이학박사 학위논문

Integrability and differentiability results for non-linear equations with measure data

(측도 데이터를 가지는 비선형 타원 방정식의 해가 가지는 적분성과 미분성에 대한 결과)

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조남경

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Integrability and differentiability results for non-linear equations with measure data

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Abstract

Integrability and differentiability results for non-linear equations with measure data

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This thesis discusses the regularity of a distribution solution to nonlinear elliptic equations when the right-hand side is a measure.

First, we establish Calderón-Zygmund type estimates for the borderline double phase problems by proving that the gradient of a solution has equivalent integrability to the 1-fractional maximal function of the given measure. Second, we obtain the maximal differentiability of the gradient of a solution to non-linear elliptic measure data problems with general growth.

Key words: regularity, measure data, Calderón-Zygmund estimate, non-standard growth, differentiability
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Contents

Chapter 1

Introduction

We shall discuss the measure data problems of the type

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, Du) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here, the domain Ω is bounded, and μ is a finite (Radon) measure on Ω, i.e., $|\mu|(\Omega) < \infty$. By considering the zero extension to $\mathbb{R}^n$, we may assume that $\mu(\cdot)$ is a measure defined on $\mathbb{R}^n$. The nonlinearity $\mathcal{A}(x, \xi) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ will be specified later for each problem.

The main purpose of this thesis is two-fold: on the one hand, we shall investigate the global integrability for the gradient of a solution to (1) when nonlinearity $\mathcal{A}$ has so-called the borderline double phase growth. In addition, our results are obtained under the optimal regularity assumptions on the coefficient and the boundary of the domain; see Chapter 2 for the details. On the other hand, we shall provide the fractional differentiability results for the gradient of a solution to (1) when the nonlinearity $\mathcal{A}$ has the so-called Orlicz growth. In particular, we focus on the limiting case of Calderón-Zygmund theory; see Chapter 3 for the details.

To explain our results in further detail, let us consider the problem

$$\begin{cases} -\mathrm{div}(|Du|^{p-2}Du) = \delta_0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{2}$$

where δ_0 is the Dirac delta function at the origin. Then the fundamental solution of (2) is

$$u(x) = c(n,p) \begin{cases} |x|^{\frac{p-n}{p-1}} - 1 & \text{if } 1 < p \neq n, \\ \log |x| & \text{if } p = n. \end{cases} \tag{3}$$

Here, a positive constant $c(n,p)$ is determined only by n and p. From (3) and a direct computation, it is straight forward to check that $u \in W^{1,q}(B_1)$ for all $q < \min\{p, \frac{n(p-1)}{n-1}\}$ and $u \in W_0^{1,1}(B_1)$ if and only if $p > 2 - \frac{1}{n}$.

Now, let us consider the elliptic measure data problems under more general conditions. Suppose that the nonlinearity $\mathcal{A}(x,\xi) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable in the first variable and differentiable in the second variable satisfying the following growth and ellipticity conditions:

$$\begin{cases} |\mathcal{A}(x,\xi)| + |\partial_\xi \mathcal{A}(x,\xi)||\xi| \leq L(s^2 + |\xi|^2)^{p-1}, \\ \nu(s^2 + |\xi|^2)^{p-2}|\zeta|^2 \leq \langle \partial_\xi \mathcal{A}(x,\xi)\zeta, \zeta\rangle, \end{cases} \tag{4}$$

for any $x, \eta, \xi \in \mathbb{R}^n$ and for some $0 < \nu \leq L$. Here, $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n \times \mathbb{R}^n$ and $p > 2 - \frac{1}{n}$.

We now introduce the notion of a very weak solution as in below.

Definition 1. *A function $u \in W_0^{1,1}(\Omega)$ is called a very weak solution to the equation* (2) *under the assumptions* (4) *if $|\mathcal{A}(x,Du)| \in L^1(\Omega)$ and*

$$\int_\Omega \langle \mathcal{A}(x,Du), D\varphi\rangle = \int_\Omega \varphi \, d\mu \quad \forall \varphi \in C_0^\infty(\Omega). \tag{5}$$

It is worth mentioning that very weak solutions may not belong to an energy solution, even for the simple homogeneous linear problems of the form $-\mathrm{div}\,(A(x)Du) = 0$, see [77] for details. To further investigate regularity results for a solution to the measure data problems, it is often required to consider a special kind of very weak solution, so-called a SOLA(Solution Obtained via Limits of Approximations). We remark that a SOLA is not the only notion of a solution when μ is (barely) a measure or a L^1 function. For instance, we refer to [38] for the definition of an entropy solution and [12] for the definition of a renormalized solution.

Definition 2. *We say that $u \in W_0^{1,1}(\Omega)$ is a SOLA to the problem* (1) *under the assumptions* (4) *if u is a very weak solution to* (2) *and there exists a sequence of functions $\{f_k\}_{k\in\mathbb{N}} \subset L^\infty(\Omega)$ and a sequence of weak solutions*

$\{u_k\}_{k\in\mathbb{N}}$ *to the following regularized problems*

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, Du_k) = f_k & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

such that $u_k \to u$ *in* $W^{1,\max\{1,p-1\}}(\Omega)$ *and* $f_k \rightharpoonup \mu$ *in measure.*

In the seminal papers [14, 15], the authors proved the existence of a SOLA with an optimal convergence results, namely

$$u_k \to u \quad \text{in} \quad W^{1,q} \quad \forall q \in \Big[1, \frac{n(p-1)}{n-1}\Big).$$

The regularity theory for the p-Laplace measure data problems has been extensively studied since then. For instance we refer [2, 11, 70] for fractional differentiability results, [46, 61, 62, 63, 69] for potential estimate results and [71, 73] for Calderón-Zygmund type estimates.

In particular, the author of [73] proved the following global estimates

$$\int_\Omega |Du|^q \, dx \le c \int_\Omega M_1(\mu)^{\frac{q}{p-1}} \, dx \quad \text{for all} \quad 0 < q < \infty,$$

where $c > 0$ is an universal constant independent on u and μ. Here, $M_1(\cdot)$ is a 1-fractional maximal operator defined by

$$M_1(\mu)(x) := \sup_{B_R(x)\subset\mathbb{R}^n} \frac{R|\mu|(B_R(x))}{|B_R|} \quad \text{for } x \in \mathbb{R}^n$$

and Ω is a (δ, R)-Reifenberg flat domain whose precise definition will be stated in Definition 1.1 in Chapter 2. Our interest is to provide similar integrability results with an elliptic equation with nonstandard growth.

Partial differential equations (PDEs) with nonstandard growth conditions have been extensively studied for the last few decades. These problems have various applications such as non-Newtonian fluids [72], electrorheological fluids [74, 75], and image restorations [3, 28]. Regularity results with a different kind of nonstandard growth conditions have been extensively investigated when given μ is not a measure. For instance, see [6, 22, 23] for an elliptic equation with a variable exponent growth, [8, 20, 34, 35] for an elliptic equation with double phase growth, and [42, 44, 45] an elliptic equation with an

Orlicz growth.

On the other hand, measure data problems with nonstandard growth are only studied quite recently. In a very interesting paper [18], the authors considered the following $p(x)$-Laplace equation

$$\begin{cases} -\mathrm{div}\,\left(|Du|^{p(x)-2}Du\right) = \mu & \text{in} \quad \Omega, \\ \qquad\qquad\qquad\quad u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

and proved that

$$\int_\Omega |Du|^q \, dx \le c \int_\Omega \left(M_1(\mu)^{\frac{q}{p(x)-1}} + 1 \right) dx \quad \text{for all} \quad 0 < q < \infty,$$

for a constant $c > 0$ independent on u under the assumptions that

$$2 - \frac{1}{n} < \gamma_1 \le p(x) \le \gamma_2 < \infty \quad \text{and} \quad p(\cdot) \text{ is a log-Hölder continuous.}$$

Measure data problems with general growth is another interesting topic. In [7], the author considered a quasilinear measure data problem whose model equation is given by

$$\begin{cases} -\mathrm{div}\,\left(\dfrac{g(|Du|)}{|Du|}Du\right) = \mu & \text{in} \quad \Omega, \\ \qquad\qquad\qquad\quad u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{7}$$

where $g \in C^1(0,\infty)$ and

$$1 \le \gamma_1 - 1 \le \frac{tg'(t)}{g(t)} \le \gamma_2 - 1 < \infty. \tag{8}$$

The author proved the following point-wise estimate

$$g(|Du(x_0)|) \le c \int_0^R \frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}} \frac{d\rho}{\rho} + cg \left(\fint_{B_R(x_0)} |Du| \, dx \right)$$

for almost all $x_0 \in \Omega$ and for every ball $B_{2R}(x_0) \subset \Omega$. In [78, Chapter 4], the author considered a solution to (7) under the weaker assumption on $g(\cdot)$, which is

$$1-\frac{1}{n}<\gamma_1-1\le\frac{tg'(t)}{g(t)}\le\gamma_2-1<\infty \tag{9}$$

and prove that there exists a positive constant $c>0$, independent of u, satisfying

$$\int_\Omega |Du|^q\,dx \le c\int_\Omega \big(g^{-1}(M_1(\mu))\big)^q\,dx \quad \text{for all} \quad 1<q<\infty.$$

Motivated by previously mentioned results, we study Calderón-Zygmund type estimates for the measure data problems with a borderline double phase growth in Chapter 2.

Next topic in this thesis is a limiting case of Calderón-Zygmund theory. To explain this result in details, let us begin with the classical Poisson problem

$$\Delta u = \operatorname{div}(Du) = \mu.$$

The classical Calderón-Zygmund theory implies that

$$\mu\in L^q_{\mathrm{loc}} \implies Du\in W^{1,q}_{\mathrm{loc}} \quad \text{whenever} \quad 1<q<\infty. \tag{10}$$

This means that in the L^q sense, we can replace divergence operator with a gradient. When $q=1$, the implication (10) fails to hold, but instead, we have

$$\mu\in L^1 \implies Du\in W^{\sigma,1} \quad \text{for all } 0<\sigma<1, \tag{11}$$

where the precise definition of fractional Sobolev spaces $W^{\sigma,1}$ will be described in Chapter 3, Section 2.3. Surprisingly, the authors of [2] proved that if u is a SOLA to

$$-\operatorname{div}\left(|Du|^{p-2}Du\right)=\mu$$

then, we have

$$|Du|^{p-2}Du\in W^{\sigma,1}_{\mathrm{loc}} \quad \text{for all } 0<\sigma<1.$$

Our interest is to generalize these results to a solution of an elliptic equation with a general growth. In Chapter 3, we have proved that if $u\in W^{1,1}$ is a SOLA to (7) under the assumption (8), then we have

$$\frac{g(|Du|)}{|Du|}Du\in W^{\sigma,1}_{\mathrm{loc}} \quad \text{for all } 0<\sigma<1. \tag{12}$$

Our method depends on a linearizion technique developed in [7], see Lemma 3.1 in Chapter 3. For this reason, we need the stronger assumption on the growth condition of $g(\cdot)$. However, it would be interesting to show that the implication (12) holds when $g(\cdot)$ satisfies the weaker assumption (9).

Chapter 2 is based on joint work with Sun-Sig Byun and Yeonghun Youn, and Chapter 3 is parts of the submitted paper co-worked with Sun-Sig Byun and Ho-Sik Lee.

Chapter 2

Global gradient estimates for a borderline case of double phase problems with measure data

1 Introduction and Main Result

This chapter aims to present a sharp Calderón-Zygmund estimate for the borderline case of double phase problems with measure data on the right-hand side. The model equation is given by

$$-\operatorname{div}\left(|Du|^{p-2}(1+a(x)\log(e+|Du|))Du\right)=\mu, \tag{1.1}$$

where μ is a Radon measure with finite mass.

The energy functional corresponding to (1.1) features one of two different energy densities according to the values of $a(x)$. In other words the growth in (1.1) varies depending on the x-variable, and so it is one of the non-standard growth problems which have attracted a lot of interest recently. In the context of mathematical modeling of strongly anisotropic materials, non-standard growth problems were first introduced in [79, 80, 81]. These problems have various applications such as non-Newtonian fluids [72], electrorheological fluids [74, 75], and image restorations [3, 28].

Two well-known examples of non-standard growth problems are the so-called variable exponent problems and double phase problems. For the variable exponent problems, many regularity results have been obtained. See, for instance, [5, 6, 37, 47] for Hölder continuity results, [4, 22, 23, 24] for

Calderón-Zygmund type estimates, [10, 16, 26, 67] for potential estimates and so on. Comparing to the variable exponent case, double phase problems drastically change their growth with respect to the x-variable, which makes them hard to analyze. We refer to [8, 20, 34, 35, 36, 39] for the regularity results for double phase problems. More recently there have been several attempts to obtain such regularity results for non-standard growth problems in a comprehensive way. See [51, 52, 53].

Let us consider a general elliptic equation of the form

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, Du) = \mu & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1.2}$$

where the mapping $\mathcal{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be C^1-regular in the second variable ξ, with $\partial_\xi \mathcal{A}(\cdot)$ being Carathéodory regular. In addition, we assume that A satisfies the following non-standard growth, ellipticity, and continuity assumptions:

$$\begin{cases} |A(x,\xi)| + |\partial_\xi \mathcal{A}(x,\xi)||\xi| \le L(1 + a(x)\log(e + |\xi|))|\xi|^{p-1}, \\ \nu|\xi|^{p-2}(1 + a(x)\log(e + |\xi|))|\eta|^2 \le \langle \partial_\xi \mathcal{A}(x,\xi)\eta, \eta\rangle, \\ |\mathcal{A}(x,\xi) - \mathcal{A}(y,\xi)| \le L\omega(|x-y|)\log(e + |\xi|)|\xi|^{p-1} \end{cases} \tag{1.3}$$

for every $x, y \in \Omega$ and ξ, $\eta \in \mathbb{R}^n$, where $0 < \nu \le L$ and $p > 2 - \frac{1}{n}$ are fixed constants, and $\omega(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is the modulus of continuity of the modulating coefficient $a(\cdot) : \Omega \to R^+$. The range of p is assumed to guarantee the existence of SOLA, a notion of distributional solutions, to (1.2). See Lemma 1.4 for the details.

Throughout this chapter, $a(\cdot)$ is assumed to be log-Hölder continuous, which means that there exists $R > 0$ such that

$$\sup_{0<r\le R} \omega(r) \log\left(\frac{1}{r}\right) \le 1. \tag{1.4}$$

Then $a(\cdot)$ is bounded, and

$$r^{-\omega(r)} = e^{-\omega(r)\log(r)} \le e \tag{1.5}$$

holds for every $0 < r \le R$. Note that this log-Hölder continuity assumption has been used in studying variable exponent problems. Furthermore, any regularity assumptions on the exponent functions in the variable exponent problems are parallel to the ones on $a(\cdot)$ in (1.2), see for instance [9].

To state our main assumptions on $(a(\cdot), A(\cdot), \Omega)$, we define

$$g(x,t) := (1 + a(x)\log(e+t))t^{p-1} \quad \text{and} \quad G(x,t) := \int_0^t g(x,s)\,ds \tag{1.6}$$

for every $x \in \Omega$ and $t \in \mathbb{R}$. Recalling that $g(x,t)$ is a monotone increasing function with respect to $t \in \mathbb{R}^+$, we define $g_x^{-1}(t) : \mathbb{R}^+ \to \mathbb{R}^+$ by the inverse function of $g(x,t)$ for each $x \in \Omega$. We will see some basic properties of g, G, and related function spaces later in Section 2.

Definition 1.1. *We say that $(a(\cdot), A(\cdot), \Omega)$ is (δ, R)-vanishing, if the followings hold for some $\delta \in (0, \frac{1}{8})$ and $R > 0$.*

1. *The modulating coefficient $a(\cdot)$ is log-Hölder continuous with the estimate*
$$\sup_{0<r\le R} \omega(r) \log\left(\frac{1}{r}\right) \le \delta.$$
2. *For any measurable set $U \subset \Omega$ and $x \in \Omega$, we set*
$$\theta(U)(x) := \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \left| \frac{A(x,\xi)}{g(x,|\xi|)} - \fint_U \frac{A(z,\xi)}{g(z,|\xi|)}\,dz \right|. \tag{1.7}$$
Then we have
$$\sup_{0<r\le R}\sup_{y\in\mathbb{R}^n} \fint_{B_r(y)} \theta(B_r(y))(x)\,dx \le \delta. \tag{1.8}$$
3. *Ω is a (δ, R)-Reifenberg flat domain. More precisely, for each $y \in \partial\Omega$ and $r \in (0, R]$, there exists a coordinate system $\{\tilde{y}_1, \cdots, \tilde{y}_n\}$ with the origin at y such that*
$$B_r(0) \cap \{\tilde{y}_n > \delta r\} \ \subset\ B_r(0) \cap \Omega \ \subset\ B_r(0) \cap \{\tilde{y}_n > -\delta r\},$$

where $B_r(0)$ is the ball with center the origin and radius r.

Some properties of (δ, R)-vanishing conditions play an important role in the proof of the main result. In the next remark, we summarize the properties which we will use in the rest of this paper.

Remark 1.2. *Note that* (1.8) *covers nonlinear elliptic equations with coefficients having small BMO-norm. From* (1.7) *and* (1.2), *we see*

$$\theta(U)(x) \leq 2L$$

for any measurable set $U \subset \Omega$ and $x \in \Omega$. It then follows from (1.8) *that*

$$\fint_{B_r(y)} [\theta(B_r(y))(x)]^l \, dx \leq (2L)^{l-1}\delta, \tag{1.9}$$

whenever $l \geq 1$.

We now turn our attention to Reifenberg flatness. It is readily check that any Lipschitz domain with small Lipschitz constant is a Reifenberg flat domain. It is worth mentioning that if Ω is (δ, R)-Reifenberg flat, then it satisfies the following measure density conditions:

$$\sup_{0<r\leq R} \sup_{y\in\Omega} \frac{|B_r(y)|}{|\Omega \cap B_r(y)|} \leq \left(\frac{2}{1-\delta}\right)^n \leq \left(\frac{16}{7}\right)^n \tag{1.10}$$

and

$$\inf_{0<r\leq R} \inf_{y\in\partial\Omega} \frac{|B_r(y) \cap \Omega^c|}{|B_r(y)|} \geq \left(\frac{1-\delta}{2}\right)^n \geq \left(\frac{7}{16}\right)^n. \tag{1.11}$$

In Subsection 3.1, we will frequently use (1.10) *and* (1.11) *to obtain comparison estimates near the boundary of Ω. For the further properties of Reifenberg flat domains, we refer to [27, 66, 74] and references therein.*

We now take a constant $R_0 \in (0, R]$ satisfying

$$R_0 = R_0(\nu, L, |\mu|(\Omega)) \leq \frac{1}{|\mu|(\Omega) + e^{\frac{L}{\nu}}}. \tag{1.12}$$

Recalling (1.5), we see

$$\sup_{0<r\leq R_0} \omega(r) \leq \frac{\nu}{2L}. \tag{1.13}$$

From (1.5) and (1.12), we have

$$(|\mu|(\Omega)+1)^{\omega(r)} \leq \left(\frac{1}{R_0}\right)^{\omega(R_0)} \leq e^{-\log(R_0)\omega(R_0)} \leq e, \tag{1.14}$$

whenever $0 < r \leq R_0$. We will use (1.12)-(1.14) frequently throughout

Subsection 3.1 for some technical reasons to handle the modulating coefficient $a(\cdot)$.

We now introduce a class of distributional solutions, the so-called SOLAs (Solutions Obtained by Limits of Approximations), to the elliptic measure data problems.

Definition 1.3. *We say that $u \in W^{1,1}(\Omega)$ is a SOLA to (1.2) if the following statement holds. There exists a sequence of weak solutions $\{u_l\}_{l\in\mathbb{N}} \subset W_0^{1,G}(\Omega)$ to*

$$-\operatorname{div} A(x, Du_l) = \mu_l \quad \text{in} \quad \Omega,$$

where $\{\mu_l\}_{l\in\mathbb{N}}$ is a sequence of bounded functions. Moreover, u_l converges to u in $W^{1,\max\{1,p-1\}}(\Omega)$, while μ_l converges to μ weakly in measure.

The notion of SOLAs to p-Laplacian type equations was first introduced in the seminar papers [14, 15]. By following the similar ideas in the papers, the existence of SOLAs to (1.2) is obtained in [25, Lemma 2.5], which we state as follows:

Lemma 1.4. *Let $p \in (2 - \frac{1}{n}, \infty)$. Under the assumptions (1.2) and (1.2), there exists $u \in W_0^{1,1}(\Omega)$ such that*

$$\int_\Omega \langle A(x, Du), D\phi\rangle \, dx = \int_\Omega \phi \, d\mu \tag{1.15}$$

for every $\phi \in C_0^\infty(\Omega)$. Moreover, $u \in W_0^{1,q}(\Omega)$ for every $q \in \left[1, \frac{n(p-1)}{n-1}\right)$.

As previously mentioned, our main result is a global Calderón-Zygmund type estimate for (1.2) in terms of the 1-fractional maximal function of μ

$$M_1(\mu)(x) := \sup_{r>0} \frac{r\, |\mu|(B_r(x))}{|B_r(x)|}. \tag{1.16}$$

Theorem 1.5. *Under the assumptions (1.2), let $u \in W_0^{1,1}(\Omega)$ be a SOLA to the problem (1.2). Suppose that $g_x^{-1}(M_1(\mu)(x)) \in L^q(\Omega)$ for some $1 \le q < \infty$. Then there exists a small constant $\delta = \delta(n, p, q, \nu, L) > 0$ such that if $(a(\cdot), A(\cdot), \Omega)$ is (δ, R)-vanishing, then $Du \in L^q(\Omega)$ with the estimate*

$$\int_\Omega |Du|^q \, dx \le c\left(\int_\Omega |Du| \, dx\right)^q + c\int_\Omega \left(g_x^{-1}(\mathcal{M}_1(\mu))\right)^q \, dx, \tag{1.17}$$

where c depends only on $n, p, q, \nu, L, \mathrm{diam}(\Omega)$, and R_0.

2 Preliminaries

2.1 Notations and auxiliary results

Throughout this chapter, $c \geq 1$ denotes a positive constant, which may vary from line to line, depending only on n, p, ν, and L. The notation $f \lesssim h$ is a shortcut meaning that there exists a universal constant $c = c(n, p, \nu, L)$ satisfying $f \leq ch$. Moreover, we write $f \approx h$ when both $h \lesssim f$ and $f \lesssim h$ hold. $B_r(x_0)$ denotes the ball centered at $x_0 \in \Omega$ with radius $r > 0$ and $\Omega_r(x_0) := \Omega \cap B_r(x_0)$. When the center x_0 is clear from the context, we simply write $B_r \equiv B_r(x_0)$ and $\Omega_r \equiv \Omega_r(x_0)$.

We define an auxiliary vector field $V : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$V(x, \xi) = (|\xi|^{p-2} + a(x)\log(e + |\xi|)|\xi|^{p-2})^{\frac{1}{2}}\xi$$

for every $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Then according to [42, Lemma 3], we have the following property of $V(\cdot)$, which we use later in the proof: for each $x \in \Omega$

$$\begin{aligned}\langle A(x,\xi_1) - A(x,\xi_2), \xi_1 - \xi_2\rangle &\approx |V(x,\xi_1) - V(x,\xi_2)|^2 \\ &\approx \frac{g(x, |\xi_1| + |\xi_2|)}{|\xi_1| + |\xi_2|}|\xi_1 - \xi_2|^2. \end{aligned} \tag{2.18}$$

It is worth pointing out that $V(\cdot)$ considered in [42, Lemma 3] does not depend on the x-variable. Indeed, it also holds in our case for each fixed $x \in \Omega$.

For the sake of convenience, we employ the following notations:

$$\begin{aligned} &g_{m,U}(t) = \min_{x\in U} g(x,t), && g_{M,U}(t) = \max_{x\in U} g(x,t), \\ &g_{\log}(t) = t^{p-1}\log(e+t), && G_{m,U}(t) = \int_0^t g_{m,U}(\tau)\,d\tau, \\ &G_{M,U}(t) = \int_0^t g_{M,U}(\tau)\,d\tau, && G_{\log}(t) = \int_0^t g_{\log}(\tau)\,d\tau \\ &a_{m,U} = \min_{x\in U} a(x), && a_{M,U} = \max_{x\in U} a(x), \end{aligned} \tag{2.19}$$

for any measurable set $U \subset \Omega$. If there is no confusion, we omit writing U, for instance, $g_m(\cdot) = g_{m,U}(\cdot)$.

In the rest of this subsection, we investigate some properties of the logarithm function. A direct calculation yields

$$\begin{aligned}\frac{d}{dt}\left(\frac{t^p}{g(x,t)}\right) &= \frac{d}{dt}\left(\frac{t}{1+a(x)\log(e+t)}\right)\\ &= \left(\frac{(1+a(x)\log(e+t)) - \frac{a(x)t}{e+t}}{(1+a(x)\log(e+t))^2}\right) \geq 0 \qquad (2.20)\end{aligned}$$

and

$$\begin{aligned}\frac{d^2}{dt^2}\left(\frac{t^p}{g(x,t)}\right) &= \frac{d^2}{dt^2}\left(\frac{t}{1+a(x)\log(e+t)}\right)\\ &= a(x)\left(\frac{2a(x)t - (2e+t)(1+a(x)\log(e+t))}{(e+t)^2(1+a(x)\log(e+t))^3}\right) \leq 0 \qquad (2.21)\end{aligned}$$

for every $x \in \mathbb{R}^n$. To obtain the last inequality, we have considered two cases, $t \geq e^2 - e$ and $0 < t \leq 2e$. Then (2.20) and (2.21) imply that $t \mapsto t^p/g(x,t)$ is an increasing and concave function for each $x \in \mathbb{R}^n$.

Recalling

$$\log(e+t) \leq \frac{1}{\alpha}(e+t)^{\alpha} \qquad (2.22)$$

for every $\alpha \in (0,\infty)$ and $t \geq 0$, we discover

$$\begin{aligned}g_{M,B_r}(t) &\leq g_{m,B_r}(t) + \omega_a(r)\log(e+t)t^{p-1}\\ &\leq g_{m,B_r}(t) + (e+t)^{\omega_a(r)}t^{p-1} \lesssim g_{m,B_r}(t) + t^{p-1+\omega_a(r)}, \qquad (2.23)\end{aligned}$$

whenever $0 < r \leq R$. The following inequalities have been often used in the regularity theory of variable exponent problems:

$$\log(e+t_1t_2) \leq \log(e+t_1) + \log(e+t_2) \qquad (2.24)$$

and

$$\log(e+t_1) \leq c(\alpha)\log(e+t_1^{\alpha}) \qquad (2.25)$$

for every $t_1, t_2 \in [0,\infty)$ and $\alpha \geq 1$. We further recall an estimate of $L\log L$-function from [4, 55, 56].

Lemma 2.1. *Let $f \in L^q(\Omega)$ for $q > 1$. Then for any $\beta \geq 1$, we have*

$$\fint_{\Omega} |f| \log^{\beta}\left(e + \frac{|f|}{\|f\|_{L^1(\Omega)}}\right) dx \leq c(q,\beta)\left(\fint_{\Omega} |f|^q dx\right)^{\frac{1}{q}}. \tag{2.26}$$

2.2 Generalized N-function and Musielak-Orlicz spaces

We say that $G : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is a generalized N-function if $\Phi(x,\cdot)$ is a convex function satisfying

$$\Phi(x,t) = 0 \iff t = 0, \quad \lim_{t\to 0} \frac{\Phi(x,t)}{t} = 0, \quad \text{and} \quad \lim_{t\to\infty} \frac{\Phi(x,t)}{t} = \infty$$

for almost every $x \in \Omega$. It is readily checked that G, G_m, and G_M, given in (1.6) and (2.19), are generalized N-functions. We define $\Phi^* : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\Phi^*(x,s) := \sup_{t\geq 0}\{st - \Phi(x,t)\},$$

which we call the complementary function of Φ.

From [25, (2.14)], for every $p > 1$ we have

$$p \leq \frac{t\partial_t G(x,t)}{G(x,t)} = \frac{tg(x,t)}{G(x,t)} \leq p+1$$

or equivalently,

$$0 < p-1 \leq \frac{t\partial_t^2 G(x,t)}{\partial_t G(x,t)} = \frac{t\partial_t g(x,t)}{g(x,t)} \leq p. \tag{2.27}$$

It then follows from [65, Lemma 1.1] that

$$\min\{\alpha^p, \alpha^{p+1}\}G(x,t) \leq G(x,\alpha t) \leq \max\{\alpha^p,\alpha^{p+1}\}G(x,t) \tag{2.28}$$

for every $t, \alpha \geq 0$ and almost every $x \in \Omega$. Recalling [76, Proposition 2.1.1] and using (2.7), we have the following well-known equivalent relation

$$G^*(x, g(x,t)) \approx G(x,t) \tag{2.29}$$

and inequalities

$$st \le \varepsilon G^*(x,t) + c(\varepsilon)G(x,t) \quad \text{and} \quad st \le \varepsilon G(x,t) + c(\varepsilon)G^*(x,t). \tag{2.30}$$

It is worth mentioning that inequalities (2.27)-(2.30) hold not only for $G(x,\cdot)$ but also for $G_m(\cdot)$ and $G_M(\cdot)$.

We end this section with introducing Musielak-Orlicz spaces. For a given generalized N-function G, we define Musielak-Orlicz space $L^G(\Omega)$ by

$$L^G(\Omega) := \left\{ u \in L^1(\Omega) : \int_\Omega G(x,|u|)\,dx < \infty \right\}$$

and the corresponding (Luxemberg) norm $\|\cdot\|_{L^G(\Omega)}$ by

$$\|u\|_{L^G(\Omega)} := \inf \left\{ \lambda \ge 0 : \int_\Omega G\left(x, \left|\frac{u}{\lambda}\right|\right)\,dx \le 1 \right\}.$$

Similarly, we define

$$W^{1,G}(\Omega) := \{u \in W^{1,1}(\Omega) : u, |Du| \in L^G(\Omega)\}$$

with the norm $\|u\|_{W^{1,G}(\Omega)} := \|u\|_{L^G(\Omega)} + \|Du\|_{L^G(\Omega)}$. In addition, we denote $W_0^{1,G}(\Omega)$ by the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,G}(\Omega)}$. For G in (1.6) under the assumption (1.4), the Musielak-Orlicz space is separable Banach space and the Lavrentiev phenomenon does not occur. We refer to [40, 49] for a further discussion on Musielak-Orlicz spaces.

3 Comparison estimates

This section concerns comparison estimates for the weak solution u to (1.2) under the assumption $\mu \in L^\infty(\Omega)$. Recalling $L^\infty(\Omega) \subset W^{-1,p'}(\Omega) \subset (W_0^{1,G}(\Omega))^*$, we may assume that $u \in W_0^{1,G}(\Omega)$. Later, by using an approximating procedure, we prove our main result in light of the lemmas presented in this section.

3.1 Boundary comparison estimates

For any $x_0 \in \Omega$ and $0 < r \leq \frac{R_0}{6}$, we denote $\Omega_r = \Omega_r(x_0) = \Omega \cap B_r(x_0)$. In this subsection, we assume that

$$B_{3r}^{+} \subset \Omega_{3r} \subset B_{3r} \cap \{x_n > -6\delta r\}.$$

Throughout this subsection, we write $g_m = g_{m,\Omega_{3r}}$ and $a_m = a_{m,\Omega_{3r}}$.

Let us consider the following homogeneous equation:

$$\begin{cases} -\operatorname{div} A(x, Dw) = 0 & \text{in} \quad \Omega_{3r}, \\ \hfill w = u & \text{on} \quad \partial\Omega_{3r}. \end{cases} \tag{3.31}$$

Lemma 3.1. *Let* $u \in W_0^{1,G}(\Omega)$ *be the weak solution to* (1.2) *and* $w \in W_0^{1,G}(\Omega_{3r})$ *be the weak solution to* (3.31). *Then there exist* $c = c(n, p, \nu, L) \geq 1$ *such that*

$$\begin{aligned} \fint_{\Omega_{3r}} |Du - Dw|\, dx &\leq c\, g_m^{-1}\left(\left[\frac{|\mu|(\Omega_{3r})}{r^{n-1}}\right]\right) \\ &\quad + c\,\chi_{[p<2]}\left[\frac{|\mu|(\Omega_{3r})}{r^{n-1}}\right]^{\frac{1}{p}} \frac{\left(\fint_{\Omega_{3r}} |Du| dx\right)}{g_m\left(\fint_{\Omega_{3r}} |Du| dx\right)^{\frac{1}{p}}}. \end{aligned} \tag{3.32}$$

Proof. We start with scaling and normalization arguments. Take a constant

$$M = g_m^{-1}\left(\frac{|\mu|(\Omega_{3r})}{r^{n-1}}\right) + \chi_{[p<2]}\left[\frac{|\mu|(\Omega_{3r})}{r^{n-1}}\right]^{\frac{1}{p}} \frac{\left(\fint_{\Omega_{3r}} |Du| dx\right)}{g_m\left(\fint_{\Omega_{3r}} |Du| dx\right)^{\frac{1}{p}}} \geq 0.$$

If $|\mu|(\Omega_{3r}) = 0$, then $u \equiv w$ and there is nothing to prove. Thus, we assume that $|\mu|(\Omega_{3r}) > 0$, which directly implies $M \neq 0$ and for the similar reason, we assume $g_m\left(\fint_{\Omega_{3r}} |Du| dx\right) > 0$.

We define

$$\tilde{u}(\tilde{x}) = \frac{u(x_0+3r\tilde{x})}{3Mr}, \quad \tilde{w}(\tilde{x}) = \frac{w(x_0+3r\tilde{x})}{3Mr},$$
$$\hat{\mu}(\tilde{x}) = \frac{r\mu(x_0+3r\tilde{x})}{g_m(M)}, \quad \tilde{A}(\tilde{x},\xi) = \frac{A(x_0+3r\tilde{x}, M\xi)}{g_m(M)},$$
$$\hat{g}(\tilde{x},t) = \frac{g(x_0+3r\tilde{x}, Mt)}{g_m(M)}, \quad \text{and} \quad \hat{g}_m(t) = \frac{g_m(Mt)}{g_m(M)}$$

where $\tilde{x} \in \tilde{\Omega}_1 := \{\tilde{x} \in \mathbb{R}^n : x_0 + 3r\tilde{x} \in \Omega_{3r}\} \subset B_1(0)$. It is readily checked that

$$\langle \partial_\xi \tilde{A}(\tilde{x},\xi)\eta,\eta\rangle \geq \nu \frac{g(x_0+3r\tilde{x}, M|\xi|)}{g_m(M)|\xi|}|\eta|^2 = \nu \frac{\hat{g}(x,|\xi|)}{|\xi|}|\eta|^2,$$

and

$$\begin{cases} |\hat{\mu}|(\tilde{\Omega}_1) \leq 1 & \text{for } p \geq 2 \\ |\hat{\mu}|(\tilde{\Omega}_1) + |\hat{\mu}|(\tilde{\Omega}_1)\dfrac{\left(\displaystyle\int_{\tilde{\Omega}_1} |D\tilde{u}|\,d\tilde{x}\right)^p}{\hat{g}_m\left(\displaystyle\int_{\tilde{\Omega}_1} |D\tilde{u}|\,d\tilde{x}\right)} \leq c & \text{for } 2-\dfrac{1}{n} < p < 2. \end{cases}$$

Therefore, under this normalization, it is enough to show that

$$\int_{\tilde{\Omega}_1} |D\tilde{u} - D\tilde{w}|\,d\tilde{x} \leq c \tag{3.33}$$

for some constant $c \geq 1$. In the rest of this proof, we omit ˜ over each characters for the simplicity of notations.

Let us define some truncation functions $T_k(t) = \max\{-k, \min\{k,t\}\}$ and $\Phi_k(t) = T_1(t - T_k(t))$ for every $k \in \mathbb{R}^+$, and the corresponding sets

$$C_k = \{x \in \Omega_1 : |u-w| \leq k\} \text{ and } D_k = \{x \in \Omega_1 : k < |u-w| \leq k+1\}.$$

We test $T_k(u-w)$ and $\Phi_k(u-w)$ to (1.2) and (3.31), respectively, and then use (2.18) to obtain

$$\int_{C_k} \frac{g(x,|Du|+|Dw|)}{|Du|+|Dw|}|Du-Dw|^2 dx \le c\,k|\mu|(C_k) \le c\,k$$

and

$$\int_{D_k} \frac{g(x,|Du|+|Dw|)}{|Du|+|Dw|}|Du-Dw|^2 dx \le c\,|\mu|(D_k) \le c.$$

For each $q > 2-\frac{1}{n}$ and $k_0 \in \mathbb{N}$, we see

$$\begin{aligned}
&\int_{\Omega_1} \left(\frac{g(x,|Du|+|Dw|)}{|Du|+|Dw|}|Du-Dw|^2\right)^{\frac{1}{q}} dx \\
&\quad \le |C_{k_0}|^{1-\frac{1}{q}}\left(\int_{C_{k_0}} \frac{g(x,|Du|+|Dw|)}{|Du|+|Dw|}|Du-Dw|^2\,dx\right)^{\frac{1}{q}} \\
&\qquad + \sum_{k=k_0}^{\infty} |D_k|^{1-\frac{1}{q}}\left(\int_{D_k} \frac{g(x,|Du|+|Dw|)}{|Du|+|Dw|}|Du-Dw|^2\,dx\right)^{\frac{1}{q}} \\
&\quad \le c(q)|\mu|(\Omega_1)^{\frac{1}{q}} k_0^{\frac{1}{q}} \\
&\qquad + c(q)|\mu|(\Omega_1)^{\frac{1}{q}} \sum_{k=k_0}^{\infty}\left(\frac{1}{k^{n'}}\int_{D_k}|u-w|^{n'}\,dx\right)^{1-\frac{1}{q}} \\
&\quad \le c(q)|\mu|(\Omega_1)^{\frac{1}{q}} k_0^{\frac{1}{q}} \\
&\qquad + c(q)|\mu|(\Omega_1)\left(\sum_{k=k_0}^{\infty}\frac{1}{k^{(q-1)n'}}\right)^{\frac{1}{q}}\left(\int_{\Omega_1}|u-w|^{n'}\,dx\right)^{1-\frac{1}{q}} \\
&\quad \le c(q)|\mu|(\Omega_1)^{\frac{1}{q}} k_0^{\frac{1}{q}} + c_*|\mu|(\Omega_1)^{\frac{1}{q}}\left(\int_{\Omega_1}|Du-Dw|\,dx\right)^{\frac{n'}{q'}},
\end{aligned} \tag{3.34}$$

where the constant

$$c_* = c(n,p,q,\nu,L)\left(\sum_{k=k_0}^{\infty}\frac{1}{k^{(q-1)n'}}\right)^{\frac{1}{q}} \tag{3.35}$$

decreases to 0 as $k_0 \to \infty$.

To prove (3.33), we shall distinguish two cases $p \geq 2$ and $2 - \frac{1}{n} < p < 2$. The following inequality which is obtained from (1.10), will be used without mentioning:

$$\left(\frac{7}{16}\right)^n \leq \frac{|\Omega_1|}{|B_1|} \leq 1.$$

The case $p \geq 2$. Using the monotonicity of $t \mapsto g(t)/t$, (3.34) with $q = 2$ gives

$$\begin{aligned}\left(\fint_{\Omega_1} |Du - Dw|\, dx\right)^{\frac{p}{2}} &\leq \fint_{\Omega_1} |Du - Dw|^{\frac{p}{2}}\, dx \\ &\leq c \fint_{\Omega_1} \left(\frac{g(x, |Du| + |Dw|)}{|Du| + |Dw|}\right)^{\frac{1}{2}} |Du - Dw|\, dx \\ &\leq c k_0^{\frac{1}{2}} + c_* \left[\fint_{\Omega_1} |Du - Dw|\, dx\right]^{\frac{n'}{2}}.\end{aligned}$$

Note that $\frac{n'}{2} \leq \frac{p}{2}$ since $n \geq 2$ and $p \geq 2$. If $n = 2$ and $p = 2$, then we can take k_0 large enough so that $c_* < \frac{1}{2}$. Otherwise, we again choose k_0 large enough to satisfy $c_* < \frac{1}{2}$ and apply Young's inequality to show (3.33).

The case $2 - \frac{1}{n} < p < 2$. By a direct calculation, we found

$$
\begin{aligned}
&|Du - Dw| \\
&\leq \left(\frac{g_m(|Du| + |Dw|)}{|Du| + |Dw|} |Du - Dw|^2 \right)^{\frac{1}{p+1}} \left(\frac{(|Du| + |Dw|)^p}{g_m(|Du| + |Dw|)} \right)^{\frac{1}{p+1}} \\
&\overset{(2.21)}{\leq} c \left(\frac{g_m(|Du| + |Dw|)}{|Du| + |Dw|} |Du - Dw|^2 \right)^{\frac{1}{p+1}} \\
&\quad \cdot \left(\frac{|Du - Dw|^p}{g_m(|Du - Dw|)} + \frac{|Du|^p}{g_m(|Du|)} \right)^{\frac{1}{p+1}} \\
&\leq c \left(\frac{g_m(|Du| + |Dw|)}{|Du| + |Dw|} |Du - Dw|^2 \right)^{\frac{1}{p+1}} |Du - Dw|^{\frac{1}{p+1}} \\
&\quad + c \left(\frac{g_m(|Du| + |Dw|)}{|Du| + |Dw|} |Du - Dw|^2 \right)^{\frac{1}{p+1}} \left(\frac{|Du|^p}{g_m(|Du|)} \right)^{\frac{1}{p+1}} \\
&\leq \frac{1}{2} |Du - Dw| + c \left(\frac{g_m(|Du| + |Dw|)}{|Du| + |Dw|} |Du - Dw|^2 \right)^{\frac{1}{p}} \\
&\quad + c \left(\frac{g_m(|Du| + |Dw|)}{|Du| + |Dw|} |Du - Dw|^2 \right)^{\frac{1}{p+1}} \left(\frac{|Du|^p}{g_m(|Du|)} \right)^{\frac{1}{p+1}} .
\end{aligned} \tag{3.36}
$$

Using (3.34) with $q = p$, we further have

$$
\begin{aligned}
&\int_{\Omega_1} |Du - Dw| \, dx \\
&\leq c k_0^{\frac{1}{p}} + c_* \left(\int_{\Omega_1} |Du - Dw| \, dx \right)^{\frac{n'}{p'}} \\
&\quad + \underbrace{c \int_{\Omega_1} \left(\frac{g_m(|Du| + |Dw|)}{|Du| + |Dw|} |Du - Dw|^2 \right)^{\frac{1}{p+1}} \left(\frac{|Du|^p}{g_m(|Du|)} \right)^{\frac{1}{p+1}} dx}_{=:I} .
\end{aligned} \tag{3.37}
$$

Again, we apply Hölder's inequality and (3.34) to estimate I as

$$
\begin{aligned}
I &\le \left[\int_{\Omega_1}\left(\frac{g_m(|Du|+|Dw|)}{|Du|+|Dw|}|Du-Dw|^2\right)^{\frac{1}{p}}dx\right]^{\frac{p}{p+1}}\left[\int_{\Omega_1}\frac{|Du|^p}{g_m(|Du|)}dx\right]^{\frac{1}{p+1}}\\
&\le \left[ck_0^{\frac{1}{p}}+c_*\left[\int_{\Omega_1}|Du-Dw|dx\right]^{\frac{n'}{p'}}\right]^{\frac{p}{p+1}}\left[|\mu|(\Omega_1)\fint_{\Omega_1}\frac{|Du|^p}{g_m(|Du|)}dx\right]^{\frac{1}{p+1}}.
\end{aligned}
$$

For $t\in\mathbb{R}^+$, the mapping $t\mapsto t^p/g_m(t)$ is concave by (2.21), and so Jensen's inequality yields

$$
|\mu|(\Omega_1)\fint_{\Omega_1}\frac{|Du|^p}{g_m(|Du|)}\,dx \le |\mu|(\Omega_1)\frac{\left(\fint_{\Omega_1}|Du|\,dx\right)^p}{g_m\left(\fint_{\Omega_1}|Du|\,dx\right)}\le c. \tag{3.38}
$$

Combining (3.37)-(3.38), we have

$$
\begin{aligned}
\int_{\Omega_1}|Du-Dw|dx \le c\,k_0^{\frac{1}{p}}+c_*\left[\int_{\Omega_1}|Du-Dw|dx\right]^{\frac{n'}{p'}}&\\
+c\left[k_0^{\frac{1}{p}}+c_*\left[\int_{\Omega_1}|Du-Dw|dx\right]^{\frac{n'}{p'}}\right]^{\frac{p}{p+1}}&.
\end{aligned}
$$

Noting $\frac{n'}{p'}<1$ for the case $2-\frac{1}{n}<p<2\le n$, we use Young's inequality to complete the proof. □

Remark 3.2. *For the p-Laplacian type equations with $p<2$, the monotonicity and concavity of the map $t\mapsto t^{2-p}$ play an important role in obtaining a similar result to Lemma 3.1. In (3.36), instead of using $t\mapsto t^{2-p}$, we have used $t\mapsto t^p/g_m(t)$, which is increasing and concave regardless of the range of p. Note that $t\mapsto t/g_m(t)$ seems to be the natural modification of $t\mapsto t^{2-p}$ to fit our setting. However, the map $t\mapsto t/g_m(t)$ generally does not have monotonicity and concavity, which we need in the proof.*

Remark 3.3. *A suitable modification of the proof of Lemma 3.1 gives*

$$\int_\Omega |Du|\,dx \le c|\mu|(\Omega)^{\frac{1}{p-1}}, \tag{3.39}$$

where c depends only on n, p, ν, and L. We provide a sketch of the proof of (3.39)*.*

Let us first assume $|\mu|(\Omega) \le 1$. As previously mentioned, if $p > n$, then $\mu \in W^{-1,p'}(\Omega)$, and there exists $u \in W_0^{1,G}(\Omega)$ satisfying (1.15)*. Testing u to* (1.2)*, we find*

$$\begin{aligned}\|Du\|^p_{L^p(\Omega)} &\lesssim \|u\|_{L^\infty(\Omega)}|\mu|(\Omega)\\ &\lesssim \|u\|_{W^{1,p}(\Omega)}\\ &\lesssim \|Du\|_{L^p(\Omega)},\end{aligned}$$

where we have used Sobolev-Morrey embedding in the second inequality and Poincaré inequality in the last one. Note that (1.11) *is required to apply Poincaré inequality in the above calculations. This directly implies*

$$\int_\Omega |Du|\,dx \le c. \tag{3.40}$$

In the case of $2-\frac{1}{n} < p \le n$, we test $T_k(u),\ \Phi_k(u) \in W_0^{1,G}(\Omega)$ to (1.2)*. By following the calculations in the proof of Lemma 3.1 with $q = p$, we discover*

$$\begin{aligned}\int_\Omega |Du|\,dx &\le \int_\Omega (g(x,|Du|)|Du|)^{\frac{1}{p}}\,dx\\ &\le c(p)|\mu|(\Omega)^{1-\frac{1}{p}}k_0^{\frac{1}{p}} + c_*|\mu|(\Omega)^{1-\frac{1}{p}}\left(\int_\Omega |Du|\,dx\right)^{\frac{n'}{p'}}\\ &\le c\,k_0^{\frac{1}{p}} + c_*\left(\int_\Omega |Du|\,dx\right)^{\frac{n'}{p'}}.\end{aligned} \tag{3.41}$$

Recall that $c_* = c_*(n,p,\nu,L,k_0) > 0$ is the constant given in (3.35) *and it decreases to 0 as $k_0 \to \infty$.*

When $p < n\,(\Leftrightarrow \frac{n'}{p'} < 1)$, Young's inequality with $k_0 = 1$ yields (3.40)*. If $p = n$,* (3.40) *follows from taking $k_0 = k_0(n,p) > 0$ sufficiently large so that $c_* \le \frac{1}{2}$.*

For the general case that $|\mu|(\Omega) < \infty$, we consider the following normalization:

$$\hat{u}(x) = \frac{u(x)}{M}, \quad \hat{\mu}(x) = \frac{\mu(x)}{M^{p-1}}, \quad \hat{A}(x,\xi) = \frac{A(x,M\xi)}{M^{p-1}},$$
$$\text{and} \quad \hat{g}(x,t) = t^{p-1} + a(x)t^{p-1}\log(e+Mt))$$

where $M = |\mu|(\Omega)^{\frac{1}{p-1}}$. One can check that

$$\langle \partial_\xi \hat{A}(x,\xi)\eta, \eta\rangle \geq \nu \hat{g}(x,|\xi|)|\eta|^2 \qquad \text{and} \qquad |\hat{\mu}|(\Omega) \leq 1.$$

Then (3.39) *follows from* (3.40) *and the normalization.*

Remark 3.4. *Recall $g_m(t) \geq t^{p-1}$, which directly implies $g_m^{-1}(t) \leq t^{\frac{1}{p-1}}$. Then we use*(3.32), (3.39), *and Young's inequality to discover*

$$\begin{aligned}
\fint_{\Omega_{3r}} |Dw|\,dx &\leq \fint_{\Omega_{3r}} |Dw - Du|\,dx + \fint_{\Omega_{3r}} |Du|\,dx \\
&\lesssim \left(\frac{|\mu|(\Omega)}{r^n}\right)^{\frac{1}{p-1}} + \left(\frac{|\mu|(\Omega)}{r^n}\right)^{\frac{1}{p}} \left(\fint_{\Omega_{3r}} |Du|\,dx\right)^{\frac{1}{p}} \\
&\quad + \fint_{\Omega_{3r}} |Du|\,dx \\
&\lesssim \left(\frac{|\mu|(\Omega)}{r^n}\right)^{\frac{1}{p-1}} + \fint_{\Omega_{3r}} |Du|\,dx \\
&\lesssim \left(\frac{|\mu|(\Omega)}{r^n}\right)^{\frac{1}{p-1}} + \frac{|\mu|(\Omega)^{\frac{1}{p-1}}}{r^n} \lesssim \frac{1}{r^{3n}},
\end{aligned} \tag{3.42}$$

where we also have used (1.12) *and the assumption $0 < 8r \leq R_0$.*

Remark 3.5. *From now on, we simply denote*

$$G_0(\cdot) = G(x_0,\cdot) \qquad \text{and} \qquad g_0(\cdot) = g(x_0,\cdot).$$

Proceeding as in [25, (3.1)], we use (1.5), (1.14) *and* (2.23) *to obtain the following localized estimate:*

$$g_m^{-1}\left(\frac{|\mu|(\Omega_{3r})}{r^{n-1}}\right) \leq c g_x^{-1}\left(\frac{|\mu|(\Omega_{3r})}{r^{n-1}}\right) \tag{3.43}$$

for all $x \in \Omega_{3r}$. *By* (3.39), (1.5) *and* (1.14), *we also discover*

$$\begin{aligned}\left(\fint_{\Omega_{3r}} |Du|\,dx\right)^{\omega(6r)} &\leq \left(\frac{1}{|\Omega_{3r}|}\int_{\Omega} |Du|\,dx\right)^{\omega(6r)} \\ &\leq \left(\frac{|\mu|(\Omega)^{\frac{1}{p-1}}}{r^n}\right)^{\omega(6r)} \leq c.\end{aligned}$$

It then follows from (2.23) *that*

$$\begin{aligned}g_0\left(\fint_{\Omega_{3r}} |Du|\,dx\right) &\lesssim g_m\left(\fint_{\Omega_{3r}} |Du|\,dx\right) + \left(\fint_{\Omega_{3r}} |Du|\,dx\right)^{p-1+\omega(6r)} \\ &\lesssim g_m\left(\fint_{\Omega_{3r}} |Du|\,dx\right).\end{aligned}$$

As a consequence, we refine (3.32) *as*

$$\begin{aligned}\fint_{\Omega_{3r}} |Du - Dw|\,dx \leq c\, g_0^{-1}&\left(\left[\frac{|\mu|(\Omega_{3r})}{r^{n-1}}\right]\right) \\ &+ c\,\chi_{[p<2]}\left[\frac{|\mu|(\Omega_{3r})}{r^{n-1}}\right]^{\frac{1}{p}} \frac{\fint_{\Omega_{3r}} |Du|\,dx}{g_0\left(\fint_{\Omega_{3r}} |Du|\,dx\right)^{\frac{1}{p}}}. \qquad (3.44)\end{aligned}$$

Similarly, (1.14) *and* (1.5) *yield*

$$g_0^{-1}\left(\left[\frac{|\mu|(\Omega_{3r})}{r^{n-1}}\right]\right)^{\omega(6r)} \leq \left[\frac{|\mu|(\Omega_{3r})}{r^{n-1}}\right]^{\frac{\omega(6r)}{p-1}} \leq r^{-\frac{\omega(6r)}{p-1}} \leq c,$$

and

$$\left(\frac{\fint_{\Omega_{3r}} |Du|\,dx}{g_0\left(\fint_{\Omega_{3r}} |Du|\,dx\right)^{\frac{1}{p}}}\right)^{\omega(6r)} \leq \left(\fint_{\Omega_{3r}} |Du|\,dx\right)^{\frac{\omega(6r)}{p}} \leq r^{-\frac{\omega(6r)(n+1)}{p}}.$$

Hence, we conclude that

$$\left(\fint_{\Omega_{3r}} |Dw|\,dx\right)^{\omega(6r)} \leq \left(\fint_{\Omega_{3r}} |Dw - Du|\,dx\right)^{\omega(6r)} + \left(\fint_{\Omega_{3r}} |Du|\,dx\right)^{\omega(6r)} \leq c. \tag{3.45}$$

We next discuss higher integrability results for solutions to (3.31). Such results have been shown by the earlier papers including [9, 21, 25]. Therefore we state the desired results as follows without their proof.

Lemma 3.6. *Let $w \in W^{1,G}(\Omega_{3r})$ be the weak solution to* (3.31)*. Then for any $q \in (0,1]$, there exist constants $\sigma_1 = \sigma_1(n,p,\nu,L) > 0$ and $c = c(n,p,q,\nu,L) \geq 1$ such that*

$$\left(\fint_{\Omega_{2\rho}(y)} G(x,|Dw|)^{1+\sigma_1}\,dx\right)^{\frac{1}{1+\sigma_1}} \leq c\left(\fint_{\Omega_{3\rho}(y)} G(x,|Dw|)^{q}\,dx\right)^{\frac{1}{q}}, \tag{3.46}$$

whenever $\Omega_{3\rho}(y) \subset \Omega_{3r}$.

Remark 3.7. *We claim that for sufficiently small $R_0 > 0$ satisfying $\omega(6r) \leq \frac{\sigma_1}{4}$ for every $0 < r \leq \frac{R_0}{6}$, there holds*

$$\left(\fint_{\Omega_{2r}} G_0(|Dw|)^{1+\sigma}\,dx\right)^{\frac{1}{1+\sigma}} \leq c\,G_0\left(\fint_{\Omega_{3r}} |Dw|\,dx\right), \tag{3.47}$$

where $\sigma = \frac{\sigma_1}{4}$. With the help of (2.27)*, one can verify that*

$$t \mapsto \int_0^t \frac{G(x,s)^{\frac{1}{2p}}}{s}\,ds =: \tilde{G}(x,t)$$

is concave for each $x \in \Omega$. Moreover, there exists $c(p) \geq 1$ such that

$$G(x,t/2)^{\frac{1}{2p}} \leq \int_0^t \frac{G(x,s)^{\frac{1}{2p}}}{s}\,ds = \tilde{G}(x,t) \leq c(p)G(x,t)^{\frac{1}{2p}}. \tag{3.48}$$

If $a_0 \geq 2\omega(6r)$, then for each $x \in \Omega_{3r}$

$$G_M(t) \leq G_0(t) + \omega(6r)G_{\log}(t) \leq 2G_0(t),$$

$$2a_0 \le 2a(x) + 2\omega(6r) \le 2a(x) + a_0,$$

and

$$G_0(t) \le 2G(x,t).$$

It then follows from (3.46) *with* $q = \frac{1}{2p}$ *and* (3.48) *that*

$$\begin{aligned}
\left(\fint_{\Omega_{2r}} G_0(|Dw|)^{1+\sigma} \, dx \right)^{\frac{1}{1+\sigma}} &\lesssim \left(\fint_{\Omega_{2r}} G(x,|Dw|)^{1+\sigma} \, dx \right)^{\frac{1}{1+\sigma}} \\
&\overset{(3.46)}{\lesssim} \left(\fint_{\Omega_{3r}} G(x,|Dw|)^{\frac{1}{2p}} \, dx \right)^{2p} \\
&\lesssim \left(\fint_{\Omega_{3r}} G_M(|Dw|)^{\frac{1}{2p}} \, dx \right)^{2p} \\
&\overset{(3.48)}{\lesssim} \left(\fint_{\Omega_{3r}} \tilde{G}_M(|Dw|)^{\frac{1}{2p}} \, dx \right)^{2p} \\
&\lesssim \tilde{G}_M \left(\fint_{\Omega_{3r}} |Dw| \, dx \right) \\
&\lesssim G_0 \left(\fint_{\Omega_{3r}} |Dw| \, dx \right), \qquad (3.49)
\end{aligned}$$

where we have also used Jensen's inequality with $t \mapsto \tilde{G}_M(t)$ *for the second last inequality.*

If $a_0 \le 2\omega(6r)$, *then it follows from the assumptions* $\sigma = \frac{\sigma_1}{4}$ *and* $\omega(6r) \le \frac{\sigma_1}{4}$ *that*

$$(p + \omega(6r))(1+\sigma) \le p(1+\sigma_1).$$

Using Lemma 3.6 and Jensen's inequality, we find

$$
\begin{aligned}
&\omega(6r)\left(\fint_{\Omega_{2r}} G_{\log}(|Dw|)^{1+\sigma}\,dx\right)^{\frac{1}{1+\sigma}}\\
&\overset{(2.22)}{\lesssim} \left(\fint_{\Omega_{2r}} |Dw|^{p(1+\sigma)} + |Dw|^{(p+\omega(6r))(1+\sigma)}\,dx\right)^{\frac{1}{1+\sigma}}\\
&\lesssim \left(\fint_{\Omega_{2r}} G(x,|Dw|)^{1+\sigma_1}\,dx\right)^{\frac{1}{1+\sigma_1}} + \left(\fint_{\Omega_{2r}} G(x,|Dw|)^{1+\sigma_1}\,dx\right)^{\frac{p+\omega(6r)}{p(1+\sigma_1)}}\\
&\overset{(3.46)}{\lesssim} \left(\fint_{\Omega_{3r}} G(x,|Dw|)^{\frac{1}{2p}}\,dx\right)^{2p} + \left(\fint_{\Omega_{3r}} G(x,|Dw|)^{\frac{1}{2p}}\,dx\right)^{2(p+\omega(6r))}\\
&\lesssim \left(\fint_{\Omega_{3r}} G_M(|Dw|)^{\frac{1}{2p}}\,dx\right)^{2p} + \left(\fint_{\Omega_{3r}} G_M(|Dw|)^{\frac{1}{2p}}\,dx\right)^{2(p+\omega(6r))}\\
&\lesssim G_M\left(\fint_{\Omega_{3r}} |Dw|\,dx\right) + G_M\left(\fint_{\Omega_{3r}} |Dw|\,dx\right)^{1+\frac{\omega(6r)}{p}},
\end{aligned}
$$

where we have used (3.48) *and Jensen's inequality with* $t \mapsto \tilde{G}_M(t)^{\frac{1}{2p}}$ *for the last inequality.*

It then follows from (2.23) *that*

$$
\begin{aligned}
&\left(\fint_{\Omega_{2r}} G_0(|Dw|)^{1+\sigma}\,dx\right)^{\frac{1}{1+\sigma}}\\
&\lesssim \left(\fint_{\Omega_{2r}} G(x,|Dw|)^{1+\sigma}\,dx\right)^{\frac{1}{1+\sigma}} + \omega(6r)\left(\fint_{\Omega_{2r}} G_{\log}(|Dw|)^{1+\sigma}\,dx\right)^{\frac{1}{1+\sigma}}\\
&\lesssim G_M\left(\fint_{\Omega_{3r}} |Dw|\,dx\right) + G_M\left(\fint_{\Omega_{3r}} |Dw|\,dx\right)^{1+\frac{\omega(6r)}{p}}.
\end{aligned}
\tag{3.50}
$$

Recalling (3.45) *and* $G_M(t) \le t^p + 3\omega(6r)G_{\log}(t) \lesssim t^p + t^{p+\omega(6r)}$, *we discover*

$$
G_M\left(\fint_{\Omega_{3r}} |Dw|\,dx\right) \lesssim \left(\fint_{\Omega_{3r}} |Dw|\,dx\right)^p \lesssim G_0\left(\fint_{\Omega_{3r}} |Dw|\,dx\right). \tag{3.51}
$$

We combine (3.50) *and* (3.51) *to conclude* (3.47).

As a direct consequence of (3.47)*, we have*

$$\left(\fint_{\Omega_{2r}} |Dw|^p\,dx\right)^{\frac{1}{p}} \le G_0^{-1}\left(\fint_{\Omega_{2r}} G_0(|Dw|)\,dx\right) \lesssim \fint_{\Omega_{3r}} |Dw|\,dx \overset{(3.42)}{\lesssim} \frac{1}{r^{3n}}. \tag{3.52}$$

Here, we have used Jensen's inequality with $t \mapsto G_0(t^{\frac{1}{p}})$*, which is an increasing convex function.*

To proceed further, we now consider two vector fields

$$\bar{A}(x,\xi) = \frac{g(x_0,|\xi|)}{g(x,|\xi|)}A(x,\xi) \quad \text{and} \quad A_0(\xi) = \fint_{B_{2r}^+} \bar{A}(x,\xi)dx \tag{3.53}$$

for every $x \in \Omega_{2r}$ and $\xi \in \mathbb{R}^n$. By a direct calculation, we see

$$\partial_\xi \bar{A}(x,\xi) = \left[\frac{1 + a(x_0)\log(e+|\xi|)}{1 + a(x)\log(e+|\xi|)}\right]\partial_\xi A(x,\xi) + \left[\frac{a(x_0)-a(x)}{|\xi|(e+|\xi|)(1+a(x)\log(e+|\xi|))^2}\right]\xi \otimes A(x,\xi).$$

Moreover, (1.2) and (1.13) imply

$$\left|\left[\frac{a(x_0)-a(x)}{|\xi|(e+|\xi|)(1+a(x)\log(e+|\xi|))^2}\right]\xi \otimes A(x,\xi)\right| \le L\omega(6r)|\xi|^{p-2} \le \frac{\nu}{2}|\xi|^{p-2}.$$

Therefore, we see that $\bar{A}$ satisfies the following ellipticity and growth conditions:

$$\begin{cases} |\bar{A}(x,\xi)| + |\xi||\partial_\xi \bar{A}(x,\xi)| \le 2L[|\xi|^{p-1}(1 + a(x_0)\log(e+|\xi|))], \\ \langle \partial_\xi \bar{A}(x,\xi)\eta,\eta\rangle \ge \frac{\nu}{2}[|\xi|^{p-2}(1+a(x_0)\log(e+|\xi|))]|\eta|^2 \end{cases} \tag{3.54}$$

for every $x,\xi,\eta \in \mathbb{R}^n$, where L and ν are the constants given in (1.2).

By the definition of A_0, (3.54) also holds when we put A_0 instead of $\bar{A}$.

Moreover, (δ, R)-vanishing condition (1.7) gives us

$$\begin{aligned}\sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \frac{|\bar{A}(x,\xi)-A_0(\xi)|}{g(x_0,|\xi|)} &= \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \left| \frac{\bar{A}(x,\xi)}{g(x_0,|\xi|)} - \fint_{B_{2r}^+} \frac{\bar{A}(z,\xi)}{g(x_0,|\xi|)}\,dz \right| \\ &= \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \left| \frac{A(x,\xi)}{g(x,|\xi|)} - \fint_{B_{2r}^+} \frac{A(z,\xi)}{g(z,|\xi|)}\,dz \right| \\ &= \theta(B_{2r}^+)(x).\end{aligned} \tag{3.55}$$

for every $x \in \mathbb{R}^n$.

Let us proceed for the desired comparison estimate between the homogeneous equation (3.31) and the following frozen equation on the flat boundary:

$$\begin{cases} -\operatorname{div} A_0(Dv) = 0 & \text{in} \quad B_{2r}^+, \\ \qquad\quad v = \eta w & \text{on} \quad \partial B_{2r}^+. \end{cases} \tag{3.56}$$

Here, $\eta = \eta(x_n) \in C^\infty(\mathbb{R})$ is a cut-off function satisfying

$$0 \le \eta \le 1, \quad \eta \equiv 1 \quad \text{on } [\delta r, 2r], \quad \eta \equiv 0 \quad \text{on } (-\infty, 0] \quad \text{and} \quad |D\eta| \le \frac{4}{\delta r}.$$

In the next lemma, we use the notation $V_0(\xi) = V(x_0,\xi)$ for $\xi \in \mathbb{R}^n$.

Lemma 3.8. *Under the assumptions in Lemma 3.1, let $v \in W^{1,G_0}(B_{2r}^+)$ be the weak solution to (3.56). For any $\varepsilon > 0$ and $\lambda > 0$, there exists a small positive constant $\delta = \delta(n,p,\nu,L,\varepsilon)$ such that if*

$$\fint_{\Omega_{3r}} |Du|\,dx \le \lambda \quad \textit{and} \quad g_0^{-1}\left(\frac{|\mu|(\Omega_{3r})}{r^{n-1}}\right) \le \delta\lambda, \tag{3.57}$$

then we have

$$\fint_{\Omega_{2r}} |V_0(Dw) - V_0(Dv)|^2\,dx \le \varepsilon\, G_0(\lambda),$$

where v is extended by zero from B_{2r}^+ to Ω_{2r}^+.

Proof. By virtue of (3.44) and (3.57), we see

$$\begin{aligned}\fint_{\Omega_{3r}} |Du - Dw|\, dx &\le \delta\lambda + \fint_{\Omega_{3r}} |Du|\, dx \frac{g_0(\delta\lambda)^{\frac{1}{p}}}{g_0\left(\fint_{\Omega_{3r}} |Du|\, dx\right)^{\frac{1}{p}}}, \\ &\le 2\delta^{\frac{p-1}{p}}\lambda. \end{aligned} \tag{3.58}$$

In the last inequality, we also have used the fact that $t \mapsto (1+a_0\log(e+t))/t$ is a decreasing function. Then we have

$$\fint_{\Omega_{3r}} |Dw|\, dx \le \fint_{\Omega_{3r}} |Du|\, dx + \fint_{\Omega_{3r}} |Du - Dw|\, dx \le 3\lambda.$$

Thus, it suffices to show that

$$\fint_{\Omega_{2r}} |V_0(Dw) - V_0(Dv)|^2\, dx \le \varepsilon\, G_0\left(\fint_{\Omega_{3r}} |Dw|\, dx\right). \tag{3.59}$$

Taking $v - \eta w$ as a test function to (3.56), we have

$$\int_{B_{2r}^+} \langle A_0(Dv), Dv\rangle\, dx = \int_{B_{2r}^+} \langle A_0(Dv), \eta Dw + wD\eta\rangle\, dx.$$

It then follows from (3.54), (2.29), and (2.30) that

$$\int_{B_{2r}^+} G_0(|Dv|)\, dx \lesssim \int_{B_{2r}^+} G_0(|Dw|)\, dx + \int_{B_{2r}^+} G_0(|D\eta||w|)\, dx. \tag{3.60}$$

Using the facts that $D\eta \equiv 0$ when $x_n > \delta r$ and $\eta = 0$ in $B'_{2r} \times \{x_n \le -4\delta r\}$, we discover

$$
\begin{aligned}
\int_{B_{2r}^+} G_0(|D\eta||w|)\,dx &\lesssim \int_{\Omega_{2r}\cap\{x_n\leq\delta r\}} G_0(|D\eta||w|)\,dx \\
&\lesssim \int_{\Omega_{2r}\cap\{x_n\leq\delta r\}} G_0\left(\frac{1}{\delta r}\left|\int_{-4\delta r}^{x_n}\frac{\partial}{\partial y}w(x',y)\,dy\right|\right)dx \\
&\lesssim \int_{\Omega_{2r}\cap\{x_n\leq\delta r\}} G_0\left(\frac{1}{\delta r}\int_{-4\delta r}^{\delta r}|Dw(x',y)|\,dy\right)dx \\
&\lesssim \frac{1}{\delta r}\int_{\Omega_{2r}\cap\{x_n\leq\delta r\}}\int_{-4\delta r}^{\delta r} G_0(|Dw(x',y)|)\,dy\,dx \\
&\lesssim \int_{\Omega_{2r}\cap\{x_n\leq\delta r\}} G_0(|Dw|)\,dx. \qquad (3.61)
\end{aligned}
$$

In the second last inequality, we also have used Jensen's inequality. Combining (3.60) and (3.61), we obtain

$$
\int_{\Omega_{2r}} G_0(|Dv|)\,dx \lesssim \int_{\Omega_{2r}} G_0(|Dw|)\,dx. \qquad (3.62)
$$

Now we use $v-\eta w$ as a test function to (3.31) and (3.56), respectively, to find

$$
\begin{aligned}
&\frac{1}{c}\fint_{\Omega_{2r}} |V_0(Dw)-V_0(Dv)|^2\,dx \\
&\leq \fint_{\Omega_{2r}} \langle A_0(Dw)-A_0(Dv), Dw-Dv\rangle\,dx \\
&= \fint_{\Omega_{2r}} \langle A_0(Dw)-A_0(Dv), D(\eta w-v)\rangle\,dx \\
&\qquad + \fint_{\Omega_{2r}} \langle A_0(Dw)-A_0(Dv), D((1-\eta)w)\rangle\,dx \\
&= \fint_{\Omega_{2r}} \langle A_0(Dw)-A(x,Dw), Dw-Dv\rangle\,dx \\
&\qquad - \fint_{\Omega_{2r}} \langle A_0(Dw)-A(x,Dw), D((1-\eta)w)\rangle\,dx \\
&\qquad + \fint_{\Omega_{2r}} \langle A_0(Dw)-A_0(Dv), D((1-\eta)w)\rangle\,dx =: I+II+III. \qquad (3.63)
\end{aligned}
$$

We first estimate I as follows:

$$
\begin{aligned}
I = & \fint_{\Omega_{2r}} \langle A_0(Dw) - \bar{A}(x, Dw), Dw - Dv \rangle \, dx \\
& + \fint_{\Omega_{2r}} \langle \bar{A}(x, Dw) - A(x, Dw), Dw - Dv \rangle \, dx =: I_1 + I_2.
\end{aligned}
$$

For an arbitrary constant $\bar{\varepsilon} \in (0,1)$, we use (3.55), (2.30), and (2.29) to see

$$
\begin{aligned}
|I_1| & \le c \fint_{\Omega_{2r}} \theta(B_{2r}^+) g_0(|Dw|)(|Dw| + |Dv|) \, dx \\
& \le c(\bar{\varepsilon}) \fint_{\Omega_{2r}} \theta(B_{2r}^+) G_0(|Dw|) \, dx + \bar{\varepsilon} \fint_{\Omega_{2r}} \theta(B_{2r}^+) G_0(|Dv|) \, dx \\
& =: I_{1,1} + I_{1,2}.
\end{aligned}
\tag{3.64}
$$

In light of using (1.9) and (3.47), $I_{1,1}$ can be estimated as

$$
\begin{aligned}
I_{1,1} & \le c(\bar{\varepsilon}) \left(\fint_{\Omega_{2r}} \theta(B_{3r}^+)^{\frac{1+\sigma}{\sigma}} \, dx \right)^{\frac{\sigma}{1+\sigma}} \left(\fint_{\Omega_{2r}} G_0(|Dw|)^{1+\sigma} \, dx \right)^{\frac{1}{1+\sigma}} \\
& \le c(\bar{\varepsilon}) \delta^{\frac{\sigma}{1+\sigma}} G_0 \left(\fint_{\Omega_{3r}} |Dw| \, dx \right),
\end{aligned}
\tag{3.65}
$$

where we also have used the following estimates:

$$
\begin{aligned}
\fint_{\Omega_{2r}} \theta(B_{2r}^+)^{\frac{1+\sigma}{\sigma}} \, dx & \le \frac{c}{|B_{2r}^+|} \left[\int_{B_{2r}^+} \theta(B_{2r}^+)^{\frac{1+\sigma}{\sigma}} \, dx + \int_{\Omega_{2r} \setminus B_{2r}^+} \theta(B_{2r}^+)^{\frac{1+\sigma}{\sigma}} \, dx \right] \\
& \le c \fint_{B_{2r}^+} \theta(B_{2r}^+)^{\frac{1+\sigma}{\sigma}} \, dx + c \frac{|\Omega_{2r} \setminus B_{2r}^+|}{|B_{2r}^+|} (2L)^{\frac{1+\sigma}{\sigma}} \lesssim \delta.
\end{aligned}
$$

Note that $\sigma = \sigma(n, p, \nu, L)$ is the exponent obtained from higher integrability estimate (3.47). Moreover, (3.62) and (3.47) directly give us

$$
I_{1,2} \le 2L\bar{\varepsilon} \fint_{\Omega_{2r}} G_0(|Dw|) \, dx \lesssim \bar{\varepsilon} \, G_0 \left(\fint_{\Omega_{3r}} |Dw| \, dx \right). \tag{3.66}
$$

Combining (3.64)-(3.66), we obtain

$$|I_1| \le \left(c(\bar\varepsilon)\delta^{\frac{\sigma}{1+\sigma}} + c\bar\varepsilon\right) G_0\left(\fint_{\Omega_{3r}} |Dw|\,dx\right). \tag{3.67}$$

From the definition of $\bar A(\cdot)$, we find

$$\begin{aligned}
|I_2| &\le \omega(4r)\fint_{\Omega_{2r}} g_{\log}(|Dw|)|Dw-Dv|\,dx\\
&\le \omega(4r)\fint_{\Omega_{2r}} g_{\log}(|Dw|)(|Dw|+|Dv|)\,dx\\
&\le \bar\varepsilon\fint_{\Omega_{2r}}|Dv|^p\,dx + c(\bar\varepsilon)\,\omega(4r)^{p'}\fint_{\Omega_{2r}} g_{\log}(|Dw|)^{p'}\,dx\\
&\quad + \omega(4r)\fint_{\Omega_{2r}} g_{\log}(|Dw|)|Dw|\,dx.
\end{aligned} \tag{3.68}$$

We now use (2.24), (2.25) (3.45), Lemma 2.1, and Remark 3.7 to discover

$$\begin{aligned}
&\fint_{\Omega_{2r}} g_{\log}(|Dw|)^{p'}\,dx\\
&= \fint_{\Omega_{2r}} |Dw|^p\log^{p'}(e+|Dw|)\,dx\\
&\lesssim \fint_{\Omega_{2r}} |Dw|^p\left[\log^{p'}\left(e+\frac{|Dw|^p}{\|Dw\|^p_{L^p(\Omega_{2r})}}\right)+\log^{p'}\left(e+\|Dw\|^p_{L^p(\Omega_{2r})}\right)\right]dx\\
&\lesssim \left[\left(\fint_{\Omega_{2r}}|Dw|^{p(1+\sigma)}\,dx\right)^{\frac{1}{1+\sigma}} + \log^{p'}\left(\frac{1}{4r}\right)\fint_{\Omega_{2r}}|Dw|^p\,dx\right]\\
&\lesssim \log^{p'}\left(\frac{1}{4r}\right)G_0\left(\fint_{\Omega_{3r}}|Dw|\,dx\right).
\end{aligned} \tag{3.69}$$

Similarly, we have

$$\begin{aligned}
\fint_{\Omega_{2r}} g_{\log}(|Dw|)|Dw|\,dx &= \fint_{\Omega_{2r}} |Dw|^p\log(e+|Dw|)\,dx\\
&\lesssim \log\left(\frac{1}{4r}\right)G_0\left(\fint_{\Omega_{3r}}|Dw|\,dx\right).
\end{aligned} \tag{3.70}$$

Moreover, Young's inequality with the convex map $t \mapsto G_0(t^{\frac{1}{p}})$ gives

$$\begin{aligned}\left(\fint_{\Omega_{2r}} |Dv|^p \,dx\right)^{\frac{1}{p}} &\le G_0^{-1}\left(\fint_{\Omega_{2r}} G_0(|Dv|)\,dx\right)\\ &\overset{(3.62)}{\lesssim} G_0^{-1}\left(\fint_{\Omega_{2r}} G_0(|Dw|)\,dx\right) \overset{(3.47)}{\lesssim} \fint_{\Omega_{3r}} |Dw|\,dx. \quad (3.71)\end{aligned}$$

Combining (3.68)-(3.71) and using Definition 1.1, we obtain

$$|I_2| \le \big(c\,\bar{\varepsilon} + c(\bar{\varepsilon})\delta\big) G_0\left(\fint_{\Omega_{3r}} |Dw|\,dx\right). \quad (3.72)$$

By (3.53), (3.55), (1.7), (2.30) with $G_0(\cdot)$ and Hölder's inequality, we have

$$\begin{aligned}|II| &\le \fint_{\Omega_{2r}} |A_0(Dw) - \bar{A}(x, Dw)||D((1-\eta)w)|\,dx\\ &\quad + \fint_{\Omega_{2r}} |\bar{A}(x, Dw) - A(x, Dw)||D((1-\eta)w)|\,dx\\ &\le \fint_{\Omega_{2r}} \theta(B_{2r}^+) g_0(|Dw|)|D((1-\eta)w)|\,dx\\ &\quad + \omega(4r) \fint_{\Omega_{2r}} g_{\log}(|Dw|)|D((1-\eta)w)|\,dx\\ &\le \bar{\varepsilon} \fint_{\Omega_{2r}} G_0(|Dw|)\,dx + c(\bar{\varepsilon}) \fint_{\Omega_{2r}} G_0(|D((1-\eta)w)|)\,dx\\ &\quad + c\,\omega(4r)^{p'} \fint_{\Omega_{2r}} g_{\log}(|Dw|)^{p'}\,dx.\end{aligned}$$

It then follows from (3.47) and (3.69) that

$$|II| \le c(\bar{\varepsilon} + \delta^{p'}) G_0\left(\fint_{\Omega_{3r}} |Dw|\,dx\right) + c(\bar{\varepsilon}) \fint_{\Omega_{2r}} G_0(|D((1-\eta)w)|)\,dx. \quad (3.73)$$

We now use (2.30) with $G_0(\cdot)$,(2.29), (3.51) and (3.62) to discover.

$$\begin{aligned}
|III| &\le c(\bar\varepsilon) \fint_{\Omega_{2r}} G_0(|D((1-\eta)w)|)\,dx \\
&\qquad + \bar\varepsilon \fint_{\Omega_{2r}} G_0(|Dw|)\,dx + \bar\varepsilon \fint_{\Omega_{2r}} G_0(|Dv|)\,dx \\
&\le c\,\bar\varepsilon \fint_{\Omega_{2r}} G_0(|Dw|)\,dx + c(\bar\varepsilon) \fint_{\Omega_{2r}} G_0(|D((1-\eta)w)|)\,dx \\
&\overset{(3.47)}{\le} c\,\bar\varepsilon G_0\bigg(\fint_{\Omega_{3r}} |Dw|\,dx \bigg) + c(\bar\varepsilon) \fint_{\Omega_{2r}} G_0(|D((1-\eta)w)|)\,dx. \qquad (3.74)
\end{aligned}$$

Summing up (3.63), (3.67), (3.72), (3.73), and (3.74), we have

$$\begin{aligned}
\fint_{\Omega_{2r}} |V_0(Dw) - V_0(Dv)|^2\,dx &\le (c\,\bar\varepsilon + c(\bar\varepsilon)\delta^{\frac{\sigma}{1+\sigma}}) G_0\bigg(\fint_{\Omega_{3r}} |Dw|\,dx \bigg) \\
&\qquad + c(\bar\varepsilon) \fint_{\Omega_{2r}} G_0(|D((1-\eta)w)|)\,dx.
\end{aligned}$$

To complete the proof, we now estimate the last term in the above inequality. A straightforward calculation gives

$$\begin{aligned}
\fint_{\Omega_{2r}} G_0(|D((1-\eta)w)|)\,dx &\lesssim \frac{1}{|B_{2r}|} \int_{\Omega_{2r}\cap\{x_n\le\delta r\}} G_0(|Dw|)\,dx \\
&\qquad + \fint_{\Omega_{2r}} G_0(|D\eta||w|)\,dx. \qquad (3.75)
\end{aligned}$$

We use Hölder's inequality and (3.47) to see

$$\begin{aligned}
&\frac{1}{|B_{2r}|} \int_{\Omega_{2r}\cap\{x_n\le\delta r\}} G_0(|Dw|)\,dx \\
&\lesssim \left(\frac{1}{|B_{2r}|} \int_{\Omega_{2r}\cap\{x_n\le\delta r\}} G_0(|Dw|)^{1+\sigma}\,dx \right)^{\frac{1}{1+\sigma}} \left(\frac{|\Omega_{2r}\cap\{x_n\le\delta r\}|}{|B_{2r}|} \right)^{\frac{\sigma}{1+\sigma}} \\
&\lesssim \delta^{\frac{\sigma}{1+\sigma}} \left(\fint_{\Omega_{2r}} G_0(|Dw|)^{1+\sigma}\,dx \right)^{\frac{1}{1+\sigma}} \\
&\lesssim \delta^{\frac{\sigma}{1+\sigma}} G_0\bigg(\fint_{\Omega_{3r}} |Dw|\,dx \bigg). \qquad (3.76)
\end{aligned}$$

This and (3.61) complete the proof. □

Note that $G_0(t)$ belongs to the class of functionals with general growth conditions. Regularity results regarding general growth conditions are now well known, see [31, 32, 65]. Especially, we refer to [33, Theorem 4.1] and [35, Theorem 2.2] for the boundary Lipschitz regularity for general growth problems, which holds for (3.56) under (1.2). The next lemma is a modified version for our case.

Lemma 3.9. *Under the same assumptions as in Lemma 3.8, we have*

$$\sup_{B_r^+} |Dv| \le c_l \lambda,$$

for some constant $c_l \ge 1$ depending only on n and p.

3.2 Interior Comparison Estimates

The interior comparison estimates are analogous to the ones for boundary cases, except for the flattening argument applied to the proof of Lemma 3.8. Hence we state interior estimate without their proofs in this subsection.

We first take a ball $B_{3r} \subset \Omega$. Likewise to Section 3.1, we first consider the homogeneous equation:

$$\begin{cases} -\mathrm{div}\, A(x, Dw) = 0 & \text{in } B_{3r}, \\ w = u & \text{on } \partial B_{3r}, \end{cases} \tag{3.77}$$

and then consider the limiting equation:

$$\begin{cases} -\mathrm{div} A_0(Dv) = 0 & \text{in } B_{2r}, \\ v = w & \text{on } \partial B_{2r}, \end{cases} \tag{3.78}$$

where the vector field $A_0 : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$A_0(\xi) = \fint_{B_{2r}} \frac{g(x_0, |\xi|)}{g(x, |\xi|)} A(x, \xi)\, dx.$$

In the following lemmas, we let $w \in u + W_0^{1,G}(B_{3r})$ and $v \in w + W_0^{1,G_0}(B_{2r})$ be the weak solutions to (3.77) and (3.78), respectively.

The following is the interior analog of Lemma 3.1.

Lemma 3.10. *For any $\varepsilon \in (0,1]$, there exists $\delta = \delta(\texttt{data}, \varepsilon) > 0$ such that if*

$$\fint_{B_{3r}} |Du|\,dx \le \lambda \qquad \text{and} \qquad g_0^{-1}\left(\frac{|\mu|(B_{3r})}{r^{n-1}}\right) \le \delta\lambda$$

for some $\lambda > 0$, then

$$\fint_{B_{3r}} |Du - Dw|\,dx \le \varepsilon\lambda.$$

The next lemma is the interior version of Lemma 3.8 and Lemma 3.9.

Lemma 3.11. *Under the assumptions of Lemma 3.10, we have*

$$G_0^{-1}\left(\fint_{B_{2r}} |V(Dw) - V(Dv)|^2\,dx\right) \le \varepsilon\lambda$$

and

$$\sup_{B_r} |Dv| \le c_l\lambda,$$

where $c_l = c_l(n,p) \ge 1$.

Note that for the sake of simplicity, we have denoted c_l by the constant obtained from Lipschitz regularity of limiting equations for both boundary and interior cases. Note that c_l obtained from Lemma 3.9 and Lemma 3.11 have the same dependence and role in the later proof.

We end this section with a remark on the sharp maximal functions of μ.

Remark 3.12. *For some $x_0 \in \Omega$ and $0 < r \le \frac{R_0}{6}$, take any point $x \in \Omega_{3r}(x_0) = B_{3r}(x_0) \cap \Omega$. We see $B_{3r}(x_0) \subset B_{6r}(x)$ and*

$$\frac{|\mu|(\Omega_{3r}(x_0))}{r^{n-1}} \le c\,\frac{r|\mu|(\Omega_{6r}(x))}{|B_{6r}|} \le c\,M_1(\mu)(x), \tag{3.79}$$

where the constant $c \ge 1$ depends only on n.

Writing $g_{M,\Omega_{3r}(x)}(t) = \sup_{z\in\Omega_{3r}(x)} g(z,t)$, we use (2.23) and (1.14) to find

$$\begin{aligned} g_{M,\Omega_{3r}(x)} \circ g_{x_0}^{-1}\left(\frac{|\mu|(\Omega_{3r}(x_0))}{r^{n-1}}\right) &\lesssim \frac{|\mu|(\Omega_{3r}(x_0))}{r^{n-1}} + \left[\frac{|\mu|(\Omega_{3r}(x_0))}{r^{n-1}}\right]^{1+\frac{\omega(6r)}{p-1}} \\ &\lesssim \frac{|\mu|(\Omega_{3r}(x_0))}{r^{n-1}}, \end{aligned}$$

and so

$$g_{x_0}^{-1}\left(\frac{|\mu|(\Omega_{3r}(x_0))}{r^{n-1}}\right) \lesssim g_{M,\Omega_{3r}(x)}^{-1}\left(\frac{|\mu|(\Omega_{3r}(x_0))}{r^{n-1}}\right) \lesssim g_x^{-1}\left(\frac{|\mu|(\Omega_{3r}(x_0))}{r^{n-1}}\right). \tag{3.80}$$

Combining (3.79) *and* (3.80), *we have*

$$g_{x_0}^{-1}\left(\frac{|\mu|(\Omega_{3r}(x_0))}{r^{n-1}}\right) \lesssim g_x^{-1}(M_1(\mu)(x)).$$

We integrate this inequality over $\Omega_{3r}(x_0)$ with respect to x, and then take the average to conclude

$$g_{x_0}^{-1}\left(\frac{|\mu|(\Omega_{3r}(x_0))}{r^{n-1}}\right) \lesssim \fint_{\Omega_{3r}(x_0)} g_x^{-1}(M_1(\mu)(x))\,dx.$$

Therefore, one can replace the assumptions on μ given in Lemma 3.8 and Lemma 3.10 by

$$\fint_{\Omega_{3r}(x_0)} g_x^{-1}(M_1(\mu)(x))\,dx \le \delta\lambda$$

in the rest of this chapter.

4 Proof of main theorem

We devote this section to the proof of Theorem 1.1. Recall that Theorem 1.1 states a global Calderón-Zygmund estimate for a SOLA $u \in W_0^{1,1}(\Omega)$ to (1.2) having a bounded Radon measure μ under appropriate structural assumptions. From the definition of SOLA, there exists a sequence of solutions $\{u_l\}_{l\in\mathbb{N}} \subset W^{1,G}(\Omega)$ and a sequence of data $\{\mu_l\}_{l\in\mathbb{N}} \subset L^\infty(\Omega)$, such that u_l solves (1.2) with $\mu = \mu_l$. Moreover, $u_l \to u$ in $W^{1,q}(\Omega)$ for every $q \in [1, n'(p-1))$, and $\mu_l \rightharpoonup \mu$ weakly in measure. To prove Theorem 1.1, we are going to apply the lemmas in Section 3 to u_l for each $l \in \mathbb{N}$, since the lemmas only work for weak solutions.

Define

$$\begin{aligned} \lambda_0 &:= \fint_{\Omega} |Du|\,dx + \frac{1}{\delta}\fint_{\Omega} g_x^{-1}(\mathcal{M}_1(\mu))\,dx \\ H &:= \left(\frac{1000\cdot \mathrm{diam}\Omega}{R_0}\right)^n \end{aligned} \tag{4.81}$$

and super level sets

$$E(\lambda) := \{x\in\Omega : |Du|(x) > \lambda\} \qquad {}^\forall\lambda > H\lambda_0,$$

where δ is determined in Lemma 3.8 for some $\bar{\varepsilon}$. Without loss of generality, we assume $\lambda_0 > 0$. Indeed, if $\lambda_0 = 0$, then there is nothing to prove, as u is a constant function.

We now state the following Vitali type covering lemma without its proof, as it can be shown by a simple modification of [24, Lemma 4.1].

Lemma 4.1. *For any $\lambda \geq H\lambda_0 > 0$, there is a set of disjoint balls $\{B_{r_i}(x^i)\}_{i\geq 1}$ with $x^i \in E(\lambda)$ and $r_i \in (0, \frac{R_0}{1000})$ such that*

$$E(\lambda)\backslash N \subset \bigcup_{i\geq 1} B_{5r_i}(x^i),$$

where N is a measure zero set. Furthermore, we have

$$\fint_{\Omega_{r_i}(x^i)} |Du|\,dx + \frac{1}{\delta}\fint_{\Omega_{r_i}(x^i)} g_x^{-1}(\mathcal{M}_1(\mu))\,dx = \lambda \tag{4.82}$$

and

$$\fint_{\Omega_\rho(x^i)} |Du|\,dx + \frac{1}{\delta}\fint_{\Omega_\rho(x^i)} g_x^{-1}(\mathcal{M}_1(\mu))\,dx \leq \lambda \quad {}^\forall\rho\in(r_i, R_0]. \tag{4.83}$$

We have taken a set of disjoint balls in Lemma 4.1 for any $\lambda \geq H\lambda_0$. From the definition of SOLA, for each $i\geq 1$ and any $\varepsilon > 0$, there exists $l_i \geq 1$ such that

$$\fint_{\Omega_{1000r_i}(x^i)} |Du - Du_{l_i}|\,dx \leq \varepsilon\lambda. \tag{4.84}$$

We are now ready to prove our main result.

Proof of Theorem 1.1. For arbitrary constant $\lambda \geq H\lambda_0$, Lemma 4.1 allows us to take a set of disjoint balls $\{B_{r_i}(x^i)\}_{i\geq 1}$ such that

$$E(K\lambda)\backslash N \subset \bigcup_{i\geq 1} B_{5r_i}(x^i), \tag{4.85}$$

where N is a measure zero set and $K = 8c_l$. Recall that c_l is the constant obtained from Lipschitz regularity of the limiting equations, see Lemma 3.9 and Lemma 3.11. To employ the lemmas obtained in the previous section, we need to distinguish the boundary case $B_{15r_i}(x^i) \not\subset \Omega$ and the interior case $B_{15r_i}(x^i) \subset \Omega$.

First, we assume $B_{15r_i}(x^i) \not\subset \Omega$. Taking a point $\tilde{x}^i \in \partial\Omega \cap B_{15r_i}(x^i)$, we can find $y^i \in \Omega$ and a new coordinate system $(\tilde{y}^i_1, \cdots, \tilde{y}^i_n)$ with the origin at y^i such that $|\tilde{x}^i - y^i| \leq 150\delta r_i \leq 20 r_i$ and

$$B^+_{150r_i}(0) \subset \Omega_{150r_i}(0) \subset B_{150r_i}(0) \cap \{\tilde{y}^i_n \geq -300\delta r\}.$$

Note that

$$\Omega_{15r_i}(x^i) \subset \Omega_{50r_i}(y^i)$$

as $|x^i - y^i| \leq |\tilde{x}^i - x^i| + |\tilde{x}^i - y^i| \leq 35r_i$. With the help of (4.83) and (4.84), we have

$$\fint_{\Omega_{150r_i}(y^i)} |Du_{l_i}|\, dx \leq (10^n\varepsilon + 1)\lambda \leq 2\lambda \tag{4.86}$$

for any constant $\varepsilon \in (0, \frac{1}{10^n})$, which we will choose sufficiently small later. Moreover, Remark 3.12 and (4.83) give us

$$g^{-1}_{y_i}\left(\frac{|\mu|(\Omega_{150r_i}(y^i))}{r_i^{n-1}}\right) \lesssim \delta\lambda. \tag{4.87}$$

Let us denote $G_i(\cdot) = G(x_i, \cdot)$ and $V_i(\cdot) = V(x_i, \cdot)$. For each $i \geq 1$, there exist the weak solution $w_i \in u_{l_i} + W^{1,G}(\Omega_{150r_i}(y^i))$ to (3.31) with $u_{l_i} = u$ and $\Omega_{150r_i}(y^i) = \Omega_{3r}$, and the weak solution $v_i \in \eta w_i + W^{1,G_i}(B^+_{100r_i}(y^i))$ to (3.56) with $w_i = w$ and $B^+_{100r_i}(y^i) = B^+_{2r}$. Here, η is a cut-off function determined in (3.56) with respect to our new coordinate system $(\tilde{y}^i_1, \cdots, \tilde{y}^i_n)$.

By a direct calculation, we find

$$\begin{aligned} |Du_{l_i}| &\leq |Du_{l_i} - Dw_i| + 2\, G_i^{-1}(|V_i(Dw_i) - V_i(Dv_i)|^2) + 2\,|Dv_i| \\ &=: J_i + 2|Dv_i|. \end{aligned} \tag{4.88}$$

It is worth mentioning that (4.86) and (4.87) imply the assumptions on Lemma 3.8 with u_{l_i} in place of u. Then (3.58), Lemma 3.8, and (4.86) yield

$$\fint_{\Omega_{5r_i}(x_i)} |J_i|\,dx \lesssim \fint_{\Omega_{50r_i}(y_i)} |J_i|\,dx \lesssim (\delta^{\frac{p-1}{p}} + \varepsilon)\lambda \lesssim \varepsilon\lambda, \tag{4.89}$$

where the constant $\delta > 0$ from Lemma 3.8 is selected small enough to satisfy $\delta \leq \varepsilon^{p'}$. From Lemma 3.9, we have Lipschitz estimate of v:

$$\|Dv_i\|_{L^\infty(\Omega_{5r_i}(x^i))} \leq \|Dv_i\|_{L^\infty(\Omega_{50r_i}(y^i))} \leq 2c_l\lambda. \tag{4.90}$$

For each $i \geq 1$ and every $x \in E(K\lambda) \cap B_{5r_i}(x_i)$, we use (4.90) to discover

$$\begin{aligned} |Du| &\leq |Du - Du_{l_i}| + J_i + 2|Dv_i| \\ &\leq |Du - Du_{l_i}| + J_i + \frac{K}{2}\lambda \leq |Du - Du_{l_i}| + J_i + \frac{1}{2}|Du|, \end{aligned}$$

and so

$$|Du| \leq 2|Du - Du_{l_i}| + 2J_i \qquad \text{in} \quad E(K\lambda) \cap B_{5r_i}(x_i).$$

It then follows from (4.84) and (4.89) that

$$\begin{aligned} \int_{E(K\lambda)\cap B_{5r_i}(x^i)} |Du|\,dx &\lesssim |\Omega_{5r_i}(x^i)| \fint_{\Omega_{5r_i}(x^i)} |Du - Du_{l_i}|\,dx \\ &\quad + |\Omega_{5r_i}(x^i)| \fint_{\Omega_{5r_i}(x^i)} |J_i|\,dx \\ &\lesssim \varepsilon\lambda|\Omega_{r_i}(x^i)|. \end{aligned} \tag{4.91}$$

We now turn our attention to the interior case, that is, $B_{15r_i}(x^i) \subset \Omega$. Let $w_i \in u_{l_i} + W^{1,G}(B_{15r_i}(x^i))$ be the weak solution to (3.77) and $v_i \in w_i + W^{1,G_i}(B_{10r_i}(x^i))$ the weak solution to (3.78). Correspondingly, in the interior case, one can similarly show (4.91) by using Lemma 3.10 and Lemma 3.11.

Taking (4.85) and (4.91) into account, we obtain

$$\int_{E(K\lambda)} |Du|\,dx \leq \sum_{i\geq 1} \int_{E(K\lambda)\cap B_{5r_i}(x^i)} |Du|\,dx \lesssim \varepsilon\lambda \sum_{i\geq 1} |\Omega_{r_i}(x^i)|. \tag{4.92}$$

To estimate $|\Omega_{r_i}(x^i)|$, we take (4.82) into account to see that either

$$\fint_{\Omega_{r_i}(x^i)} |Du|\,dx \geq \frac{\lambda}{2} \quad \text{or} \quad \frac{1}{\delta}\fint_{\Omega_{r_i}(x^i)} g_x^{-1}(\mathcal{M}_1(\mu))\,dx \geq \frac{\lambda}{2} \tag{4.93}$$

holds. If the first inequality of (4.93) holds, then we have

$$\begin{aligned} \lambda|\Omega_{r_i}(x^i)| &\leq 2\int_{\Omega_{r_i}(x^i)} |Du|\,dx \\ &\leq 2\int_{\Omega_{r_i}(x^i)\cap\{|Du|\geq\frac{\lambda}{4}\}} |Du|\,dx + \frac{1}{2}\lambda|\Omega_{r_i}(x^i)|, \end{aligned}$$

and so

$$|\Omega_{r_i}(x^i)| \leq \frac{4}{\lambda}\int_{\Omega_{r_i}(x^i)\cap\{|Du|\geq\frac{\lambda}{4}\}} |Du|\,dx.$$

Similarly, for the second inequality of (4.93), we have

$$|\Omega_{r_i}(x^i)| \leq \frac{4}{\delta\lambda}\int_{\Omega_{r_i}(x^i)\cap\{g_x^{-1}(\mathcal{M}_1(\mu))\geq\frac{\delta\lambda}{4}\}} g_x^{-1}(\mathcal{M}_1(\mu))\,dx.$$

Remembering that $\{B_{r_i}\}_{i\geq1}$ is the set of mutually disjoint balls, we have

$$\begin{aligned} \int_{E(K\lambda)} |Du|\,dx \lesssim \varepsilon &\int_{\Omega\cap\{|Du|\geq\frac{\lambda}{4}\}} |Du|\,dx \\ &+ \frac{\varepsilon}{\delta}\int_{\Omega\cap\{g_x^{-1}(\mathcal{M}_1(\mu))\geq\frac{\delta\lambda}{4}\}} g_x^{-1}(\mathcal{M}_1(\mu))\,dx. \end{aligned} \tag{4.94}$$

For any large $k \geq KH\lambda_0 > 0$, recalling the truncation operation T_k given in Lemma 3.1, we estimate

$$\begin{aligned}
&\int_\Omega T_k(|Du|)^{q-1}|Du|\,dx \\
&= (q-1)K^{q-1}\int_0^{k/K}\lambda^{q-2}\int_{E(K\lambda)}|Du|\,dx\;d\lambda \\
&= (q-1)K^{q-1}\int_0^{H\lambda_0}\lambda^{q-2}\int_{E(K\lambda)}|Du|\,dx\;d\lambda \\
&\quad + (q-1)K^{q-1}\int_{H\lambda_0}^{k/K}\lambda^{q-2}\int_{E(K\lambda)}|Du|\,dx\;d\lambda =: I_1 + I_2.
\end{aligned} \tag{4.95}$$

A straightforward calculation yields

$$I_1 \le (KH\lambda_0)^{q-1}\int_\Omega |Du|\,dx. \tag{4.96}$$

In light of (4.94), I_2 can be estimated as

$$\begin{aligned}
I_2 &\lesssim \varepsilon(q-1)K^{q-1}\int_{H\lambda_0}^{k/K}\lambda^{q-2}\int_{\Omega\cap\{|Du|\ge\frac{\lambda}{4}\}}|Du|\,dx\,d\lambda \\
&+ \varepsilon(q-1)\frac{K^{q-1}}{\delta}\int_{H\lambda_0}^{k/K}\lambda^{q-2}\int_{\Omega\cap\{g_x^{-1}(\mathcal{M}_1(\mu))\ge\frac{\delta\lambda}{4}\}} g_x^{-1}(\mathcal{M}_1(\mu))\,dx\,d\lambda \\
&=: I_3 + I_4.
\end{aligned} \tag{4.97}$$

By Fubini's theorem, we estimate I_3 and I_4 as follows:

$$\begin{aligned}
I_3 &\lesssim \varepsilon(q-1)\int_0^{4k}\lambda^{q-2}\int_{\Omega\cap\{|Du|\ge\frac{\lambda}{4}\}}|Du|\,dx\,d\lambda \\
&\lesssim \varepsilon(q-1)\int_0^{k}\lambda^{q-2}\int_{\Omega\cap\{|Du|\ge\lambda\}}|Du|\,dx\,d\lambda \\
&\lesssim \varepsilon\int_\Omega T_k(|Du|)^{q-1}|Du|\,dx
\end{aligned} \tag{4.98}$$

and

$$
\begin{aligned}
I_4 &\lesssim \varepsilon(q-1)\int_0^\infty \lambda^{q-2}\int_{\Omega\cap\{g_x^{-1}(\mathcal{M}_1(\mu))\geq\delta\lambda\}} \frac{g_x^{-1}(\mathcal{M}_1(\mu))}{\delta}\,dx\,d\lambda \\
&\lesssim \varepsilon\int_\Omega \left(\frac{g_x^{-1}(\mathcal{M}_1(\mu))}{\delta}\right)^q dx. && (4.99)
\end{aligned}
$$

Combining (4.95)-(4.99), we have

$$
\begin{aligned}
&\int_\Omega T_k(|Du|)^{q-1}|Du|\,dx \\
&\quad\leq c\varepsilon\int_\Omega T_k(|Du|)^{q-1}|Du|\,dx + c\varepsilon\int_\Omega\left(\frac{g_x^{-1}(\mathcal{M}_1(\mu))}{\delta}\right)^q dx \\
&\qquad + c(KH\lambda_0)^{q-1}\int_\Omega |Du|\,dx.
\end{aligned}
$$

At this stage, we take $\varepsilon = \varepsilon(n,p,q,\nu,L) > 0$ small enough to satisfy $c\varepsilon = \frac{1}{2}$. As a consequence, $\delta = \delta(n,p,q,\nu,L) > 0$ is also determined. Letting $k\to\infty$, we obtain

$$
\begin{aligned}
&\int_\Omega |Du|^q dx \\
&\quad\leq c(K\lambda_0)^{q-1}\int_\Omega |Du|dx + c\int_\Omega \left(g_x^{-1}(\mathcal{M}_1(\mu))\right)^q dx \\
&\quad\leq c|\Omega|(KH\lambda_0)^q + \frac{1}{2}\int_\Omega |Du|^q\,dx + c\int_\Omega \left(g_x^{-1}(\mathcal{M}_1(\mu))\right)^q dx. && (4.100)
\end{aligned}
$$

This and (4.81) finally completes the proof. □

Remark 4.2. *With an additional calculation, we have L^q estimate of Du only in terms of $g_x^{-1}(\mathcal{M}_1(\mu))$ and $|\mu|(\Omega)$. We present the calculation below.*

Using (4.81) *and recalling Remark 3.3, we have*

$$
|\Omega|(KH\lambda_0)^q \leq c\frac{(KH)^q}{|\Omega|^{q-1}}|\mu|(\Omega)^{\frac{q}{p-1}} + c(KH)^q\int_\Omega \left(g_x^{-1}(\mathcal{M}_1(\mu))\right)^q dx.
$$

It then follows from (4.100) *in the proof of Theorem 1.1 that*

$$\int_\Omega |Du|^q dx \le c \int_\Omega \left(g_x^{-1}(\mathcal{M}_1(\mu)) + \frac{|\mu|(\Omega)^{\frac{1}{p-1}}}{|\Omega|^{1/q'}} \right)^q dx.$$

Chapter 3

Maximal differentiability for a general class of quasilinear elliptic equations with right-hand side measures

1 Introduction and main result

In this chapter, we consider the following elliptic equation with a finite Radon measure μ on the right-hand side:

$$\begin{cases} -\operatorname{div} \mathcal{A}(Du) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^n, n \geq 2$. The vector field $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be C^1-regular and satisfies the following ellipticity and growth assumptions:

$$\begin{cases} |\mathcal{A}(\xi)| + |\partial \mathcal{A}(\xi)||\xi| \leq Lg(|\xi|) \\ \nu \frac{g(|\xi|)}{|\xi|}|\zeta|^2 \leq \langle \partial \mathcal{A}(\xi)\zeta, \zeta \rangle \end{cases} \tag{1.2}$$

for all $\xi, \zeta \in \mathbb{R}^n$, with constants $0 < \nu \leq L$. Here, $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a $C^2((0, \infty)) \cap C^1([0, \infty))$ function satisfying

$$1 \leq \gamma_1 - 1 \leq \frac{tg'(t)}{g(t)} \leq \gamma_2 - 1 < \infty \tag{1.3}$$

with constants $\gamma_1, \gamma_2 \in \mathbb{R}$. We point out that (1.1) reduces to p-Laplace type equation when $\gamma_1 = \gamma_2 = p$. For this reason, our equation under the assumptions (1.2) and (1.3) is regarded to be a generalized p-Laplace equation. Some typical examples of $g(t)$ satisfying (1.3) are

$$
\begin{aligned}
&g(t) = t^{p-1}\big(\log(e+t)\big)^{\beta} \quad (p \geq 2, \beta \geq 1)\\
&g(t) = t^{p-1} + a_0 t^{q-1} \quad (p, q \geq 2,\ a_0 \geq 0)\,.
\end{aligned}
$$

Our problem features a nonstandard growth condition, the so-called Orlicz growth condition. One can notice that γ_1 and γ_2 in (1.3) control the speed of the decay and growth, respectively. Thus, the rate of growth and decay of $\mathcal{A}(Du)$ varies depending on $|Du|$. It does not increase too fast nor too slow. The regularity theory of elliptic equations with Orlicz growth has been widely studied in many literatures, we refer to [33, 42, 45, 50] for an overview and a further discussion regarding the Orlicz growth condition.

When μ on the right-hand side of (1.1) is merely a bounded Radon measure, the notion of the solution so-called SOLA (Solution Obtained by Limits of Approximations) is usually employed. Its precise definition will be described in Section 2.4. A SOLA is originated from the seminal papers [14, 15], and existence and regularity results are proved there. Since then, several regularity results regarding SOLA are obtained, for instance fractional differentiability results [2, 70], Calderón-Zygmund estimates [18, 19, 73] and potential estimates [17, 26, 25, 57, 60].

In the type of the equation (1.1), much attention has been paid to the regularity of the nonlinearity $\mathcal{A}(Du)$ recently, instead of Du or $V(Du)$, where $V(\cdot)$ is defined by

$$
V(\xi) := \sqrt{\frac{g(|\xi|)}{|\xi|}}\,\xi \quad \text{for } \xi \in \mathbb{R}^n. \tag{1.4}
$$

It is proved in [29] that, when $\mathcal{A}(\xi) := \frac{g(|\xi|)}{|\xi|}\xi$, $\mu \in L^2$ implies that $\mathcal{A}(Du) \in W^{1,2}$ for $1 < \gamma_1 \leq \gamma_2 < \infty$. When $g(t) = t^{p-1}$ and the forcing term is given by divergence type, i.e. $\mu = \operatorname{div} F$, it is proved in [44] that $F \in BMO$ implies that $\mathcal{A}(Du) \in BMO$. In [13], the authors measure differentiability of $\mathcal{A}(Du)$ in the scale of Besov and Triebel-Lizorkin spaces when F belongs to the same function spaces.

CHAPTER 3. MAXIMAL DIFFERENTIABILITY FOR A GENERAL CLASS OF QUASILINEAR ELLIPTIC EQUATIONS WITH RIGHT-HAND SIDE MEASURES

In the very interesting paper [2], it is proved that the following implication

$$\mu \in L^1 \implies \mathcal{A}(Du) \in W^{\sigma,1} \tag{1.5}$$

for all $\sigma \in (0,1)$, when $g(t) = t^{p-1}$ and $p > 2 - 1/n$. This result is noteworthy because in the Poisson equation $-\Delta u = -\mathrm{div}\,(Du) = \mu$, the classical Calderón-Zygmund theory states that

$$\mu \in L^q \implies Du \in W^{1,q}$$

for all $q \in (1, \infty)$, but if $q = 1$, one can show that

$$\mu \in L^1 \;\not\Longrightarrow\; Du \in W^{1,1} \quad \text{but} \quad \mu \in L^1 \implies Du \in W^{\sigma,1}$$

for all $\sigma \in (0,1)$. Therefore the implication (1.5) generalizes a limiting case of the classical Calderón-Zygmund theory to the nonlinear equation. Then it is natural to ask whether (1.5) still holds under more general growth conditions as in (1.1). The main purpose of this chapter is to prove (1.5) under superquadratic Orlicz growth condition.

For the simplicity of notation, we write

$$\texttt{data} = \{n, \nu, L, \gamma_1, \gamma_2\}.$$

We now state the main result.

Theorem 1.1. *Let $u \in W_0^{1,g}(\Omega)$ be a SOLA to* (1.1) *under the assumptions* (1.2) *and* (1.3). *Then there holds*

$$\mathcal{A}(Du) \in W^{\sigma,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^n) \quad \forall \sigma \in (0,1).$$

Moreover, there exists $c = c(\boldsymbol{\texttt{data}}, \sigma) > 0$ satisfying

$$\begin{aligned} &\fint_{B_{R/2}} \int_{B_{R/2}} \frac{|\mathcal{A}(Du(x)) - \mathcal{A}(Du(y))|}{|x-y|^{n+\sigma}}\,dx\,dy \\ &\qquad \le \frac{c}{R^\sigma} \fint_{B_R} |\mathcal{A}(Du)|\,dx + \frac{c}{R^\sigma}\left(\frac{|\mu|(B_R)}{R^{n-1}}\right) \end{aligned} \tag{1.6}$$

for all $B_R \subset\subset \Omega$.

Let us briefly summarize the contents of this chapter. In Section 2, we provide some notations and preliminary results. Regularity results of the homogeneous equation are studied, and a variant of Caccioppoli-type inequality (3.39) is proved in Section 3. We prove comparison estimates in Section 4 using a linearization approach based on [2] and [7, Lemma 6.1]. Finally in Section 5, we complete the proof of Theorem 1.1 using the iteration and scaling argument.

2 Preliminaries

2.1 General notation

Throughout this chapter, $c \geq 1$ denotes a positive constant depending only on `data` which may vary from line to line. The notation $X \lesssim Y$ implies that there exists some constant $c \geq 1$ satisfying $X \leq cY$ and the notation $X \approx Y$ implies that $X \lesssim Y$ and $Y \lesssim X$. We denote the open ball with radius $R > 0$ and center $x_0 \in \mathbb{R}^n$ by

$$B_R(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < R\}.$$

When the center is clear from the context, we will omit it. Also if B is a ball with radius $r > 0$ and $\sigma > 0$ is a positive number, σB denotes the concentric ball with radius σr. For any measurable set $\mathcal{O} \subset \mathbb{R}^n$ and a measurable function $f : \mathcal{O} \to \mathbb{R}$, we denote the integral average by

$$(f)_{\mathcal{O}} := \fint_{\mathcal{O}} f \, dx := \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} f \, dx.$$

For an open set $\Omega \subset \mathbb{R}^n$, we will identify $L^1(\Omega)$ functions with (signed) measures by denoting

$$|\mu|(\mathcal{O}) = \int_{\mathcal{O}} |\mu| \, dx \quad (\mu \in L^1(\Omega),\, \mathcal{O} \subset \Omega).$$

2.2 Basic properties of a function $g(\cdot)$ and vector fields $V(\cdot)$ and $\mathcal{A}(\cdot)$

As a direct consequence of (1.3), we have

$$\min\{s^{\gamma_1-1}, s^{\gamma_2-1}\}g(t) \le g(st) \le \max\{s^{\gamma_1-1}, s^{\gamma_2-1}\}g(t) \quad \forall s,t>0. \tag{2.7}$$

Since g is strictly increasing, the inverse function $g^{-1}:\mathbb{R}^+ \to \mathbb{R}^+$ exists. Replacing t with $g^{-1}(t)$, s with s^{γ_1-1} and s^{γ_2-1} in (2.7), we have

$$\min\left\{s^{\frac{1}{\gamma_1-1}}, s^{\frac{1}{\gamma_2-1}}\right\} g^{-1}(t) \le g^{-1}(st) \le \max\left\{s^{\frac{1}{\gamma_1-1}}, s^{\frac{1}{\gamma_2-1}}\right\} g^{-1}(t) \tag{2.8}$$

for all $s,t>0$. Also, using the fact that $\gamma_1 \ge 2$, we have

$$\frac{d}{dt}\left(\frac{g(t)}{t}\right) = \frac{tg'(t)-g(t)}{t^2} \ge \frac{(\gamma_1-2)g(t)}{t^2} \ge 0 \quad \forall t>0,$$

which implies that the mapping

$$t \longmapsto \frac{g(t)}{t} \quad \text{is an increasing function.} \tag{2.9}$$

Moreover, we assume that $g(t)$ is a convex $C^2((0,\infty))$ function and $g(1)=1$ throughout this chapter. Otherwise, we consider a convex function $\tilde{g} \in C^2((0,\infty))$ defined as

$$\tilde{g}(t) := \left(\int_0^1 \frac{g(s)}{s}\,ds\right)^{-1} \int_0^t \frac{g(s)}{s}\,ds \approx g(t)$$

instead of $g(t)$. One can notice that $\tilde{g}$ satisfies the assumption (1.3).

We define a function $G \in C^2((0,\infty))$ by

$$G(t) := \int_0^t g(s)\,ds. \tag{2.10}$$

By (2.7), convexity of $g(t)$ and the fact that $G(t) \approx tg(t)$, we have the following subadditive property for both $G(t)$ and $g(t)$:

$$G(s+t) \lesssim G(s)+G(t) \quad \text{and} \quad g(s+t) \lesssim g(s)+g(t) \quad \forall s,t>0. \tag{2.11}$$

Now it is straightforward to show that

$$g'(s+t) \lesssim \frac{g(s+t)}{s+t} \overset{(2.11)}{\lesssim} \frac{g(s)+g(t)}{s+t} \lesssim \frac{g(s)}{s} + \frac{g(t)}{t} \approx g'(s) + g'(t). \quad (2.12)$$

We define a conjugate of G by

$$G^*(t) := \sup_{s>0}\{st - G(s)\}$$

and define $g^*(t)$ similarly. From the definition above, one can observe that

$$st \le G(t) + G^*(s) \quad \text{and} \quad st \le g(t) + g^*(s).$$

The inequality above and (2.7) give

$$st \le \varepsilon g(t) + c(\gamma_1, \gamma_2, \varepsilon) g^*(s).$$

It is well-known, see for instance [76, Section 2.3], that when $\gamma_1 > 2$, $g^*(t)$ satisfies the following inequalities:

$$\begin{aligned} 1 < \frac{\gamma_2 - 1}{\gamma_2 - 2} \le \frac{t(g^*)'(t)}{g^*(t)} \le \frac{\gamma_1 - 1}{\gamma_1 - 2} < \infty, \\ \min\left\{s^{\frac{\gamma_2-1}{\gamma_2-2}}, s^{\frac{\gamma_1-1}{\gamma_1-2}}\right\} g^*(t) \le g^*(st) \le \max\left\{s^{\frac{\gamma_2-1}{\gamma_2-2}}, s^{\frac{\gamma_1-1}{\gamma_1-2}}\right\} g^*(t). \end{aligned} \quad (2.13)$$

for all $s, t > 0$. On the other hand when $2 = \gamma_1 < \gamma_2$, the following estimates hold:

$$\begin{aligned} 1 < \frac{\gamma_2 - 1}{\gamma_2 - 2} \le \frac{t(g^*)'(t)}{g^*(t)} \\ g^*(st) \le s^{\frac{\gamma_2-1}{\gamma_2-2}} g^*(t) \quad \forall t > 0,\ 0 \le \forall s \le 1. \end{aligned} \quad (2.14)$$

Moreover, using the definition of the conjugate and (1.3), one can show that

$$G^*(g(t)) \approx G(t) \quad \text{and} \quad g^*(g'(t)) \approx g(t). \quad (2.15)$$

We refer to [76] for a further discussion of properties of g.

Next we provide some important properties of $V(\cdot)$ and $\mathcal{A}(\cdot)$. Using (1.2), we observe that for all $z_1, z_2 \in \mathbb{R}^n$,

$$|V(z_1) - V(z_2)|^2 \approx \frac{g(|z_1| + |z_2|)}{|z_1| + |z_2|}|z_1 - z_2|^2 \approx g'(|z_1| + |z_2|)|z_1 - z_2|^2, \quad (2.16)$$
$$\langle \mathcal{A}(z_1) - \mathcal{A}(z_2), z_1 - z_2 \rangle \geq cg'(|z_1| + |z_2|)|z_1 - z_2|^2, \quad (2.17)$$
$$|\mathcal{A}(z_1) - \mathcal{A}(z_2)| \leq cg'(|z_1| + |z_2|)|z_1 - z_2|. \quad (2.18)$$

For the proof of the above inequalities, we refer to [42, Lemma 3] and [7, Section 2]. Using (2.18) and (2.17), we find

$$g(|z|)|z| \leq c\langle \mathcal{A}(z), z \rangle \leq c|\mathcal{A}(z)||z|.$$

Dividing both side by $|z|$, it follows that

$$g(|z|) \leq c|\mathcal{A}(z)| \quad \forall z \in \mathbb{R}^n \setminus \{0\}. \quad (2.19)$$

Moreover, we have

$$\begin{aligned} |\mathcal{A}(z_1) - \mathcal{A}(z_2)| &\overset{(2.18)}{\leq} cg'(|z_1| + |z_2|)|z_1 - z_2| \\ &\overset{(2.9)}{\leq} cg'(|z_1 - z_2| + |z_1|)|z_1 - z_2| \\ &\overset{(2.12)}{\leq} cg'(|z_1 - z_2|)|z_1 - z_2| + cg'(|z_1|)|z_1 - z_2| \\ &\overset{(1.3)}{\leq} cg(|z_1 - z_2|) + cg'(|z_1|)|z_1 - z_2|. \end{aligned} \quad (2.20)$$

2.3 Function spaces

In this subsection, we will introduce related function spaces in this chapter.

Definition 2.1. *(Orlicz space) For a function G given in* (2.10)*, we define the Orlicz space $L^G(\Omega) := \{f \in L^1(\Omega) : \int_\Omega G(|f|)\, dx < \infty\}$, with the following (Luxemburg) norm:*

$$\|f\|_{L^G(\Omega)} := \inf\left\{\lambda > 0 : \int_\Omega G\left(\frac{|f|}{\lambda}\right)\, dx \leq 1\right\}.$$

We define an Orlicz-Sobolev space $W^{1,G}(\Omega) := \{f \in W^{1,1}(\Omega) : |Df|, f \in$

$L^G(\Omega)\}$ *with the norm* $\|\cdot\|_{W^{1,G}(\Omega)}$ *defined by*

$$\|f\|_{W^{1,G}(\Omega)} := \|f\|_{L^G(\Omega)} + \||Df|\|_{L^G(\Omega)} < \infty.$$

As in [76, Chapter III], $L^G(\Omega)$ and $W^{1,G}(\Omega)$ are Banach spaces. We similarly define Banach spaces $L^g(\Omega)$ and $W^{1,g}(\Omega)$.

Next, we introduce a fractional function space to measure differentiability.

Definition 2.2. *(Fractional Sobolev space) Let* $\alpha \in (0,1)$*,* $q \in [1,\infty)$*,* $k \in \mathbb{N}$ *and let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with* $n \geq 2$*. We say* f *belongs to the fractional Sobolev space,* $W^{\alpha,q}(\Omega)$*, if and only if* f *is a measurable function and the following Gagliardo-type norm of* f *is finite:*

$$\begin{aligned}\|f\|_{W^{\alpha,q}(\Omega)} &:= \left(\int_\Omega |f(x)|^q\,dx\right)^{1/q} + \left(\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^q}{|x-y|^{n+\alpha q}}\,dx\,dy\right)^{1/q} \\ &=: \|f\|_{L^q(\Omega)} + [f]_{W^{\alpha,q}(\Omega)} < \infty.\end{aligned}$$

From the definition of fractional Sobolev spaces, it is straightforward to show that

$$W^{1,q}(\Omega) \subsetneq W^{t,q}(\Omega) \subsetneq W^{s,q}(\Omega) \subsetneq L^q(\Omega), \quad 0 < s < t < 1.$$

Fractional Sobolev spaces have their own Poincaré inequality: for all $f \in W^{\alpha,q}(B_R)$, we have

$$\int_{B_R} \left|f - (f)_{B_R}\right|^q\,dx \leq cR^{\alpha q}\int_{B_R}\int_{B_R} \frac{|f(x)-f(y)|^q}{|x-y|^{n+\alpha q}}\,dx\,dy \tag{2.21}$$

for some constant $c = c(n,q,\alpha)$. The proof of (2.21) can be found in [54] and [68, Section 4]. For a further discussion about fractional Sobolev spaces, we refer to [41].

For a vector $h \in \mathbb{R}^n$, $x \in \Omega$ and $f \in L^1(\Omega)$, we denote $\Omega_{|h|} := \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) > |h|\}$ and define a different quotient $\tau_h f(x) := f(x+h) - f(x)$ in $\Omega_{|h|}$. The following proposition measures fractional Sobolev norm in terms of the integral of the difference quotient, whose proof can be found in [1, Chapter 7].

Proposition 2.3. *Let* $f \in L^q(\Omega), q \geq 1$*, and assume that for* $\tilde{\alpha} \in (0,1]$*,*

$S \geq 0$ and an open and bounded set $\tilde{\Omega} \subset\subset \Omega$, we have

$$\|\tau_h f\|_{L^q(\tilde{\Omega})} \leq S|h|^{\tilde{\alpha}},$$

for every $h \in \mathbb{R}^n$ with $0 < |h| \leq d$, where $d = \frac{1}{2}\operatorname{dist}(\tilde{\Omega}, \partial\Omega)$. Then $f \in W^{\alpha,q}(\tilde{\Omega})$ for all $\alpha \in (0, \tilde{\alpha})$. Moreover, the following estimate holds true:

$$\|f\|_{W^{\alpha,q}(\tilde{\Omega})} \leq c\left(\frac{d^{\tilde{\alpha}-\alpha}S}{[(\tilde{\alpha}-\alpha)q]^{1/q}} + \frac{\|f\|_{L^q(\tilde{\Omega})}}{\min\{d^{n/q+\alpha}, 1\}}\right).$$

If $f \in W^{1,q}(\Omega)$, then for any concentric balls $B_r \subset\subset B_R \subset\subset \Omega$, and a vector $h \in \mathbb{R}^n$ satisfying $|h| \leq R - r$, we have

$$\int_{B_r} |\tau_h f|^q \, dx \leq c(n,q)|h|^q \int_{B_R} |Df|^q \, dx. \tag{2.22}$$

2.4 Solution Obtained as Limit of Approximations (SOLA)

Definition 2.4. *We say that $u \in W^{1,g}(\Omega)$ is a SOLA to (1.1) under the assumptions (1.2) and (1.3), if there exists a sequence of weak solutions $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,G}(\Omega)$ to*

$$\begin{cases} -\operatorname{div} \mathcal{A}(Du_k) = \mu_k & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.23}$$

such that $u_k \to u$ in $W^{1,g}(\Omega)$. Here, $\{\mu_k\}_{k\in\mathbb{N}} \subset L^\infty(\Omega) \subset \left(W^{1,G}(\Omega)\right)^*$ is a sequence of functions which converges to μ weakly in the sense of measures and it satisfies*

$$\limsup_k |\mu_k|(B) \leq |\mu|(\bar{B})$$

for every measurable set $B \subset \Omega$.

The existence of a SOLA in Definition 2.4 can be found in [7, Section 7]. It is also worth mentioning that the uniqueness of a SOLA still remains as an open problem, even with the standard growth condition except $p = 2, n$.

We will use the following lemma later in Section 5.

Lemma 2.5. *Let $u \in W^{1,g}(\Omega)$ be a SOLA to* (1.1) *under the assumptions* (1.2) *and* (1.3)*. Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence of approximating solutions of* (2.23)*. Then we have*

$$\mathcal{A}(Du_k) \to \mathcal{A}(Du) \quad \text{strongly in } L^1(\Omega) \text{ up to a subsequence.} \tag{2.24}$$

Proof. Since $Du_k \to Du$ strongly in $L^1(\Omega)$, there exists a subsequence of $\{Du_k\}_{k\in\mathbb{N}}$ that converges to Du almost everywhere. Also, as a result of [29, Lemma 4.5], there exists a decreasing function $\eta : [0, |\Omega|] \to [0, \infty)$, depending only on $\texttt{data}$, $|\Omega|, G$ and $|\mu|(\Omega)$ so that

$$\lim_{s\to 0^+} \eta(s) = 0 \tag{2.25}$$

and

$$\int_{\mathcal{O}} |\mathcal{A}(Du_k)|\, dx \le \eta(|\mathcal{O}|) \quad \text{for every measurable set } \mathcal{O} \subset \Omega. \tag{2.26}$$

By (2.25) and (2.26), it is straightforward to show that $\{\mathcal{A}(Du_k)\}_{k\in\mathbb{N}}$ is uniformly integrable in Ω. Therefore we use Vitali convergence theorem to complete our proof. $\square$

With $B_R \subset\subset \Omega$, we will end this subsection with comparison estimates between a solution of (1.1) and the solution of the following homogeneous equation:

$$\begin{cases} -\operatorname{div} \mathcal{A}(Dv) = 0 & \text{in } B_R \\ v = u & \text{on } \partial B_R. \end{cases} \tag{2.27}$$

Lemma 2.6. *[7, Lemma 5.1] Let u and v be solutions of* (1.1) *and* (2.27)*, respectively. Then, the following comparison estimates hold true:*

$$\fint_{B_R} |Du - Dv|\, dx \le c g^{-1}\left(\frac{|\mu|(B_R)}{R^{n-1}}\right),$$
$$\fint_{B_R} g(|Du - Dv|)\, dx \le c\left(\frac{|\mu|(B_R)}{R^{n-1}}\right),$$

where $c = c(\texttt{data})$.

To ensure the existence and the uniqueness of the solution of (2.27), we need to assume that u belongs to the energy space. Therefore, until the end of Section 4, we will assume that $u \in W^{1,G}(\Omega)$ and we will provide a limiting procedure in Subsection 5.1.

3 Regularity of homogeneous equation

In this section, we study a homogeneous equation

$$-\operatorname{div} \mathcal{A}(Dw) = 0 \tag{3.28}$$

in a bounded open subset $U \subset \mathbb{R}^n$, where $\mathcal{A}(\cdot)$ satisfies the structure assumptions (1.2) and (1.3). Many regularity results of a solution of (3.28) are well-known, including De Giorgi type estimates and $C^{1,\alpha}$ regularity results, see for instance [64, 65]. We present regularity results for (3.28) from [7, Lemma 4.1].

Lemma 3.1. *Let $w \in W^{1,G}(U)$ be a solution of (3.28) under the assumptions (1.2) and (1.3). Then for every ball $B_R \subset\subset U$ the following estimate holds:*

$$\sup_{B_{R/2}} |Dw| \leq c \fint_{B_R} |Dw| \, dx. \tag{3.29}$$

In addition, $w \in C^{1,\alpha}(U)$ for some $\alpha \in (0,1)$ with the following excess decay estimates

$$\fint_{B_r} |Dw - (Dw)_{B_r}| \, dx \leq c \left(\frac{r}{R}\right)^{\alpha} \fint_{B_R} |Dw - (Dw)_{B_R}| \, dx,$$

where B_r and B_R are concentric balls with radius $0 < r < R \leq 1$, respectively. Finally, we have

$$|Dw(x_1) - Dw(x_2)| \leq c \left(\frac{r}{R}\right)^{\alpha} \fint_{B_R} |Dw| \, dx \quad \forall x_1, x_2 \in B_{r/2}. \tag{3.30}$$

Next, we present the reverse Hölder-type result of $V(Dw)$, where $V(\cdot)$ is a vector field defined in (1.4). Before that, we need the following auxiliary lemma, see for instance [48].

Lemma 3.2. *Let $f : U \to \mathbb{R}^k$ be a measurable map, and $\chi_0 > 1, c > 0$*

satisfying

$$\left(\fint_{B_{R/2}} |f|^{\chi_0}\,dx\right)^{\frac{1}{\chi_0}} \le c \fint_{B_R} |f|\,dx$$

for all $B_R \subset U$. *Then, for every* $t \in (0,1]$, *there exists some constant* $c_t = c_t(t, \boldsymbol{data}) > 0$ *such that*

$$\left(\fint_{B_{R/2}} |f|^{\chi_0}\,dx\right)^{\frac{1}{\chi_0}} \le c_t \left(\fint_{B_R} |f|^t\,dx\right)^{\frac{1}{t}}.$$

Lemma 3.3. *Let* $w \in W^{1,G}(U)$ *be a solution of* (3.28). *Then there exists a positive constant* $c_t = c_t(t, \boldsymbol{data}) > 0$ *satisfying*

$$\fint_{B_{R/2}} |V(Dw) - V(z_0)|^2\,dx \le c_t \left(\fint_{B_R} |V(Dw) - V(z_0)|^{2t}\,dx\right)^{\frac{1}{t}} \tag{3.31}$$

for all $t > 0$ *and* $B_R \subset\subset U$.

Proof. As a result of [45, Lemma 3.4], there exists some $\beta \in (0,1)$ depending only on `data` such that

$$\fint_{B_{R/2}} |V(Dw) - V(z_0)|^2\,dx \le c \left(\fint_{B_R} |V(Dw) - V(z_0)|^{2\beta}\,dx\right)^{\frac{1}{\beta}}. \tag{3.32}$$

When $0 < t < \beta$, the inequality (3.31) directly follows from (3.32) and Lemma 3.2. When $t \ge \beta$, we use Hölder's inequality to complete this lemma. □

We also provide some higher differentiability results. In general Dw is not differentiable, even with the standard growth condition, see [68] for related results. However, the nonlinear vector field $V(Dw)$ have a higher differentiability. Similar results can be found in [2, Lemma 4.1] for the p-Laplace type equations.

Lemma 3.4. *Let* $w \in W^{1,G}(U)$ *be a weak solution of* (1.1) *under the structure assumptions* (1.2) *and* (1.3). *Then we have*

$$V(Dw) \in W^{1,2}_{\mathrm{loc}}(U).$$

Moreover, for any $B_{4R} \subset\subset U$ and $t > 0$, there exist positive constants $c_t = c_t(t, \texttt{data})$ and $c = c(\texttt{data})$ satisfying

$$\fint_{B_R} |DV(Dw)|^2 \, dx \le \frac{c_t}{R^2}\Big(\fint_{B_{2R}} |V(Dw)|^t \, dx\Big)^{\frac{2}{t}} \tag{3.33}$$

and

$$\fint_{B_R} |DV(Dw)|^2 \, dx \le \frac{c}{R^2} \sup_{B_{2R}} g'(|Dw|) \fint_{B_{2R}} |Dw - z_0|^2 \, dx, \tag{3.34}$$

where z_0 is an arbitrary vector in $\mathbb{R}^n$. Furthermore, we have

$$\begin{aligned} &\sup_{0<|h|<R/16} \fint_{B_{R/2}} \frac{|\tau_h \mathcal{A}(Dw)|}{|h|} \, dx \\ &\qquad \le c \sup_{B_R} g'(|Dw|)^{\frac{1}{2}} \left(\fint_{B_R} |DV(Dw)|^2 \, dx\right)^{\frac{1}{2}} \end{aligned} \tag{3.35}$$

for some positive constant $c = c(\texttt{data})$.

Proof. As a result of [42, Lemma 11], we have

$$\Big(\fint_{B_R} |DV(Dw)|^2 \, dx\Big)^{\frac{1}{2}} \le \frac{c}{R} \left(\fint_{B_{2R}} |V(Dw)|^2 \, dx \right)^{\frac{1}{2}}. \tag{3.36}$$

By (3.31) and (3.36), we have (3.33). Next, let us show (3.34). Let $\eta \in C_0^\infty(B_{2R})$ be a cut-off function satisfying $0 \le \eta \le 1$, $\eta := 1$ on $B_{3R/2}$, $\eta := 0$ on $B_{2R} \setminus B_{7R/4}$, and $|\nabla \eta| \le 8/R$ and $P := z_0 \cdot x$ be a linear function where z_0 is an arbitrary vector in $\mathbb{R}^n$. For a given vector $h \in \mathbb{R}^n \setminus \{0\}$ with $|h| \le R/16$, we test $\tau_{-h}\left(\eta^2 \tau_h(w - P)\right)$ with (3.28) in order to find

$$\begin{aligned} 0 &= \fint_{B_{2R}} \langle \tau_h \mathcal{A}(Dw), D\left(\eta^2 \tau_h(w-P)\right)\rangle \, dx \\ &= \fint_{B_{2R}} \langle \tau_h \mathcal{A}(Dw), \eta^2 \tau_h Dw\rangle \, dx + 2 \fint_{B_{2R}} \langle \tau_h \mathcal{A}(Dw), \eta D\eta \tau_h(w-P)\rangle \, dx. \end{aligned}$$

Note that we have used the fact that $D\tau_h P = 0$. The equality above implies that

$$\begin{aligned}
&\fint_{B_{2R}} \langle \tau_h \mathcal{A}(Dw), \eta^2 \tau_h Dw \rangle \, dx \\
&\qquad = -2 \fint_{B_{2R}} \langle \tau_h \mathcal{A}(Dw), \eta D\eta \tau_h (w - P) \rangle \, dx.
\end{aligned} \tag{3.37}$$

Using Young's inequality with $\delta > 0$, Jensen's inequality and Fubini's theorem, we have the followings:

$$\begin{aligned}
&\fint_{B_{2R}} \eta^2 |\tau_h V(Dw)|^2 \, dx \\
&\overset{\substack{(2.16)\\(2.17)}}{\leq} c \fint_{B_{2R}} \langle \tau_h \mathcal{A}(Dw), \eta^2 \tau_h Dw \rangle \, dx \\
&\overset{(3.37)}{\leq} \frac{c}{R} \fint_{B_{2R}} \eta |\tau_h \mathcal{A}(Dw)| |\tau_h (w - P)| \, dx \\
&\overset{(2.18)}{\leq} \frac{c}{R} \fint_{B_{2R}} \eta g'(|Dw(x)| + |Dw(x+h)|) |Dw(x+h) - Dw(x)| |\tau_h (w - P)| \, dx \\
&\overset{(2.12)}{\leq} \delta \fint_{B_{2R}} \eta^2 g'(|Dw(x)| + |Dw(x+h)|) |Dw(x+h) - Dw(x)|^2 \, dx \\
&\qquad + \frac{c_\delta |h|^2}{R^2} \sup_{B_{2R}} g'(|Dw|) \times \\
&\qquad \fint_{B_{3R/2}} \left| \fint_0^{|h|} Dw\left(x + \frac{sh}{|h|}\right) - DP\left(x + \frac{sh}{|h|}\right) \, ds \right|^2 dx \\
&\leq \delta \fint_{B_{2R}} \eta^2 g'(|Dw(x)| + |Dw(x+h)|) |Dw(x+h) - Dw(x)|^2 \, dx \\
&\qquad + \frac{c_\delta |h|^2}{R^2} \sup_{B_{2R}} g'(|Dw|) \times \\
&\qquad \fint_{B_{3R/2}} \fint_0^{|h|} \left| Dw\left(x + \frac{sh}{|h|}\right) - DP\left(x + \frac{sh}{|h|}\right) \right|^2 ds \, dx \\
&\overset{(2.16)}{\leq} c\delta \fint_{B_{2R}} \eta^2 |\tau_h V(Dw)|^2 \, dx + \frac{c_\delta |h|^2}{R^2} \sup_{B_{2R}} g'(|Dw|) \fint_{B_{2R}} |Dw - DP|^2 \, dx \\
&\leq \frac{1}{2} \fint_{B_{2R}} \eta^2 |\tau_h V(Dw)|^2 \, dx + \frac{c|h|^2}{R^2} \sup_{B_{2R}} g'(|Dw|) \fint_{B_{2R}} |Dw - DP|^2 \, dx
\end{aligned}$$

by taking $\delta > 0$ small enough. We divide both side by $|h|^2$ to have

$$\fint_{B_R} \frac{|\tau_h V(Dw)|^2}{|h|^2}\,dx \le \frac{c}{R^2} \sup_{B_{2R}} g'(|Dw|) \fint_{B_{2R}} |Dw - DP|^2 dx.$$

Since $DP = z_0$, the inequality (3.34) follows from the classical difference quotient characterization of Sobolev functions. Finally, we discover

$$\begin{aligned}
&\fint_{B_{R/2}} |\tau_h \mathcal{A}(Dw)|\,dx \\
&\quad\le\; c \fint_{B_{R/2}} g'\big(|Dw(x)| + |Dw(x+h)|\big)|Du(x+h) - Du(x)|\,dx \\
&\quad\le\; c \sup_{B_R} g'(|Dw|)^{\frac{1}{2}} \times \\
&\qquad \fint_{B_{R/2}} g'\big(|Dw(x)| + |Dw(x+h)|\big)^{\frac{1}{2}} |Du(x+h) - Du(x)|\,dx \\
&\quad\overset{(2.16)}{\le}\; c|h| \sup_{B_R} g'(|Dw|)^{\frac{1}{2}} \fint_{B_{R/2}} \frac{|V(Dw(x+h)) - V(Dw(x))|}{|h|}\,dx \\
&\quad\le\; c|h| \sup_{B_R} g'(|Dw|)^{\frac{1}{2}} \left(\fint_{B_{R/2}} \frac{|V(Dw(x+h)) - V(Dw(x))|^2}{|h|^2}\,dx \right)^{\frac{1}{2}} \\
&\quad\overset{(2.22)}{\le}\; c|h| \sup_{B_R} g'(|Dw|)^{\frac{1}{2}} \left(\fint_{B_R} |DV(Dw)|^2\,dx \right)^{\frac{1}{2}}.
\end{aligned}$$

Dividing both side by $|h|$ and taking supremum for all $h \in \mathbb{R}^n \setminus \{0\}$ with $|h| \le R/16$, we finally obtain (3.35). □

Corollary 3.5. *Under the same assumption of Lemma 3.4, we have the following inequalities:*

$$\sup_{0<|h|<R/16} \fint_{B_{R/2}} \frac{|\tau_h \mathcal{A}(Dw)|}{|h|}\,dx \le \frac{c}{R} \sup_{B_{2R}} g'(|Dw|)^{\frac{1}{2}} \fint_{B_{2R}} |V(Dw)|\,dx, \quad (3.38)$$

$$\begin{aligned}\sup_{0<|h|<R/16} \fint_{B_{R/2}} \frac{|\tau_h \mathcal{A}(Dw)|}{|h|}\, dx & \\ &\leq \frac{c}{R} \sup_{B_{2R}} g'(|Dw|) \left(\fint_{B_{2R}} |Dw - z_0|^2 \, dx \right)^{\frac{1}{2}}.\end{aligned} \tag{3.39}$$

Proof. Using (3.31), (3.33) and (3.35), we have

$$\begin{aligned}\sup_{0<|h|<R/16} \fint_{B_{R/2}} \frac{|\tau_h \mathcal{A}(Dw)|}{|h|}\, dx &\leq c \sup_{B_R} g'(|Dw|)^{\frac{1}{2}} \left(\fint_{B_R} |DV(Dw)|^2 \, dx \right)^{\frac{1}{2}} \\ &\leq \frac{c}{R} \sup_{B_R} g'(|Dw|)^{\frac{1}{2}} \left(\fint_{B_{2R}} |V(Dw)|^2 \, dx \right)^{\frac{1}{2}} \\ &\leq \frac{c}{R} \sup_{B_R} g'(|Dw|)^{\frac{1}{2}} \fint_{B_{2R}} |V(Dw)| \, dx\end{aligned}$$

which implies (3.38). Similarly, by (3.34) and (3.35), we find

$$\begin{aligned}\sup_{0<|h|<R/16} \fint_{B_{R/2}} \frac{|\tau_h \mathcal{A}(Dw)|}{|h|}\, dx &\leq c \sup_{B_R} g'(|Dw|)^{\frac{1}{2}} \left(\fint_{B_R} |DV(Dw)|^2 \, dx \right)^{\frac{1}{2}} \\ &\leq \frac{c}{R} \sup_{B_{2R}} g'(|Dw|) \left(\fint_{B_{2R}} |Dw - z_0|^2 \, dx \right)^{\frac{1}{2}}.\end{aligned}$$

Therefore, the inequality (3.39) holds true. □

4 Comparison estimates

Throughout this section, we assume that $\mu \in L^\infty(\Omega)$ and $u \in W^{1,G}(\Omega)$. After discovering desired estimates, we use a limiting argument to complete the proof of Theorem 1.1. Our proof follows the main idea of [2, Section 5].

Let us consider

$$B_{4MR} := B_{4MR}(x_0) \subset\subset \Omega \quad \text{with } M \geq 8 \text{ and } R \leq 1. \tag{4.40}$$

Here, M is a free parameter which will be determined by `data` later. We consider the following Dirichlet problems with different domains:

$$\begin{cases} -\operatorname{div} \mathcal{A}(D\tilde{v}) = 0 & \text{in } B_{MR} \\ \tilde{v} = u & \text{on } \partial B_{MR} \end{cases} \tag{4.41}$$

and

$$\begin{cases} -\operatorname{div} \mathcal{A}(Dv) = 0 & \text{in } B_{2R} \\ v = u & \text{on } \partial B_{2R}. \end{cases} \tag{4.42}$$

As a direct consequence of Lemma 2.6, we are able to discover the followings:

$$\fint_{B_{MR}} g(|Du - D\tilde{v}|)\, dx \le c \left(\frac{|\mu|(B_{MR})}{(MR)^{n-1}} \right), \tag{4.43}$$

$$\fint_{B_{MR}} |Du - D\tilde{v}|\, dx \le c g^{-1} \left(\frac{|\mu|(B_{MR})}{(MR)^{n-1}} \right), \tag{4.44}$$

$$\fint_{B_{2R}} g(|Du - Dv|)\, dx \le c \left(\frac{|\mu|(B_{2R})}{(2R)^{n-1}} \right), \tag{4.45}$$

$$\fint_{B_{2R}} |Du - Dv|\, dx \le c g^{-1} \left(\frac{|\mu|(B_{2R})}{(2R)^{n-1}} \right). \tag{4.46}$$

We also need the following result whose proof can be found in [7, Lemma 6.1].

Lemma 4.1. *Suppose that $u \in W_0^{1,G}(\Omega)$ is a solution of* (1.1) *under the assumptions* (1.2), (1.3) *and*

$$g^{-1} \left(\frac{|\mu|(B_{MR})}{(MR)^{n-1}} \right) + g^{-1} \left(\frac{|\mu|(B_{2R})}{(2R)^{n-1}} \right) \le \lambda \tag{4.47}$$

holds for some $\lambda > 0$. Also, let v and $\tilde{v}$ be the solutions defined in (4.41) *and* (4.42) *together with the bounds*

$$\frac{\lambda}{Q} \le |Dv| \le Q\lambda \quad \text{in } B_{4R/M}, \ \frac{\lambda}{Q} \le |D\tilde{v}| \le Q\lambda \quad \text{in } B_{2R}.$$

Then there exists some constant $c_{Q,M} = c_{Q,M}(Q, M, \texttt{data})$ such that

$$\fint_{B_{2R/M}} |Du - Dv|\, dx \le c_{Q,M} \frac{\lambda}{g(\lambda)} \left(\frac{|\mu|(B_{MR})}{(MR)^{n-1}} \right).$$

The following lemma is the main result of this section.

Lemma 4.2. *Let u be a weak solution of* (1.1) *under the assumptions* (1.2) *and* (1.3). *Then it is possible to choose M satisfying* (4.40) *in terms of* $\texttt{data}$ *so that if $v \in u + W_0^{1,G}(B_{2R})$ is the solution of* (4.42), *then the inequalities*

$$\begin{aligned}&\fint_{B_{2R/M}} |\mathcal{A}(Du) - \mathcal{A}(Dv)|\,dx \\ &\leq cR^{\delta} \fint_{B_{2MR}} |\mathcal{A}(Du) - \big(\mathcal{A}(Du)\big)_{B_{2MR}}|\,dx + c\left(\frac{|\mu|(B_{2MR})}{R^{n-1+\delta(\gamma_2-2)}}\right)\end{aligned} \tag{4.48}$$

and

$$\begin{aligned}&\sup_{0<|h|<R/(8M)} \fint_{B_{R/M}} \frac{|\tau_h \mathcal{A}(Dv)|}{|h|}\,dx \\ &\leq \frac{c}{R} \fint_{B_{2MR}} |\mathcal{A}(Du) - \big(\mathcal{A}(Du)\big)_{B_{2MR}}|\,dx + c\left(\frac{|\mu|(B_{2MR})}{R^{n+\delta(\gamma_2-2)}}\right)\end{aligned} \tag{4.49}$$

hold true.

Let λ be a positive number defined as

$$g(\lambda) := \fint_{B_{2R/M}} g(|Du|)\,dx \tag{4.50}$$

and let $\sigma_1 \in (0, 1/2^n)$ and $\theta \in (0,1)$ be another free parameters depending on an appropriate choice of M. Then our proceeding argument depends on either

$$\begin{aligned}&\fint_{B_{2MR}} |\mathcal{A}(Du) - \big(\mathcal{A}(Du)\big)_{B_{2MR}}|\,dx > \theta \left|\big(\mathcal{A}(Du)\big)_{B_{2R/M}}\right| \\ &\qquad \text{or} \quad \frac{|\mu|(B_{2MR})}{(2MR)^{n-1}} > \sigma_1 g(\lambda)\,,\end{aligned} \tag{4.51}$$

or else

$$\begin{aligned}&\fint_{B_{2MR}} |\mathcal{A}(Du) - \big(\mathcal{A}(Du)\big)_{B_{2MR}}|\,dx \leq \theta \left|\big(\mathcal{A}(Du)\big)_{B_{2R/M}}\right| \\ &\qquad \text{and} \quad \frac{|\mu|(B_{2MR})}{(2MR)^{n-1}} \leq \sigma_1 g(\lambda)\,.\end{aligned} \tag{4.52}$$

The parameters θ and σ_1 will be determined by `data` and M later. Indeed, we are going to select M depending only on `data`, as a consequence, θ and σ_1 depend only on `data`. For details, see Remark 4.6.

Lemma 4.3. *Suppose that* (4.51) *holds. Then for every* $\delta \in (0,1)$*, we have*

$$\begin{aligned} &\fint_{B_{2R/M}} |\mathcal{A}(Du) - \mathcal{A}(Dv)|\,dx \\ &\qquad \le cM^{2n}R^{\delta}\left(1+\frac{1}{\theta}\right)\fint_{B_{2MR}} |\mathcal{A}(Du) - \big(\mathcal{A}(Du)\big)_{B_{2MR}}|\,dx \\ &\qquad\quad + \frac{cM^{2n}}{\sigma_1}\left(\frac{|\mu|(B_{4R})}{(4R)^{n-1+\delta(\gamma_2-2)}}\right) \end{aligned} \tag{4.53}$$

and

$$\begin{aligned} &\sup_{0<|h|<R/(8M)} \fint_{B_{R/M}} \frac{|\tau_h \mathcal{A}(Dv)|}{|h|}\,dx \\ &\qquad \le \frac{cM^{2n}}{R}\left(1+\frac{1}{\theta}\right)\fint_{B_{2MR}} |\mathcal{A}(Du) - \big(\mathcal{A}(Du)\big)_{B_{2MR}}|\,dx \\ &\qquad\quad + \frac{cM^{2n}}{\sigma_1}\left(\frac{|\mu|(B_{4R})}{(4R)^{n+\delta(\gamma_2-2)}}\right) \end{aligned} \tag{4.54}$$

for some constant $c = c(\texttt{data}) > 0$.

Proof. Let $\delta > 0$ be arbitrary given. We observe that

$$\begin{aligned} &\fint_{B_{2R/M}} |\mathcal{A}(Du) - \mathcal{A}(Dv)|\,dx \\ &\qquad \overset{(2.20)}{\le} c\fint_{B_{2R/M}} g'(|Du|)|Du-Dv|\,dx + cM^n \fint_{B_{2R}} g(|Du-Dv|)\,dx \\ &\qquad \overset{(4.45)}{\le} c\fint_{B_{2R/M}} g'(|Du|)|Du-Dv|\,dx + cM^n\left(\frac{|\mu|(B_{2R})}{(2R)^{n-1}}\right). \end{aligned} \tag{4.55}$$

For the first term, we use Young's inequality on g and g^*, (2.7), (2.13), (2.15) and (4.45) to find

$$
\begin{aligned}
&\fint_{B_{2R/M}} g'(|Du|)|Du - Dv|\,dx \\
&\qquad = \fint_{B_{2R/M}} R^{\delta(\gamma_2-2)/(\gamma_2-1)} g'(|Du|)\ \frac{|Du - Dv|}{R^{\delta(\gamma_2-2)/(\gamma_2-1)}}\,dx \\
&\qquad \le \fint_{B_{2R/M}} g^*\left(R^{\delta(\gamma_2-2)/(\gamma_2-1)} g'(|Du|)\right)\,dx + \fint_{B_{2R/M}} g\left(\frac{|Du - Dv|}{R^{\delta(\gamma_2-2)/(\gamma_2-1)}}\right)\,dx \\
&\qquad \le cR^\delta \fint_{B_{2R/M}} g(|Du|)\,dx + \frac{cM^n}{R^{\delta(\gamma_2-2)}} \fint_{B_{2R}} g(|Du - Dv|)\,dx \\
&\qquad \le cR^\delta \fint_{B_{2R/M}} g(|Du|)\,dx + \frac{cM^n}{R^{\delta(\gamma_2-2)}} \left(\frac{|\mu|(B_{2R})}{(2R)^{n-1}}\right)
\end{aligned}
\tag{4.56}
$$

when $\gamma_2 \ge \gamma_1 > 2$. If $\gamma_1 = 2$, we use $(2.14)_1$ instead of (2.13) to find (4.56). When $\gamma_1 = \gamma_2 = 2$, (1.3) implies that $g(t) = t$. Therefore, we have

$$
\begin{aligned}
\fint_{B_{2R/M}} |\mathcal{A}(Du) - \mathcal{A}(Dv)|\,dx &\overset{(2.18)}{\le} c \fint_{B_{2R/M}} |Du - Dv|\,dx \\
&\overset{(4.46)}{\le} cM^n \left(\frac{|\mu|(B_{2R})}{(2R)^{n-1}}\right) \le \frac{cM^n}{R^\delta} \left(\frac{|\mu|(B_{2R})}{(2R)^{n-1}}\right).
\end{aligned}
$$

Therefore, for each case, the inequality (4.56) holds.

When $(4.51)_1$ is in force, we have

$$
\begin{aligned}
&\fint_{B_{2R/M}} g(|Du|)\,dx \\
&\qquad \overset{(1.2)_1}{\le} c \fint_{B_{2R/M}} |\mathcal{A}(Du)|\,dx \\
&\qquad \le c \fint_{B_{2R/M}} |\mathcal{A}(Du) - (\mathcal{A}(Du))_{B_{2R/M}}|\,dx + \left|\left(\mathcal{A}(Du)\right)_{B_{2R/M}}\right| \\
&\qquad \le c \fint_{B_{2R/M}} |\mathcal{A}(Du) - (\mathcal{A}(Du))_{B_{2MR}}|\,dx + \left|\left(\mathcal{A}(Du)\right)_{B_{2R/M}}\right| \\
&\qquad \overset{(4.51)_1}{\le} cM^{2n}\left(1 + \frac{1}{\theta}\right) \fint_{B_{2MR}} \left|\mathcal{A}(Du) - \left(\mathcal{A}(Du)\right)_{B_{2MR}}\right|\,dx.
\end{aligned}
\tag{4.57}
$$

This estimate (4.57), (4.55) and (4.56) assert (4.53). On the other hand, when $(4.51)_2$ is in force, we observe that

$$g(\lambda) = \fint_{B_{2R/M}} g(|Du|)\,dx \le \frac{c}{\sigma_1}\frac{|\mu|(B_{2MR})}{(2MR)^{n-1}}. \tag{4.58}$$

Then (4.53) follows from (4.55), (4.56) and (4.58).

As we have explained in the beginning of Section 2.2, we may assume that g is a convex function. Therefore using Jensen's inequality with g, we find

$$\begin{aligned}
&\sup_{0<|h|<R/(8M)} \fint_{B_{R/M}} \frac{|\tau_h \mathcal{A}(Dv)|}{|h|}\,dx \\
&\overset{(3.38)}{\le} \frac{cM}{R}\left(\sup_{B_{3R/2M}} g'(|Dv|)\right)^{\frac12} \fint_{B_{3R/2M}} |V(Dv)|\,dx \\
&\overset{(1.4)}{\le} \frac{cM}{R}\left(\sup_{B_{3R/2M}} g'(|Dv|)\right)^{\frac12} \fint_{B_{3R/2M}} |Dv| g'(|Dv|)^{\frac12}\,dx \\
&\le \frac{cM}{R}\sup_{B_{3R/2M}} g'(|Dv|) \fint_{B_{2R/M}} |Dv|\,dx \\
&\overset{(2.9)}{\le} \frac{cM}{R} g'\left(\sup_{B_{3R/2M}} |Dv|\right) \fint_{B_{2R/M}} |Dv|\,dx \\
&\overset{(3.29)}{\le} \frac{cM}{R} g'\left(\fint_{B_{2R/M}} |Dv|\,dx\right) \fint_{B_{2R/M}} |Dv|\,dx \\
&\overset{(1.3)}{\le} \frac{cM}{R} g\left(\fint_{B_{2R/M}} |Dv|\,dx\right) \\
&\le \frac{cM}{R}\fint_{B_{2R/M}} g(|Dv|)\,dx \\
&\le \frac{cM}{R}\fint_{B_{2R/M}} g(|Du|)\,dx + \frac{cM}{R}\fint_{B_{2R/M}} g(|Du-Dv|)\,dx.
\end{aligned} \tag{4.59}$$

When $(4.51)_1$ is in force, we use (4.57). On the other hand when $(4.51)_2$ is in force, we use (4.58). In either case, combining (4.45), (4.57), (4.58) and

(4.59), we conclude that (4.54) holds. □

We next choose θ in terms of $\texttt{data}$ and M so that if $(4.52)_1$ holds, then we have a change of scale.

Lemma 4.4. *There exists a positive number $\theta := \theta(\texttt{data}, M)$ such that if $(4.52)_1$ holds, then we have*

$$\fint_{B_{2\kappa R}} g(|Du|)\,dx \le cg(\lambda) \quad \forall \kappa \in [1/M, M] \tag{4.60}$$

for some constant $c > 0$ depending only on $\texttt{data}$.

Proof. Note that

$$\begin{aligned}
&\fint_{B_{2\kappa R}} g(|Du|)\,dx \\
&\overset{(2.19)}{\le} c \fint_{B_{2\kappa R}} |\mathcal{A}(Du)|\,dx \\
&\le\ c \fint_{B_{2\kappa R}} |\mathcal{A}(Du) - (\mathcal{A}(Du))_{B_{2MR}}|\,dx \\
&\qquad + c|(\mathcal{A}(Du))_{B_{2MR}} - (\mathcal{A}(Du))_{B_{2R/M}}| + c|(\mathcal{A}(Du))_{B_{2R/M}}| \\
&\le\ c \fint_{B_{2\kappa R}} |\mathcal{A}(Du) - (\mathcal{A}(Du))_{B_{2MR}}|\,dx \\
&\qquad + c \fint_{B_{2R/M}} |\mathcal{A}(Du) - (\mathcal{A}(Du))_{B_{2MR}}|\,dx + c|(\mathcal{A}(Du))_{B_{2R/M}}| \\
&\le\ c\left[\left(\frac{M}{\kappa}\right)^n + M^{2n}\right] \fint_{B_{2MR}} |\mathcal{A}(Du) - (\mathcal{A}(Du))_{B_{2MR}}|\,dx \\
&\qquad + c|(\mathcal{A}(Du))_{B_{2R/M}}| \\
&\overset{(4.52)_1}{\le} c^*\left(1 + M^{2n}\theta\right)\left|(\mathcal{A}(Du))_{B_{2R/M}}\right|,
\end{aligned} \tag{4.61}$$

for some constant $c^* = c^*(\texttt{data})$, as $\kappa \in [1/M, M]$. We choose $\theta > 0$ small enough depending only on $\texttt{data}$ and M so that $c^* M^{2n}\theta \le 1$. Then, we finally observe that

$$\fint_{B_{2\kappa R}} g(|Du|)\,dx \overset{(4.61)}{\le} c|(\mathcal{A}(Du))_{B_{2R/M}}| \overset{(1.2)}{\le} c \fint_{B_{2R/M}} g(|Du|)\,dx \overset{(4.50)}{\le} cg(\lambda).$$

This completes the proof. □

In the following lemma and Remark 4.6, we show that it is possible to choose parameters σ_1 and M in terms of $\texttt{data}$ simultaneously so that assumptions of Lemma 4.1 are satisfied when (4.52) is in force.

Lemma 4.5. *Let $\theta = \theta(\texttt{data}, M)$ be a parameter determined in Lemma 4.4. Then it is possible to choose $\sigma_1 = \sigma_1(\texttt{data}, M) \in (0, 1/2^n)$ and $M = M(\texttt{data}) \geq 8$ so that if (4.52) is in force, we have*

$$\frac{\lambda}{\tilde{c}_l} \leq |D\tilde{v}| \quad \text{in } B_{2R} \quad \text{and} \quad |D\tilde{v}| \leq \tilde{c}_u \lambda \quad \text{in } B_{MR/2} \tag{4.62}$$

and

$$\frac{\lambda}{c_l} \leq |Dv| \quad \text{in } B_{4R/M} \quad \text{and} \quad |Dv| \leq c_u \lambda \quad \text{in } B_R, \tag{4.63}$$

for some constants $\tilde{c}_l, \tilde{c}_u, c_l$ and c_u depending on $\texttt{data}$.

Proof. We first note that $g^{-1}(\cdot)$ and $g(\cdot)$ are increasing, to discover

$$\begin{aligned}
\sup_{B_{MR/2}} |D\tilde{v}| &\overset{(3.29)}{\leq} cg^{-1}\left(\fint_{B_{MR}} g(|D\tilde{v}|)\,dx\right) \\
&\overset{(2.11)}{\leq} cg^{-1}\left(c\fint_{B_{MR}} g(|Du|)\,dx + c\fint_{B_{MR}} g(|D\tilde{v} - Du|)\,dx\right) \\
&\leq cg^{-1}\left(c\fint_{B_{MR}} g(|Du|)\,dx\right) + g^{-1}\left(c\fint_{B_{MR}} g(|D\tilde{v} - Du|)\,dx\right) \\
&\overset{(2.8)}{\leq} cg^{-1}\left(\fint_{B_{MR}} g(|Du|)\,dx\right) + cg^{-1}\left(\fint_{B_{MR}} g(|D\tilde{v} - Du|)\,dx\right) \\
&\overset{(4.60)}{\leq} c\lambda + cg^{-1}\left(\fint_{B_{MR}} g(|D\tilde{v} - Du|)\,dx\right) \\
&\overset{(4.43)}{\leq} c\lambda + cg^{-1}\left(2^{n-1}\frac{|\mu|(B_{2MR})}{(2MR)^{n-1}}\right) \\
&\overset{(4.52)_2}{\leq} c\lambda + cg^{-1}\left(g(\lambda)\right) \leq c_1\lambda,
\end{aligned}$$

for some constant $c_1 = c_1(\texttt{data})$. Taking $\tilde{c}_u := c_1$, we have the upper bound of (4.62).

Now, let us prove the lower bound of (4.62). By $(1.2)_1$, (2.19) and (4.50), we have

$$\frac{g(\lambda)}{c_2} \leq (|\mathcal{A}(Du)|)_{B_{2R/M}} \leq c_2 g(\lambda) \tag{4.64}$$

for some constant $c_2 := c_2(\mathtt{data}) > 0$. Also, observe that

$$\begin{aligned}(|\mathcal{A}(D\tilde{v})|)_{B_{2R/M}} &\geq (|\mathcal{A}(Du)|)_{B_{2R/M}} - |(\mathcal{A}(D\tilde{v}))_{B_{2R/M}} - (\mathcal{A}(Du))_{B_{2R/M}}| \\ &\overset{(4.64)}{\geq} \frac{g(\lambda)}{c_2} - \fint_{B_{2R/M}} |\mathcal{A}(D\tilde{v}) - \mathcal{A}(Du)|\,dx. \end{aligned} \tag{4.65}$$

But then,

$$
\begin{aligned}
&\fint_{B_{2R/M}} |\mathcal{A}(D\tilde{v}) - \mathcal{A}(Du)|\,dx \\
&\quad \overset{(2.20)}{\le} \; c\fint_{B_{2R/M}} g'(|D\tilde{v}|)|D\tilde{v} - Du|\,dx + c\fint_{B_{2R/M}} g(|D\tilde{v} - Du|)\,dx \\
&\quad \le \; cM^{2n}\fint_{B_{MR/2}} g'(|D\tilde{v}|)|D\tilde{v} - Du|\,dx \\
&\quad\quad + cM^{2n}\fint_{B_{MR}} g(|D\tilde{v} - Du|)\,dx \\
&\quad \overset{(4.62)}{\le} \; cM^{2n}g'(\lambda)\fint_{B_{MR}} |D\tilde{v} - Du|\,dx + cM^{2n}\fint_{B_{MR}} g(|D\tilde{v} - Du|)\,dx \\
&\quad \overset{(4.44)}{\le} \; cM^{2n}g'(\lambda)g^{-1}\left(\frac{|\mu|(B_{MR})}{(MR)^{n-1}}\right) + cM^{2n}\fint_{B_{MR}} g(|D\tilde{v} - Du|)\,dx \\
&\quad \overset{(4.43)}{\le} \; cM^{2n}g'(\lambda)g^{-1}\left(\frac{2^{n-1}|\mu|(B_{2MR})}{(2MR)^{n-1}}\right) + cM^{2n}\left(\frac{|\mu|(B_{2MR})}{(2MR)^{n-1}}\right) \\
&\quad \overset{(2.8)}{\le} \; cM^{2n}g'(\lambda)g^{-1}\left(\frac{|\mu|(B_{2MR})}{(2MR)^{n-1}}\right) + cM^{2n}\left(\frac{|\mu|(B_{2MR})}{(2MR)^{n-1}}\right) \\
&\quad \overset{(4.52)_2}{\le} \; cM^{2n}g'(\lambda)g^{-1}\left(\sigma_1 g(\lambda)\right) + cM^{2n}\sigma_1 g(\lambda) \\
&\quad \overset{(2.8)}{\le} \; cM^{2n}\left(\sigma_1^{\frac{1}{\gamma_2-1}}\lambda g'(\lambda) + \sigma_1 g(\lambda)\right) \\
&\quad \overset{(1.3)}{\le} \; c_{21}M^{2n}\left(\sigma_1^{\frac{1}{\gamma_2-1}} + \sigma_1\right) g(\lambda) \qquad (4.66)
\end{aligned}
$$

for some $c_{21} = c_{21}(\mathtt{data}) > 0$. We choose $\sigma_1 = \sigma_1(\mathtt{data}, M)$ small enough to satisfy

$$
c_{21}M^{2n}\left(\sigma_1^{\frac{1}{\gamma_2-1}} + \sigma_1\right) \le \frac{1}{2c_2}. \qquad (4.67)
$$

Combining (4.65), (4.66) and (4.67), we have

$$
(\mathcal{A}(D\tilde{v}))_{B_{2R/M}} \ge \frac{g(\lambda)}{2c_2}.
$$

This implies that there exists a point $\tilde{x}_0 \in B_{2R/M}$ so that

$$\frac{g(\lambda)}{2c_2} \leq |\mathcal{A}(D\tilde{v}(\tilde{x}_0))| \overset{(1.2)}{\leq} Lg(|D\tilde{v}(\tilde{x}_0)|),$$

equivalently, there exists $c_3 := c_3(\mathtt{data}) > 0$ satisfying

$$c_3\lambda \overset{(2.8)}{\leq} g^{-1}\left(\frac{g(\lambda)}{2Lc_2}\right) \leq |Dv(\tilde{x}_0)|. \tag{4.68}$$

By (3.30), we observe that for any $\tilde{x}_1, \tilde{x}_2 \in B_{2R}$,

$$|D\tilde{v}(\tilde{x}_1) - D\tilde{v}(\tilde{x}_2)| \leq \frac{c}{M^\alpha} \fint_{MR} g(|D\tilde{v}|)\, dx \overset{(4.62)}{\leq} \frac{\tilde{c}_*\lambda}{M^\alpha}$$

for some positive constant $\tilde{c}_* := \tilde{c}_*(\mathtt{data})$, where α is given in Lemma 3.1. Choosing $M \geq 8$ large enough to satisfy

$$M \geq \left(\frac{2\tilde{c}_*}{c_3}\right)^{\frac{1}{\alpha}}, \tag{4.69}$$

we find that for any $\tilde{x} \in B_{2R}$,

$$|D\tilde{v}(\tilde{x})| \geq |D\tilde{v}(\tilde{x}_0)| - |D\tilde{v}(\tilde{x}) - D\tilde{v}(\tilde{x}_0)| \overset{(4.68)}{\geq} c_3\lambda - \frac{\tilde{c}_*\lambda}{M^\alpha} \overset{(4.69)}{\geq} \frac{c_3\lambda}{2}.$$

Indeed, since $\alpha, \tilde{c}_*$ and c_3 only depend on $\mathtt{data}$, it is possible to choose large $M \geq 8$ only in terms of $\mathtt{data}$ so that (4.69) is true. Setting $1/\tilde{c}_l := c_3/2$, we have the lower bound of (4.62).

Now, we prove (4.63). To this end, observe that

$$\begin{aligned}
\sup_{B_R} |Dv| &\overset{(3.29)}{\leq} cg^{-1}\left(\fint_{B_{2R}} g(|Dv|)\,dx\right) \\
&\overset{(2.11)}{\leq} cg^{-1}\left(c\fint_{B_{2R}} g(|Du|)\,dx + c\fint_{B_{2R}} g(|Dv-Du|)\,dx\right) \\
&\overset{(2.8)}{\leq} cg^{-1}\left(\fint_{B_{2R}} g(|Du|)\,dx\right) + cg^{-1}\left(\fint_{B_{2R}} g(|Dv-Du|)\,dx\right) \\
&\overset{(4.60)}{\leq} c\lambda + cg^{-1}\left(\fint_{B_{2R}} g(|Dv-Du|)\,dx\right) \\
&\overset{(4.45)}{\leq} c\lambda + cg^{-1}\left(M^{n-1}\frac{|\mu|(B_{2MR})}{(2MR)^{n-1}}\right) \\
&\overset{(4.52)_2}{\leq} c\lambda + cg^{-1}(M^{n-1}\sigma_1 g(\lambda)) \\
&\overset{(2.8)}{\leq} c\lambda + cM^{\frac{n-1}{\gamma_1-1}}\sigma_1^{\frac{1}{\gamma_2-1}}\lambda \\
&\leq c_4\left(1+M^{\frac{n-1}{\gamma_1-1}}\sigma_1^{\frac{1}{\gamma_2-1}}\right)\lambda
\end{aligned}$$

for some constant $c_4 = c_4(\texttt{data})$. If we choose $\sigma_1 := \sigma_1(\texttt{data}, M) > 0$ small enough so that

$$(2M)^{\frac{n-1}{\gamma_1-1}}\sigma_1^{\frac{1}{\gamma_2-1}} \leq 1, \tag{4.70}$$

and set $c_u := 2c_4$, then the upper bound of (4.63) follows.

To prove the lower bound of (4.63), we proceed as in (4.64)-(4.65) using v instead of $\tilde{v}$ to have

$$(|\mathcal{A}(Dv)|)_{B_{4R/M}} \geq \frac{g(\lambda)}{c_5} - \fint_{B_{4R/M}} |\mathcal{A}(Dv) - \mathcal{A}(Du)|\,dx, \tag{4.71}$$

for some constant $c_5 := c_5(\texttt{data}) > 0$. Then one can find

$$\begin{aligned}
&\fint_{B_{4R/M}} |\mathcal{A}(Dv) - \mathcal{A}(Du)|\, dx \\
&\overset{(2.20)}{\leq} c \fint_{B_{4R/M}} g'(|Dv|)|Dv - Du|\, dx + c \fint_{B_{4R/M}} g(|Dv - Du|)\, dx \\
&\overset{(4.63)}{\leq} cM^n g'(\lambda) \fint_{B_{2R}} |Dv - Du|\, dx + cM^n \fint_{B_{2R}} g(|Dv - Du|)\, dx \\
&\overset{(4.46)}{\leq} cM^n g'(\lambda) g^{-1}\left(M^{n-1} \frac{|\mu|(B_{2MR})}{(2MR)^{n-1}} \right) + cM^{2n-1} \left(\frac{|\mu|(B_{2MR})}{(2MR)^{n-1}} \right) \\
&\overset{(4.52)_2}{\leq} cM^n g'(\lambda) g^{-1}\left(M^{n-1}\sigma_1 g(\lambda) \right) + cM^{2n-1}\sigma_1 g(\lambda) \\
&\overset{(2.8)}{\leq} cM^{n+\frac{n-1}{\gamma_1 - 1}} \sigma_1^{\frac{1}{\gamma_2 - 1}} g'(\lambda)\lambda + cM^{2n-1}\sigma_1 g(\lambda) \\
&\overset{(1.3)}{\leq} c_6 M^{2n-1} \sigma_1^{\frac{1}{\gamma_2 - 1}} g(\lambda)
\end{aligned}$$

for some constant $c_6 := c_6(\texttt{data}) > 0$. Note that we have used the fact that $M >> 1$ and $\sigma_1 < 1$ in the last line. If we choose $\sigma_1 := \sigma_1(\texttt{data}, M) > 0$ small enough such that

$$c_6 M^{2n-1} \sigma_1^{\frac{1}{\gamma_2 - 1}} \leq \frac{1}{2c_5}, \tag{4.72}$$

then it follows from (4.71) and (4.72) that

$$(|\mathcal{A}(Dv)|)_{B_{4R/M}} \geq \frac{g(\lambda)}{2c_5}.$$

This implies that there exists $x_0 \in B_{4R/M}$ so that

$$\frac{g(\lambda)}{2c_5} \leq |\mathcal{A}(Dv(x_0))| \leq Lg(|Dv(x_0)|).$$

In other words, we have

$$c_7\lambda \overset{(2.8)}{\leq} g^{-1}\left(\frac{g(\lambda)}{2Lc_5} \right) \leq |Dv(x_0)|$$

for some constant $c_7 := c_7(\texttt{data}) > 0$. From (3.30), we observe that for all $x_1, x_2 \in B_{4R/M}$,

$$|Dv(x_1) - Dv(x_2)| \leq \frac{c}{M^\alpha} \fint_{B_{2R}} |Dv| \, dx \overset{(4.63)}{\leq} \frac{c_* \lambda}{M^\alpha}.$$

By choosing $M \geq 8$ large enough to satisfy

$$M \geq \left(\frac{2c_*}{c_7} \right)^{\frac{1}{\alpha}}, \tag{4.73}$$

we find that for all $x \in B_{4R/M}$,

$$|Dv(x)| \geq |Dv(x_0)| - |Dv(x) - Dv(x_0)| \geq c_7 \lambda - \frac{c_* \lambda}{M^\alpha} \geq \frac{c_7 \lambda}{2}.$$

By taking $1/c_l := c_7/2$, we obtain a lower bound of (4.63). □

Remark 4.6. *(Choice of M, θ and σ_1) It is worth reminding how each constant is chosen. By Lemma 4.4, we are able to choose θ in terms of $\texttt{data}$ and M. We then choose $\sigma_1 := \sigma_1(\texttt{data}, M) \in (0, 1/2^n)$ that satisfies inequalities* (4.67)*,* (4.70) *and* (4.72)*. Next, we choose $M := M(\texttt{data}) \geq 8$ to satisfy both* (4.69) *and* (4.73)*. Since M is determined only in terms of $\texttt{data}$, both θ and σ_1 are also determined only in terms of $\texttt{data}$. Moreover, inequalities in Lemma 4.4 and Lemma 4.5 hold with the constants depending only on $\texttt{data}$ as well.*

Once the parameters θ, σ_1 and M are chosen by $\texttt{data}$ as in Remark 4.6, Lemma 4.3 implies that under the assumption (4.51), Lemma 4.2 holds.

Therefore, we are left to prove Lemma 4.2 under the assumption (4.52).

Indeed, we have

$$\begin{aligned}
& g^{-1}\left(\frac{|\mu|(B_{MR})}{(MR)^{n-1}}\right)+g^{-1}\left(\frac{|\mu|(B_{2R})}{(2R)^{n-1}}\right) \\
&\quad \leq \; g^{-1}\left(2^{n-1}\frac{|\mu|(B_{2MR})}{(2MR)^{n-1}}\right)+g^{-1}\left(M^{n-1}\frac{|\mu|(B_{2MR})}{(2MR)^{n-1}}\right) \\
&\quad \overset{(2.8)}{\leq} \left(2^{\frac{n-1}{\gamma_1-1}}+M^{\frac{n-1}{\gamma_1-1}}\right)g^{-1}\left(\frac{|\mu|(B_{2MR})}{(2MR)^{n-1}}\right) \\
&\quad \overset{(4.52)_2}{\leq} (2M)^{\frac{n-1}{\gamma_1-1}}g^{-1}\left(\sigma_1 g(\lambda)\right) \\
&\quad \overset{(2.8)}{\leq} (2M)^{\frac{n-1}{\gamma_1-1}}\sigma_1^{\frac{1}{\gamma_2-1}}\lambda \overset{(4.70)}{\leq} \lambda,
\end{aligned}$$

which implies (4.47). By Lemma 4.5, we apply Lemma 4.1 to find

$$\fint_{B_{2R/M}}|Du-Dv|\,dx \leq c\frac{\lambda}{g(\lambda)}\left(\frac{|\mu|(B_{MR})}{(MR)^{n-1}}\right). \tag{4.74}$$

From this inequality, we find

$$\begin{aligned}
&\fint_{B_{2R/M}}|\mathcal{A}(Du)-\mathcal{A}(Dv)|\,dx \\
&\quad \overset{(2.20)}{\leq} c\fint_{B_{2R/M}}g'(|Dv|)|Du-Dv|\,dx+c\fint_{B_{2R/M}}g(|Du-Dv|)\,dx \\
&\quad \overset{(4.63)}{\leq} cg'(\lambda)\fint_{B_{2R/M}}|Du-Dv|\,dx+cM^n\fint_{B_{2R}}g(|Du-Dv|)\,dx \\
&\quad \overset{\substack{(4.74)\\(4.45)}}{\leq} cg'(\lambda)\frac{\lambda}{g(\lambda)}\left(\frac{|\mu|(B_{MR})}{(MR)^{n-1}}\right)+cM^n\left(\frac{|\mu|(B_{2R})}{(2R)^{n-1}}\right) \\
&\quad \leq \; c\left(\frac{|\mu|(B_{2MR})}{(2MR)^{n-1}}\right),
\end{aligned} \tag{4.75}$$

where we have used the fact that the constant M is determined only by `data`. This inequality (4.75) implies that (4.48) holds under the assumption (4.52).

Finally, we have

$$
\begin{aligned}
\sup_{0<|h|<R/(8M)} \fint_{B_{R/M}} \frac{|\tau_h \mathcal{A}(Dv)|}{|h|}\,dx
&\overset{(3.39)}{\leq} \frac{c}{R} \sup_{B_{2R/M}} g'(|Dv|) \left(\fint_{B_{2R/M}} |Dv - z_0|^2 \; dx \right)^{\frac{1}{2}} \\
&\overset{(4.63)}{\leq} \frac{c g'(\lambda)^{\frac{1}{2}}}{R} \left(\fint_{B_{2R/M}} g'(|Dv| + |z_0|)\, |Dv - z_0|^2 \; dx \right)^{\frac{1}{2}} \\
&\overset{(2.16)}{\leq} \frac{c g'(\lambda)^{\frac{1}{2}}}{R} \left(\fint_{B_{2R/M}} |V(Dv) - V(z_0)|^2 \,dx \right)^{\frac{1}{2}} \\
&\overset{(3.31)}{\leq} \frac{c g'(\lambda)^{\frac{1}{2}}}{R} \fint_{B_{4R/M}} |V(Dv) - V(z_0)|\,dx \\
&\overset{(2.16)}{\leq} \frac{c g'(\lambda)^{\frac{1}{2}}}{R} \fint_{B_{4R/M}} g'(|Dv| + |z_0|)^{\frac{1}{2}} |Dv - z_0|\,dx \\
&\overset{(4.63)}{\leq} \frac{c}{R} \fint_{B_{4R/M}} g'(|Dv| + |z_0|) |Dv - z_0|\,dx \\
&\overset{(2.17)}{\leq} \frac{c M^{2n}}{R} \fint_{B_{2MR}} |\mathcal{A}(Dv) - \mathcal{A}(z_0)|\,dx,
\end{aligned}
\tag{4.76}
$$

for any $z_0 \in \mathbb{R}^n$. Since (2.16)-(2.18) imply that $\mathcal{A}(\cdot)$ is a locally bi-Lipschitz and monotone vector field, for any vector $y_0 \in \mathbb{R}^n$, there exists a unique z_0 satisfying $\mathcal{A}(z_0) = y_0$. Therefore, setting $z_0 \in \mathbb{R}^n$ so that $\mathcal{A}(z_0) = (\mathcal{A}(Du))_{B_{2MR}}$ in (4.76), inequality (4.49) follows.

5 Proof of the main theorem

In this section, we finally prove Theorem 1.1. For the simplicity of the notation, we set $K = 2M^2 \geq 128$ and consider the following equation

$$
\begin{cases}
-\operatorname{div} \mathcal{A}(Dv) = 0 & \text{in } B_{2\sqrt{K}R} \\
v = u & \text{on } \partial B_{2\sqrt{K}R}
\end{cases}
$$

for a ball satisfying $B_{KR} \subset\subset \Omega$. Then by Lemma 4.2, we obtain the following inequalities:

$$\begin{aligned}\fint_{B_{2R}} |\mathcal{A}(Du) - \mathcal{A}(Dv)|\,dx \leq & cR^{\delta} \fint_{B_{2KR}} |\mathcal{A}(Du) - \big(\mathcal{A}(Du)\big)_{B_{2KR}}|\,dx \\ & + c\left(\frac{|\mu|(B_{2KR})}{R^{n-1+\delta(\gamma_2-2)}}\right)\end{aligned}$$

and

$$\begin{aligned}\sup_{0<|h|<R/8} \fint_{B_R} \frac{|\tau_h \mathcal{A}(Dv)|}{|h|}\,dx \leq & \frac{c}{R} \fint_{B_{2KR}} |\mathcal{A}(Du) - \big(\mathcal{A}(Du)\big)_{B_{2KR}}|\,dx \\ & + c\left(\frac{|\mu|(B_{2KR})}{R^{n+\delta(\gamma_2-2)}}\right).\end{aligned}$$

Once these inequalities are obtained, we are able to complete the proof of Theorem 1.1 with the same spirit as in [2, Section 6].

5.1 Scaling and limiting process

To prove Theorem 1.1, it is enough to show that the inequality (1.6) holds under the assumptions that $\mu \in L^{\infty}(\Omega)$ and $u \in W^{1,G}(\Omega)$. To be precise, let $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,G}(\Omega)$ be a sequence of approximations satisfying Definition 2.4 and $\{\mu_k\}_{k\in\mathbb{N}} \subset L^{\infty}(\Omega) \subset \big(W^{1,G}(\Omega)\big)^*$ be a sequence of functions that converges to μ weakly* in the sense of measures. For a fixed ball $B_R \subset\subset \Omega$ and a fixed number $\sigma \in (0,1)$, we have

$$\begin{aligned}&\fint_{B_{R/2}} \int_{B_{R/2}} \frac{|\mathcal{A}(Du(x)) - \mathcal{A}(Du(y))|}{|x-y|^{n+\sigma}}\,dx\,dy \\ &\overset{(2.24)}{\leq} \liminf_{k\to\infty} \fint_{B_{R/2}} \int_{B_{R/2}} \frac{|\mathcal{A}(Du_k(x)) - \mathcal{A}(Du_k(y))|}{|x-y|^{n+\sigma}}\,dx\,dy \\ &\overset{(1.6)}{\leq} \limsup_{k\to\infty} \frac{c}{R^{\sigma}} \fint_{B_R} |\mathcal{A}(Du_k)|\,dx + \limsup_{k\to\infty} \frac{c}{R^{\sigma}}\left(\frac{|\mu_k|(B_R)}{R^{n-1}}\right) \\ &\overset{(2.24)}{\leq} \frac{c}{R^{\sigma}} \fint_{B_R} |\mathcal{A}(Du)|\,dx + \frac{c}{R^{\sigma}}\left(\frac{|\mu|(B_R)}{R^{n-1}}\right).\end{aligned}$$

Therefore (1.6) holds for a SOLA u to (1.1) under the assumptions (1.2) and (1.3).

We now introduce a scaling and normalization argument. Take

$$H := g^{-1}\left(\fint_{B_R} |\mathcal{A}(Du)|\,dx + |B_R|^{\frac{1}{n}} \frac{|\mu|(B_R)}{|B_R|} \right)$$

and define

$$\begin{aligned} \tilde{u}(\tilde{x}) &:= \frac{u(x_0 + R\tilde{x})}{HR}, \quad \tilde{\mu}(\tilde{x}) := \frac{R\mu(x_0 + R\tilde{x})}{g(H)}, \\ \tilde{\mathcal{A}}(z) &:= \frac{\mathcal{A}(Hz)}{g(H)}, \quad \tilde{g}(t) := \frac{g(Ht)}{g(H)} \end{aligned} \tag{5.77}$$

for $\tilde{x} \in B_1, z \in \mathbb{R}^n, t > 0$. From a direct calculation, we observe that $\tilde{\mathcal{A}}(\cdot)$ satisfies the structure assumptions (1.2) and (1.3) with $\tilde{g}$ replaced by g, see [7, Lemma 5.1] for details. Moreover,

$$\fint_{B_1} |\tilde{\mathcal{A}}(D\tilde{u})|\,dx + |B_1|^{\frac{1}{n}} \frac{|\tilde{\mu}(B_1)|}{|B_1|} = 1, \tag{5.78}$$

and $\tilde{u} \in W^{1,G}(B_1)$ is a solution of

$$-\mathrm{div}\,\tilde{\mathcal{A}}(D\tilde{u}) = \tilde{\mu}.$$

We recall the Definition 2.2 and use (5.77) to find

$$\left[\tilde{\mathcal{A}}(D\tilde{u})\right]_{W^{\sigma,1}(B_{1/2})} = \frac{(R/2)^{\sigma-n}\,[\mathcal{A}(Du)]_{W^{\sigma,1}(B_{R/2})}}{g(H)}.$$

Therefore, we are left to prove that for all $\sigma \in (0,1)$,

$$[\mathcal{A}(Du)]_{W^{\sigma,1}(B_{1/2})} = \fint_{B_{1/2}} \int_{B_{1/2}} \frac{|\mathcal{A}(Du(x)) - \mathcal{A}(Du(y))|}{|x-y|^{n+\sigma}}\,dx\,dy \le c$$

for some constant $c = c(\mathtt{data}, \sigma) > 0$ under the assumption (5.78).

5.2 Iteration and conclusion

Let Ω', Ω'' be two open subsets of B_1 satisfying $\Omega' \subset\subset \Omega'' \subset B_1$ with $d := \operatorname{dist}(\Omega', \partial\Omega'') > 0$. We then define a real-valued function

$$\omega(t) := \left(1 - \frac{1-t}{\gamma_2 - 1}\right)\left(\frac{1-t}{\gamma_2 - 1} + t\right)$$

for $t \in [0,1]$. From a direct calculation, we have

$$0 \le t < 1 \iff t < \omega(t) < 1, \quad \omega(1) = 1 \quad \text{and} \quad \omega'(t) > 0. \tag{5.79}$$

Now we show a fractional differentiability from a bootstrap result.

Lemma 5.1. *Suppose that $\mathcal{A}(Du) \in W^{t,1}(\Omega'')$ for some $t \in [0,1)$ with the estimate $[\mathcal{A}(Du)]_{W^{t,1}(\Omega'')} \le c_1$ for some positive constant c_1. Then it follows that $\mathcal{A}(Du) \in W^{\tilde{t},1}(\Omega')$ for all $\tilde{t} \in [0, \omega(t))$. Moreover, there exists another constant $c_2 := c_2(\boldsymbol{data}, d, \tilde{t}, c_1) > 0$ satisfying*

$$[\mathcal{A}(Du)]_{W^{\tilde{t},1}(\Omega')} \le c_2. \tag{5.80}$$

Proof. We first take $h \in \mathbb{R}^n$ small enough to satisfy

$$0 < |h| \le \min\left\{\left(\frac{d}{1024K}\right)^{\frac{1}{\beta}}, \frac{1}{1024K}\right\} =: d_1 < d.$$

Here, $\beta \in (0,1)$ is a small number which will be determined later. Then we choose an open ball $B := B_{8|h|^\beta}(x_0) \subset\subset \Omega''$ with $x_0 \in \Omega'$. Then by (5.77) and (5.77), we have

$$\begin{aligned}
&\int_B |\tau_h \mathcal{A}(Du)|\,dx \\
&\quad \le \; c\int_B |\tau_h \mathcal{A}(Dv)|\,dx + c\int_{2B} |\mathcal{A}(Du) - \mathcal{A}(Dv)|\,dx \\
&\quad \overset{(5.77)}{\le} c|h|\int_B \frac{|\tau_h \mathcal{A}(Dv)|}{|h|}\,dx + c|h|^{\delta\beta}\int_{KB} |\mathcal{A}(Du) - (\mathcal{A}(Du))_{KB}|\,dx \\
&\qquad + c|h|^{\beta(1-\delta(\gamma_2-2))}|\mu(KB)| \\
&\quad \overset{(5.77)}{\le} c\left(|h|^{1-\beta} + |h|^{\delta\beta}\right)\int_{KB} |\mathcal{A}(Du) - (\mathcal{A}(Du))_{KB}|\,dx \\
&\qquad + c\left(|h|^{1-\beta\delta(\gamma_2-2)} + |h|^{\beta(1-\delta(\gamma_2-2))}\right)|\mu(KB)|.
\end{aligned}$$

We set $\delta = 1-\beta$ to see that

$$|h|^{1-\beta} \le |h|^{(1-\beta)\beta} \quad \text{and} \quad |h|^{1-\beta(1-\beta)(\gamma_2-2)} \le |h|^{\beta(1-(1-\beta)(\gamma_2-2))}.$$

Then we find

$$\begin{aligned}\int_B |\tau_h \mathcal{A}(Du)|\,dx \le & c|h|^{(1-\beta)\beta} \int_{KB} |\mathcal{A}(Du) - (\mathcal{A}(Du))_{KB}|\,dx \\ & + c|h|^{\beta(1-(1-\beta)(\gamma_2-2))}|\mu(KB)|.\end{aligned}$$

To proceed, we define

$$\begin{aligned}\mu_0(B) :=& |\mu|(B) + \int_B |\mathcal{A}(Du)|\,dx \\ \mu_t(B) :=& |\mu_0|(B) + \int_B\int_B \frac{|\mathcal{A}(Du(x)) - \mathcal{A}(Du(y))|}{|x-y|^{n+t}}\,dx\,dy \quad (t>0).\end{aligned} \tag{5.81}$$

Note that μ_0 is a measure but μ_t is not. However, μ_t is still a countably super-additive set function, i.e.,

$$\sum_i \mu_t(B_i) \le \mu_t\left(\bigcup_i B_i\right)$$

whenever $\{B_i\}_{i\in I}$ is a countable family of mutually disjoint Borel subsets. By the fractional Poincaré inequality (2.21), we get

$$\begin{aligned}&\int_{KB} |\mathcal{A}(Du) - (\mathcal{A}(Du))_{KB}|\,dx \\ &\qquad \le c|h|^{\beta t} \int_{KB}\int_{KB} \frac{|\mathcal{A}(Du(x)) - \mathcal{A}(Du(y))|}{|x-y|^{n+t}}\,dx\,dy.\end{aligned} \tag{5.82}$$

Combining (5.81), (5.81) and (5.82), we have

$$\begin{aligned}\int_B |\tau_h \mathcal{A}(Du)|\,dx &\overset{\substack{(5.81)\\(5.82)}}{\le} c|h|^{(1-\beta)\beta+\beta t} \int_{KB}\int_{KB} \frac{|\mathcal{A}(Du(x)) - \mathcal{A}(Du(y))|}{|x-y|^{n+t}}\,dx\,dy \\ &\quad + c|h|^{\beta(1-(1-\beta)(\gamma_2-2))}|\mu(KB)| \\ &\overset{(5.81)}{\le} c\left(|h|^{(1-\beta)\beta+\beta t} + |h|^{\beta(1-(1-\beta)(\gamma_2-2))}\right)|\mu_t(KB)|.\end{aligned}$$

At this stage, we choose $\beta > 0$ so that $\beta(1-\beta)+\beta t = \beta\left(1-(1-\beta)(\gamma_2-2)\right)$. From a direct calculation, we have

$$\beta(t) = \frac{\gamma_2 - 2 + t}{\gamma_2 - 1} \quad \text{and} \quad \omega(t) = \beta(t)(1-\beta(t)) + \beta(t)t,$$

which implies

$$\int_B |\tau_h \mathcal{A}(Du)|\, dx \le c|h|^{\omega(t)} \mu_t(KB). \tag{5.83}$$

Now, we use a covering argument. For each vector $h \in \mathbb{R}^n \setminus \{0\}$, we cover Ω' by a family of cubes $\{Q_i\}_{i\in I}$ that have sides parallel to coordinate axes and the side length equal to $\frac{16|h|^\beta}{\sqrt{n}}$. For each Q_i, we choose the smallest open ball satisfying $Q_i \subset B_i$. Then, $\{B_i\}_{i\in I}$ covers Ω'. Moreover, KB_i intersects only finite numbers of KB_j with $j \neq i$. The maximum number of intersection, say $\mathcal{H}$, is determined by K and n. Since K depends on $\texttt{data}$, so is $\mathcal{H}$. Using (5.78), (5.81) and (5.83), we have

$$\begin{aligned}\int_{\Omega'} |\tau_h \mathcal{A}(Du)|\, dx &\le \sum_{i\in I} \int_{B_i} |\tau_h \mathcal{A}(Du)|\, dx \le \sum_{i\in I} c|h|^{\omega(t)} \mu_t(KB_i) \\ &\le c\mathcal{H}|h|^{\omega(t)} \mu_t(\Omega'') \le c|h|^{\omega(t)}\end{aligned}$$

for $0 < |h| < d_1$. Finally, (5.80) follows from Proposition 2.3. □

We are now in position to show Theorem 1.1. Define two sequences $\{s_k\}_{k\in\mathbb{N}}$ and $\{t_k\}_{k\in\mathbb{N}}$ inductively by

$$\begin{aligned}&s_1 := \omega(0)/4, \quad s_{k+1} = \omega(s_k), \\ &t_1 := \omega(0)/2, \quad t_{k+1} = \left(\omega(t_k) + \omega(s_k)\right)/2.\end{aligned}$$

Using (5.79), it is straightforward to show that

$$s_k < t_k < 1, \quad t_{k+1} < \omega(t_k) \quad \text{and} \quad s_k, t_k \nearrow 1 \quad \text{as } k \to \infty.$$

We now apply Lemma 5.1 with t_k iteratively. For any $\sigma \in (0,1)$ and for some $\varepsilon \in (0, 1/2)$, we choose $\bar{k} \in \mathbb{N}$ large enough so that $t_{\bar{k}} > \sigma$ and consider a sequence of shrinking open balls satisfying

$$B_{1-\varepsilon} \subset\subset B^{\bar{k}} \subset\subset B^{\bar{k}-1} \cdots \subset\subset B^0 := B_1 \quad \text{with} \quad \operatorname{dist}(B^{k+1}, \partial B^k) \approx \varepsilon/\bar{k}.$$

Then by the assumption (5.78), we have $\mu_0(B^0) \le c_0$ for some positive constant c_0 depending on $\texttt{data}$. Then according to Lemma 5.1, $\mu_1(B^1) \le c_8$ for some $c_8 = c_8(\texttt{data}, \varepsilon, \bar{k})$. By the iteration, we have

$$\mu_{t_{\bar{k}}}(B_{1-\varepsilon}) \le \mu_{t_{\bar{k}}}(B^{\bar{k}}) \le c_9$$

for some $c_9 = c_9(\texttt{data}, \varepsilon, \bar{k}, \sigma)$. We first take $\varepsilon = 1/4$ and use the fact that $\bar{k}$ is determined by σ. Then Proposition 2.3 and (5.81) yield

$$[\mathcal{A}(Du)]_{W^{\sigma,1}(B_{1/2})} \le c_{10}$$

for some $c_{10} = c_{10}(\texttt{data}, \sigma)$. Using a scaling and normalization argument explained in Subsection 5.1, we complete the proof of Theorem 1.1.

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국문초록

이 학위논문에서는 비선형 타원형 및 포물선 방정식에서 우변이 측도로 주어졌을 때 분포 해의 정칙성에 대해서 다루고자 한다.

우선 다항 성장조건과 로그 성장조건을 가지는 이중위상 문제에서 측도 데이터를 가지는 경우 해의 그레디언트 추정값을 구하였다. 측도 데이터의 1차 극대 함수와 대역적으로 동등한 적분성을 가지고 있다는 것을 입증하였다. 또한, 측도 데이터를 가지는 방정식의 미분성에 대해서 연구를 하였다. 연구의 목적은 선형방정식의 미분성의 결과를 비선형 방정식에서 확장하는 것으로, 오리츠 유형의 미분 방정식에서 최대 미분성에 대한 결과를 얻었다.

주요어휘: 정칙성 이론, 측도 데이타,칼데론-지그문트 추정 값, 미표준 성, 미분성

학번: 2015-20279

감사의 글

대학원에 입학한지가 엊그제 같은데 벌써 6년이라는 시간이 흘렀네요. 되돌아보면 주변에 감사의 인사를 전해야 하는 분들이 너무 많습니다. 우선 짧지 않은 시간동안 제가 대학원 생활에 집중할수있도록 배려해주신 부모님께 감사의 인사를 전하고 싶습니다. 아버지와 어머니의 헌신적인 지지와 격려 덕분에 무사히 학위 과정을 마칠수 있었습니다. 이 글을 빌어 감사의 인사를 전합니다.

또한, 대학원 생활동안 저의 논문 지도와 여러 방면에서 도을 주신 변순식 선생님께 감사의 인사를 전하고 싶습니다. 연구자로써 항상 꾸준하고 성실한 모습을 보여주시며 저희에게 모범을 보이셨습니다. 또한 학자로써 갖춰야할 덕목들을 말이 아닌 실행으로 보여주셨습니다. 약 5년전 처음 뵙을때 부터 지금까지 매 주 1시간씩 학생 면담을 하시고 바쁘신데도 불구하고 세미나에서 학생들을 지도하는 모습들이 그 당시에는 알지 못했지만 지금은 생각할수록 너무 대단하고 존경스럽게 느껴집니다. 많이 부족하지만 더욱 더 발전하기 위해서 노력하겠습니다.

바쁘신 와중에도 시간을 내어 심사를 맡아 주신 이기암 교수님, 김판기 교수님, 옥지훈 교수님 그리고 윤영훈 교수님께도 감사의 인사를 전합니다.

우리 연구실 분들도 감사의 인사를 전합니다. 특히, 우리 연구실에서 졸업 후 자리를 잡으시고 후배들에게 많은 모범을 보여 주신 윤영훈 교수님께 감사의 인사를 전합니다. 대학원 생활과 연구하는 자세 그리고 열심히 노력하는 모습을옆에서 보고 많이 배웠습니다. 제가 처음 왔을 때 잘 챙겨주시던 박정태 박사님, 모르는 질문들을 잘 받아주시고 알려주신 오재한 교수님, 항상 친절하게 의논해주신 신필수 교수님, 회사에 가셔서 산업계에 대해 이것 저것 알려주신 소형석 박사님에게도 감사의 인사를 드리고 싶습니다.

그리고 이번 디펜스에 같이 심사를 보게 된 원태형과 정민이 형에게도 감사의 인사를 드립니다. 디펜스 준비과정에서 많은 도움이 되었고 종종 같이 이야기를 하면서 힘든 시간을 버틸수 있었습니다. 또 후배이지만 배울점이 많은 송경, 항상 좋은 모습을 보여주는 호식, 여러 조언을 아끼지 않는 효진이형, 많은 이야기는 나누지 못했지만 같은 연구실에서 함께 지낸 민규형 그리고 짧지만 같이 세미나를 하면서 많은 자극을 주신 허문현님께도 감사의 인사를 드립니다. 또한, 연구실을 나가 본인의 길을 가고 있는 은비와 문연에게도 안부의 인사를 전합니다. 미지막으로 옆자리에서 열심히 연구하면서 정말 좋은 모습을 보여주는 Sumiya에게도 감사의 인사를 드리고 싶습니다. 같이 연구실 생활을 하면서 그 동안 힘든 시간을 버틸수 있었습니다.

마지막으로 오랜시간 봐온 나의 친구들과 다른 연구실에 있는 대학원 동기들에게도 감사하다는 말을 전하고싶습니다.