to see that $\widetilde{f}_{j'_k}$ in (7.1.10) corresponds to

(1) moving $-_s$ at the $(r+1)$-th coordinate of a vector in $B(\varpi_n)$ to the r-th one unless $r = n$ and s is odd,

(2) placing $(-_s, -_{s+1})$ at the last two coordinates if $r = n$ and s is odd.

Then by looking at the arrangement of $d_k(\pi)$'s in Δ_n, it follows that $1_{(s,r)}$ is located to the northeast of $1_{(s+1,r+1)}$ and to the northwest of $1_{(s,r+1)}$ for $r < n-1$,

$$
\begin{array}{ccc}
\cdots & 1_{(s,r)} & \cdots \\
1_{(s+1,r+1)} & & 1_{(s,r+1)}
\end{array}
$$

and the j'_k's corresponding to $\ldots, 1_{(3,n-1)}, 1_{(2,n-1)}, 1_{(1,n-1)}$ are $\ldots, n, n-1, n$. Hence $\mathbf{d}(\pi)$ satisfies the condition (3) and (4) for $\mathcal{D}$, and the map $\pi \mapsto \mathbf{d}(\pi)$ is well-defined.

Since the map is clearly injective, it remains to show that it is surjective. Let $\pi_0 \in \mathcal{T}'$ be a unique trail such that $d_k(\pi_0)_k = 1$ for $M+1 \leq k \leq M+L$ and 0 otherwise.

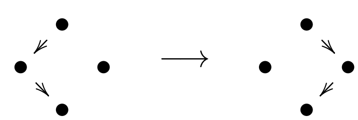
We claim that for any $\mathbf{d} \in \mathcal{D}$ there exists a sequence $\mathbf{d} = \mathbf{d}_0, \mathbf{d}_1, \ldots, \mathbf{d}_m = \mathbf{d}(\pi_0)$ in $\mathcal{D}$ such that $\mathbf{d}_{l+1}$ is obtained from $\mathbf{d}_l$ by moving an entry 1 to the right. If $\mathbf{d} \neq \mathbf{d}(\pi_0)$, then choose a minimal k such that $d_k = 0 \neq d_k(\pi_0)$ for $M+1 \leq k \leq M+L$. If $1_{(s,r)}$ denotes an entry corresponding to $d_k(\pi_0)$ in $\mathbf{d}(\pi)$, then there exists $1_{(s',r)}$ in $\mathbf{d}$ such that $s < s'$. Here we assume that s' is minimal. Then by the condition (3) for $\mathcal{D}$ and the minimality of s', we can move $1_{(s',r)}$ to the right by one position if $r < n-1$ and by two positions if $r = n-1$ to get another $\mathbf{d}' \in \mathcal{D}$ by definition of $\mathcal{D}$. Repeating this step, we obtain a required sequence. This proves our claim.

Now, let $\mathbf{d} = (d_k) \in \mathcal{D}$ be given and let $(j'_1, \ldots, j'_L)$ be the subsequence such that $d_{j'_k} = 1$. By the above claim and definition of $\mathcal{D}$, we obtain the following two reduced expressions;

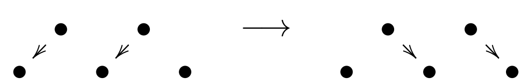
$$s_{j'_L} \cdots s_{j'_2} s_{j'_1} = s_{j_{M+L}} \cdots s_{j_{M+2}} s_{j_{M+1}},$$

where we obtain the right-hand side from the left only by applying 2-term braid move. Since $\widetilde{f}_{j_{M+L}} \cdots \widetilde{f}_{j_{M+2}} \widetilde{f}_{j_{M+1}}(+,+,+\ldots,+) = (+,+,-,\ldots,-)$, we obtain (7.1.10), which implies that there exists $\pi \in \mathcal{T}'$ such that $\mathbf{d}(\pi) = \mathbf{d}$. The proof completes. $\square$

Let $\mathcal{P}$ be the set of double paths at θ. Consider two operations on $\mathcal{P}$ which change a part of $\mathbf{p} \in \mathcal{P}$ in the following way;



(7.1.11)



(7.1.12)

where in (7.1.12) the rows denote the two rows from the bottom in Δ_n.

Lemma 7.1.13. *For $\mathbf{p} \in \mathcal{P}$, let $\mathbf{d}(\mathbf{p}) = (d_k) \in \mathcal{D}$ be given by $d_k = 0$ if $\mathbf{p}$ passes the position of d_k, and $d_k = 1$ otherwise. Then the map sending $\mathbf{p}$ to $\mathbf{d}(\mathbf{p})$ is a bijection from $\mathcal{P}$ to $\mathcal{D}$.*

Proof. Let $\mathbf{p}_0 \in \mathcal{P}$ be a unique double path at θ such that $\mathbf{p}_0$ ends at the first two dots from the left in the bottom row of Δ_n, that is, β_1 and β_n (see the first double path in Example 3.3.5). It is clear that $\mathbf{d}(\mathbf{p}_0) = \mathbf{d}(\pi_0) \in \mathcal{D}$, where $\pi_0 \in \mathcal{T}'$ is given in the proof of Lemma 7.1.12.

Let $\mathbf{p} \in \mathcal{P}$ given. Suppose that $\mathbf{p}'$ is obtained from $\mathbf{p}$ by applying either (7.1.11) or (7.1.12). If $\mathbf{d}(\mathbf{p}) \in \mathcal{D}$, then it is clear that $\mathbf{d}(\mathbf{p}') \in \mathcal{D}$. Since one can obtain $\mathbf{p}$ from $\mathbf{p}_0$ by applying (7.1.11) and (7.1.12) a finite number of times, we have $\mathbf{d}(\mathbf{p}) \in \mathcal{D}$. Hence the map $\mathbf{p} \mapsto \mathbf{d}(\mathbf{p})$ is well-defined and injective. The surjectivity follows from the fact that any $\mathbf{d} \in \mathcal{D}$ can be obtained from $\mathbf{d}(\pi_0)$ by moving an entry to the left by one or two depending on the row which it belongs to (see the proof of Lemma 7.1.12), which corresponds to (7.1.11) or (7.1.12). □

Proof of Theorem 6.2.4. By Lemmas 7.1.12 and 7.1.13, there exists a bijection from $\mathcal{T}'$ to $\mathcal{P}$. If $\pi \in \mathcal{T}'$ corresponds to $\mathbf{p} \in \mathcal{P}$, then we have $||\mathbf{c}||_\pi = ||\mathbf{c}||_{\mathbf{p}}$ for $\mathbf{c} \in \mathbf{B}^J$. Hence by Lemma 7.1.10, we have $\varepsilon_n^*(\mathbf{c}) = \max\{\, ||\mathbf{c}||_{\mathbf{p}} \,|\, \mathbf{p} \in \mathcal{P}\,\}$. □

For $1 \le l \le [\frac{n}{2}]$, let k_l be the index such that j_{k_l} belongs to the subword $(\mathbf{i}^J_{2l-1})^{*\mathrm{op}}$ of $\mathbf{j}_0$ and $j_{k_l} = n$. For $i \in I$ and an element b of a crystal, let $\widetilde{e}_i^{\max} b = \widetilde{e}_i^{\varepsilon_i(b)} b$. The following is crucial when proving Theorem 3.3.6.

Proposition 7.1.14. *For $\mathbf{c} = (c_k) \in \mathbf{B}^J$ and $1 \le l \le [\frac{n}{2}]$,*

$$\lambda_{2l-1}(\mathbf{c}) = \varepsilon_{j_{k_l}} \left(\tilde{e}^{\max}_{j_{k_l-1}} \cdots \tilde{e}^{\max}_{j_2} \tilde{e}^{\max}_{j_1} \mathbf{c} \right), \tag{7.1.13}$$

and it is equal to

$$\min_{\pi_1} \left\{ \sum_{k=1}^{M} d_k(\pi_1) c_k \right\} - \min_{\pi_2} \left\{ \sum_{k=1}^{M} d_k(\pi_2) c_k \right\}, \tag{7.1.14}$$

where π_1 and π_2 are $\mathbf{i}_0$-trails from

$$\mathrm{wt}(\underbrace{-,\cdots,-}_{2l-2},\underbrace{+,\cdots,+}_{n-2l+2}) \quad \textit{and} \quad \mathrm{wt}(\underbrace{-,\cdots,-}_{2l},\underbrace{+,\cdots,+}_{n-2l})$$

to the lowest weight element in $B(\varpi_n)$, respectively.

Proof. Let $\mathbf{c} \in \mathbf{B}^J$ given. Since $(j_1,\ldots,j_M)$ is a reduced expression of the longest element for $\mathfrak{l}$, $\tilde{e}^{\max}_{j_M} \cdots \tilde{e}^{\max}_{j_1} \mathbf{c}$ is an $\mathfrak{l}$-highest weight element, which is of the form (3.2.17). Then it is straightforward to verify (7.1.13) using Proposition 3.2.3.

On the other hand, the righthand side of (7.1.13) can be obtained by (7.1.3) letting

$$\mathbf{i} = \mathbf{j}_0, \quad \mathbf{i}' = \mathbf{i}_0, \tag{7.1.15}$$

where in this case

$$\begin{aligned} s_{j_1} \cdots s_{j_{k_l - 1}}(\varpi_{j_{k_l}}) &= s_{j_1} \cdots s_{j_{k_l - 1}}(\varpi_n) = \mathrm{wt}(\underbrace{-, \cdots, -}_{2l-2}, \underbrace{+, \cdots, +}_{n-2l+2}), \\ s_{j_1} \cdots s_{j_{k_l}}(\varpi_{j_{k_l}}) &= \mathrm{wt}(\underbrace{-, \cdots, -}_{2l}, \underbrace{+, \cdots, +}_{n-2l}). \end{aligned} \tag{7.1.16}$$

Hence the formula (7.1.3) gives (7.1.14). □

Let $1 \leq l \leq [\frac{n}{2}]$ given. Let $\mathcal{T}_l$ be the set of $\mathbf{i}_0$-trails from $s_{j_1} \cdots s_{j_{k_l}}(\varpi_n)$ (7.1.16) to $w_0\varpi_n$. Let $\mathcal{D}_l$ be the set of arrays where either 0 or 1 is placed in each r-th row of Δ_n from the top ($1 \leq r \leq n-1$) satisfying the following conditions;

(1) the entries in the first $2l$ rows are 0,

(2) the number of 1's in each r-th row is $r - 2l$ for $2l + 1 \leq r \leq n-1$,

(3) if $r > 2l + 1$ (resp. $r < n-1$) and there are two 1's in the r-th row such that the entries in the same row between them are zero, then there is exactly one 1 in the $(r-1)$-th row (resp. $(r+1)$-th row) between them,

(4) the j_k's corresponding to 1's in the $(n-1)$-th row are $n, n-1, n, \ldots$ from left to right.

Note that $\mathcal{D}_1 = \mathcal{D}$.

Lemma 7.1.15. *For $\pi \in \mathcal{T}_l$, the map sending π to $\mathbf{d}(\pi)$ is a bijection from $\mathcal{T}_l$ to $\mathcal{D}_l$, where $\mathbf{d}(\pi)$ is defined in the same way as in $\mathcal{T}'$.*

Proof. It can be shown by almost the same arguments as in Lemma 7.1.12 that the map is well-defined, and clearly injective.

It suffices to show that it is surjective. Let Δ'_{n-2l} be the set Δ_{n-2l}, which we regard as a subset of Δ_n sharing the same southwest corner with Δ_n. Let π_0 be a unique trail in $\mathcal{T}_l$ such that $d_k(\pi_0) = 1$ if and only if $d_k(\pi_0)$ is located in Δ'_{n-2l}. Then as in the proof of Lemma 7.1.12 we can check that for any $\mathbf{d} \in \mathcal{D}_l$ there exists a sequence

$\mathbf{d} = \mathbf{d}_0, \mathbf{d}_1, \ldots, \mathbf{d}_m = \mathbf{d}(\pi_0)$ in $\mathcal{D}_l$ such that $\mathbf{d}_{l+1}$ is obtained from $\mathbf{d}_l$ by moving an entry 1 to the *left*, and hence that there exists $\pi \in \mathcal{T}_l$ such that $\mathbf{d}(\pi) = \mathbf{d}$. $\square$

Let $\mathcal{P}_l$ be the set of l-tuple $\mathbf{p} = (\mathbf{p}_1, \cdots, \mathbf{p}_l)$ of mutually non-intersecting double paths in Δ_n such that each $\mathbf{p}_i$ is a double path at some point in the $(2i-1)$-th row.

Lemma 7.1.16. *The map sending* $\mathbf{p}$ *to* $\mathbf{d}(\mathbf{p})$ *is a surjective map from* $\mathcal{P}_l$ *to* $\mathcal{D}_l$*, where* $\mathbf{d}(\mathbf{p})$ *is defined in the same way as in* $\mathcal{P}$*.*

Proof. Suppose that $\underline{\mathbf{p}} = (\mathbf{p}_1, \cdots, \mathbf{p}_l) \in \mathcal{P}_l$ is given. By definition of $\mathcal{P}_l$, one can check that all the points in the first $2l$ rows in Δ_n are occupied by $\underline{\mathbf{p}}$.

Let $\underline{\mathbf{p}}^0 = (\mathbf{p}_1^0, \ldots, \mathbf{p}_l^0)$ be given such that $\mathbf{p}_i^0$ starts at $\epsilon_{2i-1} + \epsilon_n$ and ends at $\epsilon_{2i-1} + \epsilon_{2i}$ and $\epsilon_{2i} + \epsilon_{2i+1}$ for $1 \leq i \leq r$. We have $\mathbf{d}(\underline{\mathbf{p}}^0) = \mathbf{d}(\pi_0)$, where π_0 is given in the proof of Lemma 7.1.15. Applying the operations (7.1.11) and (7.1.12) on $\mathcal{P}_l$, one can obtain a sequence in $\mathcal{P}_l$ *from* $\underline{\mathbf{p}}$ *to* $\underline{\mathbf{p}}^0$, whose image under $\mathbf{d}$ lies in $\mathcal{D}_l$. Then similar arguments as in Lemma 7.1.13 implies the surjectivity. $\square$

Proof of Theorem 3.3.6. Let $\mathbf{c} \in \mathbf{B}^J$ given. For $\pi \in \mathcal{T}_l$, there exists $\underline{\mathbf{p}} = (\mathbf{p}_1, \cdots, \mathbf{p}_l) \in \mathcal{P}_l$ such that $\mathbf{d}(\underline{\mathbf{p}}) = \mathbf{d}(\pi)$ by Lemmas 7.1.15, and 7.1.16, and

$$\sum_{1 \leq k \leq M} d_k(\pi) c_k = \sum_{1 \leq k \leq M} c_k - \left(||\mathbf{c}||_{\mathbf{p}_1} + \cdots + ||\mathbf{c}||_{\mathbf{p}_l} \right). \tag{7.1.17}$$

Indeed, (7.1.17) holds for any $\underline{\mathbf{p}} \in \mathcal{P}_l$ such that $\mathbf{d}(\underline{\mathbf{p}}) = \mathbf{d}(\pi)$. Therefore, we have by (7.1.14) and (7.1.17)

$$\begin{aligned}
\lambda_{2l-1}(\mathbf{c}) &= \min_{\pi_1 \in \mathcal{T}_{l-1}} \left\{ \sum_{1 \leq k \leq M} d_k(\pi_1) c_k \right\} - \min_{\pi_2 \in \mathcal{T}_l} \left\{ \sum_{1 \leq k \leq M} d_k(\pi_2) c_k \right\} \\
&= \left(\sum_{1 \leq k \leq M} c_k - \max_{\underline{\mathbf{p}} \in \mathcal{P}_{l-1}} \{ \, ||\mathbf{c}||_{\mathbf{p}_1} + \cdots + ||\mathbf{c}||_{\mathbf{p}_{l-1}} \, \} \right) \\
&\qquad\qquad - \left(\sum_{1 \leq k \leq M} c_k - \max_{\underline{\mathbf{p}} \in \mathcal{P}_l} \{ \, ||\mathbf{c}||_{\mathbf{p}_1} + \cdots + ||\mathbf{c}||_{\mathbf{p}_l} \, \} \right) \\
&= \max_{\underline{\mathbf{p}} \in \mathcal{P}_l} \{ \, ||\mathbf{c}||_{\mathbf{p}_1} + \cdots + ||\mathbf{c}||_{\mathbf{p}_l} \, \} - \max_{\underline{\mathbf{p}} \in \mathcal{P}_{l-1}} \{ \, ||\mathbf{c}||_{\mathbf{p}_1} + \cdots + ||\mathbf{c}||_{\mathbf{p}_{l-1}} \, \}.
\end{aligned}$$

This gives the formula in Theorem 3.3.6. $\square$

7.2 In Chapter 4

7.2.1 Proof of Lemma 4.3.1

Let $\mathbf{B}$ be one of $\mathbf{T}(a)$ $(0 \leq a \leq n-1)$ and $\mathbf{T}^{\rm sp}$ in Definition 4.1.4. When $\mathbf{B} = \mathbf{T}^{\rm sp}$, we regard an element of $\mathbf{B}$ as in the sense of Remark 4.1.6 (2). For $T \in \mathbf{B}$, we define

$$\mathtt{s}_T := \max_{1 \leq s \leq {\rm ht}(T^{\tt R})} \left\{ s \mid T^{\tt L}(s - \mathfrak{r}_T - 1 + a) > T^{\tt R}(s) \right\} \cup \{ 1 \}, \tag{7.2.1}$$

where $T^{\tt L}[k] := -\infty$ for $k \leq 0$. Note that $a = \mathfrak{r}_T$ when $\mathbf{B} = \mathbf{T}^{\rm sp}$.

Let $(T, S) \in \mathbf{T}(a_2) \times \mathbf{B}$ be an admissible pair such that

$$T \in SST_{[\overline{n}]}(\lambda(a_2, b_2, c_2)), \qquad S \in SST_{[\overline{n}]}(\lambda(a_1, b_1, c_1))$$

for $a_i \in \mathbb{Z}_+$ and $b_i, c_i \in 2\mathbb{Z}_+$ $(i = 1, 2)$ with $a_1 \leq a_2$. If the pair (T, S) satisfies ${\rm ht}(T^{\tt R}) > {\rm ht}(S^{\tt L}) - a_1$, then put

$$S^{\tt L}[0] := -\infty.$$

Note that above inequality occurs only for the case $\mathfrak{r}_T \cdot \mathfrak{r}_S = 1$.

Lemma 7.2.1.

(1) *If* $\mathfrak{r}_T \cdot \mathfrak{r}_S = 0$ *(resp.* $\mathfrak{r}_T \cdot \mathfrak{r}_S = 1$*), then* $T \triangleleft S$ *(resp.* $T \triangleleft (S^{\tt L*}, S^{\tt R*})$*).*

(2) *If* $\mathfrak{r}_T \cdot \mathfrak{r}_S = 1$*, then* $T \triangleleft S$ *is equivalent to the following condition:*

$$T^{\tt R}(k) \leq S^{\tt L}(k - 1 + a_1) \quad \textit{for} \;\; \mathtt{s}_T \leq k \leq {\rm ht}(T^{\tt R}). \tag{7.2.2}$$

Proof.

(1) : If $\mathfrak{r}_T \cdot \mathfrak{r}_S = 0$ (resp. $\mathfrak{r}_T \cdot \mathfrak{r}_S = 1$), then $T \triangleleft S$ (resp. $T \triangleleft (S^{\tt L*}, S^{\tt R*})$) follows immediately from Definition 4.1.5 (iii).

(2) : Assume that $\mathfrak{r}_T \cdot \mathfrak{r}_S = 1$. The relation $T \triangleleft S$ implies

$$T^{\tt R}(k) = {}^{\tt R}T(k - 1 + a_2) \leq S^{\tt L}(k - 1 + a_1) \quad \text{for} \quad \mathtt{s}_T \leq k \leq {\rm ht}(T^{\tt R}).$$

Conversely, we assume that (7.2.2) holds. Note that by definition of $S^{\tt L*}$,

$$S^{\tt L*}(k) = S^{\tt L}(k), \tag{7.2.3}$$

if there exists k such that $1 \le k \le \mathtt{s}_S - 2 + a_1$. Now we consider two cases.

Case 1. $\mathtt{s}_T > \mathtt{s}_S$. If $\mathtt{s}_S \le k < \mathtt{s}_T$, then by Definition 4.1.5 (ii),

$$T^{\mathtt{R}}(k) = T^{\mathtt{R}*}(k) \le {}^{\mathtt{L}}S(k) = S^{\mathtt{L}}(k - 1 + a_1). \tag{7.2.4}$$

Combining (7.2.2), (7.2.3), (7.2.4) and Definition 4.1.5 (iii), we have

$${}^{\mathtt{R}}T(k + a_2 - a_1) \le S^{\mathtt{L}}(k) \quad \text{for } 1 \le k \le \mathrm{ht}(S^{\mathtt{L}}).$$

Case 2. $\mathtt{s}_T \le \mathtt{s}_S$. In this case, the relation $T \triangleleft S$ follows directly from (7.2.2), (7.2.3) and Definition 4.1.5 (iii). □

Lemma 7.2.2. *Assume that* $\mathfrak{r}_T \cdot \mathfrak{r}_S = 1$. *For* $1 \le k \le \mathrm{ht}(T^{\mathtt{R}})$,

(1) $T^{\mathtt{R}}(k) \le S^{\mathtt{L}*}(k)$.

(2) *If* $T \triangleleft S$, *then* $T^{\mathtt{R}}(k) \le S^{\mathtt{L}}(k)$.

Proof. By (7.2.4) and Definition 4.1.5 (iii), in any case, we have

$$T^{\mathtt{R}}(\mathtt{s}_S) \le S^{\mathtt{L}}(\mathtt{s}_S - 1 + a_1) \le S^{\mathtt{R}}(\mathtt{s}_S) = S^{\mathtt{L}*}(\mathtt{s}_S - 1 + a_1). \tag{7.2.5}$$

Then (1) follows from Definition 4.1.5 (ii)–(iii) and (7.2.5). By the same argument in the proof of Lemma 7.2.1 (2), we obtain (2). □

Under the map (4.3.2), put

$$\begin{aligned} (T, S) &= (U_4, U_3, U_2, U_1), \\ \mathcal{S}_2(T, S) &= (\widetilde{U}_4, \widetilde{U}_3, \widetilde{U}_2, \widetilde{U}_1). \end{aligned} \tag{7.2.6}$$

Here $\mathcal{S}_2$ is the operator given as in (4.3.3). For $1 \le i \le 4$, we regard a tableau U_i as follows:

$$\begin{aligned} U_1^{\mathtt{body}} &= U_1 \boxminus \emptyset, \quad U_2^{\mathtt{body}} = U_2 \boxminus S^{\mathtt{tail}}, \\ U_3^{\mathtt{body}} &= U_3 \boxminus \emptyset, \quad U_4^{\mathtt{body}} = U_4 \boxminus T^{\mathtt{tail}}, \end{aligned}$$

where $T^{\mathtt{tail}} = (T^{\mathtt{L}}(a_2), \ldots, T^{\mathtt{L}}(1))$ and $S^{\mathtt{tail}} = (S^{\mathtt{L}}(a_1), \ldots, S^{\mathtt{L}}(1))$. Then we consider

(7.2.6) in $\mathbb{P}_L$. For simplicity, put

$$h = \mathrm{ht}(T^{\mathtt{R}}), \quad g = \mathrm{ht}(S^{\mathtt{R}}).$$

We consider some sequences given inductively as follows:

(i) Define a sequence $v_1 < \cdots < v_h$ by

$$\begin{aligned} v_1 &= \min_{1 \leq k \leq a_2} \left\{ k \;\middle|\; T^{\mathtt{L}}(k) \leq T^{\mathtt{R}}(1) \right\}, \\ v_s &= \min_{v_{s-1}+1 \leq k \leq s+a_2} \left\{ k \;\middle|\; T^{\mathtt{L}}(k) \leq T^{\mathtt{R}}(s) \right\}. \end{aligned} \tag{7.2.7}$$

(ii) Define a sequence $w_1 < \cdots < w_h$ by

$$\begin{aligned} w_h &= \max_{h \leq k \leq h+a_1} \left\{ k \;\middle|\; T^{\mathtt{R}}(h) \leq S^{\mathtt{L}}(k) \right\}, \\ w_t &= \max_{t \leq k \leq w_{t+1}-1} \left\{ k \;\middle|\; T^{\mathtt{R}}(t) \leq S^{\mathtt{L}}(k) \right\}. \end{aligned} \tag{7.2.8}$$

(iii) Define a sequence $x_1 < \cdots < x_g$ by

$$\begin{aligned} x_1 &= \min_{1 \leq k \leq a_1} \left\{ k \;\middle|\; S^{\mathtt{L}}(k) \leq S^{\mathtt{R}}(1) \right\}, \\ x_u &= \min_{x_{u-1}+1 \leq k \leq u+a_1} \left\{ k \;\middle|\; S^{\mathtt{L}}(k) \leq S^{\mathtt{R}}(u) \right\}. \end{aligned} \tag{7.2.9}$$

Proof of Lemma 4.3.1 (1). By Lemma 7.2.1 (1), the proof for the case $\mathfrak{r}_T \cdot \mathfrak{r}_S = 0$ is identical with the argument in [66, Lemma 5.2]. So we prove the case $\mathfrak{r}_T \cdot \mathfrak{r}_S = 1$ here. We consider two cases along $\triangleleft$ (recall Definition 4.1.9).

Case 1. $T \triangleleft S$. In this case, $\mathcal{S}_2 = \mathcal{F}_2^{a_1}$. This implies that $\widetilde{U}_1 = U_1$ and $\widetilde{U}_4 = U_4$. By Lemma 7.2.2 (2),

$$(T^{\mathtt{R}},\, S^{\mathtt{L}}) \in SST_{[\overline{n}]}(\lambda(0, b, c)),$$

where $b = a_1 + c_1 - b_2 - c_2$ and $c = b_2 + c_2$. Since $T \prec S$ with $T \triangleleft S$, $c_1 - b_2 - c_2 \geq 0$. Therefore $\mathcal{F}_2^{a_1}(T, S)$ is well-defined. Now we show that $(U_4, \widetilde{U}_3)$ and $(\widetilde{U}_2, U_1)$ are semistandard along L.

(1) $(U_4, \widetilde{U}_3)$ *is semistandard along* L. For $1 \leq k \leq \mathrm{ht}(T^{\mathtt{R}})$ satisfying $v_k \geq a_2 - a_1 + 1$,

the relation $T \triangleleft S$ implies

$$v_k - a_2 + a_1 \leq w_k. \tag{7.2.10}$$

(i) Assume that $\widetilde{U}_3(k) = T^{\mathtt{R}}(k')$ for some k'. By definition of $w_{k'}$, we have

$$k = w_{k'}. \tag{7.2.11}$$

If $v_{k'} \geq a_2 - a_1 + 1$, then

$$\begin{aligned} U_4(k + a_2 - a_1) &= T^{\mathtt{L}}(w_{k'} + a_2 - a_1) && \text{by (7.2.6) and (7.2.11)} \\ &\leq T^{\mathtt{L}}(v_{k'}) && \text{by (7.2.10)} \\ &\leq T^{\mathtt{R}}(k') = \widetilde{U}_3(k) && \text{by (7.2.7)} \end{aligned} \tag{7.2.12}$$

If $v_{k'} < a_2 - a_1 + 1$, then

$$U_4(k + a_2 - a_1) = T^{\mathtt{L}}(k + a_2 - a_1) < T^{\mathtt{L}}(v_{k'}) \leq T^{\mathtt{R}}(k') = \widetilde{U}_3(k). \tag{7.2.13}$$

(ii) Assume that $\widetilde{U}_3(k) \neq T^{\mathtt{R}}(k')$ for any k'. In this case, we have

$$\widetilde{U}_3(k) = S^{\mathtt{L}}(k).$$

Then

$$\begin{aligned} U_4(k + a_2 - a_1) &= T^{\mathtt{L}}(k + a_2 - a_1) \\ &\leq {}^{\mathtt{R}}T(k + a_2 - a_1) && \text{by definition of } {}^{\mathtt{R}}T \\ &\leq S^{\mathtt{L}}(k) = \widetilde{U}_3(k) && \text{by } T \triangleleft S \end{aligned} \tag{7.2.14}$$

By (7.2.12), (7.2.13) and (7.2.14), $(U_4, \widetilde{U}_3)$ is semistandard along L.

(2) $(\widetilde{U}_2, U_1)$ is semistandard along L. By (7.2.8), (7.2.9) and Definition 4.1.5 (ii), we have

$$x_k \leq w_k \quad \text{for} \quad 1 \leq k \leq \mathtt{s}_T - 1. \tag{7.2.15}$$

Also the relation $T \triangleleft S$ implies

$$k - 1 + a_1 \leq w_k \quad \text{for} \quad k \geq \mathtt{s}_T. \tag{7.2.16}$$

(i) For $1 \leq k \leq \mathtt{s}_T - 1$,

$$\begin{aligned} \widetilde{U}_2(k) &= S^{\mathtt{L}}(w_k) && \text{by (7.2.8)} \\ &\leq S^{\mathtt{L}}(x_k) && \text{by (7.2.15)} \\ &\leq S^{\mathtt{R}}(k) = U_1(k) && \text{by (7.2.9)} \end{aligned}$$

(ii) For $k \geq \mathtt{s}_T$, (7.2.16) implies

$$\widetilde{U}_2(k) = S^{\mathtt{L}}(w_k) \leq S^{\mathtt{L}}(k-1+a_1) \leq S^{\mathtt{R}}(k) = U_1(k).$$

Note that $S^{\mathtt{L}}(k-1+a_1) \leq S^{\mathtt{R}}(k)$ holds since $\mathfrak{r}_S = 1$.

By (i)–(ii), $(\widetilde{U}_2, U_1)$ is semistandard along L.

Case 2. $T \not\trianglelefteq S$. In this case, $\mathcal{S}_2 = \mathcal{E}_2\mathcal{E}_1\mathcal{F}_2^{a_1-1}\mathcal{F}_1$. Note that by definition

$$\mathcal{F}_1(S^{\mathtt{L}}, S^{\mathtt{R}}) = (S^{\mathtt{L}*}, S^{\mathtt{R}*}).$$

By Lemma 7.2.2 (1),

$$(T^{\mathtt{R}},\, S^{\mathtt{L}*}) \in SST_{[\overline{n}]}(\lambda(0,b,c)),$$

where $b = a_1 + c_1 - b_2 - c_2 + 1$ and $c = b_2 + c_2$. Note that by Definition 4.1.5 (i),

$$b = a_1 - 1 + \{c_1 - (b_2 + c_2 - 2)\} \geq a_1 - 1.$$

Therefore, $\mathcal{F}_2^{a_1-1}\mathcal{F}_1(T,S)$ is well-defined. Put

$$\mathcal{F}_2^{a_1-1}\mathcal{F}_1(T,S) = (\dot{U}_4,\, \dot{U}_3,\, \dot{U}_2,\, \dot{U}_1).$$

Note that $\dot{U}_4 = U_4$ by definition of $\mathcal{F}_2^{a_1-1}\mathcal{F}_1$. We use sequences (w_k) and (x_k) in (7.2.8) and (7.2.9) replacing $S^{\mathtt{L}}$, $S^{\mathtt{R}}$ and a_1 with $S^{\mathtt{L}*}$, $S^{\mathtt{R}*}$ and $a_1 - 1$, respectively.

(1) $(U_4, \dot{U}_3)$ *is semistandard along* L. By Lemma 7.2.1 (1), we use the similar argument in the proof of *Case 1 (1)*.

(2) $(\dot{U}_2, \dot{U}_1)$ *is semistandard along* L. We observe

$$\mathcal{E}^{a_1}(S^{\mathtt{L}*}, S^{\mathtt{R}*}) = \mathcal{E}^{a_1}\,\mathcal{F}(S^{\mathtt{L}}, S^{\mathtt{R}}) = \mathcal{E}^{a_1-1}(S^{\mathtt{L}}, S^{\mathtt{R}}) = ({}^{\mathtt{L}}S, {}^{\mathtt{R}}S).$$

Therefore we use the similar argument in the proof of *Case 1 (2)*.

Note that (7.2.5) implies that $S^{\mathtt{R}}(\mathtt{s}_S)$ is contained in $\dot{U}_2$. Then the operator $\mathcal{E}_1$ on $(U_4, \dot{U}_3, \dot{U}_2, \dot{U}_1)$ moves $S^{\mathtt{R}}(\mathtt{s}_S)$ by one position to the right. Therefore we have

$$\widetilde{U}_1 = U_1. \tag{7.2.17}$$

On the other hand, (7.2.4) and Definition 4.1.5 (iii) implies that the operator $\mathcal{E}_2$ on $\mathcal{E}_1(U_4, \dot{U}_3, \dot{U}_2, \dot{U}_1)$ moves

$$\dot{U}_3(w_k) = T^{\mathtt{R}}(k) \quad \text{for some } k \geq \mathtt{s}_T.$$

by one position to the right. By the choice of $\mathtt{s}_T$ and $\mathtt{s}_S$ (7.2.1) with (7.2.17), $(U_4, \widetilde{U}_3)$ and $(\widetilde{U}_2, U_1)$ are semistandard along L.

We complete the proof of Lemma 4.3.1 (1). □

Proof of Lemma 4.3.1 (2). Put $\widetilde{e}_k(T, S) = (T', S')$. Then it is not difficult to check that

$$\mathfrak{r}_T = \mathfrak{r}_{T'}, \quad \mathfrak{r}_S = \mathfrak{r}_{S'}. \tag{7.2.18}$$

($\Rightarrow$) Assume that

$$\widetilde{e}_k(T, S) = (T, \widetilde{e}_k\, S) \quad \text{and} \quad \widetilde{e}_k\, S = (\widetilde{e}_k\, S^{\mathtt{L}}, S^{\mathtt{R}}) \quad (k \in J). \tag{7.2.19}$$

Otherwise it is clear that $T' \triangleleft S'$.

If $\mathfrak{r}_T \cdot \mathfrak{r}_S = 0$, then Lemma 7.2.1 (1) and (7.2.18) implies that $T' \triangleleft S'$ holds.

If $\mathfrak{r}_T \cdot \mathfrak{r}_S = 1$, then suppose $T' \not\triangleleft S'$. By Lemma 7.2.1 (2), there exists $s \geq \mathtt{s}_T$ such that

$$T^{\mathtt{R}}(s) = \overline{k}$$

which contradicts to (7.2.19) by the tensor product rule. Hence we have $T' \triangleleft S'$.

($\Leftarrow$) It follows from the similar argument of the previous proof. □

7.2.2 Proof of Lemma 4.3.3

We remark that the results in Section 5.3 are also available in proving Lemma 4.3.3, see [67, Remark 3.8].

Let us define

$$\widetilde{\mathbf{T}} = \begin{cases} (\widetilde{T}_{l-1}, \ldots, \widetilde{T}_1, \widetilde{T}_0), & \text{if } n = 2l, \\ (\widetilde{T}_l, \ldots, \widetilde{T}_1, \widetilde{T}_0), & \text{if } n = 2l+1, \end{cases} \tag{7.2.20}$$

as follows:

(1) if $n = 2l$, then let $\widetilde{T}_0 = U_1$ and let $\widetilde{T}_i \in \mathbf{T}(a_i)$ for $1 \le i \le l-1$ such that $(\widetilde{T}_i^{\mathtt{L}}, \widetilde{T}_i^{\mathtt{R}}) = (\widetilde{U}_{2i+1}, \widetilde{U}_{2i})$, given in Corollary 5.3.16(ii),

(2) if $n = 2l+1$, then let $\widetilde{T}_0 = \emptyset$, $\widetilde{T}_1 \in \mathbf{T}(0)$ and $\widetilde{T}_{i+1} \in \mathbf{T}(a_i)$ for $1 \le i \le l-1$ such that $(\widetilde{T}_1^{\mathtt{L}}, \widetilde{T}_1^{\mathtt{R}}) = (U_1, U_0)$, $(\widetilde{T}_i^{\mathtt{L}}, \widetilde{T}_i^{\mathtt{R}}) = (\widetilde{U}_{2i+1}, \widetilde{U}_{2i})$, given in Corollary 5.3.16(i) and (ii), respectively.

We have $\widetilde{\mathbf{T}} \in \widehat{\mathbf{T}}_{\widetilde{\lambda}}$. Let us show that $\widetilde{\mathbf{T}} \in \mathbf{T}_{\widetilde{\lambda}}$. For simplicity, let us assume that $n = 2l$ since the proof for $n = 2l+1$ is almost identical.

By Corollary 5.3.16(1), we have $\widetilde{T}_1 \prec \widetilde{T}_0$. So it suffices to show that $\widetilde{T}_i \prec \widetilde{T}_{i-1}$ for $2 \le i \le l-1$. This can be checked in a straightforward way using the fact that $\mathbf{T} \in \mathbf{H}(\lambda)$ and Lemma 5.3.15 as follows.

Consider a triple (T_{i+1}, T_i, T_{i-1}) in $\mathbf{T}$. Recall that each T_i satisfies (H1) and (H2). Without loss of generality, let us consider (T_3, T_2, T_1), which can be identified with

$$(U_6, U_5, U_4, U_3, U_2, U_1)$$

under the map (4.3.1). Put

$$\mathcal{S}_2\mathcal{S}_4(T_3, T_2, T_1) = (\widetilde{U}_6, \widetilde{U}_5, \widetilde{U}_4, \widetilde{U}_3, \widetilde{U}_2, \widetilde{U}_1).$$

Note that $\widetilde{U}_1 = U_1$ and $\widetilde{U}_6 = U_6$. Let $\lambda(a_j, b_j, c_j)$ be the shape of T_j for $j = 1, 2, 3$. Let $\widetilde{T}_j$ be the tableau corresponding to $(\widetilde{U}_{j+2}, \widetilde{U}_{j+1})$ for $j = 1, 2, 3$ in (7.2.20).

We consider the following four cases. The other cases can be checked in a similar manner.

Case 1. $(\mathfrak{r}_3, \mathfrak{r}_2, \mathfrak{r}_1) = (0, 0, 0)$**.** In this case, the operators $\mathcal{S}_2$ and $\mathcal{S}_4$ are just sliding $T_2^{\mathtt{tail}}$ and $T_1^{\mathtt{tail}}$ to the left horizontally. Note that $\widetilde{T}_1 \in SST(\lambda(a_1, c_1 - b_2 - c_2, b_2 + c_2))$ and $\widetilde{T}_2 \in SST(\lambda(a_2, c_2 - b_3 - c_3, b_3 + c_3))$. It is straightforward to check that $\widetilde{T}_2 \prec \widetilde{T}_1$.

Case 2. $(\mathfrak{r}_3, \mathfrak{r}_2, \mathfrak{r}_1) = (1, 1, 0)$**.** If $U_5(1) < U_4(a_2)$, then the proof is the same as in *Case 1.* So we assume that $U_5(1) > U_4(a_2)$.

Note that $\widetilde{T}_1 \in SST(\lambda(a_1, c_1 - b_2 - c_2, b_2 + c_2))$ and $\widetilde{T}_2 \in SST(\lambda(a_2, c_2 - b_3 - c_3 + 4, b_3 + c_3 - 2))$. By Corollary 5.3.16(2), $\widetilde{T}_2$ and $\widetilde{T}_1$ have residue 1 and 0 respectively. We see that Definition 4.1.5(1)-(i) holds on $(\widetilde{T}_2, \widetilde{T}_1)$.

By [67, Lemma 3.4], we have $U_5(1) = \widetilde{U}_4(1) \leq \widetilde{U}_3(a_1+1) = U_3(1)$, which together with (H1) on T_2 implies Definition 4.1.5(1)-(ii) on $(\widetilde{T}_2, \widetilde{T}_1)$. Definition 4.1.5(1)-(iii) on $(\widetilde{T}_2, \widetilde{T}_1)$ follows from the one on (T_2, T_1), (H1) and (H2) on T_2. Thus $\widetilde{T}_2 \prec \widetilde{T}_1$.

Case 3. $(\mathfrak{r}_3, \mathfrak{r}_2, \mathfrak{r}_1) = (0, 1, 1)$**.** If $U_3(1) < U_2(a_1)$, then the proof is the same as in *Case 1*. So we assume that $U_3(1) > U_2(a_1)$.

Note that $\widetilde{T}_2 \in SST(\lambda(a_1, c_1 - b_2 - c_2 + 4, b_2 + c_2 - 2))$ and $\widetilde{T}_1 \in SST(\lambda(a_2, c_2 - b_3 - c_3, b_3 + c_3))$. By Corollary 5.3.16(2), $\widetilde{T}_2$ and $\widetilde{T}_1$ have residue 0 and 1 respectively. We see that Definition 4.1.5(1)-(i) holds on $(\widetilde{T}_2, \widetilde{T}_1)$. Definition 4.1.5(1)-(ii) on $(\widetilde{T}_2, \widetilde{T}_1)$ follows from (H1) on T_2. Also, Definition 4.1.5(1)-(iii) on $(\widetilde{T}_2, \widetilde{T}_1)$ follows from the one on (T_2, T_1). Thus $\widetilde{T}_2 \prec \widetilde{T}_1$.

Case 4. $(\mathfrak{r}_3, \mathfrak{r}_2, \mathfrak{r}_1) = (1, 1, 1)$**.** If $U_3(1) < U_2(a_1)$ or $U_5(1) < U_4(a_2)$, then the proof is the same as the one of *Case 1*-*Case 3*. So we assume that $U_3(1) > U_2(a_1)$ and $U_5(1) > U_4(a_2)$.

Note that $\widetilde{T}_2 \in SST(\lambda(a_1, c_1 - b_2 - c_2 + 4, b_2 + c_2 - 2))$ and $\widetilde{T}_1 \in SST(\lambda(a_2, c_2 - b_3 - c_3 + 4, b_3 + c_3 - 2))$ and both have residue 1. Since $\mathfrak{r}_2 = 1$, we have $b_2 \geq 2$ and $c_2 + 2 \leq b_2 + c_2$, which implies Definition 4.1.5(1)-(i) on $(\widetilde{T}_2, \widetilde{T}_1)$.

Definition 4.1.5(1)-(ii) and (iii) on $(\widetilde{T}_2, \widetilde{T}_1)$ follow from the same argument as in *Case 2*. Thus $\widetilde{T}_2 \prec \widetilde{T}_1$.

Finally, since $\widetilde{\mathbf{T}}$ corresponds to $\widetilde{\mathbf{U}}$ (4.3.7), we have $\widetilde{\mathbf{T}} \in \mathbf{H}(\widetilde{\lambda})$. □

7.3 In Chapter 5

7.3.1 Outline

The proof of Theorem 5.4.4 is rather lengthy and technical, so we outline the proof. The proof of Theorem 5.4.4 is organized as follows.

In subsection 7.3.2, we consider the case of $n - 2\mu'_1 \geq 0$, which is easier to deal with than the case of $n - 2\mu'_1 < 0$.

(1) (*Well-definedness*) First, we show that $\overline{\mathbf{T}}^{\mathtt{tail}} \in \overline{\mathtt{LR}}^{\lambda'}_{\delta'\mu'}$ (Corollary 7.3.4). To do this, we study some properties of the sequences $(m_i)_{1 \leq i \leq p}$ and $(n_j)_{1 \leq j \leq q}$ associated with

$\overline{\mathbf{T}}^{\mathtt{tail}}$ with respect to sliding (Lemmas 7.3.2 and 7.3.3), which implies that $\overline{\mathbf{T}}^{\mathtt{tail}}$ satisfies (5.4.1).

(2) (*Injectivity*) Second, we show that the map

$$\begin{array}{ccc} \mathtt{LR}^{\mu}_{\lambda}(\mathfrak{d}) & \longrightarrow & \bigsqcup_{\delta \in \mathscr{P}^{(2)}_n} \overline{\mathtt{LR}}^{\lambda'}_{\delta'\mu'} \\ \mathbf{T} & \longmapsto & \overline{\mathbf{T}}^{\mathtt{tail}} \end{array}.$$

is injective by using Proposition 5.3.19 (Lemma 7.3.5).

(3) (*Surjectivity*) Finally, we prove the above map is surjective, that is, for $\mathbf{W} \in \overline{\mathtt{LR}}^{\lambda'}_{\delta'\mu'}$, there exists $\mathbf{T} \in \mathtt{LR}^{\mu}_{\lambda}(\mathfrak{d})$ such that $\overline{\mathbf{T}}^{\mathtt{tail}} = \mathbf{W}$. We use induction on n. The initial step when $n = 4$ is proved in Lemma 7.3.6. Then based on this step, we construct $\mathbf{T} \in \mathtt{LR}^{\mu}_{\lambda}(\mathfrak{d})$ in general in Lemma 7.3.8

In subsection 7.3.3, we consider the case of $n - 2\mu'_1 < 0$. The proof is almost identical to the case of $n - 2\mu'_1 \geq 0$, but the major difficulty occurs when we consider the columns with odd height in $\overline{\mathbf{T}}(0)$ and $\mathbf{T}^{\mathrm{sp}-}$ (cf. Remark 4.1.6(2)–(3)). To overcome this, we reduce the problems to the ones in the case of $n - 2\mu'_1 \geq 0$ so that we may apply the results (or the arguments in the proof) in subsection 7.3.2.

7.3.2 Proof of Theorem 5.4.4 when when $n - 2\mu'_1 \geq 0$

Let $\mu \in \mathcal{P}(\mathrm{O}_n)$ and $\lambda \in \mathscr{P}_n$ be given. We assume that $n - 2\mu'_1 \geq 0$. We keep the notations in Sections 3.1.1 and 4.1.1 and Chapter 5.

Suppose that $n = 2l + r$, where $l \geq 1$ and $r = 0, 1$. Let $\mathbf{T} \in \mathtt{LR}^{\mu}_{\lambda}(\mathfrak{d})$ be given with $\mathbf{T} = (T_l, \ldots, T_1, T_0)$ as in (5.3.4). Let us assume that $r = 0$ since the argument for $r = 1$ is almost identical. Write $\widetilde{\mathbf{T}} = (\widetilde{T}_{l-1}, \ldots, \widetilde{T}_1, \widetilde{T}_0)$. Let $\overline{\mathbf{T}}^{\mathsf{body}} = H_{(\delta')^{\pi}}$ for some $\delta \in \mathscr{P}^{(2)}$. Let $s_1 \leq \cdots \leq s_p$ denote the entries in the first row, and $t_1 \leq \cdots \leq t_q$ the entries in the second row of $\overline{\mathbf{T}}^{\mathtt{tail}}$.

Lemma 7.3.1. *Suppose that* $\overline{\mathbf{T}} = (U_{2l}, \ldots, U_1) \in \mathbf{E}^n$ *under* (4.3.1)*. If* $T^{\mathtt{R}}_{i+1}(1) < T^{\mathtt{L}}_i(a_i)$*, then*

$$U_{2i} = T^{\mathtt{L}}_i \boxminus T^{\mathtt{tail}}_i.$$

In this case, $T^{\mathtt{tail}}_i$ *is the* $(l-i+1)$*-th column of* $\overline{\mathbf{T}}^{\mathtt{tail}}$ *from the left.*

Proof. If $T^{\mathtt{R}}_{i+1}(1) < T^{\mathtt{L}}_i(a_i)$, then by definition we have $\widetilde{T}_i$ has residue 0. By Lemma 4.3.3, $\widetilde{T}^{\mathtt{R}}_i(1) < \widetilde{T}^{\mathtt{L}}_{i-1}(a_{i-1})$. Inductively, we have $U_{2i} = T^{\mathtt{L}}_i \boxminus T^{\mathtt{tail}}_i$. By applying this argument together with Lemma 4.3.3, we obtain the second statement. $\square$

For simplicity, let us put $\mathbb{T} = \widetilde{\mathbf{T}}$. Let $\widetilde{\mu} = (\mu_2, \mu_3, \dots)$ and $\zeta = (\delta_1, \dots, \delta_{n-1}) \in \mathscr{P}^{(2)}_{n-1}$. By Lemmas 5.3.6, 4.3.3 and Proposition 5.3.19, we have $\overline{\mathbb{T}}^{\mathtt{body}} = H_{(\zeta')^\pi}$ and $\overline{\mathbb{T}}^{\mathtt{tail}} \in \mathtt{LR}^{\xi'}_{\zeta'\widetilde{\mu}'}$ where ξ is given by $(H_{\zeta'} \leftarrow \overline{\mathbb{T}}^{\mathtt{tail}}) = H_{\xi'}$. Let $\widetilde{s}_1 \le \cdots \le \widetilde{s}_{p-1}$ be the entries in the first row of $\overline{\mathbb{T}}^{\mathtt{tail}}$ and let $(\widetilde{m}_i)_{1\le i\le p-1}$ (resp. $(m_i)_{1\le i\le p}$) be the sequence associated with $\overline{\mathbb{T}}^{\mathtt{tail}}$ (resp. $\overline{\mathbf{T}}^{\mathtt{tail}}$) in Definition 5.4.1. Note that $s_i = \widetilde{s}_{i-1}$ for $2 \le i \le p$. Put $\mathtt{T}_i = T_{l-i+1}$ for $1 \le i \le l$ and $\widetilde{\mathtt{T}}_j = \widetilde{T}_{l-j}$ for $1 \le j \le l-1$. Assume that $\mathtt{T}_i \in \mathbf{T}(\mathtt{a}_i)$ for $1 \le i \le l$.

Lemma 7.3.2. *Under the above hypothesis, the sequences $(m_i)_{1\le i\le p}$ and $(\widetilde{m}_i)_{1\le i\le p-1}$ satisfy the relation*

$$m_i = \widetilde{m}_{i-1} + \tau_i + 1 \quad (2 \le i \le p), \tag{7.3.1}$$

where τ_i is given by

$$\tau_i = \begin{cases} 1, & \text{if } \mathtt{T}^{\mathtt{R}}_{i-1}(1) < \mathtt{T}^{\mathtt{L}}_i(\mathtt{a}_i), \\ 0, & \text{if } \mathtt{T}^{\mathtt{R}}_{i-1}(1) > \mathtt{T}^{\mathtt{L}}_i(\mathtt{a}_i). \end{cases}$$

Proof. Fix $i \ge 2$. If $\mathtt{T}^{\mathtt{R}}_{i-1}(1) < \mathtt{T}^{\mathtt{L}}_i(\mathtt{a}_i)$, then by Lemma 5.3.15(i) and 7.3.1

$$m_i = 2i - 1, \qquad \widetilde{m}_{i-1} = 2i - 3.$$

If $\mathtt{T}^{\mathtt{R}}_{i-1}(1) > \mathtt{T}^{\mathtt{L}}_i(\mathtt{a}_i)$, then we have by Lemma 5.3.15(ii) $m_i < 2i - 1$. This implies $m_i = \widetilde{m}_{i-1} + 1$. Hence we have (7.3.1). $\square$

Lemma 7.3.3. *For $1 \le i \le q$, we have*

$$t_i = \mathtt{T}^{\mathtt{L}}_i(\mathtt{a}_i - 1) > \mathtt{T}^{\mathtt{R}}_i(1) = \delta^{\mathtt{rev}}_{n_i}.$$

Proof. By Lemma 5.3.15, it is easy to see that $t_i = \mathtt{T}^{\mathtt{L}}_i(\mathtt{a}_i - 1)$. Next we claim that $\mathtt{T}^{\mathtt{R}}_i(1) = \delta^{\mathtt{rev}}_{n_i}$, which implies the inequality since $\mathfrak{r}_{\mathtt{T}_i} \le 1$. We use induction on n. For each i, we define θ_i to be the number of j's with $i+1 \le j \le p$ such that $m_j < 2i+1$. Then we have

$$n_i = 2i + \theta_i. \tag{7.3.2}$$

If $\theta_i = 0$, then $n_i = 2i$ and $m_{i+1} = 2i + 1$, which implies that

$$\mathtt{T}_i^{\mathtt{R}}(1) < \mathtt{T}_{i+1}^{\mathtt{L}}(\mathtt{a}_{i+1}).$$

By applying Lemma 7.3.1 on $\widetilde{\mathbf{T}}$, we have $\delta_{2i}^{\mathtt{rev}} = \mathtt{T}_i^{\mathtt{R}}(1)$.

If $\theta_i > 0$, then we have by definition of θ_i,

$$\mathtt{T}_{i-1}^{\mathtt{R}}(1) > \mathtt{T}_i^{\mathtt{L}}(\mathtt{a}_i). \tag{7.3.3}$$

Let $(\widetilde{m}_i)_{1\le i\le p-1}$ and $(\widetilde{n}_i)_{1\le i\le q-1}$ be the sequences in Definition 5.4.1 associated with $\widetilde{\mathbf{T}}$. Let $\widetilde{\theta}_i$ be defined in the same way with respect to $(\widetilde{m}_i)_{1\le i\le p-1}$. By definition of $\widetilde{\theta}_i$ and Lemma 7.3.2, we have for $j \ge i+2$,

$$m_j < 2i+1 \implies \widetilde{m}_{j-1} < 2i - \tau_j \le 2i - 1.$$

Thus we have $\widetilde{\theta}_i = \theta_i - 1$. By induction hypothesis, (7.3.2), and (7.3.3), we have

$$\mathtt{T}_i^{\mathtt{R}}(1) = \widetilde{\mathtt{T}}_i^{\mathtt{R}}(1) = \widetilde{\delta}_{\widetilde{n}_i}^{\mathtt{rev}} = \widetilde{\delta}_{2i+\widetilde{\theta}_i}^{\mathtt{rev}} = \delta_{2i+\theta_i}^{\mathtt{rev}} = \delta_{n_i}^{\mathtt{rev}}.$$

□

Corollary 7.3.4. *Under the above hypothesis, we have* $\overline{\mathbf{T}}^{\mathtt{tail}} \in \overline{\mathtt{LR}}_{\mu'\delta'}^{\lambda'}$.

Proof. It follows from Remark 5.4.2 and Lemma 7.3.3. □

Lemma 7.3.5. *The map* $\mathbf{T} \longmapsto \overline{\mathbf{T}}^{\mathtt{tail}}$ *is injective on* $\mathtt{LR}_\lambda^\mu(\mathfrak{d})$.

Proof. Let $\mathbf{T}, \mathbf{S} \in \mathtt{LR}_\lambda^\mu(\mathfrak{d})$ be given. Suppose that $\overline{\mathbf{T}}^{\mathtt{tail}} = \overline{\mathbf{S}}^{\mathtt{tail}}$. We first claim that $\overline{\mathbf{T}} = \overline{\mathbf{S}}$. By Proposition 5.3.19(1), we have $\overline{\mathbf{T}}^{\mathtt{body}} = H_{\delta'}$ and $\overline{\mathbf{S}}^{\mathtt{body}} = H_{\chi'}$ for some $\delta, \chi \in \mathscr{P}^{(2)}$. Since $\overline{\mathbf{T}}^{\mathtt{tail}} = \overline{\mathbf{S}}^{\mathtt{tail}}$ and $(\overline{\mathbf{T}}^{\mathtt{tail}} \to H_{\delta'}) = (\overline{\mathbf{S}}^{\mathtt{tail}} \to H_{\chi'}) = H_{\lambda'}$, we have $\delta = \chi$. Hence $\overline{\mathbf{T}}^{\mathtt{body}} = \overline{\mathbf{S}}^{\mathtt{body}}$, which implies $\overline{\mathbf{T}} = \overline{\mathbf{S}}$. Since the map $\mathbf{T} \mapsto \overline{\mathbf{T}}$ is reversible, we have $\mathbf{T} = \mathbf{S}$. □

Now, we verify that the map in Theorem 5.4.4 is surjective. Let $\mathbf{W} \in \overline{\mathtt{LR}}_{\delta'\mu'}^{\lambda'}$ be given for some $\delta \in \mathscr{P}_n^{(2)}$. Let $\mathbf{V} = H_{(\delta')^\pi}$ and $\mathbf{X}$ be the tableaux of a skew shape η as in (4.3.8) with n columns such that

$$\mathbf{X}^{\mathtt{body}} = \mathbf{V}, \quad \mathbf{X}^{\mathtt{tail}} = \mathbf{W}.$$

The semistandardness of $\mathbf{X}$ follows from Definition 5.4.1 and Remark 5.4.2. Let V_i and W_i denote the i-th column of $\mathbf{V}$ and $\mathbf{W}$ from right, respectively.

Let us first consider the following, which is used in the proof of Lemma 7.3.8.

Lemma 7.3.6. *Assume that $n = 4$ and $\mu'_1 = 2$. Then there exists $\mathbf{T} = (T_2, T_1) \in \mathtt{LR}^{\mu}_{\lambda}(\mathfrak{d})$ such that $\overline{\mathbf{T}} = \mathbf{X}$, that is, $\overline{\mathbf{T}}^{\mathtt{body}} = \mathbf{V}$ and $\overline{\mathbf{T}}^{\mathtt{tail}} = \mathbf{W}$. In fact, $\mathbf{T} = (T_2, T_1)$ is given as follows:*

(1) If $m_2 = 3$, then

$$(T_2^{\mathtt{L}}, T_2^{\mathtt{R}}) = (V_4 \boxplus W_2, V_3), \quad (T_1^{\mathtt{L}}, T_1^{\mathtt{R}}) = (V_2 \boxplus W_1, V_1).$$

(2) If $m_2 = 2$, then

$$(T_2^{\mathtt{L}}, T_2^{\mathtt{R}}) = (V_4 \boxplus W_2, V_3^{\diamond}), \quad (T_1^{\mathtt{L}}, T_1^{\mathtt{R}}) = (V_2^{\diamond} \boxplus W_1^{\diamond}, V_1),$$

where $V_3^{\diamond}$, $V_2^{\diamond}$ and $W_1^{\diamond}$ are given by

$$\begin{aligned} &V_3^{\diamond} = (\ldots, V_3(2), V_3(1), W_1(a_1), V_2(1)) \boxplus \emptyset, \\ &V_2^{\diamond} = (\ldots, V_2(4), V_2(3)), \quad W_1^{\diamond} = (W_2(2), W_1(a_1 - 1), \ldots W_1(1)). \end{aligned}$$

Proof. By Remark 5.4.2 and definition of them, T_1 and T_2 are semistandard. Also, the residue $\mathfrak{r}_i$ of T_i is by Definition 5.4.1 at most 1 for $i = 1, 2$. It suffices to verify that $T_2 \prec T_1$ since this implies $\overline{\mathbf{T}}^{\mathtt{body}} = \mathbf{V}$ and $\overline{\mathbf{T}}^{\mathtt{tail}} = \mathbf{W}$ by construction of $\mathbf{T}$.

Let a_i be the height of W_i for $i = 1, 2$. We have $V_i = (1, 2, \ldots, \delta_i^{\mathtt{rev}})$ for $1 \le i \le 4$, with $\delta_1^{\mathtt{rev}} \le \delta_2^{\mathtt{rev}} \le \delta_3^{\mathtt{rev}} \le \delta_4^{\mathtt{rev}}$. Let $w(\mathbf{X}) = w_1 w_2 \cdots w_m$. Put

$$P_k = ((((w_1 \leftarrow w_2) \leftarrow w_3) \leftarrow \cdots) \leftarrow w_k)$$

for $k \le m$. Suppose that

$$\begin{aligned} &w(V_1)w(V_2)w(V_3) = w_1 w_2 \cdots w_s, \\ &w(V_1)w(V_2)w(V_3)w(W_1)w(V_4) = w_1 w_2 \cdots w_t, \end{aligned}$$

for some $s \le t \le m$.

Case 1. $m_2 = 3$**.** We first assume that $\mathfrak{r}_1 \mathfrak{r}_2 = 0$.

(i) $\mathfrak{r}_1 = 0,\ \mathfrak{r}_2 = 0$: It is obvious that $T_2 \prec T_1$.

(ii) $\mathfrak{r}_1 = 0,\ \mathfrak{r}_2 = 1$: Definition 4.1.5(1)-(i) follows from $\delta_2^{\mathtt{rev}} \leq \delta_3^{\mathtt{rev}}$. Also, Definition 4.1.5(1)-(ii) follows from $s_2 \geq \delta_3^{\mathtt{rev}} \geq \delta_2^{\mathtt{rev}}$. The semistandardness of $\mathbf{W}$ implies Definition 4.1.5(1)-(iii). Thus we have $T_2 \prec T_1$.

(iii) $\mathfrak{r}_1 = 1,\ \mathfrak{r}_2 = 0$: We may use the same argument as in (ii) to have $T_2 \prec T_1$.

Next, we assume that $\mathfrak{r}_1\mathfrak{r}_2 = 1$. Definition 4.1.5(1)-(i) holds by definition of $\mathbf{T}$. Since $\mathfrak{r}_1 = 1$ and $\mathfrak{r}_2 = 1$, we have

$$\delta_3^{\mathtt{rev}} < W_1(a_1) \leq \delta_4^{\mathtt{rev}}, \quad \delta_1^{\mathtt{rev}} < W_2(a_2) \leq \delta_2^{\mathtt{rev}}. \tag{7.3.4}$$

By Lemma 5.3.6,

$$(P_s \leftarrow W_1(a_1)) \text{ and } (P_t \leftarrow W_2(a_2)) \text{ are } \mathfrak{l}\text{-highest weight elements.} \tag{7.3.5}$$

By (7.3.4) and (7.3.5), we have

$$W_1(a_1) = \delta_3^{\mathtt{rev}} + 1, \quad W_2(a_2) = \delta_1^{\mathtt{rev}} + 1. \tag{7.3.6}$$

This implies Definition 4.1.5(1)-(ii). Definition 4.1.5(1)-(iii) follows from the semistandardness of $\mathbf{W}$ and (7.3.6). Thus we have $T_2 \prec T_1$.

Case 2. $m_2 = 2$**.** Since $m_2 = 2$, we have

$$\delta_2^{\mathtt{rev}} < \delta_3^{\mathtt{rev}}. \tag{7.3.7}$$

Otherwise, we have $W_1(a_1) > \delta_3^{\mathtt{rev}}$, which contradicts to $m_2 = 2$. By definition of m_2, we have

$$\delta_2^{\mathtt{rev}} < W_1(a_1) < \delta_3^{\mathtt{rev}}. \tag{7.3.8}$$

Note that if $W_1(a_1) = \delta_3^{\mathtt{rev}}$, then by (7.3.7) the tableau $(P_s \leftarrow W_1(a_1))$ cannot be an $\mathfrak{l}$-highest weight element. Since $(P_s \leftarrow W_1(a_1))$ is an $\mathfrak{l}$-highest weight element,

$$W_1(a_1) = \delta_2^{\mathtt{rev}} + 1. \tag{7.3.9}$$

In particular, we have

$$\delta_2^{\mathtt{rev}} + 2 \leq \delta_3^{\mathtt{rev}}. \tag{7.3.10}$$

Since $W_2(a_2) \leq W_1(a_1) < \delta_3^{\mathtt{rev}}$ and $W_1(a_1) = \delta_2^{\mathtt{rev}} + 1$, we have $\mathfrak{r}_2 = 1$. Also, since $\delta_3^{\mathtt{rev}} \leq \delta_4^{\mathtt{rev}}$, it is clear that $\mathfrak{r}_1 = 1$. Note that

$$\delta_1^{\mathtt{rev}} < W_2(a_2) \leq W_1(a_1) = \delta_2^{\mathtt{rev}} + 1 < \delta_3^{\mathtt{rev}}.$$

This implies that

$$W_2(a_2) = \delta_1^{\mathtt{rev}} + 1, \tag{7.3.11}$$

since $(P_t \leftarrow W_2(a_2))$ is an $\mathfrak{l}$-highest weight element.

Now, Definition 4.1.5(1)-(i) follows from (7.3.10), and Definition 4.1.5(1)-(ii) and (1)-(iii) follow from (7.3.9), (7.3.10), (7.3.11) and the semistandardness of $\mathbf{W}$. Hence we have $T_2 \prec T_1$. □

Let $\mathbb{X}$ be the tableau obtained from $\mathbf{X}$ by removing its leftmost column. Let $\widetilde{\mu} = (\mu_2, \mu_3, \dots)$ and $\zeta = (\delta_1, \dots, \delta_{n-1}) \in \mathscr{P}_{n-1}^{(2)}$. Since $\mathbb{X}$ is an $\mathfrak{l}$-highest weight element by Lemma 5.3.6, we have $\mathbb{X}^{\mathtt{body}} = H_{(\zeta')^\pi}$ and $\mathbb{X}^{\mathtt{tail}} \in \mathtt{LR}^{\xi'}_{\zeta'\widetilde{\mu}'}$, where ξ is given by $(H_{\zeta'} \leftarrow \mathbb{X}^{\mathtt{tail}}) = H_{\xi'}$.

Lemma 7.3.7. *We have* $\mathbb{X}^{\mathtt{tail}} \in \overline{\mathtt{LR}}^{\xi'}_{\zeta'\widetilde{\mu}'}$.

Proof. Let $(m_i)_{1\leq i\leq p}$ and $(n_i)_{1\leq i\leq q}$ be the sequences associated with $\mathbf{X}^{\mathtt{tail}} = \mathbf{W} \in \overline{\mathtt{LR}}^{\lambda'}_{\delta'\mu'}$. Let $\widetilde{s}_1 \leq \cdots \leq \widetilde{s}_{p-1}$ and $\widetilde{t}_1 \leq \cdots \leq \widetilde{t}_{q-1}$ be the entries in the first and second rows of $\mathbb{X}^{\mathtt{tail}}$, respectively.

We define a sequence $1 \leq \widetilde{m}_1 < \cdots < \widetilde{m}_{p-1} \leq n-1$ inductively as in Definition 5.4.1 with respect to $(\widetilde{s}_i)_{1\leq i\leq p-1}$. Note that the sequence $(\widetilde{m}_i)_{1\leq i\leq p-1}$ is well-defined by Remark 5.4.2. By Lemma 5.3.15, we observe that

$$\widetilde{m}_i = \begin{cases} 1, & \text{if } i = 1, \\ m_{i+1} - 1, & \text{if } i > 1 \text{ and } m_{i+1} < 2i+1, \\ m_{i+1} - 2, & \text{if } i > 1 \text{ and } m_{i+1} = 2i+1. \end{cases} \tag{7.3.12}$$

Let $(\widetilde{n}_i)_{1\leq i\leq q-1}$ be the sequence with respect to $(\widetilde{m}_i)_{1\leq i\leq p-1}$, that is,

$$\widetilde{n}_i = \text{the } i\text{-th smallest integer in } \{i+1, \dots, n-1\} \setminus \{\widetilde{m}_{i+1}, \dots, \widetilde{m}_{p-1}\}.$$

By (7.3.12), we obtain

$$\widetilde{n}_i \leq n_{i+1} - 1, \tag{7.3.13}$$

and hence

$$\widetilde{t}_i = t_{i+1} > \delta^{\mathtt{rev}}_{n_{i+1}} = \widetilde{\delta}^{\mathtt{rev}}_{n_{i+1}-1} \geq \widetilde{\delta}^{\mathtt{rev}}_{\widetilde{n}_i}.$$

Therefore, we have $\mathbb{X}^{\mathtt{tail}} \in \overline{\mathtt{LR}}^{\xi'}_{\zeta'\widetilde{\mu}'}$. □

Lemma 7.3.8. *There exists* $\mathbf{T} \in \mathtt{LR}^{\mu}_{\lambda}(\mathfrak{d})$ *such that* $\overline{\mathbf{T}} = \mathbf{X}$*, that is,*

$$\overline{\mathbf{T}}^{\mathsf{body}} = \mathbf{V}, \quad \overline{\mathbf{T}}^{\mathtt{tail}} = \mathbf{W}.$$

Proof. We use induction on $n \geq 2$. We may assume that $\mu'_1 \neq 0$.

Let us first consider $n = 3$. Note that $\mathbf{V} = (V_3, V_2, V_1)$, and $\mathbf{W}$ is a tableau of single-columned shape. Define (T_2, T_1) by

$$(T_2^{\mathtt{L}}, T_2^{\mathtt{R}}) = (V_3 \boxplus \mathbf{W}, V_2), \quad T_1 = V_1.$$

Clearly T_1 and T_2 are semistandard. By Definition 5.4.1, the residue of T_2 is at most 1. It is easy to check that $T_2 \prec T_1$. Therefore, $\mathbf{T} = (T_2, T_1) \in \mathtt{LR}^{\mu}_{\lambda}(\mathfrak{d})$ and $\overline{\mathbf{T}} = \mathbf{X}$. Next, consider $n = 4$. When $\mu'_1 = 1$, apply the same argument as in the case of $n = 3$. When $\mu'_1 = 2$, we apply Lemma 7.3.6.

Suppose that $n > 4$. Let us assume that $n = 2l$ is even since the argument for n odd is almost the same. By Lemma 7.3.7 and induction hypothesis, there exists $\mathbb{T} \in \mathtt{LR}^{\widetilde{\mu}}_{\xi}(\mathfrak{d})$ such that

$$\overline{\mathbb{T}}^{\mathsf{body}} = \mathbb{X}^{\mathsf{body}}, \quad \overline{\mathbb{T}}^{\mathtt{tail}} = \mathbb{X}^{\mathtt{tail}},$$

where $\widetilde{\mu}$ and ξ are as in Lemma 7.3.7.

Now, let us construct $\mathbf{T} = (T_l, \ldots, T_1) \in \mathtt{LR}^{\mu}_{\lambda}(\mathfrak{d})$ from $\mathbb{T}$, which satisfies $\overline{\mathbf{T}} = \mathbf{X}$, by applying Lemma 7.3.6 repeatedly.

Let $\mathbb{T} = (\widetilde{T}_{l-1}, \ldots, \widetilde{T}_1, \widetilde{T}_0)$ and let a_i be the height of $\widetilde{T}_i^{\mathtt{tail}}$ for $1 \leq i \leq l-1$. Put

$$\mathbb{U} = (\widetilde{U}_{2l-1}, \ldots, \widetilde{U}_2, \widetilde{U}_1),$$

where

$$\widetilde{U}_1 = \widetilde{T}_0, \quad (\widetilde{U}_{2i+1}, \widetilde{U}_{2i}) = (\widetilde{T}_i^{\mathtt{L}}, \widetilde{T}_i^{\mathtt{R}}) \quad (1 \leq i \leq l-1).$$

Let us define

$$\mathbf{U} = (U_{2l}, \dots, U_2, U_1).$$

First, let $U_1 = \widetilde{U}_1$ and let U_{2l} be the leftmost column of $\mathbf{X}$. For $1 \le i \le l-1$, let (U_{2i+1}, U_{2i}) be defined in the following way. Suppose that $a_i = 0$. Then we put

$$U_{2i+1} = \widetilde{U}_{2i+1}, \quad U_{2i} = \widetilde{U}_{2i}.$$

Suppose that $a_i \neq 0$. By Proposition 5.3.9, we have $\widetilde{T}_i^{\mathtt{L}}(a_i) \neq \widetilde{T}_i^{\mathtt{R}}(1)$ for $1 \le i \le l$. If $\widetilde{T}_i^{\mathtt{L}}(a_i) > \widetilde{T}_i^{\mathtt{R}}(1)$, then

$$U_{2i+1} = \widetilde{T}_i^{\mathtt{L}} \boxminus \widetilde{T}_i^{\mathtt{tail}}, \quad U_{2i} = \widetilde{T}_i^{\mathtt{R}} \boxplus \widetilde{T}_i^{\mathtt{tail}}. \tag{7.3.14}$$

If $\widetilde{T}_i^{\mathtt{L}}(a_i) < \widetilde{T}_i^{\mathtt{R}}(1)$, then

$$\begin{aligned} U_{2i+1} &= \left(\widetilde{T}_i^{\mathtt{L}} \cup \left\{\widetilde{T}_i^{\mathtt{R}}(1), \widetilde{T}_i^{\mathtt{L}}(a_i)\right\}\right) \boxplus \emptyset, \\ U_{2i} &= \left(\widetilde{T}_i^{\mathtt{R}} \setminus \left\{\widetilde{T}_i^{\mathtt{R}}(1), \widetilde{T}_i^{\mathtt{R}}(2)\right\}\right) \boxplus \left(\left(\widetilde{T}_i^{\mathtt{tail}} \setminus \left\{\widetilde{T}_i^{\mathtt{L}}(a_i)\right\}\right) \cup \left\{\widetilde{T}_i^{\mathtt{R}}(2)\right\}\right), \end{aligned} \tag{7.3.15}$$

where we identify a semistandard tableau of single-columned shape with the set of its entries.

Set

$$\mathbf{T} = (T_l, T_{l-1}, \dots, T_1), \text{ where } (T_i^{\mathtt{L}}, T_i^{\mathtt{R}}) = (U_{2i}, U_{2i-1}) \text{ for } 1 \le i \le l. \tag{7.3.16}$$

We can check without difficulty that T_i is semistandard, and the residue $\mathfrak{r}_i$ of T_i is at most 1 by Lemma 7.3.3 and (7.3.13).

Next we show that $T_{i+1} \prec T_i$ and $(T_{i+1}, T_i) \in \mathbf{H}^\circ((\mu'_{l-i}, \mu'_{l-i+1}), 4)$ for $1 \le i \le l-1$, which implies that $\mathbf{T} \in \mathbf{H}^\circ(\mu, n)$. The proof is similar to the case of $n = 4$ in Lemma 7.3.6.

Let us prove $T_{i+1} \prec T_i$ inductively on i. For $i = 1$, it follows from Lemma 7.3.6. Suppose that $T_i \prec T_{i-1} \prec \cdots \prec T_1$ holds for given $i \ge 2$. Consider

$$\mathbf{X}_i = (U_{2i+2}, \widetilde{U}_{2i+1}, \widetilde{U}_{2i}, U_{2i-1}).$$

The admissibility on $\mathbb{T}$ implies that $\mathbf{X}_i^{\mathtt{tail}}$ is semistandard. It follows from (7.3.14), (7.3.15), Definition 5.3.7(H1) on $\mathbb{T}$ and the induction hypothesis that $\mathbf{X}_i^{\texttt{body}}$ is equal to $H_{\rho'^{\pi}}$, for some $\rho = (2a, 2b, 2c, 2d)$ with $a \geq b \geq c \geq d \geq 0$, except the entries in the southeast corner and the next one to the left.

We remark that the map

$$\mathbf{X}_i = (U_{2i+2}, \widetilde{U}_{2i+1}, \widetilde{U}_{2i}, U_{2i-1}) \longmapsto (U_{2i+2}, U_{2i+1}, U_{2i}, U_{2i-1}) = (T_{i+1}, T_i)$$

is the same as the map $\mathbf{X} \mapsto \mathbf{T}$ in Lemma 7.3.6.

Case 1. *$\widetilde{U}_{2i+1}(a_i) < \widetilde{U}_{2i}(1)$ and $\widetilde{U}_{2i+3}(a_{i+1}) < \widetilde{U}_{2i+2}(1)$.* First, we show that $\mathfrak{r}_i = \mathfrak{r}_{i+1} = 1$. By (7.3.15) and [67, Lemma 3.4], we have

$$U_{2i+2}(a_{i+1}) = \widetilde{U}_{2i+2}(2) < \widetilde{U}_{2i+2}(1) \leq \widetilde{U}_{2i}(1) = U_{2i+1}(1). \tag{7.3.17}$$

By (7.3.15), (7.3.17) and Proposition 5.3.9, we have $\mathfrak{r}_{i+1} = 1$. Also, we have $\mathfrak{r}_i = 1$ by similar way.

Next, we verify Definition 4.1.5 (1)-(i), (ii) and (iii) for (T_{i+1}, T_i). The condition (1)-(i) follows from (7.3.15). In this case, $T_{i+1}^{\texttt{R}*}$ and ${}^{\texttt{L}}T_i$ are given by

$$T_{i+1}^{\texttt{R}*} = \left(\widetilde{U}_{2i+1}^{\texttt{body}} \cup \left\{ \widetilde{U}_{2i+1}(a_i) \right\}\right) \boxplus \emptyset, \quad {}^{\texttt{L}}T_i = \left(\widetilde{U}_{2i} \setminus \left\{ \widetilde{U}_{2i}(1) \right\}\right) \boxplus \emptyset.$$

By Proposition 5.3.9 and the admissibility on $\mathbb{T}$, we have $T_{i+1}^{\texttt{R}*}(k) \leq {}^{\texttt{L}}T_i(k)$. So the condition (1)-(ii) holds.

Now, we consider the condition (1)-(iii). In this case, ${}^{\texttt{R}}T_{i+1}$ and $T_i^{\texttt{L}*}$ are given by

$$\begin{aligned} {}^{\texttt{R}}T_{i+1} &= \left(\widetilde{U}_{2i+1}^{\texttt{body}} \cup \left\{ \widetilde{U}_{2i+1}(a_i) \right\}\right) \boxplus \left(\left(\widetilde{U}_{2i+3}^{\texttt{tail}} \setminus \left\{ \widetilde{U}_{2i+3}(a_{i+1}) \right\}\right) \cup \left\{ \widetilde{U}_{2i}(1) \right\}\right) \\ T_i^{\texttt{L}*} &= \left(\widetilde{U}_{2i} \setminus \left\{ \widetilde{U}_{2i}(1) \right\}\right) \boxplus \left(\left(\widetilde{U}_{2i+1}^{\texttt{tail}} \setminus \left\{ \widetilde{U}_{2i+1}(a_i) \right\}\right) \cup \left\{ T_i^{\texttt{R}}(1) \right\}\right), \end{aligned}$$

where

$$T_i^{\texttt{R}}(1) = \begin{cases} \widetilde{U}_{2i-1}(a_{i-1}+1), & \text{if } \widetilde{U}_{2i-1}(a_{i-1}) > \widetilde{U}_{2i-2}(1), \\ \widetilde{U}_{2i-2}(1), & \text{if } \widetilde{U}_{2i-1}(a_{i-1}) < \widetilde{U}_{2i-2}(1). \end{cases} \tag{7.3.18}$$

Note that we have by [67, Lemma 3.4] and the admissibility of $\mathbb{T}$

$${}^{\texttt{R}}T_{i+1}(a_{i+1}) = \widetilde{U}_{2i}(1) \leq T_i^{\texttt{R}}(1) = T_i^{\texttt{L}*}(a_i). \tag{7.3.19}$$

Then the condition (1)-(iii) for (T_{i+1}, T_i) follows from (7.3.19), Proposition 5.3.9 and the admissibility of $\mathbb{T}$.

Finally, we have $(T_{i+1}, T_i) \in \mathbf{H}^\circ((\mu'_{l-i}, \mu'_{l-i+1}), 4)$ by (7.3.15), induction hypothesis and Proposition 5.3.9.

Case 2. *$\widetilde{U}_{2i+1}(a_i) > \widetilde{U}_{2i}(1)$ and $\widetilde{U}_{2i+3}(a_{i+1}) < \widetilde{U}_{2i+2}(1)$.* Since $\widetilde{U}_{2i+1}(a_i) > \widetilde{U}_{2i}(1)$, we have by the admissibility of $\mathbb{T}$

$$U_{2i+2}(a_{i+1}) = \widetilde{U}_{2i+2}(2) < \widetilde{U}_{2i+2}(1) \leq \widetilde{U}_{2i+1}(a_i + 1) = U_{2i+1}(1).$$

Thus the residue $\mathfrak{r}_{i+1}$ is equal to 1. If the residue $\mathfrak{r}_i = 0$, then the admissibility of (T_{i+1}, T_i) follows immediately from the one of $\mathbb{T}$, and we have $(T_{i+1}, T_i) \in \mathbf{H}^\circ((\mu'_{l-i}, \mu'_{l-i+1}), 4)$ by (7.3.14), (7.3.15), induction hypothesis and Proposition 5.3.9.

We assume $\mathfrak{r}_i = 1$. Then ${}^{\mathtt{L}}T_i$, $T_{i+1}^{\mathtt{R}*}$, $T_i^{\mathtt{L}*}$ and ${}^{\mathtt{R}}T_{i+1}$ are given by

$$\begin{aligned}
T_{i+1}^{\mathtt{R}*} &= \left(\widetilde{U}_{2i+1}^{\mathtt{body}} \setminus \left\{ \widetilde{U}_{2i+1}(a_i+1) \right\}\right) \boxplus \emptyset\,, \qquad {}^{\mathtt{L}}T_i = \left(\widetilde{U}_{2i} \cup \left\{ \widetilde{U}_{2i+1}(a_i) \right\}\right) \boxplus \emptyset\,, \\
{}^{\mathtt{R}}T_{i+1} &= \left(\widetilde{U}_{2i+1}^{\mathtt{body}} \setminus \left\{ \widetilde{U}_{2i+1}(a_i+1) \right\}\right) \boxplus \left(\left(\widetilde{U}_{2i+3}^{\mathtt{tail}} \setminus \left\{ \widetilde{U}_{2i+3}(a_{i+1}) \right\}\right) \cup \left\{ \widetilde{U}_{2i+1}(a_i+1) \right\}\right), \\
T_i^{\mathtt{L}*} &= \left(\widetilde{U}_{2i} \cup \left\{ \widetilde{U}_{2i+1}(a_i) \right\}\right) \boxplus \left(\left(\widetilde{U}_{2i+1}^{\mathtt{tail}} \setminus \left\{ \widetilde{U}_{2i+1}(a_i) \right\}\right) \cup \left\{ T_i^{\mathtt{R}}(1) \right\}\right),
\end{aligned} \tag{7.3.20}$$

where $T_i^{\mathtt{R}}(1)$ is given as in (7.3.18). By applying [67, Lemma 3.4] on $\mathbb{T}$, we have

$${}^{\mathtt{R}}T_{i+1}(a_{i+1}) = \widetilde{U}_{2i+1}(a_i+1) \leq T_i^R(1) = T^{\mathtt{L}*}(a_i). \tag{7.3.21}$$

Now we apply a similar argument with *Case 1* to (7.3.20) with (7.3.21) to obtain the admissibility of (T_{i+1}, T_i) and $(T_{i+1}, T_i) \in \mathbf{H}^\circ((\mu'_{l-i}, \mu'_{l-i+1}), 4)$ in this case.

Case 3. *$\widetilde{U}_{2i+1}(a_i) < \widetilde{U}_{2i}(1)$ and $\widetilde{U}_{2i+3}(a_{i+1}) > \widetilde{U}_{2i+2}(1)$.* The proof of this case is almost identical with *Case 2*. We leave it to the reader.

Case 4. *$\widetilde{U}_{2i+1}(a_i) > \widetilde{U}_{2i}(1)$ and $\widetilde{U}_{2i+3}(a_{i+1}) > \widetilde{U}_{2i+2}(1)$.* In this case, the claim follows immediately from (7.3.14), and the admissibility of $\mathbb{T}$.

Therefore, we have $\mathbf{T} \in \mathbf{H}^\circ(\mu, n)$. By Lemma 5.3.15, we have $\mathbf{T} \equiv_{\mathfrak{l}} \widetilde{\mathbf{T}} \otimes U_{2l}$ and $\widetilde{\mathbf{T}} = \mathbb{T}$

since $\mathbf{T} \in \mathbf{H}^{\circ}(\mu, n)$. This implies

$$\mathbf{T} \equiv_{\mathfrak{l}} \mathbb{T} \otimes U_{2l} \equiv_{\mathfrak{l}} \mathbb{X} \otimes U_{2l} \equiv_{\mathfrak{l}} \mathbf{X}, \tag{7.3.22}$$

and hence $\mathbf{T} \in \mathtt{LR}^{\mu}_{\lambda}(\mathfrak{d})$. Since $\overline{\mathbb{T}} = \mathbb{X}$, it follows from the inductive definition of $\overline{\mathbf{T}}$ that $\overline{\mathbf{T}} = \mathbf{X}$. □

Proof of Theorem 5.4.4 when $n - 2\mu'_1 \geq 0$***.*** The map

$$\begin{array}{ccc} \mathtt{LR}^{\mu}_{\lambda}(\mathfrak{d}) & \longrightarrow & \displaystyle\bigsqcup_{\delta \in \mathscr{P}^{(2)}_n} \overline{\mathtt{LR}}^{\lambda'}_{\delta'\mu'} \\ \mathbf{T} & \longmapsto & \overline{\mathbf{T}}^{\mathtt{tail}} \end{array} \tag{7.3.23}$$

is well-defined by Corollary 7.3.4. Finally it is bijective by Lemmas 7.3.5 and 7.3.8. □

7.3.3 Proof of Theorem 5.4.4 when when $n - 2\mu'_1 < 0$

Let $\mu \in \mathcal{P}(\mathrm{O}_n)$ and $\lambda \in \mathscr{P}_n$ be given. We assume that $n - 2\mu'_1 < 0$. We also use the convention for $\mathbf{T}$ in subsection 7.3.2.

Let $\mathbf{T} \in \mathtt{LR}^{\mu}_{\lambda}(\mathfrak{d})$ be given with $\mathbf{T} = (T_l, \ldots, T_{m+1}, T_m, \ldots, T_1, T_0)$ as in (5.3.8). Then we have $\overline{\mathbf{T}}^{\mathtt{tail}} \in \mathtt{LR}^{\lambda'}_{\mu'\delta'}$ by Proposition 5.3.19(2). Let $\mathtt{L} = 2\mu'_1 - n$. Choose $\kappa = (\kappa_1, \ldots, \kappa_{\mathtt{L}}) \in \mathscr{P}^{(2)}$ such that κ_i is sufficiently large.

Let $\eta, \chi \in \mathscr{P}$ be given by

$$\begin{aligned} \eta &= \kappa \cup \lambda = (\kappa_1, \ldots, \kappa_{\mathtt{L}}, \lambda_1, \lambda_2 \ldots), \\ \xi &= \kappa \cup \delta = (\kappa_1, \ldots, \kappa_{\mathtt{L}}, \delta_1, \delta_2 \ldots). \end{aligned} \tag{7.3.24}$$

Lemma 7.3.9. *We have* $\overline{\mathbf{T}}^{\mathtt{tail}} \in \overline{\mathtt{LR}}^{\lambda'}_{\mu'\delta'}$.

Proof. Put $\mathbf{T} = (U_{2l}, \ldots, U_{2m+1}, U_{2m}, \ldots, U_0)$ under (4.3.1). Let

$$\mathbb{B} = (U^{\downarrow}_{2m}, \ldots, U^{\downarrow}_0, H_{(1^{\kappa_{\mathtt{L}}})}, \ldots, H_{(1^{\kappa_1})}),$$

where $U^{\downarrow}_i = (\ldots, U_i(3), U_i(2)) \boxplus (U_i(1))$ for $0 \leq i \leq 2m$. By the choice of κ, $\mathbb{B}$ is an $\mathfrak{l}$-highest weight element, and we note that

$$\mu'_1 = l + m + 1, \qquad (\mathtt{L} + 2m + 1) - 2(2m + 1) = 0,$$

where $\mathtt{L}+2m+1$ is the number of columns of $\mathbb{B}$ and $2m+1$ is the length of the first row of $\mathbb{B}^{\mathtt{tail}}$. Hence by Lemma 7.3.8, there exists $\mathbf{B}=(X_{2m},\ldots,X_0,Y_{\mathtt{L}},\ldots,Y_1)\in\mathtt{LR}^{\dot\mu}_{\dot\eta}(\mathfrak{d})$ such that $\overline{\mathbf{B}}=\mathbb{B}$, where $\dot\mu'=(2m+1)$ and $\dot\eta$ is determined by $\mathbb{B}\equiv_{\mathfrak{l}} H_{\dot\eta'}$.

Put $\mathbf{A}:=(U_{2l},\ldots,U_{2m+1},X_{2m},\ldots,X_0,Y_{\mathtt{L}},\ldots,Y_1)$. By construction of $\mathbf{B}$ and Corollary 5.3.17 (cf. Remark 4.1.6), it is straightforward that

$$\mathbf{A}\in\mathtt{LR}^{\mu}_{\eta}(\mathfrak{d}),\qquad \overline{\mathbf{A}}^{\mathtt{tail}}=\overline{\mathbf{T}}^{\mathtt{tail}}. \tag{7.3.25}$$

Let $(m_i)_{1\le i\le p}$ be the sequence associated with $\overline{\mathbf{A}}^{\mathtt{tail}}$, which is given as in Definition 5.4.1. Since by the construction $m_i\le\mathtt{L}$ for all $1\le i\le p$, the sequence $(m_i)_{1\le i\le p}$ can be viewed as the sequence associated with $\overline{\mathbf{T}}^{\mathtt{tail}}$ in Definition 5.4.1. Put $(n_i)_{1\le i\le q}$ to be the sequence defined in Definition 5.4.1 with respect to $(m_i)_{1\le i\le p}$. By Lemma 7.3.3 with (7.3.25), the sequence $(n_i)_{1\le i\le q}$ satisfies (5.4.1) with respect to $\overline{\mathbf{T}}^{\mathtt{tail}}$. Hence we have $\overline{\mathbf{T}}^{\mathtt{tail}}\in\overline{\mathtt{LR}}^{\lambda'}_{\mu'\delta'}$. □

Hence the map (7.3.23) is well-defined by Proposition 5.3.19(2) and Lemma 7.3.9. It is also injective since Lemma 7.3.5 still holds in this case. So it remains to verify that the map is surjective.

Let $\mathbf{W}\in\overline{\mathtt{LR}}^{\lambda'}_{\mu'\delta'}$ be given for some $\delta\in\mathscr{P}^{(2)}_n$. Let $\mathbf{V}=H_{(\delta')^\pi}$ and $\mathbf{X}$ be the tableau of a skew shape η as in (4.3.8) with n columns such that $\mathbf{X}^{\mathsf{body}}=\mathbf{V}$ and $\mathbf{X}^{\mathtt{tail}}=\mathbf{W}$. As in the case of $n-2\mu'_1\ge 0$, $\mathbf{X}$ is semistandard.

Put

$$\begin{aligned}\mathbf{Y}&=(Y_{\mathtt{L}},\ldots,Y_1),\\ \mathbf{Z}&=(X_n,\ldots,X_1,Y_{\mathtt{L}},\ldots,Y_1),\end{aligned}\tag{7.3.26}$$

where $Y_i=H_{(1^{\kappa_i})}$ for $1\le i\le\mathtt{L}$.

Lemma 7.3.10. *We have* $\mathbf{Z}^{\mathtt{tail}}\in\overline{\mathtt{LR}}^{\eta'}_{\mu'\xi'}$.

Proof. By construction of $\mathbf{Z}$, we have $\mathbf{Z}\equiv_{\mathfrak{l}} H_{\eta'}$. Let $(m_i)_{1\le i\le p}$ and $(n_i)_{1\le i\le q}$ be the sequences associated with $\mathbf{W}\in\overline{\mathtt{LR}}^{\lambda'}_{\mu'\delta'}$. Since κ_i is sufficiently large, we have $\mathbf{Z}^{\mathtt{tail}}\in\overline{\mathtt{LR}}^{\eta'}_{\mu'\xi'}$ with respect to the same sequences $(m_i)_{1\le i\le p}$ and $(n_i)_{1\le i\le q}$. □

Note that $\mathbf{Z}\in\mathbf{E}^M$ where $M=n+\mathtt{L}=2\mu'_1$. By Lemma 7.3.10, we may apply Theorem 5.4.4 for $M-2\mu'_1=0$ to conclude that there exists a unique $\mathbf{R}\in\mathbf{T}(\mu,M)$ such that

$$\overline{\mathbf{R}}=\mathbf{Z}. \tag{7.3.27}$$

Suppose that $\mathbf{R} = (R_M, \ldots, R_1) \in \mathbf{E}^M$ under (4.3.1). Put $\mathbf{S} = (R_{2\mathtt{L}}, \ldots, R_1)$. Note that $2\mathtt{L} < M$ with $M - 2\mathtt{L} = n$, and $\mathbf{S} \in \mathbf{T}((1^{\mathtt{L}}), 2\mathtt{L})$. If $\overline{\mathbf{S}} = (\overline{S}_{2\mathtt{L}} \ldots, \overline{S}_1) \in \mathbf{E}^{2\mathtt{L}}$, then we have by Corollary 5.3.17 and (7.3.27)

$$(\overline{S}_{\mathtt{L}} \ldots, \overline{S}_1) = \mathbf{Y}.$$

Now, we put

$$\mathbf{T} = (R_M, \ldots, R_{2\mathtt{L}+1}, \overline{S}_{2\mathtt{L}}, \ldots, \overline{S}_{\mathtt{L}+1}) \in \mathbf{E}^n,$$

under (4.3.1). Then it is straightforward to check that $\mathbf{T} \in \mathbf{T}(\mu, n)$. Since $\mathbf{Z} \in \mathtt{LR}^{\mu}_{\eta}(\mathfrak{d})$, we have $\mathbf{T} \in \mathtt{LR}^{\mu}_{\lambda}(\mathfrak{d})$ by construction of $\mathbf{T}$ and Lemma 5.3.6. Finally, by (7.3.27) and Corollary 5.3.17, we have

$$\overline{\mathbf{T}}^{\mathtt{body}} = \mathbf{X}^{\mathtt{body}}, \quad \overline{\mathbf{T}}^{\mathtt{tail}} = \mathbf{X}^{\mathtt{tail}}.$$

Hence, the map (7.3.23) is surjective. $\square$

Appendices

Appendix A

Index of notation, Table and Figure

A.1 Index of notation

A.1.1 Chapter 2

2.1 : $\mathbb{Z}_+$, $P^\vee$, P, $\Pi^\vee$, P^+, Π, ϖ_i, Q, Q_+, Δ, ε, S, $U_q^\pm(\mathfrak{g})$, $U_q^0(\mathfrak{g})$

2.1.2 : $x_i^{(m)}$ $(x = e,\, f)$, $\tilde{e}_i$, $\tilde{f}_i$, $\mathbb{A}_0$, L, B, ε_i, φ_i

2.2.1 : e_i', e_i'', $\tilde{e}_i$, $\tilde{f}_i$, $L(\infty)$, $B(\infty)$, Ξ_λ, $*$, $\tilde{e}_i^*$, $\tilde{f}_i^*$, ε_i^*

2.2.2 : W, s_i, $R(w)$, $\ell(w)$, T_i, f_{β_k}, $B_{\mathbf{i}}$, Φ^+

A.1.2 Chapter 3

3.1.1 : $\mathbb{N}$, $\overline{\mathbb{N}}$, $\bar{i}$, $[n]$, $[\overline{n}]$, $\mathscr{P}$, $\mathscr{P}_n$, $\ell(\lambda)$, λ^π, λ/μ, $\mathcal{A}$, $SST_{\mathcal{A}}(\lambda/\mu)$, $w(T)$, $\mathrm{sh}(T)$, H_λ H_{λ^π}, $\mathcal{W}_n$, $\mathcal{W}_n^\vee$, $\mathcal{W}$, $\mathcal{W}^\vee$, $T \leftarrow a$, $a \rightarrow T$, $T^\nwarrow$, $T^\searrow$, $P(w)^\nwarrow$, $P(w)^\searrow$

3.2.2 : $\mathcal{P}_n$, ω_λ

3.2.3 : $\mathbf{i}_0$, $\mathbf{i}^J$, $\mathbf{i}_j$

3.2.4 : $\Phi^+(J)$, Φ_J^+, $\mathbf{B}^J$, $\mathbf{B}_J$

3.3.1 :$\mathcal{T}^\searrow$, $\mathbf{c}(\mathbf{a}, \mathbf{b})$, Ω, $\kappa^\searrow$

3.3.2 : $\lambda(\mathbf{c})$, $||\mathbf{c}||_{\mathbf{p}}$

A.1.3 Chapter 4

4.1.1 : $\lambda(a,b,c)$, $T^{\mathtt{L}}$, $T^{\mathtt{R}}$, $\mathbb{P}_L$, $T^{\mathtt{body}}$, $T^{\mathtt{tail}}$, $\mathrm{ht}(U)$, $U(i)$, $U[i]$, $\boxplus$, $\boxminus$

4.1.3 : $\mathfrak{r}_T$, $\mathcal{E}$, $\mathcal{F}$, $\mathbf{T}(a)$, $\overline{\mathbf{T}}(0)$, $\mathbf{T}^{\mathtt{sp}}$, $\mathbf{T}^{\mathtt{sp}+}$, $\mathbf{T}^{\mathtt{sp}-}$, $T^{\mathtt{L}*}$, $T^{\mathtt{R}*}$, ${}^{\mathtt{L}}T$, ${}^{\mathtt{R}}T$, $\prec$, $\widehat{\mathbf{T}}_\lambda$, $\mathbf{T}_\lambda$, $\mathbf{H}_\lambda$, $\mathbf{H}(\lambda)$, $\lhd$
(For type D_∞, see Section 5.3.1)

4.3.1 : $\mathbf{E}^N$, $\mathcal{E}_j$, $\mathcal{F}_j$, $\mathcal{S}_j$

4.3.2 : $\overline{\mathbf{T}}$, $\overline{\mathbf{T}}^{\mathtt{body}}$, $\overline{\mathbf{T}}^{\mathtt{tail}}$, $\widetilde{\mathbf{U}}$

A.1.4 Chapter 5

5.1 : $\mathscr{P}^{(2)}$, $\mathscr{P}^{(1,1)}$, $\mathscr{P}^{(2,2)}$, $\mathscr{P}_\ell^{(2)}$, $\mathscr{P}_\ell^{(1,1)}$, $\mathscr{P}_\ell^{(2,2)}$, $\mathtt{LR}^\lambda_{\mu\nu}$, $\mathtt{LR}^\lambda_{\mu\nu^\pi}$, ψ, S^i, U_i

5.2.2 : $\mathcal{P}(\mathrm{O}_n)$, $\Lambda(\mu)$

5.3.1 : $\mathbf{H}(\mu,n)$, $\mathtt{LR}^\mu_\lambda(\mathfrak{d})$, $c^\mu_\lambda(\mathfrak{d})$

5.4.1 : $\delta^{\mathtt{rev}}$, $\overline{\mathtt{LR}}^{\lambda'}_{\delta'\mu'}$, $\overline{c}^\lambda_{\delta\mu}$, $\underline{\mathtt{LR}}^\lambda_{\delta\mu}$, $\underline{c}^\lambda_{\delta\mu}$

A.1.5 Chapter 6

6.3 : $\mathcal{T}$, $\mathcal{T}^{\nwarrow}$, $\kappa^{\nwarrow}$, $\mathcal{T}^s$, $\kappa^{\mathfrak{d}}$

A.2 Crystal graph

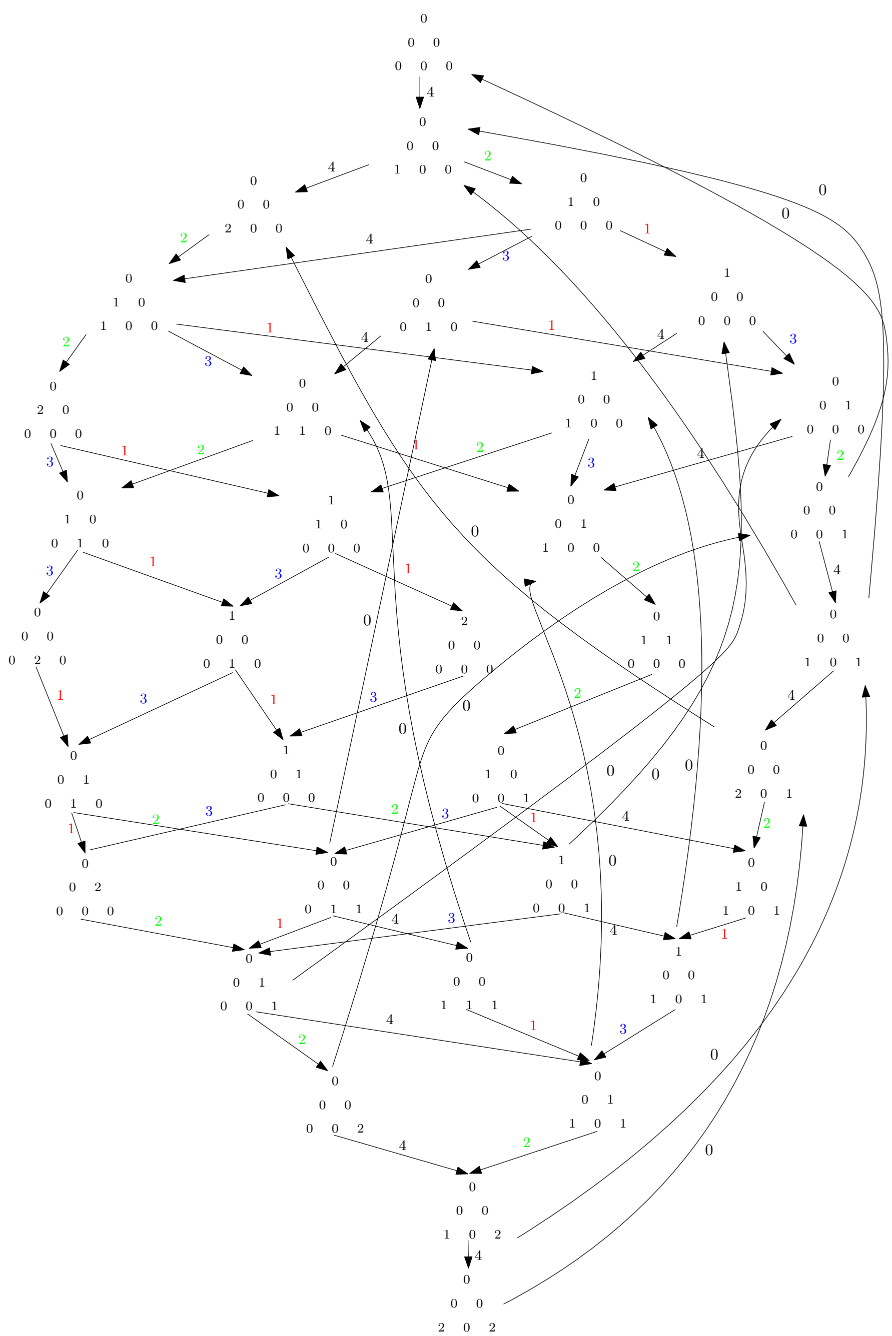


Table A.1: Crystal graph for $\mathbf{B}^{J,2}$

A.3 Table

(σ_j, σ_{j+1})	$(\widetilde{\sigma}_j, \widetilde{\sigma}_{j+1})$
$(+,+)$	$(+,+)$
$(+,-)$	$(+,-)$ or $(\cdot,\cdot)$
$(+,\cdot)$	$(+,\cdot)$ or $(\cdot,+)$
$(-,+)$	$(-,+)$
$(-,-)$	$(-,-)$
$(-,\cdot)$	$(-,\cdot)$ or $(\cdot,-)$
$(\cdot,+)$	$(\cdot,+)$ or $(+,\cdot)$
$(\cdot,-)$	$(\cdot,-)$ or $(-,\cdot)$
$(\cdot,\cdot)$	$(\cdot,\cdot)$

Table A.2: The relation between (σ_j, σ_{j+1}) and $(\widetilde{\sigma}_j, \widetilde{\sigma}_{j+1})$ when $\mathfrak{r}_{i+1}\mathfrak{r}_i = 1$

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국문초록

본 학위논문에서는 조합론적인 관점에서 D형 결정을 연구한다. 특히, 양자 군의 음의 부분의 결정 $B(\infty)$과 최고 무게가 λ인 가적 최고 무게 기약 모듈의 결정 $B(\lambda)$을 중점적으로 연구한다.

본 학위논문의 주요 결과로써, PBW기저에 의한 루스티그의 매개화를 이용하여 $B(\infty)$의 결정 구조를 명확하게 제시하고, $B(\lambda)$에서 PBW기저의 결정 구조와 양립하는 조합론적 알고리즘을 개발한다. 그리고 이러한 결과로부터 $B(\lambda)$에서 $B(\infty)$으로의 결정 매입의 조합론적 모형을 얻는다.

위 $B(\lambda)$와 $B(\infty)$의 결정 구조 연구의 응용으로 D형 로빈슨-솀스티드-커누스 대응의 아핀 결정 이론적 해석, GL_n에서 O_n으로의 분지 중복도에 대한 새로운 조합론적 공식 그리고 스핀점과 연관된 D형 키릴로프-레셰티킨 결정의 조합론적 모형을 얻는다.

주요어휘**:** 양자군, 결정 기저, 키릴로프-레셰티킨 결정, 로빈슨-솀스테드-커누스 대응, 분지 규칙, 일반화된 지수
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감사의 글

본 학위기간동안 공부를하는데 있어서 아낌없는 지원과 가르침을 주신 권재훈 교수님께 진심으로 감사의 마음을 전합니다. 여러가지로 부족했던 제가 지금까지 수학 공부를 포기하지 않고 계속 할 수 있었던 것은 권재훈 교수님의 아낌없는 가르침 덕분이었습니다. 배움에 있어서 많이 느렸던 저를 포기하지 않으시고 끝까지 지도해주시며 기다려주셔서 감사합니다. 이 은혜 잊지않고 앞으로도 더욱 연구에 매진하여 좋은 수학자가될 수 있도록 노력하겠습니다.

바쁘신 와중에 본 학위 논문을 심사해주신 오병권 교수님, 박의용 교수님, 김장수 교수님 그리고 이승진 교수님께 진심으로 감사의 마음을 전합니다. 특히 여러모로 부족한 저에게 좋은 가르침을 주셨던 박의용 교수님께 감사의 마음을 전합니다. 저의 작은 결과에 관심을 가져주시고 발표를 하게 해주신 오세진 교수님과 저의 공부에 도움이 되는 질문과 코멘트를 주셨던 오영탁 교수님께도 감사의 마음을 전합니다.

수학 공부를 막 시작하고자 했을때 아낌없는 지원과 많은 도움을 주신 양미혜 교수님, 장규환 교수님, 최원 교수님, 김인현 교수님, 이윤복 교수님, 문병수 교수님 그리고 오준석 박사님에게 감사의 마음을 전합니다. 그리고 학부 동기들, 선후배분들에게도 함께 공부를 해주어서 너무나 고마웠습니다. 감사합니다. 이분들의 도움덕분에 아무것도 모르던 제가 수학공부를 시작할 수 있었습니다.

학위 기간동안 저에게는 너무나도 어려웠던 여러 배경 내용들을 함께 공부해주고, 여러가지로 많은 도움을 주었던 최승일 박사님, 황병학 박사님 그리고 태혁, 호빈, 상준, 정우, 현세, 신명, 우루노에게도 진심으로 감사의 마음을 전합니다. 함께 했기 때문에 그 많은 내용들을 포기하지 않고 공부해나갈수 있었습니다. 감사합니다. 학회에서 맺은 인연으로 지금까지 함께하고 있는 영훈씨, 도현씨에게도 감사의 마음을 전합니다. 덕분에 더 많은 것들을 배울 수 있었고 함께한 매순간들이 즐거웠습니다.

여러가지로 어려웠던 대학원 생활에서 많은 도움을 주고 즐거움과 좋은 추억을 만들어준 동기들 탁원, 경현, 남경, 성수, 원태, 영흠, 재성, 진섭, 소민누님, 지은 그리고 이태훈 박사에게 진심으로 감사의 마음을 전합니다. 덕분에 대학원 생활을 끝까지 버틸수 있었습니다.

언제나 나의 편에서 응원해주고 아낌없는 격려를 주었던 나의 사랑하는 가족들에게 큰 감사의 마음을 전합니다. 마지막으로 연구로 힘들어하던 나를 옆에서 항상 묵묵히 기다려주며 응원해주고, 잘 웃지 못하던 나에게 웃음을 주고, 행복이 무엇인지 알려준 나의 사랑하는 아내 하연이에게 무한한 감사의 마음을 전합니다. 그리고 늘 고맙고, 미안하고, 사랑합니다.