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Dynamics on homogeneous spaces and Diophantine approximation

(균질공간에서의 동역학과 디오판틴 근사)

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Dynamics on homogeneous spaces and Diophantine approximation

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Abstract

Dynamics on homogeneous spaces and Diophantine approximation

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Dynamics of group actions on homogeneous spaces, which is referred to as "homogeneous dynamics", has a lot of connections to number theory. These connections have been intensively and extensively studied over the past decades, and have produced various and abundant number-theoretic results.

In this thesis, we focus on the relationship between homogeneous dynamics and Diophantine approximation, and consider the following three objects in Diophantine approximation: *Dirichlet non-improvable affine forms*, *badly approximable affine forms*, and *weighted singular vectors*.

We improve equidistribution results in homogeneous dynamics in terms of weak L^1 estimates, and establish local ubiquity systems for Dirichlet non-improvable affine forms using Transference Principle in Diophantine approximation. These developments imply zero-infinite phenomena for Hausdorff measures of Dirichlet non-improvable affine forms.

Next, we establish an effective version of entropy rigidity, which implies the effective upper bound of Hausdorff dimension of badly approximable affine forms by constructing "well-behaved" σ-algebras and certain invariant measures with large entropy. We further characterize full Hausdorff-dimensionality of badly approximable affine forms for fixed matrix by a Diophantine condition of singularity on average. We also consider Diophantine approximation over global function fields and have similar results in this setting.

Finally, we improve lattice point counting in geometry of numbers, which arises from the fractal structure of weighted singular vectors. Combining the improvement and the shadowing property in homogeneous dynamics, we obtain the sharp lower bound of Hausdorff dimension of weighted singular vectors.

Key words: Homogeneous dynamics, Diophantine approximation, Entropy rigidity, Geometry of numbers, Ubiquitous system, Global function field,
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Chapter 1

Introduction

After the celebrated work of Margulis [Mar87] on Oppenheim conjecture, the rigidity phenomenon in homogeneous dynamics has been intensively and extensively studied over the past decades. These extensive studies have produced various and abundant number-theoretic results: the proof of Oppenheim conjecture [Mar87] and its quantitative versions [DM93, EMM98, EMM05], the proof of Baker-Sprindžuk conjecture [KM98], proof of arithmetic quantum unique ergodicity [Lin06], an important partial result on Littlewood conjecture [EKL06], etc.

The present thesis is focused on the metric theory of Diophantine approximation, which originates from the problem of the approximation of real numbers by rational numbers. Since Dani's work [Dan85] on the relation between Diophantine approximation and homogeneous dynamics, various dynamical methods such as equidistribution, mixing, or measure rigidity have been widely used in the study of metric Diophantine approximation [KM98, KM99, KLW04, EKL06].

More precisely, the theory of Diophantine approximation is concerned with the following question: if A is an $m \times n$ real matrix (interpreted as a system of m linear forms in n variables), how small, in terms of the size of $\mathbf{q} \in \mathbb{Z}^n$, can be the distance from $A\mathbf{q}$ to $\mathbb{Z}^m$? This question can be seen in terms of homogeneous dynamics as follows. The homogeneous space associated with Diophantine approximation is $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$ for $d = m + n$, which is identified with the space of lattices in $\mathbb{R}^d$ with covolume 1. By Mahler's compactness criterion of $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$, Diophantine approximation of a matrix A can be described by cusp excursions of the orbit $(a_t\Lambda_A)_{t\geq 0}$ of the diagonal flow as follows:

$$a_t = \begin{pmatrix} e^{t/m} I_m & 0 \\ 0 & e^{-t/n} I_n \end{pmatrix} \quad \text{and} \quad \Lambda_A = \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} \mathbb{Z}^d.$$

The property that the matrix A has good Diophantine approximation is equivalent to the property that the orbit $(a_t \Lambda_A)_{t \geq 0}$ has excursions into small cusp neighborhood. This observation allows us to use various dynamical methods in the study of metric Diophantine approximation.

In the theory of Diophantine approximation, the starting point is *Dirichlet theorem*: For any $A \in M_{m,n}(\mathbb{R})$ and $T > 1$, there exist $\mathbf{p} \in \mathbb{Z}^m$ and $\mathbf{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ such that

$$\|A\mathbf{q} - \mathbf{p}\|^m \leq \frac{1}{T} \quad \text{and} \quad \|\mathbf{q}\|^n < T. \tag{1.1}$$

Dirichlet theorem implies the following corollary, which will be called *Dirichlet corollary*: For any $A \in M_{m,n}(\mathbb{R})$ there exist infinitely many $\mathbf{q} \in \mathbb{Z}^n$ such that

$$\|A\mathbf{q} - \mathbf{p}\|^m < \frac{1}{\|\mathbf{q}\|^n} \quad \text{for some } \mathbf{p} \in \mathbb{Z}^m. \tag{1.2}$$

The above two statements give a rate of approximation that works for all real matrices. However, if we replace the right-hand sides of (1.1) and (1.2) by faster decaying functions of T and $\|\mathbf{q}\|^n$ respectively, then one can ask sizes of corresponding sets of matrices satisfying the improved systems, which leads to the metric theory of Diophantine approximation.

In this thesis, we study metrical properties of the following four main objects using both dynamical methods and number-theoretical methods:

1. *Dirichlet non-improvable affine forms*, based on the joint work with Wooyeon Kim [KK22],

2. *badly approximable affine forms*, based on the joint work with Wooyeon Kim and Seonhee Lim [KKL],

3. *badly approximable affine forms on global function fields*, based on the joint work with Seonhee Lim and Frédéric Paulin [KLP],

4. *weighted singular vectors*, based on the joint work with Jaemin Park [KP].

1.1 Dirichlet non-improvable affine forms

Classically, the improvability of Dirichlet corollary, i.e. the inequality (1.2), has been studied for a long time. To consider the improvability of (1.2), let an approximating function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be given. Then we say that $A \in M_{m,n}(\mathbb{R})$ is *ψ-approximable* if there exist infinitely many $\mathbf{q} \in \mathbb{Z}^n$ such that [1]

$$\|A\mathbf{q} - \mathbf{p}\|^m < \psi(\|\mathbf{q}\|^n) \quad \text{for some } \mathbf{p} \in \mathbb{Z}^m. \tag{1.3}$$

Denote by $W_{m,n}(\psi)$ the set of ψ-approximable matrices in the unit cube $[0,1]^{mn}$. Then the set $W_{m,n}(\psi)$ satisfies the following zero-one law with respect to the Lebesgue measure.

Theorem 1.1.1 (Khintchine-Groshev Theorem)**.** *Given a non-increasing ψ, the set $W_{m,n}(\psi)$ has zero (resp. full) Lebesgue measure if and only if the series $\sum_k \psi(k)$ converges (resp. diverges).*

To distinguish between sizes of null sets, we can consider Hausdorff measure and dimension as the appropriate tools. Since the set $W_{m,n}(\psi)$ is always containing $m(n-1)$-dimensional hyperplanes, we may focus on s-dimensional Hausdorff measures with $s > m(n-1)$. The following result was proved by Jarník in 1931 for $n = 1$ and [DV97] in general.

Theorem 1.1.2 (Jarník)**.** *Let ψ be a non-increasing function. Then for $s > m(n-1)$,*

$$\mathcal{H}^s(W_{m,n}(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{m+n-1}\left(\frac{\widehat{\psi}(q)}{q}\right)^{s-m(n-1)} < \infty, \\ \mathcal{H}^s([0,1]^{mn}) & \text{if } \sum_{q=1}^{\infty} q^{m+n-1}\left(\frac{\widehat{\psi}(q)}{q}\right)^{s-m(n-1)} = \infty, \end{cases}$$

where $\widehat{\psi}(q) = \psi(q^n)^{\frac{1}{m}}$.

Here, $\mathcal{H}^s([0,1]^{mn})$ is infinity for $s < mn$. On the other hand, $\mathcal{H}^{mn}$ comparable to the mn-dimensional Lebesgue measure, hence, Theorem 1.1.2 implies Khintchine-Groshev Theorem.

It is worth mentioning that Jarník's theorem was indeed proved for any dimension functions f, not just the functions of the form $f(r) := r^s$ stated in

[1] Here, we follow the definition given in [KM99, KW19] but, in Chapter 2, we will use the slightly different definition, such as [BV10], where the inequality $\|A\mathbf{q} - \mathbf{p}\| < \psi(\|\mathbf{q}\|)$ is used instead of (1.3).

Theorem 1.1.2, see [DV97]. Regarding inhomogeneous Diophantine approximation, the analogue of Jarník's theorem for doubly metric case was proved in [HKS20] and for singly metric case in [Bug04(1)].

For similar generalizations in the setting of Dirichlet's Theorem, let us give the following definition: for a non-increasing function $\psi : [T_0, \infty) \to \mathbb{R}_+$, where $T_0 > 1$ is fixed, we say that $A \in M_{m,n}(\mathbb{R})$ is *ψ-Dirichlet* if the system

$$\|A\mathbf{q} - \mathbf{p}\|^m < \psi(T) \quad \text{and} \quad \|\mathbf{q}\|^n < T$$

has a nontrivial integral solution for all large enough T. Surprisingly, no zero-one law analogous to Khintchine-Groshev Theorem was known until recently when Kleinbock and Wadleigh [KW18] proved a zero-one law on the Lebesgue measure of Dirichlet improvable numbers, that is, $m = n = 1$. The Hausdorff measure-theoretic results for Dirichlet non-improvable numbers analogous to Theorem 1.1.2 have also been established in [HKWW18] for a general class of dimension functions f called the essentially sub-linear dimension functions. For the non-essentially sub-linear dimension functions, the relevant results are in [BHS]. For general $m, n \in \mathbb{N}$, Kleinbock, Strömbergsson, and Yu [KSY21] recently gave sufficient conditions on ψ to ensure that the set of ψ-Dirichlet $m \times n$ matrices has zero or full Lebesgue measure.

Now, we focus our attention on inhomogeneous Diophantine approximation replacing the values of a system of linear forms $A\mathbf{q}$ by those of a system of *affine forms* $\mathbf{q} \mapsto A\mathbf{q} + \mathbf{b}$, where $A \in M_{m,n}(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^m$. Let $\widetilde{M}_{m,n}(\mathbb{R}) := M_{m,n}(\mathbb{R}) \times \mathbb{R}^m$. Following [KW19], for a non-increasing function $\psi : [T_0, \infty) \to \mathbb{R}_+$, we say that a pair $(A, \mathbf{b}) \in \widetilde{M}_{m,n}(\mathbb{R})$ is *ψ-Dirichlet* if there exist $\mathbf{p} \in \mathbb{Z}^m$ and $\mathbf{q} \in \mathbb{Z}^n$ such that

$$\|A\mathbf{q} + \mathbf{b} - \mathbf{p}\|^m < \psi(T) \quad \text{and} \quad \|\mathbf{q}\|^n < T \tag{1.4}$$

whenever T is large enough. Denote by $\widehat{D}_{m,n}(\psi)$ the set of ψ-Dirichlet pairs in the unit cube $[0,1]^{mn+m}$. Note that in this definition, the case $\mathbf{q} = 0$ is allowed so that $(A, \mathbf{b})$ is always ψ-Dirichlet for any $\mathbf{b} \in \mathbb{Z}^m$.

Recently, Kleinbock and Wadleigh established the following zero-one law for the set $\widehat{D}_{m,n}(\psi)$ with respect to the Lebesgue measure.

Theorem 1.1.3. *[KW19] Given a non-increasing ψ, the set $\widehat{D}_{m,n}(\psi)$ has zero* *(resp. full) Lebesgue measure if and only if the series*

$$\sum_j \frac{1}{\psi(j)j^2} \tag{1.5}$$

diverges (resp. converges).

As mentioned in [KW19, Section 7], one can naturally ask whether Theorem 1.1.3 can be extended along two directions:

- Zero-infinity law for a Hausdorff measure,
- Singly metric case ($\mathbf{b}$ fixed).

Although Theorem 1.1.3 provides the Lebesgue measure of the set $\widehat{D}_{m,n}(\psi)$, nothing was known about the Hausdorff dimension of this set. In this thesis, we give an analogue of Theorem 1.1.3 for the Hausdorff measure by establishing the zero-infinity law analogous to Theorem 1.1.2. Let us state our main theorem.

Theorem 1.1.4. *Given a decreasing function ψ with $\lim_{T\to\infty}\psi(T) = 0$ and $0 \le s \le mn+m$, the s-dimensional Hausdorff measure of $\widehat{D}_{m,n}(\psi)^c$ is given by*

$$\mathcal{H}^s(\widehat{D}_{m,n}(\psi)^c) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \frac{1}{\psi(q)q^2}\left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}}\right)^{mn+m-s} < \infty, \\ \mathcal{H}^s([0,1]^{mn+m}) & \text{if } \sum_{q=1}^{\infty} \frac{1}{\psi(q)q^2}\left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}}\right)^{mn+m-s} = \infty. \end{cases}$$

Moreover, the convergent case still holds for every non-increasing function ψ without the assumption $\lim_{T\to\infty}\psi(T) = 0$.

On the other hand, Theorem 1.1.3 provides only the information on Lebesgue measure in the doubly metric case, i.e. it computes Lebesgue measure of the set $\widehat{D}_{m,n}(\psi) \subseteq \widetilde{M}_{m,n}$. A more refined question in inhomogeneous Diophantine approximation is fixing $\mathbf{b} \in \mathbb{R}^m$ and asking the analogous question for the slices of $\widehat{D}_{m,n}(\psi)$. For fixed $\mathbf{b} \in \mathbb{R}^m$, let $\widehat{D}^{\mathbf{b}}_{m,n}(\psi) := \{A \in M_{m,n}(\mathbb{R}) : (A, \mathbf{b}) \in \widehat{D}_{m,n}(\psi)\}$. The following theorem answers the question for the singly metric case.

Theorem 1.1.5. *Given a decreasing function ψ with $\lim_{T\to\infty}\psi(T)=0$ and $0\le s\le mn$, the s-dimensional Hausdorff measure of $\widehat{D}^{\mathbf{b}}_{m,n}(\psi)^c$ is given by*

$$\mathcal{H}^s(\widehat{D}^{\mathbf{b}}_{m,n}(\psi)^c)=\begin{cases}0 & \text{if } \sum_{q=1}^{\infty}\frac{1}{\psi(q)q^2}\left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}}\right)^{mn-s}<\infty,\\ \mathcal{H}^s([0,1]^{mn}) & \text{if } \sum_{q=1}^{\infty}\frac{1}{\psi(q)q^2}\left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}}\right)^{mn-s}=\infty.\end{cases}$$

for every $\mathbf{b}\in\mathbb{R}^m\setminus\mathbb{Z}^m$. Moreover, the convergent case still holds for every $\mathbf{b}\in\mathbb{R}^m$ and every non-increasing function ψ without the assumption $\lim_{T\to\infty}\psi(T)=0$.

Theorem 1.1.4 and Theorem 1.1.5 can be applied to compute the Hausdorff dimension of the Dirichlet non-improvable set for some specific functions explicitly. For example, let $\psi_a(q):=q^{-a}$ and $\psi_{a,b}(q):=q^{-a}(\log q)^b$ for $a>0$, $b\ge 0$. Our results directly gives the following: For $0<a\le 1$, the Hausdorff dimension of $\widehat{D}^{\mathbf{b}}_{m,n}(\psi_{a,b})^c$ is $s_a:=mn-\frac{mn(1-a)}{m+na}$ and

$$\mathcal{H}^{s_a}(\widehat{D}^{\mathbf{b}}_{m,n}(\psi_{a,b})^c)=\begin{cases}0 & \text{if } b>\frac{m+na}{m+n},\\ \mathcal{H}^{s_a}([0,1]^{mn}) & \text{if } b\le\frac{m+na}{m+n}\end{cases}$$

for every $\mathbf{b}\in\mathbb{R}^m\setminus\mathbb{Z}^m$. More explicitly, $\mathcal{H}^{s_a}([0,1]^{mn})=\mathcal{H}^{mn}([0,1]^{mn})\asymp 1$ if $a=1$, and $\mathcal{H}^{s_a}([0,1]^{mn})=\infty$ otherwise. For the doubly metric case, the Hausdorff dimension of $\widehat{D}_{m,n}(\psi_{a,b})^c$ is s_a+m and

$$\mathcal{H}^{s_a+m}(\widehat{D}_{m,n}(\psi_{a,b})^c)=\begin{cases}0 & \text{if } b>\frac{m+na}{m+n},\\ \mathcal{H}^{s_a+m}([0,1]^{mn+m}) & \text{if } b\le\frac{m+na}{m+n}.\end{cases}$$

Also, we can observe that the Hausdorff dimension is always bigger than $mn-n$ for the singly metric case and $mn+m-n$ for the doubly metric case regardless of the choice of ψ.

We remark that the above results for ψ_a can be stated in terms of *uniform Diophantine exponents*. We denote by $\widehat{w}(A,\mathbf{b})$ the supremum of the real numbers w for which, for all sufficiently large T, the inequalities

$$\|A\mathbf{q}+\mathbf{b}-\mathbf{p}\|<T^{-w}\quad\text{and}\quad\|\mathbf{q}\|<T$$

have an integral solution $\mathbf{p}\in\mathbb{Z}^m$ and $\mathbf{q}\in\mathbb{Z}^n$. For further details and references regarding the above notion, see [BL05, B16]. Considering ψ_a with $a=\frac{mw}{n}$, we have the following corollary by the definition.

Corollary 1.1.6. *For any $w > 0$,*

$$\dim_H \{(A, \mathbf{b}) \in M_{m,n}(\mathbb{R}) \times \mathbb{R}^m : \widehat{w}(A, \mathbf{b}) \leq w\} = \min\left\{mn + m - \frac{n - mw}{1 + w}, mn + m\right\},$$

$$\dim_H \{A \in M_{m,n}(\mathbb{R}) : \widehat{w}(A, \mathbf{b}) \leq w\} = \min\left\{mn - \frac{n - mw}{1 + w}, mn\right\}$$

for every $\mathbf{b} \in \mathbb{R}^m \setminus \mathbb{Z}^m$. Therefore, for any $0 < w \leq \frac{n}{m}$,

$$\dim_H \{(A, \mathbf{b}) \in M_{m,n}(\mathbb{R}) \times \mathbb{R}^m : \widehat{w}(A, \mathbf{b}) = w\} = mn + m - \frac{n - mw}{1 + w},$$

$$\dim_H \{A \in M_{m,n}(\mathbb{R}) : \widehat{w}(A, \mathbf{b}) = w\} = mn - \frac{n - mw}{1 + w}$$

for every $\mathbf{b} \in \mathbb{R}^m \setminus \mathbb{Z}^m$.

1.2 Badly approximable affine forms

The simplest way to improve Dirichlet corollary is to replace the right hand side of (1.2) by $\frac{\epsilon}{\|\mathbf{q}\|^n}$ for $0 < \epsilon < 1$. It is well known that the set of matrices satisfying that the improved system is solvable has full Lebesgue measure [Gro38] and the exceptional set has full Hausdorff dimension [Sch69]. The elements of the exceptional set are called *badly approximable linear forms*.

In this thesis, we consider the inhomogeneous Diophantine approximation: the distribution of $A\mathbf{q}$ modulo $\mathbb{Z}^m$ near a "target" $\mathbf{b} \in \mathbb{R}^m$. Although Dirichlet theorem does not hold anymore, there exist infinitely many $q \in \mathbb{Z}$ such that

$$|q\alpha - b - p| < 1/|q| \quad \text{for some } p \in \mathbb{Z}$$

for almost every $(\alpha, b) \in \mathbb{R}^2$ and moreover,

$$\liminf_{p,q \in \mathbb{Z}, |q| \to \infty} |q||q\alpha - b - p| = 0$$

for almost every $(\alpha, b) \in \mathbb{R}^2$ by the inhomogeneous Khintchine theorem ([Cas57, Theorem II in Chapter VII]).

Similarly to numbers, for an $m \times n$ real matrix $A \in M_{m,n}(\mathbb{R})$, we study $Aq \in \mathbb{R}^m$ modulo $\mathbb{Z}^m$ near the target $b \in \mathbb{R}^m$ for vectors $q \in \mathbb{Z}^n$. In this general situation as well, using inhomogeneous Khintchine-Groshev theorem ([Sch64, Theorem1] or [Spr79, Chapter1, Theorem 15]), we have

$$\liminf_{q\in\mathbb{Z}^n, \|q\|\to\infty} \|q\|^n \langle Aq-b\rangle^m = 0$$

for almost every $(A,b)\in M_{m,n}(\mathbb{R})\times\mathbb{R}^m$. Here, $\langle v\rangle := \inf_{p\in\mathbb{Z}^m}\|v-p\|$ denotes the distance from $v\in\mathbb{R}^m$ to the nearest integral vector with respect to the supremum norm $\|\cdot\|$.

The exceptional set of the above equality is our object of interest. We will consider the exceptional set with weights in the following sense. Let us first fix, throughout the paper, an m-tuple and an n-tuple of positive reals $\mathbf{r}=(r_1,\cdots,r_m)$, $\mathbf{s}=(s_1,\cdots,s_n)$ such that $\sum_{1\le i\le m} r_i = 1 = \sum_{1\le j\le n} s_j$. The special case where $r_i = 1/m$ and $s_j = 1/n$ for all $i=1,\dots,m$ and $j=1,\dots,n$ is called the unweighted case.

Define the $\mathbf{r}$-quasinorm of $\mathbf{x}\in\mathbb{R}^m$ and $\mathbf{s}$-quasinorm of $\mathbf{y}\in\mathbb{R}^n$ by

$$\|\mathbf{x}\|_{\mathbf{r}} := \max_{1\le i\le m}|x_i|^{\frac{1}{r_i}} \quad\text{and}\quad \|\mathbf{y}\|_{\mathbf{s}} := \max_{1\le j\le n}|y_j|^{\frac{1}{s_j}}.$$

Denote $\langle\mathbf{x}\rangle_{\mathbf{r}} := \inf_{p\in\mathbb{Z}^m}\|\mathbf{x}-p\|_{\mathbf{r}}$. We call A *ϵ-bad* for $b\in\mathbb{R}^m$ if

$$\liminf_{q\in\mathbb{Z}^n, \|q\|_{\mathbf{r}}\to\infty} \|q\|_{\mathbf{s}}\langle Aq-b\rangle_{\mathbf{r}} \ge \epsilon. \tag{1.6}$$

Denote

$$\begin{aligned}
\mathbf{Bad}(\epsilon) &:= \{(A,b)\in M_{m,n}(\mathbb{R})\times\mathbb{R}^m : A \text{ is } \epsilon\text{-bad for } b\},\\
\mathbf{Bad}_A(\epsilon) &:= \{b\in\mathbb{R}^m : A \text{ is } \epsilon\text{-bad for } b\},\ \ \mathbf{Bad}_A := \bigcup_{\epsilon>0}\mathbf{Bad}_A(\epsilon),\\
\mathbf{Bad}^b(\epsilon) &:= \{A\in M_{m,n}(\mathbb{R}) : A \text{ is } \epsilon\text{-bad for } b\},\ \ \mathbf{Bad}^b := \bigcup_{\epsilon>0}\mathbf{Bad}^b(\epsilon).
\end{aligned}$$

The set $\mathbf{Bad}^0$ can be seen as the set of badly approximable systems of m linear forms in n variables. This set is of Lebesgue measure zero [Gro38], but has full Hausdorff dimension mn [Sch69]. See [PV02, KTV06, KW10] for the weighted setting.

For any b, $\mathbf{Bad}^b$ also has zero Lebesgue measure [Sch66] and full Hausdorff dimension for every b [ET11]. Indeed, it is shown that $\mathbf{Bad}^b$ is a winning set [ET11] and even a hyperplane winning set [HKS20], a property which implies full Hausdorff dimension. On the other hand, the set $\mathbf{Bad}_A$ also has full Hausdorff dimension for every A [BHKV10]. See [Har12, HM17, BM17] for the weighted setting.

The sets $\mathbf{Bad}^b$ and $\mathbf{Bad}_A$ are unions of subsets $\mathbf{Bad}^b(\epsilon)$ and $\mathbf{Bad}_A(\epsilon)$ over $\epsilon > 0$, respectively, thus a more refined question is whether the Hausdorff dimension of $\mathbf{Bad}^b(\epsilon)$, $\mathbf{Bad}_A(\epsilon)$ could still be of full dimension. For the homogeneous case ($b = 0$), the Hausdorff dimension $\mathbf{Bad}^0(\epsilon)$ is less than the full dimension mn (see [BK13, Sim18] for the unweighted case and [KM19] for the weighted case). Thus, a natural question is whether $\mathbf{Bad}^b(\epsilon)$ can have full Hausdorff dimension for some b. Our first main result says that in the unweighted case, $\mathbf{Bad}^b(\epsilon)$ cannot have full Hausdorff dimension for any b. We provide an effective bound on the dimension in terms of ϵ as well.

Theorem 1.2.1. *For the unweighted case, i.e. $r_i = 1/m$ and $s_j = 1/n$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$, there exist $c > 0$ and $M_0 > 0$ depending only on d such that for any $\epsilon > 0$ and $b \in \mathbb{R}^m$,*

$$\dim_H \mathbf{Bad}^b(\epsilon) \le mn - c\epsilon^{M_0}.$$

As for the set $\mathbf{Bad}_A(\epsilon)$, it was showed in [LSS19] that Hausdorff dimension of $\mathbf{Bad}_A(\epsilon)$ is less than the full dimension m for almost every A. In fact, it was shown that one can associate to A a certain point x_A in the space of unimodular lattices $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$ such that if x_A has no escape of mass on average for a certain diagonal flow, which is satisfied by almost every point, then the Hausdorff dimension of $\mathbf{Bad}_A(\epsilon)$ is less than m.

In this thesis, we provide an effective bound on the dimension in terms of ϵ and a certain Diophantine property of A as follows. We say that an $m \times n$ matrix A is *singular on average* if for any $\epsilon > 0$

$$\lim_{N\to\infty} \frac{1}{N} \left|\left\{ l \in \{1, \cdots, N\} : \exists q \in \mathbb{Z}^n \text{ s.t. } \langle Aq \rangle_{\mathbf{r}} < \epsilon 2^{-l} \text{ and } 0 < \|q\|_{\mathbf{s}} < 2^l \right\}\right| = 1.$$

Theorem 1.2.2. *For any $A \in M_{m,n}(\mathbb{R})$ which is not singular on average, there exists a constant $c(A) > 0$ depending on A such that for any $\epsilon > 0$, $\dim_H \mathbf{Bad}_A(\epsilon) \le m - c(A)\frac{\epsilon}{\log(1/\epsilon)}$.*

On the other hand, it was showed in [BKLR21] that in the one-dimensional case ($m = n = 1$), $\mathbf{Bad}_\alpha(\epsilon)$ has full Hausdorff dimension for some $\epsilon > 0$ if and only if $\alpha \in \mathbb{R}$ is singular on average. We generalize this characterization to the general dimensional setting.

Theorem 1.2.3. *Let $A \in M_{m,n}(\mathbb{R})$ be a matrix. Then the following are equivalent:*

1. *For some $\epsilon > 0$, the set $\mathbf{Bad}_A(\epsilon)$ has full Hausdorff dimension.*
2. *A is singular on average.*

Note that the implication (1) $\implies$ (2) of Theorem 1.2.3 follows from Theorem 1.2.2. The other direction will be shown in Section 3.6.

1.3 Badly approximable affine forms over global function field

As an extension of Diophantine approximation, we can consider Diophantine approximation over a local field of positive characteristic, which goes back to E. Artin, who first introduced continued fractions over a local field of positive characteristic [Art24]. In this setting, there are numerous results on Diophantine approximation, see for instance [Las00] or [Bug04(1), Chapter 9].

On the other hand, little is known about Diophantine approximation over general global function fields. Thus, we will consider Diophantine approximation over global function fields. Let us start with the following setting of global function fields.

Let K be any global function field over a finite field $\mathbb{F}_q$ of q elements for a prime power q, that is, the function field of a geometrically connected smooth projective curve $\mathbf{C}$ over $\mathbb{F}_q$. The most studied example in Diophantine approximation in positive characteristic is the case of the field $K = \mathbb{F}_q(Z)$ of rational fractions in one variable Z over $\mathbb{F}_q$, where $\mathbf{C} = \mathbb{P}^1$ is the projective line, but we emphasize the fact that our work applies in the general situation above.

We fix a (normalized) discrete valuation v on K. Let K_v and $\mathcal{O}_v$ be the completion of K with respect to v and its valuation ring, respectively. We fix a uniformizer $\pi_v \in K$, which satisfies $v(\pi_v) = 1$. Let $k_v = \mathcal{O}_v/\pi_v\mathcal{O}_v$ be the residual field and let q_v be its cardinality. The (normalized) absolute value $|\cdot|$ associated with v is defined by $|x| = q_v^{-v(x)}$. For every $\sigma \in \mathbb{Z}_{\geq 1}$, let $\| \; \| : K_v^\sigma \to [0, +\infty[$ be the norm $(\xi_1, \dots, \xi_\sigma) \mapsto \max_{1 \leq i \leq \sigma} |\xi_i|$. We denote by $\dim_H$ the Hausdorff dimension of the subsets of K_v^σ for this standard norm.

The discrete object analogous to the set of integers $\mathbb{Z}$ in $\mathbb{R}$ is the affine algebra R_v of the curve $\mathbf{C}$ minus the point v. If $K = \mathbb{F}_q(Z)$ and $v = [1:0]$ is the standard point at infinity of $\mathbf{C} = \mathbb{P}^1$, then $R_v = \mathbb{F}_q[Z]$ is the ring of polynomials in Z over $\mathbb{F}_q$.

Let $m, n \in \mathbb{Z}_{\geq 1}$. Let us fix, throughout the paper, two *weights* consisting of a m-tuple $\mathbf{r} = (r_1, \cdots, r_m)$ and a n-tuple $\mathbf{s} = (s_1, \cdots, s_n)$ of positive integers such that we have $|\mathbf{r}| = \sum_{1\leq i\leq m} r_i = \sum_{1\leq j\leq n} s_j$. The $\mathbf{r}$-*quasinorm* of $\boldsymbol{\xi} \in K_v^m$ and $\boldsymbol{s}$-*quasinorm* of $\boldsymbol{\theta} \in K_v^n$ are given by

$$\| \boldsymbol{\xi} \|_{\mathbf{r}} = \max_{1\leq i\leq m} |\, \boldsymbol{\xi}_i \,|^{\frac{1}{r_i}} \quad \text{and} \quad \| \boldsymbol{\theta} \|_{\mathbf{s}} = \max_{1\leq j\leq n} |\, \boldsymbol{\theta}_j \,|^{\frac{1}{s_j}} \ .$$

We denote by $\langle \boldsymbol{\xi} \rangle_{\mathbf{r}} = \inf_{\mathbf{x}\in R_v^m} \| \boldsymbol{\xi} - \mathbf{x} \|_{\mathbf{r}}$ the (weighted) distance from $\boldsymbol{\xi}$ to the set R_v^m of integral vectors in K_v^m.

Let $\epsilon > 0$. A matrix $A \in \mathcal{M}_{m,n}(K_v)$ is said to be ϵ-*bad* for a vector $\boldsymbol{\theta} \in K_v^m$ if

$$\liminf_{\mathbf{x}\in R_v^n,\ \| \mathbf{x} \|_{\mathbf{s}}\to\infty} \| \mathbf{x} \|_{\mathbf{s}} \ \langle A\mathbf{x} - \boldsymbol{\theta} \rangle_{\mathbf{r}} \geq \epsilon \ . \tag{1.7}$$

Denote by $\mathbf{Bad}_A(\epsilon)$ the set of vectors $\boldsymbol{\theta} \in K_v^m$ such that A is ϵ-bad for $\boldsymbol{\theta}$. Given two subsets U and V of a given set, we denote $U - V = \{x \in U : x \notin V\}$. We say that a matrix $A \in \mathcal{M}_{m,n}(K_v)$ is $(\mathbf{r}, \mathbf{s})$-*singular on average* if for every $\epsilon > 0$, we have

(1.8)
$$\lim_{N\to\infty} \frac{1}{N} \ \mathrm{card}\{\, \ell \in \{1, \ldots, N\} : \exists\, \mathbf{y} \in R_v^n - \{0\},\ \langle A\,\mathbf{y} \rangle_{\mathbf{r}} \leq \epsilon\, q_v^{-\,\ell},\ \| \mathbf{y} \|_{\mathbf{s}} \leq q_v^{\ell} \,\} = 1 \ .$$

For the basic example of function field, when $K = \mathbb{F}_q[Z]$ and $v = [1:0]$, Bugeaud and Zhang [BZ19] found a sufficient condition (and an equivalent one when $n = m = 1$) on A for the Hausdorff dimension of $\mathbf{Bad}_A(\epsilon)$ to be full. We first strenghten and extend their result to general function fields.

Theorem 1.3.1. *Let $A \in \mathcal{M}_{m,n}(K_v)$ be a matrix. The following assertions are equivalent:*

1. *there exists $\epsilon > 0$ such that the set $\mathbf{Bad}_A(\epsilon)$ has full Hausdorff dimension,*
2. *the matrix A is $(\mathbf{r}, \mathbf{s})$-singular on average.*

We also provide an effective upper bound on the Hausdorff dimension in terms of ϵ, which is a new result even in the basic case $K = \mathbb{F}_q[Z]$ and $v = [1 : 0]$.

Theorem 1.3.2. *For every $A \in \mathcal{M}_{m,n}(K_v)$ which is not $(\mathbf{r}, \mathbf{s})$-singular on average, there exists a constant $c(A) > 0$ depending only on A, $\mathbf{r}$, $\mathbf{s}$, such that for every $\epsilon > 0$, we have $\dim_H \mathbf{Bad}_A(\epsilon) \leq m - c(A)\frac{\epsilon^{|\mathbf{r}|}}{\ln(1/\epsilon)}$.*

1.4 Weighted singular vectors

The simplest way to improve Dirichlet theorem is to replace the right hand side of (1.1) by $\frac{\epsilon}{T}$ for $0 < \epsilon < 1$ and such matrix $A \in M_{m,n}(\mathbb{R})$ is called *ϵ-Dirichlet improvable.* A matrix $A \in M_{m,n}(\mathbb{R})$ is called *singular* if it is ϵ-Dirichlet improvable for all $\epsilon > 0$, and denote by $\mathrm{Sing}(m, n)$ the set of singular $m \times n$ matrices.

The name *singular* derives from the fact that the set of singular vectors is a Lebesgue nullset. On the other hand, the computation of the Hausdorff dimension of the set of singular vectors, or more generally singular matrices, has been a challenge until the breakthrough [DFSU] using a variational principle in the parametric geometry of numbers. Historically, the first breakthrough was made in [Che11] to prove that the Hausdorff dimension of the set of 2-dimensional singular vectors is 4/3, which was extended in [CC16] to d-dimensional singular vectors. They proved that the set of d-dimensional singular vectors has Hausdorff dimension $d^2/(d+1)$. For general $m \times n$ singular matrices, it was proved in [KKLM17] that the Hausdorff dimension of $m \times n$ singular matrices is at most $mn(1-\frac{1}{m+n})$, and finally, it was shown in [DFSU] that the upper bound is sharp.

In this thesis, we consider the weighted version of the singularity as follows: Let $w = (w_1, \ldots, w_d) \in \mathbb{R}^d_{>0}$ be an ordered d-tuple of positive real numbers such that $\sum_i w_i = 1$ and $w_1 \geq \cdots \geq w_d$. We say that a vector $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ is *w-singular* if for every $\epsilon > 0$ there exists $T_0 > 1$ such that for all $T > T_0$ the system of inequalities

$$\max_{1 \leq i \leq d} |qx_i - p_i|^{\frac{1}{w_i}} < \frac{\epsilon}{T} \quad \text{and} \quad 0 < q < T \tag{1.9}$$

admits an integer solution $(\mathbf{p}, q) = (p_1, \ldots, p_d, q) \in \mathbb{Z}^d \times \mathbb{Z}$. Denote by $\mathrm{Sing}(w)$ the set of w-singular vectors in $\mathbb{R}^d$. Here and hereafter we always assume that the weight vector w satisfies the above assumption.

In the weighted setting, it was shown in [LSST20] that the set of 2-dimensional w-singular vectors has Hausdorff dimension $2 - \frac{1}{1+w_1}$. The aim of this thesis is to extend this 2-dimensional result to higher dimensions regarding the lower bound of the Hausdorff dimension.

Theorem 1.4.1. *For $d \geq 2$, the Hausdorff dimension of* $\mathrm{Sing}(w)$ *is at least* $d - \frac{1}{1+w_1}$.

One of the main ingredients of the proof of Theorem 1.4.1 is Dani's correspondence, which means that w-singular vectors correspond to certain divergent trajectories in the space $\mathcal{L}_{d+1}$ of unimodular lattices in $\mathbb{R}^{d+1}$. More precisely, let $a_t := \mathrm{diag}\left(e^{w_1 t}, \ldots, e^{w_d t}, e^{-1}\right) \in \mathrm{SL}_{d+1}(\mathbb{R})$ and let

$$h(x) := \begin{pmatrix} I_d & x \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_{d+1}(\mathbb{R}) \quad \text{for } x \in \mathbb{R}^d,$$

where I_d is the $d \times d$ identity matrix. Then x is w-singular if and only if the diagonal orbit $\left(a_t h(x)\mathbb{Z}^{d+1}\right)_{t\geq 0}$ is divergent.

Our method for the proof of Theorem 1.4.1 is basically extension of the method in [LSST20], hence we also have the following result as in [LSST20, Theorem 1.5].

Theorem 1.4.2. *For any $\Lambda \in \mathcal{L}_{d+1}$ and any nonempty open subset U in $\mathbb{R}^d$, the Hausdorff dimension of the set*

$$\{x \in U : (a_t h(x)\Lambda)_{t\geq 0} \text{ is divergent}\}$$

is at least $d - \frac{1}{1+w_1}$.

Theorem 1.4.2 implies the following corollary as in [LSST20, Corollary 1.6].

Corollary 1.4.3. *The Hausdorff dimension of the set*

$$\{\Lambda \in \mathcal{L}_{d+1} : (a_t\Lambda)_{t\geq 0} \text{ is divergent}\}$$

is at least $\dim \mathrm{SL}_{d+1}(\mathbb{R}) - \frac{1}{1+w_1} = (d+1)^2 - 1 - \frac{1}{1+w_1}$.

Recently, Solan [Sol] established a variational principle in the parametric geometry of numbers for general flows. Following his notations, we consider the following two subgroups:

$$H = \{g \in \mathrm{SL}_{d+1}(\mathbb{R}) : a_{-t} g a_t \to I_{d+1} \text{ as } t \to \infty\},$$
$$H' = \{h(x) \in \mathrm{SL}_{d+1}(\mathbb{R}) : x \in \mathbb{R}^d\}.$$

Note that H is the unstable horospherical subgroup of a_1. In the unweighted setting ($w_1 = \cdots = w_d$), the two subgroups H and H' are the same, but in general, H is bigger than H'. One of the applications of the variational principle for general flows in [Sol] is to give an upper bound of the Hausdorff dimension of the set

$$\mathrm{Sing}(H, \Lambda; a_t) = \{h \in H : (a_t h \Lambda)_{t \geq 0} \text{ is divergent}\}.$$

More precisely, [Sol, Corollary 2.34] implies that the Hausdorff dimension of $\mathrm{Sing}(H, \Lambda; a_t)$ is at most $\dim H - \frac{1}{1+w_1}$. On the other hand, Theorem 1.4.2 implies that the Hausdorff dimension of $\mathrm{Sing}(H, \Lambda; a_t)$ is at least $\dim H - \frac{1}{1+w_1}$, hence we have the following corollary.

Corollary 1.4.4. *The Hausdorff dimension of* $\mathrm{Sing}(H, \Lambda; a_t)$ *is* $\dim H - \frac{1}{1+w_1}$.

The thesis is organized as follows. In Chapter 2, we obtain some weak L^1 estimates and establish local ubiquity systems. Combining these two results and Transference Principle in Diophantine approximation, we conclude the proof of the main theorems in Section 1.1. In Chapter 3, we review classical entropy theory, interpret the entropy theory in terms of homogeneous dynamics, and establish the effective version of variational principle. By constructing some "well-behaved" σ-algebras and certain invariant measures with large entropy, we conclude the effective upper bound of the main theorems in Section 1.2. For the lower bound part, we characterize the singularity on average in terms of best approximation vectors and construct modified Bugeaud-Laurent sequences, which implies the lower bound of the main theorems in Section 1.2. In Chapter 4, we review global function fields and develop some basic Diophantine properties in the global function field setting. Following Chapter 3, we establish the effective variational principle and construct certain "well-behaved" σ-algebras and invariant measures with large entropy, which concludes the effective upper bound of the main theorem in Section 1.3. We also characterize the singularity on average and construct modified Bugeaud-Zhang sequences to obtain the lower bound part in the main theorem in Section 1.3. In Chapter 5, we review some notions : rooted trees, fractal structures, and self-affine structures. Then we calculate the lower bound of Hausdorff dimension of the fractal set related to the self-affine structure. We construct the self-affine structure related to weighted singular vectors and estimate sharp lattice point counting, which concludes the main theorems in Section 1.4.

Chapter 2

Equidistribution and Ubiquitous system

2.1 Preliminaries

2.1.1 Hausdorff measure and auxiliary lemmas

Below we give a brief introduction to Hausdorff measure and dimension. For further details, see [Fal14].

Let E be a subset of a Euclidean space $\mathbb{R}^k$. For $\delta > 0$, a cover $\mathcal{C}$ of E is called a *δ-cover* of E if $\operatorname{diam}(C) \leq \delta$ for all $C \in \mathcal{C}$. For $0 \leq s \leq k$, let

$$\mathcal{H}^s_\delta(E) = \inf \sum_{C \in \mathcal{C}} \operatorname{diam}(C)^s,$$

where the infimum is taken over all finite or countable δ-cover $\mathcal{C}$ of E. Then the *s-dimensional Hausdorff measure* of a set E is defined by

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E).$$

Finally, the *Hausdorff dimension* of E is given by

$$\dim_H(E) = \inf\{s \geq 0 : \mathcal{H}^s(E) = 0\}.$$

The following principle commonly known as the Mass Distribuiton Principle [Fal14, §4.1] will be used to show the divergent part of Theorem 1.1.4.

Lemma 2.1.1 (Mass Distribution Principle). *Let μ be a probability measure supported on $F \subset \mathbb{R}^k$. Suppose there are positive constants $c > 0$, $r_0 > 0$, and $0 \le s \le k$ such that*

$$\mu(B) \le cr^s$$

for any ball B with radius $r \le r_0$. If E is a subset of F with $\mu(E) = \lambda > 0$ then $\mathcal{H}^s(E) \ge \lambda/c$.

We state the Hausdorff measure version of the Borel-Cantelli lemma [BD99, Lemma 3.10] which will allow us to estimate the Hausdorff measure of the convergent part of Theorem 1.1.4 and Theorem 1.1.5.

Lemma 2.1.2 (Hausdorff-Cantelli). *Let $\{B_i\}_{i\ge1}$ be a sequence of measurable sets in $\mathbb{R}^k$ and suppose that for some $0 \le s \le k$,*

$$\sum_i \operatorname{diam}(B_i)^s < \infty.$$

Then

$$\mathcal{H}^s(\limsup_{i\to\infty} B_i) = 0.$$

2.1.2 Homogeneous dynamics

Our argument is based on the Dani correspondence, which forms a connection between Diophantine approximation and homogeneous dynamics. The classical Dani correspondence for homogeneous Diophantine approximation dates back to [Dan85] (See also [KM99]). The analogous correspondence between inhomogeneous Diophantine approximation and the dynamics in the space of grids have been used in [Kle99, Sha11, ET11, LSS19, GV18]. In this section, we introduce the space of grids in $\mathbb{R}^{m+n}$ and the diagonal flow on this space. For $d = m + n$, let

$$G_d = SL_d(\mathbb{R}) \quad \text{and} \quad \widehat{G}_d = ASL_d(\mathbb{R}) = G_d \ltimes \mathbb{R}^d,$$

and let

$$\Gamma_d = SL_d(\mathbb{Z}) \quad \text{and} \quad \widehat{\Gamma}_d = ASL_d(\mathbb{Z}) = \Gamma_d \ltimes \mathbb{Z}^d.$$

Elements of $\widehat{G}_d$ will be denoted by $\langle g, \mathbf{w}\rangle$, where $g \in G_d$ and $\mathbf{w} \in \mathbb{R}^d$. Denote by $X_d = G_d/\Gamma_d$ the space of unimodular lattices in $\mathbb{R}^d$ and denote by $\widehat{X}_d = \widehat{G}_d/\widehat{\Gamma}_d$ the space of unimodular grids, i.e. affine shifts of unimodular lattices in $\mathbb{R}^d$.

For simplicity, let $G := G_d$, $X := X_d$ and denote by m_X the Haar probability measure on X_d. For $t \in \mathbb{R}$, let

$$a_t := \mathrm{diag}(e^{t/m}, \dots, e^{t/m}, e^{-t/n}, \dots, e^{-t/n}).$$

Let us denote by

$$u_A := \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} \in G_d \quad \text{and} \quad u_{A,\mathbf{b}} := \left\langle \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix}, \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix} \right\rangle \in \widehat{G}_d$$

for $A \in M_{m,n}(\mathbb{R})$ and $(A, \mathbf{b}) \in \widetilde{M}_{m,n}(\mathbb{R})$. Let us also denote by

$$\Lambda_A := u_A \mathbb{Z}^d \in X \quad \text{and} \quad \Lambda_{A,\mathbf{b}} := u_{A,\mathbf{b}} \mathbb{Z}^d \in \widehat{X}_d,$$

where $u_{A,\mathbf{b}}\mathbb{Z}^d = \left\{ \begin{pmatrix} A\mathbf{q} + \mathbf{b} - \mathbf{p} \\ \mathbf{q} \end{pmatrix} : \mathbf{p} \in \mathbb{Z}^m, \mathbf{q} \in \mathbb{Z}^n \right\}$. The *expanding horospherical subgroup* of G_d with respect to $\{a_t : t > 0\}$ is given by $H := \{u_A : A \in M_{m,n}(\mathbb{R})\}$. On the other hand, the *nonexpanding horospherical subgroup* of G_d with respect to $\{a_t : t > 0\}$ is given by

$$\tilde{H} := \left\{ \begin{pmatrix} P & 0 \\ R & Q \end{pmatrix} : P \in M_{m,m}(\mathbb{R}), Q \in M_{n,n}(\mathbb{R}), R \in M_{n,m}(\mathbb{R}), \det(P)\det(Q) = 1 \right\}.$$

Note that $\tilde{H}$ is the complementary subgroup to H. We denote by m_H and $m_{\tilde{H}}$ the left-invariant Haar measure on H and $\tilde{H}$, respectively.

Let $\mathbf{d}$ be a right invariant metric on G. We can take $\mathbf{d}$ to satisfy $||g - id|| \leq \mathbf{d}(g, id)$ for g in the sufficiently small ball $B_r^G(id)$, where $||\cdot||$ is the supremum norm on $M_{d,d}(\mathbb{R})$. This metric induces metrics on $H, \tilde{H}$, and X by restriction. We let $B_r^G(id), B_r^H(id), B_r^{\tilde{H}}(id)$, and $B_r^X(id)$ denote the open ball in $G, H, \tilde{H}$, and X of radius r centered at the identity, respectively.

Following [KW19], we define the functions $\Delta : \widehat{X}_d \to [-\infty, \infty)$ by

$$\Delta(\Lambda) := \log \inf_{\mathbf{v} \in \Lambda} \|\mathbf{v}\|,$$

which can be considered as the logarithm of a height function.

Lemma 2.1.3. *[KM99, Lemma 8.3] Let $m, n \in \mathbb{N}$ and $T_0 \in \mathbb{R}_+$ be given. Suppose $\psi : [T_0, \infty) \to \mathbb{R}_+$ is a continuous, non-increasing function. Then* *there exists a unique continuous function*

$$z = z_\psi : [t_0, \infty) \to \mathbb{R},$$

where $t_0 := \frac{m}{m+n} \log T_0 - \frac{n}{m+n} \log \psi(T_0)$, *such that*

1. *the function* $t \mapsto t + nz(t)$ *is strictly increasing and unbounded;*
2. *the function* $t \mapsto t - mz(t)$ *is nondecreasing;*
3. $\psi(e^{t+nz(t)}) = e^{-t+mz(t)}$ *for all* $t \geq t_0$.

The following lemma reduces the inhomogeneous Diophantine approximation problem to the shrinking target problem on the space of grids.

Lemma 2.1.4. *[KM99, KW19] Let* $\psi : [T_0, \infty) \to \mathbb{R}_+$ *be a non-increasing continuous function, and let* $z = z_\psi$ *be the function associated to* ψ *by Lemma 2.1.3. Then* $(A, \mathbf{b}) \in \widehat{D}_{m,n}(\psi)$ *if and only if* $\Delta(a_t \Lambda_{A,\mathbf{b}}) < z_\psi(t)$ *for all sufficiently large* t.

Remark 2.1.5. *In other words, Lemma 2.1.4 means that*

$$\widehat{D}_{m,n}(\psi)^c = \limsup_{t\to\infty} \{(A, \mathbf{b}) : \Delta(a_t \Lambda_{A,\mathbf{b}}) \geq z_\psi(t)\},$$

$$\widehat{D}^{\mathbf{b}}_{m,n}(\psi)^c = \limsup_{t\to\infty} \{A : \Delta(a_t \Lambda_{A,\mathbf{b}}) \geq z_\psi(t)\}.$$

Here, the limsup sets are taken for real values $t \in \mathbb{R}$. *However, in the proof of the convergent part, we are going to work with limsup sets taken for* $t \in \mathbb{N}$ *to apply the Hausdorff-Cantelli lemma. Thus, in the Section 3, we will use the following alternative: there exists a constant* $C_0 > 0$ *satisfying*

$$\widehat{D}_{m,n}(\psi)^c \subseteq \limsup_{t\to\infty, t\in\mathbb{N}} \{(A, \mathbf{b}) : \Delta(a_t \Lambda_{A,\mathbf{b}}) \geq z_\psi(t) - C_0\},$$

$$\widehat{D}^{\mathbf{b}}_{m,n}(\psi)^c \subseteq \limsup_{t\to\infty, t\in\mathbb{N}} \{A : \Delta(a_t \Lambda_{A,\mathbf{b}}) \geq z_\psi(t) - C_0\}.$$

This alternative holds since z_ψ *is uniformly continuous by Lemma 2.1.3 and* Δ *is uniformly continuous on the set* $\Delta^{-1}([z, \infty])$ *for any* $z \in \mathbb{R}$ *([KW19, Lemma 2.1]).*

Lemma 2.1.6. *Let* $\psi : [T_0, \infty) \to \mathbb{R}_+$ *be a non-increasing continuous function, and let* $z = z_\psi$ *be the function associated to* ψ *by Lemma 2.1.3. For* $0 \leq s \leq$

mn, we have

$$\sum_{q=\lceil T_0\rceil}^{\infty} \frac{1}{\psi(q)q^2}\left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}}\right)^{mn-s} < \infty \iff \sum_{t=\lceil t_0\rceil}^{\infty} e^{-(m+n)\left(z(t)-\frac{mn-s}{mn}t\right)} < \infty.$$

Proof. Note that if $0 \le s \le mn-n$, both of the sum is infinity regardless of ψ, thus we may assume $mn-n < s \le mn$. Following [KM99] and [KW19], we replace the sums with integrals

$$\int_{T_0}^{\infty} \frac{1}{\psi(x)x^2}\left(\frac{x^{\frac{1}{n}}}{\psi(x)^{\frac{1}{m}}}\right)^{mn-s} dx \quad \text{and} \quad \int_{t_0}^{\infty} e^{-(m+n)\left(z(t)-\frac{mn-s}{mn}t\right)} dt$$

respectively. Define

$$P := -\log\circ\psi\circ\exp : [T_0,\infty) \to \mathbb{R} \quad \text{and} \quad \lambda(t) := t + nz(t).$$

Since $\psi(e^\lambda) = e^{-P(\lambda)}$, we have

$$\int_{T_0}^{\infty} \frac{1}{\psi(x)x^2}\left(\frac{x^{\frac{1}{n}}}{\psi(x)^{\frac{1}{m}}}\right)^{mn-s} dx = \int_{\log T_0}^{\infty} e^{-\left(1-\frac{mn-s}{n}\right)\lambda+\left(1+\frac{mn-s}{m}\right)P(\lambda)} d\lambda$$

Using $P(\lambda(t)) = t - mz(t)$, we also have

$$\begin{aligned}
&\int_{t_0}^{\infty} e^{-(m+n)\left(z(t)-\frac{mn-s}{mn}t\right)} dt \\
&\quad = \int_{\log T_0}^{\infty} e^{-\left(1-\frac{mn-s}{n}\right)\lambda+\left(1+\frac{mn-s}{m}\right)P(\lambda)} d\left[\frac{m}{m+n}\lambda + \frac{n}{m+n}P(\lambda)\right] \\
&\quad = \frac{m}{m+n}\int_{\log T_0}^{\infty} e^{-\left(1-\frac{mn-s}{n}\right)\lambda+\left(1+\frac{mn-s}{m}\right)P(\lambda)} d\lambda \\
&\quad + \frac{n}{m+n}\int_{\log T_0}^{\infty} e^{-\left(1-\frac{mn-s}{n}\right)\lambda+\left(1+\frac{mn-s}{m}\right)P(\lambda)} dP(\lambda).
\end{aligned}$$

The second term in the last line can be expressed by

$$\begin{aligned}
&\frac{n}{m+n}\int_{\log T_0}^{\infty} e^{-\left(1-\frac{mn-s}{n}\right)\lambda+\left(1+\frac{mn-s}{m}\right)P(\lambda)} dP(\lambda) \\
&= \frac{n}{m+n}\left(1+\frac{mn-s}{m}\right)^{-1}\int_{\log T_0}^{\infty} e^{-\left(1-\frac{mn-s}{n}\right)\lambda} d\left(e^{\left(1+\frac{mn-s}{m}\right)P(\lambda)}\right)
\end{aligned}$$

Using integration by parts, the last integral is

$$\begin{aligned}
&\int_{\log T_0}^{\infty} e^{-\left(1-\frac{mn-s}{n}\right)\lambda} d\left(e^{\left(1+\frac{mn-s}{m}\right)P(\lambda)}\right) \\
&= \left(1-\frac{mn-s}{n}\right)\int_{\log T_0}^{\infty} e^{-\left(1-\frac{mn-s}{n}\right)\lambda+\left(1+\frac{mn-s}{m}\right)P(\lambda)} d\lambda \\
&+ \left(\lim_{\lambda\to\infty} e^{-\left(1-\frac{mn-s}{n}\right)\lambda+\left(1+\frac{mn-s}{m}\right)P(\lambda)} - T_0^{-\left(1-\frac{mn-s}{n}\right)}\psi(T_0)^{-\left(1+\frac{mn-s}{m}\right)}\right).
\end{aligned}$$

Note that $\lim_{\lambda\to\infty} e^{-\left(1-\frac{mn-s}{n}\right)\lambda+\left(1+\frac{mn-s}{m}\right)P(\lambda)} = 0$ if the integral

$$\int_{\log T_0}^{\infty} e^{-\left(1-\frac{mn-s}{n}\right)\lambda+\left(1+\frac{mn-s}{m}\right)P(\lambda)} d\lambda$$

converges. Thus the convergence of

$$\int_{T_0}^{\infty} \frac{1}{\psi(x)x^2}\left(\frac{x^{\frac{1}{n}}}{\psi(x)^{\frac{1}{m}}}\right)^{mn-s} dx \text{ or } \int_{t_0}^{\infty} e^{-(m+n)\left(z(t)-\frac{mn-s}{mn}t\right)} dt$$

implies the convergence of the other one since all summands are positive except the finite value $-T_0^{-\left(1-\frac{mn-s}{n}\right)}\psi(T_0)^{-\left(1+\frac{mn-s}{m}\right)}$.

□

2.2 Equidistribution and Weak-L^1 estimate

To obtain the upper bound of Hausdorff dimension, we will basically count the number of covering balls following the ideas from [KKLM17]. We are going to use the equidistribution of expanding subgroup of the a_t-action on X to compute the Lebesgue measure of the set of points visiting the shrinking target for each time t, following the "thickening" technique of Margulis [Mar04]. We also refer to the formulation of [KM96]. However, if we apply the thickening argument for L^2 functions as usual, it does not give the optimal dimension upper bound. To obtain the optimal dimension bound, we need a $L^{1,w}$ estimate as the following Proposition 2.2.1. $L^{1,w}$ norm of a function f on X is defined by $||f||_{L^{1,w}(X)} := \sup_{M>0} M m_X(\{x \in X : |f(x)| \geq M\})$, and $L^{1,w}(X)$ is the space of measurable functions with finite $L^{1,w}$-norm.

Proposition 2.2.1. *Let $H, \tilde{H}$ be the maximal expanding, nonexpanding sub-*

group of G_d, respectively. Assume that $f \in L^{1,w}(X)$ is a nonnegative function satisfying the following condition: there exist $c, r_0 > 0$ such that $c < |\frac{f(\tilde{h}x)}{f(x)}|$ for any $\tilde{h} \in B^{\tilde{H}}_{r_0}(id)$, $x \in X$. Then for any $x \in X$, there exist a constant $K = K(x) > 0$ such that

$$m_H(\{h \in B^H_1(id) : f(a_t hx) \geq M\}) \leq \frac{K}{M}$$

for all $M > 0$, $t > 0$, i.e. for any $x \in X$, $||(a_t)_* f_x||_{L^{1,w}(B^H_1(id))}$ is uniformly bounded for all $t > 0$, where the function $f_x : H \to \mathbb{R}$ is defined by $f_x(h) = f(hx)$.

Proof. Fix $x \in X$ and let $E_{M,t} := \left\{h \in B^H_1(id) : f(a_t hx) \geq M\right\}$. For contradiction, suppose that for any $K > 0$, there exist $t, M > 0$ such that $m_H(E_{M,t}) > \frac{K}{M}$. Let $\hat{E}_{M,t} := \left\{\tilde{h}h : h \in E_{M,t}, \tilde{h} \in B^{\tilde{H}}_r(id)\right\}$, where $0 < r < r_0$ is a small real number to be determined later. Then for any $\tilde{h}h \in \hat{E}_{M,t}$,

$$f(a_t\tilde{h}hx) = f((a_t\tilde{h}a_t^{-1})a_t hx) > cf(a_t hx) \geq cM$$

since $a_t\tilde{h}a_t^{-1} \in B^{\tilde{H}}_r(id)$. We partition $B^H_1(id)$ into $D_1, \cdots D_N$ so that a map $\pi_x : G \to X$ defined by $\pi_x(g) = gx$ is injective on each D_i. Note that the number of the partition N is not depending on K. Choose r small enough so that π_x is injective on $B^{\tilde{H}}_r(id)D_i$ for all $1 \leq i \leq N$. Let $E_{M,t} = \bigsqcup_{i=1}^{N} E_i$, where $E_i = E_{M,t} \cap D_i$, then

$$\begin{aligned} m_X(\{y \in X : f(y) \geq cM\}) &= m_X(\{y \in X : f(a_t y) \geq cM\}) \\ &\geq m_X(\left\{\tilde{h}hx \in X : \tilde{h}h \in \hat{E}_{M,t}\right\}) \\ &\geq m_G(\left\{\tilde{h}h \in G : \tilde{h} \in B^{\tilde{H}}_r(id), h \in E_i\right\}) \\ &\asymp m_{\tilde{H}}(B^{\tilde{H}}_r(id))m_H(E_i) \end{aligned}$$

for all $1 \leq i \leq N$. Summing over $1 \leq i \leq N$, we have

$$\begin{aligned} Nm_X(\{y \in X : f(y) \geq cM\}) &\gg m_{\tilde{H}}(B^{\tilde{H}}_r(id))m_H(E_{M,t}) \\ &> \frac{m_{\tilde{H}}(B^{\tilde{H}}_r(id))K}{M} \end{aligned}$$

and it implies $\|f\|_{L^{1,w}(X)} = \infty$ since $K > 0$ is arbitrary and c, r, N are independent to K. It contradicts the assumption $f \in L^{1,w}(X)$. $\square$

2.3 Application to Diophantine approximation

2.3.1 Successive minima function

Let $\lambda_j(\Lambda)$ denote the j-th successive minimum of a lattice $\Lambda \subseteq \mathbb{R}^d$ i.e. the infimum of λ such that the ball $B_\lambda^{\mathbb{R}^d}(0)$ contains j independent vectors of Λ. The following inequality explains the relationship between the successive minima functions λ_1 and λ_d.

Theorem 2.3.1 (Mahler's inequality, [Cas59], Theorem VI in Chapter VIII)**.** *For any lattice $\Lambda \subseteq \mathbb{R}^d$, $1 \leq \lambda_1(\Lambda^*)\lambda_d(\Lambda) \leq d!$ holds, where Λ^* is the dual lattice of Λ.*

Note that the Haar measure m_X is invariant under the dual operation since the dual operation is induced by the transpose of the inverse of a matrix, which is an automorphism of G. Another ingredient we will use is Siegel's integral formula.

Theorem 2.3.2 (Siegel's integral formula)**.** *For a compactly supported integrable function $f \in L^1(\mathbb{R}^d)$, we define a function $\hat{f}$ on X by*

$$\hat{f}(\Lambda) = \sum_{v \in \Lambda \setminus \{0\}} f(v).$$

Then for any f as above, $\int_X \hat{f} dm_X = \int_{\mathbb{R}^d} f dm_{\mathbb{R}^d}$.

In the following Proposition 2.3.3 and 2.3.4, we will show that the function λ_d^d satisfies the assumption of Proposition 2.2.1.

Proposition 2.3.3. $\lambda_d^d \in L^{1,w}(X)$.

Proof. For any $r > 0$,

$$\begin{aligned} m_X(\left\{\Lambda : \lambda_d^d(\Lambda) \geq (d!)^d r^{-d}\right\}) &= m_X(\{\Lambda : \lambda_d(\Lambda) \geq d! r^{-1}\}) \\ &\leq m_X(\{\Lambda : \lambda_1(\Lambda^*) \leq r\}) \\ &= m_X(\{\Lambda : \lambda_1(\Lambda) \leq r\}) \\ &\leq \int_{\Lambda \in X} \widehat{\chi_{B_r(0)}}(\Lambda) dm_X(\Lambda) \\ &= \int_{\mathbb{R}^d} \chi_{B_r(0)} dm_{\mathbb{R}^d} \asymp r^d, \end{aligned} \tag{2.1}$$

thus $\lambda_d^d \in L^{1,w}(X)$. In (3.28), the second line is by Mahler's inequality, the third line is by the invariance of m_X under the dual operation, the fourth line is using the fact that $\lambda_1(\Lambda) \leq r$ implies $\widehat{\chi_{B_r(0)}}(\Lambda) \geq 1$, and the last line is by Siegel's integral formula. □

Proposition 2.3.4. *For any $0 < c < 1$, there exists $r > 0$ such that for any $g \in G$ with $\mathbf{d}(g, id) < r$, $c\lambda_d(\Lambda) < \lambda_d(g\Lambda) < c^{-1}\lambda_d(\Lambda)$ holds for any $\Lambda \in X$.*

Proof. It suffices to show the statement under the stronger assumption that both of g and g^{-1} are in the ball $B_r^G(id)$. Then there exist independent vectors $v_1, \cdots, v_d \in \Lambda$ such that $\|v_1\| \leq \|v_2\| \leq \cdots \leq \|v_d\| = \lambda_d(\Lambda)$. For each $1 \leq i \leq d$,

$$\|gv_i - v_i\| \leq d||g - id|| \|v_i\| \leq dr\lambda_d(\Lambda),$$

thus $\|gv_i\| \leq (1+dr)\lambda_d(\Lambda)$. It implies $\lambda_d(g\Lambda) \leq (1+dr)\lambda_d(\Lambda)$ since $gv_1, \cdots, gv_d$ are independent vectors. Applying this for g^{-1} and $g\Lambda$, instead of g and Λ, we have

$$\lambda_d(\Lambda) = \lambda_d(g^{-1}g\Lambda) \leq (1 + dr)\lambda_d(g\Lambda).$$

Thus for any $\Lambda \in X$ and $g \in B_r^G(id)$, $(1+dr)^{-1}\lambda_d(\Lambda) < \lambda_d(g\Lambda) < (1+dr)\lambda_d(\Lambda)$ holds. □

2.3.2 The number of covering balls

In this subsection, we will construct a sequence of coverings for $\widehat{D}_{m,n}(\psi)^c$ and $\widehat{D}^{\mathbf{b}}_{m,n}(\psi)^c$ to apply Hausdorff-Cantelli Theorem. Recall that we adopt the supremum norm $\|\cdot\|$ on $[0, 1]^{mn}$.

Proposition 2.3.5. *Let $C_0 > 0$ be a constant described in Remark 2.1.5. For $t \in \mathbb{N}$, let $Z_t := \{A \in [0, 1]^{mn} : \log(d\lambda_d(a_t\Lambda_A)) \geq z_\psi(t) - C_0\}$. Then Z_t can be covered with $Ke^{(m+n)(t-z_\psi(t))}$ balls in $M_{m,n}(\mathbb{R})$ of radius $\frac{1}{2}e^{-(\frac{1}{m}+\frac{1}{n})t}$ for a constant $K > 0$ depending only on the dimension d.*

Proof. $[0, 1]^{mn}$ can be covered with $p(\asymp e^{(m+n)t})$ cubes $D_1, D_2, \cdots, D_p$ with sides parallel to the axes of $\mathbb{R}^{mn}$ and of sidelength $r \leq e^{-(\frac{1}{m}+\frac{1}{n})t}$ and having mutually disjoint interiors.

Lemma 2.3.6. *For $t \in \mathbb{N}$, let*

$$Z'_t := \left\{A \in [0, 1]^{mn} : \log(d^2\lambda_d(a_t\Lambda_A)) \geq z_{\psi(t)} - C_0 - 1\right\}.$$

For any $t \geq 1$, if $D_i \cap Z_t \neq \phi$ for some $1 \leq i \leq p$, then $D_i \subset Z'_t$.

Proof. Assume that there exists $x \in D_i$ but $x \notin Z'_t$ for some $t > 0$. Choose a point $y \in D_i \cap Z_t$, then $\|x - y\| \leq r$ and

$$\begin{aligned} ||a_t u_{x-y} a_{-t} - id|| &= \left\| \begin{pmatrix} I_m & e^{(\frac{1}{m}+\frac{1}{n})t}(x-y) \\ & I_n \end{pmatrix} - id \right\| \\ &\leq \|e^{(\frac{1}{m}+\frac{1}{n})t}(x-y)\| \leq 1. \end{aligned}$$

Thus, for $g = a_t u_{x-y} a_{-t}$, it satisfies $||g - id|| \leq 1$ and $a_t\Lambda_y = g a_t \Lambda_x$. On the other hand, $\log(d^2\lambda_d(a_t\Lambda_x)) < z_\psi(t) - C_0 - 1$, $\log(d\lambda_d(a_t\Lambda_y)) \geq z_\psi(t) - C_0$ hold since $x \notin Z'_t$, $y \in Z_t$. We can take independent vectors $v_1, \cdots, v_d \in \mathbb{R}^d$ in the lattice $a_t\Lambda_x$ satisfying $\|v_i\| < \frac{1}{d^2} e^{z_\psi(t) - C_0 - 1}$ for all $1 \leq i \leq d$. Let $w_i = g v_i$, then w_i's are independent vectors in the lattice $a_t\Lambda_y$ and satisfy

$$\|w_i\| \leq d||g||\|v_i\| \leq 2d\|v_i\| < \frac{2}{d} e^{z_\psi(t)-C_0-1} < \frac{1}{d} e^{z_\psi(t)-C_0}$$

for all $1 \leq i \leq d$. Thus we obtain $\log(d\lambda_d(a_t\Lambda_y)) < z_\psi(t) - C_0$ but it contradicts to $y \in Z_t$. □

Let $p' := |\{D_i : D_i \cap Z_t \neq \phi\}|$ and by reordering the D_i's if necessary, we can assume that $\{D_1, \cdots, D_{p'}\} = \{D_i : D_i \cap Z_t \neq \phi\}$. Then $Z_t \subseteq \bigcup_{i=1}^{p'} D_i \subseteq Z'_t$ by Lemma 2.3.6. Now we will apply Proposition 2.2.1 for the function λ_d^d with the base point $x = \mathbb{Z}^d$. By Proposition 2.3.3 and Proposition 2.3.4, λ_d^d satisfies the conditions of Proposition 2.2.1. Then we have

$$\begin{aligned} m_{\mathbb{R}^{mn}}(Z'_t) &\leq m_{\mathbb{R}^{mn}}\left(\left\{A \in [0,1]^{mn} : \lambda_d(a_t\Lambda_A) \geq \frac{1}{d^2} e^{z_\psi(t)-C_0-1}\right\}\right) \\ &\asymp m_H\left(\left\{h \in B_1^H(id) : \lambda_d^d(a_t h\mathbb{Z}^d) \geq \frac{1}{d^{2d}} e^{d(z_\psi(t)-C_0-1)}\right\}\right) \\ &\ll e^{-dz_\psi(t)}. \end{aligned}$$

On the other hand, $m_{\mathbb{R}^{mn}}(Z'_t) \geq \sum_{i=1}^{p'} m_{\mathbb{R}^{mn}}(D_i) = p' e^{-dt}$ holds, thus we finally obtain $p' \ll e^{d(t - z_\psi(t))}$. It means that Z_t can be covered by $\ll e^{d(t-z_\psi(t))}$ many balls of r-radius since $Z_t \subseteq \bigcup_{i=1}^{p'} D_i$. □

Proposition 2.3.7. *Let* $0 \le s \le mn$. *If* $\sum_{q=1}^{\infty} \frac{1}{\psi(q)q^2} \left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}} \right)^{mn-s} < \infty$, *then* $\mathcal{H}^s(\limsup_{t\to\infty} Z_t) = 0$ *and* $\mathcal{H}^{s+m}(\limsup_{t\to\infty} Z_t \times [0,1]^m) = 0$.

Proof. By Lemma 2.1.6, the assumption $\sum_{q=1}^{\infty} \frac{1}{\psi(q)q^2} \left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}} \right)^{mn-s} < \infty$ is equivalent to $\sum_{t=1}^{\infty} e^{-(m+n)\left(z(t)-\frac{mn-s}{mn}t\right)} < \infty$. For each $t \in \mathbb{N}$, let us denote by $D_{t1}, D_{t2}, \cdots, D_{tp_t}$ the balls of radius $\frac{1}{2}e^{-(\frac{1}{m}+\frac{1}{n})t}$ covering Z_t as in Proposition 2.3.5. Note that p_t, the number of the balls, is not greater than $Ke^{(m+n)(t-z_\psi(t))}$ by Proposition 2.3.5. By applying Lemma 2.1.2 to a sequence of balls $\{D_{tj}\}_{t\in\mathbb{N},1\le j\le p_t}$, we have $\mathcal{H}^s(\limsup_{t\to\infty} Z_t) \le \mathcal{H}^s(\limsup_{t\to\infty} D_{tj}) = 0$.

We prove the second statement by a similar argument. Proposition 2.3.5 implies that $Z_t \times [0,1]^m$ can be covered with $Ke^{\frac{m+n}{n}t}e^{(m+n)(t-z_\psi(t))}$ balls of radius $\frac{1}{2}e^{-(\frac{1}{m}+\frac{1}{n})t}$. Applying Lemma 2.1.2 again, we have $\mathcal{H}^{s+m}(\limsup_{t\to\infty} Z_t \times [0,1]^m) = 0$.

□

The convergent part of Theorem 1.1.4 and 1.1.5 follows this proposition.

Proof of Theorem 1.1.4 and 1.1.5. We first prove the singly metric case, Theorem 1.1.5. We claim that $\log(d\lambda_d(a_t\Lambda_A)) \ge \Delta(a_t\Lambda_{A,\mathbf{b}})$ for every $\mathbf{b} \in \mathbb{R}^m$. Let $v_1, \cdots, v_d$ be independent vectors satisfying $\|v_i\| \le \lambda_d(a_t\Lambda_A)$ for $1 \le i \le d$. Then there exists a vector of $a_t\Lambda_{A,\mathbf{b}}$ which can be written as a form of $\sum_{i=1}^{d} \alpha_i v_i$ for some $-1 \le \alpha_i \le 1$'s, so the length of the shortest vector is $\le \sum_{i=1}^{d} \|v_i\|$. Thus, $\Delta(a_t\Lambda_{A,\mathbf{b}}) \le \log \sum_{i=1}^{d} \|v_i\| \le \log(d\lambda_d(a_t\Lambda_A))$. It implies $\widehat{D}^{\mathbf{b}}_{m,n}(\psi)^c \subseteq \limsup_{t\to\infty} \{A \in [0,1]^{mn} : \Delta(a_t\Lambda_{A,\mathbf{b}}) \ge z_\psi(t) - C_0\} \subseteq \limsup_{t\to\infty} Z_t$ by Lemma 2.1.4, thus we obtain $\mathcal{H}^s(\widehat{D}^{\mathbf{b}}_{m,n}(\psi)^c) \le \mathcal{H}^s(\limsup_{t\to\infty} Z_t) = 0$ by Proposition 2.3.7.

Similarly for the doubly metric case, together with the second statement of Proposition 2.3.7,

$$\begin{aligned}\widehat{D}_{m,n}(\psi)^c &\subseteq \limsup_{t\to\infty} \left\{(A,\mathbf{b}) \in [0,1]^{mn+m} : \Delta(a_t\Lambda_{A,\mathbf{b}}) \ge z_\psi(t) - C_0\right\} \\ &\subseteq \limsup_{t\to\infty} Z_t \times [0,1]^m\end{aligned}$$

provides the proof of Theorem 1.1.4. □

2.4 Local ubiquitous system

2.4.1 Historical Remarks

The proof of the divergent parts of Theorem 1.1.5, that is the singly metric case, is based on the ubiquity framework developed in [BDV06, BV09]. The concept of ubiquitous systems goes back to [BS70] and [DRV90] as a method of determining lower bounds for the Hausdorff dimension of limsup sets. This concept was developed by Beresnevich, Dickinson and Velani in [BDV06] to provide a very general and abstract approach for establishing the Hausdorff measure of a large class of limsup sets. In this subsection, we introduce a simplified form of ubiquitous systems to deal with the specific application as in [BDV06, Section 12.1].

We consider $[0,1]^{mn}$ with the supremum norm $\|\cdot\|$. Let $\mathcal{R} := (R_\alpha)_{\alpha\in J}$ be a family of *resonant sets* $R_\alpha \subset [0,1]^{mn}$ indexed by a countable set J. We assume that each resonant set R_α is an $(m-1)n$-dimensional, rational hyperplane following [BDV06, Section 12.1]. Let $\beta : J \to \mathbb{R}^+ : \alpha \mapsto \beta_\alpha$ be a positive function on J for which the number of $\alpha \in J$ with β_α bounded above is always finite. Given a set $S \subset [0,1]^{mn}$, let

$$\Delta(S,r) := \{X \in [0,1]^{mn} : \operatorname{dist}(X,S) < r\},$$

where $\operatorname{dist}(X,S) := \inf\{\|X-Y\| : Y \in S\}$. Fix a decreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$, which is called the *approximating function*. For $N \in \mathbb{N}$, let

$$\Delta(\Psi,N) := \bigcup_{\alpha\in J\,:\,2^{N-1}<\beta_\alpha\le 2^N} \Delta(R_\alpha,\Psi(\beta_\alpha))$$

and let

$$\Lambda(\Psi) := \limsup_{N\to\infty} \Delta(\Psi,N) = \bigcap_{M=1}^{\infty} \bigcup_{N=M}^{\infty} \Delta(\Psi,N).$$

Throughout, $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ will denote a function satisfying $\lim_{t\to\infty}\rho(t) = 0$ and is usually referred to as the *ubiquitous function*. Let

$$\Delta(\rho,N) := \bigcup_{\alpha\in J\,:\,2^{N-1}<\beta_\alpha\le 2^N} \Delta(R_\alpha,\rho(\beta_\alpha)).$$

Definition 2.4.1 (Local ubiquity)**.** *Let B be an arbitrary ball in $[0,1]^{mn}$. Suppose that there exist a ubiquitous function ρ and an absolute constant $\kappa > 0$ such that*

$$|B \cap \Delta(\rho, N)| \geq \kappa |B| \quad \text{for } N \geq N_0(B), \tag{2.2}$$

where $|\cdot|$ denotes the Lebesgue measure on $[0,1]^{mn}$. Then the pair $(\mathcal{R}, \beta)$ is said to be a locally ubiquitous system relative to ρ.

With notations in [BDV06], the Lebesgue measure on $[0,1]^{mn}$ is of type (M2) with $\delta = mn$ and the intersection conditions are also satisfied with $\gamma = (m-1)n$ (see [BDV06, Section 12.1]). These conditions are not stated here but these extra conditions exist and need to be established for the more abstract ubiquity.

Finally, a function h is said to be *2-regular* if there exists a positive constant $\lambda < 1$ such that for N sufficiently large

$$h(2^{N+1}) \leq \lambda h(2^N).$$

The following theorem is a simplified version of [BV09, Theorem 1].

Theorem 2.4.2. *[BV09, Theorem 1] Suppose that $(\mathcal{R}, \beta)$ is a local ubiquitous system relative to ρ and that Ψ is an approximating function. Furthermore, suppose that ρ is 2-regular. Then for $(m-1)n < s \leq mn$*

$$\mathcal{H}^s(\Lambda(\Psi)) = \mathcal{H}^s([0,1]^{mn}) \quad \text{if} \quad \sum_{N=1}^{\infty} \frac{\Psi(2^N)^{s-(m-1)n}}{\rho(2^N)^n} = \infty.$$

2.4.2 Transference Principle on Diophantine approximation

Let $d = m+n$ and assume that $\psi : [T_0, \infty) \to \mathbb{R}_+$ be a decreasing function satisfying $\lim_{T\to\infty} \psi(T) = 0$. Denote by $\|\cdot\|_{\mathbb{Z}}$ and $|\cdot|_{\mathbb{Z}}$ the distance to the nearest integral vector and integer, respectively. Define the function $\widetilde{\psi} : [S_0, \infty) \to \mathbb{R}_+$ by

$$\widetilde{\psi}(S) = \left(\psi^{-1}(S^{-m})\right)^{-\frac{1}{n}},$$

where $S_0 = \psi(T_0)^{-1/m}$. We associate ψ-Dirichlet non-improvability with $\widetilde{\psi}$-approximability via a transference lemma as follows.

Lemma 2.4.3 (A transference lemma, [Cas57]). *Given $(A, \mathbf{b}) \in \widetilde{M}_{m,n}(\mathbb{R})$, if the system*

$$\|{}^t A\mathbf{x}\|_{\mathbb{Z}} < d^{-1}|\mathbf{b}\cdot\mathbf{x}|_{\mathbb{Z}}\widetilde{\psi}(S) \quad \text{and} \quad \|\mathbf{x}\| < d^{-1}|\mathbf{b}\cdot\mathbf{x}|_{\mathbb{Z}}S$$

has a nontrivial solution $\mathbf{x} \in \mathbb{Z}^m$ for an unbounded set of $S \geq S_0$, then $(A, \mathbf{b}) \in \widehat{D}_{m,n}(\psi)^c$.

Proof. Using part A of Theorem XVII in Chapter V of [Cas57] with $C = \psi(T)^{1/m}$ and $X = T^{1/n}$, the fact that

$$\|A\mathbf{q} - \mathbf{b}\|_{\mathbb{Z}} \leq \psi(T)^{1/m} \quad \text{and} \quad \|\mathbf{q}\| \leq T^{1/n}$$

for some $\mathbf{q} \in \mathbb{Z}^n$ implies that

$$|\mathbf{b}\cdot\mathbf{x}|_{\mathbb{Z}} \leq d\max(T^{1/n}\|{}^t A\mathbf{x}\|_{\mathbb{Z}}, \psi(T)^{1/m}\|\mathbf{x}\|)$$

holds for all $\mathbf{x} \in \mathbb{Z}^m$. Thus the lemma follows with $S = \psi(T)^{-1/m}$ and $\widetilde{\psi}(S) = T^{-1/n}$ since $\lim_{T\to\infty}\psi(T) = 0$. □

Thus we adopt the following notations for each $S \geq S_0$ and $0 < \epsilon < 1/2$:

- Let $W_{S,\epsilon}$ be the set of $A \in [0,1]^{mn}$ such that there exists $\mathbf{x}_{A,S} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$ satisfying

$$\|{}^t A\mathbf{x}_{A,S}\|_{\mathbb{Z}} < d^{-1}\epsilon\widetilde{\psi}(S) \quad \text{and} \quad \|\mathbf{x}_{A,S}\| < d^{-1}\epsilon S.$$

- $\widehat{W}_{S,\epsilon} := \{(A, \mathbf{b}) \in [0,1]^{mn+m} : A \in W_{S,\epsilon} \text{ and } |\mathbf{b}\cdot\mathbf{x}_{A,S}|_{\mathbb{Z}} > \epsilon\}$.
- For fixed $\mathbf{b} \in \mathbb{R}^m$, let $W_{\mathbf{b},S,\epsilon}$ be the set of $A \in [0,1]^{mn}$ such that there exists $\mathbf{x} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$ satisfying
  - (i) $|\mathbf{b}\cdot\mathbf{x}|_{\mathbb{Z}} > \epsilon$,
  - (ii) $\|{}^t A\mathbf{x}\|_{\mathbb{Z}} < d^{-1}\epsilon\widetilde{\psi}(S)$ and $\|\mathbf{x}\| < d^{-1}\epsilon S$.
- $W_{\mathbf{b},\epsilon} := \limsup_{S\to\infty} W_{\mathbf{b},S,\epsilon}$.

Note that $A \in W_{S,\epsilon}$ if and only if

$$\|{}^t A\mathbf{x}_{A,S}\|_{\mathbb{Z}} < \Psi_\epsilon(U) \quad \text{and} \quad \|\mathbf{x}_{A,S}\| < U,$$

where

$$\Psi_\epsilon(U) := d^{-1}\epsilon\widetilde{\psi}(d\epsilon^{-1}U). \tag{2.3}$$

By Lemma 2.4.3, $\limsup_{S\to\infty} \widehat{W}_{S,\epsilon} \subset \widehat{D}_{m,n}(\psi)^c$ and $W_{\mathbf{b},\epsilon} \subset \widehat{D}^{\mathbf{b}}_{m,n}(\psi)^c$.

We remark that $\limsup_{S\to\infty} W_{S,\epsilon}$ is the set of Ψ_ϵ-approximable matrices, that is, $\limsup_{S\to\infty} W_{S,\epsilon} = \{A \in [0,1]^{mn} : {}^tA \in W_{n,m}(\Psi_\epsilon)\}$. Here and hereafter, as mentioned before in footnote 1, we adopt the slightly different definition for Ψ_ϵ-approximability, where the inequality $\|{}^tA\mathbf{x}\|_{\mathbb{Z}} < \Psi_\epsilon(\|\mathbf{x}\|)$ is used instead of (1.3). Then, $W_{\mathbf{b},\epsilon}$ can be considered as the set of Ψ_ϵ-approximable matrices with solutions restricted on the set $\{\mathbf{x} \in \mathbb{Z}^m : |\mathbf{b}\cdot\mathbf{x}|_{\mathbb{Z}} > \epsilon\}$.

2.4.3 Mass distributions on Ψ_ϵ-approximable matrices

In this subsection, we prove the divergent part of Theorem 1.1.4 using mass distributions on Ψ_ϵ-approximable matrices following [AB18].

Lemma 2.4.4. *For each $0 \le s \le mn$ and $0 < \epsilon < 1/2$, let $U_0 = d^{-1}\epsilon S_0$. Then,*

$$\sum_{q=\lceil T_0\rceil}^{\infty} \frac{1}{\psi(q)q^2}\left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}}\right)^{mn-s} < \infty \iff \sum_{h=\lceil U_0\rceil}^{\infty} h^{m+n-1}\left(\frac{\Psi_\epsilon(h)}{h}\right)^{s-n(m-1)} < \infty.$$

Proof. Since $\Psi_\epsilon(h) = d^{-1}\epsilon\widetilde{\psi}(d\epsilon^{-1}h)$,

$$\sum_{h=\lceil U_0\rceil}^{\infty} h^{m+n-1}\left(\frac{\Psi_\epsilon(h)}{h}\right)^{s-n(m-1)} < \infty \iff \sum_{q=\lceil S_0\rceil}^{\infty} q^{m+n-1}\left(\frac{\widetilde{\psi}(q)}{q}\right)^{s-n(m-1)} < \infty.$$

Thus, similar to Lemma 2.1.6, we may assume $mn - n < s \le mn$ and replace the sums with integrals

$$\int_{T_0}^{\infty} \frac{1}{\psi(x)x^2}\left(\frac{x^{\frac{1}{n}}}{\psi(x)^{\frac{1}{m}}}\right)^{mn-s} dx \quad \text{and} \quad \int_{S_0}^{\infty} y^{m+n-1}\left(\frac{\widetilde{\psi}(y)}{y}\right)^{s-n(m-1)} dy,$$

respectively. Since $\widetilde{\psi}(y) = \psi^{-1}(y^{-m})^{-\frac{1}{n}}$, we have

$$\begin{aligned}
&\int_{S_0}^{\infty} y^{m+n-1} \left(\frac{\widetilde{\psi}(y)}{y} \right)^{s-n(m-1)} dy \\
&\quad = \int_{S_0}^{\infty} y^{mn+m-1-s} \left(\psi^{-1}(y^{-m}) \right)^{m-1-\frac{s}{n}} dy \\
&\quad = \frac{1}{m} \int_{(S_0)^m}^{\infty} t^{n-\frac{s}{m}} \left(\psi^{-1}(t^{-1}) \right)^{m-1-\frac{s}{n}} dt \\
&\quad = \frac{1}{m} \int_{\psi^{-1}(S_0^{-m})}^{\infty} x^{m-1-\frac{s}{n}} \psi(x)^{-n+\frac{s}{m}} d\psi(x)^{-1} \\
&\quad = \frac{1}{m} \left(n+1-\frac{s}{m} \right)^{-1} \int_{T_0}^{\infty} x^{m-1-\frac{s}{n}} d \left(\psi(x)^{-1} \right)^{n+1-\frac{s}{m}}
\end{aligned}$$

Using integration by parts,

$$\begin{aligned}
&\int_{T_0}^{\infty} x^{m-1-\frac{s}{n}} d \left(\psi(x)^{-1} \right)^{n+1-\frac{s}{m}} \\
&= \left(\lim_{x\to\infty} x^{m-1-\frac{s}{n}} \psi(x)^{-n-1+\frac{s}{m}} - T_0^{m-1-\frac{s}{n}} \psi(T_0)^{-n-1+\frac{s}{m}} \right) \\
&+ \left(\frac{s-n(m-1)}{n} \right) \int_{T_0}^{\infty} \psi(x)^{-n-1+\frac{s}{m}} x^{m-2-\frac{s}{n}} dx.
\end{aligned}$$

Observe that

$$\int_{T_0}^{\infty} \psi(x)^{-n-1+\frac{s}{m}} x^{m-2-\frac{s}{n}} dx = \int_{T_0}^{\infty} \psi(x)^{-n-1+\frac{s}{m}} x^{m-1-\frac{s}{n}} d\log x.$$

Thus the convergence of $\int_{T_0}^{\infty} \psi(x)^{-n-1+\frac{s}{m}} x^{m-2-\frac{s}{n}} dx$ gives that

$$\lim_{x\to\infty} x^{m-1-\frac{s}{n}} \psi(x)^{-n-1+\frac{s}{m}} < \infty.$$

Hence the convergence of

$$\int_{T_0}^{\infty} \frac{1}{\psi(x)x^2} \left(\frac{x^{\frac{1}{n}}}{\psi(x)^{\frac{1}{m}}} \right)^{mn-s} dx \quad \text{or} \quad \int_{S_0}^{\infty} y^{m+n-1} \left(\frac{\widetilde{\psi}(y)}{y} \right)^{s-n(m-1)} dy$$

implies the convergence of the other one since all summands are positive except the finite value $-T_0^{m-1-\frac{s}{n}} \psi(T_0)^{-n-1+\frac{s}{m}}$. □

Lemma 2.4.5. *[AB18, Section 5] Assume that*

$$\sum_{q=1}^{\infty} \frac{1}{\psi(q)q^2} \left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}} \right)^{mn-s} = \infty.$$

Fix $0 < \epsilon < 1/2$. Then, for any $\eta > 1$, there exists a probability measure μ on $\limsup_{S\to\infty} W_{S,\epsilon}$ satisfying the condition that for any arbitrary ball D of sufficiently small radius $r(D)$ we have

$$\mu(D) \ll \frac{r(D)^s}{\eta},$$

where the implied constant does not depend on D or η.

Proof. Note that $\limsup_{S\to\infty} W_{S,\epsilon} = \{A \in [0,1]^{mn} : {}^tA \in W_{n,m}(\Psi_\epsilon)\}$. By Lemma 2.4.4, $\sum_{h=1}^{\infty} h^{m+n-1} \left(\frac{\Psi_\epsilon(h)}{h}\right)^{s-n(m-1)} = \infty$, which is the divergent assumption of Jarník's Theorem (Theorem 1.1.2) for $W_{n,m}(\Psi_\epsilon)$. From the proof of Jarník's Theorem in [AB18] and the construction of a probability measure in [AB18, Section 5] we can obtain a probability measure μ on $\limsup_{S\to\infty} W_{S,\epsilon}$ satisfying the above condition. □

Let us give a proof of the divergence part of Theorem 1.1.4.

Proof of Theorem 1.1.4. If $s = mn + m$, then it follows from Theorem 1.1.3. Assume that $m \leq s < mn + m$ and fix $0 < \epsilon < 1/2$. For any fixed $\eta > 1$, let μ be a probability measure on $\limsup_{S\to\infty} W_{S,\epsilon}$ as in Lemma 2.4.5 with $s - m$ instead of s. Consider the product measure $\nu = \mu \times m_{\mathbb{R}^m}$, where $m_{\mathbb{R}^m}$ is the canonical Lebesgue measure on $\mathbb{R}^m$, and let π_1 and π_2 be the natural projections from $\mathbb{R}^{mn+m}$ to $\mathbb{R}^{mn}$ and $\mathbb{R}^m$, respectively. For any fixed integer $N \geq 1$, let $V_{S,\epsilon} = W_{S,\epsilon} \setminus \bigcup_{k=N}^{S-1} W_{k,\epsilon}$ and $\widehat{V}_{S,\epsilon} = \{(A,\mathbf{b}) \in \widehat{W}_{S,\epsilon} : A \in V_{S,\epsilon}\}$ and $E_{A,S,\epsilon} = \{\mathbf{b} \in [0,1]^m : |\mathbf{b}\cdot\mathbf{x}_{A,S}|_{\mathbb{Z}} > \epsilon\}$. Note that $m_{\mathbb{R}^m}(E_{A,S,\epsilon}) \geq 1 - 2\epsilon$. Using

Fubini's theorem, we have

$$\begin{aligned}\nu(\bigcup_{S\geq N}\widehat{W}_{S,\epsilon}) &= \nu(\bigcup_{S\geq N}\widehat{V}_{S,\epsilon}) = \sum_{S\geq N}\nu(\widehat{V}_{S,\epsilon})\\ &\geq \sum_{S\geq N}(1-2\epsilon)\mu(V_{S,\epsilon}) = (1-2\epsilon)\mu(\bigcup_{S\geq N}W_{S,\epsilon})\\ &= 1-2\epsilon.\end{aligned}$$

Since $N\geq 1$ is arbitrary, we have $\nu(\limsup_{S\to\infty}\widehat{W}_{S,\epsilon})\geq 1-2\epsilon$.

For any arbitrary ball $B\subset\mathbb{R}^{mn+m}$ of sufficiently small radius $r(B)$, we have

$$\nu(B) = \mu(\pi_1(B))\times m_{\mathbb{R}^m}(\pi_2(B)) \ll \frac{r(B)^s}{\eta}, \tag{2.4}$$

where the implied constant does not depend on B or η. If $0\leq s<m$, we have (2.4) with μ in Lemma 2.4.5 with $s=0$.

By the Mass Distribution Principle (Lemma 2.1.1) and Lemma 2.4.3, we have

$$\mathcal{H}^s(\widehat{D}_{m,n}(\psi)^c)\geq\mathcal{H}^s(\limsup_{S\to\infty}\widehat{W}_{S,\epsilon})\gg(1-2\epsilon)\eta$$

and the proof is finished by taking $\eta\to\infty$.

□

2.4.4 Establishing the local ubiquity

The singly metric case is more complicated than the doubly metric case. In this subsection, we will prove Theorem 1.1.5 by establishing the ubiquitous system for $W_{\mathbf{b},\epsilon}$ with an appropriate ϵ as follows.

For $\mathbf{b}=(b_1,\dots,b_m)\in\mathbb{R}^m\setminus\mathbb{Z}^m$, define

$$\epsilon(\mathbf{b}) := \min_{1\leq j\leq m,\ |b_j|_{\mathbb{Z}}>0}\frac{|b_j|_{\mathbb{Z}}}{4}. \tag{2.5}$$

Note that the fact that $\mathbf{b}\in\mathbb{R}^m\setminus\mathbb{Z}^m$ implies $\epsilon(\mathbf{b})>0$. The following lemma will be used when we count the number of integral vectors $\mathbf{z}\in\mathbb{Z}^m$ such that

$$|\mathbf{b}\cdot\mathbf{z}|_{\mathbb{Z}}\leq\epsilon(\mathbf{b}). \tag{2.6}$$

Lemma 2.4.6. *For* $\mathbf{b} = (b_1, \dots, b_m) \in \mathbb{R}^m \setminus \mathbb{Z}^m$*, let* $\epsilon(\mathbf{b})$ *be as in* (2.5) *and* $1 \le i \le m$ *be an index such that* $\epsilon(\mathbf{b}) = \frac{|b_i|_{\mathbb{Z}}}{4}$*. Then, for any* $\mathbf{x} \in \mathbb{Z}^m$*, at most one of* $\mathbf{x}$ *and* $\mathbf{x} + \mathbf{e}_i$ *satisfies* (2.6)*, where* $\mathbf{e}_i$ *denotes the vector with a* 1 *in the* i*th coordinate and* 0*'s elsewhere.*

Proof. Observe that if $|\mathbf{b} \cdot \mathbf{x}|_{\mathbb{Z}} \le \epsilon(\mathbf{b})$, then

$$\big||\mathbf{b} \cdot (\mathbf{x} \pm \mathbf{e}_i)|_{\mathbb{Z}} - |\pm b_i|_{\mathbb{Z}}\big| \le |\mathbf{b} \cdot \mathbf{x}|_{\mathbb{Z}} \le \epsilon(\mathbf{b}).$$

By definition of $\epsilon(\mathbf{b})$, we have

$$|\mathbf{b} \cdot (\mathbf{x} \pm \mathbf{e}_i)|_{\mathbb{Z}} \ge |b_i|_{\mathbb{Z}} - \epsilon(\mathbf{b}) > \epsilon(\mathbf{b}).$$

□

Now we fix $\mathbf{b} \in \mathbb{R}^m \setminus \mathbb{Z}^m$ and write $\epsilon_0 := \epsilon(\mathbf{b})$ and $\Psi_0 := \Psi_{\epsilon_0}$ as we set in (2.3) and (2.5). Let

$$J := \{(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}^m \times \mathbb{Z}^n : \|\mathbf{y}\| \le m\|\mathbf{x}\| \text{ and } |\mathbf{b} \cdot \mathbf{x}|_{\mathbb{Z}} > \epsilon_0\}, \quad \Psi(h) := \frac{\Psi_0(h)}{h},$$
$$\alpha := (\mathbf{x}, \mathbf{y}) \in J, \quad \beta_\alpha := \|\mathbf{x}\|, \quad R_\alpha := \{A \in [0,1]^{mn} : {}^t A\mathbf{x} = \mathbf{y}\}.$$

Note that $W_{\mathbf{b},\epsilon_0} = \Lambda(\Psi)$ and the family $\mathcal{R}$ of resonant sets R_α consists of $(m-1)n$-dimensional, rational hyperplanes.

By Lemma 2.4.4, we may assume that $\sum_{h=1}^{\infty} h^{m+n-1} \left(\frac{\Psi_0(h)}{h}\right)^{s-n(m-1)} = \infty$. Then we can find a strictly increasing sequence of positive integers $\{h_i\}_{i\in\mathbb{N}}$ such that

$$\sum_{h_{i-1} < h \le h_i} h^{m+n-1} \left(\frac{\Psi_0(h)}{h}\right)^{s-(m-1)n} > 1$$

and $h_i > 2h_{i-1}$. Put $\omega(h) := i^{\frac{1}{n}}$ if $h_{i-1} < h \le h_i$. Then ω is 2-regular and

$$\sum_{h=1}^{\infty} h^{m+n-1} \left(\frac{\Psi_0(h)}{h}\right)^{s-n(m-1)} \omega(h)^{-n} = \infty.$$

For a constant $c > 0$, define the ubiquitous function $\rho_c : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\rho_c(h) := \begin{cases} ch^{-\frac{1+n}{n}} & \text{if } m = 1, \\ ch^{-\frac{m+n}{n}}\omega(h) & \text{if } m \ge 2, \end{cases} \tag{2.7}$$

Clearly the ubiquitous function is 2-regular.

Theorem 2.4.7. *The pair $(\mathcal{R}, \beta)$ is a locally ubiquitous system relative to $\rho = \rho_c$ for some constant $c > 0$.*

Proof of Theorem 1.1.5. For fixed $\mathbf{b} = (b_1, \ldots . b_m) \in \mathbb{R}^m \setminus \mathbb{Z}^m$, assume that $b_i \notin \mathbb{Z}$. If b_i is rational, then there is $0 < \epsilon < 1/2$ such that $|kb_i|_{\mathbb{Z}} > \epsilon$ for infinitely many positive integer k. If b_i is irrational, then the set $\{kb_i \pmod 1 : k \in \mathbb{Z}\}$ is dense in $[0, 1]$. Hence, for any fixed $0 < \epsilon < 1/2$, $|kb_i|_{\mathbb{Z}} > \epsilon$ holds for infinitely many positive integer k. Let us denote that increasing sequence by $(k_j)_{j=1}^{\infty}$. This observation implies that the set $\{A \in [0, 1]^{mn} : \|{}^tAk_j\mathbf{e}_i\|_{\mathbb{Z}} = 0\}$, which is the finite union of $(m-1)n$-dimensional hyperplanes, is a subset of $W_{\mathbf{b},\epsilon}$ for each $j \in \mathbb{N}$. Hence for any $0 \le s \le (m-1)n$

$$\mathcal{H}^s(D^{\mathbf{b}}_{m,n}(\psi)^c) \ge \mathcal{H}^s(W_{\mathbf{b},\epsilon}) = \mathcal{H}^s([0, 1]^{mn}).$$

Now assume that $(m-1)n < s \le mn$. It follows from Theorem 2.4.2 and Theorem 2.4.7 that

$$\mathcal{H}^s(D^{\mathbf{b}}_{m,n}(\psi)^c) \ge \mathcal{H}^s(W_{\mathbf{b},\epsilon_0}) = \mathcal{H}^s([0, 1]^{mn}).$$

Here, we use the fact that the divergence and convergence of the sums

$$\sum_{N=1}^{\infty} 2^{N\alpha} f(2^N) \quad \text{and} \quad \sum_{h=1}^{\infty} h^{\alpha-1} f(h) \quad \text{coincide}$$

for any monotonic function $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $\alpha \in \mathbb{R}$. $\square$

Recall that we adopt the supremum norm $\|\cdot\|$ on $[0, 1]^{mn}$. We consider $m = 1$ and $m \ge 2$, separately.

Proof of Theorem 2.4.7 for $m = 1$. Note that, for $(x, \mathbf{y}) \in J$, the resonant set $R_{(x,\mathbf{y})}$ is the one point set $\{\frac{\mathbf{y}}{x} := (\frac{y_1}{x}, \ldots, \frac{y_n}{x})\}$ and $\Delta(R_{x,\mathbf{y}}, \rho(2^N)) = B(\frac{\mathbf{y}}{x}, \rho(2^N))$, the ball of radius $\rho(2^N)$ centered at $\frac{\mathbf{y}}{x}$. We basically follow the strategy in [Tho04, Chapter 3].

Let B an arbitrary square in $[0, 1]^n$ and write $B = \prod_{i=1}^n [l_i, u_i]$, $\mathbf{l} = (l_1, \ldots, l_n)$, $\mathbf{u} = (u_1, \ldots, u_n)$. We restrict $\mathbf{y}$ to $\gcd(x, \mathbf{y}) = 1$ and $\frac{\mathbf{y}}{x} \in B$.

Observe that

$$|B\cap\Delta(\rho,N)| \geq \left| \bigcup_{\substack{2^{N-1}<x\leq 2^N\\ |b\cdot x|_{\mathbb{Z}}>\epsilon_0}} \bigcup_{\substack{x\mathbf{l}\leq \mathbf{y}\leq x\mathbf{u}\\ \gcd(x,\mathbf{y})=1}} B\left(\frac{\mathbf{y}}{x},\rho(2^N)\right)\right| + O(\rho(2^N)). \tag{2.8}$$

Here, $x\mathbf{l} < \mathbf{y} < x\mathbf{u}$ means that $xl_i < y_i < xu_i$ for all $1 \leq i \leq n$. Let

$$T(N) := \left\{\frac{\mathbf{y}}{x} \in \mathbb{Q}^n : (x,\mathbf{y}) \in J,\ \gcd(x,\mathbf{y}) = 1,\ x\mathbf{l} \leq \mathbf{y} \leq x\mathbf{u},\ 2^{N-1} < x \leq 2^N\right\},$$
$$G(N) := \left\{\frac{\mathbf{y}}{x} \in T(N) : B\left(\frac{\mathbf{y}}{x},\rho(2^N)\right) \cap B\left(\frac{\mathbf{s}}{t},\rho(2^N)\right) = \varnothing,\ \forall\frac{\mathbf{s}}{t}\left(\neq\frac{\mathbf{y}}{x}\right) \in T(N)\right\}.$$

Lemma 2.4.8. *For N large enough*

1. $\#T(N) \geq c_1|B|2^{N(n+1)}$ *for some constant* $0 < c_1 < 1$.
2. $\#G(N) \geq \frac{1}{2}\#T(N)$.

Thus, it follows from Lemma 2.4.8 that for N large enough

$$\begin{aligned}\text{r.h.s. of (2.8)} &\geq \left|\bigsqcup_{\frac{\mathbf{y}}{x}\in G(N)} B\left(\frac{\mathbf{y}}{x},\rho(2^N)\right)\right| + O(\rho(2^N))\\ &= \#G(N)\times 2^n\rho(2^N)^n + O(\rho(2^N))\\ &\geq \frac{1}{2}\#T(N)\times 2^n\rho(2^N)^n + O(\rho(2^N))\\ &\geq c^n c_1 2^{n-1}|B| + O(\rho(2^N)) \geq \frac{1}{2}c^n c_1 2^{n-1}|B|.\end{aligned}$$

Thus the local ubiquity follows from (2.8).

Proof of (1) *in Lemma 2.4.8.* Note that for $\alpha > 0$ and $\ell \in \mathbb{N}$

$$\begin{aligned}\sum_{\substack{1\leq k\leq\alpha\ell\\ \gcd(k,\ell)=1}} 1 &= \sum_{1\leq k\leq\alpha\ell}\sum_{d|\gcd(k,\ell)}\mu(d) = \sum_{d|\ell}\mu(d)\sum_{1\leq k'\leq\alpha\ell/d} 1\\ &= \sum_{d|\ell}\mu(d)\lfloor\alpha\ell/d\rfloor = \alpha\varphi(\ell) + O(\tau(\ell)).\end{aligned} \tag{2.9}$$

where $\tau(\ell) = \sum_{d|\ell} 1$, the number of divisors of ℓ. Here and hereafter, μ, φ, and $\lfloor\cdot\rfloor$ stand for the Möbius function, Euler function, and floor function, respectively.

Fix small $0 < \epsilon < \frac{3}{\pi^2} - \frac{1}{4}$. Note that $\frac{1}{N^2}\sum_{q=1}^{N}\varphi(q) \to \frac{3}{\pi^2}$ as $N \to \infty$ (see [HW60, Theorem 330]) and $\tau(h) = O(h^\delta)$ for any $\delta > 0$ (see [HW60, Theorem 315]). Thus, for N large enough and for $\delta > 0$ small enough,

$$\begin{aligned}
\#T(N) &= \sum_{\substack{2^{N-1}<x\leq 2^N \\ |b\cdot x|_{\mathbb{Z}}>\epsilon_0}} \sum_{\substack{x\mathbf{l}\leq\mathbf{y}\leq x\mathbf{u} \\ \gcd(x,\mathbf{y})=1}} 1 \geq \sum_{\substack{2^{N-1}<x\leq 2^N \\ |b\cdot x|_{\mathbb{Z}}>\epsilon_0}} \sum_{\substack{xl_i\leq y_i\leq xu_i \\ i=2,\dots,n}} \sum_{\substack{xl_1\leq y_1\leq xu_1 \\ \gcd(x,y_1)=1}} 1 \\
&\geq \sum_{\substack{2^{N-1}<x\leq 2^N \\ |b\cdot x|_{\mathbb{Z}}>\epsilon_0}} \left(|B|\varphi(x)x^{n-1} + O(x^{n-1}\tau(x))\right) \\
&\geq \sum_{\substack{2^{N-1}<x\leq 2^N \\ |b\cdot x|_{\mathbb{Z}}>\epsilon_0}} |B|\varphi(x)2^{(N-1)(n-1)} + O(2^{N(n+\delta)}) \\
&\geq |B|2^{(N-1)(n-1)}\left(\sum_{2^{N-1}<x\leq 2^N}\varphi(x) - \sum_{\substack{2^{N-1}<x\leq 2^N \\ |b\cdot x|_{\mathbb{Z}}\leq\epsilon_0}} x\right) + O(2^{N(n+\delta)}) \\
&\geq |B|2^{(N-1)(n-1)}\left(\frac{3}{\pi^2} - \frac{1}{4} - \epsilon\right)(2^{2N} - 2^{2(N-1)}) = c_1|B|2^{N(n+1)}.
\end{aligned}$$

The second line is by (2.9) and the fifth line is by Lemma 2.4.6. □

Proof of (2) *in Lemma 2.4.8.* Let $B(N) := T(N) \setminus G(N)$. By definition, $\frac{\mathbf{y}}{x} \in B(N)$ if and only if there is a point $\frac{\mathbf{s}}{t}\left(\neq \frac{\mathbf{y}}{x}\right) \in T(N)$ such that

$$B\left(\frac{\mathbf{y}}{x}, \rho(2^N)\right) \cap B\left(\frac{\mathbf{s}}{t}, \rho(2^N)\right) \neq \varnothing.$$

The coprimeness condition ensures that the centers $\frac{\mathbf{y}}{x}$ and $\frac{\mathbf{s}}{t}$ of the balls are distinct. Thus, we have $0 < \left\|\frac{\mathbf{y}}{x} - \frac{\mathbf{s}}{t}\right\| \leq 2\rho(2^N)$, or, equivalently,

$$0 < \|t\mathbf{y} - x\mathbf{s}\| \leq 2xt\rho(2^N).$$

It follows that the associated 4-tuple $(\mathbf{y}, x, \mathbf{s}, t)$ is an element of the set

$$\begin{aligned}
V(N) := \{(\mathbf{y}, x, \mathbf{s}, t) : 0 < \|t\mathbf{y} - x\mathbf{s}\| \leq 2^{2N+1}\rho(2^N),\ \gcd(x, \mathbf{y}) = \gcd(t, \mathbf{s}) = 1,& \\
2^{N-1} < x, t \leq 2^N,\ x\mathbf{l} \leq \mathbf{y} \leq x\mathbf{u},\ t\mathbf{l} \leq \mathbf{s} \leq t\mathbf{u}\}&
\end{aligned}$$

Hence, $\#B(N) \leq \#V(N)$ and it is enough to show that $\#V(N) < \frac{1}{2}\#T(N)$. Observe that if $n = 1$, then $V(N)$ is empty by taking $c < \frac{1}{2}$. We consider $n = 2$

and $n > 2$, separately.

Case n = 2. Note that $2^{2N+1}\rho(2^N) = 2c2^{N/2}$. If $(\mathbf{y}, x, \mathbf{s}, t) \in V(N)$, then there exist a_1, a_2 with $|a_i| \le 2c2^{N/2}$ and at least one of a_i's being nonzero, such that $ty_i - xs_i = a_i$ for all $i = 1, 2$. Let us denote by $V(a_1, a_2, N)$ the set of the above $(\mathbf{y}, x, \mathbf{s}, t) \in V(N)$ for given a_1, a_2.

We first consider the case either $a_1 = 0$ or $a_2 = 0$. Given $2^{N-1} < x, t \le 2^N$ and a, the number of solutions $(y, s) \in [1, 2^N]^2$ of the equation $ty - xs = a$ is less than $2\mathrm{gcd}(x, t)$ since the general solution of this equation is of the form $(y_0 + p\frac{x}{\mathrm{gcd}(x,t)}, s_0 + p\frac{t}{\mathrm{gcd}(x,t)})$ for $p \in \mathbb{Z}$. It follows that the number of elements $(\mathbf{y}, x, \mathbf{s}, t) \in V(N)$ such that either $a_1 = 0$ or $a_2 = 0$ is bounded above by

(2.10)
$$\begin{aligned}
&\sum_{1\le|a_1|\le 2c2^{\frac{N}{2}}} \#V(a_1, 0, N) + \sum_{1\le|a_2|\le 2c2^{\frac{N}{2}}} \#V(0, a_2, N) \\
&\le 4 \sum_{\substack{1\le a_1\le 2c2^{\frac{N}{2}} \\ (x,t)\in(2^{N-1},2^N]^2}} \#\left\{(y_1, s_1) : ty_1 - xs_1 = a_1\right\} \#\left\{(y_2, s_2) : ty_2 - xs_2 = 0\right\} \\
&\le 4 \sum_{1\le a_1\le 2c2^{\frac{N}{2}}} \sum_{\substack{(x,t)\in(2^{N-1},2^N]^2 \\ \mathrm{gcd}(x,t)|a_1}} (2\mathrm{gcd}(x, t))^2 \\
&= 16 \sum_{1\le d\le 2^N} \sum_{\substack{1\le a_1\le 2c2^{\frac{N}{2}} \\ d|a_1}} d^2 \# \left\{(x, t) \in (2^{N-1}, 2^N]^2 : \mathrm{gcd}(x, t) = d\right\} \\
&\ll \sum_{1\le d\le 2^N} \frac{2^{\frac{N}{2}}}{d} d^2 \left(\frac{2^{N-1}}{d}\right)^2 = O(N2^{\frac{5}{2}N}).
\end{aligned}$$

We now consider the case $a_1 \ne 0$ and $a_2 \ne 0$. Note that if $(\mathbf{y}, x, \mathbf{s}, t) \in V(a_1, a_2, N)$, then we have

$$a_1 y_2 - a_2 y_1 = kx \tag{2.11}$$

for some $k \in \mathbb{Z}$. Thus we will count the set of $(a_1, a_2, k, x, y_1, y_2)$ satisfying the equation (2.11) where $2^{N-1} < x \le 2^N$, $l_i x \le y_i \le u_i x$, and $1 \le |a_i| \le 2c2^{N/2}$ for $i = 1, 2$. Let us denote by $\bar{V}(N)$ the above set. We will only present the counting for the case $a_1 > 0$ and $a_2 > 0$, but the counting estimates remains the same for the cases of the other signs, and the proof also still works similarly.

For fixed $a_1 > 0$ and $a_2 > 0$, let us count the set of (k, x, y_1, y_2) such that $(a_1, a_2, k, x, y_1, y_2) \in \bar{V}(N)$. It follows from the equation (2.11) and $l_i x \le y_i \le$

$u_i x$ for $i = 1, 2$ that

$$a_1 l_2 - a_2 u_1 \le k \le a_1 u_2 - a_2 l_1.$$

Denoting by $d = \gcd(a_1, a_2)$, it follows from the equation (2.11) that $d|kx$. Thus we can write $d = d_1 d_2$, where $d_1|k$ and $d_2|x$, and denote by $a_i' = a_i/d$ for $i = 1, 2$, $k' = k/d_1$, and $x' = x/d_2$. Then we have

$$a_1' y_2 - a_2' y_1 = k'x'. \tag{2.12}$$

If $(\bar{y}_1, \bar{y}_2)$ is a solution of (2.12), then the general solution of (2.12) is of the form $(\bar{y}_1 + pa_1', \bar{y}_2 + pa_2')$ with $p \in \mathbb{Z}$. Hence the number of solution (y_1, y_2) of (2.12) with $l_i x \le y_i \le u_i x$ for $i = 1, 2$ is at most

$$\min\left(\left\lceil \frac{(u_1 - l_1)x}{a_1'} \right\rceil, \left\lceil \frac{(u_2 - l_2)x}{a_2'} \right\rceil\right) \le 2\min\left(\frac{(u_1 - l_1)x}{a_1'}, \frac{(u_2 - l_2)x}{a_2'}\right)$$

since $(u_i - l_i)x/a_i' \ge 1$ with $i = 1, 2$ for all large enough N. Hence it follows that for any small enough $\delta > 0$,

$$\begin{aligned}
&\sum_{1 \le a_1, a_2 \le 2c2^{N/2}} \#\{(k, x, y_1, y_2) : (a_1, a_2, k, x, y_1, y_2) \in \bar{V}(N)\} \\
&\le \sum_{1 \le a_1, a_2 \le 2c2^{N/2}} \sum_{d = d_1 d_2} \sum_{\substack{d_1|k, d_2|x \\ 2^{N-1} < x \le 2^N \\ a_1 l_2 - a_2 u_1 \le k \le a_1 u_2 - a_2 l_1}} 2\min\left(\frac{(u_1 - l_1)x}{a_1'}, \frac{(u_2 - l_2)x}{a_2'}\right).
\end{aligned}$$

Note that

$$\begin{aligned}
&\sum_{\substack{d_1|k, d_2|x \\ 2^{N-1} < x \le 2^N \\ a_1 l_2 - a_2 u_1 \le k \le a_1 u_2 - a_2 l_1}} 2\min\left(\frac{(u_1 - l_1)x}{a_1'}, \frac{(u_2 - l_2)x}{a_2'}\right) \\
&\le 2\left\lceil \frac{a_2(u_1 - l_1) + a_1(u_2 - l_2)}{d_1} \right\rceil \left\lceil \frac{2^N}{d_2} \right\rceil \min\left(\frac{(u_1 - l_1)x}{a_1'}, \frac{(u_2 - l_2)x}{a_2'}\right) \\
&\le 4\left(\frac{a_2(u_1 - l_1) + a_1(u_2 - l_2)}{d_1} + 1\right) \frac{2^N}{d_2} \min\left(\frac{(u_1 - l_1)2^N}{a_1'}, \frac{(u_2 - l_2)2^N}{a_2'}\right) \\
&\le 2^{2N+2}\left((a_2(u_1 - l_1) + a_1(u_2 - l_2))\min\left(\frac{u_1 - l_1}{a_1}, \frac{u_2 - l_2}{a_2}\right) + \frac{(u_1 - l_1)d_1}{a_1}\right).
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\sum_{1\le a_1,a_2\le 2c2^{N/2}} \#\{(k,x,y_1,y_2):(a_1,a_2,k,x,y_1,y_2)\in \bar{V}(N)\}\\
&\le \sum_{1\le a_1,a_2\le 2c2^{N/2}} \tau(d)\left(2^{2N+3}(u_1-l_1)(u_2-l_2)+d2^{2N+2}\frac{(u_1-l_1)}{a_1}\right)\\
&\ll \sum_{1\le d\le 2c2^{N/2}}\ \sum_{1\le a_1',a_2'\le \frac{2c2^{N/2}}{d}} d^{\delta}\left(2^{2N+3}|B|+2^{2N+2}\frac{(u_1-l_1)}{a_1'}\right)\\
&\ll \sum_{1\le d\le 2c2^{N/2}} \left(\frac{1}{d^{2-\delta}}c^2 2^{3N}|B|+\frac{1}{d^{1-\delta}}N2^{2N+\frac{N}{2}}\right)\\
&\ll c^2|B|2^{3N}+O(N2^{2N+\frac{(1+\delta)N}{2}}).
\end{aligned}$$

Combining with the cases of other signs, we have

$$\#\bar{V}(N)\ll c^2|B|2^{3N}+O(N2^{2N+\frac{(1+\delta)N}{2}}). \tag{2.13}$$

We next claim that $\sum_{1\le|a_1|,|a_2|\le 2c2^{N/2}}\#V(a_1,a_2,N)\le 2\#\bar{V}(N)$ by showing that for fixed $(a_1,a_2,k,x,y_1,y_2)\in\bar{V}(N)$, there are at most two pairs of (s_1,s_2,t) such that $(\mathbf{y},x,\mathbf{s},t)\in V(N)$. To see this, observe that $ty_i\equiv a_i \pmod x$ for $i=1,2$ and $\gcd(\mathbf{y},x)=1$. Since $\gcd(\mathbf{y},x)=1$, there exist $\alpha_1,\alpha_2\in\mathbb{Z}$ such that $\alpha_1y_1+\alpha_2y_2\equiv 1 (\mathrm{mod}\ x)$. It follows that $t\equiv a_1\alpha_1+a_2\alpha_2$ is uniquely determined modulo x for fixed a_i, y_i and x. Since $t<2x$, the number of possible t is at most two. Once t is determined, then s_1 and s_2 are also determined uniquely, thus the claim follows.

Hence, combining (2.10), (2.13), and the above claim, we have

$$\#V(N)\ll c^2|B|2^{3N}+O(N2^{2N+\frac{(1+\delta)N}{2}}+N2^{\frac{5}{2}N}).$$

By taking $\delta<1$, for all large enough N, $\#V(N)\le Cc^2|B|2^{3N}$ for some absolute constant $C>0$. It follows that $\#V(N)<\frac{1}{2}\#T(N)$ for sufficiently large N by choosing $c<(\frac{c_1}{2C})^{1/2}$.

Case $\mathbf{n>2}$. For fixed $2^{N-1}<x,t\le 2^N$, we denote by

$$d=\gcd(x,t),\ x'=\frac{x}{d},\ t'=\frac{t}{d},\ A=2^{2N+1}\rho(2^N),\ \text{and } A'=\frac{A}{d}.$$

We will count the following set: for $0 \leq a \leq A'$,

$$V_{x,t}(a) := \{(y, s) : t'y - x's = a,\ x\ell \leq y \leq xu,\ t\ell \leq s \leq tu\}.$$

Claim 1. $\#V_{x,t}(0) \leq \max(\lceil d(u-\ell) \rceil, 1)$.

Proof. Since $(t', x') = 1$, $x'|y$ holds. Thus

$$\#\{y : x'|y,\ x\ell \leq y \leq xu\} \leq \max\left(\left\lceil \frac{x(u-\ell)}{x'} \right\rceil, 1\right) = \max(\lceil d(u-\ell) \rceil, 1),$$

which concludes the claim since s is uniquely determined by y. □

Now assume that $a \neq 0$ and $A' \geq 1$. Let $y_0 = y_0(x,t)$ and $s_0 = s_0(x,t)$ be the integers with the smallest absolute value such that

$$t'y_0 \equiv a_0 \pmod{x'} \quad \text{and} \quad x's_0 \equiv -b_0 \pmod{t'},$$

for some $0 < a_0 = a_0(x,t) \leq A'$ and $0 < b_0 = b_0(x,t) \leq A'$. We remark that such y_0 and s_0 are unique since $a_0 \neq 0$.

Claim 2. $a_0 = b_0$ and $t'y_0 - x's_0 = a_0$.

Proof. Let y, s be such that $t'y_0 - x's = a_0$ and $t'y - x's_0 = b_0$. Then

$$\begin{aligned} |s| = \frac{|t'y_0 - a_0|}{x'} \leq \frac{a_0 + t'|y_0|}{x'} \leq \frac{a_0 + t'|y|}{x'} &= \frac{a_0 + |b_0 + x's_0|}{x'} \\ &\leq \frac{a_0 + b_0}{x'} + |s_0|. \end{aligned}$$

Since $n > 2$, for all large enough N,

$$\frac{a_0 + b_0}{x'} \leq \frac{2A'}{x'} = \frac{2A}{x} \leq \frac{2c2^{(1-\frac{1}{n})N}}{2^{N-1}} < 1.$$

Hence we have $|s| = |s_0|$, and similarly $|y| = |y_0|$. If $s = -s_0$, then on one hand, $x's \equiv b_0 (\bmod\ t')$; on the other hand, since $t'y_0 - x's = a_0$, we have $x's \equiv -a_0 \equiv x' - a_0 (\bmod\ t')$. It cannot happen that $b_0 = x' - a_0$ since $x' > 2A' \geq a_0 + b_0$. Hence we get $s = s_0$, and similarly $y = y_0$. It concludes the claim. □

Claim 3. $1 \leq |y_0|, |s_0| \leq \left\lceil \frac{2x'}{A'} \right\rceil = \left\lceil \frac{2x}{A} \right\rceil \leq 2^{\frac{N}{n}+1}$.

Proof. Consider the set $P = \left\{t', 2t', \ldots, \left\lceil \frac{2x'}{A'} \right\rceil t'\right\}$ modulo x'. Partition $[1, x'] \cap \mathbb{N}$ into $\lfloor A' \rfloor$ consecutive integers. Then the number of the partitions is at most $\left\lceil \frac{x'}{\lfloor A' \rfloor} \right\rceil$. It follows from $A' \geq 1$ that $2\lfloor A' \rfloor \geq A'$, hence, $\left\lceil \frac{x'}{\lfloor A' \rfloor} \right\rceil \leq \left\lceil \frac{2x'}{A'} \right\rceil$. By the pigeonhole principle, there are at least two elements of P in the same partition, say it' and jt' with $i \neq j$. Then $(i-j)t' \pmod{x'}$ is contained in $[1, \lfloor A' \rfloor]$ or $[-\lfloor A' \rfloor, -1]$. The fact that $|i-j| \leq \left\lceil \frac{2x'}{A'} \right\rceil$ and the minimality of y_0 imply the claim for y_0. Similarly, we can conclude the claim for s_0. □

Claim 4. If $|y_0| \leq (u-\ell)a_0$ or $|s_0| \leq (u-\ell)a_0$, then

$$\sum_{a=1}^{A'} \#V_{x,t}(a) \leq 10A(u-\ell).$$

Proof. It suffices to show the case $|y_0| \leq (u-l)a_0$. Let

$$\mathcal{A}_k := \{xl \leq y \leq xu : y \equiv k (\text{mod } y_0)\} = \{z_k, z_k + |y_0|, \cdots, z_k + \alpha_k |y_0|\}.$$

for $0 \leq k \leq |y_0| - 1$. Then $z_k \in \mathbb{N}$ is the element such that $xl \leq z_k < xl + |y_0|$ and $z_k \equiv k (\text{mod } y_0)$, and $\alpha_k \in \mathbb{N}$ satisfies $\alpha_k \leq \frac{x(u-l)}{|y_0|}$. Partition $\mathcal{A}_k$ into $M = \lfloor \frac{x'}{a_0} \rfloor$ consecutive integers. Recall that $\frac{x'}{a_0} \geq \frac{x'}{A'} \geq 1$ holds. Then the number of the partitions is at most $\lceil \frac{\alpha_k + 1}{M} \rceil \leq \frac{\frac{x(u-l)}{|y_0|} + 1}{\frac{x'}{2a_0}} + 1$.

Let $P = \{z'_k, z'_k + |y_0|, \cdots, z'_k + (M-1)|y_0|\}$ be a partition of M consecutive integers, where $z'_k \in \mathcal{A}_k$. To count the number of $y \in P$ such that $t'y \equiv a \pmod{x'}$ with $1 \leq a \leq A'$, we see the set $t'P$ modulo x'. Write $t'z'_k \equiv w (\text{mod } x')$ for some $0 \leq w < x'$. Then the elements of $t'P$ can be written $\{w, w + a_0, \cdots, w + (M-1)a_0\}$ or $\{w - a_0, \cdots, w - (M-1)a_0\} \pmod{x'}$ depending on the sign of y_0. Since $Ma_0 = \lfloor \frac{x'}{a_0} \rfloor a_0 \leq x'$, there are at most $\lceil \frac{A'}{a_0} \rceil$ elements in $t'P$ which are congruent to a modulo x' for some $1 \leq a \leq A'$.

To sum up, for each $\mathcal{A}_k$, there are at most

$$\begin{aligned} \lceil \frac{A'}{a_0} \rceil \cdot \lceil \frac{\alpha_k + 1}{M} \rceil &\leq \frac{2A'}{a_0} \left(\frac{2a_0 x(u-l) + 2a_0 |y_0|}{x' |y_0|} + 1 \right) \\ &= \frac{4A' d(u-l)}{|y_0|} + \frac{4A'}{x'} + \frac{2A'}{a_0} \end{aligned}$$

number of y such that $y \in \mathcal{A}_k$ and $t'y \equiv a (\text{mod } x')$ for some $1 \leq a \leq A'$. Since there are $|y_0|$ number of $\mathcal{A}_k$'s and s is uniquely determined by y, we have

$$\begin{aligned}\sum_{a=1}^{A'} \#V_{x,t}(a) &\leq |y_0| \left(\frac{4A'd(u-l)}{|y_0|} + \frac{4A'}{x'} + \frac{2A'}{a_0} \right) \\ &= 4A(u-l) + \frac{4A|y_0|}{x} + \frac{2A'|y_0|}{a_0} \\ &\leq 4A(u-l) + 4A(u-l)\frac{a_0}{x} + 2A'(u-l) \leq 10A(u-l).\end{aligned}$$

Here we used the assumption $|y_0| \leq (u-l)a_0$ in the last line. □

We remark that under the assumption of **Claim 4**, the counting of $V_{x,t}(a)$'s is good enough for our purpose. Thus we will count the set of x, t's such that y_0, s_0, a_0 may not satisfy the assumption of **Claim 4**.

Note that $\gcd(y_0, s_0) = 1$, otherwise it contradicts to the minimality of y_0, s_0. Through **Claim 3**, we consider the following sets and the map:

$$\begin{aligned} S_{\text{good}} &= \{(y_0, s_0, a_0) : |y_0| \leq 2^{\frac{N}{n}+1},\ |s_0| \leq 2^{\frac{N}{n}+1},\ \gcd(y_0, s_0) = 1, \\ &\qquad (u-\ell)^{-1}\min(|y_0|, |s_0|) \leq a_0 \leq A'\}, \\ S_{\text{bad}} &= \{(y_0, s_0, a_0) : |y_0| \leq 2^{\frac{N}{n}+1},\ |s_0| \leq 2^{\frac{N}{n}+1},\ \gcd(y_0, s_0) = 1, \\ &\qquad 1 \leq a_0 < (u-\ell)^{-1}\min(|y_0|, |s_0|)\}, \\ \pi &: (2^{N-1}, 2^N]^2 \ni (x,t) \mapsto (y_0(x,t), s_0(x,t), a_0(x,t)) \in S_{\text{good}} \cup S_{\text{bad}}. \end{aligned}$$

Let us first count the set $\pi^{-1}(S_{\text{bad}})$. For $(y_0, s_0, a_0) \in S_{\text{bad}}$, assume that there exists t'_0, x'_0 such that $t'_0 y_0 - x'_0 s_0 = a_0$. Since $\gcd(y_0, s_0) = 1$, all solutions of $t'y_0 - x's_0 = a_0$ can be represented in the form

$$(t', x') = (t'_0 + ks_0, x'_0 + ky_0), \quad k \in \mathbb{Z}.$$

Thus, for each $d \geq 1$,

$$\begin{aligned} &\#\left\{(x,t) \in (2^{N-1}, 2^N]^2 : y_0(x,t) = y_0,\ s_0(x,t) = s_0,\ \gcd(x,t) = d\right\} \\ &\leq \min\left(\frac{2^N}{d|s_0|}, \frac{2^N}{d|y_0|} \right). \end{aligned}$$

Summing over $1 \le d \le 2^N$, we have

$$\begin{aligned}&\#\left\{(x,t) \in (2^{N-1}, 2^N]^2 : y_0(x,t) = y_0,\ s_0(x,t) = s_0\right\}\\ &\le \sum_{1\le d\le 2^N} \frac{1}{d} \min\left(\frac{2^N}{|s_0|}, \frac{2^N}{|y_0|}\right) \ll N \min\left(\frac{2^N}{|s_0|}, \frac{2^N}{|y_0|}\right).\end{aligned}$$

Since $n > 2$, it follows that for all small enough $\delta > 0$,

$$\begin{aligned}\#\pi^{-1}(S_{\mathrm{bad}}) &\ll \sum_{\substack{|y_0|,|s_0|\le 2^{\frac{N}{n}+1}\\ 1\le a_0<(u-\ell)^{-1}\min(|y_0|,|s_0|)}} N \min\left(\frac{2^N}{|s_0|}, \frac{2^N}{|y_0|}\right)\\ &\le \sum_{|y_0|,|s_0|\le 2^{\frac{N}{n}+1}} N(u-\ell)^{-1} \min\left(\frac{2^N}{|s_0|}, \frac{2^N}{|y_0|}\right) \min(|y_0|, |s_0|)\\ &\le (u-\ell)^{-1} \sum_{|y_0|,|s_0|\le 2^{\frac{N}{n}+1}} N2^N \min\left(\frac{|y_0|}{|s_0|}, \frac{|s_0|}{|y_0|}\right)\\ &\ll (u-\ell)^{-1} \frac{N^2}{n} 2^{N+\frac{2N}{n}} \ll 2^{(2-\delta)N}.\end{aligned} \tag{2.14}$$

Now, for each $1 \le i \le n$ and $0 \le a \le A'$, let us denote by

$$V^i_{x,t}(a) := \{(y_i, s_i) : t'y_i - x's_i = a,\ x\ell_i \le y_i \le xu_i,\ t\ell_i \le s_i \le tu_i\}.$$

Then we have

$$\#V(N) \le \sum_{(x,t)\in(2^{N-1},2^N]^2} \prod_{i=1}^{n} \sum_{a=0}^{A'} \#V^i_{x,t}(a).$$

For $(x,t) \in \pi^{-1}(S_{\mathrm{good}})$ and sufficiently large N,

$$\sum_{a=1}^{A'} \#V^i_{x,t}(a) + \#V^i_{x,t}(0) \le 10A(u_i - l_i) + \max(A(u_i - l_i), 1) \le 11A(u_i - l_i). \tag{2.15}$$

We applied **Claim 4** for the first term, and **Claim 1** for the second term.

For each $0 \le a \le A'$, the number of solutions (y,s) of

$$t'y - x's = a,\ 1 \le y \le x,\ 1 \le s \le t$$

is at most d, hence $\#V^i_{x,t}(a) \le d$. For $(x,t) \in \pi^{-1}(S_{\text{bad}})$, it follows that

$$\sum_{a=0}^{A'} \#V^i_{x,t}(a) \le (A'+1)d \le 2A. \tag{2.16}$$

Therefore, combining (2.14), (2.15), and (2.16), we have

$$\begin{aligned}\#V(N) &\ll \left(\#\pi^{-1}(S_{\text{good}})\right) \prod_{i=1}^{n} A(u_i - \ell_i) + \left(\#\pi^{-1}(S_{\text{bad}})\right)(2A)^n \\ &\ll 2^{2N} A^n \prod_{i=1}^{n} (u_i - \ell_i) + O(2^{N(n+1-\delta)}) \ll c^n |B| 2^{N(n+1)},\end{aligned}$$

hence $\#V(N) \le Cc^n|B|2^{N(n+1)}$ for sufficiently large N and some absolute constant $C > 0$. It follows that $\#V(N) < \frac{1}{2}\#T(N)$ for sufficiently large N by choosing $c < (\frac{c_1}{2C})^{1/n}$. □

This proves Theorem 2.4.7 for $m = 1$. □

Proof of Theorem 2.4.7 for $m \ge 2$. Note that it suffices to show that

$$|\Delta(\rho, N)| \to 1 \quad \text{as} \quad N \to \infty \tag{2.17}$$

for the local unbiquity. Instead of the strategy for $m = 1$, we will use mean and variance techniques in [DV97] using the auxiliary function ω in (2.7).

Without loss of generality we may assume that $\epsilon_0 = \epsilon(\mathbf{b}) = \frac{|b_1|_{\mathbb{Z}}}{4}$. Let $I(N)$ denote the set of vectors $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}^m$ such that

1. $N \le x_1 \le 2N$,
2. For $i = 2, \dots, m$,
$$1 \le x_i \le \frac{N}{\omega(2N)^{\frac{1}{2(m-1)}}},$$
3. $\gcd(\mathbf{x}) = 1$,
4. $|\mathbf{b} \cdot \mathbf{x}|_{\mathbb{Z}} > \epsilon_0$.

Denote by $J(N) := \{(\mathbf{x}, \mathbf{y}) \in I(N) \times \mathbb{Z}^n : \|\mathbf{y}\| \le m\|\mathbf{x}\|\}$. Then $J(N) \subset J$.

Let $\chi_{\Delta_{(\mathbf{x},\mathbf{y})}}$ be the characteristic function

$$\chi_{\Delta_{(\mathbf{x},\mathbf{y})}}(A) := \begin{cases} 1 & \text{if } A \in \Delta_{(\mathbf{x},\mathbf{y})}, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\Delta_{(\mathbf{x},\mathbf{y})} := \Delta(R_{(\mathbf{x},\mathbf{y})}, N^{-\frac{m}{n}}\omega(2N)\|\mathbf{x}\|^{-1}).$$

Also, for a matrix $A \in [0,1]^{mn}$, define

$$\nu_N(A) := \sum_{(\mathbf{x},\mathbf{y})\in J(N)} \chi_{\Delta_{(\mathbf{x},\mathbf{y})}}({}^tA).$$

Thus $\nu_N(A)$ is the number of resonant sets $R_{(\mathbf{x},\mathbf{y})}$ for $(\mathbf{x},\mathbf{y}) \in J(N)$ which are 'close' to ${}^tA$, i.e. such that $\|{}^tA\mathbf{x} - \mathbf{y}\| < \delta(N)$, where $\delta(N) := N^{-\frac{m}{n}}\omega(2N)$. Denote by μ_N and σ_N^2 the mean and variance respectively, that is

$$\mu_N := \int_{[0,1]^{mn}} \nu_N(A)dA \quad \text{and} \quad \sigma_N^2 := \int_{[0,1]^{mn}} \nu_N^2(A)dA - \mu_N^2.$$

Since $\|\mathbf{x}\|^{-1} \le N^{-1}$ for any $\mathbf{x} \in I(N)$, we have

$$\Delta(R_{(\mathbf{x},\mathbf{y})}, N^{-\frac{m}{n}}\omega(2N)\|\mathbf{x}\|^{-1}) \subset \Delta(R_{(\mathbf{x},\mathbf{y})}, \rho(2N))$$

by taking $c < 2^{\frac{m+n}{n}}$. Thus, we claim that

$$|Z_N| \to 0 \quad \text{as} \quad N \to \infty,$$

where $Z_N := \nu_N^{-1}(0) = \{A \in [0,1]^{mn} : \nu_N(A) = 0\}$, which implies (2.17) by replacing N with 2^{N-1}.

Lemma 2.4.9. *For N large enough, $\sigma_N^2 \le \mu_N$ and $\mu_N \ge c_0\omega(2N)^{\frac{1}{2}}$ for some positive constant c_0 independent of N.*

Proof. Suppose that N is large enough so that $\delta(N) = N^{-\frac{m}{n}}\omega(2N) < \frac{1}{2}$. By Lemma 8 in [Spr79], for $\mathbf{x} \in I(N)$,

$$\begin{aligned}\sum_{\mathbf{y}:(\mathbf{x},\mathbf{y})\in J(N)} \int_{[0,1]^{mn}} \chi_{\Delta_{(\mathbf{x},\mathbf{y})}}(A)dA &= |\{A \in [0,1]^{mn} : \|{}^tA\mathbf{x}\|_{\mathbb{Z}} < \delta(N)\}| \\ &= (2\delta(N))^n = 2^n N^{-m}\omega(2N)^n.\end{aligned}$$

Hence

$$\mu_N = \sum_{(\mathbf{x},\mathbf{y})\in J(N)} \int_{[0,1]^{mn}} \chi_{\Delta_{(\mathbf{x},\mathbf{y})}}(A)dA = \sum_{\mathbf{x}\in I(N)} 2^n N^{-m}\omega(2N)^n.$$

Let $\mathcal{S}(i)$ denote the set of vectors $\mathbf{x}\in\mathbb{Z}^m$ satisfying the condition (i) in the definition $I(N)$ for each $i=1,2,3,4$. Note that

$$\sum_{\mathbf{x}\in I(N)} 1 \geq \sum_{\mathbf{x}\in\mathcal{S}(1)\cap\mathcal{S}(2)\cap\mathcal{S}(3)} 1 - \sum_{\mathbf{x}\in\mathcal{S}(1)\cap\mathcal{S}(2)\cap\mathcal{S}(4)^c} 1.$$

Following [Spr79, p.40],

$$\begin{aligned}
&\sum_{\mathbf{x}\in\mathcal{S}(1)\cap\mathcal{S}(2)\cap\mathcal{S}(3)} 1 = \sum_{\mathbf{x}\in\mathcal{S}(1)\cap\mathcal{S}(2)} \sum_{d|\gcd(\mathbf{x})} \mu(d) \\
&= \sum_{N\leq x_1\leq 2N} \sum_{d|x_1} \mu(d) \prod_{i=2}^{m} \left| \left\{ x_i\in\mathbb{Z} : d|x_i,\ 1\leq x_i\leq \frac{N}{\omega(2N)^{\frac{1}{2(m-1)}}} \right\} \right| \\
&= \sum_{N\leq x_1\leq 2N} \sum_{d|x_1} \mu(d) \left\lfloor \frac{N}{d\omega(2N)^{\frac{1}{2(m-1)}}} \right\rfloor^{m-1} \\
&= \sum_{N\leq x_1\leq 2N} \left(\frac{N^{m-1}}{\omega(2N)^{\frac{1}{2}}} \sum_{d|x_1} \frac{\mu(d)}{d^{m-1}} + O\left(N^{m-2} \sum_{d|x_1} \frac{|\mu(d)|}{d^{m-2}} \right) \right) \\
&= \begin{cases} \displaystyle\sum_{N\leq x_1\leq 2N} \frac{N}{\omega(2N)^{\frac{1}{2}}} \frac{\varphi(x_1)}{x_1} + O(\tau(x_1)) & \text{if } m=2, \\ \displaystyle\sum_{N\leq x_1\leq 2N} \frac{N^{m-1}}{\omega(2N)^{\frac{1}{2}}} \prod_{\substack{p|x_1 \\ p \text{ prime}}} \left(1-\frac{1}{p^{m-1}}\right) + O(N^{m-2}\tau(x_1)) & \text{if } m\geq 3. \end{cases}
\end{aligned}$$

Fix small $0<\epsilon<\frac{6}{\pi^2}-\frac{1}{2}$. Note that $\frac{1}{N}\sum_{q=1}^{N}\frac{\varphi(q)}{q}\to\frac{6}{\pi^2}$ as $N\to\infty$ (see [Har98, Lemma 2.4]) and $\tau(h)=O(h^\delta)$ for any $\delta>0$ (see [HW60, Theorem 315]). In the case $m=2$, we have

$$\sum_{\mathbf{x}\in\mathcal{S}(1)\cap\mathcal{S}(2)\cap\mathcal{S}(3)} 1 \geq \left(\frac{6}{\pi^2}-\epsilon\right)\frac{N^2}{\omega(2N)^{\frac{1}{2}}}$$

for all large enough N. If $m \geq 3$, then

$$\prod_{\substack{p|x_1 \\ p \text{ prime}}} \left(1 - \frac{1}{p^{m-1}}\right) > \prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right) = \frac{6}{\pi^2},$$

hence we have that for all large enough N,

$$\sum_{\mathbf{x} \in \mathcal{S}(1) \cap \mathcal{S}(2) \cap \mathcal{S}(3)} 1 \geq \left(\frac{6}{\pi^2} - \epsilon\right) \frac{N^m}{\omega(2N)^{\frac{1}{2}}}.$$

On the other hand, it follows from Lemma 2.4.6 that

$$\sum_{\mathbf{x} \in \mathcal{S}(1) \cap \mathcal{S}(2) \cap \mathcal{S}(4)^c} 1 \leq \frac{1}{2} \frac{N^m}{\omega(2N)^{\frac{1}{2}}}.$$

Taking $c_0 = 2^n \left(\frac{6}{\pi^2} - \frac{1}{2} - \epsilon\right) > 0$, it follows that

$$\mu_N = 2^n N^{-m} \omega(2N)^n \sum_{\mathbf{x} \in I(N)} 1 \geq c_0 \omega(2N)^{\frac{1}{2}}.$$

To prove that $\sigma_N^2 \leq \mu_N$, we note that, for $\mathbf{x} \neq \mathbf{x}' \in I(N)$,

$$\begin{aligned}
&\sum_{\mathbf{y}:(\mathbf{x},\mathbf{y}) \in J(N)} \sum_{\mathbf{y}':(\mathbf{x}',\mathbf{y}') \in J(N)} \int_{[0,1]^{mn}} \chi_{\Delta_{(\mathbf{x},\mathbf{y})}}(A) \chi_{\Delta_{(\mathbf{x}',\mathbf{y}')}}(A) dA \\
&= |\{A \in [0,1]^{mn} : \|{}^t A\mathbf{x}\|_{\mathbb{Z}} < \delta(N)\}| \times |\{A \in [0,1]^{mn} : \|{}^t A\mathbf{x}'\|_{\mathbb{Z}} < \delta(N)\}| \\
&= 2^{2n} N^{-2m} \omega(2N)^{2n}.
\end{aligned}$$

by Lemma 9 in [Spr79]. Thus we have

$$\begin{aligned}
&\int_{[0,1]^{mn}} \nu_N^2(A) dA \\
&= \sum_{\mathbf{x} \in I(N)} \sum_{\mathbf{x}' \in I(N)} \sum_{\mathbf{y}:(\mathbf{x},\mathbf{y}) \in J(N)} \sum_{\mathbf{y}':(\mathbf{x}',\mathbf{y}') \in J(N)} \int_{[0,1]^{mn}} \chi_{\Delta_{(\mathbf{x},\mathbf{y})}}(A) \chi_{\Delta_{(\mathbf{x}',\mathbf{y}')}}(A) dA \\
&= \mu_N + 2^{2n} N^{-2m} \omega(2N)^{2n} \sum_{\mathbf{x} \neq \mathbf{x}' \in I(N)} 1 \leq \mu_N + \mu_N^2.
\end{aligned}$$

By definition of σ_N^2, we have

$$\sigma_N^2 \leq \mu_N.$$

□

Note that

$$\sigma_N^2 = \int_{[0,1]^{mn}} (\nu_N(A) - \mu_N)^2 \, dA \geq \int_{Z_N} (\nu_N(A) - \mu_N)^2 \, dA = \mu_N^2 |Z_N|.$$

This together with Lemma 2.4.9 implies that

$$|Z_N| \leq \frac{1}{\mu_N} \to 0 \quad \text{as} \quad N \to \infty.$$

□

Chapter 3

Entropy rigidity and Best approximation vectors

3.1 General entropy theory

In this section, we recall the definitions and basic properties of the entropy and the relative entropy for σ-algebras we use in the later sections. We refer the reader to [ELW, Chapter 1 & 2] for details.

Definition 3.1.1. *Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system on a Borel probability space, and let $\mathcal{A}, \mathcal{C} \subseteq \mathcal{B}$ be sub-σ-algebras. Suppose that $\mathcal{C}$ is countably generated. Note that there exists an $\mathcal{A}$-measurable conull set $X' \subset X$ and a system $\left\{\mu_x^{\mathcal{A}} | x \in X'\right\}$ of measures on X, referred to as* conditional measures, *given for instance by [ELW, Theorem 2.2]. The* information function *of $\mathcal{C}$ given $\mathcal{A}$ with respect to μ is defined by*

$$I_\mu(\mathcal{C}|\mathcal{A})(x) = -\log \mu_x^{\mathcal{A}}([x]_{\mathcal{C}}),$$

where $[x]_{\mathcal{C}}$ is the atom of $\mathcal{C}$ containing x.

1. *The* conditional (static) entropy of $\mathcal{C}$ given $\mathcal{A}$ *is defined by*

$$H_\mu(\mathcal{C}|\mathcal{A}) := \int_X I_\mu(\mathcal{C}|\mathcal{A})(x) d\mu(x),$$

which is the average of the information. If the σ-algebra $\mathcal{A}$ is trivial, then we denote by $H_\mu(\mathcal{C}) = H_\mu(\mathcal{C}|\mathcal{A})$, which is called the (static) entropy of

$\mathcal{C}$. Note that the entropy of the countable partition $\xi = \{A_1, A_2, \dots\}$ of X is given by

$$H_\mu(\xi) = H(\mu(A_1), \dots) = -\sum_{i\geq 1} \mu(A_i) \log \mu(A_i) \in [0, \infty],$$

where $0 \log 0 = 0$.

2. *Let $\mathcal{A} \subseteq \mathcal{B}$ be a sub-σ-algebra such that $T^{-1}\mathcal{A} = \mathcal{A}$. For any countable partition ξ of X, let*

$$h_\mu(T, \xi) := \lim_{n\to\infty} \frac{1}{n} H_\mu(\xi_0^{n-1}) = \inf_{n\geq 1} \frac{1}{n} H_\mu(\xi_0^{n-1}),$$

$$h_\mu(T, \xi|\mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} H_\mu(\xi_0^{n-1}|\mathcal{A}) = \inf_{n\geq 1} \frac{1}{n} H_\mu(\xi_0^{n-1}|\mathcal{A}),$$

where $\xi_0^{n-1} = \bigvee_{i=0}^{n-1} T^{-i}\xi$. Then the (dynamical) entropy of T *is*

$$h_\mu(T) := \sup_{\xi: H_\mu(\xi)<\infty} h_\mu(T, \xi).$$

Moreover, the conditional (dynamical) entropy of T given $\mathcal{A}$ *is*

$$h_\mu(T|\mathcal{A}) := \sup_{\xi: H_\mu(\xi)<\infty} h_\mu(T, \xi|\mathcal{A}).$$

We gather the basic properties for the entropy.

Proposition 3.1.2 (Additivity, Monotoniciy, Invariance, and Continuity). *Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system on a Borel probability space, let $\mathcal{A}$, $\mathcal{C}_1$, and $\mathcal{C}_2$ be sub-σ-algebras of $\mathcal{B}$, and suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are countably-generated. Then,*

1. $H_\mu(\mathcal{C}_1 \vee \mathcal{C}_2|\mathcal{A}) = H_\mu(\mathcal{C}_1|\mathcal{A}) + H_\mu(\mathcal{C}_2|\mathcal{C}_1 \vee \mathcal{A})$
2. $H_\mu(\mathcal{C}_2|\mathcal{C}_1 \vee \mathcal{A}) \leq H_\mu(\mathcal{C}_2|\mathcal{A})$
3. $H_\mu(\mathcal{C}_1 \vee \mathcal{C}_2|\mathcal{A}) \leq H_\mu(\mathcal{C}_1|\mathcal{A}) + H_\mu(\mathcal{C}_2|\mathcal{A})$
4. $H_\mu(\mathcal{C}_1|\mathcal{A}) = H_\mu(T^{-1}\mathcal{C}_1|T^{-1}\mathcal{A})$

5. *Let $\mathcal{C}_n \nearrow \mathcal{C}$ be an increasing sequence of countably generated sub-σ-algebras of $\mathcal{B}$ and $\mathcal{C}$ be countably generated. Then*

$$H_\mu(\mathcal{C}|\mathcal{A}) \nearrow H_\mu(\mathcal{C}|\mathcal{A})$$

as $n \to \infty$.

6. *Let $\mathcal{A}_n \nearrow \mathcal{A}_\infty$ be an increasing (or $\mathcal{A}_n \searrow \mathcal{A}_\infty$ a decreasing) sequence of sub-σ-algebras of $\mathcal{B}$. If ξ is a finite partition, then we have*

$$H_\mu(\xi|\mathcal{A}_n) \to H_\mu(\xi|\mathcal{A}_\infty)$$

as $n \to \infty$.

We refer the reader to Propositions 2.12, 2.13 and Lemma 2.17 of [ELW].

Proposition 3.1.3 (Basic properties). *Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system on a Borel probability space, let ξ and η be countable partitions of X with finite entropy, and let $\mathcal{A} = T^{-1}\mathcal{A} \subseteq \mathcal{B}$ be a strictly invariant sub-σ-algebra. Then,*

1. $h_\mu(T, \xi|\mathcal{A}) \leq H_\mu(\xi|\mathcal{A}) \leq H_\mu(\xi)$ *and* $h_\mu(T, \xi|\mathcal{A}) \leq h_\mu(T, \xi)$;
2. $h_\mu(T, \xi \vee \eta|\mathcal{A}) \leq h_\mu(T, \xi|\mathcal{A}) + h_\mu(T, \eta|\mathcal{A})$;
3. $h_\mu(T, \eta|\mathcal{A}) \leq h_\mu(T, \xi|\mathcal{A}) + H_\mu(\eta|\xi \vee \mathcal{A}) \leq h_\mu(T, \xi|\mathcal{A}) + H_\mu(\eta|\xi)$;
4. $h_\mu(T, \xi|\mathcal{A}) = h_\mu(T, \xi_0^k|\mathcal{A})$ *for all* $k \geq 1$;
5. $h_\mu(T, \xi|\mathcal{A}) = h_\mu(T^{-1}, \xi|\mathcal{A}) = h_\mu(T, \xi_{-k}^k|\mathcal{A})$ *for all* $k \geq 1$ *if* T *is invertible;*
6. $h_\mu(T^k|\mathcal{A}) = k h_\mu(T|\mathcal{A})$ *for all* $k \geq 1$*; and*
7. $h_\mu(T|\mathcal{A}) = h_\mu(T^{-1}|\mathcal{A})$ *if* T *is invertible.*

Moreover,

6. **(Future formula)**

$$h_\mu(T, \xi|\mathcal{A}) = H_\mu(\xi|\xi_1^\infty \vee \mathcal{A}).$$

7. **(Additivity)** *If T is invertible,*

$$\begin{aligned} h_\mu(T, \xi \vee \eta|\mathcal{A}) &= h_\mu(T, \xi|\mathcal{A}) + h_\mu(T, \eta|\xi_{-\infty}^\infty \vee \mathcal{A}) \\ &= h_\mu(T, \xi|\mathcal{A}) + H_\mu(\eta|\eta_1^\infty \vee \xi_{-\infty}^\infty \vee \mathcal{A}). \end{aligned}$$

Proposition 3.1.4 (**Kolmogorov-Sinaĭ for sequence of partitions**). *Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system on a Borel probability space. Suppose that (ξ_k) is a sequence of partitions of finite entropy with the property that*

- $\mathcal{B} = \bigvee_{k=1}^{\infty}(\xi_k)_0^\infty$ *mod* μ *and* $(\xi_k)_0^\infty \subseteq (\xi_{k+1})_0^\infty$ *for all* $k \in \mathbb{N}$*, or*
- $\mathcal{B} = \bigvee_{k=1}^{\infty}(\xi_k)_{-\infty}^\infty$ *mod* μ *and* $(\xi_k)_{-\infty}^\infty \subseteq (\xi_{k+1})_{-\infty}^\infty$ *for all* $k \in \mathbb{N}$ *if* T *is invertible.*

If $\mathcal{A} = T^{-1}\mathcal{A} \subseteq \mathcal{B}$ is a strictly invariant sub-σ-algebra, then

$$h_\mu(T|\mathcal{A}) = \sup_k h_\mu(T, \xi_k|\mathcal{A}) = \lim_{k\to\infty} h_\mu(T, \xi_k|\mathcal{A}).$$

Proposition 3.1.5 (**Entropy and ergodic decomposition**). *Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system on a Borel probability space, with ergodic decomposition*

$$\mu = \int_X \mu_x^{\mathcal{E}} d\mu(x)$$

as in [ELW, Theorem 2.7]. Let $\mathcal{A} \subseteq \mathcal{B}$ be a strictly T-invariant sub-σ-algebra. Then

$$h_\mu(T, \xi|\mathcal{A}) = \int_X h_{\mu_x^{\mathcal{E}}}(T, \xi|\mathcal{A}) d\mu(x)$$

for any partition ξ with $H_\mu(\xi) < \infty$, and

$$h_\mu(T|\mathcal{A}) = \int_X h_{\mu_x^{\mathcal{E}}}(T|\mathcal{A}) d\mu(x).$$

3.2 Entropy on homogeneous spaces

3.2.1 General setup

Let G be a closed real linear group (or connected, simply connected real Lie group) and let $\Gamma < G$ be a lattice. We consider the quotient $Y = G/\Gamma$ with a G-invariant probability measure m_Y and call it Haar measure on Y. Let d_G be a right invariant metric on G, which induces the metric d_Y on the space $Y = G/\Gamma$. Then Y is locally isometric to G, that is, for every $y \in Y$ there exists some $r > 0$ such that the map $g \mapsto gy$ is an isometry from the open r-ball B_r^G around the identity in G onto the open r-ball $B_r^Y(y)$ around $y \in Y$. Let r_y be the maximal injectivity radius at $y \in Y$ which is the supremum of

$r > 0$ such that the above map can be an isometry. For any $r > 0$, we denote by $Y(r) = \{y \in Y : r_y \geq r\}$. It follows from the continuity of the injectivity radius that $Y(r)$ is compact. Let us denote by

$$r_{\max} = \inf\{r > 0 : r_y \leq r \text{ for all } y \in Y\}.$$

Since Γ is a lattice, $r_{\max} < \infty$. Hence we now assume that $r_{\max} \leq 1$ by rescaling the right invariant metric d_G on G. Note that for any $r > 1$, $Y(r) = \varnothing$.

For any closed subgroup $L < G$, we consider the right invariant metric d_L by restricting d_G on L, and similarly denote by B_r^L the open r-ball around the identity in L.

In this section, we fix an element $a \in G$ which is Ad-diagonalizable over $\mathbb{R}$. Let $G^+ = \left\{g \in G | a^k g a^{-k} \to id \text{ as } k \to -\infty\right\}$ be the unstable (resp. stable) horospherical subgroup associated to a (resp. a^{-1}), which is always a closed subgroup of G in our setting.

Let $L < G^+$ be a closed subgroup normalized by a and let $\mathfrak{l}$ denote the Lie algebra of L. We can take a basis $\{v_1, \ldots, v_{\dim(\mathfrak{l})}\}$ of $\mathfrak{l}$ so that the adjoint map Ad_a on $\mathfrak{l}$ can be considered as the expansion $(v_i) \mapsto (e^{c_i} v_i)$ for some $c_i > 0$. Now assume that $c_i \leq 1$ for all i. Then for $\mathbf{c} = (c_1, \ldots, c_{\dim(\mathfrak{l})})$, we define the quasinorm $\|\cdot\|_{\mathbf{c}}$ by $\|x\|_{\mathbf{c}} = \max_i |x_i|^{1/c_i}$ for $x = \sum_i x_i v_i \in \mathfrak{l}$. We remark that for $x, y \in \mathfrak{l}$ and $k \in \mathbb{Z}$,

- using the convexity of the function $s \mapsto s^{1/c_i}$,

$$\|x + y\|_{\mathbf{c}} \leq 2^{\frac{1-\min \mathbf{c}}{\min \mathbf{c}}} (\|x\|_{\mathbf{c}} + \|y\|_{\mathbf{c}}); \tag{3.1}$$

- and

$$\| \operatorname{Ad}_{a^k} x\|_{\mathbf{c}} = e^k \|x\|_{\mathbf{c}}.$$

The quasinorm $\|\cdot\|_{\mathbf{c}}$ induces the quasi-metric $d_{\mathfrak{l},\mathbf{c}}$ on the Lie algebra $\mathfrak{l}$, thus induces the quasi-metric $d_{L,\mathbf{c}}$ locally on L using the logarithm map from L to $\mathfrak{l}$ (see Subsection **??** for the definition of quasi-metric). We similary denote by $B_r^{L,\mathbf{c}}$ the open r-ball around the identity in L with respect to the quasi-metric $d_{L,\mathbf{c}}$. For any $y \in Y$, we also denote by $d_{L,\mathbf{c}}$ the induced quasi-metric on the fiber $B_{r_y}^L \cdot y$.

3.2.2 Construction of a^{-1}-descending, subordinate algebra and its entropy properties

In this subsection, our goal is to strengthen results of [EL10, §7] for our quantitative purposes.

Definition 3.2.1 (7.25. of [EL10])**.** *Let $G^+ < G$ be the unstable horospherical subgroup associated to a. Let μ be an a-invariant measure on Y and $L < G^+$ be a closed subgroup normalized by a.*

1. *We say that a countably generated σ-algebra $\mathcal{A}$ is* subordinate to L *(mod μ) if for μ-a.e. y, there exists $\delta > 0$ such that*

$$B_\delta^L \cdot y \subset [y]_{\mathcal{A}} \subset B_{\delta^{-1}}^L \cdot y. \tag{3.2}$$

2. *We say that $\mathcal{A}$ is a^{-1}*-descending *if $(a^{-1})^{-1}\mathcal{A} = a\mathcal{A} \subseteq \mathcal{A}$.*

For each $L < G^+$ and a-invariant ergodic probability measure μ on Y, there exists a countably generated σ-algebra $\mathcal{A}$ which is a^{-1}-descending and subordinate to L [EL10, Proposition 7.37]. We will prove that such a σ-algebra can be constructed so that we also have an explicit upper bound of the measure of the set violating (3.2) for fixed $\delta > 0$. In order to prove an effective version of the variational principle later, we need this quantitative estimate independent of μ.

We first introduce some notations that will be used in this subsection. For a subset $B \subset Y$ and $\delta > 0$, we denote by $\partial_\delta B$ the δ-neighborhood of the boundary of B, i.e.

$$\partial_\delta B = \{y \in Y : \inf_{z \in B} d_Y(y,z) + \inf_{z \notin B} d_Y(y,z) < \delta\}.$$

We also define the neighborhood of the boundary of a countable partition $\mathcal{P}$ by

$$\partial_\delta \mathcal{P} = \bigcup_{P \in \mathcal{P}} \partial_\delta P.$$

Here, we deal with the entropy with respect to a^{-1}, so we write for any extended integers $\ell \le \ell'$ in $\mathbb{Z} \cup \{\pm\infty\}$,

$$\mathcal{P}_\ell^{\ell'} = \bigvee_{k=\ell}^{\ell'} a^k \mathcal{P},$$

for a given partition $\mathcal{P}$ of Y. We will use this notation also for σ-algebras.

We first construct a finite partition which has small measures on neighborhoods of the boundary. The following lemma is the main ingredient of the effectivization in this section. A key feature is that the measure estimate below is independent of μ.

Lemma 3.2.2. *There exists a constant $c > 0$ only depending on $\dim G$ such that the following holds. Let μ be a probability measure on Y. For any $r > 0$ and any measurable subset $\Omega \subset Y(2r)$, there exist a measurable subset $K \subset Y$ and a partition $\mathcal{P} = \{P_1, \cdots, P_N\}$ of K such that*

1. $\Omega \subseteq K \subseteq B^G_{\frac{11}{10}r}\Omega$,
2. *For each $1 \leq i \leq N$, there exists $z_i \in B^G_{\frac{r}{10}}\Omega$ such that*
$$B^G_{\frac{r}{5}} \cdot z_i \subseteq P_i \subseteq B^G_r \cdot z_i, \qquad K = \bigcup_{i=1}^{N} B^G_r \cdot z_i,$$
3. $\mu(\partial_\delta \mathcal{P}) \leq \left(\frac{\delta}{r}\right)^{\frac{1}{2}} \mu(B^G_{\frac{12}{10}r}\Omega)$ *for any $0 < \delta < cr$.*

Proof. Choose a maximal $\frac{9}{10}r$-separated set $\{y_1, \cdots, y_N\}$ of Ω.

Claim There exist a constant $c > 0$ depending only on $\dim G$, and $\{g_i\}_{i=1}^N \subset B^G_{\frac{r}{10}}$ such that for $z_i = g_i y_i$ and for any $0 < \delta < cr$,

$$\sum_i \Big(\mu(\partial_\delta(B^G_r \cdot z_i)) + \mu(\partial_\delta(B^G_{\frac{r}{2}} \cdot z_i))\Big) \leq \left(\frac{\delta}{r}\right)^{\frac{1}{2}} \mu(B^G_{\frac{12}{10}r}\Omega). \tag{3.3}$$

Proof of Claim. To prove this claim, we randomly choose each g_i with the independent uniform distribution on $B^G_{\frac{r}{10}}$. Then we have

$$\begin{aligned}
\mathbb{E}\left(\sum_i \mu(\partial_\delta(B^G_r \cdot z_i))\right) &= \sum_i \frac{1}{m_G(B^G_{\frac{r}{10}})} \int_{B^G_{\frac{r}{10}}} \int_Y \mathbb{1}_{B^G_{r+\delta}\cdot g_i y_i \setminus B^G_{r-\delta}\cdot g_i y_i}(y)\, d\mu(y)\, dm_G(g_i) \\
&\asymp \sum_i \frac{1}{r^{\dim G}} \int_Y m_G\left(\left\{g_i \in B^G_{\frac{r}{10}} : r - \delta \leq d(g_i y_i, y) < r + \delta\right\}\right) d\mu(y) \\
&\ll \sum_i \frac{1}{r^{\dim G}} \int_{B^G_{\frac{11}{10}r+\delta}\cdot y_i} \delta r^{\dim G - 1} d\mu \leq \frac{\delta}{r} \int_{B^G_{\frac{12}{10}r}\Omega} \sum_i \mathbb{1}_{B_{\frac{12}{10}r}\cdot y_i}(y)\, d\mu(y).
\end{aligned}$$

For any $y \in B^G_{\frac{12}{10}r}\Omega$, the number of y_i's contained in $B_{\frac{12}{10}r} \cdot y$ is at most $10^{\dim G}$ since y_i's are $\frac{9}{10}r$-separated. It implies that $\sum_i \mathbb{1}_{B_{\frac{12}{10}r}\cdot y_i}(y) \leq 10^{\dim G}$ for any $y \in B^G_{\frac{12}{10}r}\Omega$. It follows that

$$\mathbb{E}\left(\sum_i \mu(\partial_\delta(B^G_r \cdot z_i))\right) \ll \frac{\delta}{r}\int_{B^G_{\frac{12}{10}r}\Omega} 10^{\dim G} d\mu(y) \ll \frac{\delta}{r}\mu(B^G_{\frac{12}{10}r}\Omega),$$

where the implied constant is an absolute constant only depending on $\dim G$.

Applying the same argument for $\partial_\delta(B^G_{\frac{r}{2}} \cdot z_i)$ instead of $\partial_\delta(B^G_r \cdot z_i)$,

$$\mathbb{E}\left(\sum_i \left(\mu(\partial_\delta(B^G_r \cdot z_i)) + \mu(\partial_\delta(B^G_{\frac{r}{2}} \cdot z_i))\right)\right) \ll \frac{\delta}{r}\mu(B^G_{\frac{12}{10}r}\Omega).$$

It follows that for any $0 < \delta < \frac{r}{10}$,

$$\mathbb{P}\left(\sum_i \left(\mu(\partial_\delta(B^G_r \cdot z_i)) + \mu(\partial_\delta(B^G_{\frac{r}{2}} \cdot z_i))\right) \geq \frac{1}{2}\left(\frac{\delta}{r}\right)^{\frac{1}{2}} \mu(B^G_{\frac{12}{10}r}\Omega)\right) \ll \left(\frac{\delta}{r}\right)^{\frac{1}{2}}.$$

Hence, for any $0 < \delta < \frac{r}{10}$, we have

(3.4)

$$\mathbb{P}\left(\bigcap_{k\geq 0}\left\{\sum_i \left(\mu(\partial_{2^{-k}\delta}(B^G_r \cdot z_i)) + \mu(\partial_{2^{-k}\delta}(B^G_{\frac{r}{2}} \cdot z_i))\right) < \frac{1}{2}\left(\frac{2^{-k}\delta}{r}\right)^{\frac{1}{2}} \mu(B^G_{\frac{12}{10}r}\Omega)\right\}\right)$$
$$> 1 - O\left(\left(\frac{\delta}{r}\right)^{\frac{1}{2}}\right).$$

Thus, there exists $c > 0$ so that the right-hand side of (3.4) is positive for any $\delta < cr$. It follows that we can find $\{g_i\}_{i=1}^N$ such that $z_i = g_i y_i$'s satisfy (3.3) for any $0 < \delta < cr$. □

Let $c > 0$ and $\{g_i\}_{i=1}^N \subset B^G_{\frac{r}{10}}$ be as in **Claim**. The set $\{z_i = g_i y_i\}_{i=1}^N$ is $\frac{7}{10}r$-separated since $\{y_i\}_{i=1}^N$ is $\frac{9}{10}r$-separated. Let $K := \bigcup_{i=1}^N B^G_r \cdot z_i$. Since $B^G_{\frac{9}{10}r} \cdot y_i \subseteq B^G_r \cdot z_i \subseteq B^G_{\frac{11}{10}r} \cdot y_i$, we have

$$\Omega \subseteq \bigcup_{i=1}^N B^G_{\frac{9}{10}r} \cdot y_i \subseteq K \subseteq \bigcup_{i=1}^N B^G_{\frac{11}{10}r} \cdot y_i \subseteq B^G_{\frac{11}{10}r}\Omega.$$

Now we define a partition $\mathcal{P}$ of K inductively as follows:

$$P_i = B_r^G \cdot z_i \setminus \left(\bigcup_{j=1}^{i-1} P_j \cup \bigcup_{j=i+1}^{N} B_{\frac{r}{2}}^G \cdot z_j \right)$$

for $1 \leq i \leq N$. It is clear from the construction that $B_{\frac{r}{5}}^G \cdot z_i \subseteq P_i \subseteq B_r^G \cdot z_i$ and $z_i \in B_{\frac{r}{10}}^G \Omega$ for $1 \leq i \leq N$. We also observe that the δ-neighborhood of $\mathcal{P}$ is contained in $\bigcup_{i=1}^{N} \left(\partial_\delta (B_r^G \cdot z_i) \cup \partial_\delta (B_{\frac{r}{2}}^G \cdot z_i) \right)$. Hence it follows from **Claim** that for any $0 < \delta < cr$,

$$\mu(\partial_\delta \mathcal{P}) \leq \sum_i \left(\mu(\partial_\delta (B_r^G \cdot z_i)) + \mu(\partial_\delta (B_{\frac{r}{2}}^G \cdot z_i)) \right) \leq \left(\frac{\delta}{r} \right)^{\frac{1}{2}} \mu(B_{\frac{12}{10}r}^G \Omega).$$

□

We need the following thickening lemma.

Lemma 3.2.3. *For any $r > \delta > 0$, we have*

$$B_\delta^G Y(r) \subset Y(r-\delta) \quad \textit{and} \quad B_\delta^G Y(r)^c \subset Y(r+\delta)^c.$$

Proof. For any $g \in B_\delta^G$ and $y \in Y(r)$, we need to show $r_{gy} \geq r - \delta$. Suppose that $r_{gy} < r - \delta$. Then there exist $g_1, g_2 \in B_{r-\delta}^G$ such that $g_1 gy = g_2 gy$, or equivalently, $g^{-1} g_2^{-1} g_1 gy = y$. But it follows from $y \in Y(r)$ that $g^{-1} g_2^{-1} g_1 g \notin B_r^G$, hence using the triangular inequality and the right invariance of d_G,

$$\begin{aligned} r &\leq d_G(g^{-1} g_2^{-1} g_1 g, id) = d_G(g_1 g, g_2 g) \leq d_G(g_1 g, id) + d_G(g_2 g, id) \\ &\leq d_G(g_1, id) + d_G(g_2, id) + 2 d_G(g, id) < r, \end{aligned}$$

which is a contradiction. This concludes the first assertion. The second assertion follows similarly. □

Using Lemma 3.2.2 inductively, we have the following partition of Y with its subpartition having small boundary measures. Recall that $Y(r) = \varnothing$ for any $r > 1$ by our choice of the right invariant metric d_G on G.

Lemma 3.2.4. *Let μ be a probability measure on Y. There exists a partition $\{K_k\}_{k=1}^{\infty}$ of Y such that for each $k \geq 1$, the following statements hold:*

1. $K_k \subseteq Y(2^{-k}) \setminus Y(2^{-k+2})$;

2. *there exist a partition* $\mathcal{P}_k = \{P_{k1}, \cdots, P_{kN_k}\}$ *of* K_k *and a point* $z_i \in B^G_{\frac{2^{-k-1}}{10}} K_k$ *for each* $1 \leq i \leq N_k$ *satisfying*

$$B^G_{\frac{1}{5}2^{-k-1}} \cdot z_i \subseteq P_{ki} \subseteq B^G_{2^{-k-1}} \cdot z_i;$$

3. $\mu(\partial_\delta \mathcal{P}_k) \leq (2^{k+4}\delta)^{\frac{1}{2}} \mu(Y(2^{-k-1}) \setminus Y(2^{-k+3}))$ *for any* $0 < \delta < c2^{-k-2}$, *where* $c > 0$ *is the constant in Lemma 3.2.2.*

Proof. We will construct $\{K_k\}_{k\geq 1}$ and $\{\mathcal{P}_k\}_{k\geq 1}$ using Lemma 3.2.2 inductively. For each $k \geq 1$, let us say that $K_k \subset Y$ and $\mathcal{P}_k$ satisfy ($\spadesuit_k$) if they satisfy the three conditions in the statement. We will also need auxiliary bounded sets $K'_k \subset Y$'s and corresponding partitions $\mathcal{P}'_k$'s during the inductive procedure. Let us say that K'_k and a partition $\mathcal{P}'_k$ of K'_k satisfy ($\clubsuit_k$) if they satisfy the following three conditions.

1. $Y(2^{-k+1}) \setminus \bigcup_{j=1}^{k-1} K_j \subseteq K'_k \subseteq B^G_{\frac{11}{10}2^{-k-1}}(Y(2^{-k+1}) \setminus \bigcup_{j=1}^{k-1} K_j)$,

2. For each $1 \leq i \leq N_k$, there exists $z_{ki} \in B_{\frac{2^{-k-1}}{10}} K'_k$ such that

$$B^G_{\frac{1}{5}2^{-k-1}} \cdot z_{ki} \subseteq P'_{ki} \subseteq B^G_{2^{-k-1}} \cdot z_{ki}, \quad K'_k = \bigcup_{i=1}^{N} B^G_{2^{-k-1}} \cdot z_{ki},$$

3. $\mu(\partial_\delta \mathcal{P}'_k) \leq (2^{k+1}\delta)^{\frac{1}{2}} \mu(Y(2^{-k}) \setminus Y(2^{-k+3}))$ for any $0 < \delta < c2^{-k-1}$.

Here, $\bigcup_{j=1}^{0} K_j$ means the empty set.

Let us start with the initial step. We first choose $\Omega_1 = Y(1)$ and apply Lemma 3.2.2 with $r = 2^{-2}$ and $\Omega = \Omega_1 \subset Y(\frac{1}{2})$. Then we have a subset $K'_1 \subset Y$ and a partition $\mathcal{P}'_1$ of K'_1 satisfying (1), (2) of ($\clubsuit_1$), and $\mu(\partial_\delta \mathcal{P}'_1) \leq (2^2\delta)^{\frac{1}{2}} \mu(B^G_{\frac{12}{11}2^{-2}}\Omega_1)$ for any $0 < \delta < c2^{-2}$. It follows from Lemma 3.2.3 that $B^G_{\frac{12}{10}2^{-2}} Y(1) \subset Y(\frac{1}{2})$, which implies (3) of ($\clubsuit_1$) since $Y(4) = \varnothing$. Also note that $K'_1 \subset B^G_{\frac{11}{10}2^{-2}} Y(1) \subset Y(\frac{1}{2})$.

Now let $\Omega_2 = Y(\frac{1}{2}) \setminus K'_1$ and apply Lemma 3.2.2 again with $r = 2^{-3}$ and $\Omega = \Omega_2 \subset Y(\frac{1}{4})$. We have a subset $K'_2 \subset Y$ and a partition $\mathcal{P}'_2$ of K'_2 satisfying $\Omega_2 \subset K'_2 \subset B^G_{\frac{11}{10}2^{-3}}\Omega_2$, (2) of ($\clubsuit_2$), and $\mu(\partial_\delta \mathcal{P}'_2) \leq (2^3\delta)^{\frac{1}{2}} \mu(B^G_{\frac{12}{10}2^{-3}}\Omega_2)$ for any $0 < \delta < c2^{-3}$. Set $K_1 = K'_1 \setminus K'_2$, then (1) of ($\clubsuit_2$) and (1) of ($\spadesuit_1$) follow since

$Y(2) = \varnothing$. Since $K_1' \supset Y(1)$, it follows from Lemma 3.2.3 that $B^G_{\frac{12}{10}2^{-3}}\Omega_2 \subset Y(\frac{1}{4})\setminus Y(2)$, which implies (3) of ($\clubsuit_2$). Define a partition $\mathcal{P}_1 = \{P_{11}, \ldots, P_{1N_1}\}$ from $\mathcal{P}_1' = \{P_{11}', \ldots, P_{1N_1}'\}$ by $P_{1i} = P_{1i}' \setminus K_2'$ for each $1 \le i \le N_1$. For each $1 \le i \le N_1$ and $y \in B^G_{\frac{2^{-2}}{5}} \cdot z_{1i}$, observe that $y \notin K_2'$ since $B^G_{2^{-2}} \cdot z_{1i} \subseteq K_1'$ and $K_2' \subseteq B^G_{\frac{11}{10}2^{-3}}\Omega_2 \subset B^G_{\frac{11}{10}2^{-3}}(Y \setminus K_1')$. Hence, $B^G_{\frac{2^{-2}}{5}} \cdot z_{1i} \subset P_{1i}$ holds, so (2) of ($\spadesuit_1$) follows. Since $P_{1i} = P_{1i}' \setminus K_2'$ for each $1 \le i \le N_1$, we have

$$\begin{aligned}\mu(\partial_\delta \mathcal{P}_1) &\le \mu(\partial_\delta \mathcal{P}_1') + \mu(\partial_\delta \mathcal{P}_2') \\ &\le (2^2\delta)^{\frac{1}{2}}\mu(Y(2^{-1}) \setminus Y(2^2)) + (2^3\delta)^{\frac{1}{2}}\mu(Y(2^{-2}) \setminus Y(2)) \\ &\le (2^5\delta)^{\frac{1}{2}}\mu(Y(2^{-2}) \setminus Y(2^2))\end{aligned}$$

for any $0 < \delta < c2^{-3}$. Hence (3) of ($\spadesuit_1$) follows.

Our desired disjoint sets $\{K_k\}_{k\ge1}$ and partitions $\{\mathcal{P}_k\}_{k\ge1}$ will be obtained by applying this procedure repeatedly.

Claim For $k \ge 2$, suppose that we have disjoint bounded sets K_j of Y and corresponding partitions $\mathcal{P}_j$ satisfying ($\spadesuit_j$) for $j = 1, \ldots, k-1$, and a subset $K_k' \subset Y$ and a partition $\mathcal{P}_k'$ satisfying ($\clubsuit_k$). Then we can find $K_k \subseteq K_k'$ and a partition $\mathcal{P}_k$ of K_k satisfying ($\spadesuit_k$), and $K_{k+1}' \subset Y$ and a partition $\mathcal{P}_{k+1}'$ of K_{k+1}' satisfying ($\clubsuit_{k+1}$).

Proof of Claim. Note that $K_j \subset Y(2^{-j}) \subset Y(2^{-k})$ for each $j = 1, \ldots, k-1$ and $K_k' \subset B^G_{\frac{11}{10}2^{-k-1}}Y(2^{-k+1}) \subset Y(2^{-k})$. Let $\Omega_{k+1} = Y(2^{-k}) \setminus (\bigcup_{j=1}^{k-1} K_j \cup K_k')$ and apply Lemma 3.2.2 with $r = 2^{-k-2}$ and $\Omega = \Omega_{k+1} \subset Y(2^{-k-1})$. Then we can find $K_{k+1}' \subset Y$ and a partition $\mathcal{P}_{k+1}' = \left\{P_{(k+1)1}', \cdots, P_{(k+1)N_{k+1}}'\right\}$ of K_{k+1}' satisfying $\Omega_{k+1} \subset K_{k+1}' \subset B^G_{\frac{11}{10}2^{-k-2}}\Omega_{k+1}$, (2) of ($\clubsuit_{k+1}$), and $\mu(\partial_\delta \mathcal{P}_{k+1}') \le (2^{k+2}\delta)^{\frac{1}{2}}\mu(B^G_{\frac{12}{10}2^{-k-2}}\Omega_{k+1})$ for any $0 < \delta < c2^{-k-2}$. We set $K_k = K_k' \setminus K_{k+1}'$, then (1) of ($\clubsuit_{k+1}$) follows. Since $\bigcup_{j=1}^{k-1} K_j \supset Y(2^{-k+2})$ and $K_k \subset K_k' \subset Y(2^{-k})\setminus\bigcup_{j=1}^{k-1} K_j$, (1) of ($\spadesuit_k$) follows. It follows from $\bigcup_{j=1}^{k-1} K_j \cup K_k' \supset Y(2^{-k+1})$ and Lemma 3.2.3 that $B^G_{\frac{12}{10}2^{-k-2}}\Omega_{k+1} \subset Y(2^{-k-1}) \setminus Y(2^{-k+2})$, which implies (3) of ($\clubsuit_{k+1}$). Define a partition $\mathcal{P}_k = \{P_{k1}, \cdots, P_{kN_k}\}$ from $\mathcal{P}_k' = \left\{P_{k1}', \cdots, P_{kN_k}'\right\}$ by $P_{ki} = P_{ki}' \setminus K_{k+1}'$ for any $1 \le i \le N_k$. For each $1 \le i \le N_k$ and $y \in B^G_{\frac{2^{-k-1}}{5}} \cdot z_{ki}$, observe that $y \notin K_{k+1}'$ since $B^G_{2^{-k-1}} \cdot z_{ki} \subseteq K_k'$ and $K_{k+1}' \subseteq B^G_{\frac{11}{10}2^{-k-2}}\Omega_{k+1} \subset B^G_{\frac{11}{10}2^{-k-2}}(Y \setminus K_k')$. Hence, $B^G_{\frac{2^{-k-1}}{5}} \cdot z_{ki} \subset P_{ki}$ holds, so (2) of ($\spadesuit_k$) follows. Since $P_{ki} = P'_{ki} \setminus K'_{k+1}$ for each $1 \le i \le N_1$, we have

$$\begin{aligned}\mu(\partial_\delta \mathcal{P}_k) &\le \mu(\partial_\delta \mathcal{P}'_k) + \mu(\partial_\delta \mathcal{P}'_{k+1}) \\ &\le (2^{k+1}\delta)^{\frac{1}{2}} \mu(Y(2^{-k}) \setminus Y(2^{-k+3})) + (2^{k+2}\delta)^{\frac{1}{2}} \mu(Y(2^{-k-1}) \setminus Y(2^{-k+2})) \\ &\le (2^{k+4}\delta)^{\frac{1}{2}} \mu(Y(2^{-k-1}) \setminus Y(2^{-k+3}))\end{aligned}$$

for any $0 < \delta < c2^{-k-2}$. Hence (3) of ($\spadesuit_k$) follows. □

This claim concludes the proof of Lemma 3.2.4. □

By [EL10, Lemma 7.29 and 7.45], there are constants $\alpha > 0$ and $d_0 > 0$ depending on a and G such that for every $r \in (0, 1]$,

$$a^{-k} B_r^{G^+} a^k \subset B^G_{d_0 e^{-k\alpha} r} \tag{3.5}$$

for any $k \in \mathbb{Z}$.

The following lemma is a quantitative modification of [EL10, Lemma 7.31]. We remark that the constants below are independent of μ while the set E_δ depends on μ.

Lemma 3.2.5. *Given a-invariant probability measure μ on Y, there exists a countable partition $\mathcal{P}$ of Y such that the following holds.*

1. *For any $P \in \mathcal{P}$ there exists $j \ge 1$ such that $P \subseteq Y(2^{-j}) \setminus Y(2^{-j+2})$. Moreover, there exists $z \in P$ such that*

$$B^G_{\frac{1}{5}2^{-j-1}} \cdot z \subseteq P \subseteq B^G_{2^{-j-1}} \cdot z.$$

2. *Let $c > 0$ and $d_0 > 0$ be the constants in Lemma 3.2.2 and (3.5). For any $0 < \delta < \min((\frac{c}{16 d_0})^2, 1)$, there exists $E_\delta \subset Y$ such that*

$$\mu(E_\delta) < \mu(Y \setminus Y(C_1 \delta^{\frac{1}{2}})) + C_2 \delta^{\frac{1}{2}}$$

and $B^{G^+}_\delta \cdot y \subset [y]_{\mathcal{P}_0^\infty}$ for any $y \in Y \setminus E_\delta$, where $C_1, C_2 > 0$ are constants only depending on a and G.

Proof. Let $\{K_j\}_{j\ge1}$ and $\{\mathcal{P}_j\}_{j\ge1}$ be the sets and the partitions we constructed in Lemma 3.2.4. We set $\mathcal{P} = \bigcup_{j=1}^\infty \mathcal{P}_j$. Then $\mathcal{P}$ is a countable partition of Y and the condition (1) directly follows from Lemma 3.2.4.

Now we set $E_\delta = \bigcup_{k=0}^{\infty} a^k \partial_{d_0 e^{-k\alpha}\delta}\mathcal{P}$ and split E_δ into two subsets

$$E'_\delta = \bigcup_{k=0}^{\infty} a^k \left(\bigcup_{i=2+\lceil \frac{\alpha}{\log 2}k - \frac{\log\delta}{2\log 2}\rceil}^{\infty} \partial_{d_0 e^{-k\alpha}\delta}\mathcal{P}_i \right),$$

$$E''_\delta = \bigcup_{k=0}^{\infty} a^k \left(\bigcup_{i=1}^{1+\lceil \frac{\alpha}{\log 2}k - \frac{\log\delta}{2\log 2}\rceil} \partial_{d_0 e^{-k\alpha}\delta}\mathcal{P}_i \right).$$

We claim that $E'_\delta \subset Y \setminus Y((d_0 + d_0^2)\delta^{\frac{1}{2}})$. To see this, let $y \in E'_\delta$. Then there exist $k \geq 0$ and $P \in \mathcal{P}_i$ for some $i \geq 2 + \lceil \frac{\alpha}{\log 2}k - \frac{\log\delta}{2\log 2}\rceil$ such that $y \in a^k \partial_{d_0 e^{-k\alpha}\delta} P$. By Lemma 3.2.4, $P \subset K_i \subset Y(2^{-i}) \setminus Y(2^{-i+2}) \subset Y(2^{-i+2})^c$. It follows from Lemma 3.2.3 that

$$\partial_{d_0 e^{-k\alpha}\delta} P \subset B^G_{d_0 e^{-k\alpha}\delta} P \subset B^G_{d_0 e^{-k\alpha}\delta} Y(2^{-i+2})^c \subset Y(2^{-i+2} + d_0 e^{-k\alpha}\delta)^c. \tag{3.6}$$

Using (3.5), it can be easily checked that $a^k Y(r)^c \subset Y(d_0 e^{k\alpha} r)^c$ for any $0 < r < 1$. Hence, it follows from (3.6) and $i \geq 2 + \lceil \frac{\alpha}{\log 2}k - \frac{\log\delta}{2\log 2}\rceil$ that

$$a^k \partial_{d_0 e^{-k\alpha}\delta} P \subset a^k Y(2^{-i+2} + d_0 e^{-k\alpha}\delta)^c \subset Y((d_0 + d_0^2)\delta^{\frac{1}{2}})^c.$$

This proves the claim.

The above claim implies that

$$\mu(E'_\delta) \leq \mu(Y \setminus Y(C_1 \delta^{\frac{1}{2}})) \tag{3.7}$$

for some constant $C_1 > 0$ only depending on a and G.

Next we estimate $\mu(E''_\delta)$. It follows from the a-invariance of μ that

$$\mu(E''_\delta) \leq \sum_{k=0}^{\infty} \sum_{i=1}^{1+\lceil \frac{\alpha}{\log 2}k - \frac{\log\delta}{2\log 2}\rceil} \mu(\partial_{d_0 e^{-k\alpha}\delta}\mathcal{P}_i) = \sum_{i=1}^{\infty} \sum_{k=k_i}^{\infty} \mu(\partial_{d_0 e^{-k\alpha}\delta}\mathcal{P}_i), \tag{3.8}$$

where $k_i \in \mathbb{N}$ denotes the smallest number of k such that $1 + \lceil \frac{\alpha}{\log 2}k - \frac{\log\delta}{2\log 2}\rceil \geq i$. Note that $k_i \geq \frac{\log 2}{\alpha}(i-2) + \frac{\log\delta}{2\alpha}$.

On the other hand, by Lemma 3.2.4 we have

$$\mu(\partial_{d_0 e^{-k\alpha}\delta}\mathcal{P}_i) \leq (2^{i+4} d_0 e^{-k\alpha}\delta)^{\frac{1}{2}} \mu(Y(2^{-i-1}) \setminus Y(2^{-i+3})) \tag{3.9}$$

for any $k \geq k_i$, since $d_0 e^{-k\alpha}\delta \leq d_0 2^{-i+2}\delta^{\frac{1}{2}} < c2^{-i-2}$. Hence, we obtain from (3.8) and (3.9)

$$\begin{aligned}
\mu(E''_\delta) &\leq \sum_{i=1}^{\infty}\sum_{k=k_i}^{\infty} \mu(\partial_{d_0 e^{-k\alpha}\delta}\mathcal{P}_i) \\
&\leq \sum_{i=1}^{\infty}\sum_{k=k_i}^{\infty} (2^{i+4} d_0 e^{-k\alpha}\delta)^{\frac{1}{2}} \mu(Y(2^{-i-1}) \setminus Y(2^{-i+3})) \\
&\ll \sum_{i=1}^{\infty} (2^{i+4} e^{-k_i\alpha}\delta)^{\frac{1}{2}} \mu(Y(2^{-i-1}) \setminus Y(2^{-i+3})) \\
&\ll \delta^{\frac{1}{2}} \sum_{i=1}^{\infty} \mu(Y(2^{-i-1}) \setminus Y(2^{-i+3})) \ll \delta^{\frac{1}{2}}.
\end{aligned} \tag{3.10}$$

In the last line we used the fact that $Y(2^{-i-1}) \setminus Y(2^{-i+3})$'s can be overlapped at most four times. Combining (3.7) and (3.10), we finally have

$$\mu(E_\delta) < \mu(Y \setminus Y(C_1\delta^{\frac{1}{2}})) + C_2\delta^{\frac{1}{2}}$$

for some constants $C_1, C_2 > 0$ only depending on a and G.

It remains to check that $B_\delta^{G^+} \cdot y \subset [y]_{\mathcal{P}_0^\infty}$ for any $y \in Y \setminus E_\delta$. Let $h \in B_\delta^{G^+}$ and suppose $[hy]_{\mathcal{P}_0^\infty} \neq [y]_{\mathcal{P}_0^\infty}$. Then there is some $k \geq 0$ such that $a^{-k}hy$ and $a^{-k}y$ belong to different elements of the partition $\mathcal{P}$. Since $a^{-k}ha^k \in a^{-k}B_\delta^{G^+}a^k \subset B^G_{d_0 e^{-k\alpha}\delta}$ by (3.5), we have

$$d_Y(a^{-k}hy, a^{-k}y) \leq d_G(a^{-k}ha^k, id) \leq d_0 e^{-k\alpha}\delta.$$

It follows that both $a^{-k}hy$ and $a^{-k}y$ belong to $\partial_{d_0 e^{-k\alpha}\delta}\mathcal{P}$, hence $y \in E_\delta$. It concludes that $B_\delta^{G^+} \cdot y \subset [y]_{\mathcal{P}_0^\infty}$ for any $y \in Y \setminus E_\delta$. □

The following proposition is a quantitative version of [EL10, Proposition 7.37]. Given a-invariant measure μ, it provides a σ-algebra which is a^{-1}-descending and subordinate to L in the following quantitative sense.

Proposition 3.2.6. *Let μ be an a-invariant probability measure on Y, and $L < G^+$ be a closed subgroup normalized by a. There exists a countably generated sub-σ-algebra $\mathcal{A}^L$ of Borel σ-algebra of Y satisfying*

1. *$a\mathcal{A}^L \subset \mathcal{A}^L$, that is, $\mathcal{A}^L$ is a^{-1}-descending,*
2. *$[y]_{\mathcal{A}^L} \subset B^L_{2^{-k+1}} \cdot y$ for any $y \in Y(2^{-k}) \setminus Y(2^{-k+2})$ with $k \geq 1$,*

3. *if* $0 < \delta < \min((\frac{c}{16d_0})^2, 1)$*, then* $B^L_\delta \cdot y \subset [y]_{\mathcal{A}^L}$ *for any* $y \in Y \setminus E_\delta$*, where* $c, D > 0$ *are the constants in Lemma 3.2.2 and* (3.5)*, and* E_δ *is the set in Lemma 3.2.5.*

In particular, the σ*-algebra* $\mathcal{A}^L$ *is* L*-subordinate modulo* μ*.*

Proof. For a given a-invariant probability measure μ on Y, let $\mathcal{P}$ be the countable partition of Y constructed in Lemma 3.2.5. We will construct a countably generated σ-algebra $\mathcal{P}^L$ by taking L-plaque in each $P \in \mathcal{P}$ as an atom of $\mathcal{P}^L$. Then $\mathcal{A}^L := (\mathcal{P}^L)_0^\infty$ will be the desired σ-algebra.

For each $P \in \mathcal{P}$, by Lemma 3.2.5(1), there exist $j \geq 1$ and $z \in P$ such that $P \in Y(2^{-j}) \setminus Y(2^{-j+2})$ and $B^G_{\frac{2^{-j-1}}{5}} \cdot z \subseteq P \subseteq B^G_{2^{-j-1}} \cdot z$. We can find $B_P \subset G$ with $\operatorname{diam}(B_P) \leq 2^{-j}$ such that $P = \pi_Y(B_P)$, where $\pi_Y : G \to Y$ is the natural quotient map. Define the σ-algebra

$$\mathcal{P}^L = \sigma\left(\{\pi_Y(B_P \cap S) : P \in \mathcal{P},\ S \in \mathcal{B}_{G/L}\}\right).$$

Then $\mathcal{P}^L$ is a refinement of $\mathcal{P}$ so that atoms of $\mathcal{P}^L$ are open L-plaques, i.e. for any $y \in P \in \mathcal{P}$, $[y]_{\mathcal{P}^L} = [y]_{\mathcal{P}} \cap B^L_{2^{-j}} \cdot y = V_y \cdot y$, where $V_y \subset B^L_{2^{-j}}$ is an open bounded set.

It is clear that $\mathcal{P}^L$ is countably generated, hence $\mathcal{A}^L = (\mathcal{P}^L)_0^\infty$ is also countably generated. By construction, we have $a\mathcal{A}^L = (\mathcal{P}^L)_1^\infty \subset \mathcal{A}^L$, which proves the assertion (1).

For any $y \in Y(2^{-k}) \setminus Y(2^{-k+2})$ with $k \geq 1$, take $P \in \mathcal{P}$ such that $y \in P$. By Lemma 3.2.5(1), there exist $j \geq 1$ and $z \in P$ such that $P \in Y(2^{-j}) \setminus Y(2^{-j+2})$ and $P \subseteq B^G_{2^{-j-1}} \cdot z$. Observe that $2^{-j+2} > 2^{-k}$ and $2^{-j} < 2^{-k+2}$, that is, $j - 2 < k < j + 2$. Hence we have

$$[y]_{\mathcal{A}^L} \subset [y]_{\mathcal{P}^L} = V_y \cdot y \subset B^L_{2^{-j}} \cdot y \subset B^L_{2^{-k+1}} \cdot y,$$

which proves the assertion (2).

For a given $0 < \delta < \min((\frac{c}{16d_0})^2, 1)$ and $y \in Y \setminus E_\delta$, assume that $z = hy$ with $h \in B^L_\delta$. By Lemma 3.2.5(2), $B^{G^+}_\delta \cdot y \subset [y]_{\mathcal{P}_0^\infty}$. Hence it follows that for any $k \geq 0$, $a^{-k}y$ and $a^{-k}z$ belong to the same atom $P \subset \mathcal{P}$. Then we have

$$a^{-k}y,\ a^{-k}z = a^{-k}ha^k \cdot (a^{-k}y)\ \in P.$$

Since $a^{-k}ha^k \in B^L_\delta$, $a^{-k}y$ and $a^{-k}z$ belong to the same atom of $\mathcal{P}^L$. This proves the assertion (3). □

As in [LSS19, Lemma 3.4], we need to compare the dynamical entropy and the static entropy. In [LSS19], the σ-algebra $\pi^{-1}(\mathcal{B}_X)$ is used to deal with the entropy relative to X, where $\mathcal{B}_X$ is the Borel σ-algebra of X. In order to deal with the entropy relative to the general closed subgroup $L < G^+$ normalized by a, we consider the following tail σ-algebra with respect to $\mathcal{A}^L$ in Proposition 3.2.6: Denote by

$$\mathcal{A}^L_\infty := \bigcap_{k=1}^{\infty} a^k \mathcal{A}^L = \bigcap_{k=1}^{\infty} (\mathcal{P}^L)_k^\infty . \tag{3.11}$$

This tail σ-algebra may not be countably generated but it satisfies strictly a-invariant, i.e. $a\mathcal{A}^L_\infty = \mathcal{A}^L_\infty = a^{-1}\mathcal{A}^L_\infty$.

Lemma 3.2.7. *Let μ be an a-invariant probability measure on Y, $L < G^+$ be a closed subgroup normalized by a, and $\mathcal{A}^L$ be as in Proposition 3.2.6. Then the σ-algebra $(\mathcal{A}^L)_{-\infty}^\infty$ is the Borel σ-algebra of Y modulo μ.*

Proof. Let $\mathcal{P}^L$ be as in the proof of Proposition 3.2.6. Since $(\mathcal{A}^L)_{-\infty}^\infty = (\mathcal{P}^L)_{-\infty}^\infty$ and $Y = \bigcup_{k\geq 1} Y(2^{-k}) \setminus Y(2^{-k+2})$, it is enough to show that for each $k \geq 1$ and for μ-a.e. $y \in Y(2^{-k}) \setminus Y(2^{-k+2})$, we have $[y]_{(\mathcal{P}^L)_{-\infty}^\infty} = \{y\}$.

For fixed $k \geq 1$, it follows from Poincaré recurrence (e.g. see [EW11, Theorem 2.11]) that for μ-a.e. $y \in Y(2^{-k}) \setminus Y(2^{-k+2})$, there exists an increasing sequence $(k_i)_{i\geq 1} \subset \mathbb{N}$ such that

$$a^{k_i} y \in Y(2^{-k}) \setminus Y(2^{-k+2}) \quad \text{and} \quad k_i \to \infty \text{ as } i \to \infty.$$

By Proposition 3.2.6(2), it follows that for each $i \geq 1$

$$[a^{k_i} y]_{\mathcal{A}^L} = [a^{k_i} y]_{(\mathcal{P}^L)_0^\infty} \subset B^L_{2^{-k+1}} \cdot a^{k_i} y.$$

Since $[a^{k_i} y]_{(\mathcal{P}^L)_0^\infty} = a^{k_i}[y]_{a^{-k_i}(\mathcal{P}^L)_0^\infty} = a^{k_i}[y]_{(\mathcal{P}^L)_{-k_i}^\infty}$, using (3.5), we have

$$[y]_{(\mathcal{P}^L)_{-k_i}^\infty} \subset a^{-k_i} B^L_{2^{-k+1}} \cdot a^{k_i} y = a^{-k_i} B^L_{2^{-k+1}} a^{k_i} \cdot y \subset B^L_{d_0 e^{-\alpha k_i} 2^{-k+1}} \cdot y.$$

Taking $i \to \infty$, we conclude that $[y]_{(\mathcal{P}^L)_{-\infty}^\infty} = \{y\}$. □

Proposition 3.2.8. *Let μ be an a-invariant probability measure on Y and $L < G^+$ be a closed subgroup normalized by a. Let $\mathcal{A}^L$ be as in Proposition*

3.2.6 and $\mathcal{A}^L_\infty$ *be as in* (4.48)*. Then we have*

$$h_\mu(a|\mathcal{A}^L_\infty) = h_\mu(a^{-1}|\mathcal{A}^L_\infty) = H_\mu(\mathcal{A}^L|a\mathcal{A}^L). \tag{3.12}$$

Moreover, (3.12) *holds for almost every ergodic component of* μ*.*

Proof. Let $\mathcal{P}^L$ be as in the proof of Proposition 3.2.6. Since $\mathcal{P}^L$ is countably generated, we can take an increasing sequence of finite partitions $(\mathcal{P}^L_k)_{k\geq 1}$ of Y such that $\mathcal{P}^L_k \nearrow \mathcal{P}^L$. By Lemma 3.2.7, we have $\mathcal{B}_Y = (\mathcal{P}^L)^\infty_{-\infty} = \bigvee_{k=1}^\infty (\mathcal{P}^L_k)^\infty_{-\infty}$ modulo μ, where $\mathcal{B}_Y$ is the Borel σ-algebra of Y. It is clear that $(\mathcal{P}^L_k)^\infty_{-\infty} \subseteq (\mathcal{P}^L_{k+1})^\infty_{-\infty}$ for all $k \in \mathbb{N}$. Hence it follow from Kolmogorov-Sinaĭ Theorem [ELW, Proposition 2.20] that

$$h_\mu(a^{-1}|\mathcal{A}^L_\infty) = \lim_{k\to\infty} h_\mu(a^{-1}, \mathcal{P}^L_k|\mathcal{A}^L_\infty).$$

Using the future formula [ELW, Proposition 2.19 (8)], we have

$$\lim_{k\to\infty} h_\mu(a^{-1}, \mathcal{P}^L_k|\mathcal{A}^L_\infty) = \lim_{k\to\infty} H_\mu(\mathcal{P}^L_k|(\mathcal{P}^L_k)^\infty_1 \vee \mathcal{A}^L_\infty)$$

It follows from monotonicity and continuity of entropy [ELW, Proposition 2.10, 2.12, and 2.13] that for any fixed $k \geq 1$

$$\lim_{\ell\to\infty} H_\mu(\mathcal{P}^L_k|(\mathcal{P}^L_\ell)^\infty_1 \vee \mathcal{A}^L_\infty) \leq H_\mu(\mathcal{P}^L_k|(\mathcal{P}^L_k)^\infty_1 \vee \mathcal{A}^L_\infty) \leq \lim_{\ell\to\infty} H_\mu(\mathcal{P}^L_\ell|(\mathcal{P}^L_k)^\infty_1 \vee \mathcal{A}^L_\infty),$$

hence we have

$$H_\mu(\mathcal{P}^L_k|(\mathcal{P}^L)^\infty_1 \vee \mathcal{A}^L_\infty) \leq H_\mu(\mathcal{P}^L_k|(\mathcal{P}^L_k)^\infty_1 \vee \mathcal{A}^L_\infty) \leq H_\mu(\mathcal{P}^L|(\mathcal{P}^L_k)^\infty_1 \vee \mathcal{A}^L_\infty).$$

Taking $k \to \infty$, it follows that

$$\lim_{k\to\infty} H_\mu(\mathcal{P}^L_k|(\mathcal{P}^L_k)^\infty_1 \vee \mathcal{A}^L_\infty) = H_\mu(\mathcal{P}^L|(\mathcal{P}^L)^\infty_1 \vee \mathcal{A}^L_\infty) = H_\mu(\mathcal{A}^L|a\mathcal{A}^L),$$

which concludes (3.12).

Note that $\mathcal{B}_Y = (\mathcal{P}^L)^\infty_{-\infty} = \bigvee_{k=1}^\infty (\mathcal{P}^L_k)^\infty_{-\infty}$ modulo almost every ergodic component of μ. Thus following the same argument as above, we can conclude (3.12) for almost every ergodic component of μ. □

The quantity $H_\mu(\mathcal{A}^L|a\mathcal{A}^L)$ is called *empirical entropy* and is the average of the *conditional information function*

$$I_\mu(\mathcal{A}^L|a\mathcal{A}^L)(x) = -\log \mu_x^{a\mathcal{A}^L}([x]_{\mathcal{A}}),$$

and indeed the *entropy contribution* of L (see [EL10, 7.8] for definition).

3.2.3 Effective variational principle

We first recall the variational principle. Combining [EL10, Proposition 7.34] and [EL10, Theorem 7.9], we have the following upper bound of an empirical entropy (or entropy contribution), and the entropy rigidity.

Theorem 3.2.9 ([EL10])**.** *Let $L < G^+$ be a closed subgroup normalized by a, and let $\mathfrak{l}$ denote the Lie algebra of L. Let μ be an a-invariant ergodic probability measure on Y. If $\mathcal{A}$ is a countably generated sub-σ-algebra of the Borel σ-algebra which is a^{-1}-descending and L-subordinate, then*

$$H_\mu(\mathcal{A}|a\mathcal{A}) \leq \log|\det(Ad_a|_{\mathfrak{l}})|$$

and equality holds if and only if μ is L-invariant.

This subsection is to effectivize the variational principle. Let $L < G^+$ be a closed subgroup normalized by a, m_L be the Haar measure on L, and μ be an a-invariant probability measure on Y. Let $\mathcal{A}$ be a countably generated sub-σ-algebra of Borel σ-algbera which is a^{-1}-descending and L-subordinate modulo μ. Note that for any $j \in \mathbb{Z}_{\geq 0}$, the sub-$\sigma$-algebra $a^j\mathcal{A}$ is also countably generated, a^{-1}-descending, and L-subordinate modulo μ.

For $y \in Y$, denote by $V_y \subset L$ the shape of the $\mathcal{A}$-atom at $y \in Y$ so that $V_y \cdot y = [y]_{\mathcal{A}}$. It has positive m_L-measure for μ-a.e. $y \in Y$ since $\mathcal{A}$ is L-subordinate modulo μ. Note that for any $j \in \mathbb{Z}_{\geq 0}$, we have $[y]_{a^j\mathcal{A}} = a^j V_{a^{-j}y} a^{-j} \cdot y$.

As in [EL10, 7.55] which is the proof of [EL10, Theorem 7.9], let us define $\tau_y^{a^j\mathcal{A}}$ for μ-a.e $y \in Y$ to be the normalized push forward of $m_L|_{a^j V_{a^{-j}y} a^{-j}}$ under the orbit map, i.e.,

$$\tau_y^{a^j\mathcal{A}} = \frac{1}{m_L(a^j V_{a^{-j}y} a^{-j})} m_L|_{a^j V_{a^{-j}y} a^{-j}} \cdot y,$$

which is a probability measure on $[y]_{a^j\mathcal{A}}$.

The following proposition is an effective version of Theorem 3.2.9.

Proposition 3.2.10. *Let $L < G^+$ be a closed subgroup normalized by a and μ be an a-invariant ergodic probability measure on Y. Fix $j \in \mathbb{N}$ and denote by $J \geq 0$ the maximal entropy contribution of L for a^j, that is,*

$$J = \log|\det(Ad_{a^j}|_{\mathfrak{l}})|.$$

Let $\mathcal{A}$ be a countably generated sub-σ-algebra of Borel σ-algbera which is a^{-1}-descending and L-subordinate. Suppose that there exist a measurable set $K \subset Y$ and $r > 0$ such that $[y]_{\mathcal{A}} \subset B_r^{L,\mathbf{c}} \cdot y$ for any $y \in K$, where $B_r^{L,\mathbf{c}}$ is as in Subsection 3.2.1. Then we have

$$H_\mu(\mathcal{A}|a^j\mathcal{A}) \leq J + \int_Y \log \tau_y^{a^j\mathcal{A}}((Y \setminus K) \cup B_r^{L,\mathbf{c}}\mathrm{Supp}\,\mu)d\mu(y).$$

Proof. By for instance [EL10, Theorem 5.9], for μ-a.e. $y \in Y$, $\mu_y^{a^j\mathcal{A}}$ is a probability measure on $[y]_{a^j\mathcal{A}} = a^jV_{a^{-j}y}a^{-j} \cdot y$, and $H_\mu(\mathcal{A}|a^j\mathcal{A})$ can be written as

$$H_\mu(\mathcal{A}|a^j\mathcal{A}) = -\int_Y \log \mu_y^{a^j\mathcal{A}}([y]_{\mathcal{A}})d\mu(y).$$

Note that $m_L(a^jBa^{-j}) = e^J m_L(B)$ for any measurable $B \subset L$. Let

$$p(y) := \mu_y^{a^j\mathcal{A}}([y]_{\mathcal{A}}) \quad \text{and} \quad p^{Haar}(y) := \tau_y^{a^j\mathcal{A}}([y]_{\mathcal{A}}).$$

Then we have

$$p^{Haar}(y) = \frac{m_L(V_y)}{m_L(a^jV_{a^{-j}y}a^{-j})} = \frac{m_L(V_y)}{m_L(V_{a^{-j}y})}e^{-J},$$

hence, applying the ergodic theorem, we have $-\int_Y \log p^{Haar}(y)d\mu(y) = J$.

Now we estimate an upper bound of $H_\mu(\mathcal{A}|a^j\mathcal{A}) - J$ following the computation in [EL10, 7.55]. Following [EL10, 7.55], we can partition $[y]_{a^j\mathcal{A}}$ into a countable union of $\mathcal{A}$-atoms as follows:

$$[y]_{a^j\mathcal{A}} = \bigcup_{i=1}^{\infty}[x_i]_{\mathcal{A}} \cup N_y,$$

where N_y is a null set with respect to $\mu_y^{a^j\mathcal{A}}$. Note that $\mu_y^{a^j\mathcal{A}}$ is supported on $\mathrm{Supp}\,\mu$ for μ-a.e y. Let us denote by $Z = (Y \setminus K) \cup B_r^{L,\mathbf{c}}\mathrm{Supp}\,\mu$. If $x_i \in Y \setminus Z = K \setminus B_r^{L,\mathbf{c}}\mathrm{Supp}\,\mu$, then we have $\mu_y^{a^j\mathcal{A}}([x_i]_{\mathcal{A}}) = 0$ since $[x_i]_{\mathcal{A}} \subset B_r^{L,\mathbf{c}} \cdot x_i \subseteq$

$K \setminus \operatorname{Supp}\mu$. Thus we have

$$\begin{aligned}
H_\mu(\mathcal{A}|a^j\mathcal{A}) - J &= -\int_Y \left(\log p(z) - \log p^{Haar}(z)\right) d\mu(z) \\
&= \int_Y \int_Y \left(\log p^{Haar}(z) - \log p(z)\right) d\mu_y^{a^j\mathcal{A}}(z) d\mu(y) \\
&= \int_Y \sum_{x_i \in Z} \int_{z\in[x_i]_\mathcal{A}} \left(\log p^{Haar}(z) - \log p(z)\right) d\mu_y^{a^j\mathcal{A}}(z) d\mu(y) \\
&= \int_Y \sum_{x_i\in Z} \log\left(\frac{\tau_y^{a^j\mathcal{A}}([x_i]_\mathcal{A})}{\mu_y^{a^j\mathcal{A}}([x_i]_\mathcal{A})}\right) \mu_y^{a^j\mathcal{A}}([x_i]_\mathcal{A}) d\mu(y) \\
&\le \int_Y \log\left(\sum_{x_i\in Z} \tau_y^{a^j\mathcal{A}}([x_i]_\mathcal{A})\right) d\mu(y) \\
&\le \int_Y \log \tau_y^{a^j\mathcal{A}}(Z) d\mu(y).
\end{aligned}$$

The fifth inequality is by the convexity of the logarithm. This proves the proposition. □

In particular, if $\mathcal{A} = \mathcal{A}^L$ then Proposition 3.2.10 still holds without assuming the ergodicity of μ.

Corollary 3.2.11. *Let $L < G^+$ be a closed subgroup normalized by a, μ be an a-invariant probability measure on Y, and $\mathcal{A}^L$ be as in Proposition 3.2.6. Then Proposition 3.2.10 holds with $\mathcal{A} = \mathcal{A}^L$.*

Proof. Writing the ergodic decomposition $\mu = \int \mu_z^\mathcal{E} d\mu(z)$, we have

$$h_\mu(a^j|\mathcal{A}_\infty^L) = \int h_{\mu_z^\mathcal{E}}(a^j|\mathcal{A}_\infty^L) d\mu(z),$$

where $\mathcal{A}_\infty^L$ is the σ-algebra as in (4.48). By Proposition 3.2.8, we also have

$$H_\mu(\mathcal{A}^L|a^j\mathcal{A}^L) = \int H_{\mu_z^\mathcal{E}}(\mathcal{A}^L|a^j\mathcal{A}^L) d\mu(z).$$

Applying Proposition 3.2.10 for each $\mu_z^{\mathcal{E}}$ we obtain

$$\begin{aligned}\int H_{\mu_z^{\mathcal{E}}}(\mathcal{A}^L|a^j\mathcal{A}^L)d\mu(z) &\le J + \int_Y\int_Y \log \tau_y^{a^j\mathcal{A}}((Y\setminus K)\cup B_r^{L,\mathbf{c}}\operatorname{Supp}\mu_z^{\mathcal{E}})d\mu_z^{\mathcal{E}}(y)d\mu(z)\\ &\le J + \int_Y \log \tau_y^{a^j\mathcal{A}}((Y\setminus K)\cup B_r^{L,\mathbf{c}}\operatorname{Supp}\mu)d\mu(y).\end{aligned}$$

□

3.3 Preliminaries for the upper bound

From now on, we fix the following notations:

$$d = m+n,\ G = \mathrm{ASL}_d(\mathbb{R}),\ \Gamma = \mathrm{ASL}_d(\mathbb{Z}),\ \text{and } Y = G/\Gamma.$$

We use all notations in Subsection 3.2.1 with this setting. In particular, we choose a right invariant metric d_G on G so that $r_{max} \le 1$. Denote by d_∞ the metric on G induced from the max norm on $M_{d+1,d+1}(\mathbb{R})$. Note that d_G and d_∞ are locally bi-Lipschitz.

Recall the notations a_t, $a = a_1$, U, and W in the introduction. Then the subgroups U and W are closed subgroups in G^+ normalized by a, where G^+ is the unstable horospherical subgroup associated to a. Denote by $\mathfrak{u}$ and $\mathfrak{w}$ the Lie algebras of U and W, respectively. We can take standard basis for $\mathfrak{u}$ and $\mathfrak{w}$ so that $\mathfrak{u} = \mathbb{R}^{mn} = M_{m,n}(\mathbb{R})$ and $\mathfrak{w} = \mathbb{R}^m$ with the associated quasinorms given by

$$\|A\|_{\mathbf{r}\otimes\mathbf{s}} = \max_{\substack{1\le i\le m\\ 1\le j\le n}} |A_{ij}|^{\frac{1}{r_i+s_j}} \quad \text{and} \quad \|b\|_{\mathbf{r}} = \max_{1\le i\le m} |b_i|^{\frac{1}{r_i}},$$

respectively, for any $A \in M_{m,n}(\mathbb{R})$ and $b \in \mathbb{R}^m$. We call these quasinorms $\mathbf{r}\otimes\mathbf{s}$-quasinorm and $\mathbf{r}$-quasinorm, respectively. It is also satisfies that

$$\|\operatorname{Ad}_{a_t} A\|_{\mathbf{r}\otimes\mathbf{s}} = e^t\|A\|_{\mathbf{r}\otimes\mathbf{s}} \quad \text{and} \quad \|\operatorname{Ad}_{a_t} b\|_{\mathbf{r}} = e^t\|b\|_{\mathbf{r}},$$

for any $A \in M_{m,n}(\mathbb{R})$ and $b \in \mathbb{R}^m$. These quasinorms induce the quasi-metrics $d_{\mathbf{r}\otimes\mathbf{s}}$ and $d_{\mathbf{r}}$ on $\mathfrak{u}$ and $\mathfrak{w}$, respectively. For simplicity, we keep the notations $d_{\mathbf{r}\otimes\mathbf{s}}$ and $d_{\mathbf{r}}$ as locally defined quasi-metrics on U and L, respectively.

As in Theorem 3.2.9, we can explicitly compute the maximum entropy contributions for $L = U$ and W. For $L = U$, the restricted adjoint map is the expansion $\mathrm{Ad}_a : (A_{ij}) \mapsto (e^{r_i+s_j}A_{ij})$ of $A \in M_{m,n}$, hence

$$\log|\det(Ad_a|_{\mathfrak{u}})| = \sum_{i,j}(r_i+s_j) = m+n.$$

For $L = W$, the restricted adjoint map is the expansion $Ad_a : (b_i) \mapsto (e^{r_i}b_i)$ of $b \in \mathbb{R}^m$, hence

$$\log|\det(Ad_a|_{\mathfrak{w}})| = \sum_i r_i = 1.$$

Denote by $X = \mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$ and by $\pi : Y \to X$ the natural projection sending a translated lattice $x+v$ to the lattice x. Equivalently, it is defined by $\pi\left(\begin{pmatrix} g & v \\ 0 & 1\end{pmatrix}\Gamma\right) = g\,\mathrm{SL}_d(\mathbb{Z})$ for $g \in \mathrm{SL}_d(\mathbb{R})$ and $v \in \mathbb{R}^d$. We also use the following notation: $w(v) = \begin{pmatrix} I_d & v \\ 0 & 1\end{pmatrix}$ for $v \in \mathbb{R}^d$.

3.3.1 Dimensions with quasinorms

Let Z be a space endowed with a *quasi-metric* d_Z, which is a symmetric, positive definite map $d_Z : Z \times Z \to \mathbb{R}_{\geq 0}$ such that, for some constant C, for all $x, y \in Z$, $d_Z(x,y) \leq C(d_Z(x,z) + d_Z(z,y))$. For a bounded subset $S \subset Z$, the lower Minkowski dimension $\underline{\dim}_{d_Z} S$ with respect to the quasi-metric d_Z is defined by

$$\underline{\dim}_{d_Z} S := \liminf_{\delta\to 0} \frac{\log N_{d_Z}(S,\delta)}{\log 1/\delta},$$

where $N_{d_Z}(S,\delta)$ is the maximal cardinality of a δ-separated subset of S for d_Z. If S is unbounded, we let $\underline{\dim}_{d_Z} S = \sup\{\underline{\dim}_{d_Z} S \cap K \ ; \ K \text{ compact}\}$.

At the begining of this section, we consider Lie algebras $\mathfrak{u}$ and $\mathfrak{w}$ endowed with $\mathbf{r}\otimes\mathbf{s}$-quasinorm and $\mathbf{r}$-quasinorm, which induce the quasi-metrics $d_{\mathbf{r}\otimes\mathbf{s}}$ and $d_{\mathbf{r}}$ on $\mathfrak{u}$ and $\mathfrak{w}$, repectively.

Now, for subsets $S \subset \mathfrak{u} = \mathbb{R}^{mn}$ and $S' \subset \mathfrak{w} = \mathbb{R}^m$, we denote the lower Minkowski dimensions of these subsets as follows:

$$\underline{\dim}_{\mathbf{r}\otimes\mathbf{s}} S := \underline{\dim}_{d_{\mathbf{r}\otimes\mathbf{s}}} S, \qquad \underline{\dim}_M S := \underline{\dim}_{d_E} S,$$

$$\underline{\dim}_{\mathbf{r}} S' := \underline{\dim}_{d_{\mathbf{r}}} S', \qquad \underline{\dim}_M S' := \underline{\dim}_{d_E} S'$$

where d_E is the standard metric. We will also consider Hausdorff dimensions

$\dim_H S$ and $\dim_H S'$, always defined with respect to the standard metric. We refer the reader to [Fal14] for general properties of Minkowski or Hausdorff dimensions, such as the inequality

$$\underline{\dim}_M S \geq \dim_H S.$$

Following [LSS19], we will relate dimension $\underline{\dim}_M$ to entropy, and further to Hausdorff dimension using $\underline{\dim}_{\mathbf{r}\otimes\mathbf{s}}$ and $\underline{\dim}_{\mathbf{r}}$ via the following lemma.

Lemma 3.3.1. *[LSS19, Lemma 2.2] For subsets $S \subset \mathfrak{u}$ and $S' \subset \mathfrak{w}$,*

1. $\underline{\dim}_{\mathbf{r}\otimes\mathbf{s}}\mathfrak{u} = \sum_{i,j}(r_i + s_j) = m + n$ *and* $\underline{\dim}_{\mathbf{r}}\mathfrak{w} = \sum_i r_i = 1$,
2. $\underline{\dim}_{\mathbf{r}\otimes\mathbf{s}}S \geq (m+n) - (r_1 + s_1)(mn - \underline{\dim}_M S)$,
3. $\underline{\dim}_{\mathbf{r}}S' \geq 1 - r_1(m - \underline{\dim}_M S')$.

3.3.2 Correspondence with dynamics

For $y = \begin{pmatrix} g & v \\ 0 & 1 \end{pmatrix}\Gamma \in Y$ with $g \in \mathrm{SL}_d(\mathbb{R})$ and $v \in \mathbb{R}^d$, denote by Λ_y the corresponding unimodular grid $g\mathbb{Z}^d + v$ in $\mathbb{R}^d$. We denote the $(\mathbf{r},\mathbf{s})$-quasinorm of $v = (\mathbf{x},\mathbf{y}) \in \mathbb{R}^m \times \mathbb{R}^n$ by $\|v\|_{\mathbf{r},\mathbf{s}} = \max\{\|\mathbf{x}\|_{\mathbf{r}}^{\frac{d}{m}}, \|\mathbf{y}\|_{\mathbf{s}}^{\frac{d}{n}}\}$. Let

$$\mathcal{L}_\epsilon := \{y \in Y : \forall v \in \Lambda_y, \|v\|_{\mathbf{r},\mathbf{s}} \geq \epsilon\},$$

which is a (non-compact) closed subset of Y. Following [Kle99, Section 1.3], we say that the pair $(A, b) \in M_{m,n}(\mathbb{R}) \times \mathbb{R}^m$ is *rational* if there exists some $(p, q) \in \mathbb{Z}^m \times \mathbb{Z}^n$ such that $Aq - b + p = 0$, and *irrational* otherwise.

Proposition 3.3.2. *For any irrational pair $(A, b) \in M_{m,n}(\mathbb{R}) \times \mathbb{R}^m$, $(A, b) \in$ **Bad**(ϵ) if and only if the a_t-orbit of the point $y_{A,b}$ is eventually in $\mathcal{L}_\epsilon$, i.e., there exists $T \geq 0$ such that $a_t y_{A,b} \in \mathcal{L}_\epsilon$ for all $t \geq T$.*

Proof. Suppose that there exist arbitrarily large t's satisfying $a_t y_{A,b} \notin \mathcal{L}_\epsilon$. Denote $e^{\mathbf{r}t} := \mathrm{diag}(e^{r_1 t}, \cdots, e^{r_m t}) \in M_{m,m}(\mathbb{R})$ and $e^{\mathbf{s}t} := \mathrm{diag}(e^{s_1 t}, \cdots, e^{s_n t}) \in M_{n,n}(\mathbb{R})$. Then the vectors in the grid $\Lambda_{a_t y_{A,b}}$ can be represented as

$$a_t\left(\begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} -b \\ 0 \end{pmatrix}\right) = \begin{pmatrix} e^{\mathbf{r}t}(Aq + p - b) \\ e^{-\mathbf{s}t}q \end{pmatrix}$$

for $(p, q) \in \mathbb{Z}^m \times \mathbb{Z}^n$. Therefore $a_t x_{A,b} \notin \mathcal{L}_\epsilon$ implies that for some $q \in \mathbb{Z}^n$,

$$e^t \langle Aq - b \rangle_{\mathbf{r}} < \epsilon^{\frac{m}{d}} \quad \text{and} \quad e^{-t}\|q\|_{\mathbf{s}} < \epsilon^{\frac{n}{d}}, \tag{3.13}$$

thus $\|q\|_{\mathbf{s}}\langle Aq - b\rangle_{\mathbf{r}} < \epsilon$. Since $\langle Aq - b\rangle_{\mathbf{r}} \neq 0$ for all q, we use the condition $\langle Aq - b\rangle_{\mathbf{r}} < e^{-t}\epsilon^{\frac{m}{d}}$ for arbitrarily large t to conclude that $\|q\|_{\mathbf{s}}\langle Aq - b\rangle_{\mathbf{r}} < \epsilon$ holds for infinitely many q's. This is a contradiction to the assumption that $(A, b) \in \mathbf{Bad}(\epsilon)$.

On the other hand, if $(A, b) \notin \mathbf{Bad}(\epsilon)$, then since (A, b) is irrational, there are infinitely many $q \in \mathbb{Z}^n$ such that $\|q\|_{\mathbf{s}}\langle Aq - b\rangle_{\mathbf{r}} < \epsilon$. Thus we can choose arbitrarily large t so that (3.13) hold, which contradicts to the assumption that the a_t-orbit of the point $y_{A,b}$ is eventually in $\mathcal{L}_\epsilon$. □

We claim that for a fixed $b \in \mathbb{R}^m$, the subset $\mathbf{Bad}^b_0(\epsilon)$ of $\mathbf{Bad}^b(\epsilon)$ such that (A, b) is rational is a subset of $\mathbf{Bad}^0(\epsilon)$. Indeed, if $A \in \mathbf{Bad}^b(\epsilon)$ for some b and (A, b) is rational, then $\langle Aq_0 - b\rangle_{\mathbf{r}} = 0$ for some $q_0 \in \mathbb{Z}^m$ and $\liminf_{\|q\|_{\mathbf{s}}\to\infty} \|q\|_{\mathbf{s}}\langle Aq - b\rangle_{\mathbf{r}} \geq \epsilon$, thus $\liminf_{\|q\|_{\mathbf{s}}\to\infty} \|q\|_{\mathbf{s}}\langle A(q - q_0)\rangle_{\mathbf{r}} \geq \epsilon$. Therefore, we have

$$\dim_H \mathbf{Bad}^b_0(\epsilon) \leq \dim_H \mathbf{Bad}^0(\epsilon) = mn - c_{m,n}\frac{\epsilon}{\log 1/\epsilon} < mn$$

for some constant $c_{m,n} > 0$ [KM19].

For a fixed $A \in M_{m,n}(\mathbb{R})$, the subset of $\mathbf{Bad}_A(\epsilon)$ such that (A, b) is rational is of the form $Aq + p$ for some $q, p \in \mathbb{Z}^m$ thus has Hausdorff dimension zero.

In the rest of the article, we will focus on the elements $y_{A,b}$ that are eventually in $\mathcal{L}_\epsilon$.

3.3.3 Covering counting lemma

To construct measures of large entropy in Proposition 3.4.1 and Proposition 3.5.3, we will need the following counting lemma, which is a generalization of [LSS19, Lemma 2.4].

Here, we consider two cases: $L = U$ and $L = W$. Fix a standard basis $\{e_i : i = 1, \dots, \dim \mathfrak{l}\}$ on $\mathfrak{l}$. Denote by $\|\cdot\|_{\mathbf{c}}$ both of $\|\cdot\|_{\mathbf{r}\otimes\mathbf{s}}$ and $\|\cdot\|_{\mathbf{r}}$, for simplicity. Let J_L be the maximal entropy contribution for L, that is, $J_L = \sum_i c_i$. Recall that $J_U = m + n$, and $J_W = 1$.

Before state the main result in this subsection, we fix the following notations. Let $Q^0_\infty \subset X$ be such that $X \smallsetminus Q^0_\infty$ has compact closure. Set $Q_\infty = \pi^{-1}(Q^0_\infty)$ and denote by $r_0 > 0$ the injectivity radius on $Y \smallsetminus Q_\infty$. Note that

$Y \smallsetminus Q_\infty \subset Y(r_0)$. For any $D > J_L$, choose large enough $T_D \in \mathbb{N}$ so that

$$\lceil e^{c_i T_D} \rceil \leq e^{c_i T_D} e^{\frac{D-J_L}{\dim \mathfrak{l}}} \tag{3.14}$$

for all $i = 1, \dots, \dim \mathfrak{l}$. Fix $0 < r_D = r_D(Q^0_\infty) < 1/2$ small enough so that

- d_G and d_∞ are bi-Lipschitz on $B^G_{r_D}$, that is, there is $C_\infty \geq 1$ such that for any $x, y \in B^G_{r_D}$,

$$\frac{1}{C_\infty} d_\infty(x,y) \leq d_G(x,y) \leq C_\infty d_\infty(x,y). \tag{3.15}$$

- The following inclusions hold:

$$B^{L,\mathbf{c}}_{C_\infty r_D^{\frac{1}{\max \mathbf{c}}} T_D} \subset B^L_{\frac{1}{2}r_0} \quad \text{and} \quad B^G_{r_D}(Y \smallsetminus Q_\infty) \subset Y(\frac{1}{2}r_0). \tag{3.16}$$

Lemma 3.3.3. *For any $D > J_L$, we fix the above notations. Let $y \in Y \smallsetminus Q_\infty$ and $I = \{t \in \mathbb{N} \mid a_t y \in Q_\infty\}$. For any non-negative integer T, let*

$$E_{y,T} = \{z \in B^L_{r_D} \cdot y \mid \forall t \in \{1, \dots, T\} \smallsetminus I,\ d_Y(a_t y, a_t z) \leq r_D\}.$$

The set $E_{y,T}$ can be covered by $Ce^{D|I \cap \{1,\dots,T\}|}$ $d_{L,\mathbf{c}}$-balls of radius $r_D^{\frac{1}{\max \mathbf{c}}} e^{-T}$, where C is a constant depending on Q^0_∞ and D, but independent of T.

Proof. For $s \in \{0, \dots, T_D - 1\}$ and $k \in \mathbb{Z}_{\geq 0}$, let us denote by $I_{s,k}(T_D) = \{s, s+T_D, \dots, s+kT_D\}$ and

$$E^s_{y,k} = \{z \in B^L_{r_D} \cdot y : \forall t \in I_{s,k}(T_D) \smallsetminus I, d_Y(a_t y, a_t z) \leq r_D\}.$$

Following the proof of [LSS19, Lemma2.4] with $E^s_{y,k}$ instead of $E_{y,T}$, we can obtain the following claim:

Claim The set $E^s_{y,k}$ can be covered by $C_s e^{(J_L(T_D-1)+D)|I \cap I_{s,k}(T_D)|}$ $d_{L,\mathbf{c}}$-balls of radius $C_\infty r_D^{\frac{1}{\max \mathbf{c}}} e^{-(s+kT_D)}$, where C_s is a constant depending on Q^0_∞, D and s, but independent of k.

Proof. We prove the claim by induction on k. Since the number of $d_{L,\mathbf{c}}$-balls of radius $C_\infty r_D^{\frac{1}{\max \mathbf{c}}} e^{-s}$ needed to cover $B^L_{r_D} \cdot y$ is bounded by a constant C_s depending on Q^0_∞, D and s, the claim holds for $k = 0$.

Suppose that $E^s_{y,k-1}$ can be covered by $N_{k-1} = C_s e^{(J_L(T_D-1)+D)|I\cap I_{s,k-1}(T_D)|}$ $d_{L,\mathbf{c}}$-balls of radius $C_\infty r_D^{\frac{1}{\max \mathbf{c}}} e^{-(s+(k-1)T_D)}$. By the inequality (3.14), any $d_{L,\mathbf{c}}$-ball of radius $C_\infty r_D^{\frac{1}{\max \mathbf{c}}} e^{-(s+(k-1)T_D)}$ can be covered by

$$\begin{aligned}\prod_{i=1}^{\dim \mathfrak{l}} \left\lceil \frac{e^{-(s+(k-1)T_D)c_i}}{e^{-(s+kT_D)c_i}} \right\rceil &= \prod_{i=1}^{\dim \mathfrak{l}} \lceil e^{T_D c_i} \rceil \leq \prod_{i=1}^{\dim \mathfrak{l}} e^{c_i T_D} e^{\frac{D-J_L}{\dim \mathfrak{l}}} \\ &= e^{J_L T_D} e^{D-J_L} = e^{J_L(T_D-1)+D},\end{aligned}$$

$d_{L,\mathbf{c}}$-balls of radius $C_\infty r_D^{\frac{1}{\max \mathbf{c}}} e^{-(s+kT_D)}$. Thus if $s+kT_D \in I$, then $E^s_{y,k}$ can be covered by $N_k = e^{J_L(T_D-1)+D} N_{k-1}$ $d_{L,\mathbf{c}}$-balls of radius $C_\infty r_D^{\frac{1}{\max \mathbf{c}}} e^{-(s+kT_D)}$.

Suppose that $s+kT_D \notin I$. Denote by $\{B_j : j=1,\dots,N_{k-1}\}$ the above covering of $E^s_{y,k-1}$. Since $E^s_{y,k} \subset E^s_{y,k-1}$, the set $\{E^s_{y,k} \cap B_j : j=1,\dots,N_{k-1}\}$ covers $E^s_{y,k}$. We now claim that for any $x_1, x_2 \in E^s_{y,k} \cap B_j$

$$d_{L,\mathbf{c}}(x_1,x_2) \leq 2C_\infty r_D^{\frac{1}{\max \mathbf{c}}} e^{-(s+kT_D)}.$$

Indeed, since $x_1, x_2 \in B_j \subset B^L_{r_D}\cdot y$ and B_j is a $d_{L,\mathbf{c}}$-ball of radius $C_\infty r_D^{\frac{1}{\max \mathbf{c}}} e^{-(s+(k-1)T_D)}$, there are $h \in B^L_{r_D}$ and $h_1, h_2 \in B^{L,\mathbf{c}}_{C_\infty r_D^{\frac{1}{\max \mathbf{c}}} e^{-(s+(k-1)T_D)}}$ such that $x_1 = h_1 h y$ and $x_2 = h_2 h y$. It follows from (3.16) that

$$\begin{aligned} a^{s+kT_D} h_1 h_2^{-1} a^{-(s+kT_D)} &\subset a^{s+kT_D} B^{L,\mathbf{c}}_{C_\infty r_D^{\frac{1}{\max \mathbf{c}}} e^{-(s+(k-1)T_D)}} a^{-(s+kT_D)} \\ &= B^{L,\mathbf{c}}_{C_\infty r_D^{\frac{1}{\max \mathbf{c}}} e^{T_D}} \subset B^L_{\frac{1}{2}r_0}. \end{aligned}$$

Since $hy \subset Y(\frac{1}{2}r_0)$ by (3.16), we have

$$d_Y(a^{s+kT_D}x_1, a^{s+kT_D}x_2) = d_L(a^{s+kT_D}h_1h_2^{-1}a^{-(s+kT_D)}, id).$$

It follows from (3.15) that

$$\begin{aligned} d_L(a^{s+kT_D}h_1h_2^{-1}a^{-(s+kT_D)}, id) &\geq \frac{1}{C_\infty} d_\infty(a^{s+kT_D}h_1h_2^{-1}a^{-(s+kT_D)}, id) \\ &= \frac{1}{C_\infty} \max_{i=1,\dots,\dim \mathfrak{l}} e^{c_i(s+kT_D)} |(\log h_1h_2^{-1})_i|, \end{aligned}$$

where $(\log h_1h_2^{-1})_i$ is the i-th coordinate of $\log h_1h_2^{-1}$ with respect to the standard basis $\{e_i : 1 \leq i \leq \dim \mathfrak{l}\}$.

On the other hand, since $s+kT_D \notin I$, we have $d_Y(a^{s+kT_D}y, a^{s+kT_D}x_\ell) \leq r_D$ for each $\ell = 1, 2$. Thus $d_Y(a^{s+kT_D}x_1, a^{s+kT_D}x_2) \leq 2r_D$. Since $L = U$ or $L = W$, i.e. commutative subgroups of G, for each $i = 1, \ldots, \dim \mathfrak{l}$, we have

$$|(\log h_1 h_2^{-1})_i| = |(\log h_1 - \log h_2)_i| \leq 2r_D C_\infty e^{-c_i(s+kT_D)}.$$

Note that

$$d_{L,\mathbf{c}}(x_1, x_2) = d_{L,\mathbf{c}}(h_1, h_2) = \max_{i=1,\ldots,\dim \mathfrak{l}} |(\log h_1 - \log h_2)_i|^{\frac{1}{c_i}}$$

Therefore, we have

$$d_{L,\mathbf{c}}(x_1, x_2) \leq \max_{i=1,\ldots,\dim \mathfrak{l}} (2r_D C_\infty)^{\frac{1}{c_i}} e^{-(s+kT_D)} \leq 2C_\infty r_D^{\frac{1}{\max \mathbf{c}}} e^{-(s+kT_D)}.$$

By the claim, $E^s_{y,k} \cap B_j$ is contained in a single $d_{L,\mathbf{c}}$-ball of radius $C_\infty r_D^{\frac{1}{\max \mathbf{c}}} e^{-(s+kT_D)}$ for each $j = 1, \ldots, N_{k-1}$. Hence $E^s_{y,k}$ can be covered by $N_k = N_{k-1}$ $d_{L,\mathbf{c}}$-balls of radius $C_\infty r_D^{\frac{1}{\max \mathbf{c}}} e^{-(s+kT_D)}$. □

Now, for any non-negative integer T, we can find $s \in \{0, \ldots, T_D - 1\}$ and $k \in \mathbb{Z}_{\geq 0}$ such that

$$T_D |I \cap I_{s,k}(T_D)| \leq |I \cap \{1, \ldots, T\}| \quad \text{and} \quad T - T_D < s + kT_D \leq T$$

from the pigeon hole principle. By the above observation, $E_{y,T} \subset E^s_{y,k}$ can be covered by $C_s e^{(J_L(T_D-1)+D)|I \cap I_{s,k}(T_D)|}$ $d_{L,\mathbf{c}}$-balls of radius $C_\infty r_D^{\frac{1}{\max \mathbf{c}}} e^{-(s+kT_D)}$. Since $T - T_D < s+kT_D \leq T$ and $D > J_L$, $E_{y,T}$ can be covered by $(\max_s C_s) e^{D|I \cap \{1,\ldots,T\}|}$ $d_{L,\mathbf{c}}$-balls of radius $C_\infty e^{T_D} r_D^{\frac{1}{\max \mathbf{c}}} e^{-T}$. Hence there exists a constant $C > 0$ depending on Q^0_∞, r, and D, but independent of T such that $E_{y,T}$ can be covered by $C e^{D|I \cap \{1,\ldots,T\}|}$ $d_{L,\mathbf{c}}$-balls of radius $r_D^{\frac{1}{\max \mathbf{c}}} e^{-T}$. □

3.4 Upper bound for Hausdorff dimension of $\mathbf{Bad}_A(\epsilon)$

In this section, we will prove Theorem 1.2.2 by constructing a-invariant probability measure on Y with large entropy. Here and next section, we will consider the dynamical entropy of a instead of a^{-1} contrary to Section 5.1. Hence let us use the following notation: For a given partition $\mathcal{Q}$ of Y and a integer $q \geq 1$, we denote by

$$\mathcal{Q}^{(q)} = \bigvee_{i=0}^{q-1} a^{-i}\mathcal{Q}.$$

3.4.1 Constructing measure with entropy lower bound

Let us denote by $\overline{X}$ and $\overline{Y}$ the one-point compactifications of X and Y, respectively. Let $\mathcal{A}$ be a given countably generated σ-algebra of X or Y. We denote by $\overline{\mathcal{A}}$ the σ-algebra generated by $\mathcal{A}$ and $\{\infty\}$. The diagonal action a_t is extended to the action on $\overline{X}$ and $\overline{Y}$ by $a_t(\infty) = \infty$ for $t \in \mathbb{R}$. For a finite partition $\mathcal{Q} = \{Q_1, \cdots, Q_N, Q_\infty\}$ of Y which has only one non-compact element Q_∞, denote by $\overline{\mathcal{Q}}$ the finite partition $\{Q_1, \cdots, Q_N, \overline{Q_\infty} = Q_\infty \cup \{\infty\}\}$ of $\overline{Y}$. Note that $\overline{\mathcal{Q}^{(q)}} = \overline{\mathcal{Q}}^{(q)}$ for any $M \in \mathbb{N}$. We also denote by $\mathscr{P}(X)$ the space of probability measures on X, and use similar notations for Y, $\overline{X}$, and $\overline{Y}$.

In this subsection, we construct an a-invariant measure on $\overline{Y}$ with a lower bound on the conditional entropy for the proof of Theorem 1.2.2. Here, the conditional entropy will be computed with respect to the σ-algebras constructed in Section 5.1. If x_A has no escape of mass, such measure was constructed in [LSS19, Proposition 2.3]. The following proposition generalizes the measure construction for x_A's with some escape of mass.

Proposition 3.4.1. *For $A \in M_{m,n}(\mathbb{R})$ fixed, let*

$$\eta_A = \sup\{\eta : x_A \textit{ has } \eta\textit{-escape of mass on average}\}.$$

Then there exists $\mu_A \in \mathscr{P}(\overline{X})$ with $\mu_A(X) = 1 - \eta_A$ such that for any $\epsilon > 0$, there exists an a-invariant measure $\overline{\mu} \in \mathscr{P}(\overline{Y})$ satisfying

1. $\operatorname{Supp}\overline{\mu} \subset \mathcal{L}_\epsilon \cup (\overline{Y} \setminus Y)$,

2. *$\pi_*\overline{\mu} = \mu_A$, in particular, there exists a-invariant measure $\mu \in \mathscr{P}(Y)$ such that*
$$\overline{\mu} = (1-\eta_A)\mu + \eta_A\delta_\infty,$$
where δ_∞ is the dirac delta measure on $\overline{Y} \setminus Y$.

3. *Let $\mathcal{A}^W$ be as in Proposition 3.2.6 for μ and $L = W$, and let $\mathcal{A}^W_\infty$ be as in (4.48). Then we have*
$$h_{\overline{\mu}}(a|\overline{\mathcal{A}^W_\infty}) \geq 1 - \eta_A - r_1(m - \dim_H \mathbf{Bad}_A(\epsilon)).$$

Remark 3.4.2.

1. *Note that if $\eta_A > 0$ then x_A has η_A-escape of mass on average.*
2. *One can check that $\eta_A = 0$ if and only if x_A is heavy, which is defined in [LSS19, Definition 1.1].*

Proof. Since x_A has η_A-escape of mass on average but no more than η_A, we may fix an increasing sequence of integers $\{k_i\}_{i\geq 1}$ such that

$$\frac{1}{k_i}\sum_{k=0}^{k_i-1}\delta_{a^k x_A} \xrightarrow{\text{w}^*} \mu_A \in \mathscr{P}(\overline{X})$$

with $\mu_A(X) = 1-\eta_A$.

Let us denote by $\mathbb{T}^m = [0,1]^m/\sim$ the torus in $\mathbb{R}^m$, where the equivalence relation is modulo 1. Let

$$R^{A,T} := \{b \in \mathbb{T}^m | \forall t \geq T, a_t y_{A,b} \in \mathcal{L}_\epsilon\} \cap \mathbf{Bad}_A(\epsilon).$$

As explained in Subsection 3.3.2, the subset of $\mathbf{Bad}_A(\epsilon)$ such that (A,b) is rational has Hausdorff dimension zero. Hence, by Proposition 3.3.2, $\bigcup_{T=1}^{\infty} R^{A,T}$ has Hausdorff dimension equal to $\dim_H \mathbf{Bad}_A(\epsilon)$. For any $\gamma > 0$, it follows that there exists $T_\gamma \in \mathbb{N}$ satisfying $\dim_H R^{A,T_\gamma} \geq \dim_H \mathbf{Bad}_A(\epsilon) - \gamma$.

Let $\phi_A : \mathbb{T}^m \to Y$ be the map defined by $\phi_A(b) = y_{A,b}$. Note that ϕ_A is an one-to-one Lipschitz map between $\mathbb{T}^m$ and $\phi_A(\mathbb{T}^m)$, so we may consider a quasinorm on $\phi_A(\mathbb{T}^m)$ induced from the $\mathbf{r}$-quasinorm on $\mathbb{R}^m$ and denote it again by $\|\cdot\|_{\mathbf{r}}$.

For each $k_i \geq T_\gamma$, let S_i be a maximal e^{-k_i}-separated subset of R^{A,T_γ} with respect to the $\mathbf{r}$-quasinorm. By Lemma 3.3.1,

$$\liminf_{i\to\infty}\frac{\log|S_i|}{k_i} \geq \underline{\dim}_{\mathbf{r}}(R^{A,T_\gamma}) \geq 1 - r_1(m+\gamma-\dim_H \mathbf{Bad}_A(\epsilon)). \tag{3.17}$$

Let $\nu_i = \frac{1}{|S_i|}\sum_{b\in S_i}\delta_{y_{A,b}}$ be the normalized counting measure on the set $D_i := \phi_A(S_i) = \{y_{A,b} : b \in S_i\} \subset Y$. Extracting a subsequence if necessary, we may assume without loss of generality that

$$\mu_i = \frac{1}{k_i}\sum_{k=0}^{k_i-1} a_*^k \nu_i \xrightarrow{\text{w}^*} \mu^\gamma \in \mathscr{P}(\overline{Y}).$$

The measure μ^γ is a-invariant since $a_*\mu_i - \mu_i$ goes to zero measure.

Choose any sequence of positive real numbers $(\gamma_j)_{j\geq 1}$ converging to zero and let $\{\mu^{\gamma_j}\}$ be a family of a-invariant probability measures on $\overline{Y}$ obtained from the above construction for each γ_j. Extracting a subsequence again if necessary, we may take a weak*-limit measure $\overline{\mu} \in \mathscr{P}(\overline{Y})$ of $\{\mu^{\gamma_j}\}$. We prove that $\overline{\mu}$ is the desired measure. The measure $\overline{\mu}$ is clearly a-invariant.
(1) We show that for all $\gamma > 0$, $\mu^\gamma(Y \setminus \mathcal{L}_\epsilon) = 0$. For any $b \in S_i \subseteq R^{A,T_\gamma}$, $a^T y_{A,b} \in \mathcal{L}_\epsilon$ holds for $T > T_\gamma$. Thus we have

$$\begin{aligned}\mu_i(Y \setminus \mathcal{L}_\epsilon) &= \frac{1}{k_i}\sum_{k=0}^{k_i-1} a_*^k \nu_i(Y \setminus \mathcal{L}_\epsilon) = \frac{1}{k_i}\sum_{k=0}^{T_\gamma} a_*^k \nu_i(Y \setminus \mathcal{L}_\epsilon)\\ &= \frac{1}{k_i|S_i|}\sum_{y\in D_i, 0\leq k\leq T_\gamma} \delta_{a^k y}(Y \setminus \mathcal{L}_\epsilon) \leq \frac{T_\gamma}{k_i}.\end{aligned}$$

By taking $k_i \to \infty$, we have $\mu^\gamma(Y \setminus \mathcal{L}_\epsilon) = 0$ for arbitrary $\gamma > 0$, hence

$$\overline{\mu}(Y \setminus \mathcal{L}_\epsilon) = \lim_{j\to\infty} \mu^{\gamma_j}(Y \setminus \mathcal{L}_\epsilon) = 0.$$

(2) For all $\gamma > 0$, $\pi_*\mu^\gamma = \mu_A$ holds since $\pi_*\nu_i = \delta_{x_A}$ for all $i \geq 1$. It follows that $\pi_*\overline{\mu} = \mu_A$. Hence,

$$\overline{\mu}(\overline{Y} \setminus Y) = \lim_{j\to\infty} \mu^{\gamma_j}(\overline{Y} \setminus Y) = \mu_A(\overline{X} \setminus X) = \eta_A,$$

so we have a decomposition $\overline{\mu} = (1-\eta_A)\mu + \eta_A\delta_\infty$ for some a-invariant $\mu \in \mathscr{P}(Y)$.
(3) We first fix any $D > J_W = 1$. Recall the notations in Subsection 3.3.3. Suppose that $\mathcal{Q}$ is any finite partition of Y satisfying:

- $\mathcal{Q}$ contains an atom Q_∞ of the form $\pi^{-1}(Q_\infty^0)$, where $X \smallsetminus Q_\infty^0$ has compact closure,
- $\forall Q \in \mathcal{Q} \smallsetminus \{Q_\infty\}$, $\operatorname{diam} Q < r_D = r_D(Q_\infty^0)$, where r_D is as in Subsection 3.3.3,

- $\forall Q \in \mathcal{Q}, \forall j \geq 1,\ \mu^{\gamma_j}(\partial Q) = 0.$

We will first prove the following statement. For all $q \geq 1$,

$$\frac{1}{q} H_{\overline{\mu}}(\overline{\mathcal{Q}}^{(q)} | \overline{\mathcal{A}_\infty^W}) \geq 1 - r_1(m - \dim_H \mathbf{Bad}_A(\epsilon)) - D\overline{\mu}(\overline{Q_\infty}). \tag{3.18}$$

It is clear if $\overline{\mu}(Q_\infty) = 1$, so assume that $\overline{\mu}(Q_\infty) < 1$, hence for all large enough $j \geq 1$, $\mu^{\gamma_j}(Q_\infty) < 1$. Now, we fix such $j \geq 1$ and write temporarily $\gamma = \gamma_j$.

Let $\rho > 0$ be small enough so that $\beta = \mu^\gamma(Q_\infty) + \rho < 1$. For large enough $i \geq 1$, we have

$$\begin{aligned} \beta = \mu^\gamma(Q_\infty) + \rho > \mu_i(Q_\infty) &= \frac{1}{k_i |S_i|} \sum_{y \in D_i, 0 \leq k < k_i} \delta_{a^k y}(Q_\infty) \\ &= \frac{1}{k_i} \sum_{0 \leq k < k_i} \delta_{a^k x_A}(Q_\infty^0). \end{aligned}$$

In other words, there exist at most βk_i number of $a^k x_A$'s in Q_∞^0, thus for any $y \in D_i$, we have

$$|\{k \in \{0, \ldots, k_i - 1\} : a^k y \in Q_\infty\}| < \beta k_i.$$

From Lemma 3.3.3 with $L = W$, if Q is any non-empty atom of $\mathcal{Q}^{(k_i)}$, fixing any $y \in Q$, the set

$$D_i \cap Q = D_i \cap [y]_{\mathcal{Q}^{(k_i)}} \subset E_{y, k_i - 1}$$

can be covered $Ce^{D\beta k_i}$ many $r_D^{1/r_1} e^{-k_i}$-balls for $d_{\mathbf{r}}$, where C is a constant depending on Q_∞^0 and D, but not on k_i. Since D_i is e^{-k_i}-separated with respect to $d_{\mathbf{r}}$ and $r_D^{1/r_1} < \frac{1}{2}$, we get

$$\operatorname{Card}(D_i \cap Q) \leq Ce^{D\beta k_i}. \tag{3.19}$$

Now let $\mathcal{A}^W = (\mathcal{P}^W)_0^\infty = \bigvee_{i=0}^\infty a^i \mathcal{P}^W$ be as in Proposition 3.2.6 for μ and $L = W$, and let $\mathcal{A}_\infty^W$ be as in (4.48). Using the continuity of entropy, we have

$$H_{\nu_i}(\mathcal{Q}^{(k_i)} | \mathcal{A}_\infty^W) = \lim_{\ell \to \infty} H_{\nu_i}(\mathcal{Q}^{(k_i)} | (\mathcal{P}^W)_\ell^\infty). \tag{3.20}$$

Claim $H_{\nu_i}(\mathcal{Q}^{(k_i)} | (\mathcal{P}^W)_\ell^\infty) = H_{\nu_i}(\mathcal{Q}^{(k_i)})$ for all large enough $\ell \geq 1$.

Proof. Observe that $\mu(\phi_A(\mathbb{T}^m)) > 0$. If not, it follows from the a-invariance of μ that $\mu(a^k\phi_A(\mathbb{T}^m)) = 0$ for all $k \geq 0$. It implies that $\mu_i(a^k\phi_A(\mathbb{T}^m)) = 0$ for all $k \geq 0$, but it contradict to $\mu_i(\bigcup_{k\geq 0} a^k\phi_A(\mathbb{T}^m)) = 1$. Let us denote by $\delta_0 = \mu(\phi_A(\mathbb{T}^m)) > 0$. Take $0 < \delta_1 < \min((\frac{c}{16d_0})^2, 1)$ small enough so that $\mu(Y \setminus Y(C_1\delta_1^{\frac{1}{2}})) + C_2\delta_1^{\frac{1}{2}} < \delta_0$, where $c, D > 0$ are the constants in Lemma 3.2.2 and (3.5), and $C_1, C_2 > 0$ are the constants in Lemma 3.2.5. Since $\mu(E_{\delta_1}) < \delta_0$ by Lemma 3.2.5, there exists $y \in \phi_A(\mathbb{T}^m) \cap Y \setminus E_{\delta_1}$. Hence, it follows from (3.5) and Proposition 3.2.6 that

$$[y]_{(\mathcal{P}^W)_\ell^\infty} = a^\ell[a^{-\ell}y]_{(\mathcal{P}^W)_0^\infty} = a^\ell[a^{-\ell}y]_{\mathcal{A}^W} \supset a^\ell B_{\delta_1}^W a^{-\ell}y \supset B_{d_0e^{\alpha\ell}\delta_1}^W y.$$

Since the support of ν_i is a set of finite points on a single compact W-orbit $\phi_A(\mathbb{T}^m)$, ν_i is supported on a single atom of $(\mathcal{P}^W)_\ell^\infty$ for all large enough $\ell \geq 1$. This proves the claim. ☐

Combining (3.19), (3.20), and **Claim**, it follows that

$$\begin{aligned} H_{\nu_i}(\mathcal{Q}^{(k_i)}|\mathcal{A}_\infty^W) &= \lim_{\ell\to\infty} H_{\nu_i}(\mathcal{Q}^{(k_i)}|(\mathcal{P}^W)_\ell^\infty) = H_{\nu_i}(\mathcal{Q}^{(k_i)}) \\ &\geq \log|S_i| - D\beta k_i - \log C. \end{aligned} \tag{3.21}$$

For any $q \geq 1$, write the Euclidean division of large enough $k_i - 1$ by q as

$$k_i - 1 = qk' + s \text{ with } s \in \{0, \cdots, q-1\}.$$

By subadditivity of the entropy with respect to the partition, for each $p \in \{0, \cdots, q-1\}$,

$$H_{\nu_i}(\mathcal{Q}^{(k_i)}|\mathcal{A}_\infty^W) \leq H_{a^p\nu_i}(\mathcal{Q}^{(q)}|\mathcal{A}_\infty^W) + \cdots + H_{a^{p+qk'}\nu_i}(\mathcal{Q}^{(q)}|\mathcal{A}_\infty^W) + 2q\log|\mathcal{Q}|.$$

Summing those inequalities for $p = 0, \cdots, q-1$, and using the concave property of entropy with respect to the measure, we obtain

$$\begin{aligned} qH_{\nu_i}(\mathcal{Q}^{(k_i)}|\mathcal{A}_\infty^W) &\leq \sum_{k=0}^{k_i-1} H_{a^k\nu_i}(\mathcal{Q}^{(q)}|\mathcal{A}_\infty^W)_0^M + 2q^2\log|\mathcal{Q}| \\ &\leq k_iH_{\mu_i}(\mathcal{Q}^{(q)}|\mathcal{A}_\infty^W) + 2q^2\log|\mathcal{Q}|, \end{aligned}$$

and it follows from (3.21) that

$$\begin{aligned}\frac{1}{q}H_{\mu_i}(\mathcal{Q}^{(q)}|\mathcal{A}_\infty^W) &\geq \frac{1}{k_i}H_{\nu_i}(\mathcal{Q}^{(k_i)}|\mathcal{A}_\infty^W) - \frac{2q\log|\mathcal{Q}|}{k_i}\\ &\geq \frac{1}{k_i}\Big(\log|S_i| - D\beta k_i - \log C - 2q\log|\mathcal{Q}|\Big).\end{aligned}$$

Now we can take $i \to \infty$ because the atoms Q of $\overline{\mathcal{Q}}$ and hence of $\overline{\mathcal{Q}}^{(q)}$, satisfy $\mu^\gamma(\partial Q) = 0$. Also, the constants C and $|\mathcal{Q}|$ are independent to k_i. Thus we obtain

$$\frac{1}{q}H_{\mu^\gamma}(\overline{\mathcal{Q}}^{(q)}|\overline{\mathcal{A}_\infty^W}) \geq 1 - r_1(m+\gamma - \dim_H \mathbf{Bad}_A(\epsilon)) - D\beta,$$

and by taking $\rho \to 0$, we have

$$\frac{1}{q}H_{\mu^\gamma}(\overline{\mathcal{Q}}^{(q)}|\overline{\mathcal{A}_\infty^W}) \geq 1 - r_1(m+\gamma - \dim_H \mathbf{Bad}_A(\epsilon)) - D\mu^\gamma(\overline{Q_\infty}).$$

Recall that $\gamma = \gamma_j$, and by taking $j \to \infty$ so that $\gamma_j \to 0$, we finally have (3.18), i.e.,

$$\frac{1}{q}H_{\overline{\mu}}(\overline{\mathcal{Q}}^{(q)}|\overline{\mathcal{A}_\infty^W}) \geq 1 - r_1(m - \dim_H \mathbf{Bad}_A(\epsilon)) - D\overline{\mu}(\overline{Q_\infty}).$$

As explained in [LSS19, Proof of Theorem 4.2, Claim 2], we can construct a finite partition $\mathcal{Q}$ of Y satisfying the bullet-requirements above. Hence,

$$h_{\overline{\mu}}(a|\overline{\mathcal{A}^W}) \geq 1 - r_1(m - \dim_H \mathbf{Bad}_A(\epsilon)) - D\overline{\mu}(\overline{Q_\infty}),$$

for any Q_∞ of $\mathcal{Q}$ satisfying the bullet-requirements. Moreover, we may take $Q_\infty^0 \subset X$ sufficiently small so that $\overline{\mu}(\overline{Q_\infty})$ is sufficiently close to $\overline{\mu}(\overline{Y}\setminus Y) = \eta_A$. It completes the proof by taking $D \to 1$. □

3.4.2 The proof of Theorem 1.2.2

In this subsection, we will estimate the dimension upper bound in Theorem 1.2.2 using a-invariant measure with large relative entropy constructed in Proposition 3.4.1 and the effective variational principle in Proposition 3.2.10. To use the effective variational principle, we need the following lemma.

For $x \in X$ and $H \geq 1$ we set:

$$\mathrm{ht}(x) = \sup\left\{ \|gv\|^{-1} : x = gSL_d(\mathbb{Z}), v \in \mathbb{Z}^d \setminus \{0\} \right\},$$

$$X_{\leq H} = \{x \in X : \mathrm{ht}(x) \leq H\}, \quad Y_{\leq H} = \pi^{-1}(X_{\leq H}).$$

Note that $\mathrm{ht}(x) \geq 1$ for any $x \in X$ by Minkowski's theorem and $X_{\leq H}$ and $Y_{\leq H}$ are compact sets for all $H \geq 1$ by Mahler's compact criterion.

Lemma 3.4.3. *Let $\mathcal{A}$ be a countably generated sub-σ-algebra of Borel σ-algebra which is a^{-1}-descending and W-subordinate. Let us fix $y \in Y_{\leq H}$ and suppose that $B_\delta^{W,\mathbf{r}} \cdot y \subset [y]_{\mathcal{A}} \subset B_r^{W,\mathbf{r}} \cdot y$ for some $0 < \delta < r$. For any $0 < \epsilon < 1$, if $j_1 \geq \log((2dH^{d-1})^{\frac{1}{r_m}}\delta^{-1})$ and $j_2 \geq \log((dH^{d-1})^{\frac{1}{s_n}}\epsilon^{-\frac{n}{d}})$, then $\tau_y^{a^{j_1}\mathcal{A}}(a^{-j_2}\mathcal{L}_\epsilon) \leq 1 - e^{-j_1-j_2}r^{-1}\epsilon^{\frac{m}{d}}$, where $\tau_y^{a^{j_1}\mathcal{A}}$ is as in Subsection 3.2.3.*

Proof. For $x = \pi(y) \in X_{\leq H}$, there exists $g \in SL_d(\mathbb{R})$ such that $x = gSL_d(\mathbb{Z})$ and $\inf_{v \in \mathbb{Z}^d \setminus \{0\}} \|gv\| \geq H^{-1}$. By Minkowski's second theorem with a convex body $[-1,1]^d$, we can choose vectors $gv_1, \cdots, gv_d$ in $g\mathbb{Z}^d$ so that $\prod_{i=1}^{d} \|gv_i\| \leq 1$. Then for any $1 \leq i \leq d$,

$$\|gv_i\| \leq \prod_{j \neq i} \|gv_j\|^{-1} \leq H^{d-1}.$$

Let $\Delta \subset \mathbb{R}^d$ be the parallelepiped generated by $gv_1, \cdots, gv_d$, then $\|b\| \leq dH^{d-1}$ for any $b \in \Delta$. It follows that $\|b^+\|_{\mathbf{r}} \leq (dH^{d-1})^{\frac{1}{r_m}}$ and $\|b^-\|_{\mathbf{s}} \leq (dH^{d-1})^{\frac{1}{s_n}}$ for any $b = (b^+, b^-) \in \Delta$, where $b^+ \in \mathbb{R}^m$ and $b^- \in \mathbb{R}^n$. Note that the set $\pi^{-1}(x) \subset Y$ is parametrized as follows:

$$\pi^{-1}(x) = \{w(b)g\Gamma \in Y : b \in \Delta\}.$$

Write $y = w(b_0)g\Gamma$ for some $b_0 = (b_0^+, b_0^-) \in \Delta$. Denote by $V_y \subset W$ the shape of $\mathcal{A}$-atom so that $V_y \cdot y = [y]_{a^{j_1}\mathcal{A}}$, and $\Xi \subset \mathbb{R}^m$ the corresponding set to V_y containing 0 given by the canonical bijection between W and $\mathbb{R}^m$. Since a^{j_1} expands the $\mathbf{r}$-quasinorm with the ratio e^{j_1}, we have $B_{e^{j_1}\delta}^{W,\mathbf{r}} \cdot y \subset [y]_{a^{j_1}\mathcal{A}} \subset B_{e^{j_1}r}^{W,\mathbf{r}} \cdot y$, i.e. $B_{e^{j_1}\delta}^{\mathbb{R}^m,\mathbf{r}} \subset \Xi \subset B_{e^{j_1}r}^{\mathbb{R}^m,\mathbf{r}}$. Then the atom $[y]_{a^{j_1}\mathcal{A}}$ is parametrized as follows:

$$[y]_{a^{j_1}\mathcal{A}} = \left\{ w(b)g\Gamma : b = (b^+, b_0^-), b^+ \in b_0^+ + \Xi \right\},$$

and $\tau_y^{a^{j_1}\mathcal{A}}$ can be considered as the normalized Lebesgue measure on the set

$b_0^+ + \Xi \subset \mathbb{R}^m$.

Let us consider the following sets:

$$\Theta^+ = \left\{ b^+ \in \mathbb{R}^m : \|b^+\|_{\mathbf{r}} \le e^{-j_2}\epsilon^{\frac{m}{d}} \right\} \quad \text{and} \quad \Theta^- = \left\{ b^- \in \mathbb{R}^n : \|b^-\|_{\mathbf{s}} \le e^{j_2}\epsilon^{\frac{n}{d}} \right\}.$$

If $b = (b^+, b^-) \in \Theta^+ \times \Theta^-$, then $\|e^{\mathbf{r}j_2}b^+\|_{\mathbf{r}} \le \epsilon^{\frac{m}{d}}$ and $\|e^{-\mathbf{s}j_2}b^-\|_{\mathbf{s}} \le \epsilon^{\frac{n}{d}}$, where $e^{\mathbf{r}j_2}b^+$ and $e^{-\mathbf{s}j_2}b^-$ denote the vectors such that $a^{j_2}b = (e^{\mathbf{r}j_2}b^+, e^{-\mathbf{s}j_2}b^-)$. It follows that $w(b)g\Gamma \notin a^{-j_2}\mathcal{L}_\epsilon$ since

$$a^{j_2}w(b^+, b^-)g\Gamma = w(e^{\mathbf{r}j_2}b^+, e^{-\mathbf{s}j_2}b^-)a^{j_2}g\Gamma \notin \mathcal{L}_\epsilon$$

by the definition of $\mathcal{L}_\epsilon$.

Now we claim that the set $\Theta^+ \times \left\{b_0^-\right\}$ is contained in the intersection of $(b_0^+ + \Xi) \times \left\{b_0^-\right\}$ and $\Theta^+ \times \Theta^-$. It is enough to show that $\Theta^+ \subset b_0^+ + \Xi$ and $b_0^- \in \Theta^-$. Since $\|b_0^-\|_s \le (dH^{d-1})^{\frac{1}{s_n}}$, the latter assertion follows from the assumption $j_2 \ge \log((dH^{d-1})^{\frac{1}{s_n}}\epsilon^{-\frac{n}{d}})$. To show the former assertion, fix any $b^+ \in \Theta^+$. By the quasi-metric property of $\|\cdot\|_{\mathbf{r}}$ as in (3.1), it follows from the assumptions $j_1 \ge \log((2dH^{d-1})^{\frac{1}{r_m}}\delta^{-1})$ and $j_2 \ge \log((dH^{d-1})^{\frac{1}{s_n}}\epsilon^{-\frac{n}{d}})$ that

$$\begin{aligned}
\|b^+ - b_0^+\|_{\mathbf{r}} &\le 2^{\frac{1-r_m}{r_m}}(\|b^+\|_{\mathbf{r}} + \|b_0^+\|_{\mathbf{r}}) \le 2^{\frac{1-r_m}{r_m}}(e^{-j_2}\epsilon^{\frac{m}{d}} + (dH^{d-1})^{\frac{1}{r_m}}) \\
&\le 2^{\frac{1-r_m}{r_m}}((dH^{d-1})^{-\frac{1}{s_n}}\epsilon + (dH^{d-1})^{\frac{1}{r_m}}) \le 2^{\frac{1-r_m}{r_m}+1}(dH^{d-1})^{\frac{1}{r_m}} \\
&\le e^{j_1}\delta.
\end{aligned}$$

Thus we have $b^+ \in b_0^+ + B^{\mathbb{R}^m,\mathbf{r}}_{e^{j_1}\delta} \subset b_0^+ + \Xi$, which concludes the former assertion.

By the above claim, we obtain

$$\begin{aligned}
1 - \tau_y^{a^{j_1}\mathcal{A}}(a^{-j_2}\mathcal{L}_\epsilon) = \tau_y^{a^{j_1}\mathcal{A}}(Y \setminus a^{-j_2}\mathcal{L}_\epsilon) &\ge \frac{m_{\mathbb{R}^m}(\Theta^+)}{m_{\mathbb{R}^m}(b_0^+ + \Xi)} \\
&\ge \frac{m_{\mathbb{R}^m}(B^{\mathbb{R}^m,\mathbf{r}}_{e^{-j_2}\epsilon^{\frac{m}{d}}})}{m_{\mathbb{R}^m}(B^{\mathbb{R}^m,\mathbf{r}}_{e^{j_1}r})} = \frac{e^{-j_2}\epsilon^{\frac{m}{d}}}{e^{j_1}r}.
\end{aligned}$$

This proves the lemma. □

Proof of Theorem 1.2.2. Suppose that $A \in M_{m,n}(\mathbb{R})$ is not singular on average, and let

$$\eta_A = \sup\{\eta : x_A \text{ has } \eta\text{-escape of mass}\} < 1.$$

By Proposition 3.4.1, there is an a-invariant measure $\overline{\mu} \in \mathscr{P}(\overline{Y})$ such that

$$\operatorname{Supp} \overline{\mu} \subset \mathcal{L}_\epsilon \cup (\overline{Y} \setminus Y),\ \pi_* \overline{\mu} = \mu_A \in \mathscr{P}(\overline{X}),\ \text{and } \overline{\mu}(\overline{Y} \setminus Y) = \mu_A(\overline{X} \setminus X) = \eta_A.$$

This measure can be represented by the linear combination

$$\overline{\mu} = (1 - \eta_A)\mu + \eta_A \delta_\infty,$$

where δ_∞ is the dirac delta measure on $\overline{Y} \setminus Y$ and $\mu \in \mathscr{P}(Y)$ is a-invariant. There is a compact set $K \subset X$ such that $\mu_A(K) > 0.99\mu_A(X)$. We can choose $0 < r < 1$ such that $Y(r) \supset \pi^{-1}(K)$ and $\mu(Y(r)) > 0.99$. Note that the choice of r is independent of ϵ since μ_A is only determined by fixed A.

For a-invariant probability measure μ on Y, let $\mathcal{A}^W$ be a countably generated σ-algebra as in Proposition 3.2.6. With respect to this σ-algebra, we have

$$h_{\overline{\mu}}(a|\overline{\mathcal{A}^W_\infty}) \geq (1 - \eta_A) - r_1(m - \dim_H \mathbf{Bad}_A(\epsilon))$$

by (3) of Proposition 3.4.1. Since the entropy function is linear with respect to the measure, it follows that

$$h_\mu(a|\mathcal{A}^W_\infty) = \frac{1}{1-\eta_A} h_{\overline{\mu}}(a|\overline{\mathcal{A}^W_\infty}) \geq 1 - \frac{r_1}{1-\eta_A}(m - \dim_H \mathbf{Bad}_A(\epsilon)).$$

By Proposition 3.2.8, we obtain

$$H_\mu(\mathcal{A}^W | a\mathcal{A}^W) \geq 1 - \frac{r_1}{1-\eta_A}(m - \dim_H \mathbf{Bad}_A(\epsilon)). \tag{3.22}$$

By Lemma 3.2.5, there exists $0 < \delta < \frac{c}{2}$ such that $\mu(E_\delta) < 0.01$. Note that the constants $C_1, C_2 > 0$ in Lemma 3.2.5 depend only on a and G, hence δ is independent of ϵ even if the set E_δ might depend on ϵ. It follows from Proposition 3.2.6 that $B^W_\delta \cdot y \subset [y]_{\mathcal{A}^W} \subset B^W_r \cdot y$ for any $y \in Y(r) \setminus E_\delta$. We write $Z = Y(r) \setminus E_\delta$ for simplicity. Note that $\mu(Z) \geq \mu(Y(r)) - \mu(E_\delta) > 0.98$.

To apply Lemma 3.4.3, choose $H \geq 1$ such that $Y(r) \subset Y_{\leq H}$. Note that the constant H depends only on r. Set

$$j_1 = \lceil \log((2dH^{d-1})^{\frac{1}{r_m}} \delta'^{-1}) \rceil \quad \text{and} \quad j_2 = \lceil \log((dH^{d-1})^{\frac{1}{s_n}} \epsilon^{-\frac{n}{d}}) \rceil,$$

where $\delta' > 0$ will be determined below.

Let $\mathcal{A} = a^{-k}\mathcal{A}^W$ for $k = \lceil \log(2^{\frac{1}{r_m}} r^{\frac{1}{r_1}} \epsilon^{-\frac{m}{d}}) \rceil + j_2$. By Proposition 3.2.6, for any $y \in Z$, we have $B^W_\delta \cdot y \subset [y]_{\mathcal{A}^W} \subset B^W_r \cdot y$, which implies that

$$B^{W,\mathbf{r}}_{\delta^{\frac{1}{r_m}}} \cdot y \subset [y]_{\mathcal{A}^W} \subset B^{W,\mathbf{r}}_{r^{\frac{1}{r_1}}} \cdot y.$$

Thus, for any $y \in Z$,

$$B^{W,\mathbf{r}}_{\delta^{\frac{1}{r_m}} e^{-k}} \cdot (a^{-k}y) \subset [a^{-k}y]_{a^{-k}\mathcal{A}^W} = [a^{-k}y]_{\mathcal{A}} \subset B^{W,\mathbf{r}}_{r^{\frac{1}{r_1}} e^{-k}} \cdot (a^{-k}y).$$

Finally, it follows that for any $y \in a^k Z$,

$$B^{W,\mathbf{r}}_{\delta'} \cdot y \subset [y]_{\mathcal{A}} \subset B^{W,\mathbf{r}}_{r'} \cdot y,$$

where

$$r' = 2^{-\frac{1}{r_m}} e^{-j_2} \epsilon^{\frac{m}{d}} \quad \text{and} \quad \delta' = e^{-1} r^{-\frac{1}{r_1}} \delta^{\frac{1}{r_m}} r'.$$

Now we will use Corollary 3.2.11 with $L = W$, $K = a^k Z$, and $r = r'$. Note that the maximal relative entropy of a^{j_1} with respect to $\mathcal{A}^W$ is j_1, and μ is supported on $a^{-j_2}\mathcal{L}_\epsilon$ since $\operatorname{Supp}\mu \subseteq \mathcal{L}_\epsilon$ and μ is a-invariant. We also have

$$B^{W,\mathbf{r}}_{r'} a^{-j_2}\mathcal{L}_\epsilon = a^{-j_2} B^{W,\mathbf{r}}_{e^{j_2} r'} \mathcal{L}_\epsilon = a^{-j_2} B^{W,\mathbf{r}}_{2^{-\frac{1}{r_m}} \epsilon^{\frac{m}{d}}} \mathcal{L}_\epsilon \subseteq a^{-j_2} \mathcal{L}_{2^{-\frac{d}{m r_m}} \epsilon}$$

by using the triangular inequality of $\mathbf{r}$-quasinorm as in (3.1) and the definition of $\mathcal{L}_\epsilon$ for the last inclusion. Applying Corollary 3.2.11, it follows that

(3.23)
$$\begin{aligned} H_\mu(\mathcal{A}^W | a^{j_1}\mathcal{A}^W) &\le j_1 + \int_Y \log \tau_y^{a^{j_1}\mathcal{A}^W}((Y \setminus a^k Z) \cup B^{W,\mathbf{r}}_{r'} a^{-j_2}\mathcal{L}_\epsilon) d\mu(y) \\ &\le j_1 + \int_Y \log \tau_y^{a^{j_1}\mathcal{A}^W}((Y \setminus a^k Z) \cup a^{-j_2}\mathcal{L}_{2^{-\frac{d}{m r_m}} \epsilon}) d\mu(y) \\ &\le j_1 + \int_{a^k Z \cap Y_{\le H}} \log \tau_y^{a^{j_1}\mathcal{A}^W}(a^{-j_2}\mathcal{L}_{2^{-\frac{d}{m r_m}} \epsilon}) d\mu(y) \end{aligned}$$

By Lemma 3.4.3 with $\delta = \delta'$ and $r = r'$, for any $y \in a^k Z \cap Y_{\le H}$,

$$\tau_y^{a^{j_1}\mathcal{A}^W}(a^{-j_2}\mathcal{L}_{2^{-\frac{d}{m r_m}} \epsilon}) \le 1 - 2^{-\frac{1}{r_m}} e^{-j_1 - j_2} r'^{-1} \epsilon^{\frac{m}{d}} = 1 - e^{-j_1},$$

hence $-\log \tau_y^{a^{j_1}\mathcal{A}^W}(a^{-j_2}\mathcal{L}_{2^{-\frac{d}{m r_m}} \epsilon}) \ge e^{-j_1}$. Since $\mu(a^k Z \cap Y_{\le H}) \ge \frac{1}{2}$, it follows from (3.23) that

$$
\begin{aligned}
1 - H_\mu(\mathcal{A}^W|a\mathcal{A}^W) &= 1 - \frac{1}{j_1}H_\mu(\mathcal{A}^W|a^{j_1}\mathcal{A}^W) = 1 - \frac{1}{j_1}H_\mu(\mathcal{A}|a^{j_1}\mathcal{A}) \\
&\geq -\frac{1}{j_1}\int_{a^kZ\cap Y_{\leq H}} \log \tau_y^{a^{j_1}\mathcal{A}}(a^{-j_2}\mathcal{L}_{2^{-\frac{d}{mr_m}}\epsilon})d\mu(y) \\
&\geq \frac{e^{-j_1}}{2j_1}.
\end{aligned} \tag{3.24}
$$

Recall that j_1 is chosen by

$$
\begin{aligned}
j_1 &= \lceil \log((2dH^{d-1})^{\frac{1}{r_m}}e(2r)^{\frac{1}{r_1}}\delta^{-\frac{1}{r_m}}2^{\frac{1}{r_m}}e^{j_2}\epsilon^{-\frac{m}{d}})\rceil \\
&\leq \lceil \log((2dH^{d-1})^{\frac{1}{r_m}+\frac{1}{s_n}}e^2(2r)^{\frac{1}{r_1}}\delta^{-\frac{1}{r_m}}2^{\frac{1}{r_m}}\epsilon^{-\frac{n}{d}}\epsilon^{-\frac{m}{d}})\rceil \\
&\leq \log((2dH^{d-1})^{\frac{1}{r_m}+\frac{1}{s_n}}e^3(2r)^{\frac{1}{r_1}}\delta^{-\frac{1}{r_m}}2^{\frac{1}{r_m}}) - \log\epsilon
\end{aligned}
$$

Here, the constants H, r, and δ are only depending on fixed $A \in M_{m,n}(\mathbb{R})$, not on ϵ. Combining (3.22) and (3.24), we obtain

$$
m - \dim_H \mathbf{Bad}_A(\epsilon) \geq c(A)\frac{\epsilon}{\log(1/\epsilon)},
$$

where the constant $c(A) > 0$ only depends on d, $\mathbf{r}$, $\mathbf{s}$, and $A \in M_{m,n}(\mathbb{R})$ since η_A is also only depending on A. It completes the proof. □

3.5 Upper bound for Hausdorff dimension of $\mathbf{Bad}^b(\epsilon)$

In this section, as explained in the introduction, we only consider the unweighted setting, that is,

$$
\mathbf{r} = (1/m, \ldots, 1/m) \quad \text{and} \quad \mathbf{s} = (1/n, \ldots, 1/n).
$$

3.5.1 Constructing measure with entropy lower bound

Similar to Subsection 3.4.1, we will construct an a-invariant measure on Y with a lower bound on the conditional entropy to the σ-algebra $\mathcal{A}_\infty^U$ obtained in (4.48) and Proposition 3.2.6 with $L = U$. To control the amount of escape of mass for the desired measure, we need a modification of [KKLM17, Theorem 1.1] as Proposition 3.5.2 below.

For any compact set $\mathfrak{S} \subset X$ and positive integer $k > 0$, and any $0 < \eta < 1$, let

$$F_{\eta,\mathfrak{S}} = \left\{ A \in \mathbb{T}^{mn} \subset M_{m,n}(\mathbb{R}) : \frac{1}{k}\sum_{i=0}^{k-1} \delta_{a^i x_A}(X \setminus \mathfrak{S}) < \eta \text{ for infinitely many } k \right\},$$

$$F^k_{\eta,\mathfrak{S}} = \left\{ A \in \mathbb{T}^{mn} \subset M_{m,n}(\mathbb{R}) : \frac{1}{k}\sum_{i=0}^{k-1} \delta_{a^i x_A}(X \setminus \mathfrak{S}) < \eta \right\}.$$

Given a compact set $\mathfrak{S}$ of X, $k \in \mathbb{N}, \eta \in (0,1)$, and $t \in \mathbb{N}$, define the set

$$Z(\mathfrak{S}, k, t, \eta) := \left\{ A \in \mathbb{T}^{mn} : \frac{1}{k}\sum_{i=0}^{k-1} \delta_{a^{ti} x_A}(X \setminus \mathfrak{S}) \geq \eta \right\};$$

in other words, the set of $A \in \mathbb{T}^{mn}$ such that up to time k, the proportion of times i for which the orbit point $a^{ti}x_A$ is in the complement of $\mathfrak{S}$ is at least η. The following theorem is one of the main results in [KKLM17].

Theorem 3.5.1. *[KKLM17, Theorem 1.5] There exists $t_0 > 0$ and $C > 0$ such that the following holds. For any $t > t_0$ there exists a compact set $\mathfrak{S} := \mathfrak{S}(t)$ of X such that for any $k \in \mathbb{N}$ and $\eta \in (0,1)$, the set $Z(\mathfrak{S}, k, t, \eta)$ can be covered with $Ct^{3k}e^{(m+n-\eta)mntk}$ balls in $\mathbb{T}^{mn}$ of radius $e^{-(m+n)tk}$.*

The following proposition is a slightly stronger variant of [KKLM17, Theorem 1.1] which will be needed later. We prove this using Theorem 3.5.1.

Proposition 3.5.2. *There exists a familiy of compact sets $\{\mathfrak{S}_\eta\}_{0<\eta<1}$ of X such that the following is true. For any $0 < \eta \leq 1$,*

$$\dim_H(\mathbb{T}^{mn} \setminus \limsup_{k\to\infty} \bigcap_{\eta' \geq \eta} F^k_{\eta',\mathfrak{S}_{\eta'}}) \leq mn - \frac{\eta mn}{2(m+n)}. \tag{3.25}$$

Proof. For $\eta \in (0,1)$, let $t_\eta \geq 3$ be the smallest integer such that $\frac{3\log t_\eta}{t_\eta} < \frac{\eta mn}{4}$, and $\mathfrak{S}'_\eta$ be the set $\mathfrak{S}(t_\eta)$ of Theorem 3.5.1. For $l \geq 3$, denote by $\eta_l > 0$ the smallest real number such that $t_{\eta_l} = l$. Then $\eta_l \geq \frac{2\eta_{l-1}}{3}$ for any $l \geq 4$. We note that these $\mathfrak{S}'_\eta$ can be chosen to satisfy $\mathfrak{S}'_{\eta'} \subseteq \mathfrak{S}'_\eta$ for any $0 < \eta \leq \eta'$. Hence, we can find a family of compact sets $\mathfrak{S}''_\eta$ such that $\mathfrak{S}'_{\eta_l} \subseteq \mathfrak{S}''_{\eta'}$ for any $l \geq 4$ and $\eta_l \leq \eta' < \eta_{l-1}$. For any $\eta \in (0,1)$, we can choose $\mathfrak{S}_\eta$ to be a compact set so that for any $-t_\eta \leq t \leq t_\eta$ and $x \in \mathfrak{S}''_\eta$, $a^t x \in \mathfrak{S}_\eta$.

Now we will prove that this family of compact sets $\{\mathfrak{S}_\eta\}_{0<\eta<1}$ satisfies (3.25). Since $\frac{1}{k}\sum_{i=0}^{k-1}\delta_{a^i x_A}(X\setminus\mathfrak{S}_\eta)\geq\eta$ implies

$$\frac{1}{\lceil\frac{k}{t_\eta}\rceil}\sum_{i=0}^{\lceil\frac{k}{t_\eta}\rceil-1}\delta_{a^{t_\eta i}x_A}(X\setminus\mathfrak{S}''_\eta)\geq\eta,$$

$\mathbb{T}^{mn}\setminus F^k_{\eta,\mathfrak{S}_\eta}\subseteq Z(\mathfrak{S}''_\eta,\lceil\frac{k}{t_\eta}\rceil,t_\eta,\eta)$ for any $0<\eta<1$ and $k\in\mathbb{N}$.

For any $\eta_{l+1}<\eta'\leq\eta_l$, we have $t_{\eta'}=l$ and the set $Z(\mathfrak{S}''_{\eta'},\lceil\frac{k}{t_\eta}\rceil,t_{\eta'},\eta')$ is contained in $Z(\mathfrak{S}'_{\eta_l},\lceil\frac{k}{t_{\eta_l}}\rceil,l,\eta_l)$. It follows that for any $0<\eta<1$

$$\mathbb{T}^{mn}\setminus\bigcup_{\eta'\geq\eta}F^k_{\eta',\mathfrak{S}^k_{\eta'}}\subseteq\bigcup_{\eta'\geq\eta}Z(\mathfrak{S}''_{\eta'},\lceil\frac{k}{t_{\eta'}}\rceil,t_{\eta'},\eta')\subseteq\bigcup_{l=3}^{t_\eta}Z(\mathfrak{S}'_{\eta_l},\lceil\frac{k}{l}\rceil,l,\eta_l),$$

hence

$$\mathbb{T}^{mn}\setminus\limsup_{k\to\infty}\bigcap_{\eta'\geq\eta}F^k_{\eta',\mathfrak{S}_{\eta'}}\subseteq\bigcup_{k_0\geq 1}\bigcap_{k=k_0}^{\infty}\bigcup_{l=3}^{t_\eta}Z(\mathfrak{S}'_{\eta_l},\lceil\frac{k}{l}\rceil,l,\eta_l).$$

By Theorem 3.5.1, the set $\bigcup_{l=3}^{t_\eta}Z(\mathfrak{S}'_{\eta_l},\lceil\frac{k}{l}\rceil,l,\eta_l)$ can be covered with

$$\begin{aligned}\sum_{l=3}^{t_\eta}Cl^{3\lceil\frac{k}{l}\rceil}e^{(m+n-\eta_l)mn\lceil\frac{k}{l}\rceil l}&\leq\sum_{l=3}^{t_\eta}Ct_\eta^3e^{\frac{3\log l}{l}k}e^{(m+n-\eta_l)mn(k+t_\eta)}\\&\leq\sum_{l=3}^{t_\eta}Ct_\eta^3e^{(m+n)mnt_\eta}e^{(m+n-\frac{3\eta_l}{4})mnk}\\&\leq Ct_\eta^4e^{(m+n)mnt_\eta}e^{(m+n-\frac{\eta}{2})mnk}\end{aligned}$$

balls in $\mathbb{T}^{mn}$ of radius $e^{-(m+n)k}$. Here we used $\eta_{t_\eta}\geq\frac{2\eta}{3}$ which follows from

$\eta_l \geq \frac{2\eta_{l-1}}{3}$ for any $l \geq 4$. Thus, for any sufficiently large $k_0 \in \mathbb{N}$

$$\dim_H\left(\bigcap_{k=k_0}^{\infty}\bigcup_{l=3}^{t_\eta} Z(\mathfrak{S}'_{\eta_l},\lceil\frac{k}{l}\rceil,l,\eta_l)\right) \leq \limsup_{k\to\infty}\frac{\log(Ct_\eta^4 e^{(m+n)mnt_\eta}e^{(m+n-\frac{\eta}{2})mnk})}{-\log(e^{-(m+n)k})}$$
$$= \limsup_{k\to\infty}\frac{\log(Ct_\eta^4 e^{(m+n)mnt_\eta}) + (m+n-\frac{\eta}{2})mnk}{(m+n)k} = mn - \frac{\eta mn}{2(m+n)},$$

hence we get $\dim_H(\mathbb{T}^{mn} \setminus \limsup_{k\to\infty}\bigcap_{\eta'\geq\eta} F^k_{\eta',\mathfrak{S}_{\eta'}}) \leq mn - \frac{\eta mn}{2(m+n)}$. □

The construction will basically follow the construction in Proposition 3.4.1. However, the additional step using Theorem 3.5.2 is necessary to control the escape of mass since we will allow a small amount of escape of mass.

Proposition 3.5.3. *Let $\{\mathfrak{S}_\eta\}_{0<\eta<1}$ be the familiy of compact sets of X as in Proposition 3.5.2. For b fixed and $\epsilon > 0$, assume that $\dim_H \mathbf{Bad}^b(\epsilon) > \dim_H \mathbf{Bad}^0(\epsilon)$. Let $\eta_0 := 2(m+n)(1 - \frac{\dim_H \mathbf{Bad}^b(\epsilon)}{mn})$. Then there exist an a-invariant measure $\overline{\mu} \in \mathscr{P}(\overline{Y})$ such that*

1. $\operatorname{Supp}\overline{\mu} \subseteq \mathcal{L}_\epsilon \cup (\overline{Y} \setminus Y)$,
2. *$\pi_*\overline{\mu}(\overline{X} \setminus \mathfrak{S}_{\eta'}) \leq \eta'$ for any $\eta_0 \leq \eta' < 1$, in particular, there exist $\mu \in \mathscr{P}(Y)$ and $0 \leq \widehat{\eta} \leq \eta_0$ such that*
$$\overline{\mu} = (1-\widehat{\eta})\mu + \widehat{\eta}\delta_\infty,$$
where δ_∞ is the dirac delta measure on $\overline{Y}\setminus Y$.
3. *Let $\mathcal{A}^U$ be as in Proposition 3.2.6 for μ and $L = U$, and let $\mathcal{A}^U_\infty$ be as in* (4.48). *Then we have*
$$h_{\overline{\mu}}(a|\overline{\mathcal{A}^U_\infty}) \geq (1-\widehat{\eta}^{\frac{1}{2}})(d - \frac{1}{2}\eta_0 - d\widehat{\eta}^{\frac{1}{2}}).$$

Proof. For $\epsilon > 0$, denote by R the set $\mathbf{Bad}^b(\epsilon) \setminus \mathbf{Bad}^b_0(\epsilon)$, and let

$$R^T := \{A \in R \cap \mathbb{T}^{mn} \subset M_{m,n}(\mathbb{R}) | \forall t \geq T, a_t x_{A,b} \in \mathcal{L}_\epsilon\}.$$

The sequence $\{R^T\}_{T\geq 1}$ is increasing, and $R = \bigcup_{T=1}^{\infty} R^T$ by Proposition 3.3.2. Since $\dim_H \mathbf{Bad}^b(\epsilon) > \dim_H \mathbf{Bad}^0(\epsilon) \geq \dim_H \mathbf{Bad}^b_0(\epsilon)$, it follows that $\dim_H R =$

$\dim_H \mathbf{Bad}^b(\epsilon)$. Thus for any $0 < \gamma < \frac{mn}{2(m+n)} - (mn - \dim_H \mathbf{Bad}^b(\epsilon))$, there exists $T_\gamma \geq 1$ satisfying

$$\dim_H R^{T_\gamma} > \dim_H \mathbf{Bad}^b(\epsilon) - \gamma. \tag{3.26}$$

Let $\eta = 2(m+n)(1 - \frac{\dim_H \mathbf{Bad}^b(\epsilon)-\gamma}{mn})$. Note that $0 < \eta < 1$ in the above range of γ. For $k \in \mathbb{N}$, write $\widetilde{F}_\eta^k := \bigcap_{\eta' \geq \eta} F^k_{\eta', \mathfrak{S}_{\eta'}}$ for simplicity. Recall that we have

$$\dim_H(\mathbb{T}^{mn} \setminus \limsup_{k\to\infty} \widetilde{F}_\eta^k) \leq mn - \frac{\eta mn}{2(m+n)} = \dim_H \mathbf{Bad}^b(\epsilon) - \gamma \tag{3.27}$$

by Theorem 3.5.2. It follows from (3.26) and (3.27) that

$$\dim_H(R^{T_\gamma} \cap \limsup_{k\to\infty} \widetilde{F}_\eta^k) > \dim_H \mathbf{Bad}^b(\epsilon) - \gamma.$$

Since $R^{T_\gamma} \cap \limsup_{k\to\infty} \widetilde{F}_\eta^k = \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} (R^{T_\gamma} \cap \widetilde{F}_\eta^k)$, we can find an increasing sequence of positive integers $\{k_i\} \to \infty$ such that

$$\dim_H(R^{T_\gamma} \cap \widetilde{F}_\eta^{k_i}) > \dim_H \mathbf{Bad}^b(\epsilon) - \gamma.$$

For each $k_i \geq T_\gamma$ let S_i be a maximal e^{-k_i}-separated subset of $R^{T_\gamma} \cap \widetilde{F}_\eta^{k_i}$ with respect to the quasi-distance $d_{\mathbf{r}\otimes\mathbf{s}}$. Then by Lemma 3.3.1,

$$\begin{aligned} \liminf_{i\to\infty} \frac{\log |S_i|}{k_i} &\geq \underline{\dim}_{\mathbf{r}\otimes\mathbf{s}}(R^{T_\gamma} \cap \widetilde{F}_\eta^{k_i}) \\ &> m + n - (r_1 + s_1)(mn - \dim_H \mathbf{Bad}^b(\epsilon) + \gamma) \\ &= m + n - \frac{m+n}{mn}(mn - \dim_H \mathbf{Bad}^b(\epsilon) + \gamma) \\ &= \frac{m+n}{mn}(\dim_H \mathbf{Bad}^b(\epsilon) - \gamma). \end{aligned} \tag{3.28}$$

Let $\nu_i = \frac{1}{|S_i|} \sum_{y \in D_i} \delta_y = \frac{1}{|S_i|} \sum_{A \in S_i} \delta_{y_{A,b}}$ be the normalized counting measure on the set $D_i := \{y_{A,b} : A \in S_i\} \subset Y$ and let

$$\mu_i = \frac{1}{k_i} \sum_{k=0}^{k_i - 1} a_*^k \nu_i \xrightarrow{\text{w}^*} \mu^\gamma \in \mathscr{P}(\overline{Y})$$

By extracting a subsequence if necessary, there exists a probability measure μ^γ which is a weak*-accumulation point of $\{\mu_i\}$. The measure μ^γ is clearly an a-invariant measure since $a_*\mu_i - \mu_i$ goes to zero measure.

Choose any sequence of positive real numbers $(\gamma_j)_{j\geq 1}$ converging to zero and $(\eta_j)_{j\geq 1}$ be the corresponding sequence such that

$$\eta_j = 2(m+n)\left(1 - \frac{\dim_H \mathbf{Bad}^b(\epsilon) - \gamma_j}{mn}\right).$$

Let $\{\mu^{\gamma_j}\}$ be a family of a-invariant probability measures on $\overline{Y}$ obtained from the above construction for each γ_j. Extracting a subsequence again if necessary, we may take a weak*-limit measure $\overline{\mu} \in \mathscr{P}(\overline{Y})$ of $\{\mu^{\gamma_j}\}$. We prove that $\overline{\mu}$ is the desired measure. The measure $\overline{\mu}$ is clearly a-invariant.
(1) We show that for any γ, $\mu^\gamma(Y \setminus \mathcal{L}_\epsilon) = 0$. For any $A \in S_i \subseteq R^{T_\gamma}$, $a^T y_{A,b} \in \mathcal{L}_\epsilon$ holds for $T > T_\gamma$. Thus

$$\mu_i(Y \setminus \mathcal{L}_\epsilon) = \frac{1}{k_i}\sum_{k=0}^{k_i-1}(a^k)_*\nu_i(Y \setminus \mathcal{L}_\epsilon) = \frac{1}{k_i}\sum_{k=0}^{T_\gamma}(a^k)_*\nu_i(Y \setminus \mathcal{L}_\epsilon) \leq \frac{T_\gamma}{k_i}.$$

By taking limit for $k_i \to \infty$, we have $\mu^\gamma(Y \setminus \mathcal{L}_\epsilon) = 0$ for arbitrary γ, hence,

$$\overline{\mu}(Y \setminus \mathcal{L}_\epsilon) = \lim_{j\to\infty} \mu^{\gamma_j}(Y \setminus \mathcal{L}_\epsilon) = 0.$$

(2) For any γ, if $A \in S_i \subset \widetilde{F}^{k_i}_\eta = \bigcap_{\eta' \geq \eta} F^{k_i}_{\eta', \mathfrak{S}_{\eta'}}$, then for all $i \in \mathbb{N}$ and $\eta \leq \eta' \leq 1$, $\frac{1}{k_i}\sum_{k=0}^{k_i-1} \delta_{a^k x_A}(X \setminus \mathfrak{S}_{\eta'}) < \eta'$. Therefore for all $i \in \mathbb{N}$ and $\eta \leq \eta' \leq 1$,

$$\begin{aligned}\pi_*\mu_i(X \setminus \mathfrak{S}_{\eta'}) &= \frac{1}{|S_i|}\sum_{y\in D_i}\frac{1}{k_i}\sum_{k=0}^{k_i-1}\pi_*\delta_{a^k y}(X \setminus \mathfrak{S}_{\eta'}) \\ &= \frac{1}{|S_i|}\sum_{A\in S_i}\frac{1}{k_i}\sum_{k=0}^{k_i-1}\delta_{a^k x_A}(X \setminus \mathfrak{S}_{\eta'}) < \eta',\end{aligned}$$

hence $\pi_*\mu^\gamma(\overline{X} \setminus \mathfrak{S}_{\eta'}) = \lim_{i\to\infty} \pi_*\mu_i(X \setminus \mathfrak{S}_{\eta'}) \leq \eta'$. Since η_j converges to η_0, we have

$$\pi_*\overline{\mu}(\overline{X} \setminus \mathfrak{S}_{\eta'}) \leq \eta'$$

for any $\eta' > \eta_0$. Hence,

$$\overline{\mu}(\overline{Y} \setminus Y) \leq \lim_{\eta' \to \eta_0} \pi_* \overline{\mu}(\overline{X} \setminus \mathfrak{S}_{\eta'}) \leq \eta_0,$$

so we have a decomposition $\overline{\mu} = (1 - \widehat{\eta})\mu + \widehat{\eta}\delta_\infty$ for some $\mu \in \mathscr{P}(Y)$ and $0 \leq \widehat{\eta} \leq \eta_0$.

For the rest of the proof, let us check the condition (3).

(3) Suppose that $\mathcal{Q}$ is any finite partition of Y satisfying:

- $\mathcal{Q}$ contains an atom Q_∞ of the form $\pi^{-1}(Q^0_\infty)$, where $X \smallsetminus Q^0_\infty$ has compact clousre,
- $\forall Q \in \mathcal{Q} \smallsetminus \{Q_\infty\}$, $\operatorname{diam} Q < r$, with $r \in (0, \frac{1}{2})$ such that any $d_{\mathbf{r}\otimes\mathbf{s}}$-ball of radius $3r$ has Euclidean diameter smaller than the injectivity radius on $Y \setminus Q_\infty$,
- $\forall Q \in \mathcal{Q}, \forall j \geq 1, \mu^{\gamma_j}(\partial Q) = 0$,

We will first prove the following statement. For all $q \geq 1$,

(3.29)
$$\frac{1}{q} H_{\overline{\mu}}(\overline{\mathcal{Q}}^{(q)} | \overline{\mathcal{A}^U_\infty}) \geq (m+n)(1 - \overline{\mu}(\overline{Q_\infty})^{\frac{1}{2}}) \left(\frac{\dim_H \mathbf{Bad}^b(\epsilon)}{mn} - \overline{\mu}(\overline{Q_\infty})^{\frac{1}{2}} \right).$$

It is clear if $\overline{\mu}(Q_\infty) = 1$, so assume that $\overline{\mu}(Q_\infty) < 1$, hence for all large enough $j \geq 1$, $\mu^{\gamma_j}(Q_\infty) < 1$. Now we fix such $j \geq 1$ and write temporarily $\gamma = \gamma_j$.

Let $\rho > 0$ be small enough so that $\beta := \mu^\gamma(Q_\infty) + \rho < 1$. Then

$$\beta = \mu(Q_\infty) + \rho > \mu_i(Q_\infty) = \frac{1}{k_i |S_i|} \sum_{y \in D_i, 0 \leq k < k_i} \delta_{a^k y}(Q_\infty)$$

holds for large enough i. In other words, there exist at most $\beta k_i |S_i|$ number of $a^k y$'s in Q_∞ with $y \in D_i$ and $0 \leq k < k_i$.

Let $S'_i \subset S_i$ be the set of $A \in S_i$'s such that

$$|\{0 \leq k \leq k_i - 1 : a^k y_{A,b} \in Q_\infty\}| \leq \beta^{\frac{1}{2}} k_i. \tag{3.30}$$

Then we have $|S_i \setminus S'_i| \leq \beta^{\frac{1}{2}} |S_i|$ by the pigeonhole principle, hence

$$|S'_i| \geq (1 - \beta^{\frac{1}{2}}) |S_i|. \tag{3.31}$$

Let $\nu_i' = \frac{1}{|S_i'|}\sum_{y\in S_i'}\delta_y$ be the normalized counting measure on D_i', where $D_i' := \{y_{A,b} : A \in S_i'\} \subset Y$, then $\nu_i(Q) \geq \frac{|S_i'|}{|S_i|}\nu_i'(Q)$ for all measurable set $Q \subseteq Y$. Thus, for any arbitrary countable partition $\mathcal{Q}$ fo Y,

$$
\begin{aligned}
H_{\nu_i}(\mathcal{Q}) &= -\sum_{\nu_i(Q)\leq\frac{1}{e}} \log(\nu_i(Q))\nu_i(Q) - \sum_{\nu_i(Q)>\frac{1}{e}} \log(\nu_i(Q))\nu_i(Q) \\
&\geq -\sum_{\nu_i(Q)\leq\frac{1}{e}} \log(\frac{|S_i'|}{|S_i|}\nu_i'(Q))\frac{|S_i'|}{|S_i|}\nu_i'(Q) \\
&= -\frac{|S_i'|}{|S_i|}\sum_{\nu_i(Q)\leq\frac{1}{e}} \log(\nu_i'(Q))\nu_i'(Q) - \frac{|S_i'|}{|S_i|}\log\frac{|S_i'|}{|S_i|}\sum_{\nu_i(Q)\leq\frac{1}{e}}\nu_i'(Q) \qquad (3.32)\\
&\geq \frac{|S_i'|}{|S_i|}\Big\{H_{\nu_i'}(\mathcal{Q}) + \sum_{\nu_i(Q)>\frac{1}{e}} \log(\nu_i'(Q))\nu_i'(Q)\Big\} \\
&\geq (1-\beta^{\frac{1}{2}})(H_{\nu_i'}(\mathcal{Q}) - \frac{2}{e}).
\end{aligned}
$$

In the last inequality, we use the fact that ν_i' is a probability measure, thus there can be at most two elements A of the partition for which $\nu_i'(A) > \frac{1}{e}$.

From Lemma 3.3.3 with $L = U$ and (3.30), if Q is any non-empty atom of $\mathcal{Q}^{(k_i)}$, fixing any $y \in Q$, for any $D > m+n$,

$$D_i' \cap Q = D_i' \cap [y]_{\mathcal{Q}^{(k_i)}} \subset E_{y,k_i-1}$$

can be covered $Ce^{D\sqrt{\beta}k_i}$ many $r^{\frac{1}{r_1+s_1}}e^{-k_i}$-balls for $d_{\mathbf{r}\otimes\mathbf{s}}$, where C is a constant depending on Q_∞^0, r, and D, but not on k_i. Since D_i' is e^{-k_i}-separated with respect to $d_{\mathbf{r}\otimes\mathbf{s}}$ and $r^{\frac{1}{r_1+s_1}} < \frac{1}{2}$, we get

$$\mathrm{Card}(D_i' \cap Q) \leq Ce^{D\sqrt{\beta}k_i},$$

hence we have

$$H_{\nu_i'}(\mathcal{Q}^{(k_i)}) \geq \log|S_i'| - D\beta^{\frac{1}{2}}k_i - \log C. \qquad (3.33)$$

Now let $\mathcal{A}^U = (\mathcal{P}^U)_0^\infty = \bigvee_{i=0}^\infty a^i\mathcal{P}^U$ be as in Proposition 3.2.6 for μ and

$L = U$, and let $\mathcal{A}^U_\infty$ be as in (4.48). Using the continuity of entropy, we have

$$H_{\nu_i}(\mathcal{Q}^{(k_i)}|\mathcal{A}^U_\infty) = \lim_{\ell\to\infty} H_{\nu_i}(\mathcal{Q}^{(k_i)}|(\mathcal{P}^U)^\infty_\ell). \tag{3.34}$$

Since the support of ν_i is a set of finite points on a single compact U-orbit, $H_{\nu_i}(\mathcal{Q}^{(k_i)}|(\mathcal{P}^U)^\infty_\ell) = H_{\nu_i}(\mathcal{Q}^{(k_i)})$ for all large enough $\ell \geq 1$. Combining (3.31), (3.32), (3.33), and (3.34), we have

$$\begin{aligned} H_{\nu_i}(\mathcal{Q}^{(k_i)}|\mathcal{A}^U_\infty) &= \lim_{\ell\to\infty} H_{\nu_i}(\mathcal{Q}^{(k_i)}|(\mathcal{P}^U)^\infty_\ell) = H_{\nu_i}(\mathcal{Q}^{(k_i)}) \\ &\geq (1-\beta^{\frac{1}{2}})(H_{\nu_i'}(\mathcal{Q}^{(k_i)}) - \frac{2}{e}) \\ &\geq (1-\beta^{\frac{1}{2}})(\log|S_i| - D\beta^{\frac{1}{2}}k_i - \log C - \frac{2}{e} + \log(1-\beta^{\frac{1}{2}})). \end{aligned} \tag{3.35}$$

By the same argument in the proof of Proposition 3.4.1, it follows from (3.35) that

$$\begin{aligned} \frac{1}{q}H_{\mu_i}(\mathcal{Q}^{(q)}|\mathcal{A}^U_\infty) &\geq \frac{1}{k_i}H_{\nu_i}(\mathcal{Q}^{(k_i)}|\mathcal{A}^U_\infty) - \frac{2q\log|\mathcal{Q}|}{k_i} \\ &\geq \frac{1}{k_i}\Big((1-\beta^{\frac{1}{2}})(\log|S_i| - D\beta^{\frac{1}{2}}k_i - \log C - \frac{2}{e} + \log(1-\beta^{\frac{1}{2}})) - 2q\log|\mathcal{Q}|\Big). \end{aligned}$$

Now we can take $i \to \infty$ because the atoms Q of $\mathcal{Q}$ and hence of $\mathcal{Q}^{(q)}$, satisfy $\mu^\gamma(\partial Q) = 0$. Also, the constants C, β, and $|\mathcal{Q}|$ are independent to k_i. Thus it follows from the inequality (3.28) that

$$\frac{1}{q}H_{\mu^\gamma}(\overline{\mathcal{Q}}^{(q)}|\overline{\mathcal{A}^U_\infty}) \geq (1-\beta^{\frac{1}{2}})\left(\frac{m+n}{mn}(\dim_H \mathbf{Bad}^b(\epsilon) - \gamma) - D\beta^{\frac{1}{2}}\right),$$

and by taking $\rho \to 0$ and $D \to m+n$ we have

$$\frac{1}{q}H_{\mu^\gamma}(\overline{\mathcal{Q}}^{(q)}|\overline{\mathcal{A}^U_\infty}) \geq (m+n)(1-\mu^\gamma(\overline{Q_\infty})^{\frac{1}{2}})\left(\frac{\dim_H \mathbf{Bad}^b(\epsilon) - \gamma}{mn} - \mu^\gamma(\overline{Q_\infty})^{\frac{1}{2}}\right).$$

Recall that $\gamma = \gamma_j$, and by taking $j \to \infty$ so that $\gamma_j \to 0$, we finally have (3.29), i.e.,

$$\frac{1}{q}H_{\overline{\mu}}(\overline{\mathcal{Q}}^{(q)}|\overline{\mathcal{A}^U_\infty}) \geq (m+n)(1-\overline{\mu}(\overline{Q_\infty})^{\frac{1}{2}})\left(\frac{\dim_H \mathbf{Bad}^b(\epsilon)}{mn} - \overline{\mu}(\overline{Q_\infty})^{\frac{1}{2}}\right).$$

As we did in the proof of Proposition 3.4.1, we take a finite partition $\mathcal{Q}$ of Y satisfying the three bullet-conditions above, and also take $Q^0_\infty \subset X$ sufficiently small so that $\overline{\mu}(Q^0_\infty)$ is sufficiently close to $\widehat{\eta}$. It follows that

$$\begin{aligned} h_{\overline{\mu}}(a|\overline{\mathcal{A}^U_\infty}) &\geq (m+n)(1-\widehat{\eta}^{\frac{1}{2}})(\frac{1}{mn}\dim_H \mathbf{Bad}^b(\epsilon) - \widehat{\eta}^{\frac{1}{2}}) \\ &= (1-\widehat{\eta}^{\frac{1}{2}})(d - \frac{1}{2}\eta_0 - d\widehat{\eta}^{\frac{1}{2}}). \end{aligned}$$

□

3.5.2 Effective equidistribution and the proof of Theorem 1.2.1

In this subsection, we recall some effective equidistribution results which are necessary for the proof of Theorem 1.2.1. Let $\mathfrak{g} = Lie\, G(\mathbb{R})$ and choose an orthonormal basis for $\mathfrak{g}$. Define the (left) differentiation action of $\mathfrak{g}$ on $C_c^\infty(X)$ by $Zf(x) = \frac{d}{dt}f(\exp(tZ)x)|_{t=0}$ for $f \in C_c^\infty(X)$ and Z in the orthonormal basis. This also defines for any $l \in \mathbb{N}$, L^2-Sobolev norms $\mathcal{S}_l$ on $C_c^\infty(Y)$:

$$\mathcal{S}_l(f)^2 = \sum_{\mathcal{D}} \|\mathrm{ht} \circ \pi^l \mathcal{D}(f)\|^2_{L^2}, \tag{3.36}$$

where $\mathcal{D}$ ranges over all the monomials in the chosen basis of degree $\leq l$ and ht $\circ\, \pi$ is the function assigning 1 over the smallest length of a vector in the corresponding lattice of the given grid. Let us define the function $\zeta : (\mathbb{T}^d \setminus \mathbb{Q}^d) \times \mathbb{R}^+ \to \mathbb{N}$ by

$$\zeta(b,T) := \min\left\{ N \in \mathbb{N} : \min_{1\leq q\leq N} \|qb\|_{\mathbb{Z}} \leq \frac{T^2}{N} \right\}.$$

Then there exists a sufficiently large $l \in \mathbb{N}$ such that the following equidistribution theorems hold.

Theorem 3.5.4. *[Kim, Theorem 1.3] Let K be a bounded subset in $SL_d(\mathbb{R})$ and $V \subset U$ be a fixed neighborhood of the identity in U with smooth boundary and compact closure. Then, for any $t \geq 0$, $f \in C_c^\infty(Y)$, and $y = gw(b)\Gamma$ with $g \in K$ and $b \in \mathbb{T}^d \setminus \mathbb{Q}^d$, there exists a constant $\alpha_1 > 0$ only depending on d and V so that*

$$\frac{1}{m_U(V)} \int_V f(a_t uy) dm_U(u) = \int_Y f dm_Y + O(\mathcal{S}_l(f)\zeta(b, e^{\frac{t}{2m}})^{-\alpha_1}). \tag{3.37}$$

The implied constant in (3.37) *only depends on d, V, and K.*

For $q \in \mathbb{N}$, define

$$X_q := \left\{ gw(\mathbf{p}/q)\Gamma \in Y : g \in SL_d(\mathbb{R}), \mathbf{p} \in \mathbb{Z}^d, \gcd(\mathbf{p}, q) = 1 \right\},$$
$$\Gamma_q := \{\gamma \in SL_d(\mathbb{Z}) : \gamma e_1 \equiv e_1 \pmod{q}\}.$$

Lemma 3.5.5. *The subspace $X_q \subset Y$ can be identified with the quotient space $SL_d(\mathbb{R})/\Gamma_q$. In particular, this identification is locally bi-Lipschitz.*

Proof. The action $SL_d(\mathbb{R})$ on X_q by the left multiplication is transitive and $Stab_{SL_d(\mathbb{R})}(w(e_1/q)\Gamma) = \Gamma_q$. To see the transitivity, it is enough to show that $SL_d(\mathbb{Z})e_1 \equiv \{\mathbf{p} \in \mathbb{Z}^d : \gcd(\mathbf{p}, q) = 1\} \pmod{q}$. Write $D = \gcd(\mathbf{p})$ and $\mathbf{p}' = \mathbf{p}/D$. Since $\gcd(D, q) = 1$, there are $a, b \in \mathbb{Z}$ such that $aD + bq = 1$. Take $A \in GL_d(\mathbb{Z})$ such that $\det(A) = D$ and $Ae_1 = \mathbf{p}$. If we set $\mathbf{u} = b\mathbf{p}' + (a-1)Ae_2$, then we have $\mathbf{p} + q\mathbf{u} = (A + \mathbf{u} \times {}^t(qe_1 + e_2))e_1$ and $A + \mathbf{u} \times {}^t(qe_1 + e_2) \in SL_d(\mathbb{Z})$, which concludes the transitivity. Bi-Lipshitz property of the identification follows trivially since both X_q and $SL_d(\mathbb{R})/\Gamma_q$ are locally isometric to $SL_d(\mathbb{R})$. □

Theorem 3.5.6. *[KM12, Theorem 2.3] For $q \in \mathbb{N}$, let $SL_d(\mathbb{R})/\Gamma_q \simeq X_q \subset Y$. Let K be a bounded subset in $SL_d(\mathbb{R})$ and $V \subset U$ be a fixed neighborhood of the identity in U with smooth boundary and compact closure. Then, for any $t \geq 0$, $f \in C_c^\infty(Y)$, and $y = gw(\frac{\mathbf{p}}{q})\Gamma$ with $g \in K$ and $\mathbf{p} \in \mathbb{Z}^d$, there exists a constant $\alpha_2 > 0$ only depending on d and V so that*

$$\frac{1}{m_U(V)} \int_V f(a_t u y) dm_U(u) = \int_{X_q} f dm_{X_q} + O(\mathcal{S}_l(f)[\Gamma_1 : \Gamma_q]^{\frac{1}{2}} e^{-\alpha_2 t}). \tag{3.38}$$

The implied constant in (3.38) *only depends on d, V, and K.*

Proof. This result was obtained in [KM12, Theorem 2.3] in the case $q = 1$. For general q, we refer the reader to [KM, Theorem 5.4] which gave a sketch of required modification. [KM, Theorem 5.4] is actually stated for different congruence subgroups from our Γ_q, but the modification still works and the additional factor in the error term is also given by $[\Gamma_1 : \Gamma_q]^{\frac{1}{2}}$. □

Recall the definition of $\mathcal{L}_\epsilon$ in Subsection 3.3.2. Since we assume the unweighted setting, $\mathcal{L}_\epsilon = \{y \in Y : \forall v \in \mu_y,\ \|v\| \geq \epsilon^{1/d}\}$.

Lemma 3.5.7. *For any small enough $\epsilon > 0$ and $q \in \mathbb{N}$, $m_Y(Y_{\leq \epsilon^{-1}} \setminus \mathcal{L}_\epsilon) \asymp \epsilon$ and $m_{X_q}(Y_{\leq \epsilon^{-1}} \setminus \mathcal{L}_\epsilon) \gg q^{-d}\epsilon$.*

Proof. Using Siegel integral formula [MM11, Lemma 2.1] with $f = \mathbb{1}_{B_{\epsilon^{1/d}}(0)}$, which is the indicator function on $\epsilon^{1/d}$-ball centered at 0 in $\mathbb{R}^d$, we have $m_Y(Y_{\leq \epsilon^{-1}} \setminus \mathcal{L}_\epsilon) \ll \epsilon$. On the other hands, by [Ath15, Theorem 1] with $A = B_{\epsilon^{1/d}}(0)$, we have $m_Y(\mathcal{L}_\epsilon) < \frac{1}{1+2^d\epsilon}$. It follows from Siegel integral formula on X that $m_Y(Y_{>\epsilon^{-1}}) = m_X(X_{>\epsilon^{-1}}) \leq 2^d\epsilon^d$. Since $d \geq 2$, we have

$$m_Y(Y_{\leq \epsilon^{-1}} \setminus \mathcal{L}_\epsilon) \geq m_Y(Y \setminus \mathcal{L}_\epsilon) - m_Y(Y_{>\epsilon^{-1}}) > \frac{2^d\epsilon}{1+2^d\epsilon} - 2^d\epsilon^d \gg \epsilon$$

for small enough $\epsilon > 0$, which concludes the first assertion.

To prove the second assertion, observe that for any $x \in X_{>\epsilon^{-1/d}}$, $|\pi_q^{-1}(x) \cap (Y \setminus \mathcal{L}_\epsilon)| \geq 1$, where $\pi_q : X_q \to X$ is the natural projection. Since $|\pi_q^{-1}(x)| \leq q^d$ and $m_X(x \in X : \epsilon^{-1/d} < \mathrm{ht}(x) \leq \epsilon^{-1}) \asymp \epsilon$, we have

$$m_{X_q}(Y_{\leq \epsilon^{-1}} \setminus \mathcal{L}_\epsilon) \geq q^{-d} m_X(x \in X : \epsilon^{-1/d} < \mathrm{ht}(x) \leq \epsilon^{-1}) \gg q^{-d}\epsilon.$$

□

Proposition 3.5.8. *There exist $M, M' > 0$ such that the following holds. Let $\mathcal{A}$ be a countably generated sub-σ-algebra of the Borel σ-algebra which is a^{-1}-descending and U-subordinate. Fix a compact set $K \subset Y$. Let $1 < R' < R$, $k = \lfloor \frac{mn \log R'}{4d} \rfloor$, and $y \in a^{4k}K$. Suppose that $B^U_{R'} \cdot y \subset [y]_\mathcal{A} \subset B^U_R \cdot y$ holds. For $\epsilon > 0$ let $\Omega \subset Y$ be a set satisfying $\Omega \cup a^{-3k}\Omega \subseteq \mathcal{L}_{\frac{\epsilon}{2}}$. If $R' \geq \epsilon^{-M'}$, then*

$$1 - \tau_y^\mathcal{A}(\Omega) \gg \left(\frac{R'}{R}\right)^{mn} \epsilon^{dM+1},$$

where the implied constant only depends on K.

Proof. Denote by $V_y \subset U$ the shape of $\mathcal{A}$-atom of y so that $V_y \cdot y = [y]_\mathcal{A}$. Set $V = B^U_1$. We have $B^U_{e^{-\frac{4d}{mn}}R'} \subseteq a^{4k}Va^{-4k} \subseteq V_y$ since $\frac{mn \log R'}{d} - 4 \leq 4k \leq \frac{mn \log R'}{d}$. It follows that

(3.39)
$$\begin{aligned}
1 - \tau_y^\mathcal{A}(\Omega) &= \frac{1}{m_U(V_y)} \int_{V_y} \mathbb{1}_{Y\setminus\Omega}(uy)dm_U(u) \geq \frac{1}{m_U(B^U_R)} \int_{a^{4k}Va^{-4k}} \mathbb{1}_{Y\setminus\Omega}(uy)dm_U(u) \\
&\geq e^{-4d}\left(\frac{R'}{R}\right)^{mn} \left(\frac{1}{m_U(a^{4k}Va^{-4k})} \int_{a^{4k}Va^{-4k}} \mathbb{1}_{Y\setminus\Omega}(uy)dm_U(u)\right) \\
&= e^{-4d}\left(\frac{R'}{R}\right)^{mn} \left(\frac{1}{m_U(V)} \int_V \mathbb{1}_{Y\setminus\Omega}(a^{4k}ua^{-4k}y)dm_U(u)\right).
\end{aligned}$$

Let $a^{-4k}y = g_0w(b_0)\Gamma$. For the constants α_1 in Theorem 3.5.4 and α_2 in Theorem 3.5.6, let $\alpha = \min(\alpha_1, \alpha_2)$ and $M = \frac{1}{\alpha}\left(2 + l + \frac{\dim G}{2d}\right)$. By [KM96, Lemma 2.4.7(b)] with $r = C\epsilon^{\frac{1}{d}} < 1$, we can take the approximation function $\theta \in C_c^\infty(G)$ of the identity such that $\theta \geq 0$, $\operatorname{Supp}\theta \subseteq B_r^G(id)$, $\int_G \theta = 1$, and $\mathcal{S}_l(\theta) \ll \epsilon^{-\frac{1}{d}(l+\frac{\dim G}{2})}$. Let $\psi = \theta * \mathbb{1}_{Y_{\leq \epsilon^{-1}} \setminus \mathcal{L}_{\frac{\epsilon}{4}}}$, then we have $\mathbb{1}_{Y_{\leq (2\epsilon)^{-1}} \setminus \mathcal{L}_{\frac{\epsilon}{2}}} \leq \psi \leq \mathbb{1}_{Y_{\leq 2\epsilon^{-1}} \setminus \mathcal{L}_{\frac{\epsilon}{8}}}$. Moreover, using Young's inequality, its Sobolev norm is bounded as follows:

$$\begin{aligned} \mathcal{S}_l(\psi)^2 &= \sum_{\mathcal{D}} \|\mathrm{ht} \circ \pi^l \mathcal{D}(\psi)\|_{L^2}^2 \ll \epsilon^{-l} \sum_{\mathcal{D}} \|\mathcal{D}(\theta) * \mathbb{1}_{Y_{\leq \epsilon^{-1}} \setminus \mathcal{L}_{\frac{\epsilon}{4}}}\|_{L^2}^2 \\ &\ll \epsilon^{-l} \|\mathbb{1}_{Y_{\leq \epsilon^{-1}} \setminus \mathcal{L}_{\frac{\epsilon}{4}}}\|_{L^1}^2 \sum_{\mathcal{D}} \|\mathcal{D}(\theta)\|_{L^2}^2 \ll \epsilon^{-l} \mathcal{S}_l(\theta)^2, \end{aligned} \tag{3.40}$$

hence $\mathcal{S}_l(\psi) \ll \epsilon^{-\frac{l}{2}} \mathcal{S}_l(\theta) \leq \epsilon^{-(l+\frac{\dim G}{2d})}$.

In the following two cases, we apply Theorem 3.5.4 and 3.5.6 respectively:

(i) $\zeta(b_0, e^{\frac{2k}{m}}) \geq \epsilon^{-M}$

(ii) $\zeta(b_0, e^{\frac{2k}{m}}) < \epsilon^{-M}$

Case (i): Applying Theorem 3.5.4, we have

$$\begin{aligned} &\frac{1}{m_U(V)} \int_V \mathbb{1}_{Y \setminus \Omega}(a^{4k}ua^{-4k}y)dm_U(u) \geq \frac{1}{m_U(V)} \int_V \psi(a^{4k}ua^{-4k}y)dm_U(u) \\ &= \frac{1}{m_U(V)} \int_V \psi(a^{4k}ug_0w(b_0)\Gamma)dm_U(u) = \int_Y \psi dm_Y + O(\mathcal{S}_l(\psi)\zeta(b_0, e^{\frac{2k}{m}})^{-\alpha}) \\ &= m_Y(Y_{\leq \epsilon^{-1}} \setminus \mathcal{L}_{\frac{\epsilon}{4}}) + O(\epsilon^{-(l+\frac{\dim G}{2d})}\epsilon^{M\alpha}). \end{aligned} \tag{3.41}$$

It follows from Lemma 3.5.7 and $M\alpha = 2 + (l + \frac{\dim G}{2d})$ that

$$\frac{1}{m_U(V)} \int_V \mathbb{1}_{Y \setminus \Omega}(a^{4k}ua^{-4k}y)dm_U(u) \geq m_Y(Y_{\leq \epsilon^{-1}} \setminus \mathcal{L}_{\frac{\epsilon}{4}}) + O(\epsilon^2) \asymp \epsilon. \tag{3.42}$$

Hence, $1 - \tau_y^{\mathcal{A}}(\Omega) \gg \epsilon \left(\frac{R'}{R}\right)^{mn}$ by (3.39) and (3.42).

Case (ii): The assumption $\zeta(b_0, e^{\frac{2k}{m}}) < \epsilon^{-M}$ implies that there exists $q \leq \epsilon^{-M}$ such that $\|qb_0\|_{\mathbb{Z}} \leq q^2 e^{-\frac{2k}{m}}$, whence $\|b_0 - \frac{\mathbf{p}}{q}\| \leq qe^{-\frac{2k}{m}} \leq \epsilon^{-M}e^{-\frac{2k}{m}}$ for some $\mathbf{p} \in \mathbb{Z}^d$. Let $y' = a^{4k} g_0 w(\frac{\mathbf{p}}{q})\Gamma$. Then for any $u \in B_1^U$,

$$\begin{aligned}\mathbf{d}^Y(a^k u a^{-4k} y, a^k u a^{-4k} y') &\ll e^{\frac{k}{m}} \mathbf{d}^Y(a^{-4k} y, a^{-4k} y') \\ &= e^{\frac{k}{m}} \mathbf{d}^Y(g_0 w(b_0)\Gamma, g_0 w(\frac{\mathbf{p}}{q})\Gamma) \\ &\asymp e^{\frac{k}{m}} \|b_0 - \frac{\mathbf{p}}{q}\| \le \epsilon^{-M} e^{-\frac{k}{m}},\end{aligned}$$

hence

$$\begin{aligned}|\psi(a^k u a^{-4k} y) - \psi(a^k u a^{-4k} y')| &\ll \mathcal{S}_l(\psi) \mathbf{d}^Y(a^k u a^{-4k} y, a^k u a^{-4k} y') \\ &\ll \mathcal{S}_l(\psi) \epsilon^{-M} e^{-\frac{k}{m}}.\end{aligned} \tag{3.43}$$

Since we are assuming $a^{-3k}\Omega \subseteq \mathcal{L}_{\frac{\epsilon}{2}}$, we have

$$\begin{aligned}&\frac{1}{m_U(V)} \int_V \mathbb{1}_{Y \setminus \Omega}(a^{4k} u a^{-4k} y) dm_U(u) \\ &= \frac{1}{m_U(V)} \int_V \mathbb{1}_{Y \setminus a^{-3k}\Omega}(a^k u a^{-4k} y) dm_U(u) \\ &\ge \frac{1}{m_U(V)} \int_V \psi(a^k u a^{-4k} y) dm_U(u) \\ &\ge \frac{1}{m_U(V)} \int_V \psi(a^k u a^{-4k} y') dm_U(u) + O(\mathcal{S}_l(\psi)\epsilon^{-M} e^{-\frac{k}{m}}) \\ &= \int_{X_q} \psi \, dm_Y + O(\mathcal{S}_l(\psi) q^{\frac{d}{2}} e^{-\alpha k} + \mathcal{S}_l(\psi)\epsilon^{-M} e^{-\frac{k}{m}}) \\ &\ge m_{X_q}(Y_{\le (2\epsilon)^{-1}} \setminus \mathcal{L}_{\frac{\epsilon}{8}}) + O(\epsilon^{-(l+\frac{\dim G}{2d}) - \frac{dM}{2}} e^{-\alpha k} + \epsilon^{-(l+\frac{\dim G}{2d}) - M} e^{-\frac{k}{m}}).\end{aligned} \tag{3.44}$$

We are using (3.43) for the third line, and Theorem 3.5.6 for the fourth line. Let $M' = \min\left(\frac{4d}{\alpha}(l + \frac{\dim G}{2d} + \frac{3dM}{2} + 2), 4dm(l + \frac{\dim G}{2d} + (d+1)M + 2)\right)$. If $R' > \epsilon^{-M'}$, then $e^{-4dk} < e^{4d}\epsilon^{M'}$, so $\epsilon^{-(l+\frac{\dim G}{2d}) - \frac{dM}{2}} e^{-\alpha k} \ll \epsilon^{dM+2}$ and $\epsilon^{-(l+\frac{\dim G}{2d}) - M} e^{-\frac{k}{m}} \ll \epsilon^{dM+2}$. Combining this with Lemma 3.5.7, it follows that

$$\begin{aligned}\frac{1}{m_U(V)} \int_V \mathbb{1}_{Y \setminus \Omega}(a^{4k} u a^{-4k} y) dm_U(u) &\gg q^{-d}\epsilon + O(\epsilon^{dM+2}) \\ &\gg \epsilon^{dM+1} + O(\epsilon^{dM+2}) \gg \epsilon^{dM+1}.\end{aligned} \tag{3.45}$$

Hence, $1 - \tau_y^{\mathcal{A}}(\Omega) \gg \epsilon^{dM+1} \left(\frac{R'}{R}\right)^{mn}$ by (3.39) and (3.45). □

Proof of Theorem 1.2.1. For fixed b, let $\eta_0 = 2(m+n)(1 - \frac{\dim_H \mathbf{Bad}^b(\epsilon)}{mn})$ as in Proposition 3.5.3. It is enough to consider the case that $\mathbf{Bad}^b(\epsilon)$ is sufficiently close to the full dimension mn, so we may assume $\dim_H \mathbf{Bad}^b(\epsilon) > \dim_H \mathbf{Bad}^0(\epsilon)$ and $\eta_0 \le 0.01$. By Proposition 3.5.3, there is an a-invariant measure $\overline{\mu} \in \mathscr{P}(\overline{Y})$ such that $\operatorname{Supp}\overline{\mu} \subseteq \mathcal{L}_\epsilon \cup (\overline{Y} \setminus Y)$, and $\pi_*\overline{\mu}(\overline{X} \setminus \mathfrak{S}_{\eta'}) \le \eta'$ for any $\eta_0 \le \eta' \le 1$. We also have $\mu \in \mathscr{P}(Y)$ and $0 \le \widehat{\eta} \le \eta_0$ such that

$$\overline{\mu} = (1-\widehat{\eta})\mu + \widehat{\eta}\delta_\infty.$$

In particular, for $\eta' = 0.01$, we have $\mu(\pi^{-1}(\mathfrak{S}_{0.01})) \ge 0.99$. We can choose $0 < r < 1$ such that $Y(r) \supset \pi^{-1}(\mathfrak{S}_{0.01})$. Note that the choice of r is independent of ϵ and b since $\mathfrak{S}_{0.01}$ is constructed in Proposition 3.5.2 independent to ϵ and b.

For such $0 < r < 1$ and a-invariant probability measure μ on Y, let $\mathcal{A}^U$ be and a countably generated σ-algebra as in Lemma 3.2.2 and 3.2.6, respectively. With respect to this σ-algebra, we have

$$h_{\overline{\mu}}(a|\overline{\mathcal{A}^U_\infty}) \ge (1-\widehat{\eta}^{\frac{1}{2}})(d - \widehat{\eta} - d\widehat{\eta}^{\frac{1}{2}})$$

by (3) of Proposition 3.5.3. By the linearlity of the entropy function with respect to the measure, we have

$$\begin{aligned} h_\mu(a|\mathcal{A}^U_\infty) &\ge (1+\widehat{\eta}^{\frac{1}{2}})^{-1}(d - \frac{1}{2}\eta_0 - d\widehat{\eta}^{\frac{1}{2}}) \\ &\ge d - 2d\widehat{\eta}^{\frac{1}{2}} - \frac{1}{2}\eta_0. \end{aligned} \tag{3.46}$$

On the other hand, we shall get an upper bound of $h_\mu(a|\mathcal{A}^U_\infty)$ from Proposition 3.2.8 and Corollary 3.2.11. By Lemma 3.2.5, there exists $0 < \delta < \frac{c}{2}$ such that $\mu(E_\delta) < 0.01$. Note that the constant $C_1, C_2 > 0$ in Lemma 3.2.5 depends only on a and G, hence δ is independent of ϵ even if the set E_δ depends on ϵ. It follows from Proposition 3.2.6 that $B^U_\delta \cdot y \subset [y]_{\mathcal{A}^U} \subset B^U_r \cdot y$ for any $y \in Y(r) \setminus E_\delta$. We write $Z = Y(r) \setminus E_\delta$ for simplicity. Note that $\mu(Z) \ge \mu(Y(r)) - \mu(E_\delta) > 0.98$. We also have $\mu(a^{4k}Y(r)) > 0.99$ Since μ is a-invariant.

Let M and M' be the constants in Proposition 3.5.8, $r' = (1-2^{\frac{1}{d}})\epsilon^{\frac{1}{d}}$, $R' = \epsilon^{-M'}$, $R = e^{\frac{mn}{d}}\frac{r}{\delta}R'$, and $k = \lfloor \frac{mn \log R'}{4d} \rfloor$. Let $\mathcal{A}_1 = a^{-j_1}\mathcal{A}^U$ and $\mathcal{A}_2 = a^{j_2}\mathcal{A}^U$, where

$$j_1 = \lceil -\frac{mn}{d}\log\left(r^{-1}(1-2^{\frac{1}{d}})\epsilon^{\frac{1}{d}}\right)\rceil,$$
$$j_2 = \lceil -\frac{mn}{d}\log(\delta\epsilon^{M'})\rceil.$$

Then for $y \in Z$, the atoms with respect to $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy

$$[y]_{\mathcal{A}_1} \subset B^U_{r'}\cdot y,$$

$$B^U_{R'}\cdot y \subset [y]_{\mathcal{A}_2} \subset B^U_R \cdot y.$$

For $\Omega = B^U_{r'}\mathrm{Supp}\,\mu$, note that $\Omega \subseteq B^U_r\mathcal{L}_\epsilon \subseteq \mathcal{L}_{\frac{\epsilon}{2}}$ and

$$a^{-3k}\Omega = (a^{-3k}B^U_{r'}a^{3k})a^{-3k}\mathrm{Supp}\,\mu \subseteq (a^{-3k}B^U_{r'}a^{3k})\mathcal{L}_\epsilon \subseteq \mathcal{L}_{\frac{\epsilon}{2}}$$

since $\mathrm{Supp}\,\mu$ is a-invariant set. Applying Proposition 3.5.8 with $K = Y(r)$, $\mathcal{A} = \mathcal{A}_2$, and the same R', R, Ω as we just defined, the following holds. For any $\epsilon > 0$ and $y \in a^{4k}Y(r)\cap Z$,

$$1 - \tau_y^{\mathcal{A}_2}(\Omega) \gg \epsilon^{dM+1} \tag{3.47}$$

since $\frac{R'}{R}$ is bounded below by a constant independent of ϵ. By Proposition 3.2.8, Corollary 3.2.11, and (3.47), we have

$$\begin{aligned}(j_1+j_2)(d-h_\mu(a|\mathcal{A}^U_\infty)) &= (j_1+j_2)d - H_\mu(\mathcal{A}_1|\mathcal{A}_2)\\ &\geq -\int_Y \log\tau_y^{\mathcal{A}_2}(\Omega)d\mu(y)\\ &\geq \int_{a^{4k}Y(r)\cap Z}(1-\tau_y^{\mathcal{A}_2}(\Omega))d\mu(y)\\ &\gg \mu(a^{4k}Y(r)\cap Z)\epsilon^{dM+1} > 0.9\epsilon^{dM+1}.\end{aligned}$$

It follows from (3.46) and $j_1+j_2 \asymp \log(1/\epsilon)$ that

$$\eta_0^{\frac{1}{2}} \gg \frac{1}{0.9}(2d\hat{\eta}^{\frac{1}{2}} + \frac{1}{2}\eta_0) \geq d - h_\mu(a|\mathcal{A}^U_\infty) \gg \epsilon^{dM+2}.$$

Since $\eta_0 = 2(m+n)(1-\frac{\dim_H \mathbf{Bad}^b(\epsilon)}{mn})$, we have

$$mn - \dim_H \mathbf{Bad}'(\epsilon) \geq c_0\epsilon^{2(dM+2)}$$

for some constant $c_0 > 0$ only depending on d. □

3.6 Characterization of singular on average property and Dimension esitimates

In this section, we will show (2) $\implies$ (1) in Theorem 1.2.3. Let $A \in M_{m,n}(\mathbb{R})$ and consider two subgroups

$$G(A) := A\mathbb{Z}^n + \mathbb{Z}^m \subset \mathbb{R}^m \quad \text{and} \quad G({}^tA) := {}^tA\mathbb{Z}^m + \mathbb{Z}^n \subset \mathbb{R}^n.$$

If we view alternatively $G(A)$ as a subgroup of classes modulo $\mathbb{Z}^m$, lying in the m-dimensional torus $\mathbb{T}^m$, Kronecker's theorem asserts that $G(A)$ is dense in $\mathbb{T}^m$ if and only if the group $G({}^tA)$ has maximal rank $m+n$ over $\mathbb{Z}$ (See [Cas57, Chapter III, Theorem IV]). Thus, if $\text{rank}_{\mathbb{Z}}(G({}^tA)) < m+n$, then $\text{Bad}_A(\epsilon)$ has full Hausdorff dimension for any $\epsilon > 0$. Hence, throughout this section, we consider only matrices A for which $\text{rank}_{\mathbb{Z}}(G({}^tA)) = m+n$.

3.6.1 Best approximations

We set up a weighted version of the best approximations following [CGGMS20]. (See also [BL05] and [BKLR21] for the unweighted setting.)

Given $A \in M_{m,n}(\mathbb{R})$, we denote

$$M(\mathbf{y}) = \inf_{\mathbf{q}\in\mathbb{Z}^n} \|{}^tA\mathbf{y} - \mathbf{q}\|_{\mathbf{s}}.$$

Our assumption that $\text{rank}_{\mathbb{Z}}(G({}^tA))$ equals $m+n$ guarantees that $M(\mathbf{y}) > 0$ for all non-zero $\mathbf{y} \in \mathbb{Z}^m$. One can construct a sequence of $\mathbf{y}_i \in \mathbb{Z}^n$ called *a sequence of weighted best approximations to* ${}^tA$, which satisfies the following properties:

1. Setting $Y_i = \|\mathbf{y}_i\|_{\mathbf{r}}$ and $M_i = M(\mathbf{y}_i)$, we have

$$Y_1 < Y_2 < \cdots \quad \text{and} \quad M_1 > M_2 > \cdots,$$

2. $M(\mathbf{y}) \geq M_i$ for all non-zero $\mathbf{y} \in \mathbb{Z}^m$ with $\|\mathbf{y}\|_{\mathbf{r}} < Y_{i+1}$.

The sequence $(Y_i)_{i\geq 1}$ has at least geometric growth.

Lemma 3.6.1. *[CGGMS20, Lemma 4.3] There exists a positive integer V such that for all $i \geq 1$,*

$$Y_{i+V} \geq 2Y_i.$$

In particular, there exist $c > 0$ and $\gamma > 1$ such that

$$Y_i \geq c\gamma^i$$

for all $i \geq 1$.

Remark 3.6.2.

1. *The first statement in the above lemma can be found in the proof of [CGGMS20, Lemma 4.3].*
2. *From the weighted Dirichlet's Theorem (see [Kle98, Theorem 2.2]), one can check that $M_k Y_{k+1} \leq 1$ for all $k \geq 1$.*

3.6.2 Characterization of singular on average property

In this section, we will characterize the singular on average property in terms of best approximations. At first, we will show A is singular on average if and only if ${}^tA$ is singular on average. To do this, following [Cas57, Chapter V], we prove a transference principle between two homogeneous approximations with weights. See also [GE15, Ger20].

Definition 3.6.3. *Given positive numbers $\lambda_1, \ldots, \lambda_d$, consider the parallelepiped*

$$\mathcal{P} = \left\{ \mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d : |z_i| \leq \lambda_i,\ i = 1, \ldots, d \right\}.$$

We call the parallelepiped

$$\mathcal{P}^* = \left\{ \mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d : |z_i| \leq \frac{1}{\lambda_i} \prod_{j=1}^{d} \lambda_j,\ i = 1, \ldots, d \right\}$$

the pseudo-compound of $\mathcal{P}$.

Theorem 3.6.4. *[GE15] Let $\mathcal{P}$ be as in Definition 3.6.3 and let Λ be a full-rank lattice in $\mathbb{R}^d$. Then*

$$\mathcal{P}^* \cap \Lambda^* \neq \{\mathbf{0}\} \implies c\mathcal{P} \cap \Lambda \neq \{\mathbf{0}\},$$

where $c = d^{\frac{1}{2(d-1)}}$ *and* Λ^* *is the dual lattice of* Λ.

Corollary 3.6.5. *For positive integer* m, n *let* $d = m+n$ *and let* $A \in M_{m,n}(\mathbb{R})$ *and* $0 < \epsilon < 1$ *be given. For all large enough* $X \geq 1$, *if there exists a nonzero* $\mathbf{q} \in \mathbb{Z}^n$ *such that*

$$\langle A\mathbf{q}\rangle_{\mathbf{r}} \leq \epsilon X^{-1} \quad \text{and} \quad \|\mathbf{q}\|_{\mathbf{s}} \leq X, \tag{3.48}$$

then there exists a nonzero $\mathbf{y} \in \mathbb{Z}^m$ *such that*

$$\langle {}^t A\mathbf{y}\rangle_{\mathbf{s}} \leq c^{(\frac{1}{r_m}+\frac{1}{s_n})}\epsilon^{\frac{r_m s_n}{s_n + r_1(1-s_n)}} Y^{-1} \quad \text{and} \quad \|\mathbf{y}\|_{\mathbf{r}} \leq Y, \tag{3.49}$$

where c *is as in Theorem 3.6.4 and* $Y = c^{\frac{1}{r_m}}\epsilon^{-\frac{r_m(1-s_n)}{s_n+r_1(1-s_n)}}X$.

Proof. Consider the following two parallelepipeds:

$$\mathcal{Q} = \left\{ \mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d : \begin{array}{ll} |z_i| \leq \epsilon^{r_i} X^{-r_i}, & i = 1, \ldots, m \\ |z_{m+j}| \leq X^{s_j}, & j = 1, \ldots, n \end{array} \right\},$$

$$\mathcal{P} = \left\{ \mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d : \begin{array}{ll} |z_i| \leq Z^{r_i}, & i = 1, \ldots, m \\ |z_{m+j}| \leq \delta^{s_j} Z^{-s_j}, & j = 1, \ldots, n \end{array} \right\},$$

where

$$\delta = \epsilon^{\frac{r_m s_n}{s_n + r_1(1-s_n)}} \quad \text{and} \quad Z = \epsilon^{-\frac{r_m(1-s_n)}{s_n+r_1(1-s_n)}}X.$$

Observe that the pseudo-compound of $\mathcal{P}$ is given by

$$\mathcal{P}^* = \left\{ \mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d : \begin{array}{ll} |z_i| \leq \delta Z^{-r_i}, & i = 1, \ldots, m \\ |z_{m+j}| \leq \delta^{1-s_j} Z^{s_j}, & j = 1, \ldots, n \end{array} \right\}$$

and that $\mathcal{Q} \subset \mathcal{P}^*$ since $\epsilon^{r_i} X^{-r_i} \leq \delta Z^{-r_i}$ and $X^{s_j} \leq \delta^{1-s_j} Z^{s_j}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Now, the existence of a nonzero solution $\mathbf{q} \in R_v^n$ of the inequalities (3.48) is equivalent to

$$\mathcal{Q} \cap \begin{pmatrix} I_m & A \\ & I_n \end{pmatrix} \mathbb{Z}^d \neq \{\mathbf{0}\},$$

which implies that

$$\mathcal{P}^* \cap \begin{pmatrix} I_m & A \\ & I_n \end{pmatrix} \mathbb{Z}^d \neq \{\mathbf{0}\}.$$

By Theorem 3.6.4, we have

$$c\mathcal{P} \cap \begin{pmatrix} I_m & \\ -{}^tA & I_n \end{pmatrix} \mathbb{Z}^d \neq \{\mathbf{0}\},$$

which concludes the proof of Corollary 3.6.5.

□

Corollary 3.6.6. *Let m, n be positive integers and $A \in M_{m,n}(\mathbb{R})$. Then A is singular on average if and only if ${}^tA$ is singular on average.*

Proof. It follows from Corollary 3.6.5. □

Now, we will characterize the singular on average property in terms of best approximations. Let $A \in M_{m,n}(\mathbb{R})$ be a matrix and $(\mathbf{y}_k)_{k\geq 1}$ be a sequence of weighted best approximations to ${}^tA$ and write

$$Y_k = \|\mathbf{y}_k\|_{\mathbf{r}}, \quad M_k = \inf_{\mathbf{q}\in\mathbb{Z}^n} \|{}^tA\mathbf{y}_k - \mathbf{q}\|_{\mathbf{s}}.$$

Proposition 3.6.7. *Let $A \in M_{m,n}(\mathbb{R})$ be a matrix and let $(\mathbf{y}_k)_{k\geq 1}$ be a sequence of best approximations to ${}^tA$. Then the following are equivalent:*

1. *${}^tA$ is singular on average.*
2. *For all $\epsilon > 0$,*

$$\lim_{k\to\infty} \frac{1}{\log Y_k} |\{i \leq k : M_i Y_{i+1} > \epsilon\}| = 0.$$

Proof. (1) $\Longrightarrow$ (2) : Let $0 < \epsilon < 1$. Observe that for each integer X with $Y_k \leq X < Y_{k+1}$, the inequalities

$$\|{}^tA\mathbf{p} - \mathbf{q}\|_{\mathbf{s}} \leq \epsilon X^{-1} \quad \text{and} \quad 0 < \|\mathbf{p}\|_{\mathbf{r}} \leq X \tag{3.50}$$

have a solution if and only if $X \leq \frac{\epsilon}{M_k}$. Thus, for each integer $\ell \in [\log_2 Y_k, \log_2 Y_{k+1})$ the inequalities (3.50) have no solutions for $X = 2^\ell$ if and only if

$$\log_2 \epsilon - \log_2 M_k < \ell < \log_2 Y_{k+1}. \tag{3.51}$$

Now we assume that ${}^tA$ is singular on average. For given $\delta > 0$, if the set $\{k \in \mathbb{N} : M_k Y_{k+1} > \delta\}$ is finite, then it is done. Suppose the set $\{k \in \mathbb{N} :$

$M_k Y_{k+1} > \delta\}$ is infinite and let

$$\{k \in \mathbb{N} : M_k Y_{k+1} > \delta\} = \{j(1) < j(2) < \cdots < j(k) < \cdots : k \in \mathbb{N}\}.$$

Set $\epsilon = \delta/2$ and fix a positive integer V in Lemma 3.6.1. For an integer ℓ in $[\log_2 Y_{j(k)+1} - 1, \log_2 Y_{j(k)+1})$, observe that

$$\log_2 \epsilon - \log_2 M_{j(k)} < \log_2 Y_{j(k)+1} - 1.$$

Hence the inequalities (3.50) have no solutions for $X = 2^\ell$ by (3.51). By Lemma 3.6.1, $\log_2 Y_{j(k)+1+V} - 1 \geq \log_2 Y_{j(k)+1}$. So, we have $\log_2 Y_{j(k+V)+1} - 1 \geq \log_2 Y_{j(k)+1}$. Now fix $i = 0, \cdots, V-1$. Then the intervals

$$[\log_2 Y_{j(i+sV)+1} - 1, \log_2 Y_{j(i+sV)+1}), \quad s = 1, \cdots, k$$

are disjoint. Thus, for an integer $N \in [\log_2 Y_{j(i+kV)+1}, \log_2 Y_{j(i+(k+1)V)+1})$, the number of ℓ in $\{1, \cdots, N\}$ such that (3.50) have no solutions for $X = 2^\ell$ is at least k. Since ${}^tA$ is singular on average,

$$\frac{k}{\log_2 Y_{j(i+(k+1)V)+1}} \leq \frac{1}{N}\left|\left\{\ell \in \{1, \cdots, N\} : (3.50) \text{ have no solutions for } X = 2^\ell\right\}\right|$$

tends to 0 with k, which gives $\frac{i+1+kV}{\log_2 Y_{j(i+1+kV)}}$ tends to 0 with k for all $i = 0, \cdots, V-1$. Thus, we have $\frac{k}{\log_2 Y_{j(k)}}$ tends to 0 with k.

For any $k \geq 1$, there is an unique positive integer s_k such that

$$j(s_k) \leq k < j(s_k + 1),$$

and observe that $s_k = |\{i \leq k : M_i Y_{i+1} > \delta\}|$. Thus, by the monotonicity of Y_k, we have

$$\lim_{k\to\infty} \frac{1}{\log_2 Y_k} |\{i \leq k : M_i Y_{i+1} > \delta\}| \leq \lim_{k\to\infty} \frac{s_k}{\log_2 Y_{j(s_k)}} = 0.$$

(2) $\Longrightarrow$ (1) : Given $0 < \epsilon < 1$, the number of integers ℓ in $[\log_2 Y_k, \log_2 Y_{k+1})$ such that (3.50) have no solutions for $X = 2^\ell$ is at most

$$\lceil \log_2 M_k Y_{k+1} - \log_2 \epsilon \rceil \leq \log_2 M_k Y_{k+1} - \log_2 \epsilon + 1.$$

Thus, for an integer N in $[\log_2 Y_k, \log_2 Y_{k+1})$, we have

$$\begin{aligned}\frac{1}{N}|\{\ell \in \{1,\cdots,N\} : (3.50) \text{ have no solutions for } X = 2^\ell\}| \\ \leq \frac{1}{N}\sum_{i=1}^{k} \max\left(0, \log_2 M_i Y_{i+1} - \log_2 \epsilon + 1\right) \\ \leq \frac{1}{\log_2 Y_k}\sum_{i=1}^{k} \max\left(0, \log_2 M_i Y_{i+1} - \log_2 \epsilon + 1\right).\end{aligned}$$

Since $M_i Y_{i+1} \leq 1$ for each $i \geq 1$,

$$\begin{aligned}\frac{1}{\log_2 Y_k}\sum_{i=1}^{k} \max\left(0, \log_2 M_i Y_{i+1} - \log_2 \epsilon + 1\right) \\ \leq \frac{1}{\log_2 Y_k}\left(-\log_2 \epsilon + 1\right)|\{i \leq k : M_i Y_{i+1} > \epsilon/2\}|.\end{aligned}$$

Therefore, ${}^tA$ is singular on average.

□

3.6.3 Modified Bugeaud-Laurent sequence

In this subsection we construct the following modified Bugeaud-Laurent sequence assuming the singular on average property. We refer the reader to [BL05, Section 5] for the original version of the Bugeaud-Laurent sequence.

Proposition 3.6.8. *Let $A \in M_{m,n}(\mathbb{R})$ be such that ${}^tA$ is singular on average and let $(\mathbf{y}_k)_{k\geq 1}$ be a sequence of weighted best approximations to ${}^tA$. For each $S > R > 1$, there exists an increasing function $\varphi : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ satisfying the following properties:*

1. *for any integer $i \geq 1$,*

$$Y_{\varphi(i+1)} \geq R Y_{\varphi(i)} \quad \text{and} \quad M_{\varphi(i)} Y_{\varphi(i+1)} \leq R. \tag{3.52}$$

2.

$$\limsup_{k\to\infty} \frac{k}{\log Y_{\varphi(k)}} \leq \frac{1}{\log S}. \tag{3.53}$$

Proof. The function φ is constructed in the following way. Fix a positive integer V in Lemma 3.6.1 and let $\mathcal{J} = \{j \in \mathbb{Z}_{\geq 1} : M_j Y_{j+1} \leq R/S^3\}$. Since ${}^tA$ is singular on average, by Proposition 3.6.7 with $\epsilon = R/S^3$, we have

$$\lim_{k\to\infty} \frac{1}{\log Y_k} |\{i \leq k : i \in \mathcal{J}^c\}| = 0. \tag{3.54}$$

If the set $\mathcal{J}$ is finite, then we have $\lim_{k\to\infty} Y_k^{1/k} = \infty$ by (3.54), hence the proof of [BKLR21, Theorem 2.2] implies that there exists a function $\varphi : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ for which

$$Y_{\varphi(i+1)} \geq RY_{\varphi(i)} \quad \text{and} \quad Y_{\varphi(i)+1} \geq R^{-1} Y_{\varphi(i+1)}.$$

The fact that $M_i Y_{i+1} \leq 1$ for all $i \geq 1$ implies $M_{\varphi(i)} Y_{\varphi(i+1)} \leq R$. Equation (3.53) follows from $\lim_{k\to\infty} Y_k^{1/k} = \infty$, which concludes the proof of Proposition 3.6.8.

Now, suppose that $\mathcal{J}$ is infinite. Then there are two possible cases:

(i) $\mathcal{J}$ contains all sufficiently large positive integers.

(ii) There are infinitely many positive integers in $\mathcal{J}^c$.

Case (i). Assume the first case and let $\psi(1) = \min\{j : \mathcal{J} \supset \mathbb{Z}_{\geq j}\}$. Define the auxiliary increasing sequence $(\psi(i))_{i\geq 1}$ by

$$\psi(i+1) = \min\{j \in \mathbb{Z}_{\geq 1} : SY_{\psi(i)} \leq Y_j\},$$

which is well defined since $(Y_i)_{i\geq 1}$ is increasing. Note that $\psi(i+1) \leq \psi(i) + \lceil \log_2 S \rceil V$ since $Y_{\psi(i)+\lceil \log_2 S\rceil V} \geq SY_{\psi(i)}$ by Lemma 3.6.1. Let us now define the sequence $(\varphi(i))_{i\geq 1}$ by, for each $i \geq 1$,

$$\varphi(i) = \begin{cases} \psi(i) & \text{if } M_{\psi(i)} Y_{\psi(i+1)} \leq R/S, \\ \psi(i+1) - 1 & \text{otherwise.} \end{cases}$$

Then the sequence $(\varphi(i))_{i\geq 1}$ is increasing and $\varphi \geq \psi$.

Now we claim that for each $i \geq 1$,

$$Y_{\varphi(i+1)} \geq SY_{\varphi(i)} \quad \text{and} \quad M_{\varphi(i)} Y_{\varphi(i+1)} \leq R, \tag{3.55}$$

which implies Equation (3.53) since $Y_{\varphi(k)} \geq S^{k-1} Y_{\varphi(1)}$ for all $k \geq 1$. Thus, the claim concludes the proof of Proposition 3.6.8.

Proof of Equation (3.55). There are four possible cases on the values of $\varphi(i)$ and $\varphi(i+1)$.

• Assume that $\varphi(i) = \psi(i)$ and $\varphi(i+1) = \psi(i+1)$. By the definition of $\psi(i+1)$, we have

$$Y_{\varphi(i+1)} = Y_{\psi(i+1)} \geq SY_{\psi(i)} = SY_{\varphi(i)}.$$

If $\psi(i) \neq \psi(i+1) - 1$, then by the definition of $\varphi(i)$, we have

$$M_{\varphi(i)} Y_{\varphi(i+1)} = M_{\psi(i)} Y_{\psi(i+1)} \leq R/S \leq R.$$

If $\psi(i) = \psi(i+1) - 1$, then $\varphi(i+1) = \varphi(i) + 1$, hence

$$M_{\varphi(i)} Y_{\varphi(i+1)} = M_{\varphi(i)} Y_{\varphi(i)+1} \leq 1 \leq R.$$

This proves Equation (3.55).

• Assume that $\varphi(i) = \psi(i)$ and $\varphi(i+1) = \psi(i+2) - 1$. By the definition of $\psi(i+1)$, we have

$$Y_{\varphi(i+1)} = Y_{\psi(i+2)-1} \geq Y_{\psi(i+1)} \geq SY_{\psi(i)} = SY_{\varphi(i)}.$$

It follows from the minimality of $\psi(i+2)$ that $SY_{\psi(i+1)} > Y_{\psi(i+2)-1}$. If $\psi(i+1) > \psi(i) + 1$, then $M_{\psi(i)} Y_{\psi(i+1)} \leq R/S$ by the definition of $\varphi(i)$. Hence, we have

$$M_{\varphi(i)} Y_{\varphi(i+1)} = M_{\psi(i)} Y_{\psi(i+2)-1} \leq SM_{\psi(i)} Y_{\psi(i+1)} \leq R.$$

If $\psi(i+1) = \psi(i) + 1$, then $M_{\psi(i)} Y_{\psi(i)+1} \leq R/S^3$ since $\psi(i) \in \mathcal{J}$. Hence,

$$M_{\varphi(i)} Y_{\varphi(i+1)} = M_{\psi(i)} Y_{\psi(i+2)-1} \leq SM_{\psi(i)} Y_{\psi(i)+1} \leq R/S^2 \leq R.$$

This proves Equation (3.55).

• Assume that $\varphi(i) = \psi(i+1)-1$ and $\varphi(i+1) = \psi(i+1)$. Since $\psi(i+1)-1 \in \mathcal{J}$, we have

$$M_{\varphi(i)} Y_{\varphi(i+1)} = M_{\psi(i+1)-1} Y_{\psi(i+1)} \leq R/S^3 \leq R.$$

If $\psi(i+1)-1=\psi(i)$, then by the definition of $\psi(i+1)$, we have

$$\frac{Y_{\varphi(i+1)}}{Y_{\varphi(i)}}=\frac{Y_{\psi(i+1)}}{Y_{\psi(i+1)-1}}=\frac{Y_{\psi(i+1)}}{Y_{\psi(i)}}\geq S.$$

If $\psi(i+1)-1>\psi(i)$, then we have $M_{\psi(i)}Y_{\psi(i+1)}>R/S$ by the definition of $\varphi(i)$, and we have $Y_{\psi(i+1)-1}<SY_{\psi(i)}\leq SY_{\psi(i)+1}$ from the minimality of $\psi(i+1)$. We also have $M_{\psi(i)}Y_{\psi(i)+1}\leq R/S^3$ since $\psi(i)\in\mathcal{J}$. Therefore

$$\frac{Y_{\varphi(i+1)}}{Y_{\varphi(i)}}=\frac{Y_{\psi(i+1)}}{Y_{\psi(i+1)-1}}=\frac{M_{\psi(i)}Y_{\psi(i+1)}}{M_{\psi(i)}Y_{\psi(i+1)-1}}\geq\frac{R/S}{SM_{\psi(i)}Y_{\psi(i)+1}}\geq\frac{R/S}{R/S^2}=S.$$

This proves Equation (3.55).

• Assume that $\varphi(i)=\psi(i+1)-1$ and $\varphi(i+1)=\psi(i+2)-1$. By the previous case computations, we have

$$\frac{Y_{\varphi(i+1)}}{Y_{\varphi(i)}}=\frac{Y_{\psi(i+2)-1}}{Y_{\psi(i+1)-1}}\geq\frac{Y_{\psi(i+1)}}{Y_{\psi(i+1)-1}}\geq S.$$

We have $SY_{\psi(i+1)}>Y_{\psi(i+2)-1}$ from the minimality of $\psi(i+2)$. Thus since $\psi(i+1)-1\in\mathcal{J}$, we have

$$M_{\varphi(i)}Y_{\varphi(i+1)}=M_{\psi(i+1)-1}Y_{\psi(i+2)-1}=M_{\psi(i+1)-1}Y_{\psi(i+1)}\left(\frac{Y_{\psi(i+2)-1}}{Y_{\psi(i+1)}}\right)\leq R.$$

This proves Equation (3.55). □

Case (ii). Now we assume the second case and let $j_0=\min\mathcal{J}$. Partition $\mathbb{Z}_{\geq j_0}$ into disjoint subset

$$\mathbb{Z}_{\geq j_0}=C_1\sqcup D_1\sqcup C_2\sqcup D_2\sqcup\cdots$$

where $C_i\subset\mathcal{J}$ and $D_j\subset\mathcal{J}^c$ are sets of consecutive integers with

$$\max C_i<\min D_i\leq\max D_i<\min C_{i+1}$$

for all $i\geq 1$. We consider the following two subcases.
(ii) - 1. If there is $i_0\geq 1$ such that $|C_i|<3\lceil\log_2 S\rceil V$ for all $i\geq i_0$, then we

have, for $k_0 = \min C_{i_0}$,

$$\frac{k}{\log Y_k} \leq \frac{k_0 + (3\lceil \log_2 S \rceil V + 1)\, |\{i \leq k : i \in \mathcal{J}^c\}|}{\log Y_k},$$

since there exists an element of $\mathcal{J}^c$ in any finite sequence of $3\lceil \log_2 S \rceil V + 1$ consecutive integers at least k_0. Therefore $\lim_{k\to\infty} Y_k^{1/k} = \infty$ by (3.54) and this concludes the proof of Proposition 3.6.8 following the proof when $\mathcal{J}$ is finite at the beginning.

(ii) - 2. The remaining case is that the set

$$\{i : |C_i| \geq 3\lceil \log_2 S \rceil V\} = \{i(1) < i(2) < \cdots < i(k) < \cdots : k \in \mathbb{N}\}$$

is infinte.

For each $k \geq 1$, let us define an increasing finite sequence $(\psi_k(i))_{1\leq i \leq m_k+1}$ of positive integers by setting $\psi_k(1) = \min C_{i(k)}$ and by induction

$$\psi_k(i+1) = \min\{j \in C_{i(k)} : SY_{\psi_k(i)} \leq Y_j\},$$

as long as this set is nonempty. Since $C_{i(k)}$ is a finite sequence of consecutive positive integers with length at least $3\lceil \log_2 S \rceil V$ and $Y_{i+\lceil \log_2 S \rceil V} \geq SY_i$ for every $i \geq 1$ by Lemma 3.6.1, there exists an integer $m_k \geq 2$ such that $\psi_k(i)$ is defined for $i = 1, \ldots, m_k + 1$. Note that $\psi_k(i)$ belongs to $\mathcal{J}$ since $C_{i(k)} \subset \mathcal{J}$.

As in **Case (i)**, let us define an increasing finite sequence $(\varphi_k(i))_{1\leq i\leq m_k}$ of positive integers by

$$\varphi_k(i) = \begin{cases} \psi_k(i) & \text{if } M_{\psi_k(i)} Y_{\psi_k(i+1)} \leq R/S, \\ \psi_k(i+1) - 1 & \text{otherwise.} \end{cases}$$

Following the proof of **Case (i)**, we have for each $i = 1, \ldots, m_k - 1$,

$$Y_{\varphi_k(i+1)} \geq SY_{\varphi_k(i)} \quad \text{and} \quad M_{\varphi_k(i)} Y_{\varphi_k(i+1)} \leq R. \tag{3.56}$$

Note that $\varphi_k(m_k) < \varphi_{k+1}(1)$. Let us define an increasing finite sequence $(\varphi'_k(i))_{1\leq i\leq n_k+1}$ of positve integers to interpolate between $\varphi_k(m_k)$ and $\varphi_{k+1}(1)$. Let $j_0 = \varphi_{k+1}(1)$. If the set $\{j \in \mathbb{Z}_{\geq \varphi_k(m_k)} : Y_{j_0} \geq RY_j\}$ is empty, then we set $n_k = 0$ and $\varphi'_k(1) = j_0 = \varphi_{k+1}(1)$. Otherwise, follwing [BKLR21, Theorem 2.2], by decreasing induction, let $n_k \in \mathbb{Z}_{\geq 1}$ be the maximal positive integer such that there exists $j_1, \ldots, j_{n_k} \in \mathbb{Z}_{\geq 1}$ such that for $\ell = 1, \ldots, n_k$, the set

$\{j \in \mathbb{Z}_{\geq \varphi_k(m_k)} : Y_{j_{\ell-1}} \geq RY_j\}$ is nonempty and for $\ell = 1, \ldots, n_k + 1$, the integer j_ℓ is its largest element. Set $\varphi'_k(i) = j_{n_k+1-i}$ for $i = 1, \ldots, n_k + 1$. Then the sequence $(\varphi'_k(i))_{1 \leq i \leq n_k+1}$ is contained in $[\varphi_k(m_k), \varphi_{k+1}(1)]$ and satisfies that for $i = 1, \ldots, n_k$,

$$Y_{\varphi'_k(i+1)} \geq RY_{\varphi'_k(i)} \quad \text{and} \quad M_{\varphi'_k(i)} Y_{\varphi'_k(i+1)} \leq R \tag{3.57}$$

from the proof of [BKLR21, Theorem 2.2].

Now, putting alternatively together the sequences $(\varphi_k(i))_{1 \leq i \leq m_k - 1}$ and $(\varphi'_k(i))_{1 \leq i \leq r_k}$ as k ranges over $\mathbb{Z}_{\geq 1}$, we define $N_k = \sum_{\ell=1}^{k-1} (m_\ell - 1 + n_\ell)$ and

$$\varphi(i) = \begin{cases} \varphi_k(i - N_k) & \text{if } 1 + N_k \leq i \leq m_k - 1 + N_k, \\ \varphi'_k(i + 1 - m_k - N_k) & \text{if } m_k + N_k \leq i \leq r_k - 1 + m_k + N_k. \end{cases}$$

Here, we use the standard convention that an empty sum is zero. With Equation (3.56) for $i = 1, \ldots, m_k - 2$ and Equation (3.57) for $i = 1, \ldots, n_k$, since $\varphi'_k(n_k + 1) = \varphi_{k+1}(1)$, it is enough to show the following lemma to prove that the map φ satisfies Equation (3.52).

Lemma 3.6.9. *For every $k \in \mathbb{Z}_{\geq 1}$, we have*

$$Y_{\varphi'_k(1)} \geq RY_{\varphi_k(m_k-1)} \quad \textit{and} \quad M_{\varphi_k(m_k-1)} Y_{\varphi'_k(1)} \leq R. \tag{3.58}$$

Proof. Since $\varphi'_k(1) \geq \varphi_k(m_k)$ and Equation (3.56) with $i = m_k - 1$, we have

$$Y_{\varphi'_k(1)} \geq Y_{\varphi_k(m_k)} \geq SY_{\varphi_k(m_k-1)} \geq RY_{\varphi_k(m_k-1)},$$

which prove the left hand side of Equation (3.58). If $\varphi'_k(1) = \varphi_k(m_k)$, then Equation (3.56) with $i = m_k - 1$ gives the right hand side of Equation (3.58).

Now assume that $\varphi'_k(1) > \varphi_k(m_k)$. By the maximality of n_k, we have $Y_{\varphi'_k(1)} \leq RY_{\varphi_k(m_k)}$. First, we will prove that $\varphi_k(m_k) = \psi_k(m_k)$. For a contradiction, assume that $\varphi_k(m_k) = \psi_k(m_k + 1) - 1 > \phi_k(m_k)$. Following the third subcase of the proof of Equation (3.55), we have

$$\frac{Y_{\psi_k(m_k+1)}}{Y_{\psi_k(m_k+1)-1}} = \frac{M_{\psi_k(m_k)} Y_{\psi_k(m_k+1)}}{M_{\psi_k(m_k)} Y_{\psi_k(m_k+1)-1}} \geq S.$$

Hence by the construction of $\varphi'_k(1)$, we have $\varphi'_k(1) = \varphi_k(m_k)$, which is a contradiction to our assumption $\varphi'_k(1) > \varphi_k(m_k)$.

To show the right hand side of Equation (3.58), we consider two possible values of $\varphi_k(m_k-1)$.

Assume that $\varphi_k(m_k-1)=\psi_k(m_k-1)$. If $\psi_k(m_k-1)>\psi_k(m_k)-1$, then by the definition of $\varphi_k(m_k-1)$, we have $M_{\psi_k(m_k-1)}Y_{\psi_k(m_k)}\leq R/S$. If $\psi_k(m_k-1)=\psi_k(m_k)-1$, then $M_{\psi_k(m_k-1)}Y_{\psi_k(m_k)}\leq R/S^3\leq R/S$ since $\psi_k(m_k)-1\in\mathcal{J}$. Since $\varphi_k(m_k)=\psi_k(m_k)$, we have

$$M_{\varphi_k(m_k-1)}Y_{\varphi'_k(1)}=M_{\psi_k(m_k-1)}Y_{\psi_k(m_k)}\left(\frac{Y_{\varphi'_k(1)}}{Y_{\varphi_k(m_k)}}\right)\leq R,$$

which proves the right hand side of Equation (3.58).

Assume that $\varphi_k(m_k-1)=\psi_k(m_k)-1$. Since $\varphi_k(m_k)=\psi_k(m_k)$ and $\psi_k(m_k)-1\in\mathcal{J}$, we have

$$M_{\varphi_k(m_k-1)}Y_{\varphi'_k(1)}=M_{\psi_k(m_k)-1}Y_{\psi_k(m_k)}\left(\frac{Y_{\varphi'_k(1)}}{Y_{\varphi_k(m_k)}}\right)\leq R,$$

which proves the right hand side of Equation (3.58), and concludes the proof of Lemma 3.6.9. □

Finally, we will show Equation (3.53) for the map φ. Since there exists an element of $\mathcal{J}^c$ in any finite sequence of $3\lceil\log_2 S\rceil V+1$ consecutive integers in the complement of $\bigcup_{k\geq 1}C_{i(k)}$, there exists $c_0\geq 0$ such that for every $k\geq 1$, we have

$$\frac{|\{j\leq\varphi(k):j\notin\bigcup_{k\geq 1}C_{i(k)}\}|}{\log Y_{\varphi(k)}}\leq\frac{c_0+(3\lceil\log_2 S\rceil V+1)\,|\{j\leq\varphi(k):j\in\mathcal{J}^c\}|}{\log Y_{\varphi(k)}},$$

which converges to 0 as $k\to+\infty$ by (3.54). Let us define

$$n(k)=|\{i\leq k:Y_{\varphi(i)}\geq SY_{\varphi(i+1)}\}|.$$

For each integer $\ell\geq 1$, since $Y_{i+\lceil\log_2 S\rceil V}\geq SY_i$ for every $i\geq 1$ by Lemma 3.6.1, and by the maximality of m_ℓ in the construction of $(\varphi_\ell(i))_{1\leq i\leq m_\ell}$, we have $|\{j\in C_{i(\ell)}:j\geq\varphi_\ell(m_\ell)\}|\leq 2\lceil\log_2 S\rceil V$. If $\varphi(i)$ belongs to $C_{i(\ell)}$ but $\varphi(i+1)$ does not, then $\varphi(i)\geq\varphi_\ell(m_\ell)$. If $\varphi(i)$ and $\varphi(i+1)$ belong to $C_i(\ell)$, then φ and φ_ℓ coincide on i and $i+1$. Thus, by Equation (3.56), we have

$$\begin{aligned} k - n(k) &= |\{i \leq k : Y_{\varphi(i)} < SY_{\varphi(i+1)}\}| \\ &\leq (2\lceil \log_2 S \rceil V) \left|\{j \leq \varphi(k) : j \notin \bigcup_{k \geq 1} C_{i(k)}\}\right|. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \limsup_{k\to\infty} \frac{k}{\log Y_{\varphi(k)}} &= \limsup_{k\to\infty} \frac{n(k) + k - n(k)}{\log Y_{\varphi(k)}} = \limsup \frac{n(k)}{\log Y_{\varphi(k)}} \\ &\leq \limsup_{k\to\infty} \frac{n(k)}{\log S^{n(k)-1} Y_{\varphi(1)}} = \frac{1}{\log S}. \end{aligned}$$

This proves Equation (3.53) and concludes the proof of Proposition 3.6.8. □

3.6.4 Dimension estimates

Following the notation in [BHKV10], given a sequence $\{\mathbf{y}_i\}$ in $\mathbb{Z}^m \setminus \{\mathbf{0}\}$ and $\alpha \in (0, 1/2)$, let

$$\mathrm{Bad}^{\alpha}_{\{\mathbf{y}_i\}} := \{\theta \in \mathbb{R}^m : |\theta \cdot \mathbf{y}_i|_{\mathbb{Z}} \geq \alpha \text{ for all } i \geq 1\}.$$

Proposition 3.6.10. *[CGGMS20] Let $A \in M_{m,n}(\mathbb{R})$ be a matrix and let $(\mathbf{y}_k)_{k\geq1}$ be a sequence of weighted best approximations to ${}^tA$ and let $R > 1$ and $\alpha \in (0, 1/2)$ be given. Suppose that there exists an increasing function $\varphi : \mathbb{Z}_{\geq1} \to \mathbb{Z}_{\geq1}$ such that for any integer $i \geq 1$*

$$M_{\varphi(i)} Y_{\varphi(i+1)} \leq R.$$

Then $\mathrm{Bad}^{\alpha}_{\{\mathbf{y}_{\varphi(i)}\}}$ is a subset of $\mathrm{Bad}_A(\epsilon)$ where $\epsilon = \frac{1}{R}\left(\frac{\alpha^2}{4mn}\right)^{1/\delta}$ and $\delta = \min\{r_i, s_j : 1 \leq i \leq m, 1 \leq j \leq n\}$.

Proof. In the proof of [CGGMS20, Theorem 1.11], the condition $Y_{\varphi(i)+1} \geq R^{-1} Y_{\varphi(i+1)}$ is used. However, the assumption $M_{\varphi(i)} Y_{\varphi(i+1)} \leq R$ also implies the same conclusion. □

Proposition 3.6.11. *[CGGMS20] For any $\alpha \in (0, 1/2)$, there exists $R(\alpha) > 1$ with the following property. Let $(\mathbf{y}_k)_{k\geq1}$ be a sequence in $\mathbb{Z}^m \setminus \{\mathbf{0}\}$ such that*

$\|\mathbf{y}_{k+1}\|_{\mathbf{r}}/\|\mathbf{y}_k\|_{\mathbf{r}} \geq R(\alpha)$ for all $k \geq 1$. Then

$$\dim_H \left(\mathrm{Bad}^{\alpha}_{\{\mathbf{y}_i\}}\right) \geq m - C \limsup_{k\to\infty} \frac{k}{\log \|\mathbf{y}_k\|_{\mathbf{r}}}.$$

for some positive constant $C = C(\alpha)$.

Proof. The proof of [CGGMS20, Theorem 6.1] concludes this proposition. □

The two propositions are used in [BKLR21, Theorem 5.1] in the unweighted setting.

Proof of Theorem 1.2.3 (2) $\Longrightarrow$ (1). Suppose A is singular on average. By Corollary 3.6.6, ${}^tA$ is also singular on average. Let $(\mathbf{y}_k)_{k\geq 1}$ be a sequence of weighted best approximations to ${}^tA$. Then, by Proposition 3.6.8, Proposition 3.6.10, and Proposition 3.6.11, for each $S > R(\alpha) > 1$, we have

$$\begin{aligned}\dim_H \left(\mathrm{Bad}_A(\epsilon)\right) &\geq \dim_H \left(\mathrm{Bad}^{\alpha}_{\{\mathbf{y}_{\varphi(i)}\}}\right)\\ &\geq m - C \limsup_{k\to\infty} \frac{k}{\log Y_{\varphi(k)}}\\ &\geq m - \frac{C}{\log S}\end{aligned}$$

where $\epsilon = \frac{1}{R(\alpha)}\left(\frac{\alpha^2}{4mn}\right)^{1/\delta}$. Taking $S \to \infty$, we have $\dim_H \left(\mathrm{Bad}_A(\epsilon)\right) = m$ for $\epsilon = \frac{1}{R(\alpha)}\left(\frac{\alpha^2}{4mn}\right)^{1/\delta}$.

□

Chapter 4

Diophantine approximation over global function fields

4.1 Background material for the lower bound

4.1.1 On global function fields

We refer for instance to [Gos96, Ros02], as well as [BPP19, §14.2], for the content of this section. Let $\mathbb{F}_q$ be a finite field with q elements, where q is a positive power of a positive prime. Let K be the function field of a geometrically connected smooth projective curve $\mathbf{C}$ over $\mathbb{F}_q$, or equivalently an extension of $\mathbb{F}_q$ with transcendence degree 1, in which $\mathbb{F}_q$ is algebraically closed. We denote by g the genus of $\mathbf{C}$. There is a bijection between the set of closed points of $\mathbf{C}$ and the set of normalized discrete valuations v of K, the valuation of a given element $f \in K$ being the order of the zero or the opposite of the order of the pole of f at the given closed point. We fix such an element v throughout this paper, and use the notation K_v, $\mathcal{O}_v$, π_v, k_v, q_v, $|\cdot|$ defined in the introduction. We furthermore denote by $\deg v$ the degree of v, so that

$$q_v = q^{\deg v} .$$

We denote by vol_v the normalized Haar measure on the locally compact additive group K_v such that $\mathrm{vol}_v(\mathcal{O}_v) = 1$. For any positive integer d, let vol_v^d be the normalized Haar measure on K_v^d such that $\mathrm{vol}_v^d(\mathcal{O}_v^d) = 1$. Note that for every $g \in \mathrm{GL}_d(K_v)$ we have

$$d\,\mathrm{vol}_v^d(gx) = |\det(g)|\, d\,\mathrm{vol}_v^d(x)\ ,$$

where det is the determinant of a matrix. For every discrete additive subgroup Λ of $K_v^{\,d}$, we again denote by vol_v^d (and simply vol_v when $d = 1$) the measured induced on $K_v^{\,d}/\Lambda$ by vol_v^d.

Note that the completion K_v of K for v is the field $k_v((\pi_v))$ of Laurent series $x = \sum_{i\in\mathbb{Z}} x_i(\pi_v)^i$ in the variable π_v over k_v, where $x_i \in k_v$ is zero for $i \in \mathbb{Z}$ small enough. We have

$$|\,x\,| = q_v^{-\sup\{j\in\mathbb{Z}\,:\,\forall i<j,\ x_i=0\}}\ ,$$

and $\mathcal{O}_v = k_v[[\pi_v]]$ is the local ring of power series $x = \sum_{i\in\mathbb{Z}_{\geq 0}} x_i(\pi_v)^i$ in the variable π_v over k_v.

Recall that the affine algebra R_v of the affine curve $\mathbf{C} - \{v\}$ consists of the elements of K whose only poles are at the closed point v of $\mathbf{C}$. Its field of fractions is equal to K, hence we can write elements of K as x/y with $x, y \in R_v$ and $y \neq 0$. By for instance [BPP19, Eq. (14.2)], we have

$$R_v \cap \mathcal{O}_v = \mathbb{F}_q\ . \tag{4.1}$$

For every $\xi \in K_v$, we denote by

$$|\langle\xi\rangle| = \inf_{x\in R_v} \|\,\xi - x\,\|$$

the distance in K_v from ξ to the set R_v of integral points of K_v.

For instance, if $\mathbf{C}$ is the projective line $\mathbb{P}^1$, if $\infty = [1:0]$ is its usual point at infinity and if Z is a variable name, then $g = 0$, $K = \mathbb{F}_q(Z)$, $\pi_\infty = Z^{-1}$, $K_\infty = \mathbb{F}_q((Z^{-1}))$, $\mathcal{O}_\infty = \mathbb{F}_q[[Z^{-1}]]$, $k_\infty = \mathbb{F}_q$, $q_\infty = q$ and $R_\infty = \mathbb{F}_q[Z]$. In this setting, there are numerous results on Diophantine approximation in the fields of formal power series, see for instance [Las00], [Bug04(2), Chap. 9]. On the other hand, little is known about Diophantine approximation over general global function fields, see for instance [KST17] (for a single valuation in positive characteristic) for the ground work on the geometry of number for function fields.

4.1.2 On the geometry of numbers and Dirichlet's theorem

Let d be a positive integer. An R_v*-lattice* Λ in $K_v^{\,d}$ is a discrete R_v-submodule in $K_v^{\,d}$ that generates $K_v^{\,d}$ as a K_v-vector space. The *covolume* of Λ, denoted by $\mathrm{Covol}(\Lambda)$, is defined as the measure of the (compact) quotient space $K_v^{\,d}/\Lambda$:

$$\mathrm{Covol}(\Lambda) = \mathrm{vol}_v^d(K_v^{\,d}/\Lambda)\ .$$

For example, R_v^d is an R_v-lattice in $K_v^{\,d}$, and by for instance [BPP19, Lem. 14.4)], we have

$$\mathrm{Covol}(R_v^d) = q^{(g-1)d}\ . \tag{4.2}$$

Let $\overline{B}(0,r)$ be the closed ball of radius r centered at zero in $K_v^{\,d}$ with respect to the norm $\|\cdot\| : (\xi_1,\dots,\xi_d) \mapsto \max_{1\le i\le d}|\,\xi_i\,|$. For every integer $k \in \{1,\dots,d\}$, the *k-th minimum* of an R_v-lattice Λ is defined by

$$\lambda_k(\Lambda) = \min\{r>0 : \dim_{K_v}(\mathrm{span}_{K_v}(\overline{B}(0,r)\cap\Lambda)) \ge k\},$$

where span_{K_v} denotes the K_v-linear span of a subset of a K_v-vector space and $\dim_{K_v}$ is the dimension of a K_v-vector space. Note that $\lambda_1(\Lambda),\dots,\lambda_d(\Lambda) \in q_v^{\mathbb{Z}}$. The next result follows from [KST17, Theo. 4.4] and Equation (4.2).

Theorem 4.1.1. (Minkowski's theorem) *For every R_v-lattice Λ in $K_v^{\,d}$, we have*

$$q^{-(g-1)d}\ \mathrm{Covol}(\Lambda) \le \lambda_1(\Lambda)\dots\lambda_d(\Lambda) \le q_v^d\ \mathrm{Covol}(\Lambda)\ . \quad \square$$

Since $\lambda_1(\Lambda) \le \dots \le \lambda_d(\Lambda)$, the following result follows immediately from Minkowski's theorem 4.1.1.

Corollary 4.1.2. *For every R_v-lattice Λ in $K_v^{\,d}$, we have*

$$\lambda_1(\Lambda) \le q_v\ \mathrm{Covol}(\Lambda)^{\frac{1}{d}}\ . \quad \square$$

The following result generalizes [GG17, Theo. 2.1], which is proved only when $K = \mathbb{F}_q(Z)$ and $v = \infty$, to all function fields K and valuations v. See also [KW08, Theo. 1.3] in the case of the field $\mathbb{Q}$.

Theorem 4.1.3. (Dirichlet's theorem) *For every matrix $A \in \mathcal{M}_{m,n}(K_v)$ whose rows are denoted by $A_1,\dots,A_m$, for all $(r'_1,\dots,r'_m) \in \mathbb{Z}_{\ge 0}^m$ and $(s'_1,\dots,s'_n) \in$*

$\mathbb{Z}_{\geq 0}^n$ *with*

$$r'_i > 1 + \frac{g-1}{\deg v} \text{ and } \sum_{i=1}^{m} r'_i = \sum_{j=1}^{n} s'_j \ ,$$

there exists an element $\mathbf{y} = (y_1, \dots, y_n) \in R_v^n - \{0\}$ *such that, for all* $i = 1, \dots, m$ *and* $j = 1, \dots, n$, *we have*

$$|\langle\, A_i\, \mathbf{y}\,\rangle| \leq q_v\, q^{g-1}\, {q_v}^{-r'_i} \text{ and } |\, y_j\,| \leq q_v\, q^{g-1}\, {q_v}^{s'_j} \ .$$

Proof. With A, $r'_1, \dots, r'_m$ and $s'_1, \dots, s'_n$ as in the statement, we apply Corollary 4.1.2 with $d = m+n$ to the R_v-lattice

$$\Lambda = \begin{pmatrix} \pi_v^{-r'_1} & & & & & 0 \\ & \ddots & & & & \\ & & \pi_v^{-r'_m} & & & \\ & & & \pi_v^{s'_1} & & \\ & & & & \ddots & \\ 0 & & & & & \pi_v^{s'_n} \end{pmatrix} \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} R_v^{\,d} \ ,$$

where I_k is the $k \times k$ identity matrix. Since the above two matrices have determinant 1 by the assumption $\sum_{i=1}^m r'_i = \sum_{j=1}^n s'_j$, and by Equation (4.2), we have $\operatorname{Covol}(\Lambda) = q^{(g-1)d}$. Corollary 4.1.2 hence says that there exists $(\mathbf{x} = (x_1, \dots, x_m), \mathbf{y} = (y_1, \dots, y_n)) \in R_v^d - \{0\}$ such that

$$\max\Big\{ \max_{i=1,\dots,m} |\, \pi_v^{-r'_i}(x_i + A_i \mathbf{y})\,|,\ \max_{j=1,\dots,n} |\, \pi_v^{s'_j} y_j\,|\ \Big\} \leq q_v \operatorname{Covol}(\Lambda)^{\frac{1}{d}} = q_v\, q^{g-1} \ .$$

Assume for a contradiction that $\mathbf{y} = 0$. Then for all $i = 1, \dots, m$, since $|\, \pi_v\,| = q_v^{-1}$, we have the inequality $|\, x_i\,| \leq q_v\, q^{g-1}\, {q_v}^{-r'_i}$. Since $r'_i > 1 + \frac{g-1}{\deg v}$, this would imply that $|\, x_i\,| < 1$. By Equation (4.1), we have $\{z \in R_v : |\, z\,| < 1\} = \{0\}$. Since $x_i \in R_v$, we would have that $\mathbf{x} = 0$, contradicting the fact that $(\mathbf{x}, \mathbf{y}) \neq 0$. Therefore $\mathbf{y} \neq 0$ and the result follows. □

The following corollary is due to [Kri06, Theo. 1.1] (see also [BZ19, Theo. 3.2] where the assumption that $c\,m$ is divisible by n is implicit) in the special case when $K = \mathbb{F}_q(Z)$ and $v = \infty$ and without weights.

Let $\min \mathbf{r} = \min_{1 \leq i \leq m} r_i$ and similarly for $\min \mathbf{s}$, $\max \mathbf{r}$ and $\max \mathbf{s}$.

Corollary 4.1.4. *For all $A \in \mathcal{M}_{m,n}(K_v)$ and $\alpha \in \mathbb{Z}_{\geq 0}$ with $\alpha > \frac{1}{\min \mathbf{r}} + \frac{g-1}{(\min \mathbf{r})(\deg v)}$, there exists $\mathbf{y} \in R_v^n - \{0\}$ such that*

$$\langle A\,\mathbf{y}\rangle_{\mathbf{r}} \leq q^{\frac{\deg v + g - 1}{\min \mathbf{r}}}\; q_v^{-\alpha} \text{ and } \|\,\mathbf{y}\,\|_{\mathbf{s}} \leq q^{\frac{\deg v + g - 1}{\min \mathbf{s}}}\; q_v^{\alpha}\ .$$

Proof. We apply Theorem 4.1.3 with $r'_i = \alpha\, r_i > 1 + \frac{g-1}{\deg v}$ for $i = 1, \dots, m$ and $s'_j = \alpha\, s_j$ for $j = 1, \dots, n$, noting that $\sum_{i=1}^m r'_i = \sum_{j=1}^n s'_j$ since $\sum_{i=1}^m r_i = \sum_{j=1}^n s_j$. □

Remark 4.1.5. *When $\mathbf{r} = (n, n, \dots, n)$ and $\mathbf{s} = (m, m, \dots, m)$, the above result says that for every integer $\alpha > \frac{1}{n} + \frac{g-1}{n \deg v}$, there exists $\mathbf{y} \in R_v^n - \{0\}$ such that*

$$\min_{\mathbf{x} \in R_v^m} \|A\,\mathbf{y} - \mathbf{x}\| \leq q_v\, q^{g-1}\; q_v^{-\alpha\, n} \text{ and } \|\,\mathbf{y}\,\| \leq q_v\, q^{g-1}\; q_v^{\alpha\, m}\ ,$$

where $\|\cdot\|$ is the sup norm.

4.1.3 Best approximation sequences with weights

In this subsection, we construct a version with weights, valid for all function fields, of the best approximation sequences associated with a completely irrational matrix by Bugeaud-Zhang [BZ19].

A matrix $A \in \mathcal{M}_{m,n}(K_v)$ is said to be *completely irrational* if $\langle A\,\mathbf{y}\rangle_{\mathbf{r}} \neq 0$ for every $\mathbf{y} \in R_v^n - \{0\}$. Note that this does not depend on the weight $\mathbf{r}$, and that the fact that A is completely irrational might not necessarily imply that ${}^tA$ is completely irrational.

Remark 4.1.6. *Let $A \in \mathcal{M}_{m,n}(K_v)$ be such that ${}^tA$ is not completely irrational.*

(1) The matrix ${}^tA$ is $(\mathbf{s}, \mathbf{r})$-singular on average.

(2) For every $\epsilon > 0$ small enough, the set $\mathbf{Bad}_A(\epsilon)$ has full Hausdorff dimension.

Proof. By assumption, there exist $\mathbf{x} \in R_v^n$ and $\mathbf{y} = (y_1, \dots, y_m) \in R_v^m - \{0\}$ such that ${}^tA\,\mathbf{y} - \mathbf{x} = 0$.

(1) For every $\epsilon > 0$, if $\ell_0 = \lceil \log_{q_v} \| \mathbf{y} \|_{\mathbf{r}} \rceil$ then for all integers $N \geq \ell_0$ and $\ell \in \{\ell_0, \dots, N\}$, we have $\langle {}^t\!A\, \mathbf{y} \rangle_{\mathbf{s}} = 0 \leq \epsilon\ q_v^{-\ell}$ and $\| \mathbf{y} \|_{\mathbf{r}} \leq q_v^{\ell}$, hence ${}^tA$ is $(\mathbf{s}, \mathbf{r})$-singular on average (see Equation (1.8)).

(2) For every $\boldsymbol{\theta} = (\theta_1, \dots, \theta_m) \in K_v^{\,m}$, let

$$\mathbf{y} \cdot \boldsymbol{\theta} = \sum_{j=1}^{m} y_i\, \theta_i \ \in K_v\ .$$

For every $\epsilon \in \,]0, \frac{1}{\| \mathbf{y} \|_{\mathbf{r}}}]$, let $U_{\mathbf{y},\epsilon} = \big\{ \boldsymbol{\theta} \in K_v^{\,m} : |\langle \mathbf{y} \cdot \boldsymbol{\theta} \rangle| \geq (\epsilon\ \| \mathbf{y} \|_{\mathbf{r}})^{\min \mathbf{r}} \big\}$. If ϵ is small enough, then the set $U_{\mathbf{y},\epsilon}$ contains a closed ball of positive radius: For instance, let $j_0 \in \{1, \dots, m\}$ be such that $y_{j_0} \neq 0$; define $\theta_{0,j} = 0$ if $j \neq j_0$, $\theta_{0,j_0} = \frac{\pi_v}{y_{j_0}}$ and $\boldsymbol{\theta}_0 = (\theta_{0,1}, \dots, \theta_{0,m})$; then it is easy to check using the ultrametric inequality that the closed ball $\overline{B}(\boldsymbol{\theta}_0, \frac{1}{q_v^2\, \| \mathbf{y} \|})$ is contained in $U_{\mathbf{y},\epsilon}$ if $\epsilon < q_v^{-\frac{1}{\min \mathbf{r}}}\, \| \mathbf{y} \|_{\mathbf{r}}^{-1}$.

Let us prove that $\mathbf{Bad}_A(\epsilon)$ contains $U_{\mathbf{y},\epsilon}$, which implies that $\dim_H \big(\mathbf{Bad}_A(\epsilon)\big) = m$ if ϵ is small enough. Let $\boldsymbol{\theta} \in U_{\mathbf{y},\epsilon}$ and $(\mathbf{y}', \mathbf{x}') \in R_v^{\,m} \times (R_v^{\,n} - \{0\})$.

If $\| \mathbf{y} \|_{\mathbf{r}} \| A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta} \|_{\mathbf{r}} \geq 1$, then since $\mathbf{x}' \in R_v^{\,n} - \{0\}$ so that $\| \mathbf{x}' \|_{\mathbf{s}} \geq 1$, we have

$$\| \mathbf{x}' \|_{\mathbf{s}}\ \| A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta} \|_{\mathbf{r}} \geq \frac{1}{\| \mathbf{y} \|_{\mathbf{r}}}\ \| \mathbf{y} \|_{\mathbf{r}}\ \| A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta} \|_{\mathbf{r}} \geq \frac{1}{\| \mathbf{y} \|_{\mathbf{r}}} \geq \epsilon\ .$$

If $\| \mathbf{y} \|_{\mathbf{r}}\ \| A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta} \|_{\mathbf{r}} \leq 1$, then since $\mathbf{y} \cdot (A\mathbf{x}' + \mathbf{y}') = ({}^tA\mathbf{y}) \cdot \mathbf{x}' + \mathbf{y} \cdot \mathbf{y}' = \mathbf{x} \cdot \mathbf{x}' + \mathbf{y} \cdot \mathbf{y}' \in R_v$, and since $\boldsymbol{\theta} \in U_{\mathbf{y},\epsilon}$, we have

$$\begin{aligned}
\| \mathbf{x}' \|_{\mathbf{s}}\ \| A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta} \|_{\mathbf{r}} &\geq \frac{1}{\| \mathbf{y} \|_{\mathbf{r}}}\ \| \mathbf{y} \|_{\mathbf{r}}\ \| A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta} \|_{\mathbf{r}} \\
&\geq \frac{1}{\| \mathbf{y} \|_{\mathbf{r}}}\ \big(\max_{1 \leq j \leq m} \| \mathbf{y} \|_{\mathbf{r}}^{r_j} \| A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta} \|_{\mathbf{r}}^{r_j} \big)^{\frac{1}{\min \mathbf{r}}} \\
&\geq \frac{1}{\| \mathbf{y} \|_{\mathbf{r}}}\ \big| \mathbf{y} \cdot (A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta}) \big|^{\frac{1}{\min \mathbf{r}}} \geq \frac{1}{\| \mathbf{y} \|_{\mathbf{r}}}\ |\langle \mathbf{y} \cdot \boldsymbol{\theta} \rangle|^{\frac{1}{\min \mathbf{r}}} \geq \epsilon\ .
\end{aligned}$$

Therefore $\boldsymbol{\theta} \in \mathbf{Bad}_A(\epsilon)$, as wanted. □

For every matrix $A \in \mathcal{M}_{m,n}(K_v)$, a *best approximation sequence* for A with weights $(\mathbf{r}, \mathbf{s})$ is a sequence $(\mathbf{y}_i)_{i \geq 1}$ in $R_v^{\,n}$ such that, with $Y_i = \| \mathbf{y}_i \|_{\mathbf{s}}$ and $M_i = \langle A\, \mathbf{y}_i \rangle_{\mathbf{r}}$,

- the sequence $(Y_i)_{i \geq 1}$ is positive and strictly increasing,

• the sequence $(M_i)_{i\geq 1}$ is positive and strictly decreasing, and
• for every $\mathbf{y} \in R_v^n - \{0\}$ with $\|\,\mathbf{y}\,\|_{\mathbf{s}} < Y_{i+1}$, we have $\langle A\,\mathbf{y}\rangle_{\mathbf{r}} \geq M_i$.

We denote by $\operatorname{lcm} \mathbf{r}$ the least common multiple of $r_1, \dots, r_m$, and similarly for $\operatorname{lcm} \mathbf{s}$.

Lemma 4.1.7. *Assume that $A \in \mathcal{M}_{m,n}(K_v)$ is completely irrational.*

(1) There exists a best approximation sequence $(\mathbf{y}_i)_{i\geq 1}$ for A with weights $(\mathbf{r}, \mathbf{s})$.

(2) If $(\mathbf{y}_i)_{i\geq 1}$ is a best approximation sequence for A with weights $(\mathbf{r}, \mathbf{s})$, then

i) we have $M_i \in q_v^{\frac{1}{\operatorname{lcm}\mathbf{r}}\mathbb{Z}}$ and $M_i \in q_v^{\frac{1}{\operatorname{lcm}\mathbf{r}}\mathbb{Z}_{\leq 0}}$ if i is large enough,

ii) we have $Y_i \in q_v^{\frac{1}{\operatorname{lcm}\mathbf{s}}\mathbb{Z}_{\geq 0}}$ and $Y_i \geq q_v^{\frac{i-1}{\operatorname{lcm}\mathbf{s}}}$ for every $i \geq 1$,

iii) the sequence $\big(M_i\, Y_{i+1}\big)_{i\geq 1}$ is uniformly bounded.

Note that a best approximation sequence might be not unique (and the terminology "best", though traditional, is not very appropriate). When $m = n = r_1 = s_1 = 1$, $K = \mathbb{F}_q(Z)$ and $v = \infty$, then $A \in K_v$ is completely irrational if and only if $A \in K_v - K$, and with $\big(\frac{P_k}{Q_k}\big)_{k\geq 0}$ the sequence of convergents of A (see for instance [Las00]), we may take $y_i = Q_{i-1}$ for all $i \geq 1$.

If $A \in \mathcal{M}_{m,n}(K_v)$ is not completely irrational, a *best approximation sequence* for A with weights $(\mathbf{r}, \mathbf{s})$ is a finite sequence $(\mathbf{y}_i)_{1\leq i\leq i_0}$ in R_v^n, such that, with $Y_i = \|\,\mathbf{y}_i\,\|_{\mathbf{s}}$ and $M_i = \langle A\,\mathbf{y}_i\rangle_{\mathbf{r}}$,
• $1 = Y_1 < \cdots < Y_{i_0}$,
• $M_1 > \cdots > M_{i_0} = 0$,
• for all $i \in \{1, \dots, i_0 - 1\}$ and $\mathbf{y} \in R_v^n - \{0\}$ with $\|\,\mathbf{y}\,\|_{\mathbf{s}} < Y_{i+1}$, we have $\langle A\,\mathbf{y}\rangle_{\mathbf{r}} \geq M_i$, and
• which stops at the first i_0 such that there exists $\mathbf{z} \in R_v^n$ with $0 < \|\,\mathbf{z}\,\|_{\mathbf{s}} \leq Y_{i_0}$ and $\langle A\,\mathbf{z}\rangle_{\mathbf{r}} = 0$.

The proof of Lemma 4.1.7 is similar to the one given after [BZ19, Def. 3.3] in the particular case when $K = \mathbb{F}_q(Z)$, $v = \infty$ and without weights.

Proof. (1) Let us prove by induction on $i \geq 1$ that there exist $\mathbf{y}_1, \dots, \mathbf{y}_i$ in R_v^n such that, with $Y_j = \|\,\mathbf{y}_j\,\|_{\mathbf{s}}$ and $M_j = \langle A\,\mathbf{y}_j\rangle_{\mathbf{r}}$ for every $1 \leq j \leq i$, we have $1 = Y_1 < \cdots < Y_i$, $M_1 > \cdots > M_i > 0$, and (using $M_0 = +\infty$ by convention)
(a_i) we have $\langle A\,\mathbf{y}\rangle_{\mathbf{r}} \geq M_{i-1}$ for every $\mathbf{y} \in R_v^n - \{0\}$ with $\|\,\mathbf{y}\,\|_{\mathbf{s}} < Y_i$,
(b_i) we have $\langle A\,\mathbf{y}\rangle_{\mathbf{r}} \geq M_i$ for every $\mathbf{y} \in R_v^n - \{0\}$ with $\|\,\mathbf{y}\,\|_{\mathbf{s}} \leq Y_i$.

Note that $\{x \in R_v : |\,x\,| \leq 1\} = R_v \cap \mathcal{O}_v = \mathbb{F}_q$ by Equation (4.1). Hence the elements with smallest $\mathbf{s}$-quasinorm in $R_v^n - \{0\}$ are the elements in the finite set $\mathbb{F}_q^n - \{0\}$, which is the set of elements in R_v^n with $\mathbf{s}$-quasinorm 1. Furthermore, the set $\{\|\,\mathbf{y}\,\|_\mathbf{s} : \mathbf{y} \in R_v^n - \{0\}\}$ is contained in $q_v^{\bigcup_{j=1}^n \frac{1}{s_j}\mathbb{Z}_{\geq 0}} \subset q_v^{\frac{1}{\operatorname{lcm}\mathbf{s}}\mathbb{Z}_{\geq 0}}$. Similarly, for every $\mathbf{x} \in K_v^m - \{0\}$, we have $\langle \mathbf{x} \rangle_\mathbf{r} \in q_v^{\frac{1}{\operatorname{lcm}\mathbf{r}}\mathbb{Z}}$.

Therefore there exists an element $\mathbf{y}_1 \in R_v^n$ with $\|\,\mathbf{y}_1\,\|_\mathbf{s} = 1$ such that

$$\langle A\,\mathbf{y}_1 \rangle_\mathbf{r} = \min\{\, \langle A\,\mathbf{y} \rangle_\mathbf{r} : \mathbf{y} \in R_v^n,\ \ \|\,\mathbf{y}\,\|_\mathbf{s} = 1\,\}\ .$$

We thus have $Y_1 = \|\,\mathbf{y}_1\,\|_\mathbf{s} = 1$ and $M_1 = \langle A\,\mathbf{y}_1 \rangle_\mathbf{r} > 0$ since A is completely irrational. There is no $\mathbf{y} \in R_v^n - \{0\}$ with $\|\,\mathbf{y}\,\|_\mathbf{s} < Y_1$, and if $\|\,\mathbf{y}\,\|_\mathbf{s} = Y_1$, then $\langle A\,\mathbf{y} \rangle_\mathbf{r} \geq M_1$, hence the claims (a_1) and (b_1) are satisfied.

Assume by induction that $\mathbf{y}_1, \dots, \mathbf{y}_i$ as above are constructed. Let

$$S = \{\, \mathbf{y} \in R_v^n :\ \|\,\mathbf{y}\,\|_\mathbf{s} > Y_i,\ \ \langle A\,\mathbf{y} \rangle_\mathbf{r} < M_i\,\}\ .$$

Note that the set $\{\mathbf{z} \in R_v^n,\ \ 0 < \|\,\mathbf{z}\,\|_\mathbf{s} \leq Y_i\}$ is finite by the discreteness of R_v^n, and $\epsilon_i = \min\{\, \langle A\,\mathbf{z} \rangle_\mathbf{r} : \mathbf{z} \in R_v^n,\ \ 0 < \|\,\mathbf{z}\,\|_\mathbf{s} \leq Y_i\,\}$ is positive, since A is completely irrational. Corollary 4.1.4 of Dirichlet's theorem implies in particular, by taking in its statement α large enough, that for every $\epsilon > 0$, there exists $\mathbf{y} \in R_v^n - \{0\}$ such that $\langle A\,\mathbf{y} \rangle_\mathbf{r} < \epsilon$. Applying this with $\epsilon = \min\{M_i, \epsilon_i\} > 0$ proves that the set S is nonempty. Hence the set $S_{\min}$ of elements of S with minimal $\mathbf{s}$-quasinorm, which is finite again by the discreteness of R_v^n, is nonempty. Therefore there exists $\mathbf{y}_{i+1} \in S_{\min}$ such that

$$\langle A\,\mathbf{y}_{i+1} \rangle_\mathbf{r} = \min\{\, \langle A\,\mathbf{z} \rangle_\mathbf{r} : \mathbf{z} \in S_{\min}\,\}\ .$$

Then $Y_{i+1} = \|\,\mathbf{y}_{i+1}\,\|_\mathbf{s} = \min \|\,S\,\|_\mathbf{s} > Y_i$ by the definition of the set S. We also have that $M_{i+1} = \langle A\,\mathbf{y}_{i+1} \rangle_\mathbf{r} < M_i$ since $\mathbf{y}_{i+1} \in S_{\min} \subset S$, and again by the definition of S.

Let us now prove that $\mathbf{y}_{i+1}$ satisfies the properties (a_{i+1}) and (b_{i+1}).

• Let $\mathbf{y} \in R_v^n - \{0\}$ be such that $\|\,\mathbf{y}\,\|_\mathbf{s} < Y_{i+1}$. If $\|\,\mathbf{y}\,\|_\mathbf{s} \leq Y_i$, then by the induction hypothesis (b_i), we have $\langle A\,\mathbf{y} \rangle_\mathbf{r} \geq M_i$, as wanted for Property (a_{i+1}). If $\|\,\mathbf{y}\,\|_\mathbf{s} > Y_i$, then by the definition of S, we have $\langle A\,\mathbf{y} \rangle_\mathbf{r} \geq M_i$ as wanted for Property (a_{i+1}), otherwise $\mathbf{y}$ would be an element of S with $\mathbf{s}$-quasinorm strictly less than the minimum $\mathbf{s}$-quasinorm of the elements of S, a contradiction.

• Let $\mathbf{y} \in R_v^n - \{0\}$ be such that $\|\,\mathbf{y}\,\|_{\mathbf{s}} \leq Y_{i+1}$. Either $\|\,\mathbf{y}\,\|_{\mathbf{s}} < Y_{i+1}$, in which case, as just seen, $\langle A\,\mathbf{y}\rangle_{\mathbf{r}} \geq M_i \geq M_{i+1}$, as wanted for Property (b_{i+1}). Or $\|\,\mathbf{y}\,\|_{\mathbf{s}} = Y_{i+1} > Y_i$, in which case either $\langle A\,\mathbf{y}\rangle_{\mathbf{r}} \geq M_i \geq M_{i+1}$, as wanted for Property (b_{i+1}), or $\langle A\,\mathbf{y}\rangle_{\mathbf{r}} < M_i$, so that $\mathbf{y}$ belongs to $S_{\min}$, hence $\langle A\,\mathbf{y}\rangle_{\mathbf{r}} \geq \min\{\,\langle A\,\mathbf{z}\rangle_{\mathbf{r}} : \mathbf{z} \in S_{\min}\,\} = M_{i+1}$.

By induction, this proves Assertion (1) of Lemma 4.1.7.

(2) i) This follows from the facts that $M_i \in q_v^{\frac{1}{\operatorname{lcm}\mathbf{r}}\mathbb{Z}}$ and that $M_{i+1} < M_i$.

ii) Since $Y_1 = 1$, this follows by induction from the facts that $Y_i \in q_v^{\frac{1}{\operatorname{lcm}\mathbf{s}}\mathbb{Z}}$ and that $Y_{i+1} > Y_i$.

iii) Let $\alpha = \big\lfloor \log_{q_v}(q^{-\frac{\deg v+g-1}{\min \mathbf{s}}}\,Y_{i+1})\big\rfloor - 1$, which satisfies $\alpha > \frac{1}{\min \mathbf{r}} + \frac{g-1}{(\min \mathbf{r})(\deg v)}$ if i is large enough, by Assertion (2) ii). By Corollary 4.1.4, there exists $\mathbf{y} \in R_v^n - \{0\}$ such that

$$\|\,\mathbf{y}\,\|_{\mathbf{s}} \leq q^{\frac{\deg v+g-1}{\min \mathbf{s}}}\, q_v^{\alpha} < q^{\frac{\deg v+g-1}{\min \mathbf{s}}}\, q_v^{\log_{q_v}(q^{-\frac{\deg v+g-1}{\min \mathbf{s}}}\,Y_{i+1})} = Y_{i+1}$$

and

$$\begin{aligned}\langle A\,\mathbf{y}\rangle_{\mathbf{r}} &\leq q^{\frac{\deg v+g-1}{\min \mathbf{r}}}\, q_v^{-\alpha}\\ &\leq q^{\frac{\deg v+g-1}{\min \mathbf{r}}}\, q_v^{-\left(\log_{q_v}(q^{-\frac{\deg v+g-1}{\min \mathbf{s}}}\,Y_{i+1})-2\right)} = q^{(\deg v+g-1)\left(\frac{1}{\min \mathbf{r}}+\frac{1}{\min \mathbf{s}}\right)+2\deg v}\,(Y_{i+1})^{-1}\;.\end{aligned}$$

Since $M_i \leq \min\{\,\langle A\,\mathbf{y}\rangle_{\mathbf{r}} : \mathbf{y} \in R_v^n,\ 0 < \|\,\mathbf{y}\,\|_{\mathbf{s}} < Y_{i+1}\,\}$ by the definition of a best approximation sequence, the result follows. □

4.1.4 Transference theorems with weights

In this section, we will show that a matrix $A \in \mathcal{M}_{m,n}(K_v)$ is singular on average if and only if its transpose ${}^tA$ is singular on average. To do this, following [Cas57, Chap. V], we prove a transference principle between two problems of homogeneous approximations with weights. See also [GE15, Ger20] in the disjoint case of the field $\mathbb{Q}$.

Let $d \in \mathbb{Z}_{\geq 2}$ be a positive integer at least 2. For all $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d)$ and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_d)$ in K_v^d, we denote

$$\boldsymbol{\xi} \cdot \boldsymbol{\theta} = \sum_{k=1}^{d} \xi_k\, \theta_k\;.$$

Let $\alpha_1, \ldots, \alpha_d \in \mathbb{Z}$ be integers and let $\alpha = \sum_{k=1}^{d} \alpha_k$. We consider the parallelepiped

$$\mathcal{P} = \left\{\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d) \in K_v^{\,d} : \forall\, k = 1, \ldots, d, \;\; |\,\xi_k\,| \le q_v^{\alpha_k}\right\} .$$

Following Schmidt's terminology [Sch80, page 109] in the case of the field $\mathbb{Q}$ (building on Mahler's compound one), we call the parallelepiped

$$\mathcal{P}^* = \left\{\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d) \in K_v^{\,d} : \forall\, k = 1, \ldots, d, \;\; |\,\xi_k\,| \le \frac{1}{q_v^{\alpha_k}} \prod_{i=1}^{d} q_v^{\alpha_i} = q_v^{\alpha-\alpha_k}\right\}$$

the *pseudocompound* of $\mathcal{P}$. Note that $\mathcal{P}$ and $\mathcal{P}^*$ are preserved by the multiplication of the components of their elements by elements of $\mathcal{O}_v$.

Theorem 4.1.8. *With $\mathcal{P}$ and $\mathcal{P}^*$ as above, for every $F \in \mathrm{SL}_d(K_v)$,*

$$\textit{if} \;\; \mathcal{P}^* \cap {}^tF^{-1}(R_v^{\,d}) \ne \{0\}, \quad \textit{then} \;\; \pi_v^{-\beta_d}\mathcal{P} \cap F(R_v^{\,d}) \ne \{0\},$$

where

$$\beta_d = \left\lceil \frac{1}{d-1}\Big(d + 1 + \frac{(g-1)d}{\deg v}\Big)\right\rceil .$$

Remark 4.1.9. *The R_v-lattice ${}^tF^{-1}(R_v^{\,d})$ is called the dual lattice of the R_v-lattice $F(R_v^{\,d})$ since we have $\mathbf{z}\cdot\mathbf{w} \in R_v$ for all $\mathbf{z} \in {}^tF^{-1}(R_v^{\,d})$ and $\mathbf{w} \in F(R_v^{\,d})$. They have the same covolume as $R_v^{\,d}$, since* $\det(F) = 1$.

Proof. Let $\mathbf{z} = (z_1, \ldots, z_d) \in \mathcal{P}^* \cap {}^tF^{-1}(R_v^{\,d}) - \{0\}$ and $\kappa_0 = \max\{k \in \mathbb{Z}_{\ge 0} : \mathbf{z} \in \pi_v^k\mathcal{P}^*\}$. Up to permuting the coordinates, we may assume that, for all $k = 2, \ldots, d$, we have

$$|\,z_1\,| = q_v^{\alpha-\alpha_1-\kappa_0} \quad \text{and} \quad |\,z_k\,| \le q_v^{\alpha-\alpha_k-\kappa_0} . \tag{4.3}$$

With F_k the k-th row of F, let us consider the R_v-lattice $\Lambda = M(R_v^{\,d})$ where

$$M = \begin{pmatrix} \pi_v^{-1}\sum_{k=1}^{d} z_k F_k \\ \pi_v^{\beta_d+\alpha_2} F_2 \\ \vdots \\ \pi_v^{\beta_d+\alpha_d} F_d \end{pmatrix} .$$

By subtracting to the first row a linear combination of the other rows, and since $\det F = 1$, the determinant of the above matrix M is equal to $\pi_v^{(d-1)\beta_d+\alpha-\alpha_1-1}\, z_1$. By Equations (4.3) and (4.2), we thus have

$$\operatorname{Covol}(\Lambda) = \det(M)\operatorname{Covol}(R_v^d) = q_v^{1-\kappa_0-(d-1)\beta_d}\, q^{(g-1)d} .$$

Since $d \geq 2$ and $\beta_d \geq \frac{1}{d-1}\big(d+1+\frac{(g-1)d}{\deg v}\big)$, Corollary 4.1.2 applied to the R_v-lattice Λ gives that

$$\lambda_1(\Lambda) \leq q_v \operatorname{Covol}(\Lambda)^{\frac{1}{d}} \leq 1.$$

Hence, by the definition of the first minimum $\lambda_1(\Lambda)$, there exists $\mathbf{w} \in R_v^d - \{0\}$ such that for every $k = 2, \dots, d$, we have

$$|\,\mathbf{z}\cdot F(\mathbf{w})\,| \leq q_v^{-1} < 1 \quad \text{and} \quad |\,F_k(\mathbf{w})\,| \leq q_v^{\beta_d+\alpha_k} . \tag{4.4}$$

Since $\mathbf{z} \in {}^tF^{-1}(R_v^d)$ and $\mathbf{w} \in R_v^d$, we have $\mathbf{z}\cdot F(\mathbf{w}) \in R_v$ by the above Remark. The first inequality of Equation (4.4) hence implies that $\mathbf{z}\cdot F(\mathbf{w}) = 0$, which means that

$$z_1 F_1(\mathbf{w}) = -\sum_{k=2}^{d} z_k F_k(\mathbf{w}) .$$

By the ultrametric property of $|\cdot|$, by Equations (4.3) and (4.4), we have

$$\begin{aligned} q_v^{\alpha-\alpha_1-\kappa_0}|\,F_1(\mathbf{w})\,| = |\,z_1F_1(\mathbf{w})\,| &\leq \max_{2\leq k\leq d} |\,z_kF_k(\mathbf{w})\,| \\ &\leq \max_{2\leq k\leq d} q_v^{\alpha-\alpha_k-\kappa_0} q_v^{\beta_d+\alpha_k} = q_v^{\alpha+\beta_d-\kappa_0} . \end{aligned}$$

Therefore $|\,F_1(\mathbf{w})\,| \leq q_v^{\beta_d+\alpha_1}$ and with the second inequality of Equation (4.4), we conclude that $F(\mathbf{w}) \in \pi_v^{\beta_d}\mathcal{P}$. □

Corollary 4.1.10. *There exist $\kappa_1, \kappa_2, \kappa_3, \kappa_4 \geq 0$ with $\kappa_2 > 0$, depending only on m, n, g, $\deg v$, $\mathbf{r}$ and $\mathbf{s}$, such that for all $A \in \mathcal{M}_{m,n}(K_v)$ and $\epsilon \in q_v^{\mathbb{Z}_{\leq -1}}$, for every large enough $Y \in q_v^{\mathbb{Z}_{\geq 1}}$, if there exists $\mathbf{y} \in R_v^n - \{0\}$ such that*

$$\langle A\mathbf{y}\rangle_{\mathbf{r}} \leq \epsilon\, Y^{-1} \quad \textit{and} \quad \|\,\mathbf{y}\,\|_{\mathbf{s}} \leq Y , \tag{4.5}$$

then there exists $\mathbf{x} \in R_v^m - \{0\}$ such that

$$\langle\, {}^tA\mathbf{x}\rangle_{\mathbf{s}} \leq q_v^{\kappa_1}\, \epsilon^{\kappa_2}\, X^{-1} \quad \textit{and} \quad \|\,\mathbf{x}\,\|_{\mathbf{r}} \leq X , \tag{4.6}$$

where $X = q_v^{\kappa_3}\,\epsilon^{-\kappa_4}\,Y$.

Proof. Let $|\mathbf{s}| = \sum_{j=1}^n s_j$. Denoting $\alpha_\epsilon = -\log_{q_v}\epsilon \in \mathbb{Z}_{\geq 1}$ and $\alpha_Y = \log_{q_v} Y \in \mathbb{Z}_{\geq 1}$, we define $\delta = q_v^{-\alpha_\delta}$ and $Z = q_v^{\alpha_Z}\,Y$ where

$$\alpha_\delta = \Big\lfloor \frac{\alpha_\epsilon - 1}{|\mathbf{s}|\big(\frac{1}{\min \mathbf{r}} + \frac{1}{\min \mathbf{s}}\big) - 1} \Big\rfloor \text{ and } \alpha_Z = \Big\lceil \Big(\frac{|\mathbf{s}|}{\min \mathbf{s}} - 1\Big)\alpha_\delta \Big\rceil\,. \tag{4.7}$$

Note that α_δ is well defined since $\frac{|\mathbf{s}|}{\min \mathbf{s}} \geq 1$, and that α_δ and α_Z are nonnegative. We have

$$\begin{aligned} \Big(|\mathbf{s}|\big(\frac{1}{\min \mathbf{r}} + \frac{1}{\min \mathbf{s}}\big) - 1\Big)\,\alpha_\delta &\leq \alpha_\epsilon - 1\,, \\ \text{hence } \Big(\frac{|\mathbf{s}|}{\min \mathbf{s}} - 1\Big)\,\alpha_\delta + 1 &\leq \alpha_\epsilon - \frac{|\mathbf{s}|}{\min \mathbf{r}}\,\alpha_\delta\,, \\ \text{therefore } \Big(\frac{|\mathbf{s}|}{\min \mathbf{s}} - 1\Big)\,\alpha_\delta \leq \alpha_Z &\leq \alpha_\epsilon - \frac{|\mathbf{s}|}{\min \mathbf{r}}\,\alpha_\delta\,. \quad (4.8) \end{aligned}$$

Let $d = m + n \geq 2$. Let us consider the following parallelepipeds

$$\mathcal{Q} = \left\{ \xi = (\xi_1, \dots, \xi_d) \in K_v^d : \begin{array}{ll} \forall\, i = 1, \dots, m, & |\,\xi_i\,| \leq \epsilon^{r_i}\,Y^{-r_i} \\ \forall\, j = 1, \dots, n, & |\,\xi_{m+j}\,| \leq Y^{s_j} \end{array} \right\},$$

$$\mathcal{P} = \left\{ \xi = (\xi_1, \dots, \xi_d) \in K_v^d : \begin{array}{ll} \forall\, i = 1, \dots, m, & |\,\xi_i\,| \leq Z^{r_i} \\ \forall\, j = 1, \dots, n, & |\,\xi_{m+j}\,| \leq \delta^{s_j}\,Z^{-s_j} \end{array} \right\}.$$

Since $\sum_{i=1}^m r_i = \sum_{j=1}^n s_j$, the pseudocompound $\mathcal{P}^*$ of $\mathcal{P}$ is equal to

$$\mathcal{P}^* = \left\{ \xi = (\xi_1, \dots, \xi_d) \in K_v^d : \begin{array}{ll} \forall\, i = 1, \dots, m, & |\,\xi_i\,| \leq \delta^{|\mathbf{s}|}\,Z^{-r_i} \\ \forall\, j = 1, \dots, n, & |\,\xi_{m+j}\,| \leq \delta^{|\mathbf{s}|-s_j}\,Z^{s_j} \end{array} \right\}.$$

By the right inequality of Equation (4.8), for every $i = 1, \dots, m$, we have

$$\delta^{|\mathbf{s}|}Z^{-r_i} = q_v^{-|\mathbf{s}|\alpha_\delta - r_i\alpha_Z}\;Y^{-r_i} \geq q_v^{-r_i(\alpha_Z + \frac{|\mathbf{s}|}{\min \mathbf{r}}\alpha_\delta)}\;Y^{-r_i} \geq \epsilon^{r_i}\,Y^{-r_i}\,.$$

By the left inequality of Equation (4.8), for every $j = 1, \dots, n$, we have

$$\delta^{|\mathbf{s}|-s_j}Z^{s_j} = q_v^{-(|\mathbf{s}|-s_j)\alpha_\delta + s_j\alpha_Z}\;Y^{s_j} \geq q_v^{s_j(\alpha_Z - (\frac{|\mathbf{s}|}{\min \mathbf{s}} - 1)\alpha_\delta)}\;Y^{s_j} \geq Y^{s_j}\,.$$

Therefore $\mathcal{Q}$ is contained in $\mathcal{P}^*$.

Now, by the assumption of Corollary 4.1.10, let $\mathbf{y} \in R_v^n - \{0\}$ be such that the inequalities (4.5) are satisfied. Then there exists $(\mathbf{x}', \mathbf{y}) \in R_v^m \times (R_v^n - \{0\})$ such that

$$\|A\mathbf{y} - \mathbf{x}'\|_{\mathbf{r}} \leq \epsilon\, Y^{-1} \quad \text{and} \quad \|\,\mathbf{y}\,\|_{\mathbf{s}} \leq Y\ .$$

Therefore

$$\mathcal{Q} \cap \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} R_v^d \neq \{0\}\ .$$

Since $\mathcal{Q} \subset \mathcal{P}^*$, this implies that

$$\mathcal{P}^* \cap \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} R_v^d \neq \{0\}\ .$$

By Theorem 4.1.8, we have

$$\pi_v^{-\beta_d}\mathcal{P} \cap \begin{pmatrix} I_m & 0 \\ -\,{}^tA & I_n \end{pmatrix} R_v^d \neq \{0\}\ .$$

Then there exists $(\mathbf{x}, \mathbf{y}') \in (R_v^m \times R_v^n) - \{0\}$ such that

$$\|\pi_v^{\beta_d}\,\mathbf{x}\|_{\mathbf{r}} \leq Z \quad \text{and} \quad \|\pi_v^{\beta_d}(-\,{}^tA\mathbf{x} - \mathbf{y}')\|_{\mathbf{s}} \leq \delta\, Z^{-1}\ . \tag{4.9}$$

The above inequality on the left-hand side and the two equalities of Equation (4.7) give

$$\begin{aligned}\|\,\mathbf{x}\,\|_{\mathbf{r}} &\leq q_v^{\frac{\beta_d}{\min \mathbf{r}}}\, Z = q_v^{\frac{\beta_d}{\min \mathbf{r}} + \alpha_Z}\, Y \leq q_v^{\frac{\beta_d}{\min \mathbf{r}} + 1 + (\frac{|\mathbf{s}|}{\min \mathbf{s}} - 1)\frac{\alpha_\epsilon - 1}{|\mathbf{s}|(\frac{1}{\min \mathbf{r}} + \frac{1}{\min \mathbf{s}}) - 1}}\, Y \\ &\leq q_v^{\frac{\beta_d}{\min \mathbf{r}} + 1}\, \epsilon^{-\frac{\frac{|\mathbf{s}|}{\min \mathbf{s}} - 1}{|\mathbf{s}|(\frac{1}{\min \mathbf{r}} + \frac{1}{\min \mathbf{s}}) - 1}}\, Y\ .\end{aligned}$$

If $\kappa_3 = \frac{\beta_d}{\min \mathbf{r}} + 1 > 0$ and $\kappa_4 = \frac{\frac{|\mathbf{s}|}{\min \mathbf{s}} - 1}{|\mathbf{s}|(\frac{1}{\min \mathbf{r}} + \frac{1}{\min \mathbf{s}}) - 1} \geq 0$, this proves the right inequality in Equation (4.6) with $X = q_v^{\kappa_3}\, \epsilon^{-\kappa_4}\, Y$.

The right inequality in Equation (4.9), since $\beta_d \geq 0$ and by using the left inequality in Equation (4.8) and the definition (4.7) of α_δ, gives

$$
\begin{aligned}
\langle {}^t\!A\mathbf{x}\rangle_{\mathbf{s}} &\leq q_v^{\frac{\beta_d}{\min \mathbf{s}}}\, \delta\, Z^{-1} = q_v^{\frac{\beta_d}{\min \mathbf{s}} - \alpha_\delta - \alpha_Z}\, Y^{-1} \leq q_v^{\frac{\beta_d}{\min \mathbf{s}} - \frac{|\mathbf{s}|}{\min \mathbf{s}}\alpha_\delta + \kappa_3}\, \epsilon^{-\kappa_4}\, X^{-1} \\
&\leq q_v^{\frac{\beta_d}{\min \mathbf{s}} - \frac{|\mathbf{s}|}{\min \mathbf{s}}\Big(\frac{\alpha_\epsilon - 1}{|\mathbf{s}|\left(\frac{1}{\min \mathbf{r}} + \frac{1}{\min \mathbf{s}}\right) - 1} - 1\Big) + \kappa_3 + \frac{\frac{|\mathbf{s}|}{\min \mathbf{s}} - 1}{|\mathbf{s}|(\frac{1}{\min \mathbf{r}} + \frac{1}{\min \mathbf{s}}) - 1}\alpha_\epsilon}\; X^{-1} \\
&= q_v^{\frac{\beta_d}{\min \mathbf{s}} + \frac{|\mathbf{s}|}{\min \mathbf{s}}\Big(\frac{1}{|\mathbf{s}|\left(\frac{1}{\min \mathbf{r}} + \frac{1}{\min \mathbf{s}}\right) - 1} + 1\Big) + \kappa_3}\; \epsilon^{\frac{1}{|\mathbf{s}|(\frac{1}{\min \mathbf{r}} + \frac{1}{\min \mathbf{s}}) - 1}}\, X^{-1}\,.
\end{aligned}
$$

This proves the left inequality in Equation (4.6) for appropriate positive constants κ_1 and κ_2.

If $\mathbf{x} = 0$, then we have $\mathbf{y}' \neq 0$ and $\|\, \mathbf{y}' \|_{\mathbf{s}} \leq q_v^{\kappa_1 - \kappa_3}\, \epsilon^{\kappa_2 + \kappa_4}\, Y^{-1}$, which contradicts the fact that $\mathbf{y}' \in R_v^n$ if Y is large enough. This concludes the proof of Corollary 4.1.10. □

Corollary 4.1.11. *Let m, n be positive integers and $A \in \mathcal{M}_{m,n}(K_v)$. Then A is $(\mathbf{r}, \mathbf{s})$-singular on average if and only if ${}^t\!A$ is $(\mathbf{s}, \mathbf{r})$-singular on average.*

Proof. This follows from Corollary 4.1.10. □

It follows from this corollary and from Remark 4.1.6 that if $A \in \mathcal{M}_{m,n}(K_v)$ is such that ${}^t\!A$ is not completely irrational, then A is $(\mathbf{r}, \mathbf{s})$-singular on average.

4.2 Characterisation of singular on average property

In this section, we give a characterisation of the singular on average property with weights in terms of an asymptotic property in average of the best approximation sequence with weights. In the real case, the relation between the singular property and the best approximation sequence has been studied in [Che11, Chev13, CC16, LSST20]. Also in the real case, and with weights, the relation (similar to the one below) between the singular on average property and the best approximation sequence has been studied in [KKL, Prop. 6.7].

For the sake of later applications, we work with transposes of matrices.

Theorem 4.2.1. *Let $A \in \mathcal{M}_{m,n}(K_v)$ and let $(\mathbf{y}_i)_{i \geq 1}$ be a best approximation sequence in K_v^m for ${}^t\!A$ with weights $(\mathbf{s}, \mathbf{r})$. The following statements are equivalent.*

1. *For all $a>1$ and $\epsilon>0$, we have*

$$\lim_{N\to\infty}\frac{1}{N}\;\#\{\,\ell\in\{1,\dots,N\}:\exists\,\mathbf{y}\in R_v^m-\{0\},\ \langle {}^t\!A\,\mathbf{y}\rangle_{\mathbf{s}}\le\epsilon\,a^{-\ell},\ \|\,\mathbf{y}\,\|_{\mathbf{r}}\le a^{\ell}\,\}=1\,.$$

2. *The matrix ${}^tA$ is $(\mathbf{s},\mathbf{r})$-singular on average.*

3. *There exists $a>1$ such that for every $\epsilon>0$, we have*

$$\lim_{N\to\infty}\frac{1}{N}\;\#\{\,\ell\in\{1,\dots,N\}:\exists\,\mathbf{y}\in R_v^m-\{0\},\ \langle {}^t\!A\,\mathbf{y}\rangle_{\mathbf{s}}\le\epsilon\,a^{-\ell},\ \|\,\mathbf{y}\,\|_{\mathbf{r}}\le a^{\ell}\,\}=1\,.$$

4. *For every $\epsilon'>0$, we have*

$$\lim_{k\to\infty}\frac{1}{\log_{q_v}Y_k}\;\operatorname{card}\big\{i\le k: M_i\,Y_{i+1}>\epsilon'\big\}=0\,.$$

Proof. Since Assertion (2) is Assertion (1) for $a=q_v>1$, it is immediate that (1) implies (2) implies (3).

Let us first prove that Assertion (3) implies Assertion (4). Let $a>1$ be as in Assertion (3) and let $\epsilon'\in\,]0,1[\,$. Let $\epsilon=\frac{\epsilon'}{a}>0$.

We may assume that the set $I=\{i\in\mathbb{Z}_{\ge1}: M_iY_{i+1}>\epsilon'\}$ is infinite, otherwise Assertion (4) is clear since $\lim_{k\to\infty}Y_k=+\infty$. We consider the increasing sequence $(i_j)_{j\in\mathbb{Z}_{\ge1}}$ of positive integers such that $I=\{i_j: j\ge1\}$. For every $j\ge1$, by taking the logarithm in base a, we thus have $\log_a\epsilon'-\log_a M_{i_j}<\log_a Y_{i_j+1}$, hence

$$\log_a\epsilon-\log_a M_{i_j}<\log_a Y_{i_j+1}-1\,. \tag{4.10}$$

Note that for every $i\ge1$ and $X\in[Y_i,Y_{i+1}[\,$, the system of inequalities

$$\langle {}^t\!A\,\mathbf{y}\rangle_{\mathbf{s}}\le\epsilon\,X^{-1}\quad\text{and}\quad 0<\|\,\mathbf{y}\,\|_{\mathbf{r}}\le X \tag{4.11}$$

has a solution $\mathbf{y}\in R_v^m$ if and only if $M_i\le\epsilon\,X^{-1}$. Indeed, if the latter inequality is satisfied, then $\mathbf{y}_i$ is a solution of the system (4.11) since $M_i=\langle {}^t\!A\,\mathbf{y}_i\rangle_{\mathbf{s}}$ and $X\ge Y_i=\|\,\mathbf{y}_i\,\|_{\mathbf{r}}$. Conversely, if this system has a solution, then since

$$M_i\le\min\{\,\langle {}^t\!A\,\mathbf{y}\rangle_{\mathbf{s}}:\mathbf{y}\in R_v^m,\ 0<\|\,\mathbf{y}\,\|_{\mathbf{r}}<Y_{i+1}\,\}$$

by the definition of a best approximation sequence, the inequality $M_i\le\epsilon\,X^{-1}$ holds since $X<Y_{i+1}$. Hence, for every integer $\ell\in[\log_a Y_i,\log_a Y_{i+1}[\,$, the

system of inequalities (4.11) has no integral solutions for $X = a^\ell$ if and only if

$$\log_a \epsilon - \log_a M_i < \ell < \log_a Y_{i+1} \ . \tag{4.12}$$

There exists an integer $j_0 \geq 1$ such that for every integer $j \geq j_0$, we have $\log_a Y_{i_j+1} \geq 2$ by Lemma 4.1.7 (2) ii). If ℓ is the integer in the interval $[\log_a Y_{i_j+1} - 1, \log_a Y_{i_j+1}[$ (which is half-open and has length 1, hence does contain one and only one integer), then $\ell \geq 1$ and by Equations (4.10) and (4.12), the system (4.11) has no integral solutions for $X = a^\ell$.

Let $u = \lceil (\operatorname{lcm} \mathbf{r})(\log_{q_v} a) \rceil$, which belongs to $\mathbb{Z}_{\geq 1}$. By Lemma 4.1.7 (2) ii), for every $k \in \mathbb{Z}_{\geq 1}$, since the sequence $(i_j)_{j \in \mathbb{Z}_{\geq 1}}$ is increasing, we have

$$Y_{i_{k+u}+1} \geq q_v^{\frac{u}{\operatorname{lcm} \mathbf{r}}} \ Y_{i_k+1} \geq a \ Y_{i_k+1} \ .$$

The intervals $[\log_a Y_{i_{uj}+1} - 1, \log_a Y_{i_{uj}+1}[$ and $[\log_a Y_{i_{u(j+1)}+1} - 1, \log_a Y_{i_{u(j+1)}+1}[$ are hence disjoint for every $j \in \mathbb{Z}_{\geq 1}$. Thus, if j is large enough, with $N_j = \lceil \log_a Y_{i_{uj}+1} \rceil$, the number $n(N_j)$ of integers $\ell \in \{1, \dots, N_j\}$ such that the system of inequalities (4.11) has no integral solutions for $X = a^\ell$ is at least $j - j_0$. Therefore $\frac{j-j_0}{\lceil \log_a Y_{i_{uj}+1} \rceil} \leq \frac{n(N_j)}{N_j}$ tends to 0 as $j \to +\infty$, by Assertion (3). This implies that $\frac{j}{\log_a Y_{i_j}}$ tends to 0 as $j \to +\infty$.

For every integer $k \geq 1$, let $j(k) \geq 1$ be the unique positive integer such that we have $i_{j(k)} \leq k < i_{j(k)+1}$, so that $j(k) = \operatorname{card}\{i \leq k : M_i Y_{i+1} > \epsilon'\}$. Hence, since $(Y_i)_{i \geq 1}$ is increasing, we have

$$\lim_{k \to \infty} \frac{1}{\log_{q_v} Y_k} \ \operatorname{card}\big\{i \leq k : M_i Y_{i+1} > \epsilon'\big\} \leq \frac{\ln q_v}{\ln a} \lim_{k \to \infty} \frac{j(k)}{\log_a Y_{i_{j(k)}}} = 0 \ ,$$

which proves Assertion (4).

Let us now prove that Assertion (4) implies Assertion (1). Let $a > 1$ and $\epsilon \in \,]0, 1[$. By Lemma 4.1.7 (2) iii), let $c \geq 1$ be such that for every $i \geq 1$, we have $M_i Y_{i+1} \leq a^c$. By Equation (4.12), since the number of integer points in an open interval is at most equal to its length, for every $i \geq 1$, the number of integers $\ell \in [\log_a Y_i, \log_a Y_{i+1}[$ such that the system of inequalities (4.11) has no integral solutions for $X = a^\ell$ is at most

$$\log_a Y_{i+1} - (\log_a \epsilon - \log_a M_i) = \big(\log_a M_i Y_{i+1} - \log_a \epsilon\big) \ .$$

For every $N \geq 1$ large enough, let $k_N \geq 1$ be such that $N \in [\log_a Y_{k_N}, \log_a Y_{k_N+1}[$

and let $n'(N)$ be the number of integers $\ell \in \{1, \ldots, N\}$ such that the system of inequalities (4.11) has no integral solutions for $X = a^\ell$. Then

$$\begin{aligned}\frac{n'(N)}{N} &\leq \frac{1}{N}\sum_{i=1}^{k_N} \max\big\{0, \log_a M_i Y_{i+1} - \log_a \epsilon\big\} \\ &\leq \big(c - \log_a \epsilon\big)\, \frac{1}{\log_a Y_{k_N}}\ \operatorname{card}\big\{i \leq k_N : M_i Y_{i+1} > \epsilon\big\}\ .\end{aligned}$$

This last term tends to 0 as $N \to +\infty$ by Assertion (4) applied with $\epsilon' = \epsilon$. Therefore $\lim_{N\to+\infty} \frac{n'(N)}{N} = 0$, thus proving Assertion (1). □

4.3 Full Hausdorff dimension for singular on average matrices

4.3.1 Modified Bugeaud-Zhang sequences

In this subsection, we construct a subsequence with controlled growth of the best approximation sequence with weights of a matrix, assuming that its transpose is singular on average for those weights. We use as inspiration [BZ19, page 470] in the special case of $K = \mathbb{F}_q(Z)$ and $v = v_\infty$, and the first claim of the proof of [BKLR21, Theo. 2.2] in the case of the field $\mathbb{Q}$ (with characteristic zero).

Proposition 4.3.1. *Let $A \in \mathcal{M}_{m,n}(K_v)$ be such that ${}^tA$ is completely irrational and $(\mathbf{s}, \mathbf{r})$-singular on average. Let $(\mathbf{y}_i)_{i\in\mathbb{Z}_{\geq 1}}$ be a best approximation sequence in K_v^m for ${}^tA$ with weights $(\mathbf{s}, \mathbf{r})$, and let $c > 0$ be such that $M_i Y_{i+1} \leq q_v^c$ for every $i \in \mathbb{Z}_{\geq 1}$. For all $a > b > 0$, there exists an increasing map $\varphi : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ such that*

(1) for every $i \in \mathbb{Z}_{\geq 1}$, we have

$$Y_{\varphi(i+1)} \geq q_v^b\, Y_{\varphi(i)} \quad \text{and} \quad M_{\varphi(i)}\, Y_{\varphi(i+1)} \leq q_v^{b+c}\ , \tag{4.13}$$

(2) we have

$$\limsup_{k\to\infty} \frac{k}{\log_{q_v} Y_{\varphi(k)}} \leq \frac{1}{a}\ . \tag{4.14}$$

Proof. Let A, $(\mathbf{y}_i)_{i\in\mathbb{Z}_{\geq 1}}$ and a, b be as in the statement. We start by proving a particular case, that will be useful in two of the four cases below.

Lemma 4.3.2. *If furthermore we have* $\lim_{k\to\infty} Y_k^{\frac{1}{k}} = +\infty$*, then there exists an increasing map* $\varphi : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ *such that Equations* (4.13) *and* (4.14) *are satisfied.*

Proof. The fact that $\lim_{k\to\infty} Y_k^{\frac{1}{k}} = +\infty$ implies that the set

$$\mathcal{J}_0 = \{j \in \mathbb{Z}_{\geq 1} : Y_{j+1} \geq q_v^b \, Y_j\}$$

is infinite. We construct the increasing sequence $(\varphi(i))_{i\in\mathbb{Z}_{\geq 1}}$ of positive integers by stacks $\{\varphi(i_k+1), \ldots, \varphi(i_{k+1})\}$ with $i_{k+1} > i_k$, by induction on $k \in \mathbb{Z}_{\geq 0}$. For $k = 0$, let $i_0 = 0$, let $i_1 = 1$ and let $\varphi(1)$ be the smallest element of $\mathcal{J}_0$.

For $k \in \mathbb{Z}_{\geq 0}$, assume that i_k and $\varphi(i_k)$ are constructed such that $\varphi(i_k) \in \mathcal{J}_0$ and Equation (4.13) holds for every $i \leq i_k - 1$. Let us construct i_{k+1} and $\varphi(i_k + 1), \ldots, \varphi(i_{k+1})$ such that $\varphi(i_{k+1}) \in \mathcal{J}_0$ and Equation (4.13) holds for every $i \leq i_{k+1} - 1$. Let j_0 be the smallest element of $\mathcal{J}_0$ greater than $\varphi(i_k)$. Let $r' = 0$ if the set $\{j > \varphi(i_k) : Y_{j_0} \geq q_v^b \, Y_j\}$ is empty. Otherwise, let $r' \in \mathbb{Z}_{\geq 1}$ be the maximal integer such that by induction there exist $j_1, j_2, \ldots, j_{r'} \in \mathbb{Z}_{\geq 1}$ such that for $\ell = 1, \ldots, r'$, the set $\{j > \varphi(i_k) : Y_{j_{\ell-1}} \geq q_v^b \, Y_j\}$ is nonempty and for $\ell = 1, \ldots, r'+1$ the integer j_ℓ is its largest element. Since the sequence $(Y_i)_{i\in\mathbb{Z}_{\geq 1}}$ is increasing, this in particular implies that $j_{\ell-1} > j_\ell$ for $\ell = 1, \ldots, r'+1$, which itself ensures the finiteness of r'. Now we define $i_{k+1} = i_k + r' + 1$ and

$$\varphi(i_k + 1) = j_{r'}, \ \varphi(i_k + 2) = j_{r'-1}, \ \ldots, \ \varphi(i_k + r') = j_1, \ \varphi(i_{k+1}) = j_0 \ .$$

By construction, for $\ell = 1, \ldots, r'$, we have

$$Y_{\varphi(i_k+\ell+1)} = Y_{j_{r'-\ell}} \geq q_v^b \, Y_{j_{r'-\ell+1}} = q_v^b \, Y_{\varphi(i_k+\ell)} \ .$$

As $\varphi(i_k + 1) = j_{r'} > \varphi(i_k)$, we have $Y_{\varphi(i_k+1)} \geq Y_{\varphi(i_k)+1} \geq q_v^b \, Y_{\varphi(i_k)}$ since $\varphi(i_k) \in \mathcal{J}_0$. Note that $\varphi(i_{k+1}) = j_0 \in \mathcal{J}_0$. This proves the claim on the left hand side of Equation (4.13) for $i \leq i_{k+1} - 1$.

By the maximality property of $j_{r'-\ell}$ in the above construction, for every $\ell = 1, \ldots, r'$, we have $Y_{\varphi(i_k+\ell+1)} = Y_{j_{r'-\ell}} < q_v^b \, Y_{j_{r'-\ell+1}+1} = q_v^b \, Y_{\varphi(i_k+\ell)+1}$. By the maximality of r' in the above construction, we have $Y_{\varphi(i_k+1)} < q_v^b \, Y_{\varphi(i_k)+1}$. Hence, by the definition of c, for every $\ell = 0, \ldots, r'$, we have

$$M_{\varphi(i_k+\ell)} Y_{\varphi(i_k+\ell+1)} \leq M_{\varphi(i_k+\ell)} \, Y_{\varphi(i_k+\ell)+1} \, q_v^b \leq q_v^{b+c} \ .$$

This proves the claim on the right hand side of Equation (4.13) for $i \leq i_{k+1}-1$.

Since $\lim_{k\to\infty} \frac{k}{\log_{q_v} Y_k} = 0$, Equation (4.14) is satisfied for φ, and this concludes the proof of Lemma 4.3.2. □

Now in what follows, we will discuss four cases on the configuration in $\mathbb{Z}_{\geq 1}$ of the set

$$\mathcal{J} = \{j \in \mathbb{Z}_{\geq 1} : M_j \, Y_{j+1} \leq q_v^{b+c-3a}\} \, .$$

By Theorem 4.2.1 (4) applied with $\epsilon' = q_v^{b+c-3a}$, we have

$$\lim_{k\to\infty} \frac{1}{\log_{q_v} Y_k} \ \operatorname{card}\big\{i \leq k : i \in {}^c\mathcal{J}\big\} = 0 \, . \tag{4.15}$$

Case 1. Assume first that $\mathcal{J}$ is finite.

By Equation (4.15), we then have $\lim_{k\to\infty} \frac{k}{\log_{q_v} Y_k} = 0$, hence Proposition 4.3.1 follows from Lemma 4.3.2.

Case 2. Let us now assume that there exists $j_* \in \mathbb{Z}_{\geq 1}$ such that $j \in \mathcal{J}$ for every $j \geq j_*$.

Let us consider the auxiliary increasing sequence $(\psi(i))_{i\in\mathbb{Z}_{\geq 1}}$ of positive integers defined by induction by setting $\psi(1) = \min\{j_* \in \mathbb{Z}_{\geq 1} : \ \forall \, j \geq j_*, \ j \in \mathcal{J}\}$ and, for every $i \geq 1$,

$$\psi(i+1) = \min\{j \in \mathbb{Z}_{\geq 1} : q_v^a \, Y_{\psi(i)} \leq Y_j\} \, .$$

Since the sequence $(Y_i)_{i\in\mathbb{Z}_{\geq 1}}$ is increasing and converges to $+\infty$, this is well defined, and ψ is increasing, hence takes value in $\mathcal{J}$ by the assumption of Case 2. Let us now define the sequence $(\varphi(i))_{i\in\mathbb{Z}_{\geq 1}}$ by, for every $i \in \mathbb{Z}_{\geq 1}$,

$$\varphi(i) = \begin{cases} \psi(i) & \text{if } M_{\psi(i)} \, Y_{\psi(i+1)} \leq q_v^{b+c-a} \, , \\ \psi(i+1)-1 & \text{otherwise.} \end{cases}$$

Note that the sequence $(\varphi(i))_{i\in\mathbb{Z}_{\geq 1}}$ is increasing with $\varphi \geq \psi$.

Let $i \in \mathbb{Z}_{\geq 1}$. Let us prove that

$$Y_{\varphi(i+1)} \geq q_v^a \, Y_{\varphi(i)} \quad \text{and} \quad M_{\varphi(i)} Y_{\varphi(i+1)} \leq q_v^{b+c} \, , \tag{4.16}$$

by discussing on the values of $\varphi(i)$ and $\varphi(i+1)$. This implies that Equation (4.13) is satisfied since $a \geq b$, and that Equation (4.14) is satisfied since by induction $Y_{\varphi(k)} \geq q_v^{a\,(k-1)} \, Y_{\varphi(1)}$ for every $k \in \mathbb{Z}_{\geq 1}$.

• Assume that $\varphi(i) = \psi(i)$ and $\varphi(i+1) = \psi(i+1)$. By the definition of $\psi(i+1)$, we have

$$Y_{\varphi(i+1)} = Y_{\psi(i+1)} \geq q_v^a \, Y_{\psi(i)} = q_v^a \, Y_{\varphi(i)} \, .$$

If $\psi(i) \neq \psi(i+1) - 1$, then by the definition of $\varphi(i)$, we have

$$M_{\varphi(i)} \, Y_{\varphi(i+1)} = M_{\psi(i)} \, Y_{\psi(i+1)} \leq q_v^{b+c-a} \leq q_v^{b+c} \, .$$

If $\psi(i) = \psi(i+1) - 1$, then $\varphi(i+1) = \varphi(i) + 1$ and by the definition of c, we have

$$M_{\varphi(i)} \, Y_{\varphi(i+1)} = M_{\varphi(i)} \, Y_{\varphi(i)+1} \leq q_v^c \leq q_v^{b+c} \, .$$

This proves Equation (4.16).

• Assume that $\varphi(i) = \psi(i)$ and $\varphi(i+1) = \psi(i+2) - 1$. Since the sequence $(Y_i)_{i \in \mathbb{Z}_{\geq 1}}$ is increasing and by the definition of $\psi(i+1)$, we have

$$Y_{\varphi(i+1)} = Y_{\psi(i+2)-1} \geq Y_{\psi(i+1)} \geq q_v^a \, Y_{\psi(i)} = q_v^a \, Y_{\varphi(i)} \, .$$

We have $q_v^a \, Y_{\psi(i+1)} > Y_{\psi(i+2)-1}$ by the minimality property of $\psi(i+2)$. If $\psi(i+1) > \psi(i) + 1$, then $M_{\psi(i)} \, Y_{\psi(i+1)} \leq q_v^{b+c-a}$ by the dichotomy in the definition of $\varphi(i)$. Hence

$$M_{\varphi(i)} \, Y_{\varphi(i+1)} = M_{\psi(i)} \, Y_{\psi(i+2)-1} \leq M_{\psi(i)} \, Y_{\psi(i+1)} \, q_v^a \leq q_v^{b+c-a} q_v^a = q_v^{b+c} \, .$$

If $\psi(i+1) = \psi(i) + 1$, then $M_{\psi(i)} \, Y_{\psi(i)+1} \leq q_v^{b+c-3a}$ since $\psi(i) \in \mathcal{J}$. Hence

$$M_{\varphi(i)} \, Y_{\varphi(i+1)} = M_{\psi(i)} \, Y_{\psi(i+2)-1} \leq M_{\psi(i)} \, Y_{\psi(i)+1} \, q_v^a \leq q_v^{b+c-3a} q_v^a \leq q_v^{b+c} \, .$$

This proves Equation (4.16).

• Assume that $\varphi(i) = \psi(i+1) - 1$ and $\varphi(i+1) = \psi(i+1)$. Since $\psi(i+1) - 1 \in \mathcal{J}$, we have

$$M_{\varphi(i)} Y_{\varphi(i+1)} = M_{\psi(i+1)-1} Y_{\psi(i+1)} \leq q_v^{b+c-3a} \leq q_v^{b+c} \, .$$

If $\psi(i+1) - 1 = \psi(i)$, then by the definition of $\psi(i+1)$, we have

$$\frac{Y_{\varphi(i+1)}}{Y_{\varphi(i)}} = \frac{Y_{\psi(i+1)}}{Y_{\psi(i+1)-1}} = \frac{Y_{\psi(i+1)}}{Y_{\psi(i)}} \geq q_v^a \, .$$

If $\psi(i+1)-1 > \psi(i)$, then we have $M_{\psi(i)}\ Y_{\psi(i+1)} > q_v^{b+c-a}$ by the dichotomy in the definition of $\varphi(i)$, we have $Y_{\psi(i+1)-1} < q_v^a\ Y_{\psi(i)} \leq q_v^a\ Y_{\psi(i)+1}$ by the minimality property of $\psi(i+1)$, and we have $M_{\psi(i)}\ Y_{\psi(i)+1} \leq q_v^{b+c-3a}$ since $\psi(i) \in \mathscr{J}$. Therefore

$$\frac{Y_{\varphi(i+1)}}{Y_{\varphi(i)}} = \frac{Y_{\psi(i+1)}}{Y_{\psi(i+1)-1}} = \frac{M_{\psi(i)}\ Y_{\psi(i+1)}}{M_{\psi(i)}\ Y_{\psi(i+1)-1}} \geq \frac{q_v^{b+c-a}}{M_{\psi(i)}\ Y_{\psi(i)+1}\ q_v^a} \geq \frac{q_v^{b+c-a}}{q_v^{b+c-3a}\ q_v^a} = q_v^a\ .$$

This proves Equation (4.16).

• Assume that $\varphi(i) = \psi(i+1)-1$ and $\varphi(i+1) = \psi(i+2)-1$. By the previous case computations, we have

$$\frac{Y_{\varphi(i+1)}}{Y_{\varphi(i)}} = \frac{Y_{\psi(i+2)-1}}{Y_{\psi(i+1)-1}} \geq \frac{Y_{\psi(i+1)}}{Y_{\psi(i+1)-1}} \geq q_v^a\ .$$

We have $q_v^a\ Y_{\psi(i+1)} > Y_{\psi(i+2)-1}$ by the minimality property of $\psi(i+2)$. Hence since $\psi(i+1)-1 \in \mathscr{J}$, we have

$$\begin{aligned} M_{\varphi(i)} Y_{\varphi(i+1)} &= M_{\psi(i+1)-1} Y_{\psi(i+2)-1} = M_{\psi(i+1)-1} Y_{\psi(i+1)} \Big(\frac{Y_{\psi(i+2)-1}}{Y_{\psi(i+1)}}\Big) \\ &\leq q_v^{b+c-3a}\ q_v^a \leq q_v^{b+c}\ . \end{aligned}$$

This proves Equation (4.16) and concludes the proof of Case 2.

Case 3. Let us now assume that $\mathscr{J}$ and ${}^c\mathscr{J}$ are both infinite, and that the number of sequences of consecutive elements of $\mathscr{J}$ with length at least $3a$ is finite.

Let $j_0 = \min \mathscr{J}$. Let us write the set $\mathbb{Z}_{\geq j_0} = \bigcup_{i \in \mathbb{Z}_{\geq 1}} C_i \cup D_i$ as the disjoint union of nonempty finite sequences C_i of consecutive integers in $\mathscr{J}$ and finite nonempty sequences D_i of consecutive integers in ${}^c\mathscr{J}$ with $\max C_i < \min D_i \leq \max D_i < \min C_{i+1}$ for all $i \in \mathbb{Z}_{\geq 1}$. Under the assumption of Case 3, let $i_0 \in \mathbb{Z}_{\geq 1}$ be such that card $C_i < 3a$ for every $i \geq i_0$. Let $k_0 = \min C_{i_0}$.

Then there exists an element of ${}^c\mathscr{J}$ in any finite sequence of $3\lceil a \rceil + 1$ consecutive integers at least k_0, so that for every $k \in \mathbb{Z}_{\geq 1}$ we have

$$\frac{k}{\log_{q_v} Y_k} \leq \frac{k_0 + (3\lceil a \rceil + 1)\ \operatorname{card}\big\{i \leq k : i \in {}^c\mathscr{J}\big\}}{\log_{q_v} Y_k}\ ,$$

which converges to 0 as $k \to +\infty$ by Equation (4.15) and since $\lim_{k\to\infty}\ Y_k =$

$+\infty$. Therefore $\lim_{k\to\infty} Y_k^{\frac{1}{k}} = +\infty$, and Lemma 4.3.2 implies Proposition 4.3.1.

Case 4. Let us finally assume that $\mathscr{J}$ and $\mathscr{C}\mathscr{J}$ are both infinite, and that there are infinitely many sequences of consecutive elements of $\mathscr{J}$ with length at least $3a$.

With the notation $(C_i)_{i\in\mathbb{Z}_{\geq 1}}$ and $(D_i)_{i\in\mathbb{Z}_{\geq 1}}$ of the beginning of Case 3, let $(i_k)_{k\in\mathbb{Z}_{\geq 1}}$ be the increasing sequence of positive integers such that $\{i \in \mathbb{Z}_{\geq 1} : \text{card } C_i \geq 3a\} = \{i_k : k \in \mathbb{Z}_{\geq 1}\}$.

For every $k \in \mathbb{Z}_{\geq 1}$, let us define an increasing finite sequence $(\psi_k(i))_{1\leq i\leq m_k+1}$ of positive integers by setting $\psi_k(1) = \min C_{i_k}$ and by induction

$$\psi_k(i+1) = \min\{\, j \in C_{i_k} : q_v^a \, Y_{\psi_k(i)} \leq Y_j \,\} ,$$

as long as this set is nonempty. Since C_{i_k} is a finite sequence of consecutive positive integers with length at least $3a$ and $Y_{i+1} \geq q_v^{\frac{1}{\min \mathbf{r}}} \, Y_i$ for every $i \in \mathbb{Z}_{\geq 1}$, there exists $m_k \in \mathbb{Z}_{\geq 2}$ such that $\psi_k(i)$ is defined for $i = 1, \ldots, m_k + 1$. Note that $\psi_k(i)$ belongs to $\mathscr{J}$ for $i = 1, \ldots, m_{k+1}$ since $C_{i_k} \subset \mathscr{J}$.

As in Case 2, let us define an increasing finite sequence $(\varphi_k(i))_{1\leq i\leq m_k}$ of positive integers by

$$\varphi_k(i) = \begin{cases} \psi_k(i) & \text{if } M_{\psi_k(i)} \, Y_{\psi_k(i+1)} \leq q_v^{b+c-a} , \\ \psi_k(i+1) - 1 & \text{otherwise.} \end{cases}$$

As in the proof of Case 2, since for $i = 1, \ldots, m_k$, the integers $\psi_k(i)$, $\psi_k(i+1)$ as well as $\psi_k(i+1) - 1$ belong to $\mathscr{J}$, we have, for every $i = 1, \ldots, m_k - 1$,

$$Y_{\varphi_k(i+1)} \geq q_v^a \, Y_{\varphi_k(i)} \quad \text{and} \quad M_{\varphi_k(i)} \, Y_{\varphi_k(i+1)} \leq q_v^{b+c} \,. \tag{4.17}$$

Since $\varphi_k(m_k) \in C_{i_k}$ and $\varphi_{k+1}(1) \in C_{i_{k+1}}$, we have $\varphi_k(m_k) < \varphi_{k+1}(1)$. Let us define an increasing finite sequence $(\varphi'_k(i))_{1\leq i\leq r_k+1}$ of positive integers that will allow us to interpolate between $\varphi_k(m_k)$ and $\varphi_{k+1}(1)$. Let $j_0 = \varphi_{k+1}(1)$. If $\{j \in \mathbb{Z}_{\geq \varphi_k(m_k)} : Y_{j_0} \geq q_v^b \, Y_j\}$ is empty, let $r'_k = 0$ and $\varphi'_k(1) = j_0 = \varphi_{k+1}(1)$. Otherwise, by decreasing induction, let $r'_k \in \mathbb{Z}_{\geq 1}$ be the maximal positive integer such that there exist $j_1, \ldots, j_{r'_k} \in \mathbb{Z}_{\geq 1}$ such that for $\ell = 1, \ldots, r'_k$, the set $\{j \in \mathbb{Z}_{\geq \varphi_k(m_k)} : Y_{j_{\ell-1}} \geq q_v^b \, Y_j\}$ is nonempty and for $\ell = 1, \ldots, r'_k + 1$, the integer j_ℓ is its largest element. As in the part of the proof of Case 1 that does not need some belonging to $\mathscr{J}_0$, the sequence $(\varphi'_k(i) = j_{r'_k+1-i})_{1\leq i\leq r'_k+1}$ is well defined, it is contained in $[\varphi_k(m_k), \varphi_{k+1}(1)]$, and for $i = 1, \ldots, r'_k$, we have

$$Y_{\varphi'_k(i+1)} \geq q_v^b \, Y_{\varphi'_k(i)} \quad \text{and} \quad M_{\varphi'_k(i)} \, Y_{\varphi'_k(i+1)} \leq q_v^{b+c} \,. \tag{4.18}$$

Putting alternatively together the sequences $(\varphi_k(i))_{1 \leq i \leq m_k - 1}$ and $(\varphi'_k(i))_{1 \leq i \leq r'_k}$ as k ranges over $\mathbb{Z}_{\geq 1}$, we now define (with the standard convention that an empty sum is zero) $N_k = \sum_{\ell=1}^{k-1} (m_\ell - 1 + r'_\ell)$ and

$$\varphi(i) = \begin{cases} \varphi_k\big(i - N_k\big) & \text{if } 1 + N_k \leq i \leq m_k - 1 + N_k \\ \varphi'_k\big(i + 1 - m_k - N_k\big) & \text{if } m_k + N_k \leq i \leq r'_k - 1 + m_k + N_k \,. \end{cases}$$

By Equation (4.17) for $i = 1, \ldots, m_k - 2$, by Equation (4.18) for $i = 1, \ldots, r'_k$, and since $\varphi'_k(r'_k + 1) = \varphi_{k+1}(1)$, in order to prove that the map φ satisfies Equation (4.13), hence Assertion (1) of Proposition 4.3.1, we only have to prove the following lemma.

Lemma 4.3.3. *For every $k \in \mathbb{Z}_{\geq 1}$, we have*

$$Y_{\varphi'_k(1)} \geq q_v^b \, Y_{\varphi_k(m_k - 1)} \quad \text{and} \quad M_{\varphi_k(m_k - 1)} \, Y_{\varphi'_k(1)} \leq q_v^{b+c} \,. \tag{4.19}$$

Proof. Since $\varphi'_k(1) \geq \varphi_k(m_k)$, hence $Y_{\varphi'_k(1)} \geq Y_{\varphi_k(m_k)}$, the left hand side of Equation (4.19) follows from the left hand side of Equation (4.17) with $i = m_k - 1$. If $\varphi'_k(1) = \varphi_k(m_k)$, then the right hand side of Equation (4.19) follows from the right hand side of Equation (4.17) with $i = m_k - 1$.

Let us hence assume that $\varphi'_k(1) > \varphi_k(m_k)$, so that

$$Y_{\varphi'_k(1)} \leq q_v^b \, Y_{\varphi_k(m_k)} \leq q_v^a \, Y_{\varphi_k(m_k)} \tag{4.20}$$

by the maximality of r'_k. Let us prove that $\varphi_k(m_k) = \psi_k(m_k)$. For a contradiction, assume otherwise that $\varphi_k(m_k) = \psi_k(m_k{+}1) - 1 > \psi_k(m_k)$. As in the third subcase of Case 2, we have $M_{\psi_k(m_k)} \, Y_{\psi_k(m_k+1)} > q_v^{b+c-a}$ by the dichotomy in the definition of $\varphi_k(m_k)$, we have $Y_{\psi_k(m_k+1)-1} < q_v^a \, Y_{\psi_k(m_k)} \leq q_v^a \, Y_{\psi_k(m_k)+1}$ by the minimality property of $\psi_k(m_k + 1)$, and we have $M_{\psi_k(m_k)} \, Y_{\psi_k(m_k)+1} \leq q_v^{b+c-3a}$ since $\psi_k(m_k) \in \mathcal{J}$. Therefore, as in the third subcase of Case 2, we have

$$\frac{Y_{\psi_k(m_k+1)}}{Y_{\psi_k(m_k+1)-1}} = \frac{M_{\psi_k(m_k)} \, Y_{\psi_k(m_k+1)}}{M_{\psi_k(m_k)} \, Y_{\psi_k(m_k+1)-1}} \geq q_v^a \,.$$

Hence by the construction of $\varphi'_k(1)$, we have $\varphi'_k(1) = \varphi_k(m_k)$, a contradiction to our assumption that $\varphi'_k(1) > \varphi_k(m_k)$. We now discuss on the two possible values of $\varphi_k(m_k-1)$.

First assume that $\varphi_k(m_k-1)=\psi_k(m_k-1)$. If $\psi_k(m_k-1)\neq\psi_k(m_k)-1$ then $M_{\psi_k(m_k-1)}\ Y_{\psi_k(m_k)}\leq q_v^{b+c-a}$ by the dichotomy in the definition of $\varphi_k(m_k-1)$. If on the contrary $\psi_k(m_k-1)=\psi_k(m_k)-1$ then $M_{\psi_k(m_k-1)}\ Y_{\psi_k(m_k)}\leq q_v^{b+c-3a}\leq q_v^{b+c-a}$ since the integer $\psi_k(m_k)-1$ belong to $\mathcal{J}$ as $m_k\geq 2$. Since $\varphi_k(m_k)=\psi_k(m_k)$ by Equation (4.20), we have

$$M_{\varphi_k(m_k-1)}\ Y_{\varphi'_k(1)}=M_{\psi_k(m_k-1)}\ Y_{\psi_k(m_k)}\Big(\frac{Y_{\varphi'_k(1)}}{Y_{\varphi_k(m_k)}}\Big)\leq q_v^{b+c-a}\ q_v^{a}=q_v^{b+c}\ .$$

This proves the right hand side of Equation (4.19).

Now assume that $\varphi_k(m_k-1)=\psi_k(m_k)-1$. Again since $\varphi_k(m_k)=\psi_k(m_k)$, since the integer $\psi_k(m_k)-1$ belongs to $\mathcal{J}$ as $m_k\geq 2$, and by Equation (4.20), we have

$$M_{\varphi_k(m_k-1)}\ Y_{\varphi'_k(1)}=M_{\psi_k(m_k)-1}\ Y_{\psi_k(m_k)}\Big(\frac{Y_{\varphi'_k(1)}}{Y_{\varphi_k(m_k)}}\Big)\leq q_v^{b+c-3a}\ q_v^{a}\leq q_v^{b+c}\ .$$

This proves the right hand side of Equation (4.19), and concludes the proof of Lemma 4.3.3. □

Finally, let us prove Assertion (2) of Proposition 4.3.1. Since there exists an element of $\mathscr{J}$ in any finite sequence of $3\lceil a\rceil+1$ consecutive integers in the complement of $\bigcup_{k\in\mathbb{Z}_{\geq 1}}C_{i_k}$, there exists $c_0\geq 0$ such that, for every $k\in\mathbb{Z}_{\geq1}$, we have

$$\frac{\operatorname{card}\{j\leq\varphi(k):j\notin\bigcup_{k\in\mathbb{Z}_{\geq 1}}C_{i_k}\}}{\log_{q_v}Y_{\varphi(k)}}\leq\frac{c_0+(3\lceil a\rceil+1)\ \operatorname{card}\big\{j\leq\varphi(k):j\in\ \mathscr{J}\big\}}{\log_{q_v}Y_{\varphi(k)}}\ ,$$

which converges to 0 as $k\to+\infty$ as seen at the end of the proof of Case 3. Let us define $n(k)=\operatorname{card}\{i\leq k:Y_{\varphi(i)}\geq q_v^a\ Y_{\varphi(i+1)}\}$. For every $\ell\in\mathbb{Z}_{\geq1}$, since $Y_{j+1}\geq q_v^{\frac{1}{\min\mathbf{r}}}\,Y_j$ for every $j\in\mathbb{Z}_{\geq1}$, and by the maximality of m_ℓ in the construction of $\big(\varphi_\ell(i)\big)_{1\leq i\leq m_\ell}$, we have $\operatorname{card}\{j\in C_{i_\ell}:j\geq\varphi_\ell(m_\ell)\}\leq 2\lceil a\rceil\min\mathbf{r}$. If $\varphi(i)$ belongs to C_{i_ℓ} but $\varphi(i+1)$ does not, then $\varphi(i)\geq\varphi_\ell(m_\ell)$. Since when $\varphi(i)$ and $\varphi(i+1)$ belong to C_{i_ℓ} for some $\ell\in\mathbb{Z}_{\geq1}$, then φ and φ_ℓ coincide on i and $i+1$, and since Equation (4.17) holds, we hence have

$$k-n(k)=\#\{i\leq k:Y_{\varphi(i)}<q_v^a\ Y_{\varphi(i+1)}\}\leq 2\lceil a\rceil\ \min\mathbf{r}\ \#\{j\leq\varphi(k):j\notin\bigcup_{k\in\mathbb{Z}_{\geq1}}C_{i_k}\}\ .$$

Hence

$$\limsup_{k\to+\infty} \frac{k}{\log_{q_v} Y_{\varphi(k)}} = \limsup_{k\to+\infty} \frac{n(k)+k-n(k)}{\log_{q_v} Y_{\varphi(k)}} = \limsup_{k\to+\infty} \frac{n(k)}{\log_{q_v} Y_{\varphi(k)}}$$
$$\leq \limsup_{k\to+\infty} \frac{n(k)}{\log_{q_v} q_v^{a(n(k)-1)} Y_{\varphi(1)}} = \frac{1}{a}\ .$$

This proves Equation (4.14) and concludes the proof of Proposition 4.3.1. □

4.3.2 Lower bound on the Hausdorff dimension of $\mathbf{Bad}_A(\epsilon)$

In this subsection, we use the scheme of proof in the real case of [CGGMS20, Theo. 6.1], which is a weighted version of [BKLR21, Theo. 5.1], in order to estimate the lower bound on the Hausdorff dimension of the ϵ-bad sets of $(\mathbf{r},\mathbf{s})$-singular in average matrices.

For a given sequence $(\mathbf{y}_i)_{i\geq 1}$ in $R_v^m - \{0\}$ and for every $\delta > 0$, let

$$\mathbf{Bad}^{\delta}_{(\mathbf{y}_i)_{i\geq 1}} = \{\boldsymbol{\theta} \in (\pi_v\mathcal{O}_v)^m :\ \forall\, i\geq 1,\ \ |\langle \boldsymbol{\theta}\cdot \mathbf{y}_i\rangle| \geq \delta\,\}\ .$$

Proposition 4.3.4. *Let $A\in\mathcal{M}_{m,n}(K_v)$ be such that ${}^tA$ is completely irrational and let $(\mathbf{y}_i)_{i\geq 1}$ be a best approximation sequence in K_v^m for ${}^tA$ with weights $(\mathbf{s},\mathbf{r})$. Suppose that there exist $b,c>0$ and an increasing function $\varphi:\mathbb{Z}_{\geq 1}\to\mathbb{Z}_{\geq 1}$ such that*

$$\forall\, i\in\mathbb{Z}_{\geq 1},\quad M_{\varphi(i)}Y_{\varphi(i+1)} \leq q_v^{b+c}\ .$$

Then for every $\delta\in\,]0,1]$, if $\epsilon = \delta^{\frac{1}{\min \mathbf{r}}+\frac{1}{\min \mathbf{s}}}\ q_v^{-b-c}$, then the set $\mathbf{Bad}^{\delta}_{(\mathbf{y}_{\varphi(i)})_{i\geq 1}}$ is contained in the set $\mathbf{Bad}_A(\epsilon)$.

Proof. Fix $\delta\in\,]0,1]$ and $\boldsymbol{\theta}\in\mathbf{Bad}^{\delta}_{(\mathbf{y}_{\varphi(i)})_{i\geq 1}}$. Let $\epsilon_1 = \delta^{\frac{1}{\min\mathbf{s}}}\ q_v^{-b-c}$. For every $(\mathbf{y}',\mathbf{x}')$ in $R_v^m\times R_v^n$ such that $\|\,\mathbf{x}'\,\|_{\mathbf{s}} \geq \epsilon_1 Y_{\varphi(1)}$, let k be the unique element of $\mathbb{Z}_{\geq 1}$ for which

$$Y_{\varphi(k)} \leq \epsilon_1^{-1}\|\,\mathbf{x}'\,\|_{\mathbf{s}} < Y_{\varphi(k+1)}\ ,$$

which exists since $\|\,\mathbf{x}'\,\|_{\mathbf{s}} \geq \epsilon_1 Y_{\varphi(1)}$ and since the sequence $(Y_{\varphi(i)})_{i\geq 1}$ is increasing, converging to $+\infty$. Let $\mathbf{x}_{\varphi(k)}\in R_v^n$ be such that $M_{\varphi(k)} = \|\,{}^tA\mathbf{y}_{\varphi(k)} - \mathbf{x}_{\varphi(k)}\|_{\mathbf{s}}$. Then by the ultrametric inequality, the assumption of the proposition, the fact that $\epsilon_1\, q_v^{b+c} = \delta^{\frac{1}{\min \mathbf{s}}} \le 1$ and the definition of $\mathbf{Bad}^{\delta}_{(\mathbf{y}_{\varphi(i)})_{i\ge1}}$, we have

$$|(\,{}^t\!A\mathbf{y}_{\varphi(k)} - \mathbf{x}_{\varphi(k)})\cdot \mathbf{x}'| \le \max_{1\le i\le n} M_{\varphi(k)}^{s_i} \|\,\mathbf{x}'\,\|_{\mathbf{s}}^{s_i} < \max_{1\le i\le n} (\epsilon_1 M_{\varphi(k)} Y_{\varphi(k+1)})^{s_i}$$

$$\le (\epsilon_1\, q_v^{b+c})^{\min \mathbf{s}} = \delta \le \min_{\ell'\in R_v} |\,\mathbf{y}_{\varphi(k)}\cdot \boldsymbol{\theta} - \ell'\,|\,. \tag{4.21}$$

Observe that

$$\begin{aligned}\mathbf{y}_{\varphi(k)}\cdot\boldsymbol{\theta} &= \mathbf{y}_{\varphi(k)}\cdot(A\,\mathbf{x}') + \mathbf{y}_{\varphi(k)}\cdot\mathbf{y}' - \mathbf{y}_{\varphi(k)}\cdot(A\,\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta})\\ &= (\,{}^t\!A\,\mathbf{y}_{\varphi(k)})\cdot\mathbf{x}' - \mathbf{x}_{\varphi(k)}\cdot\mathbf{x}' + \ell - \mathbf{y}_{\varphi(k)}\cdot(A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta}),\end{aligned}$$

where $\ell = \mathbf{x}_{\varphi(k)}\cdot\mathbf{x}' + \mathbf{y}_{\varphi(k)}\cdot\mathbf{y}' \in R_v$. Thus we have, using the equality case of the ultrametric inequality for the second equality below with the strict inequality in Equation (4.21), and again the definition of $\mathbf{Bad}^{\delta}_{(\mathbf{y}_{\varphi(i)})_{i\ge1}}$ for the last inequality below,

$$\begin{aligned}|\mathbf{y}_{\varphi(k)}\cdot(A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta})| &= |(\,{}^t\!A\mathbf{y}_{\varphi(k)} - \mathbf{x}_{\varphi(k)})\cdot\mathbf{x}' - \mathbf{y}_{\varphi(k)}\cdot\boldsymbol{\theta} + \ell\,|\\ &= \max\big\{|(\,{}^t\!A\mathbf{y}_{\varphi(k)} - \mathbf{x}_{\varphi(k)})\cdot\mathbf{x}'|, |\,\mathbf{y}_{\varphi(k)}\cdot\boldsymbol{\theta} - \ell\,|\big\}\\ &= |\,\mathbf{y}_{\varphi(k)}\cdot\boldsymbol{\theta} - \ell\,| \ge |\langle\mathbf{y}_{\varphi(k)}\cdot\boldsymbol{\theta}\rangle| \ge \delta.\end{aligned}$$

Hence, we have

$$\delta \le |\mathbf{y}_{\varphi(k)}\cdot(A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta})| \le \max_{1\le j\le m} Y_{\varphi(k)}^{r_j}\ \|A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta}\|_{\mathbf{r}}^{r_j},$$

which implies, since $\delta \le 1$, that

$$Y_{\varphi(k)}\|A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta}\|_{\mathbf{r}} \ge \min_{1\le j\le m} \delta^{\frac{1}{r_j}} = \delta^{\frac{1}{\min \mathbf{r}}}.$$

Finally, for every $(\mathbf{y}', \mathbf{x}')$ in $R_v^m \times R_v^n$ such that $\|\,\mathbf{x}'\,\|_{\mathbf{s}} \ge \epsilon_1 Y_{\varphi(1)}$, we have

$$\|\,\mathbf{x}'\,\|_{\mathbf{s}}\ \|A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta}\|_{\mathbf{r}} \ge \epsilon_1\ Y_{\varphi(k)}\ \|A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta}\|_{\mathbf{r}} \ge \delta^{\frac{1}{\min\mathbf{r}} + \frac{1}{\min\mathbf{s}}}\ q_v^{-b-c}\,.$$

By Equation (1.7), this implies that $\boldsymbol{\theta} \in \mathbf{Bad}_A(\epsilon)$ for $\epsilon = \delta^{\frac{1}{\min\mathbf{r}} + \frac{1}{\min\mathbf{s}}}\ q_v^{-b-c}$. □

Proposition 4.3.5. *For every $\delta \in\ ]0, \frac{1}{q_v^{3m}}[$, there exist $b = b(\delta) > 0$ and $C = C(\delta) > 0$ such that for every sequence $(\mathbf{y}_i)_{i\in\mathbb{Z}_{\ge1}}$ in $R_v^m - \{0\}$ satisfying*

$\| \mathbf{y}_{i+1} \|_{\mathbf{r}} \geq q_v^b \| \mathbf{y}_i \|_{\mathbf{r}}$ *for all* $i \in \mathbb{Z}_{\geq 1}$*, we have*

$$\dim_H \mathbf{Bad}^{\delta}_{(\mathbf{y}_i)_{i\geq 1}} \geq m - C \limsup_{k\to\infty} \frac{k}{\log_{q_v} \| \mathbf{y}_k \|_{\mathbf{r}}} .$$

Proof. Fix $\delta \in \,]0, \frac{1}{q_v^{3m}}[$. Let

$$b = b(\delta) = \frac{-\log_{q_v} \delta}{\min \mathbf{r}} , \tag{4.22}$$

which is positive since $\delta < 1$. By the mass distribution principle (see for instance [Fal14, page 60]), it is enough to prove that there exist a (Borel, positive) measure μ, supported on $\mathbf{Bad}^{\delta}_{(\mathbf{y}_i)_{i\geq 1}}$, and constants $C, C_0, r_0 > 0$, with C depending only on δ, such that, for every closed ball B of radius $r < r_0$, we have

$$\mu(B) \leq C_0 \; r^{m - C \limsup_{k\to\infty} \frac{k}{\log_{q_v} \| \mathbf{y}_k \|_{\mathbf{r}}}} .$$

We adapt by modifying it quite a lot the measure construction in the proof of [CGGMS20, Theo. 6.1].

By convention, let $Y_0 = 1$ and $n_{0,j} = 0$ for $j = 1, \dots, m$. For every $k \in \mathbb{Z}_{\geq 1}$, define $Y_k = \| \mathbf{y}_k \|_{\mathbf{r}}$, which is at least 1 since $\mathbf{y}_k \in R_v^m - \{0\}$, and for every $j = 1, \dots, m$, let $n_{k,j} \in \mathbb{Z}_{\geq 0}$ be such that

$$q_v^{-n_{k,j}} \leq Y_k^{-r_j} < q_v^{-n_{k,j}+1} . \tag{4.23}$$

Note that the sequence $(n_{k,j})_{k\in\mathbb{Z}_{\geq 0}}$ is nondecreasing, for all $j = 1, \dots, m$.

For every $k \in \mathbb{Z}_{\geq 0}$, let us consider the polydisc

$$\Pi(Y_k) = \overline{B}(0, \frac{1}{q_v} Y_k^{-r_1}) \times \cdots \times \overline{B}(0, \frac{1}{q_v} Y_k^{-r_m}) = \overline{B}(0, q_v^{-n_{k,1}-1}) \times \cdots \times \overline{B}(0, q_v^{-n_{k,m}-1}) ,$$

where $\overline{B}(0, r')$ is the closed ball of radius $r' > 0$ and center 0 in K_v. Note that $\Pi(Y_0) = (\pi_v \mathcal{O}_v)^m$ is the open unit ball of K_v^m and that $\Pi(Y_k)$ is an additive subgroup of K_v^m. Since the residual field $k_v = \mathcal{O}_v / \pi_v \mathcal{O}_v$ lifts as a subfield of order q_v of K_v, for every $\ell \in \mathbb{Z}_{\geq 0}$, we have a disjoint union

$$\overline{B}(0, q_v^{-\ell}) = \bigsqcup_{a \in k_v} \big(a\, \pi_v^{\ell} + \overline{B}(0, q_v^{-\ell-1}) \big) .$$

Hence by induction, the polydisc $\Pi(Y_k)$ is the disjoint union of

$$\Delta_{k+1} = \prod_{1\leq j\leq m} q_v^{n_{k+1,j}-n_{k,j}}$$

translates of the polydisc $\Pi(Y_{k+1})$. Note that

$$\Delta_{k+1} \geq \prod_{1\leq j\leq m} Y_{k+1}^{r_j} Y_k^{-r_j} q_v^{-1} = q_v^{-m} \left(Y_{k+1}Y_k^{-1}\right)^{|\mathbf{r}|} . \tag{4.24}$$

For every $k \in \mathbb{Z}_{\geq 0}$, let us fix some elements $\theta_{1,k+1}, \ldots, \theta_{\Delta_{k+1},k+1}$ in $(\pi_v\mathcal{O}_v)^m$ (which are not unique in the ultrametric space K_v^m) such that

$$\Pi(Y_k) = \bigsqcup_{i=1}^{\Delta_{k+1}} \left(\theta_{i,k+1} + \Pi(Y_{k+1})\right) .$$

By convention, let us define $Z_{0,\delta} = \emptyset$ and $I_0 = \{\Pi(Y_0)\}$. For every $k \in \mathbb{Z}_{\geq 1}$, let us define

$$Z_{k,\delta} = \{\boldsymbol{\theta} \in (\pi_v\mathcal{O}_v)^m : |\langle \mathbf{y}_k \cdot \boldsymbol{\theta}\rangle| < \delta\}$$

and

$$I_k = \left\{\theta_{i_1,1} + \cdots + \theta_{i_k,k} + \Pi(Y_k) : \forall\, j \in \{1, \ldots, k\},\ \ 1 \leq i_j \leq \Delta_j\right\} .$$

Lemma 4.3.6. *For every $k \in \mathbb{Z}_{\geq 1}$, we have*

(1) for every $I' \in I_{k+1}$, if $I' \cap Z_{k,\delta} \neq \emptyset$ then $I' \subset Z_{k,\delta}$,

(2) for every $I \in I_k$, we have $\mathrm{vol}_v^m(I \cap Z_{k,\delta}) \leq \delta\, Y_k^{-|\mathbf{r}|}$.

Proof. (1) If $I' \in I_{k+1}$ and $I' \cap Z_{k,\delta} \neq \emptyset$, let $\boldsymbol{\theta} \in I' \cap Z_{k,\delta}$. Then for every $\boldsymbol{\theta}' \in I'$, if $x, x' \in R_v$ are such that $|\langle \mathbf{y}_k \cdot \boldsymbol{\theta}\rangle| = |(\mathbf{y}_k \cdot \boldsymbol{\theta}) - x|$ and $|\langle \mathbf{y}_k \cdot (\boldsymbol{\theta}' - \boldsymbol{\theta})\rangle| = |(\mathbf{y}_k \cdot (\boldsymbol{\theta}' - \boldsymbol{\theta})) - x'|$, then by the ultrametric inequality, since $\boldsymbol{\theta} \in Z_{k,\delta}$ and $\boldsymbol{\theta}' - \boldsymbol{\theta} \in \Pi(Y_{k+1})$, by the assumption of Proposition 4.3.5, and by the definition of b, we have

$$\begin{aligned} |\langle \mathbf{y}_k \cdot \boldsymbol{\theta}'\rangle| &\leq |\mathbf{y}_k \cdot (\boldsymbol{\theta} + (\boldsymbol{\theta}' - \boldsymbol{\theta})) - (x + x')| \leq \max\left\{|\langle \mathbf{y}_k \cdot \boldsymbol{\theta}\rangle|,\ |\langle \mathbf{y}_k \cdot (\boldsymbol{\theta}' - \boldsymbol{\theta})\rangle|\right\} \\ &\leq \max\left\{\delta, \max_{1\leq j\leq m} Y_k^{r_j} \frac{1}{q_v}\, Y_{k+1}^{-r_j}\right\} \leq \max\left\{\delta, q_v^{-1-b\min \mathbf{r}}\right\} = \delta\ . \end{aligned}$$

This inequality $|\langle \mathbf{y}_k \cdot \boldsymbol{\theta}'\rangle| \leq \delta$ is actually strict, since $|\langle \mathbf{y}_k \cdot \boldsymbol{\theta}\rangle| < \delta$ and by

Equation (4.22), we have $q_v^{-1-b\min\mathbf{r}} = q_v^{-1}\delta < \delta$. Since I' is contained in $\Pi(Y_0) = (\pi_v\mathcal{O}_v)^m$, we thus have that $\boldsymbol{\theta}' \in Z_{k,\delta}$ and this proves Assertion (1).

(2) Let $j_0 \in \{1,\dots,m\}$ be such that $Y_k = |y_{k,j_0}|^{1/r_{j_0}}$ where $\mathbf{y}_k = (y_{k,1},\dots,y_{k,m})$. In particular, y_{k,j_0} is nonzero. For every $z \in R_v$, let

$$L_k(z) = \{\boldsymbol{\theta} \in K_v^m : \mathbf{y}_k \cdot \boldsymbol{\theta} = z\}\ ,$$

which is an affine hyperplane of K_v^m transverse to the j_0-axis, and let

$$\mathcal{N}(k,z) = \{\boldsymbol{\theta}' \in (\pi_v\mathcal{O}_v)^m : \exists\ \mathbf{u}' \in L_k(z),\ |\theta'_{j_0} - u'_{j_0}| \leq \delta Y_k^{-r_{j_0}} \text{ and } \forall j \neq j_0, \theta'_j = u'_j\},$$

which is the intersection with the open unit ball in K_v^m of the $(\delta\ Y_k^{-r_{j_0}})$-thickening along the j_0-axis of the affine hyperplane $L_k(z)$.

Fix $I \in I_k$. Since $\operatorname{vol}_v(\overline{B})(0,r')) = q_v^{\lfloor \log_{q_v} r' \rfloor} \leq r'$ for all $r' > 0$, and by Fubini's theorem, we have

$$\operatorname{vol}_v^m(I \cap \mathcal{N}(k,z)) \leq \delta\ Y_k^{-r_{j_0}} \prod_{j \neq j_0} Y_k^{-r_j} = \delta\ Y_k^{-|\mathbf{r}|}\ . \tag{4.25}$$

Claim 1. Let us prove that the set $Z_{k,\delta}$ is contained in the union of the sets $\mathcal{N}(k,z)$ for $z \in R_v$.

Proof. Let $\boldsymbol{\theta} = (\theta_1,\dots,\theta_m) \in Z_{k,\delta}$ and let $z \in R_v$ be such that $|\langle \mathbf{y}_k \cdot \boldsymbol{\theta}\rangle| = |\,\mathbf{y}_k \cdot \boldsymbol{\theta} - z\,|$. Let us define $u_j = \theta_j$ if $j \neq j_0$,

$$u_{j_0} = \frac{z - \sum_{j \neq j_0} y_{k,j}\theta_j}{y_{k,j_0}}$$

and $\mathbf{u} = (u_1,\dots,u_m)$, which is the projection of $\boldsymbol{\theta}$ on the affine hyperplane $L_k(z)$ along the j_0-axis. Then, since $\boldsymbol{\theta} \in Z_{k,\delta}$, we have

$$|\theta_{j_0} - u_{j_0}| = \frac{|\,\mathbf{y}_k \cdot \boldsymbol{\theta} - z\,|}{|y_{k,j_0}|} = \frac{|\langle \mathbf{y}_k \cdot \boldsymbol{\theta}\rangle|}{|y_{k,j_0}|} \leq \delta\ Y_k^{-r_{j_0}}\ .$$

Since $Z_{k,\delta}$ is contained in $(\pi_v\mathcal{O}_v)^m$, this proves Claim 1. □

Claim 2. Let us prove that there exists a unique $z \in R_v$ such that $I \cap Z_{k,\delta}$ is contained in $I \cap \mathcal{N}(k,z)$.

Proof. By Claim 1, the set $I\cap Z_{k,\delta}$ is contained in $\bigcup_{z\in R_v} I\cap\mathcal{N}(k,z)$. Assume for a contradiction that there exist two distinct elements z, z' in R_v such that there exist $\boldsymbol{\theta}\in I\cap\mathcal{N}(k,z)$ and $\boldsymbol{\theta}'\in I\cap\mathcal{N}(k,z')$. Let $\mathbf{u}\in L_k(z)$ and $\mathbf{u}'\in L_k(z')$ be the projections of $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ along the j_0-axis on $L_k(z)$ and $L_k(z')$ respectively.

Let $j\in\{1,\dots,m\}$. Note that $\boldsymbol{\theta}-\boldsymbol{\theta}'\in\Pi(Y_k)$ since $I\in I_k$. If $j\neq j_0$, then

$$|u_j-u'_j|=|\theta_j-\theta'_j|\leq\frac{1}{q_v}\,Y_k^{-r_j}\ .$$

Furthermore, by the ultrametric inequality, since $\boldsymbol{\theta}$ (respectively $\boldsymbol{\theta}'$) is contained in the $(\delta\ Y_k^{-r_{j_0}})$-thickening along the j_0-axis of $L_k(z)$ (respectively $L_k(z')$), and since $\delta\leq\frac{1}{q_v}$, we have

$$\begin{aligned}|u_{j_0}-u'_{j_0}|&=|(u_{j_0}-\theta_{j_0})+(\theta_{j_0}-\theta'_{j_0})+(\theta'_{j_0}-u'_{j_0})|\\&\leq\max\{|u_{j_0}-\theta_{j_0}|,\ |\theta_{j_0}-\theta'_{j_0}|,\ |\theta'_{j_0}-u'_{j_0}|\}\\&\leq\max\{\delta\,Y_k^{-r_{j_0}},\ \frac{1}{q_v}\,Y_k^{-r_{j_0}}\}=\frac{1}{q_v}\,Y_k^{-r_{j_0}}\ .\end{aligned}$$

This implies since $\mathbf{u}\in L_k(z)$ and $u'\in L_k(z)$ that

$$1\leq|z-z'|=|\mathbf{y}_k\cdot\mathbf{u}-\mathbf{y}_k\cdot\mathbf{u}'|\leq\max_{1\leq j\leq m}|y_{k,j}|\,|u_j-u'_j|\leq\max_{1\leq j\leq m}Y_k^{r_j}\,\frac{1}{q_v}\,Y_k^{-r_j}=\frac{1}{q_v}\ ,$$

which is a contradiction since $q_v>1$. This proves Claim 2. □

By Equation (4.25), Claim 2 concludes the proof of Assertion (2) of Lemma 4.3.6. □

Since every element I' of I_{k+1} is a translate of $\Pi(Y_{k+1})$, and by Equation (4.23), we have

$$\mathrm{vol}_v^m(I')=\mathrm{vol}_v^m(\Pi(Y_{k+1}))=\prod_{j=1}^m q_v^{-n_{k+1,\,j}-1}\geq q_v^{-2m}\,Y_{k+1}^{-|\mathbf{r}|}\ .$$

For every $I\in I_k$, there are Δ_{k+1} elements $I'\in I_{k+1}$ contained in I, they are pairwise disjoint and they have the same volume $\mathrm{vol}_v^m(\Pi(Y_{k+1}))$. Among them, those who meet $Z_{k,\delta}$ are actually contained in $I\cap Z_{k,\delta}$ by Lemma 4.3.6 (1), thus their number is at most $\frac{\mathrm{vol}_v^m(I\cap Z_{k,\delta})}{\mathrm{vol}_v^m(\Pi(Y_{k+1}))}$. Therefore, by Equation (4.24) and Lemma 4.3.6 (2), we have

$$\begin{aligned}\operatorname{card}\{I'\in I_{k+1}: I'\subset I,\ \ I'\cap Z_{k,\delta}=\emptyset\} &\geq \Delta_{k+1}-\frac{\operatorname{vol}_v^m(I\cap Z_{k,\delta})}{\operatorname{vol}_v^m(\Pi(Y_{k+1}))}\\ &\geq q_v^{-m}(Y_{k+1}Y_k^{-1})^{|\mathbf{r}|}-\frac{\delta\, Y_k^{-|\mathbf{r}|}}{q_v^{-2m}Y_{k+1}^{-|\mathbf{r}|}}\\ &= c_1\,(Y_{k+1}Y_k^{-1})^{|\mathbf{r}|}\ , \end{aligned}\tag{4.26}$$

where $c_1=q_v^{-m}-q_v^{2m}\delta$ belongs to $]0,1[$ by the assumption on δ.

Now, let us define by induction $J_0=I_0$ and for every $k\in\mathbb{Z}_{\geq 0}$,

$$J_{k+1}=\bigcup_{J\in J_k}\{I\in I_{k+1}: I\subset J,\ \ I\cap Z_{k,\delta}=\emptyset\}.$$

By Equation (4.26) and by induction, we have

$$\operatorname{card}\ J_{k+1}\geq\prod_{j=1}^{k}c_1\,(Y_{j+1}Y_j^{-1})^{|\mathbf{r}|}=c_1^k\,(Y_{k+1}Y_1^{-1})^{|\mathbf{r}|}\ .\tag{4.27}$$

By Lemma 4.3.6 (1) and by induction, we have

$$J_{k+1}=\{J\in I_{k+1}:\forall j\in\{1,\dots,k\},\ J\cap Z_{j,\delta}=\emptyset\}=\{J\in I_{k+1}: J\subset\bigcap_{j=1}^{k}{}^cZ_{j,\delta}\},$$

where ^{c} denotes the complement in $(\pi_v\mathcal{O}_v)^m$. Hence $\big(\bigcup J_k\big)_{k\geq 1}$ is a decreasing sequence of compact subsets of $(\pi_v\mathcal{O}_v)^m$, whose intersection is contained in $\bigcap_{k\geq 1}{}^cZ_{k,\delta}=\mathbf{Bad}^{\delta}_{(\mathbf{y}_i)_{i\geq 1}}$.

For every $k\in\mathbb{Z}_{\geq 0}$, let us define a measure

$$\mu_k=\big(\operatorname{vol}_v^m(\Pi(Y_k))\ \operatorname{card}\ J_k\big)^{-1}\sum_{J\in J_k}\operatorname{vol}_v^m|_J\ ,$$

which is a probability measure with support $\bigcup J_k$. By the compactness of $(\pi_v\mathcal{O}_v)^m$, any weakstar accumulation point μ of the sequence $(\mu_k)_{k\geq 1}$ is a probability measure with support in $\mathbf{Bad}^{\delta}_{(\mathbf{y}_i)_{i\geq 1}}$.

For every closed ball B in $(\pi_v\mathcal{O}_v)^m$ with radius $r'\in\,]0,r_0=Y_1^{-\min\mathbf{r}}]$, let

$k \in \mathbb{Z}_{\geq 1}$ be such that

$$Y_{k+1}^{-\min \mathbf{r}} < r' \leq Y_k^{-\min \mathbf{r}} . \tag{4.28}$$

Note that $\lceil t \rceil \leq t+1 \leq q_v\, t$ if $t \geq 1$, and that $r'\, q_v^{n_{k+1,j}+1} \geq Y_{k+1}^{-\min \mathbf{r}}\, Y_{k+1}^{r_j}\, q_v \geq 1$ for every $j = 1, \ldots, m$, by Equation (4.23). Then B can be covered by a subset of I_{k+1} with cardinality at most

$$\prod_{j=1}^{m} \lceil r'\, q_v^{n_{k+1,j}+1} \rceil \leq (r')^m\, q_v^{3m}\, Y_{k+1}^{|\mathbf{r}|} .$$

Let $C = \frac{-\log_{q_v} c_1}{\min \mathbf{r}} > 0$, which depends (besides on m, q_v and $\mathbf{r}$) only on δ. Defining $C_0 = q_v^{3m}\, Y_1^{|\mathbf{r}|}$, by Equations (4.27) and (4.28), we thus have

$$\begin{aligned} \mu_{k+1}(B) &\leq q_v^{3m}\, (r')^m\, Y_{k+1}^{|\mathbf{r}|}\, (\text{card } J_{k+1})^{-1} \leq q_v^{3m}\, (r')^m\, c_1^{-k}\, Y_1^{|\mathbf{r}|} \\ &\leq C_0\, (r')^{m - C \frac{k}{\log_{q_v} Y_k}} . \end{aligned}$$

Therefore, since the ball B is closed and open and since $r' \leq r_0 \leq 1$, we have

$$\mu(B) \leq \limsup_{k \to \infty} C_0\, (r')^{m - C \frac{k}{\log_{q_v} Y_k}} = C_0\, (r')^{m - C \limsup_{k \to \infty} \frac{k}{\log_{q_v} Y_k}},$$

which concludes the proof of Proposition 4.3.5. □

4.3.3 Proof that Assertion (2) implies Assertion (1) in Theorem 1.3.1

Suppose that A is $(\mathbf{r}, \mathbf{s})$-singular on average. Then by Corollary 4.1.11, the matrix ${}^tA$ is also $(\mathbf{s}, \mathbf{r})$-singular on average. By Remark 4.1.6 (2), in order to prove that there exists $\epsilon > 0$ such that $\mathbf{Bad}_A(\epsilon)$ has full Hausdorff dimension, we may assume that the matrix ${}^tA$ is completely irrational.

By Lemma 4.1.7, let $(\mathbf{y}_k)_{k \in \mathbb{Z}_{\geq 1}}$ be a best approximation sequence in K_v^m for the matrix ${}^tA$ with weights $(\mathbf{s}, \mathbf{r})$, and let $c > 0$ be such that $M_i Y_{i+1} \leq q_v^c$ for every $i \in \mathbb{Z}_{\geq 1}$. Fix some $\delta \in \left]0, \frac{1}{q_v^{3m}}\right[$ and let $b = b(\delta) > 0$ and $C = C(\delta) > 0$ as in Proposition 4.3.5. By Proposition 4.3.1, for every $a > b$, we have a subsequence $(\mathbf{y}_{\varphi(k)})_{k \geq 1}$ such that the properties (4.13) and (4.14) are satisfied. Proposition 4.3.4, whose assumption is satisfied by the second inequality in Equation (4.13) and where $\epsilon = \delta^{\frac{1}{\min \mathbf{r}} + \frac{1}{\min \mathbf{s}}}\, q_v^{-b-c}$, gives that

$\mathbf{Bad}_A(\epsilon)$ contains $\mathbf{Bad}^{\delta}_{(\mathbf{y}_{\varphi(i)})_{i\geq 1}}$. Therefore, using Proposition 4.3.5 applied to the sequence $(\mathbf{y}_{\varphi(i)})_{i\geq 1}$, whose assumption is satisfied by the first inequality in Equation (4.13), and using Equation (4.14) for the last inequality, we have

$$\dim_H \mathbf{Bad}_A(\epsilon) \geq \dim_H \mathbf{Bad}^{\delta}_{(\mathbf{y}_{\varphi(i)})_{i\geq 1}} \geq m - C \limsup_{k\to\infty} \frac{k}{\log_{q_v} Y_{\varphi(k)}} \geq m - \frac{C}{a}\ .$$

Letting a tend to $+\infty$, this concludes the proof that Assertion (2) implies Assertion (1) in Theorem 1.3.1. □

4.4 Background material for the upper bound

4.4.1 Homogeneous dynamics

Let $K_v, \mathcal{O}_v, \pi_v, R_v, q_v$ be as in Subsection 4.1.1. Let $m, n \in \mathbb{N} - \{0\}$ and $d = m + n$. We fix some weights $\mathbf{r} = (r_1, \ldots, r_m)$ and $\mathbf{s} = (s_1, \ldots, s_n)$ as in the introduction. In this subsection, we introduce the space of unimodular grids $\mathcal{Y}$ in $K_v^{\,d}$ and the diagonal flow $(\mathfrak{a}^\ell)_{\ell\in\mathbb{Z}}$ acting on this space. Let

$$G_0 = \mathrm{SL}_d(K_v) \quad \text{and} \quad G = \mathrm{ASL}_d(K_v) = \mathrm{SL}_d(K_v) \ltimes K_v^{\,d},$$

and let

$$\Gamma_0 = \mathrm{SL}_d(R_v) \quad \text{and} \quad \Gamma = \mathrm{ASL}_d(R_v) = \mathrm{SL}_d(R_v) \ltimes R_v^{\,d}.$$

The product in G is given by

$$(g, u) \cdot (g', u') = (gg', u + gu') \tag{4.29}$$

for all $g, g' \in G_0$ and $u, u' \in K_v^{\,d}$. We also view G as a subgroup of $\mathrm{SL}_{d+1}(K_v)$ by

$$G = \left\{ \begin{pmatrix} g & u \\ 0 & 1 \end{pmatrix} : g \in \mathrm{SL}_d(K_v),\ u \in K_v^{\,d} \right\}\ .$$

We shall identify G_0 with the corresponding subgroup of G. We consider the one-parameter diagonal subgroup $(\mathfrak{a}^\ell)_{\ell\in\mathbb{Z}}$ of G_0, where $\mathfrak{a} = \mathrm{diag}\,(()\,\mathfrak{a}_-, \mathfrak{a}_+)$ and

$$\mathfrak{a}_- = \mathrm{diag}\left(\pi_v^{-r_1}, \ldots, \pi_v^{-r_m}\right) \in \mathrm{GL}_m(K_v) \ \text{ and } \ \mathfrak{a}_+ = \mathrm{diag}\left(\pi_v^{s_1}, \ldots, \pi_v^{s_n}\right) \in \mathrm{GL}_n(K_v)\ .$$

Note that for all $\boldsymbol{\theta} \in K_v^m$, $\boldsymbol{\xi} \in K_v^n$ and $\ell \in \mathbb{Z}$, we have

$$\| \mathfrak{a}_-^\ell \, \boldsymbol{\theta} \, \|_{\mathbf{r}} = q_v^\ell \, \| \, \boldsymbol{\theta} \, \|_{\mathbf{r}} \quad \text{and} \quad \| \, \mathfrak{a}_+^\ell \, \boldsymbol{\xi} \, \|_{\mathbf{s}} = q_v^{-\ell} \, \| \, \boldsymbol{\xi} \, \|_{\mathbf{s}} \, . \tag{4.30}$$

We denote by G^+ the unstable horospherical subgroup for $\mathfrak{a}$ in G and by U the unipotent radical of G, that is,

$$G^+ = \{ g \in G : \lim_{\ell \to -\infty} \mathfrak{a}^\ell g \, \mathfrak{a}^{-\ell} = I_{d+1} \} \quad \text{and} \quad U = \left\{ \begin{pmatrix} I_d & u \\ 0 & 1 \end{pmatrix} : u \in K_v^{\,d} \right\} .$$

Let $U^+ = G^+ \cap U = \left\{ \begin{pmatrix} I_m & 0 & w \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix} : w \in K_v^{\,m} \right\}$, which is a closed subgroup in G^+ normalized by $\mathfrak{a}$.

Let us define

$$\mathcal{X} = G_0 / \Gamma_0 \quad \text{and} \quad \mathcal{Y} = G / \Gamma \, .$$

Even though we have $\operatorname{Covol}(R_v^d) = q^{(g-1)d}$ by Equation (4.2), we say that an R_v-lattice Λ in K_v^d is *unimodular* if $\operatorname{Covol}(\Lambda) = \operatorname{Covol}(R_v^d)$. A translate in the affine space K_v^d of an unimodular lattice is called an *unimodular grid*. We identify the homogeneous space $\mathcal{X} = \operatorname{SL}_d(K_v) / \operatorname{SL}_d(R_v)$ with the space of unimodular lattices in K_v^d by the equivariant homeomorphism

$$x = g \, \Gamma_0 \mapsto \Lambda_x = g \, R_v^d \, ,$$

and the homogeneous space $\mathcal{Y} = \operatorname{ASL}_d(K_v) / \operatorname{ASL}_d(R_v)$ with the space of unimodular grids by the equivariant homeomorphism

$$y = \begin{pmatrix} g & u \\ 0 & 1 \end{pmatrix} \Gamma \mapsto \widetilde{\Lambda}_y = g \, R_v^d + u \, . \tag{4.31}$$

We denote by $\pi : \mathcal{Y} \to \mathcal{X}$ the natural projection map (forgetting the translation factor), which is a proper map. Note that the fibers of π are exactly the orbits of U in $\mathcal{Y}$, and in particular each orbit under U^+ in $\mathcal{Y}$ is contained in some fiber of π (see Lemma 4.4.3 for a precise understanding of the U^+-orbits).

For every $N \in \mathbb{N} - \{0\}$, we denote by $d_{\operatorname{SL}_N(K_v)}$ the right-invariant distance on $\operatorname{SL}_N(K_v)$ defined by for all $g, h \in \operatorname{SL}_N(K_v)$

$$d_{\operatorname{SL}_N(K_v)}(g, h) = \max\{ \ln(1 + \|\!| \, g h^{-1} - \operatorname{id} \, |\!\|), \; \ln(1 + \|\!| \, h g^{-1} - \operatorname{id} \, |\!\|) \} \, ,$$

where $|||\ \ |||$ is the operator norm on $\mathcal{M}_N(K_v)$ defined by the sup norm $\|\ \|$ on K_v^N. We endow every closed subgroup H of G with the right-invariant distance d_H on H, which is the restriction to H of the distance $d_{\mathrm{SL}_{d+1}(K_v)}$. For instance, identifying the additive group K_v^m with U^+ by the map $w \mapsto \widehat{w} = \begin{pmatrix} I_m & 0 & w \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we have

$$\forall\, w, w' \in K_v^m, \quad d_{U^+}(\widehat{w}, \widehat{w'}) = \ln(1 + \|w - w'\|), \tag{4.32}$$

We also consider the distance $d_{U^+,m}$ on U^+ induced from the norm $\|\cdot\|$ on K_v^m, that is,

$$\forall\, w, w' \in K_v^m, \quad d_{U^+,m}(\widehat{w}, \widehat{w'}) = \|w - w'\|. \tag{4.33}$$

Then it is clear that $(U^+, d_{U^+,m})$ is isometric to $(K_v^m, \|\cdot\|)$. On the other hand, observe that $(K_v^m, \|\cdot\|)$ or $(U^+, d_{U^+,m})$ are locally bi-Lipschitz to (U^+, d_{U^+}). So, we fix small $0 < r_0 < 1$ such that for any $w, w' \in K_v^m$

$$d_{U^+}(\widehat{w}, \widehat{w'}) < r_0 \quad \Longrightarrow \quad \frac{1}{2}\|w - w'\| \leq d_{U^+}(\widehat{w}, \widehat{w'}) \leq \|w - w'\|. \tag{4.34}$$

We endow $\mathcal{Y} = G/\Gamma$ with the quotient distance $d_{\mathcal{Y}}$ of the distance d_G on G, defined by

$$\forall\, y, y' \in \mathcal{Y}, \quad d_{\mathcal{Y}}(y, y') = \min_{\gamma \in \Gamma}\ d_G(\,\widetilde{y}\,\gamma,\, \widetilde{y}\,')$$

for any representatives $\widetilde{y}$ and $\widetilde{y}\,'$ of the classes y and y' in G/Γ respectively. This is a well defined distance since the canonical projection $G \to \mathcal{Y}$ is a covering map and the distance d_G on G is right-invariant. Given any closed subgroup H of G, we denote by $B_H(x, r)$ (respectively $B_{\mathcal{Y}}(x, r)$) the open ball of center x and radius $r > 0$ for the distance d_H (respectively $d_{\mathcal{Y}}$), and by B_r^H the open ball $B_H(\mathrm{id}, r)$. Note that for all $y \in \mathcal{Y}$ and $r > 0$, we have (for the left action of subsets of G on $\mathcal{Y}$)

$$B_{\mathcal{Y}}(y, r) = B_r^G y\ .$$

In particular, we denote by $B_r^{U^+,m}$ the open ball of center id and radius $r > 0$ for the distance $d_{U^+,m}$ on U^+.

Lemma 4.4.1. *For all* $\epsilon > 0$ *and* $k \in \mathbb{Z}_{\geq 0}$*, we have*

$$\mathfrak{a}^{-k} B_\epsilon^{U^+} \mathfrak{a}^k \subset B^{U^+}_{\ln(1+\epsilon\, q_v^{-k \min \mathbf{r}})} \quad \textit{and} \quad \mathfrak{a}^{-k} B_\epsilon^{U^+,m} \mathfrak{a}^k \subset B^{U^+,m}_{\epsilon\, q_v^{-k \min \mathbf{r}}} .$$

Similary, we have

$$\mathfrak{a}^{k} B_\epsilon^{U^+} \mathfrak{a}^{-k} \subset B^{U^+}_{\ln(1+\epsilon\, q_v^{k \max \mathbf{r}})} \quad \textit{and} \quad \mathfrak{a}^{k} B_\epsilon^{U^+,m} \mathfrak{a}^{-k} \subset B^{U^+,m}_{\epsilon\, q_v^{k \max \mathbf{r}}} .$$

Proof. The proof of the second claim being similar, we only prove the first one. For every $w = (w_1, \dots, w_m) \in K_v^m$, we have $\mathfrak{a}^{-k} \widehat{w}\, \mathfrak{a}^k = \widehat{\mathfrak{a}_-^{-k} w}$ and

$$\| \,\mathfrak{a}_-^{-k} w \,\| = \max_{1 \leq i \leq m} |\, \pi_v^{r_i k} w_i \,| \leq q_v^{-k \min \mathbf{r}} \;\| \,w\, \| .$$

The result hence follows from Equations (4.32) and (4.33). □

Given a point x in $\mathcal{Y}$ (and similarly for x in $\mathcal{X}$), we define the *injectivity radius* of $\mathcal{Y}$ at x to be

$$\operatorname{inj}(x) = \sup \big\{ r > 0 : \forall\, \gamma \in \Gamma - \{\mathrm{id}\},\;\; B_G(\widetilde{x}, r) \cap B_G(\widetilde{x}\,\gamma, r) = \emptyset \big\} ,$$

which does not depend on the choice of $\widetilde{x} \in G$ such that $x = \widetilde{x}\,\Gamma$, and is positive and finite since the canonical projection $G \to \mathcal{Y}$ is a nontrivial covering map. For every $r > 0$, we denote the *r-thick part* of $\mathcal{Y}$ by

$$\mathcal{Y}(r) = \{ x \in \mathcal{Y} : \operatorname{inj}(x) \geq r \} .$$

It follows from the finiteness of a (quotient) Haar measure of $\mathcal{Y}$ that $\mathcal{Y}(r)$ is a compact subset of $\mathcal{Y}$ for every $r > 0$, and that the Haar measure of the *r-thin part* $\mathcal{Y} - \mathcal{Y}(r)$ tends to 0 as r goes to 0. For every compact subset K of $\mathcal{Y}$, there exists $r > 0$ such that $K \subset \mathcal{Y}(r)$.

4.4.2 Dani correspondence

In this subsection, we give an interpretation of the property for a matrix $A \in \mathcal{M}_{m,n}(K_v)$ to be $(\mathbf{r}, \mathbf{s})$-singular on average in terms of dynamical properties of the action of the one-parameter diagonal subgroup $(\mathfrak{a}^\ell)_{\ell \in \mathbb{Z}}$ on the space of unimodular lattices, as originally developped by Dani (see for instance [Kle99, §4]). For every $A \in \mathcal{M}_{m,n}(K_v)$, let $u_A = \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} \in G_0$.

Proposition 4.4.2. *A matrix $A \in \mathcal{M}_{m,n}(K_v)$ is $(\mathbf{r}, \mathbf{s})$-singular on average if and only if the forward orbit $\{\mathfrak{a}^\ell u_A R_v^d : \ell \in \mathbb{Z}_{\geq 0}\}$ in $\mathcal{X}$ of the lattice $u_A R_v^d$ under $\mathfrak{a}$* diverges on average *in $\mathcal{X}$, that is, if and only if for any compact subset Q of $\mathcal{X}$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \;\mathrm{card}\{\ell \in \{1, \cdots, N\} : \mathfrak{a}^\ell \, u_A \, \Gamma_0 \in Q\} = 0\ .$$

Proof. Let Q be a compact subset of $\mathcal{X}$. By Mahler's compactness criterion (see for instance [KST17, Theo. 1.1]), there exists $\varepsilon \in \,]0, 1[$ such that Q is contained in

$$\mathcal{X}_{>\epsilon} = \{g\, R_v^d \in \mathcal{X} : \forall\, (\boldsymbol{\theta}, \boldsymbol{\xi}) \in g\, R_v^d - \{0\} \subset K_v^{\,m} \times K_v^{\,n},\ \ \max\{\|\,\boldsymbol{\theta}\,\|_{\mathbf{r}}, \|\,\boldsymbol{\xi}\,\|_{\mathbf{s}}\} > \varepsilon\}\ ,$$

which is the subset of $\mathcal{X}$ consisting of the unimodular lattices with systole (for an appropriate quasinorm) larger than ϵ. Observe that by Equation (4.30), for all sufficiently large $\ell \in \mathbb{Z}_{\geq 1}$, there exists an element $\mathbf{y} \in R_v^n - \{0\}$ such that $\langle A\,\mathbf{y}\rangle_{\mathbf{r}} \leq \varepsilon q_v^{-\ell}$ and $\|\,\mathbf{y}\,\|_{\mathbf{s}} \leq \varepsilon q_v^{\ell}$ if and only if we have $\mathfrak{a}^\ell u_A R_v^d = \begin{pmatrix} \mathfrak{a}_-^\ell & 0 \\ 0 & \mathfrak{a}_+^\ell \end{pmatrix} \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} R_v^d \in \mathcal{X} - \mathcal{X}_{>\varepsilon}$.

With $\ell_\epsilon = \lfloor -\log_{q_v} \varepsilon \rfloor$, it follows that

$$\begin{aligned}
0 \leq\ & \mathrm{card}\{\,\ell \in \{1, \cdots, N\} : \mathfrak{a}^\ell\, u_A\, R_v^d \in Q\,\} \\
\leq\ & \mathrm{card}\{\,\ell \in \{1, \cdots, N\} : \mathfrak{a}^\ell\, u_A\, R_v^d \in \mathcal{X}_{>\varepsilon}\,\} \\
=\ & \mathrm{card}\{\ell \in \{1, \cdots, N\} : \nexists\, \mathbf{y} \in R_v^n - \{0\},\ \ \langle A\,\mathbf{y}\rangle_{\mathbf{r}} \leq \varepsilon q_v^{-\ell},\ \ \|\,\mathbf{y}\,\|_{\mathbf{s}} \leq \varepsilon q_v^{\ell}\} \\
\leq\ & \mathrm{card}\{\ell \in \{1, \cdots, N\} : \nexists\, \mathbf{y} \in R_v^n - \{0\},\ \ \langle A\,\mathbf{y}\rangle_{\mathbf{r}} \leq \frac{\varepsilon^2}{q_v} q_v^{-(\ell-\ell_\epsilon)},\ \ \|\,\mathbf{y}\,\|_{\mathbf{s}} \leq q_v^{\ell-\ell_\epsilon}\} \\
\leq\ & \ell_\epsilon + \mathrm{card}\{\ell \in \{1, \cdots, N-\ell_\epsilon\} : \nexists \mathbf{y} \in R_v^n - \{0\},\ \langle A\,\mathbf{y}\rangle_{\mathbf{r}} \leq \frac{\varepsilon^2}{q_v} q_v^{-\ell},\ \|\,\mathbf{y}\,\|_{\mathbf{s}} \leq q_v^{\ell}\}.
\end{aligned}$$

After dividing by N (or equivalently by $N - \ell_\epsilon$) this last expression, its limit as N tends to 0 exists and is equal to 0 if A is $(\mathbf{r}, \mathbf{s})$-singular on average (see Equation (1.8)). Hence we have $\lim_{N\to\infty} \frac{1}{N}$ card$\{\ell \in \{1, \cdots, N\} : \mathfrak{a}^\ell\, u_A\, \Gamma_0 \in Q\} = 0$ by the above string of (in)equalities.

The converse implication follows similarly by taking for the compact set Q the subset $\mathcal{X}_{>\varepsilon}$. □

We denote by $\| \ \|_{\mathbf{s},\mathbf{r}}$ the quasi-norm on $K_v^d = K_v^m \times K_v^n$ defined by

$$\| (\boldsymbol{\theta}, \boldsymbol{\xi}) \|_{\mathbf{r},\mathbf{s}} = \max \left\{ \| \boldsymbol{\theta} \|_{\mathbf{r}}^{\frac{d}{m}}, \ \| \boldsymbol{\xi} \|_{\mathbf{s}}^{\frac{d}{n}} \right\} .$$

Let $\varepsilon > 0$. We define

$$\mathcal{L}_\varepsilon = \{ y \in \mathcal{Y} : \forall \, u \in \widetilde{\Lambda}_y, \ \ \| u \|_{\mathbf{r},\mathbf{s}} \geq \varepsilon \} . \tag{4.35}$$

By Mahler's compactness criterion (see for instance [KST17, Theo. 1.1]) and since the natural projection $\pi : \mathcal{Y} \to \mathcal{X}$ is proper, the subset $\mathcal{L}_\varepsilon$ is compact.

For every $\boldsymbol{\theta} \in K_v^m$, we denote by $y_{A,\boldsymbol{\theta}}$ the unimodular grid $u_A R_v^d - \binom{\boldsymbol{\theta}}{0}$.

Lemma 4.4.3. *For every $A \in \mathcal{M}_{m,n}(K_v)$, the map $K_v^m \to \mathcal{Y}$ defined by $\boldsymbol{\theta} \mapsto y_{A,\boldsymbol{\theta}}$ induces a local bi-Lipschitz map ϕ_A from $\mathbb{T}^m = K_v^m / R_v^m$ endowed with the quotient distance $d_{\mathbb{T}^m}$ of the distance on K_v^m defined by the standard norm $\| \ \|$, and the U^+-orbit $U^+ y_{A,0}$ endowed with the restriction of the distance $d_{\mathcal{Y}}$ of $\mathcal{Y}$. In particular, the map ϕ_A is isometry onto the U^+-orbit $U^+ y_{A,0}$ endowed with the distance $d_{U^+,m}$ of U^+ in Equation* (4.33)*.*

Proof. The map $K_v^m \to \mathcal{Y}$ defined by $\boldsymbol{\theta} \mapsto y_{A,\boldsymbol{\theta}}$ is clearly invariant under translations by R_v^m, and induces a bijection

$$\phi_A : \boldsymbol{\theta} \mod R_v^m \mapsto y_{A,\boldsymbol{\theta}} \tag{4.36}$$

from $\mathbb{T}^m = K_v^m / R_v^m$ to the orbit $U^+ y_{A,0}$. This orbit is contained in the fiber $\pi^{-1}(x_A)$ of $x_A = u_A \, R_v^m$ for the natural projection $\pi : \mathcal{Y} \to \mathcal{X}$, as already seen.

For all $A \in \mathcal{M}_{m,n}(K_v)$ and $\boldsymbol{\theta} \in K_v^m$, let $u_{A,\boldsymbol{\theta}} = \begin{pmatrix} I_m & A & \boldsymbol{\theta} \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G$, so that we have $y_{A,\boldsymbol{\theta}} = u_{A,-\boldsymbol{\theta}} \Gamma$. For all $\boldsymbol{\theta}, \boldsymbol{\theta}' \in \mathbb{T}^m$, denoting lifts of them to K_v^m by $\widetilde{\boldsymbol{\theta}}, \widetilde{\boldsymbol{\theta}}'$ respectively, identifying K_v^d with $K_v^m \times K_v^n$, and using Equation (4.29) and right-invariance of d_G,

$$\begin{aligned} d_{\mathcal{Y}}(\phi_A(\boldsymbol{\theta}), \phi_A(\boldsymbol{\theta}')) &= \inf_{x \in R_v^m} d_{U^+}\big((\mathrm{id}, (x - \widetilde{\boldsymbol{\theta}}, 0)), (\mathrm{id}, (-\widetilde{\boldsymbol{\theta}}', 0))\big) \\ &= \inf_{x \in R_v^m} \ln(1 + \| \widetilde{\boldsymbol{\theta}} - \widetilde{\boldsymbol{\theta}}' - x \|). \end{aligned}$$

Thus it follows from Equation (4.34) that if $d_{\mathbb{T}^m}(\boldsymbol{\theta}, \boldsymbol{\theta}') < q_v^{-\ell_0}$, then

$$\frac{1}{2} d_{\mathbb{T}^m}(\boldsymbol{\theta}, \boldsymbol{\theta}') \leq d_{\mathcal{Y}}(\phi_A(\boldsymbol{\theta}), \phi_A(\boldsymbol{\theta}')) \leq d_{\mathbb{T}^m}(\boldsymbol{\theta}, \boldsymbol{\theta}') \ .$$

□

Proposition 4.4.4. *Let $\varepsilon > 0$. For every $(A, \boldsymbol{\theta}) \in \mathcal{M}_{m,n}(K_v) \times K_v^m$ such that $\boldsymbol{\theta} \in \mathbf{Bad}_A(\varepsilon)$, one of the following statements holds.*

1. *There exists $\mathbf{y} \in R_v^n$ such that $\langle A\,\mathbf{y} - \boldsymbol{\theta} \rangle_{\mathbf{r}} = 0$. Note that given A, there are only countably many $\boldsymbol{\theta}$ satisfying this statement.*
2. *The forward $\mathfrak{a}$-orbit of the point $y_{A,\boldsymbol{\theta}}$ is eventually in $\mathcal{L}_\varepsilon$, that is, there exists $T \geq 0$ such that for every $\ell \geq T$, we have $\mathfrak{a}^\ell\, y_{A,\boldsymbol{\theta}} \in \mathcal{L}_\varepsilon$.*

Proof. Assume for a contradiction that both statements do not hold. Then there exist infinitely many $\ell \in \mathbb{Z}_{\geq 1}$ such that $\mathfrak{a}^\ell y_{A,\boldsymbol{\theta}} \notin \mathcal{L}_\varepsilon$, hence such that there exists $\mathbf{y}_\ell \in R_v^n$ with $\langle A\,\mathbf{y}_\ell - \boldsymbol{\theta} \rangle_{\mathbf{r}} < q_v^{-\ell} \varepsilon^{\frac{m}{d}}$ and $\|\,\mathbf{y}_\ell\,\|_{\mathbf{s}} < q_v^{\ell} \varepsilon^{\frac{n}{d}}$. Since the statement (1) does not hold, the inequality

$$\|\,\mathbf{y}\,\|_s \langle A\,\mathbf{y} - \boldsymbol{\theta} \rangle_{\mathbf{r}} < \varepsilon$$

has infinitely many solutions $\mathbf{y} \in R_v^n$, which contradicts the assumption $\boldsymbol{\theta} \in \mathbf{Bad}_A(\varepsilon)$. □

4.4.3 Entropy, partition construction, and effective variational principle

In this subsection, after recalling the basic definitions and properties about entropy (using [ELW] as a general reference, and in particular its Chapter 2), we give the preliminary constructions of σ-algebras and results on entropy that will be needed in Section 4.5. In particular, we give an effective and positive characteristic version of the variational principle for conditional entropy of [EL10, §7.55], adapting to the function field case the result of [KKL].

Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space. For every set E of subsets of X, we denote by $\sigma(E)$ the σ-algebra of subsets of X generated by E. Let $\mathcal{P}$ be a (finite or) countable $\mathcal{B}$-measurable partition of X. Let $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{C}'$ be sub-σ-algebras of $\mathcal{B}$. Suppose that $\mathcal{C}$ and $\mathcal{C}'$ are countably generated.

For every $x \in X$, we denote by $[x]_\mathcal{P}$ the *atom* of x for $\mathcal{P}$, which is the element of the partition $\mathcal{P}$ containing x. We denote by $[x]_\mathcal{C}$ the *atom* of x for $\mathcal{C}$, which is the intersection of all elements of $\mathcal{C}$ containing x. Note that $[x]_{\sigma(\mathcal{P})} = [x]_{\mathcal{P}}$. We denote by $(\mu_x^{\mathcal{A}})_{x\in X}$ an $\mathcal{A}$-measurable family of (Borel probability) *conditional measures* of μ with respect to $\mathcal{A}$ on X, given for instance by [EL10, Theo. 5.9].

Using the standard convention $0\log_{q_v} 0 = 0$ and using $\log_{q_v}$ instead of log for computational purposes in the field K_v, the *entropy* of the partition $\mathcal{P}$ with respect to μ is defined by

$$H_\mu(\mathcal{P}) = -\sum_{P\in\mathcal{P}} \mu(P)\log_{q_v}\mu(P) \ \in [0,\infty] \ .$$

Recall the (logarithmic) cardinality majoration

$$H_\mu(\mathcal{P}) \leq \log_{q_v}(\operatorname{card}\mathcal{P}) \ . \tag{4.37}$$

The *information function* of $\mathcal{C}$ given $\mathcal{A}$ with respect to μ is the measurable map $I_\mu(\mathcal{C}|\mathcal{A}) : X \to [0,\infty]$ defined by

$$\forall\, x\in X, \quad I_\mu(\mathcal{C}|\mathcal{A})(x) = -\log_{q_v}\mu_x^{\mathcal{A}}([x]_{\mathcal{C}}) \ .$$

The *conditional entropy* of $\mathcal{C}$ given $\mathcal{A}$ with respect to μ is defined by

$$H_\mu(\mathcal{C}|\mathcal{A}) = \int_X I_\mu(\mathcal{C}|\mathcal{A})\, d\mu \ . \tag{4.38}$$

Recall the additivity property $H_\mu(\mathcal{C}\vee\mathcal{C}'\,|\,\mathcal{A}) = H_\mu(\mathcal{C}\,|\,\mathcal{C}'\vee\mathcal{A}) + H_\mu(\mathcal{C}'\,|\,\mathcal{A})$ (see for instance [ELW, Prop. 2.13]) so that if $\mathcal{A}\subset\mathcal{C}'\subset\mathcal{C}$, we have

$$H_\mu(\mathcal{C}\,|\,\mathcal{A}) = H_\mu(\mathcal{C}\,|\,\mathcal{C}') + H_\mu(\mathcal{C}'\,|\,\mathcal{A}) \ . \tag{4.39}$$

Let $T : (X,\mathcal{B},\mu) \to (X,\mathcal{B},\mu)$ be a measure-preserving transformation. Assume that the σ-algebra $\mathcal{A}$ is strictly T-invariant, i.e., $T^{-1}\mathcal{A} = \mathcal{A}$. If the partition $\mathcal{P}$ has finite entropy with respect to μ, let

$$h_\mu(T,\mathcal{P}|\mathcal{A}) = \lim_{n\to\infty}\frac{1}{n}H_\mu\Big(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{P}|\mathcal{A}\Big) = \inf_{n\geq 1}\frac{1}{n}H_\mu\Big(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{P}|\mathcal{A}\Big) \ .$$

The *conditional (dynamical) entropy* of T given $\mathcal{A}$ is

$$h_\mu(T|\mathcal{A}) = \sup_{\mathcal{P}}\ h_\mu(T,\mathcal{P}|\mathcal{A}) \ ,$$

where the upper bound is taken on all countable $\mathcal{B}$-measurable partitions $\mathcal{P}$ of X with finite entropy with respect to μ.

With the above notations, the following result is proven in [KKL, Prop. 2.2 and Appendix A].

Proposition 4.4.5 (Entropy and ergodic decomposition). *If $T^{-1}\mathcal{A} \subset \mathcal{A}$, then for every countable $\mathcal{B}$-measurable partition $\mathcal{P}$ with finite entropy with respect to μ, we have*

$$h_\mu(T, \mathcal{P}|\mathcal{A}) = \int_X h_{\mu_x^{\mathcal{E}}}(T, \mathcal{P}|\mathcal{A})\, d\mu(x) \quad \text{and} \quad h_\mu(T|\mathcal{A}) = \int_X h_{\mu_x^{\mathcal{E}}}(T|\mathcal{A})\, d\mu(x)\ . \quad \square$$

We now work in the standard Borel space $\mathcal{Y}$ of unimodular grids, endowed with the distance $d_{\mathcal{Y}}$ (see Section 4.4.1). Let $\delta > 0$. For every subset B of $\mathcal{Y}$, we define the *δ-boundary* $\partial_\delta B$ of B by

$$\partial_\delta B = \big\{\, y \in \mathcal{Y} : \inf_{y' \in B} d_{\mathcal{Y}}(y, y') + \inf_{y'' \in \mathcal{Y} - B} d_{\mathcal{Y}}(y, y'') < \delta \,\big\}$$

if B and $\mathcal{Y} - B$ are nonempty, and $\partial_\delta B = \emptyset$ otherwise. Note that for all subsets B and B' of $\mathcal{Y}$, we have

$$\partial_\delta(B \cup B') \subset \partial_\delta B \cup \partial_\delta B' \quad \text{and} \quad \partial_\delta(B - B' \cap B) \subset \partial_\delta B \cup \partial_\delta B'\ . \tag{4.40}$$

We also have $\partial_\delta B \subset \partial_{\delta'} B$ if $\delta \le \delta'$. Given any set $\mathcal{P}$ of subsets of $\mathcal{Y}$, we define the *δ-boundary* $\partial_\delta \mathcal{P}$ of $\mathcal{P}$ by

$$\partial_\delta \mathcal{P} = \bigcup_{B \in \mathcal{P}} \partial_\delta B\ .$$

Lemma 4.4.6. *For every $r > 0$, there exist $\delta_r \in\,]0, r]$ and a finite measurable partition $\mathcal{P} = \{P_1, \dots, P_N, P_\infty\}$ by closed and open subsets of $\mathcal{Y}$ such that*

1. *the subset P_∞ is contained in the r-thin part $\mathcal{Y} - \mathcal{Y}(r)$,*
2. *for every $i \in \{1, \dots, N\}$, there exists $y_i \in \mathcal{Y}(r)$ such that $B^G_{\frac{r}{2}} y_i \subset P_i \subset B^G_r y_i$,*
3. *the set $\partial_{\delta_r} \mathcal{P}$ is empty.*

Proof. Choose a finite maximal r-separated subset $\{y_1, \dots, y_N\}$ of $\mathcal{Y}(r)$ for the distance $d_{\mathcal{Y}}$, which exists by the compactness of $\mathcal{Y}(r)$. By induction on

$i = 1, \dots, N$, we define a Borel subset P_i of $\mathcal{Y}$ by

$$P_i = B_r^G y_i - \Big(\bigcup_{j=1}^{i-1} P_j \cup \bigcup_{j=i+1}^{N} B_{\frac{r}{2}}^G y_j \Big) .$$

Define $P_\infty = \mathcal{Y} - \bigcup_{j=1}^N P_j$, which is also a Borel subset of $\mathcal{Y}$.

By construction, we have $P_i \subset B_r^G y_i$. Since the set $\{y_1, \dots, y_N\}$ is ϵ-separated, the intersection of open balls $B_{\frac{r}{2}}^G y_i \cap B_{\frac{r}{2}}^G y_j = B_{\mathcal{Y}}(y_i, \frac{r}{2}) \cap B_{\mathcal{Y}}(y_j, \frac{r}{2})$ is empty if $j > i$. By construction, the intersection $B_{\frac{r}{2}}^G y_i \cap P_j$ is empty if $j < i$. Therefore P_i contains $B_{\frac{r}{2}}^G y_i$, and Assertion (ii) follows.

By construction, we have $\bigcup_{j=1}^N P_j \subset \bigcup_{j=1}^N B_r^G y_j = \bigcup_{j=1}^N B_{\mathcal{Y}}(y_j, r)$, and the later union contains $\mathcal{Y}(r)$, since the ϵ-separated set $\{y_1, \dots, y_N\}$ is maximal. Assertion (i) follows.

For every $s > 0$, let $n_s = \Big\lceil \frac{\ln(e^s - 1)}{\ln q_v} \Big\rceil \in \mathbb{Z}$ and $\delta'_s = \ln \big(\frac{1 + q_v^{n_s}}{1 + q_v^{n_s - 1}} \big) > 0$. For all $\delta > 0$ and $y \in \mathcal{Y}$, assume that there exists a point $z \in \partial_\delta B_{\mathcal{Y}}(y, s)$. Let $z' \in B_{\mathcal{Y}}(y, s)$ and $z'' \notin B_{\mathcal{Y}}(y, s)$ be such that $d_{\mathcal{Y}}(z, z') + d_{\mathcal{Y}}(z, z'') < \delta$. Since the operator norm on $\mathcal{M}_{d+1}(K_v)$ has values in $\{0\} \cup q_v^{\mathbb{Z}}$, the set $\{d_{\mathcal{Y}}(y, y') : y, y' \in \mathcal{Y}\}$ of values of the distance function $d_{\mathcal{Y}}$ on $\mathcal{Y}$ is contained in $\{0\} \cup \{\ln(1 + q_v^n) : n \in \mathbb{Z}\}$. Since $s \in \,]\ln(1 + q_v^{n_s - 1}), \ln(1 + q_v^{n_s})]$, we hence have $d_{\mathcal{Y}}(y, z') \leq \ln(1 + q_v^{n_s - 1})$ since $z' \in B_{\mathcal{Y}}(y, s)$ and $d_{\mathcal{Y}}(y, z'') \geq \ln(1 + q_v^{n_s})$ since $z'' \notin B_{\mathcal{Y}}(y, s)$. Therefore by the triangle inequality and the inverse triangle inequality, we have

$$\begin{aligned} \delta > d_{\mathcal{Y}}(z, z') + d_{\mathcal{Y}}(z, z'') &\geq d_{\mathcal{Y}}(z', z'') \geq d_{\mathcal{Y}}(y, z'') - d_{\mathcal{Y}}(y, z') \\ &\geq \ln(1 + q_v^{n_s}) - \ln(1 + q_v^{n_s - 1}) = \delta'_s \, . \end{aligned}$$

Hence $\partial_\delta B_{\mathcal{Y}}(y, s)$ is empty for every $\delta \in \,]0, \delta'_s]$.

By Equation (4.40), for every $\delta > 0$, we have

$$\partial_\delta \mathcal{P} \subset \bigcup_{j=1}^N \partial_\delta (B_r^G y_j) \cup \bigcup_{j=1}^N \partial_\delta (B_{\frac{r}{2}}^G y_j) \, .$$

Hence Assertion (iii) follows with $\delta_r = \min\{\delta'_{\frac{r}{2}}, r\}$.

Note that since the distance d_G has values in $\{0\} \cup \{\ln(1 + q_v^n) : n \in \mathbb{Z}\}$, the open balls in G are open and compact, and since the canonical projection $G \to \mathcal{Y}$ is open and continuous, the subsets P_i of $\mathcal{Y}$ are by construction open and compact, and P_∞ is closed and open. □

Let $\mathcal{C}$ be a countably generated σ-algebra of subsets of $\mathcal{Y}$. Note that for every $j \in \mathbb{Z}$, the σ-algebra $\mathfrak{a}^j\mathcal{C}$ is also countably generated and

$$[y]_{a^j\mathcal{C}} = \mathfrak{a}^j\,[\mathfrak{a}^{-j}y]_{\mathcal{C}}\ .$$

We say that $\mathcal{C}$ is $\mathfrak{a}^{-1}$-*descending* if $\mathfrak{a}\mathcal{C}$ is contained in $\mathcal{C}$. In particular, for all $y \in \mathcal{Y}$ and $j \in \mathbb{Z}_{\geq 0}$, we have

$$[y]_{\mathcal{C}} \subset [y]_{\mathfrak{a}^j\mathcal{C}}\ .$$

Given a Borel probability measure μ on $\mathcal{Y}$ and a closed subgroup H of G, we say that $\mathcal{C}$ is H-*subordinated modulo* μ if for μ-almost every $y \in \mathcal{Y}$, there exists $r = r_y \in\ ]0, 1]$ such that we have

$$B^H_r y \subset [y]_{\mathcal{C}} \subset B^H_{1/r} y\ .$$

If $\mathcal{C}$ is U^+-subordinated modulo μ and if furthermore μ is $\mathfrak{a}$-invariant, since $\mathfrak{a}$ normalises U^+ and by Lemma 4.4.1, for every $j \in \mathbb{Z}$, the σ-algebra $\mathfrak{a}^j\mathcal{C}$ is also U^+-subordinated modulo μ.

For every σ-algebra $\mathcal{A}$ of subsets of $\mathcal{Y}$, for all a, b in $\mathbb{Z} \cup \{\pm\infty\}$ with $a < b$, we define a σ-algebra $\mathcal{A}^b_a$ of subsets of $\mathcal{Y}$ by

$$\mathcal{A}^b_a = \bigvee_{i=a}^{b} \mathfrak{a}^i\mathcal{A} = \sigma\Big(\bigcup_{a\leq i\leq b} \mathfrak{a}^i\mathcal{A}\Big)\ .$$

Note that if $\mathcal{A}$ is countably generated, then so is $\mathcal{A}^b_a$.

Proposition 4.4.7. *For every $r \in\ ]0, 1[\,$, there exists a countably generated sub-σ-algebra $\mathcal{A}^{U^+}$ of the Borel σ-algebra of $\mathcal{Y}$ such that*

1. *the countably generated σ-algebra $\mathcal{A}^{U^+}$ is $\mathfrak{a}^{-1}$-descending,*
2. *for every $y \in \mathcal{Y}(r)$, we have $[y]_{\mathcal{A}^{U^+}} \subset B^{U^+}_r y$,*
3. *for every $y \in \mathcal{Y}$, we have $B^{U^+}_{\delta_r} y \subset [y]_{\mathcal{A}^{U^+}}$, where $\delta_r \in\ ]0, r]$ is as in Lemma 4.4.6.*

Let μ be a Borel $\mathfrak{a}$-invariant ergodic probability measure on $\mathcal{Y}$ with $\mu(\mathcal{Y}(r)) > 0$. Then $\mathcal{A}^{U^+}$ is U^+-subordinated modulo μ.

Proof. Fix $r \in \,]0,1[\,$. Let $\mathcal{P} = \{P_1, \dots, P_N, P_\infty\}$ be a partition given by Lemma 4.4.6 for this r. We prove a preliminary result on the countably generated sub-σ-algebra $\sigma(\mathcal{P})_0^\infty$.

Lemma 4.4.8. *For every $y \in \mathcal{Y}$, we have $B_{\delta_r}^{U^+} y \subset [y]_{\sigma(\mathcal{P})_0^\infty}$.*

Proof. Let $h \in B_{\delta_r}^{U^+}$. Assume for a contradiction that $hy \notin [y]_{\sigma(\mathcal{P})_0^\infty}$. Then there exists $k \in \mathbb{Z}_{\geq 0}$ such that $\mathfrak{a}^{-k} hy$ and $\mathfrak{a}^{-k} y$ belong to different atoms of the partition $\mathcal{P}$. Let $\alpha = \min \mathbf{r} > 0$. By Lemma 4.4.1, we have

(4.41)
$$d_{\mathcal{Y}}(\mathfrak{a}^{-k} hy, \mathfrak{a}^{-k} y) \leq d_G(\mathfrak{a}^{-k} h \mathfrak{a}^k, \mathrm{id}) = d_{U^+}(\mathfrak{a}^{-k} h \mathfrak{a}^k, \mathrm{id}) < q_v^{-k\alpha} \delta_r \leq \delta_r \leq r \,.$$

It follows that both $\mathfrak{a}^{-k} hy$ and $\mathfrak{a}^{-k} y$ belong to the δ_r-boundary $\partial_{\delta_r} \mathcal{P}$ of $\mathcal{P}$. But the set $\partial_{\delta_r} \mathcal{P}$ is empty by Lemma 4.4.6 (3), which gives a contradiction. □

By Lemma 4.4.6, for every $i \in \{1, \dots, N\}$, there exist $y_i \in \mathcal{Y}(r)$ and a Borel subset V_i of $\mathcal{Y}$ contained in B_r^G such that $P_i = V_i y_i$. Let $\mathcal{P}^{U^+}$ be the sub-σ-algebra of the Borel σ-algebra of $\mathcal{Y}$ generated by the subsets $P_\infty \cap \pi^{-1}(W)$, where W is a Borel subset of $\mathcal{X}$, and the subsets $((U^+ B) \cap V_i) y_i$, where $i \in \{1, \dots, N\}$ and B is a Borel subset of G. Then $\mathcal{P}^{U^+}$ is countably generated, since the Borel σ-algebra of $\mathcal{X}$ is countably generated and U^+ is a closed subgroup of G. For every $y \in \mathcal{Y}$, the atom of y for $\mathcal{P}^{U^+}$ is equal to

$$(4.42) \qquad [y]_{\mathcal{P}^{U^+}} = \begin{cases} Uy & \text{if } \ y \in P_\infty \\ P_i \cap (B_r^{U^+} y) & \text{if } \ \exists\, i \in \{1, \dots, N\}, \ y \in P_i \,. \end{cases}$$

Let us now define $\mathcal{A}^{U^+} = (\mathcal{P}^{U^+})_0^\infty$, which is a countably generated sub-σ-algebra of the Borel σ-algebra of $\mathcal{Y}$, since so is $\mathcal{P}^{U^+}$. Note that $\mathfrak{a}\,\mathcal{A}^{U^+} = (\mathcal{P}^{U^+})_1^\infty \subset \mathcal{A}^{U^+}$, which proves Assertion (1).

For every $y \in \mathcal{Y}(r)$, since $P_\infty \subset \mathcal{Y} - \mathcal{Y}(r)$ by Lemma 4.4.6 (1) and by Equation (4.42), we have $[y]_{\mathcal{A}^{U^+}} \subset [y]_{\mathcal{P}^{U^+}} \subset B_r^{U^+} y$, which proves Assertion (2).

In order to prove the last Assertion (3), let us take $y \in \mathcal{Y}$ and $h \in B_{\delta_r}^{U^+}$ and let us prove that $hy \in [y]_{\mathcal{A}^{U^+}}$. Since we have $hy \in [y]_{\sigma(\mathcal{P})_0^\infty}$ by Lemma 4.4.8, for every $k \geq 0$, there exists $i \in \{1, \dots, N, \infty\}$ such that the points $\mathfrak{a}^{-k} y$ and $\mathfrak{a}^{-k} hy = \mathfrak{a}^{-k} h \mathfrak{a}^k (\mathfrak{a}^{-k} y)$ both belong to $P_i \in \mathcal{P}$. If $i = \infty$, then by Equation (4.42), the points $\mathfrak{a}^{-k} y$ and $\mathfrak{a}^{-k} hy$ lie in the same atom $[\mathfrak{a}^{-k} y]_{\mathcal{P}^{U^+}} = U \mathfrak{a}^{-k} y$ since $\mathfrak{a}^{-k} h \mathfrak{a}^k \in U^+$. Assume that $1 \leq i \leq N$. Since $h \in B_{\delta_r}^{U^+}$, it follows from

Equation (4.41) that $\mathfrak{a}^{-k}h\mathfrak{a}^k \in B_r^{U^+}$. Hence by Equation (4.42), the points $\mathfrak{a}^{-k}y$ and $\mathfrak{a}^{-k}hy$ lie in the same atom $[\mathfrak{a}^{-k}y]_{\mathcal{P}^{U^+}} = P_i \cap (B_r^{U^+}\mathfrak{a}^{-k}y)$ of $\mathcal{P}^{U^+}$. This proves Assertion (3).

Now let μ be an $\mathfrak{a}$-invariant ergodic probability measure on $\mathcal{Y}$ with $\mu(\mathcal{Y}(r)) > 0$. By ergodicity, for μ-almost every $y \in \mathcal{Y}$, there exists $k \in \mathbb{Z}_{\geq 1}$ such that $\mathfrak{a}^{-k}y \in \mathcal{Y}(r)$. Since $\mathfrak{a}^k\mathcal{A}^{U^+} \subset \mathcal{A}^{U^+}$, by Assertion (1) and by Lemma 4.4.1, we have

$$[y]_{\mathcal{A}^{U^+}} \subset [y]_{\mathfrak{a}^k\mathcal{A}^{U^+}} = \mathfrak{a}^k[\mathfrak{a}^{-k}y]_{\mathcal{A}^{U^+}} \subset \mathfrak{a}^k B_r^{U^+}\mathfrak{a}^{-k}y \subset B^{U^+}_{\ln(1+q_v^{k\max\mathbf{r}})}y\ .$$

With Assertion (3), this proves that $\mathcal{A}^{U^+}$ is U^+-subordinated modulo μ. $\square$

Let us introduce some material before stating and proving our next Lemma 4.4.9. The map $d_{K_v^m,\mathbf{r}} : K_v^m \times K_v^m \to [0,+\infty[$ defined by

$$\forall\, \boldsymbol{\xi},\boldsymbol{\xi}' \in K_v^m, \quad d_{K_v^m,\mathbf{r}}(\boldsymbol{\xi},\boldsymbol{\xi}') = \|\,\boldsymbol{\xi}-\boldsymbol{\xi}'\,\|_{\mathbf{r}} \tag{4.43}$$

is an ultrametric distance on K_v^m, since the $\mathbf{r}$-pseudonorm $\|\ \|_{\mathbf{r}}$ satisfies the ultrametric inequality : for all $\boldsymbol{\xi},\boldsymbol{\xi}' \in K_v^m$, we have

$$\|\,\boldsymbol{\xi}+\boldsymbol{\xi}'\,\|_{\mathbf{r}} \leq \max\{\|\,\boldsymbol{\xi}\,\|_{\mathbf{r}},\ \|\,\boldsymbol{\xi}'\,\|_{\mathbf{r}}\}\ , \tag{4.44}$$

with equality if $\|\,\boldsymbol{\xi}\,\|_{\mathbf{r}} \neq \|\,\boldsymbol{\xi}'\,\|_{\mathbf{r}}$. Note that the map similar to $d_{K_v^m,\mathbf{r}}$ in the real case of [KKL] is not a distance if $m \geq 2$ for general $\mathbf{r}$. For every $\epsilon > 0$, we denote by $B_\epsilon^{K_v^m,\mathbf{r}}$ the open ball of center 0 and radius ϵ in K_v^m for $d_{K_v^m,\mathbf{r}}$. Note that the distance $d_{K_v^m,\mathbf{r}}$ is bihölder equivalent to the standard one: For all $\boldsymbol{\xi},\boldsymbol{\xi}' \in K_v^m$ such that $\|\,\boldsymbol{\xi}-\boldsymbol{\xi}'\,\| \leq 1$, we have

$$\|\,\boldsymbol{\xi}-\boldsymbol{\xi}'\,\|^{\frac{1}{\min\mathbf{r}}} \leq d_{K_v^m,\mathbf{r}}(\boldsymbol{\xi},\boldsymbol{\xi}') \leq \|\,\boldsymbol{\xi}-\boldsymbol{\xi}'\,\|^{\frac{1}{\max\mathbf{r}}}\ . \tag{4.45}$$

We also endow the quotient space $\mathbb{T}^m = K_v^m/R_v^m$ with the quotient distance $d_{\mathbb{T}^m,\mathbf{r}}$ of the distance $d_{K_v^m,\mathbf{r}}$ on K_v^m defined by Equation (4.43). For every $A \in \mathcal{M}_{m,n}(K_v)$, we denote by $d_{U^+y_{A,0},\mathbf{r}}$ the distance on the orbit $U^+y_{A,0} = \phi_A(\mathbb{T}^m)$ induced from $d_{\mathbb{T}^m,\mathbf{r}}$, that is,

$$d_{U^+y_{A,0},\mathbf{r}}(\phi_A(\boldsymbol{\theta}),\phi_A(\boldsymbol{\theta}')) = d_{\mathbb{T}^m,\mathbf{r}}(\boldsymbol{\theta},\boldsymbol{\theta}')).$$

Then the homeomorphism ϕ_A defined in Lemma 4.4.3 is also isometry for the distances $d_{\mathbb{T}^m,\mathbf{r}}$ and $d_{U^+y_{A,0},\mathbf{r}}$.

Using the identification $w \mapsto \widehat{w}$ between K_v^m and U^+ (see Subsection 4.4.1), for every $\epsilon > 0$, we denote by $B_\epsilon^{U^+,\mathbf{r}}$ the open ball of radius ϵ in U^+ centered at the identity element for the distance $d_{U^+,\mathbf{r}}$ on U^+ induced from the distance $d_{K_v^m,\mathbf{r}}$ on K_v^m. The map $u \mapsto u\, y_{A,0}$ from U^+ onto $U^+ y_{A,0}$ is 1-Lipschitz and locally isometric for the distances $d_{U^+,\mathbf{r}}$ and $d_{U^+ y_{A,0},\mathbf{r}}$. Improving Lemma 4.4.1, for all $\epsilon > 0$ and $k \in \mathbb{Z}$, we have

$$\mathfrak{a}^{-k} B_\epsilon^{U^+,\mathbf{r}} \mathfrak{a}^k = B_{\epsilon\, q_v^{-k}}^{U^+,\mathbf{r}} \;. \tag{4.46}$$

Again using the (locally compact) topological group identification $w \mapsto \widehat{w}$ between $(K_v^m, +)$ and U^+, we endow U^+ with the Haar measure m_{U^+} which corresponds to the normalized Haar measure vol_v^m of K_v^m (see Section 4.1.1). For every $j \in \mathbb{Z}$, the Jacobian Jac_j with respect to the measure m_{U^+} of the homeomorphism $\varphi_j : u \mapsto \mathfrak{a}^j\; u\; \mathfrak{a}^{-j}$ from U^+ to U^+ (which is constant since φ_j is a group automorphism and m_{U^+} is bi-invariant) is easy to compute: we have

$$\mathrm{Jac}_j = q_v^{j\,|\mathbf{r}|} \;. \tag{4.47}$$

We consider the following tail σ-algebra:

$$\mathcal{A}_\infty^{U^+} = \bigcap_{k=1}^{\infty} \bigvee_{i=k}^{\infty} \mathfrak{a}^i \mathcal{A}^{U^+} = \lim_{k\to\infty} (\mathcal{A}^{U^+})_k^\infty = \lim_{k\to\infty} (\mathcal{P}^{U^+})_k^\infty . \tag{4.48}$$

This σ-algebra may not be countably generated, but it is strictly $\mathfrak{a}$-invariant, i.e., $\mathfrak{a}\mathcal{A}_\infty^{U^+} = \mathcal{A}_\infty^{U^+} = \mathfrak{a}^{-1}\mathcal{A}_\infty^{U^+}$, hence we will use this σ-algebra to observe the entropy relative to U^+.

Lemma 4.4.9. *For every $r \in \,]0,1[\,$, let $\mathcal{A}^{U^+}$ be as in Proposition 4.4.7 and $\mathcal{A}_\infty^{U^+}$ be as in Equation* (4.48)*. Let μ be an $\mathfrak{a}$-invariant ergodic probability measure on $\mathcal{Y}$. Then*

$$h_\mu(\mathfrak{a}^{-1} |\, \mathcal{A}_\infty^{U^+}) \leq |\mathbf{r}| \;.$$

Furthermore, if $\mu(\mathcal{Y}(r)) > 0$, then

$$h_\mu(\mathfrak{a}^{-1} |\, \mathcal{A}_\infty^{U^+}) = H_\mu(\mathcal{A}^{U^+} |\, \mathfrak{a}\, \mathcal{A}^{U^+}) \;.$$

Proof. Let us prove the first assertion. By [EL10, Prop. 7.44], there exists a countable Borel-measurable partition $\mathcal{G}$ with finite entropy which is a generator for $\mathfrak{a}$ modulo μ, such that $\sigma(\mathcal{G})_0^\infty$ is $\mathfrak{a}^{-1}$-descending and G^+-subordinated modulo μ. Following the proof of [LSS19, Lemma 3.4], it follows from [ELW, Prop. 2.19 (8) and Theo. 2.20] that

$$h_\mu(\mathfrak{a}^{-1}|\mathcal{A}_\infty^{U^+}) = H_\mu(\sigma(\mathcal{G})|\sigma(\mathcal{G})_1^\infty \vee \mathcal{A}_\infty^{U^+}).$$

Using the continuity and monotonicity of entropy [ELW, Prop. 2.12 and Prop. 2.13], we have

$$\begin{aligned} H_\mu(\sigma(\mathcal{G})|\sigma(\mathcal{G})_1^\infty \vee \mathcal{A}_\infty^{U^+}) &= \lim_{\ell\to\infty} H_\mu(\sigma(\mathcal{G})|\sigma(\mathcal{G})_1^\infty \vee (\mathcal{A}^{U^+})_\ell^\infty) \\ &\leq \lim_{\ell\to\infty} H_\mu(\sigma(\mathcal{G})_0^\infty \vee (\mathcal{A}^{U^+})_\ell^\infty | \mathfrak{a}(\sigma(\mathcal{G})_0^\infty \vee (\mathcal{A}^{U^+})_\ell^\infty)). \end{aligned}$$

Note that for each $\ell \geq 1$ the σ-algebra $\sigma(\mathcal{G})_0^\infty \vee (\mathcal{A}^{U^+})_\ell^\infty$ is countably generated, $\mathfrak{a}^{-1}$-descending, and U^+-subordinated since $[y]_{(\mathcal{A}^{U^+})_\ell^\infty} \subset Uy$ for all $y \in \mathcal{Y}$ and since $\sigma(\mathcal{G})_0^\infty$ is G^+-subordinated. Thus by [EL10, Prop. 7.34] (recalling that we are using logarithms with base q_v), we have

$$H_\mu(\sigma(\mathcal{G})_0^\infty \vee (\mathcal{A}^{U^+})_\ell^\infty |\, \mathfrak{a}\,(\sigma(\mathcal{G})_0^\infty \vee (\mathcal{A}^{U^+})_\ell^\infty)) = \lim_{k\to\infty} \frac{\log_{q_v} \mu_x^{U^+}(\mathfrak{a}^k B_1^{U^+} \mathfrak{a}^{-k})}{k} \ ,$$

where $\mu_x^{U^+}$ is the leaf-wise measure of μ at $x \in \mathcal{Y}$ with respect to U^+ as defined in [EL10, Theo. 6.3]. By [EL10, Theo. 6.30] (which applies since U^+ is abelian, hence unimodular) and by Equation (4.47) (see also [EL10, §7.42]), we have

$$\limsup_{k\to\infty} \frac{\mu_x^{U^+}(\mathfrak{a}^k B_1^{U^+} \mathfrak{a}^{-k})}{k^2\, q_v^{k|\mathbf{r}|}} = 0 \ ,$$

hence we have

$$\lim_{k\to\infty} \frac{\log_{q_v} \mu_x^{U^+}(\mathfrak{a}^k B_1^{U^+} \mathfrak{a}^{-k})}{k} \leq |\mathbf{r}| \ .$$

This proves the first assertion of the lemma.

To prove the second assertion, let us take a sequence of finite partitions $(\mathcal{P}_k^{U^+})_{k\geq 1}$ of $\mathcal{Y}$ such that $\sigma(\mathcal{P}_k^{U^+}) \nearrow \mathcal{P}^{U^+}$, which is possible since $\mathcal{P}^{U^+}$ is countably generated. Since μ is ergodic and $\mu(\mathcal{Y}(r)) > 0$, for μ almost every $y \in \mathcal{Y}$, there exists an increasing sequence of positive integers $(k_i)_{i\geq 1}$ such that $\mathfrak{a}^{k_i} y \in \mathcal{Y}(r)$. By Proposition 4.4.7(2), we have $[\mathfrak{a}^{k_i} y]_{\mathcal{A}^{U^+}} \subset B_r^{U^+} \mathfrak{a}^{k_i} y$ for all $i \geq 1$. Hence it follows from Lemma 4.4.1 that

$$[y]_{(\mathcal{P}^{U^+})_{-k_i}^\infty} = \mathfrak{a}^{-k_i}[\mathfrak{a}^{k_i} y]_{(\mathcal{P}^{U^+})_0^\infty} \subset \mathfrak{a}^{-k_i} B_r^{U^+} \mathfrak{a}^{k_i} y \subset B^{U^+}_{\ln(1+r q_v^{-k_i \min \mathbf{r}})} y.$$

Taking $i \to \infty$, we have $[y]_{(\mathcal{P}^{U^+})_{-\infty}^{\infty}} = \{y\}$ for μ almost every $y \in \mathcal{Y}$. It means that $(\mathcal{P}^{U^+})_{-\infty}^{\infty} = \mathcal{B}_{\mathcal{Y}}$ modulo μ, where $\mathcal{B}_{\mathcal{Y}}$ is the Borel σ-algebra of $\mathcal{Y}$. It follows that $\bigvee_{k=1}^{\infty}(\mathcal{P}_k^{U^+})_{-\infty}^{\infty} = (\mathcal{P}^{U^+})_{-\infty}^{\infty} = \mathcal{B}_{\mathcal{Y}}$ modulo μ, and $(\mathcal{P}_k^{U^+})_{-\infty}^{\infty} \subseteq (\mathcal{P}_{k+1}^{U^+})_{-\infty}^{\infty}$ for each $k \geq 1$. Again using [ELW, Prop. 2.19 (8) and Theo. 2.20] and the continuity of entropy, we have

$$\begin{aligned} h_\mu(\mathfrak{a}^{-1}|\mathcal{A}_\infty^{U^+}) &= \lim_{k\to\infty} h_\mu(\mathfrak{a}^{-1}, \mathcal{P}_k^{U^+}|\mathcal{A}_\infty^{U^+}) \\ &= \lim_{k\to\infty} H_\mu(\mathcal{P}_k^{U^+}|(\mathcal{P}_k^{U^+})_1^{\infty} \vee \mathcal{A}_\infty^{U^+}) \\ &= H_\mu(\mathcal{P}^{U^+}|(\mathcal{P}^{U^+})_1^{\infty} \vee \mathcal{A}_\infty^{U^+}) \\ &= H_\mu(\mathcal{P}^{U^+}|(\mathcal{P}^{U^+})_1^{\infty}) = H_\mu(\mathcal{A}^{U^+}|\mathfrak{a}\mathcal{A}^{U^+}). \end{aligned}$$

This proves the second assertion of the lemma. □

Let us introduce some more material before stating and proving our final Proposition 4.4.10 of Subsection 4.4.3. Let $\mathcal{A}$ be a countably generated sub-σ-algebra of the Borel σ-algebra of $\mathcal{Y}$. For all $j \in \mathbb{Z}_{\geq 0}$ and $y \in \mathcal{Y}$, let

$$V_y^{\mathfrak{a}^j\mathcal{A}} = \{u \in U^+ : u\, y \in [y]_{\mathfrak{a}^j\mathcal{A}}\}\ , \tag{4.49}$$

which is a Borel subset of U^+, called the *U^+-shape of the atom* $[y]_{\mathfrak{a}^j\mathcal{A}}$. Note that for every $j \in \mathbb{Z}_{\geq 0}$, we have

$$V_y^{\mathfrak{a}^j\mathcal{A}} = \mathfrak{a}^j\ V_{\mathfrak{a}^{-j}y}^{\mathcal{A}}\ \mathfrak{a}^{-j}\ .$$

Let us define a Borel-measurable family $\big(\tau_y^{\mathfrak{a}^j\mathcal{A}}\big)_{y\in\mathcal{Y}}$ of Borel measures on $\mathcal{Y}$, that we call the *U^+-subordinated Haar measure of* $\mathfrak{a}^j\mathcal{A}$, as follows:

- if $m_{U^+}(V_y^{\mathfrak{a}^j\mathcal{A}})$ is equal to 0 or ∞, we set $\tau_y^{\mathfrak{a}^j\mathcal{A}} = 0$,
- otherwise, $\tau_y^{\mathfrak{a}^j\mathcal{A}}$ is the push-forward of the normalized measure

$$\frac{1}{m_{U^+}(V_y^{\mathfrak{a}^j\mathcal{A}})}\ m_{U^+}|_{V_y^{\mathfrak{a}^j\mathcal{A}}}$$

by the map $u \mapsto u\, y$.

Now let μ be a Borel $\mathfrak{a}$-invariant probability measure on $\mathcal{Y}$, such that $\mathcal{A}$ is U^+-subordinated modulo μ. In particular, for μ-almost every $y \in \mathcal{Y}$, the atom $V_y^{\mathfrak{a}^j\mathcal{A}}$ has positive and finite m_{U^+}-measure, hence the measure $\tau_y^{\mathfrak{a}^j\mathcal{A}}$ is a probability measure with support in $[y]_{\mathfrak{a}^j\mathcal{A}}$. Furthermore, if $z \in [y]_{\mathfrak{a}^j\mathcal{A}}$ then there exists $u \in U^+$ such that $z = u\, y$, $V_z^{\mathfrak{a}^j\mathcal{A}} = V_y^{\mathfrak{a}^j\mathcal{A}}\, u^{-1}$, and $\tau_z^{\mathfrak{a}^j\mathcal{A}} = \tau_y^{\mathfrak{a}^j\mathcal{A}}$, by the right-invariance of m_{U^+}.

The following proposition is a function field analog of the effective real case version [KKL, Prop. 2.10, §2.4] of [EL10, §7.55].

Proposition 4.4.10. *Let μ be a Borel $\mathfrak{a}$-invariant ergodic probability measure on $\mathcal{Y}$ and let $\mathcal{A}$ be a countably generated sub-σ-algebra of the Borel σ-algebra of $\mathcal{Y}$ which is $\mathfrak{a}^{-1}$-descending and U^+-subordinated modulo μ. Fix $j \in \mathbb{Z}_{\geq 1}$ and a U^+-saturated Borel subset K' of $\mathcal{Y}$. Suppose that there exists $\epsilon > 0$ such that $[z]_\mathcal{A} \subset B_\epsilon^{U^+,\mathbf{r}} z$ for every $z \in K'$. Then we have*

$$H_\mu(\mathcal{A}|\mathfrak{a}^j\mathcal{A}) \leq j\,|\mathbf{r}| + \int_\mathcal{Y} \log \tau_y^{\mathfrak{a}^j\mathcal{A}}((\mathcal{Y} - K') \cup B_\epsilon^{U^+,\mathbf{r}}\operatorname{Supp}\mu)\; d\mu(y).$$

Proof. We fix μ, $\mathcal{A}$, j, K' and ϵ as in the statement. By for instance [EL10, Theo. 5.9], let $\big(\mu_y^{\mathfrak{a}^j\mathcal{A}}\big)_{y\in\mathcal{Y}}$ be a measurable family of conditional measures of μ with respect to $\mathfrak{a}^j\mathcal{A}$, so that for μ-almost every $y \in \mathcal{Y}$, the measure $\mu_y^{\mathfrak{a}^j\mathcal{A}}$ is a probability measure on $\mathcal{Y}$ giving full measure to the atom $[y]_{\mathfrak{a}^j\mathcal{A}}$, with $\mu_z^{\mathfrak{a}^j\mathcal{A}} = \mu_y^{\mathfrak{a}^j\mathcal{A}}$ if $z \in [y]_{\mathfrak{a}^j\mathcal{A}}$, and such that the following *disintegration formula* holds true:

$$\mu = \int_{y\in\mathcal{Y}} \mu_y^{\mathfrak{a}^j\mathcal{A}}\; d\mu(y)\;. \tag{4.50}$$

Let $p_\mu : y \mapsto \mu_y^{\mathfrak{a}^j\mathcal{A}}([y]_\mathcal{A})$ and $p_\tau : y \mapsto \tau_y^{\mathfrak{a}^j\mathcal{A}}([y]_\mathcal{A})$, which are nonnegative and measurable functions on $\mathcal{Y}$. Since $\mathcal{A}$ is $\mathfrak{a}^{-1}$-descending and U^+-subordinated modulo μ, the atom $[y]_\mathcal{A}$ contains an open neighborhood of y in the atom $[y]_{\mathfrak{a}^j\mathcal{A}}$ for μ-almost every $y \in \mathcal{Y}$. In particular, the function p_τ is μ-almost everywhere positive.

Since $\mathcal{A}$ is countably generated and $\mathfrak{a}^{-1}$-descending, for every $y \in \mathcal{Y}$, the atom of y for $\mathfrak{a}^j\mathcal{A}$ is countably partitioned into atoms for $\mathcal{A}$ up to measure 0, that is, there exist a finite or countable subset I_y of $[y]_{\mathfrak{a}^j\mathcal{A}}$ and a $\mu_y^{\mathfrak{a}^j\mathcal{A}}$-measure zero subset N_y of $[y]_{\mathfrak{a}^j\mathcal{A}}$ such that

$$[y]_{\mathfrak{a}^j\mathcal{A}} = N_y \sqcup \bigsqcup_{x\in I_y} [x]_\mathcal{A}\;. \tag{4.51}$$

Let $I'_y = \{x \in I_y : [x]_\mathcal{A} \cap \operatorname{Supp}\mu \neq \emptyset\}$.

Lemma 4.4.11. *Let $x \in I_y$*

(1) If $x \notin I'_y$, then $\mu_y^{\mathfrak{a}^j\mathcal{A}}([x]_{\mathcal{A}}) = 0$.

(2) If $x \in I'_y$, then $[x]_{\mathcal{A}}$ is contained in $(\mathcal{Y} - K') \cup B_\epsilon^{U^+,\mathbf{r}}\,\mathrm{Supp}\,\mu$.

Proof. (1) This follows since $\mathrm{Supp}\,\mu_y^{\mathfrak{a}^j\mathcal{A}}$ is contained in $\mathrm{Supp}\,\mu$.

(2) If $x \in I'_y$, there exists $z \in [x]_{\mathcal{A}} \cap \mathrm{Supp}\,\mu$. For every $z' \in [x]_{\mathcal{A}}$, we have either $z' \in \mathcal{Y} - K'$ or $z' \in K'$. In the second case, since $\mathcal{A}$ is U^+-subordinated and K' is U^+-saturated, we have $z \in [x]_{\mathcal{A}} = [z']_{\mathcal{A}} \subset U^+z' \subset K'$. Hence by the assumption of Proposition 4.4.10, we have $z' \in [x]_{\mathcal{A}} = [z]_{\mathcal{A}} \subset B_\epsilon^{U^+,\mathbf{r}} z \subset B_\epsilon^{U^+,\mathbf{r}}\,\mathrm{Supp}\,\mu$, which proves the result. □

By the definition of the U^+-subordinated Haar measure of $\mathfrak{a}^j\mathcal{A}$, for μ-almost every $y \in \mathcal{Y}$, we have

$$p_\tau(y) = \frac{m_{U^+}(V_y^{\mathcal{A}})}{m_{U^+}(V_y^{\mathfrak{a}^j\mathcal{A}})} = \frac{m_{U^+}(V_y^{\mathcal{A}})}{m_{U^+}(\mathfrak{a}^j\ V_{\mathfrak{a}^{-j}y}^{\mathcal{A}}\ \mathfrak{a}^{-j})} = \frac{m_{U^+}(V_y^{\mathcal{A}})}{\mathrm{Jac}_j\ m_{U^+}(V_{\mathfrak{a}^{-j}y}^{\mathcal{A}})}\ .$$

Hence, by the $\mathfrak{a}$-invariance of μ and by Equation (4.47), we have

$$\int_{z\in\mathcal{Y}} \log_{q_v} p_\tau(z)\ d\mu(z) = -\log_{q_v} \mathrm{Jac}_j = -\,j\,|\mathbf{r}|\ .$$

We have

$$\begin{aligned}
&H_\mu(\mathcal{A}\,|\,\mathfrak{a}^j\mathcal{A})\ -j\ |\mathbf{r}| \\
&= -\int_{z\in\mathcal{Y}} \big(\log_{q_v} p_\mu(z) - \log_{q_v} p_\tau(z)\big)\ d\mu(z) \\
&= \int_{y\in\mathcal{Y}}\int_{z\in\mathcal{Y}} \big(\log_{q_v} p_\tau(z) - \log_{q_v} p_\mu(z)\big)\ d\mu_y^{\mathfrak{a}^j\mathcal{A}}(z)\ d\mu(y) \\
&= \int_{y\in\mathcal{Y}}\sum_{x\in I'_y}\int_{z\in[x]_{\mathcal{A}}} \big(\log_{q_v} p_\tau(z) - \log_{q_v} p_\mu(z)\big)\ d\mu_y^{\mathfrak{a}^j\mathcal{A}}(z)\ d\mu(y) \\
&= \int_{y\in\mathcal{Y}}\sum_{x\in I'_y} \log_{q_v} \frac{\tau_y^{\mathfrak{a}^j\mathcal{A}}([x]_{\mathcal{A}})}{\mu_y^{\mathfrak{a}^j\mathcal{A}}([x]_{\mathcal{A}})}\ \mu_y^{\mathfrak{a}^j\mathcal{A}}([x]_{\mathcal{A}})\ d\mu(y) \\
&\leq \int_{y\in\mathcal{Y}} \log_{q_v}\Big(\sum_{x\in I'_y} \tau_y^{\mathfrak{a}^j\mathcal{A}}([x]_{\mathcal{A}})\Big)\ d\mu(y) \\
&\leq \int_{y\in\mathcal{Y}} \log_{q_v}\big(\tau_y^{\mathfrak{a}^j\mathcal{A}}((\mathcal{Y}-K') \cup B_\epsilon^{U^+,\mathbf{r}}\mathrm{Supp}\,\mu)\big)\ d\mu(y)\ ,
\end{aligned}$$

- by the definition of the conditional entropy in Equation (4.38),
- by the disintegration formula (4.50),
- since $\mu_y^{\mathfrak{a}^j\mathcal{A}}$ gives full measure to $[y]_{\mathfrak{a}^j\mathcal{A}}$ which is partitionned as in Equation (4.51), and by Lemma 4.4.11 (1),
- since when z varies in $[x]_{\mathcal{A}} \subset [y]_{\mathfrak{a}^j\mathcal{A}}$, the values $p_\mu(z) = \mu_z^{\mathfrak{a}^j\mathcal{A}}([z]_{\mathcal{A}}) = \mu_y^{\mathfrak{a}^j\mathcal{A}}([x]_{\mathcal{A}})$ and $p_\tau(z) = \tau_z^{\mathfrak{a}^j\mathcal{A}}([z]_{\mathcal{A}}) = \tau_y^{\mathfrak{a}^j\mathcal{A}}([x]_{\mathcal{A}})$ are constant,
- by the concavity property of the logarithm,
- by Lemma 4.4.11 (2).

This proves the result. □

4.5 Upper bound on the Hausdorff dimension of $\mathbf{Bad}_A(\epsilon)$

4.5.1 Constructing measures with large entropy

In this subsection, we construct, as in [KKL, Prop. 4.1] in the real case, an $\mathfrak{a}$-invariant probability measure on $\mathcal{Y}$ giving an appropriate lower bound on the conditional entropy of $\mathfrak{a}$ relative to the σ-algebra $\mathcal{A}_\infty^{U^+}$ defined in Equation (4.48) with respect to the σ-algebra $\mathcal{A}^{U^+}$ constructed in Proposition 4.4.7.

For any point x in a measurable space, we denote by Δ_x the unit Dirac measure at x. We denote by $\overset{*}{\rightharpoonup}$ the weak-star convergence of Borel measures on any locally compact space.

Let us denote by $\overline{\mathcal{X}} = \mathcal{X} \cup \{\infty_{\mathcal{X}}\}$ and $\overline{\mathcal{Y}} = \mathcal{Y} \cup \{\infty_{\mathcal{Y}}\}$ the one-point compactifications of $\mathcal{X}$ and $\mathcal{Y}$, respectively. We denote by $\overline{\pi} : \overline{\mathcal{Y}} \to \overline{\mathcal{X}}$ the unique continuous extension of the natural projection $\pi : \mathcal{Y} \to \mathcal{X}$, mapping $\infty_{\mathcal{Y}}$ to $\infty_{\mathcal{X}}$. The left actions of $\mathfrak{a}$ on $\mathcal{X}$ and $\mathcal{Y}$ continuously extend to actions on $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$ fixing the points at infinity $\infty_{\mathcal{X}}$ and $\infty_{\mathcal{Y}}$. For every countably generated σ-algebra $\mathcal{A}$ of subsets of $\mathcal{X}$ or $\mathcal{Y}$, we denote by $\overline{\mathcal{A}}$ the countably generated σ-algebra of subsets of $\overline{\mathcal{X}}$ or $\overline{\mathcal{Y}}$ generated by $\mathcal{A}$ and its point at infinity. For a finite partition $\mathcal{Q} = \{Q_1, \dots, Q_N, Q_\infty\}$ of $\mathcal{Y}$ with only one unbounded atom Q_∞, we denote by $\overline{\mathcal{Q}}$ the finite partition $\{Q_1, \dots, Q_N, \overline{Q}_\infty = Q_\infty \cup \{\infty_{\mathcal{Y}}\}\}$ of $\overline{\mathcal{Y}}$. Note that $\overline{\bigvee_{i=a}^b \mathfrak{a}^{-i}\mathcal{Q}} = \bigvee_{i=a}^b \mathfrak{a}^{-i}\,\overline{\mathcal{Q}}$ for all a, b in $\mathbb{Z}$ with $a < b$.

For every $\eta \in [0,1]$, we say that an element $x \in \mathcal{X}$ has *η-escape of mass on average* under the action of $\mathfrak{a}$ if for every compact subset Q of $\mathcal{X}$,

$$\liminf_{N\to\infty} \frac{1}{N} \operatorname{card} \big\{\ell \in \{1, \cdots, N\} : \mathfrak{a}^\ell x \notin Q\big\} \geq \eta\,.$$

When $\eta = 1$, as defined in the Introduction and in Proposition 4.4.2, we say that x diverges on average in $\mathcal{X}$ under the action of $\mathfrak{a}$. For every $A \in \mathcal{M}_{m,n}(K_v)$, we denote by $x_A = u_A R_v^m \in \mathcal{X}$ its associated unimodular lattice (see Section 4.4.2), and by $\eta_A \in [0,1]$ the upper bound of the elements $\eta \in [0,1]$ such that x_A has η-escape of mass on average. Note that this upper bound is actually a maximum.

Proposition 4.5.1. *For every $A \in \mathcal{M}_{m,n}(K_v)$, there exists a Borel probability measure μ_A on $\overline{\mathcal{X}}$ with $\mu_A(\mathcal{X}) = 1 - \eta_A$ such that for every $\epsilon > 0$, there exists an $\mathfrak{a}$-invariant Borel probability measure $\overline{\mu}$ on $\overline{\mathcal{Y}}$ satisfying the following properties.*

1. *The support of $\overline{\mu}$ is contained in $\mathcal{L}_\epsilon \cup \{\infty_\mathcal{Y}\}$, where $\mathcal{L}_\epsilon$ is defined in Equation* (4.35)*.*

2. *We have $\overline{\pi}_*\overline{\mu} = \mu_A$. In particular, there exists an $\mathfrak{a}$-invariant Borel probability measure μ on $\mathcal{Y}$ such that*

$$\overline{\mu} = (1-\eta_A)\mu + \eta_A \Delta_{\infty_\mathcal{Y}}.$$

3. *For every $r \in \,]0,1[\,$, let $\mathcal{A}^{U^+}$ be the σ-algebra of subsets of $\mathcal{Y}$ constructed in Proposition 4.4.7 and let $\mathcal{A}^{U^+}_\infty$ be as in Equation* (4.48)*. Then*

$$h_{\overline{\mu}}\big(\mathfrak{a}^{-1}|\,\overline{\mathcal{A}^{U^+}_\infty}\,\big) = h_{\overline{\mu}}\big(\mathfrak{a}|\,\overline{\mathcal{A}^{U^+}_\infty}\,\big) \geq |\mathbf{r}|(1-\eta_A) - \max \mathbf{r}\,\left(m - \dim_H \mathbf{Bad}_A(\epsilon)\right).$$

Proof. Since x_A has η_A-escape of mass on average but does not have $(\eta_A+\delta)$-escape of mass on average for any $\delta > 0$, there exists an increasing sequence of positive integers $(k_i)_{i\in\mathbb{Z}_{\geq 1}}$ such that, for the weak-star convergence of Borel probability measures on the compact space $\overline{\mathcal{X}}$, as $i \to +\infty$, we have

$$\frac{1}{k_i}\sum_{k=0}^{k_i-1} \Delta_{\mathfrak{a}^k x_A} \overset{*}{\rightharpoonup} \mu_A \,, \tag{4.52}$$

and μ_A is a Borel probability measure on $\overline{\mathcal{X}}$ with $\mu_A(\mathcal{X}) = 1 - \eta_A$. This is equivalent to $\mu_A(\{\infty_\mathcal{X}\}) = \eta_A$.

Let $\epsilon > 0$. For every $T \in \mathbb{Z}_{\geq 0}$, with the notation of Subsection 4.4.2 (see in particular Equations (4.35) and (4.36)), let

$$R_T = \{\boldsymbol{\theta} \in \mathbb{T}^m : \forall k \geq T, \mathfrak{a}^k \phi_A(\boldsymbol{\theta}) \in \mathcal{L}_\epsilon\} \cap \mathbf{Bad}_A(\epsilon)\,.$$

By Proposition 4.4.4, since a countable subset of K_ν^m has Hausdorff dimension 0, we have $\dim_H\big(\bigcup_{T=1}^\infty R_T\big) = \dim_H \mathbf{Bad}_A(\epsilon)$. Thus, for every $j \in \mathbb{Z}_{\geq 1}$, there exists $T_j \in \mathbb{Z}_{\geq 0}$ satisfying

$$\dim_H R_{T_j} \geq \dim_H \mathbf{Bad}_A(\epsilon) - \frac{1}{j} \ .$$

For all $i, j \in \mathbb{Z}_{\geq 1}$ such that $k_i \geq T_j$, let $S_{i,j}$ be a maximal $q_v^{-k_i}$-separated subset of R_{T_j} for the distance $d_{\mathbb{T}^m,\mathbf{r}}$ defined after Equation (4.45). Then R_{T_j} can be covered by card $S_{i,j}$ open balls of radius $q_v^{-k_i}$ for $d_{\mathbb{T}^m,\mathbf{r}}$. Each open ball of radius $q_v^{-k_i}$ for $d_{\mathbb{T}^m,\mathbf{r}}$ can be covered by $\prod_{j=1}^m {q_v}^{-k_i r_j}/q_v^{-k_i \max \mathbf{r}} = q_v^{k_i(m \max \mathbf{r} - |\mathbf{r}|)}$ open balls of radius $q_v^{-k_i \max \mathbf{r}}$ with respect to the standard distance $d_{\mathbb{T}^m}$ (defining the Hausdorff dimension of subsets of $\mathbb{T}^m$). Since the lower Minskowski dimension is at least equal to the Hausdorff dimension, we have

$$\liminf_{i\to\infty} \frac{\log_{q_v}\big(q_v^{k_i(m\max\mathbf{r}-|\mathbf{r}|)}\ \mathrm{card}\ \ S_{i,j}\big)}{-\log_{q_v}\big({q_v}^{-k_i\max\mathbf{r}}\big)} \geq \dim_H R_{T_j} \geq \dim_H \mathbf{Bad}_A(\epsilon) - \frac{1}{j}\ ,$$

which implies that

$$\liminf_{i\to\infty} \frac{\log_{q_v}\mathrm{card}\ \ S_{i,j}}{k_i} \geq |\mathbf{r}| - \max\mathbf{r}\ \Big(m + \frac{1}{j} - \dim_H \mathbf{Bad}_A(\epsilon)\Big)\ . \tag{4.53}$$

Let us define the Borel probability measures

$$\nu_{i,j} = \frac{1}{\mathrm{card}\ \ S_{i,j}} \sum_{\boldsymbol{\theta}\in S_{i,j}} \Delta_{\phi_A(\boldsymbol{\theta})}\ ,$$

which is the normalized counting measure on the finite subset $\phi_A(S_{i,j})$ of the U^+-orbit $\phi_A(\mathbb{T}^m) = U^+ y_{A,0} \subset \pi^{-1}(x_A)$, and

$$\widetilde{\nu}_{i,j} = \frac{1}{k_i} \sum_{0\leq k\leq k_i-1} \mathfrak{a}_*^k \nu_{i,j}\ ,$$

which is the average of the previous one on the first k_i points of the $\mathfrak{a}$-orbit. Since $\overline{\mathcal{Y}}$ is compact, extracting diagonally a subsequence if necessary, we may assume that $\widetilde{\nu}_{i,j}$ weak-star converges as $i \to +\infty$ towards an $\mathfrak{a}$-invariant Borel probability measure $\widetilde{\mu}_j$, and that $\widetilde{\mu}_j$ weak-star converges as $j \to +\infty$ towards an $\mathfrak{a}$-invariant Borel probability measure $\overline{\mu}$. Let us prove that $\overline{\mu}$ satisfies the three assertions of Proposition 4.5.1.

(1) For all $k \geq T_j$ and $\boldsymbol{\theta} \in S_{i,j} \subset R_{T_j}$, we have $\mathfrak{a}^k \phi_A(\boldsymbol{\theta}) \in \mathcal{L}_\epsilon$ by the definition of R_{T_j}. Since $\mathfrak{a}_*^k \nu_{i,j}$ is a probability measure, we hence have

$$\widetilde{\nu}_{i,j}(\mathcal{Y} - \mathcal{L}_\epsilon) = \frac{1}{k_i} \sum_{k=0}^{k_i-1} \mathfrak{a}_*^k \nu_{i,j}(\mathcal{Y} - \mathcal{L}_\epsilon) = \frac{1}{k_i} \sum_{k=0}^{T_j} \mathfrak{a}_*^k \nu_{i,j}(\mathcal{Y} - \mathcal{L}_\epsilon) \leq \frac{T_j}{k_i} \ .$$

Since $\mathcal{L}_\epsilon \cup \{\infty_\mathcal{Y}\}$ is closed in $\overline{\mathcal{Y}}$ and by taking limits first as $i \to +\infty$ then as $j \to +\infty$, we therefore have $\overline{\mu}(\mathcal{Y} - \mathcal{L}_\epsilon) = 0$. This proves Assertion (1).

(2) Since $\phi_A(S_{i,j})$ is contained in the fiber above x_A of $\overline{\pi}$ and since $\nu_{i,j}$ is a probability measure, we have $\overline{\pi}_* \nu_{i,j} = \Delta_{x_A}$. By the linearity and equivariance of $\overline{\pi}_*$, we hence have

$$\overline{\pi}_* \widetilde{\nu}_{i,j} = \frac{1}{k_i} \sum_{0 \leq k \leq k_i - 1} \mathfrak{a}_*^k \, \overline{\pi}_* \, \nu_{i,j} = \frac{1}{k_i} \sum_{0 \leq k \leq k_i - 1} \Delta_{\mathfrak{a}^k x_A} \ .$$

By the weak-star continuity of $\overline{\pi}_*$ and Equation (4.52), we thus have

$$\overline{\pi}_* \overline{\mu} = \lim_{j \to +\infty} \lim_{i \to +\infty} \overline{\pi}_* \widetilde{\nu}_{i,j} = \lim_{j \to +\infty} \mu_A = \mu_A \ .$$

Note that the point at infinity $\infty_\mathcal{Y}$ is an isolated point in the support of $\overline{\mu}$ by Assertion (1), since $\mathcal{L}_\epsilon$ is compact. We hence have

$$\overline{\mu}(\{\infty_\mathcal{Y}\}) = \overline{\mu}(\,\overline{\pi}^{-1}(\{\infty_\mathcal{X}\})) = \mu_A(\{\infty_\mathcal{X}\}) = \eta_A \ . \tag{4.54}$$

(3) Suppose that $\mathcal{Q}$ is any finite Borel-measurable partition of $\mathcal{Y}$ satisfying

(i) the partition $\mathcal{Q}$ contains an atom Q_∞ of the form $\pi^{-1}(Q_\infty^*)$, where $\mathcal{X} - Q_\infty^*$ has compact closure,

(ii) there exists $\ell_0 \geq 1$ such that for every atom $Q \in \mathcal{Q}$ different from Q_∞ and for any $y \in Q$, diam $(U^+ y \cap Q) < q_v^{-\ell_0 \max \mathbf{r}}$ for the distance $d_{U^+,m}$.

(iii) for all $Q \in \mathcal{Q}$ and $j \in \mathbb{Z}_{\geq 1}$, we have $\widetilde{\mu}_j(\partial Q) = 0$ and $\overline{\mu}(\partial Q) = 0$.

We first prove the following entropy bound: For every $M \in \mathbb{Z}_{\geq 1}$,

$$\frac{1}{M} H_{\overline{\mu}}\big(\,\overline{\sigma(\mathcal{Q}^{(M)})}\,|\,\overline{\mathcal{A}_\infty^{U^+}}\,\big) \geq |\mathbf{r}|(1 - \overline{\mu}(\,\overline{Q}_\infty)) - \max \mathbf{r}\ (m - \dim_H \mathbf{Bad}_A(\epsilon)) \ , \tag{4.55}$$

where $\mathcal{Q}^{(M)} = \bigvee_{k=0}^{M-1} \mathfrak{a}^{-k} \mathcal{Q}$. Since Equation (4.55) is clear if $\overline{\mu}(\,\overline{Q}_\infty) = 1$, we may assume that $\overline{\mu}(\overline{Q}_\infty) < 1$, hence that $\widetilde{\mu}_j(\overline{Q}_\infty) < 1$ for all large enough $j \geq 1$. Now, we fix such a $j \geq 1$.

Take $\rho > 0$ small enough so that $\widetilde{\mu}_j(\overline{Q}_\infty) + \rho < 1$ and let

$$\beta = \widetilde{\mu}_j(\overline{Q}_\infty) + \rho \; . \tag{4.56}$$

Then for all large enough $i \in \mathbb{Z}_{\geq 1}$, since $\phi_A(S_{i,j}) \subset \pi^{-1}(x_A)$ and $Q_\infty = \pi^{-1}(Q^*_\infty)$ by Property (i) of $\mathcal{Q}$, we have

$$\begin{aligned}\beta = \widetilde{\mu}_j(\overline{Q}_\infty) + \rho > \widetilde{\nu}_{i,j}(Q_\infty) &= \frac{1}{k_i \operatorname{card}\ S_{i,j}} \sum_{k=0}^{k_i-1} \sum_{\boldsymbol{\theta} \in S_{i,j}} \Delta_{\mathfrak{a}^k \phi_A(\boldsymbol{\theta})}(Q_\infty)\\ &= \frac{1}{k_i} \sum_{k=0}^{k_i-1} \Delta_{\mathfrak{a}^k x_A}(Q^*_\infty) \; .\end{aligned}$$

Thus, for every $\boldsymbol{\theta} \in \mathbb{T}^m$, since $\mathfrak{a}^k \phi_A(\boldsymbol{\theta}) \in Q_\infty$ implies that $\mathfrak{a}^k x_A \in Q^*_\infty$ by Property (i) of $\mathcal{Q}$, we have

$$\operatorname{card}\{k \in \{0, \dots, k_i - 1\} : \mathfrak{a}^k \phi_A(\boldsymbol{\theta}) \in Q_\infty\} < \beta\, k_i \; . \tag{4.57}$$

Let us prove the following counting lemma inspired by [ELMV12, Lem. 4.5] and [LSS19, Lem. 2.4], where ℓ_0 is given by Property (ii) of $\mathcal{Q}$.

Lemma 4.5.2. *There exists a constant $C > 0$ depending only on $\mathbf{r}$ and ℓ_0 such that for all $A \in \mathcal{M}_{m,n}(K_v)$, $\boldsymbol{\theta} \in \mathbb{T}^m$ and $T \in \mathbb{Z}_{\geq 0}$, defining $y = \phi_A(\boldsymbol{\theta})$, $I = \{k \in \mathbb{Z}_{\geq 0} : \mathfrak{a}^k y \in Q_\infty\}$, and*

$$E_{y,T} = \{z \in U^+ y : \forall\, k \in \{0, \dots, T\} - I,\;\; d_{U^+,m}(\mathfrak{a}^k y, \mathfrak{a}^k z) < q_v^{-\ell_0 \max \mathbf{r}}\ \} \; ,$$

the set $E_{y,T}$ can be covered by $C\, q_v^{|\mathbf{r}| \operatorname{card}(I \cap \{0,\dots,T\})}$ closed balls of radius $q_v^{-(\ell_0+T)}$ for the distance $d_{U^+y,\mathbf{r}}$.

Proof. As in the proof of [LSS19, Lemma 2.4], we proceed by induction on T.

By the compactness of $\mathbb{T}^m$, there exists a constant $C \in \mathbb{Z}_{\geq 1}$ depending only on $\mathbf{r}$ and ℓ_0 such that the metric space $(\mathbb{T}^m, d_{\mathbb{T}^m,\mathbf{r}})$ can be covered by C closed balls of radius $q_v^{-\ell_0}$. Since $\phi_A : \mathbb{T}^m \to U^+ y$ is an isometry for the distances $d_{\mathbb{T}^m,\mathbf{r}}$ and $d_{U^+y,\mathbf{r}}$, the orbit $U^+ y$ can be covered by C closed balls for $d_{U^+y,\mathbf{r}}$ of radius $q_v^{-\ell_0}$. Thus the lemma holds for $T = 0$. Let $N_T = C\, q_v^{|\mathbf{r}| \operatorname{card}(I \cap \{0,\dots,T\})}$.

Assume by induction that $E_{y,T-1}$ can be covered by N_{T-1} balls for $d_{U^+y,\mathbf{r}}$

of radius $q_v^{-(\ell_0+T-1)}$. Note that for every $k \in \mathbb{Z}$, since $\pi_v^k \mathcal{O}_v/(\pi_v^{k+1}\mathcal{O}_v)$ has order q_v, every closed ball in K_v of radius q_v^{-k} is the disjoint union of q_v closed ball of radius q_v^{-k-1}. Hence every closed ball for $d_{U^+y,\mathbf{r}}$ of radius $q_v^{-(\ell_0+T-1)}$ in U^+y can be covered by $q_v^{|\mathbf{r}|}$ closed balls for $d_{U^+y,\mathbf{r}}$ of radius $q_v^{-(\ell_0+T)}$. Therefore, if $T \in I$, then $E_{y,T} = E_{y,T-1}$ can be covered by $N_T = q_v^{|\mathbf{r}|} N_{T-1}$ closed balls for $d_{U^+y,\mathbf{r}}$ of radius $q_v^{-(\ell_0+T)}$.

Suppose conversely that $T \notin I$, so that in particular $N_T = N_{T-1}$. Denote the above covering of $E_{y,T-1}$ by $\{B_i : i = 1, \dots, N_{T-1}\}$. Since we have $E_{y,T} \subset E_{y,T-1}$, the set $\{E_{y,T} \cap B_i : i = 1, \dots, N_{T-1}\}$ is a covering of $E_{y,T}$.

Claim. For all $i = 1, \dots, N_{T-1}$ and $z_1, z_2 \in E_{y,T} \cap B_i$, we have $d_{U^+y,\mathbf{r}}(z_1, z_2) \leq q_v^{-(\ell_0+T)}$.

Proof. Since $T \notin I$, we have $d_{U^+,m}(\mathfrak{a}^T y, \mathfrak{a}^T z_j) < q_v^{-\ell_0 \max \mathbf{r}}$ for each $j = 1, 2$. Thus we have $d_{U^+,m}(\mathfrak{a}^T z_1, \mathfrak{a}^T z_2) < q_v^{-\ell_0 \max \mathbf{r}}$ by the ultrametric inequality property of $\|\cdot\|$. Note that since $z_1, z_2 \in U^+y = U^+y_{A,\boldsymbol{\theta}}$, there exist $\boldsymbol{\theta}_1 = (\theta_{1,1}, \dots, \theta_{1,m})$ and $\boldsymbol{\theta}_2 = (\theta_{2,1}, \dots, \theta_{2,m})$ in $\mathbb{T}^m$ such that (denoting in the same way lifts of $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ to K_v^m) we have $z_1 = y_{A,\boldsymbol{\theta}_1}$ and $z_2 = y_{A,\boldsymbol{\theta}_2}$. With $|\langle\ \rangle|$ the map defined after Equation (4.1), it follows that we have

$$\max_{1 \leq i \leq m} q_v^{r_i T} |\langle \theta_{1,i} - \theta_{2,i} \rangle| = d_{\mathbb{T}^m}(\mathfrak{a}_-^T \boldsymbol{\theta}_1, \mathfrak{a}_-^T \boldsymbol{\theta}_2) = d_{U^+,m}(\mathfrak{a}^T y_{A,\boldsymbol{\theta}_1}, \mathfrak{a}^T y_{A,\boldsymbol{\theta}_2})$$
$$= d_{U^+,m}(\mathfrak{a}^T z_1, \mathfrak{a}^T z_2) < q_v^{-\ell_0 \max \mathbf{r}} .$$

Hence, we have

$$d_{U^+y,\mathbf{r}}(z_1, z_2) = d_{\mathbb{T}^m,\mathbf{r}}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = \max_{1 \leq i \leq m} |\langle \theta_{1,i} - \theta_{2,i} \rangle|^{\frac{1}{r_i}} < q_v^{-(\ell_0+T)} ,$$

which concludes the claim. □

By the above claim, the intersection $E_{y,T} \cap B_i$ is contained in a single ball for $d_{U^+y,\mathbf{r}}$ of radius $q_v^{-(\ell_0+T)}$ for each $i = 1, \dots, N_{T-1}$. Thus $E_{y,T}$ can be covered by $N_T = N_{T-1}$ balls for $d_{U^+y,\mathbf{r}}$ of radius $q_v^{-(\ell_0+T)}$. □

Recall that as constructed in the proof of Proposition 4.4.7, there exist a Borel-measurable partition $\mathcal{P} = \{P_1, \dots, P_N, P_\infty\}$ of $\mathcal{Y}$ with $N+1$ elements, and a countably generated Borel-measurable σ-algebra $\mathcal{P}^{U^+}$ of subsets of $\mathcal{Y}$, with $[y]_{\mathcal{P}^{U^+}} = [y]_{\mathcal{P}} \cap B_r^{U^+} y$ for every $y \in \mathcal{Y}(r)$ by Equation (4.42), such that we have $\mathcal{A}^{U^+} = (\mathcal{P}^{U^+})_0^\infty$. We now consider the sequence of σ-algebras $\{(\mathcal{P}^{U^+})_\ell^\infty\}_{\ell\geq 1}$, which is decreasing sequence conversing to $\mathcal{A}_\infty^{U^+}$, i.e., $(\mathcal{P}^{U^+})_\ell^\infty \searrow \mathcal{A}_\infty^{U^+}$. Note that for each $\ell \geq 1$, the σ-algebra $(\mathcal{P}^{U^+})_\ell^\infty$ is countably generated.

If Q is any atom of the finite partition $\mathcal{Q}^{(k_i)} = \bigvee_{k=0}^{k_i-1} \mathfrak{a}^{-k}\mathcal{Q}$ of $\mathcal{Y}$, then fixing any $y \in Q$, by Property (ii) of $\mathcal{Q}$, the intersection $\phi_A(S_{i,j}) \cap Q$ is contained in E_{y,k_i-1} with the notation of Lemma 4.5.2. It follows from Lemma 4.5.2 and Equation (4.57) that $\phi_A(S_{i,j}) \cap Q$ can be covered by $C\, q_v^{|\mathbf{r}|\beta k_i}$ closed balls for $d_{U^+y_{A,0},\mathbf{r}}$ of radius $q_v^{-(\ell_0+k_i-1)} = q_v^{-\ell_0+1}q_v^{-k_i}$, where C depends only on $\mathbf{r}$ and ℓ_0. Since $S_{i,j}$ is $q_v^{-k_i}$-separated (hence $q_v^{-\ell_0+1}q_v^{-k_i}$-separated since $\ell_0 \geq 1$) with respect to $d_{\mathbb{T}^m,\mathbf{r}}$, and since $\phi_A : (\mathbb{T}^n, d_{\mathbb{T}^m,\mathbf{r}}) \to (U^+y_{A,0}, d_{U^+y_{A,0},\mathbf{r}})$ is an isometry, we have

$$\operatorname{card}(\phi_A(S_{i,j}) \cap Q) \leq C\, q_v^{|\mathbf{r}|\beta k_i}\ .$$

Since $\nu_{i,j}$ is the normalised counting measure on $\phi_A(S_{i,j})$, for all large enough $\ell \in \mathbb{Z}_{\geq 1}$, we have $H_{\nu_{i,j}}(\sigma(\mathcal{Q}^{(k_i)})|(\mathcal{P}^{U^+})_\ell^\infty) = H_{\nu_{i,j}}(\sigma(\mathcal{Q}^{(k_i)}))$. Since the map $\Psi = -\log_{q_v}$ is nonincreasing, it hence follows that

$$\begin{aligned}
H_{\nu_{i,j}}(\sigma(\mathcal{Q}^{(k_i)})|(\mathcal{P}^{U^+})_\ell^\infty) = H_{\nu_{i,j}}(\sigma(\mathcal{Q}^{(k_i)})) &= \sum_{Q\in\mathcal{Q}^{(k_i)}} \nu_{i,j}(Q)\Psi\big(\nu_{i,j}(Q)\big) \\
&= \sum_{Q\in\mathcal{Q}^{(k_i)}} \nu_{i,j}(Q)\Psi\Big(\frac{\operatorname{card}(\phi_A(S_{i,j})\cap Q)}{\operatorname{card}\ S_{i,j}}\Big) \\
&\geq \Psi\Big(\frac{C\, q_v^{|\mathbf{r}|\beta k_i}}{\operatorname{card}\ S_{i,j}}\Big) \sum_{Q\in\mathcal{Q}^{(k_i)}} \nu_{i,j}(Q) \\
&= \log_{q_v}(\operatorname{card}\ S_{i,j}) - |\mathbf{r}|\,\beta\, k_i - \log_{q_v} C.
\end{aligned}$$

By taking $\ell \to \infty$ it follows from the continuity of entropy that

$$H_{\nu_{i,j}}(\sigma(\mathcal{Q}^{(k_i)})|\mathcal{A}_\infty^{U^+}) \geq \log_{q_v}(\operatorname{card}\ S_{i,j}) - |\mathbf{r}|\,\beta\, k_i - \log_{q_v} C. \tag{4.58}$$

Since $\mathcal{A}_\infty^{U^+}$ is strictly $\mathfrak{a}$-invariant, by the subadditivity and concavity properties of the entropy as in the proof of [LSS19, Eq. (2.9)], for every $M \in \mathbb{Z}_{\geq 1}$, we have

$$\frac{1}{M} H_{\widetilde{\nu}_{i,j}}(\sigma(\mathcal{Q}^{(M)})|\mathcal{A}_\infty^{U^+}) \geq \frac{1}{k_i} H_{\nu_{i,j}}(\sigma(\mathcal{Q}^{(k_i)})|\mathcal{A}_\infty^{U^+}) - \frac{2\,M\ \log_{q_v}(\operatorname{card}\ \mathcal{Q})}{k_i}. \tag{4.59}$$

Therefore, since $\nu_{i,j}(\infty_{\mathcal{Y}}) = 0$, it follows from Equations (4.59) and (4.58) that

$$\frac{1}{M} H_{\widetilde{\nu}_{i,j}}\big(\overline{\sigma(\mathcal{Q}^{(M)})} | \overline{\mathcal{A}_\infty^{U^+}} \big) = \frac{1}{M} H_{\widetilde{\nu}_{i,j}}\big(\sigma(\mathcal{Q}^{(M)}) | \mathcal{A}_\infty^{U^+}\big)$$
$$\geq \frac{1}{k_i} \big(\log_{q_v}(\text{card } S_{i,j}) - |\mathbf{r}|\beta k_i - \log_{q_v} C - 2\, M \log_{q_v}(\text{card } \mathcal{Q})\big) .$$

Now we can take $i \to \infty$ since the atoms Q of the partition $\overline{\mathcal{Q}}$ and hence of the partition $\overline{\mathcal{Q}^{(M)}}$, satisfy $\widetilde{\mu}_j(\partial Q) = 0$ by the property (iii) of $\mathcal{Q}$. Also, the constants C and card $\mathcal{Q}$ are independent of k_i. Thus it follows from Equation (4.53) that

$$\frac{1}{M} H_{\widetilde{\mu}_j}\big(\overline{\sigma(\mathcal{Q}^{(M)})} | \overline{\mathcal{A}_\infty^{U^+}} \big) \geq |\mathbf{r}|(1-\beta) - \max \mathbf{r}\ (m + \frac{1}{j} - \dim_H \mathbf{Bad}_A(\epsilon)) .$$

By taking $\rho \to 0$ in Equation (4.56), we have

$$\frac{1}{M} H_{\widetilde{\mu}_j}\big(\overline{\sigma(\mathcal{Q}^{(M)})} | \overline{\mathcal{A}_\infty^{U^+}} \big) \geq |\mathbf{r}|(1-\widetilde{\mu}_j(\overline{Q}_\infty)) - \max \mathbf{r}\ (m + \frac{1}{j} - \dim_H \mathbf{Bad}_A(\epsilon)) .$$

Hence, it follows by taking $j \to \infty$ and by using the property (iii) of $\mathcal{Q}$ that

$$\frac{1}{M} H_{\overline{\mu}}\big(\overline{\sigma(\mathcal{Q}^{(M)})} | \overline{\mathcal{A}_\infty^{U^+}} \big) \geq |\mathbf{r}|(1-\overline{\mu}(\overline{Q}_\infty)) - \max \mathbf{r}\ (m - \dim_H \mathbf{Bad}_A(\epsilon)) ,$$

which proves Equation (4.55).

Hence, by taking $M \to \infty$, we have

$$h_{\overline{\mu}}(\mathfrak{a}^{-1} | \overline{\mathcal{A}_\infty^{U^+}}) = h_{\overline{\mu}}(\mathfrak{a} | \overline{\mathcal{A}_\infty^{U^+}}) \geq |\mathbf{r}|(1-\overline{\mu}(\overline{Q}_\infty)) - \max \mathbf{r}\ (m - \dim_H \mathbf{Bad}_A(\epsilon)) ,$$

provided that we have a partition $\mathcal{Q}$ satisfying the above requirements (i), (ii) and (iii). After taking a sufficiently small neighborhood of infinity Q_∞^* in $\mathcal{X}$, so that if $Q_\infty = \pi^{-1}(Q_\infty^*)$, then $\overline{\mu}(\overline{Q}_\infty)$ is sufficiently close to $\overline{\mu}(\infty_{\mathcal{Y}}) = \eta_A$, we can indeed construct a finite Borel-measurable partition $\mathcal{Q}$ of $\mathcal{Y}$ satisfying Properties (i), (ii) and (iii), by following the procedure in [LSS19, Proof of Theorem 4.2, Claim 2]. This proves Assertion (3). □

4.5.2 Effective upper bound on $\dim_H \mathbf{Bad}_A(\epsilon)$

For every $\ell \in \mathbb{Z}_{\leq 1}$, with λ_1 the shortest length function of a nonzero vector of an R_v-lattice (see Subsection 4.1.2), we define

$$\mathcal{X}^{\geq q_v^\ell} = \{x \in \mathcal{X} : \lambda_1(x) \geq q_v^\ell\} \quad \text{and} \quad \mathcal{Y}^{\geq q_v^\ell} = \pi^{-1}(\mathcal{X}^{\geq q_v^\ell}) .$$

Note that by Corollary 4.1.2, we have $\lambda_1(x) \leq q_v$ for all $x \in \mathcal{X}$, thus $\mathcal{X} = \bigcup_{\ell=-\infty}^{1} \mathcal{X}^{\geq q_v^\ell}$. By Mahler's compactness criterion (see for instance [KST17, Theo. 1.1]), the subsets $\mathcal{X}^{\geq q_v^\ell}$ and $\mathcal{Y}^{\geq q_v^\ell}$ are compact.

Lemma 4.5.3. *Let μ' be an $\mathfrak{a}$-invariant Borel probability measure on $\mathcal{Y}$ and let $\mathcal{A}$ be a countably generated sub-σ-algebra of the Borel σ-algebra of $\mathcal{Y}$ which is $\mathfrak{a}^{-1}$-descending and U^+-subordinated modulo μ'. For all $r' \geq \delta' > 0$, $\epsilon \in\,]0,1]$ and $\ell \in \mathbb{Z}_{\leq 0}$, let j_1, j_2 be integers satisfying*

$$j_1 > \frac{d-(d-1)\ell}{\min \mathbf{r}} - \log_{q_v} \delta' \quad \text{and} \quad j_2 > \frac{d-(d-1)\ell}{\min \mathbf{s}} - \frac{n}{d} \log_{q_v} \epsilon .$$

If $y \in \mathcal{Y}^{\geq q_v^\ell}$ satisfies $B_{\delta'}^{U^+,\mathbf{r}} \mathfrak{a}^{-j_1} y \subset [\mathfrak{a}^{-j_1} y]_{\mathcal{A}} \subset B_{r'}^{U^+,\mathbf{r}} \mathfrak{a}^{-j_1} y$, then we have

$$\tau_y^{\mathfrak{a}^{j_1}\mathcal{A}}(\mathfrak{a}^{-j_2}\mathcal{L}_\epsilon) \leq 1 - \left(q_v^{-(j_1+j_2)}(r')^{-1}\epsilon^{\frac{m}{d}}\right)^{|\mathbf{r}|} .$$

Proof. Let $x = \pi(y)$, which belongs to $\mathcal{X}^{\geq q_v^\ell}$. Since x is a unimodular R_v-lattice, by Minkowski's theorem 4.1.1, we hence have

$$q_v^{(d-1)\ell}\lambda_d(x) \leq (\lambda_1(x))^{d-1}\lambda_d(x) \leq \lambda_1(x)\lambda_2(x)\cdots\lambda_d(x) \leq q_v^d ,$$

therefore $\lambda_d(x) \leq q_v^{d-(d-1)\ell}$. There are linearly independent vectors $v_1, \ldots, v_d$ in the R_v-lattice x such that $\|v_i\| \leq q_v^{d-(d-1)\ell}$. Let Δ be the parallelepiped in K_v^d generated by $v_1, \ldots, v_d$, that is,

$$\Delta = \{t_1 v_1 + \cdots + t_d v_d \in K_v^d : \ \forall\, i = 1, \ldots, d, \ \ |\,t_i\,| \leq 1\} .$$

We identify K_v^d with $K_v^m \times K_v^n$. Then for every $\mathbf{b} = (\mathbf{b}^-, \mathbf{b}^+) \in \Delta$ with $\mathbf{b}^- \in K_v^m$ and $\mathbf{b}^+ \in K_v^n$, we have $\|\,\mathbf{b}\,\| \leq q_v^{d-(d-1)\ell}$, hence $\|\,\mathbf{b}^-\|_{\mathbf{r}} \leq q_v^{\frac{d-(d-1)\ell}{\min \mathbf{r}}}$ and $\|\,\mathbf{b}^+\|_{\mathbf{s}} \leq q_v^{\frac{d-(d-1)\ell}{\min \mathbf{s}}}$ since $\ell \leq 0$. Note that the fiber $\pi^{-1}(x)$ can be parametrized as follows: Fixing $g \in G_0$ with $x = g\Gamma_0$, since Δ is a fondamental domain for the action of R_v^d on K_v^d, we have

$$\pi^{-1}(x) = \{w(\mathbf{b})g\Gamma : \mathbf{b} \in \Delta\}, \text{ where } w(\mathbf{b}) = \begin{pmatrix} I_d & \mathbf{b} \\ 0 & 1 \end{pmatrix} .$$

In particular, there exists $\mathbf{b}_0 = (\mathbf{b}_0^-, \mathbf{b}_0^+) \in \Delta$ such that $y = w(\mathbf{b}_0)g\Gamma$.

With a slightly simplified notation, let V_y be the U^+-shape of the atom $[y]_{\mathfrak{a}^{j_1}\mathcal{A}}$ (see Equation (4.49)), so that we have $V_y y = [y]_{\mathfrak{a}^{j_1}\mathcal{A}}$. Let $\Xi = \{\boldsymbol{\theta} \in K_v^m : w(\boldsymbol{\theta}, 0) \in V_y\}$ be the Borel set corresponding to V_y by the canonical bijection $\boldsymbol{\theta} \mapsto w(\boldsymbol{\theta}, 0)$ (see above Equation (4.33)) between K_v^m and U^+. Note that $0 \in \Xi$ as $I_{d+1} \in V_y$. Since $\mathfrak{a}_-^{j_1}$ expands the $\mathbf{r}$-quasinorm on K_v^m with ratio exactly $q_v^{j_1}$ (see Equation (4.30)), and by the assumption on y in the statement of Lemma 4.5.3, we have $B^{U^+,\mathbf{r}}_{q_v^{j_1}\delta'} y \subset [y]_{\mathfrak{a}^{j_1}\mathcal{A}} \subset B^{U^+,\mathbf{r}}_{q_v^{j_1}r'} y$, hence

$$B^{K_v^m,\mathbf{r}}_{q_v^{j_1}\delta'} \subset \Xi \subset B^{K_v^m,\mathbf{r}}_{q_v^{j_1}r'} . \tag{4.60}$$

The atom $[y]_{\mathfrak{a}^{j_1}\mathcal{A}}$ can be parametrized by

$$[y]_{\mathfrak{a}^{j_1}\mathcal{A}} = \left\{w(\mathbf{b})g\Gamma : \exists\, \mathbf{b}^- \in \mathbf{b}_0^- + \Xi, \;\; \mathbf{b} = (\mathbf{b}^-, \mathbf{b}_0^+)\right\} ,$$

and $\tau_y^{\mathfrak{a}^{j_1}\mathcal{A}}$ is the pushforward measure of the normalized Haar measure on the Borel set (with positive measure) $\mathbf{b}_0^- + \Xi$ of K_v^m.

Let us consider the sets

$$\Theta^- = \{\mathbf{b}^- \in K_v^m : \|\, \mathbf{b}^- \|_{\mathbf{r}} < q_v^{-j_2}\epsilon^{\frac{m}{d}}\} \text{ and } \Theta^+ = \{\mathbf{b}^+ \in K_v^n : \|\, \mathbf{b}^+ \|_{\mathbf{s}} < q_v^{j_2}\epsilon^{\frac{n}{d}}\} .$$

If $\mathbf{b} = (\mathbf{b}^-, \mathbf{b}^+) \in \Theta^- \times \Theta^+$, then $\|\mathfrak{a}_-^{j_2}\mathbf{b}^-\|_{\mathbf{r}} < \epsilon^{\frac{m}{d}}$ and $\|\mathfrak{a}_+^{j_2}\mathbf{b}^+\|_{\mathbf{s}} < \epsilon^{\frac{n}{d}}$ by Equation (4.30). By the definition of $\mathcal{L}_\epsilon$ in Equation (4.35), and since the grid $\mathfrak{a}^{j_2}gR_v^m + (\mathfrak{a}_-^{j_2}\mathbf{b}^-, \mathfrak{a}_+^{j_2}\mathbf{b}^+)$ contains the vector $(\mathfrak{a}_-^{j_2}\mathbf{b}^-, \mathfrak{a}_+^{j_2}\mathbf{b}^+)$, we have

$$\mathfrak{a}^{j_2}w(\mathbf{b})g\Gamma = w(\mathfrak{a}_-^{j_2}\mathbf{b}^-, \mathfrak{a}_+^{j_2}\mathbf{b}^+)\mathfrak{a}^{j_2}g\Gamma \notin \mathcal{L}_\epsilon .$$

Hence we have $w(\mathbf{b})g\Gamma \notin \mathfrak{a}^{-j_2}\mathcal{L}_\epsilon$, so that

$$[y]_{\mathfrak{a}^{j_1}\mathcal{A}} - \mathfrak{a}^{-j_2}\mathcal{L}_\epsilon \supset w\big(\,((\mathbf{b}_0^- + \Xi) \times \{\mathbf{b}_0^+\}) \cap (\Theta^- \times \Theta^+)\big)g\Gamma . \tag{4.61}$$

Claim. We have the inclusion $\Theta^- \times \{\mathbf{b}_0^+\} \subset \big((\mathbf{b}_0^- + \Xi) \times \{\mathbf{b}_0^+\}\big) \cap (\Theta^- \times \Theta^+)$.

Proof. We only have to prove that $\mathbf{b}_0^+ \in \Theta^+$ and that $\Theta^- \subset \mathbf{b}_0^- + \Xi$. Since

$(\mathbf{b}_0^-,\mathbf{b}_0^+)\in\Delta$, we have $\|\,\mathbf{b}_0^+\,\|_{\mathbf{s}}\le q_v^{\frac{d-(d-1)\ell}{\min\mathbf{s}}}$, hence the former assertion follows from the assumption that $j_2>\frac{d-(d-1)\ell}{\min\mathbf{s}}-\frac{n}{d}\log_{q_v}\epsilon$.

In order to prove the latter assertion, let us fix $\mathbf{b}^-\in\Theta^-$. Recall that the $\mathbf{r}$-quasinorm $\|\cdot\|_{\mathbf{r}}$ satisfies the ultrametric inequality property, see Equation (4.44). Hence, it follows from the assumptions $j_2>\frac{d-(d-1)\ell}{\min\mathbf{s}}-\frac{n}{d}\log_{q_v}\epsilon$ and $j_1>\frac{d-(d-1)\ell}{\min\mathbf{r}}-\log_{q_v}\delta'$, since $\epsilon\le 1$, that

$$\begin{aligned}\|\,\mathbf{b}^--\mathbf{b}_0^-\,\|_{\mathbf{r}}&\le\max\{\|\,\mathbf{b}^-\,\|_{\mathbf{r}},\|\,\mathbf{b}_0^-\,\|_{\mathbf{r}}\}\le\max\big\{q_v^{-j_2}\epsilon^{\frac{m}{d}},q_v^{\frac{d-(d-1)\ell}{\min\mathbf{r}}}\big\}\\&\le\max\big\{q_v^{-\frac{d-(d-1)\ell}{\min\mathbf{s}}}\epsilon,q_v^{\frac{d-(d-1)\ell}{\min\mathbf{r}}}\big\}=q_v^{\frac{d-(d-1)\ell}{\min\mathbf{r}}}<q_v^{j_1}\delta'\,.\end{aligned}$$

Hence by the left inclusion in Equation (4.60), we have $\mathbf{b}^-\in\mathbf{b}_0^-+B^{K_v^m,\mathbf{r}}_{q_v^{j_1}\delta'}\subset\mathbf{b}_0^-+\Xi$, which concludes the latter assertion. □

Now by Equation (4.61), by the above claim and by the right inclusion in Equation (4.60), we have

$$\begin{aligned}1-\tau_y^{\mathfrak{a}^{j_1}\mathcal{A}}(\mathfrak{a}^{-j_2}\mathcal{L}_\epsilon)&=\tau_y^{\mathfrak{a}^{j_1}\mathcal{A}}([y]_{\mathfrak{a}^{j_1}\mathcal{A}}-\mathfrak{a}^{-j_2}\mathcal{L}_\epsilon)\ge\frac{m_{K_v^m}(\Theta^-)}{m_{K_v^m}(\mathbf{b}_0^-+\Xi)}\ge\frac{m_{K_v^m}\big(B^{K_v^m,\mathbf{r}}_{q_v^{-j_2}\epsilon^{\frac{m}{d}}}\big)}{m_{K_v^m}\big(B^{K_v^m,\mathbf{r}}_{q_v^{j_1}r'}\big)}\\&=\Big(\frac{q_v^{-j_2}\epsilon^{\frac{m}{d}}}{q_v^{j_1}r'}\Big)^{|\mathbf{r}|}=\big(q_v^{-(j_1+j_2)}(r')^{-1}\epsilon^{\frac{m}{d}}\big)^{|\mathbf{r}|}\,.\end{aligned}$$

This proves the lemma. □

Proof of Theorem 1.3.2. We fix a matrix $A\in\mathcal{M}_{m,n}(K_v)$ which is not $(\mathbf{r},\mathbf{s})$-singular on average, or equivalently by Proposition 4.4.2 and the definition of η_A just before Lemma 4.5.1, we assume that $\eta_A<1$. We also fix $\epsilon\in\,]0,1]$ and $r_0\in\,]0,1[$ which is in Equation (4.34).

By Proposition 4.5.1, there exist an $\mathfrak{a}$-invariant Borel probability measure $\overline{\mu}$ on $\overline{\mathcal{Y}}$ (depending on ϵ) and an $\mathfrak{a}$-invariant Borel probability measure μ on $\mathcal{Y}$ (unique since $\eta_A<1$) such that

$$\operatorname{Supp}\overline{\mu}\subset\mathcal{L}_\epsilon\cup\{\infty_{\mathcal{Y}}\},\ \ \overline{\pi}_*\overline{\mu}=\mu_A,\ \text{ and }\ \overline{\mu}=(1-\eta_A)\mu+\eta_A\Delta_{\infty_{\mathcal{Y}}}\,.$$

Take a compact subset K_0 of $\mathcal{X}$ such that $\mu_A(K_0)>0.99\,\mu_A(\mathcal{X})=0.99\,(1-\eta_A)$. Write $K=\pi^{-1}(K_0)$ and choose $r\in\,]0,r_0[$ such that $K\subset\mathcal{Y}(r)$. Then

$\mu(\mathcal{Y}(r)) \geq \mu(K) > 0.99$ since $\eta_A < 1$. Note that the choices of K and r are independent of ϵ since the measure μ_A depends only on A (see Proposition 4.5.1 and Equation (4.52)).

For such an $r > 0$, let $\mathcal{A}^{U^+}$ be the σ-algebra of subsets of $\mathcal{Y}$ constructed in Proposition 4.4.7. Proposition 4.5.1 (3) gives the inequality

$$h_{\overline{\mu}}(\mathfrak{a}^{-1}|\,\overline{\mathcal{A}_\infty^{U^+}}\,) \geq |\mathbf{r}|(1-\eta_A) - \max\mathbf{r}\,(m - \dim_H \mathbf{Bad}_A(\epsilon))\,.$$

By the linearity of entropy (and since the entropy of $\mathfrak{a}^{-1}$ vanishes on the fixed set $\{\infty_\mathcal{Y}\}$), we have

(4.62)
$$h_\mu(\mathfrak{a}^{-1}|\,\mathcal{A}_\infty^{U^+}) = \frac{1}{1-\eta_A} h_{\overline{\mu}}(\mathfrak{a}^{-1}|\,\overline{\mathcal{A}_\infty^{U^+}}\,) \geq |\mathbf{r}| - \frac{\max\mathbf{r}}{1-\eta_A}\,(m - \dim_H \mathbf{Bad}_A(\epsilon))\,.$$

In order to use Lemma 4.4.9 and Proposition 4.4.10, we need an ergodicity assumption on the measures that appear in these statements. We will choose an appropriate ergodic component of μ. Let us denote the ergodic decomposition of μ by

$$\mu = \int_{y\in\mathcal{Y}} \mu_y^{\mathcal{E}}\, d\mu(y).$$

Let $E = \{y \in \mathcal{Y} : \mu_y^{\mathcal{E}}(K) > 0.9\}$. It follows from $\mu(K) > 0.99$ that

$$0.99 < \int_\mathcal{Y} \mu_y^{\mathcal{E}}(K)\, d\mu(y) \leq \mu(E) + 0.9\,\mu(\mathcal{Y} - E) = 0.9 + 0.1\,\mu(E)\,,$$

hence $\mu(E) > 0.9$. By Equation (4.62), we have

$$\int_\mathcal{Y} h_{\mu_y^{\mathcal{E}}}(\mathfrak{a}^{-1}|\,\mathcal{A}_\infty^{U^+})\, d\mu(y) = h_\mu(\mathfrak{a}^{-1}|\,\mathcal{A}_\infty^{U^+}) \geq |\mathbf{r}| - \frac{\max\mathbf{r}}{1-\eta_A}\,(m - \dim_H \mathbf{Bad}_A(\epsilon))\,.$$

Since $h_{\mu_y^{\mathcal{E}}}(\mathfrak{a}^{-1}|\,\mathcal{A}_\infty^{U^+}) \leq |\mathbf{r}|$ for every $y \in \mathcal{Y}$ by Lemma 4.4.9, we have

$$\int_{\mathcal{Y}-E} h_{\mu_y^{\mathcal{E}}}(\mathfrak{a}^{-1}|\,\mathcal{A}_\infty^{U^+})\, d\mu(y) \leq |\mathbf{r}|\,\mu(\mathcal{Y} - E)\,.$$

Hence

$$\begin{aligned}\int_E h_{\mu_y^{\mathcal{E}}}(\mathfrak{a}^{-1}|\,\mathcal{A}_\infty^{U^+})\, d\mu(y) &\geq |\mathbf{r}|\,\mu(E) - \frac{\max\mathbf{r}}{1-\eta_A}\,(m - \dim_H \mathbf{Bad}_A(\epsilon)) \\ &\geq \mu(E)\Big(|\mathbf{r}| - \frac{\max\mathbf{r}}{0.9\,(1-\eta_A)}\,(m - \dim_H \mathbf{Bad}_A(\epsilon))\Big)\,.\end{aligned}$$

Therefore, there exists $z \in \mathcal{Y}$ such that $\mu_z^{\mathcal{E}}(K) > 0.9$ and

$$h_{\mu_z^{\mathcal{E}}}(\mathfrak{a}^{-1} | \mathcal{A}_\infty^{U^+}) \geq |\mathbf{r}| - \frac{\max \mathbf{r}}{0.9\,(1-\eta_A)}\,(m - \dim_H \mathbf{Bad}_A(\epsilon))\ .$$

We denote $\lambda = \mu_z^{\mathcal{E}}$ for such a $z \in \mathcal{Y}$. Then λ is an $\mathfrak{a}$-invariant ergodic Borel probability measure on $\mathcal{Y}$ and $\operatorname{Supp}\lambda \subset \operatorname{Supp}\mu \subset \mathcal{L}_\epsilon$. By Lemma 4.4.9, we have

$$H_\lambda(\mathcal{A}^{U^+} | \,\mathfrak{a}\,\mathcal{A}^{U^+}) \geq |\mathbf{r}| - \frac{\max \mathbf{r}}{0.9\,(1-\eta_A)}\,(m - \dim_H \mathbf{Bad}_A(\epsilon))\ . \tag{4.63}$$

We will apply Lemma 4.5.3 with $\mu' = \lambda$ and $\mathcal{A} = \mathfrak{a}^{-k}\mathcal{A}^{U^+}$ for some $k \geq 1$. Take an integer $\ell \leq 0$ such that $K \subset \mathcal{Y}^{\geq q_v^\ell}$, which depends only on A. Set

$$j_1 = \left\lceil \frac{d-(d-1)\ell}{\min \mathbf{r}} - \log_{q_v} \delta' \right\rceil + 1 \ \text{ and } \ j_2 = \left\lceil \frac{d-(d-1)\ell}{\min \mathbf{s}} - \frac{n}{d}\log_{q_v} \epsilon \right\rceil + 1\ ,$$

where δ' will be determined later on.

Let $k = \left\lceil \log_{q_v} \left(r^{\frac{1}{\max \mathbf{r}}}\, \epsilon^{-\frac{m}{d}} \right) \right\rceil + j_2 + 1$ and $\mathcal{A} = \mathfrak{a}^{-k}\mathcal{A}^{U^+}$. By the properties of $\mathcal{A}^{U^+}$ given in Proposition 4.4.7 and since $K \subset \mathcal{Y}(r)$, for every $y \in K$, we have

$$B_{\delta_r}^{U^+} y \subset [y]_{\mathcal{A}^{U^+}} \subset B_r^{U^+} y\ .$$

It follows from Equations (4.34) and (4.45) that since $r \leq r_0$, for any $y \in K$,

$$B^{U^+,\mathbf{r}}_{(\delta_r/2)^{\frac{1}{\min \mathbf{r}}}} y \subset B^{U^+,m}_{\delta_r/2} y \subset [y]_{\mathcal{A}^{U^+}} \subset B^{U^+,m}_{\delta_r} y \subset B^{U^+,\mathbf{r}}_{r^{\frac{1}{\max \mathbf{r}}}} y\ .$$

Hence, by Equation (4.46), we have

$$B^{U^+,\mathbf{r}}_{q_v^{-k}(\delta_r/2)^{\frac{1}{\min \mathbf{r}}}} \mathfrak{a}^{-k} y \subset [\mathfrak{a}^{-k} y]_{\mathfrak{a}^{-k}\mathcal{A}^{U^+}} = [\mathfrak{a}^{-k} y]_{\mathcal{A}} \subset B^{U^+,\mathbf{r}}_{q_v^{-k} r^{\frac{1}{\max \mathbf{r}}}} \mathfrak{a}^{-k} y\ .$$

Thus for every $y \in \mathfrak{a}^k K$, we have

$$B^{U^+,\mathbf{r}}_{\delta'} y \subset [y]_{\mathcal{A}} \subset B^{U^+,\mathbf{r}}_{r'} y\ , \tag{4.64}$$

where, by the definition of k, we take

$$r' = q_v^{-j_2-1} \epsilon^{\frac{m}{d}} \quad \text{and} \quad \delta' = q_v^{-1}\, r^{-\frac{1}{\max \mathbf{r}}}\, (\delta_r/2)^{\frac{1}{\min \mathbf{r}}}\, r'\ .$$

Equation (4.64) implies that for every $y \in \mathfrak{a}^{j_1+k}K$, we have

$$B^{U^+,\mathbf{r}}_{\delta'}\mathfrak{a}^{-j_1}y \subset [\mathfrak{a}^{-j_1}y]_{\mathcal{A}} \subset B^{U^+,\mathbf{r}}_{r'}\mathfrak{a}^{-j_1}y\ . \tag{4.65}$$

Now, we will use Proposition 4.4.10 with $j = j_1$, $K' = \mathfrak{a}^k K$ (which is U^+-saturated since so is K and as $\mathfrak{a}$ normalizes U^+), and $\epsilon = r'$ (which satisfies the assumption of Proposition 4.4.10 by Equation (4.64)). We claim that

$$B^{U^+,\mathbf{r}}_{q_v^{-1}\epsilon^{\frac{m}{d}}}\mathcal{L}_\epsilon \subset \mathcal{L}_\epsilon\ . \tag{4.66}$$

Indeed, for all $y \in \mathcal{L}_\epsilon$ and $\boldsymbol{\theta} \in K_v^m$ such that $\|\,\boldsymbol{\theta}\,\|_{\mathbf{r}} \leq q_v^{-1}\epsilon^{\frac{m}{d}}$, for every vector $u = (u^-, u^+)$ in the grid $w(\boldsymbol{\theta}, \mathbf{0})y$, we can write $u = v + (\boldsymbol{\theta}, \mathbf{0})$ for some $v = (v^-, v^+)$ in the grid $\widetilde{\Lambda}_y$ associated with y (see Equation (4.31)). Since $y \in \mathcal{L}_\varepsilon$, we have (see Equation (4.35)) $\|\,v\,\|_{\mathbf{r},\mathbf{s}} = \max\{\|\,v^-\,\|_{\mathbf{r}}^{\frac{d}{m}}, \|\,v^+\,\|_{\mathbf{s}}^{\frac{d}{n}}\} \geq \epsilon$. Since $u^+ = v^+$, if $\|\,v^+\,\|_{\mathbf{s}}^{\frac{d}{n}} \geq \epsilon$, then $w(\boldsymbol{\theta}, \mathbf{0})y \in \mathcal{L}_\epsilon$. Otherwise $\|\,v^-\,\|_{\mathbf{r}}^{\frac{d}{m}} \geq \epsilon$. We then have $\|\,\boldsymbol{\theta}\,\|_{\mathbf{r}} \leq q_v^{-1}\epsilon^{\frac{m}{d}} < \epsilon^{\frac{m}{d}} \leq \|\,v^-\,\|_{\mathbf{r}}$. It follows from the equality case of the ultrametric inequality property of $\|\ \|_{\mathbf{r}}$ that

$$\|\,u^-\,\|_{\mathbf{r}} = \|\,\boldsymbol{\theta} + v^-\,\|_{\mathbf{r}} = \max\big\{\|\,\boldsymbol{\theta}\,\|_{\mathbf{r}},\ \|\,v^-\,\|_{\mathbf{r}}\big\} = \|\,v^-\,\|_{\mathbf{r}} \geq \epsilon^{\frac{m}{d}}\ .$$

Hence $w(\boldsymbol{\theta}, \mathbf{0})y \in \mathcal{L}_\epsilon$, which proves Equation (4.66).

By Proposition 4.4.7, the σ-algebra $\mathcal{A}^{U^+}$ is $\mathfrak{a}^{-1}$-descending and U^+-subordinated modulo λ, and so is $\mathcal{A} = \mathfrak{a}^{-k}\mathcal{A}^{U^+}$ since $\mathfrak{a}$ normalizes U^+. Note that $\operatorname{Supp}\lambda \subset \mathfrak{a}^{-j_2}\mathcal{L}_\epsilon$ since λ is $\mathfrak{a}$-invariant. By Equations (4.46) and (4.66), we have

$$B^{U^+,\mathbf{r}}_{r'}\mathfrak{a}^{-j_2}\mathcal{L}_\epsilon = \mathfrak{a}^{-j_2}B^{U^+,\mathbf{r}}_{q_v^{j_2}r'}\mathcal{L}_\epsilon = \mathfrak{a}^{-j_2}B^{U^+,\mathbf{r}}_{q_v^{-1}\epsilon^{\frac{m}{d}}}\mathcal{L}_\epsilon \subset \mathfrak{a}^{-j_2}\mathcal{L}_\epsilon\ .$$

Note that we have

$$\tau_y^{\mathfrak{a}^{j_1}\mathcal{A}}(\mathcal{Y} - \mathfrak{a}^k K) = 0$$

for λ-almost every $y \in \mathfrak{a}^k K$, since then (see just above Proposition 4.4.10) the support $\operatorname{Supp}\tau_y^{\mathfrak{a}^{j_1}\mathcal{A}}$ is contained in $[y]_{\mathfrak{a}^{j_1}\mathcal{A}}$, which is contained in U^+y, hence in $\mathfrak{a}^k K$ since $\mathfrak{a}$ normalizes U^+ and $K = \pi^{-1}(K_0)$ is U^+-saturated. Therefore, it follows from Proposition 4.4.10 for the first line, from the fact that the integrated function is nonpositive (hence its integral on a smaller domain is larger) for the third line, that

$$\begin{aligned}
H_\lambda(\mathcal{A}|\mathfrak{a}^{j_1}\mathcal{A}) &\leq j_1|\mathbf{r}| + \int_{\mathcal{Y}} \log_{q_v} \tau_y^{\mathfrak{a}^{j_1}\mathcal{A}}((\mathcal{Y}-\mathfrak{a}^k K)\cup B_{r'}^{U^+,\mathbf{r}}\operatorname{Supp}\lambda)\; d\lambda(y)\\
&\leq j_1|\mathbf{r}| + \int_{\mathcal{Y}} \log_{q_v} \tau_y^{\mathfrak{a}^{j_1}\mathcal{A}}((\mathcal{Y}-\mathfrak{a}^k K)\cup \mathfrak{a}^{-j_2}\mathcal{L}_\epsilon)\; d\lambda(y)\\
&\leq j_1|\mathbf{r}| + \int_{\mathfrak{a}^k K\cap\mathfrak{a}^{j_1+k}K\cap\mathcal{Y}^{\geq q_v^\ell}} \log_{q_v} \tau_y^{\mathfrak{a}^{j_1}\mathcal{A}}((\mathcal{Y}-\mathfrak{a}^k K)\cup \mathfrak{a}^{-j_2}\mathcal{L}_\epsilon)\; d\lambda(y)\\
&= j_1|\mathbf{r}| + \int_{\mathfrak{a}^k K\cap\mathfrak{a}^{j_1+k}K\cap\mathcal{Y}^{\geq q_v^\ell}} \log_{q_v} \tau_y^{\mathfrak{a}^{j_1}\mathcal{A}}(\mathfrak{a}^{-j_2}\mathcal{L}_\epsilon)\; d\lambda(y)\;.
\end{aligned}$$

We now apply Lemma 4.5.3 with as said above $\mu' = \lambda$ and $\mathcal{A} = \mathfrak{a}^{-k}\mathcal{A}^{U^+}$, and with $y \in \mathfrak{a}^{j_1+k}K\cap\mathcal{Y}^{\geq q_v^\ell}$ which satisfies the assumption of Lemma 4.5.3 by Equation (4.65). Thus

$$\tau_y^{\mathfrak{a}^{j_1}\mathcal{A}}(\mathfrak{a}^{-j_2}\mathcal{L}_\epsilon) \leq 1 - \Big(q_v^{-(j_1+j_2)}r'^{-1}\epsilon^{\frac{m}{d}}\Big)^{|\mathbf{r}|} = 1 - q_v^{-(j_1-1)|\mathbf{r}|}\;.$$

Hence

$$-\log_{q_v}\tau_y^{\mathfrak{a}^{j_1}\mathcal{A}}(\mathfrak{a}^{-j_2}\mathcal{L}_\epsilon) \geq -\log_{q_v}\big(1-q_v^{-(j_1-1)|\mathbf{r}|}\big) \geq \frac{q_v^{-(j_1-1)|\mathbf{r}|}}{\ln q_v}\;.$$

Note that $\lambda(\mathfrak{a}^kK\cap\mathfrak{a}^{j_1+k}K\cap\mathcal{Y}^{\geq q_v^\ell}) \geq \frac{1}{2}$ since λ is $\mathfrak{a}$-invariant, $K\subset\mathcal{Y}^{\geq q_v^\ell}$ and $\lambda(K) > 0.9$, so that the three sets $\mathfrak{a}^kK$, $\mathfrak{a}^{j_1+k}K$ and $\mathcal{Y}^{\geq q_v^\ell}$ have λ-measure > 0.9, hence their pairwise intersections have λ-measure $> 2\times 0.9 - 1 = 0.8$, and their triple intersection has λ-measure $> 2\times 0.8 - 1 = 0.6$. It follows from Equation (4.39) and the invariance under $\mathfrak{a}$ of λ, hence of the conditional entropy, that

$$\begin{aligned}
|\mathbf{r}| - H_\lambda(\mathcal{A}^{U^+}|\,\mathfrak{a}\,\mathcal{A}^{U^+}) = |\mathbf{r}| - \frac{1}{j_1}H_\lambda(\mathcal{A}^{U^+}|\,\mathfrak{a}^{j_1}\mathcal{A}^{U^+}) &= |\mathbf{r}| - \frac{1}{j_1}H_\lambda(\mathcal{A}|\,\mathfrak{a}^{j_1}\mathcal{A})\\
&\geq -\frac{1}{j_1}\int_{\mathfrak{a}^k K\cap\mathfrak{a}^{j_1+k}K\cap\mathcal{Y}^{\geq q_v^\ell}} \log_{q_v}\tau_y^{\mathfrak{a}^{j_1}\mathcal{A}}(\mathfrak{a}^{-j_2}\mathcal{L}_\epsilon)\; d\lambda(y)\\
&\geq \frac{q_v^{|\mathbf{r}|}}{2\ln q_v}\frac{q_v^{-j_1|\mathbf{r}|}}{j_1}\;.
\end{aligned}$$

Therefore, by Equation (4.63), we have

$$\frac{\max \mathbf{r}}{0.9\,(1-\eta_A)}\,(m-\dim_H \mathbf{Bad}_A(\epsilon)) \geq |\mathbf{r}| - H_\lambda(\mathcal{A}^{U^+}\,|\,\mathfrak{a}\,\mathcal{A}^{U+}) \geq \frac{q_v^{|\mathbf{r}|}}{2\ln q_v}\frac{q_v^{-j_1|\mathbf{r}|}}{j_1}\ .$$

Observe that

$$\begin{aligned}
j_1 &= \Big\lceil\, \frac{d-(d-1)\ell}{\min \mathbf{r}} - \log_{q_v} \delta' \,\Big\rceil + 1 \\
&= \Big\lceil\, \frac{d-(d-1)\ell}{\min \mathbf{r}} - \log_{q_v} \Big(\, \frac{(\delta_r/2)^{\frac{1}{\min \mathbf{r}}}}{q_v^2\ r^{\frac{1}{\max \mathbf{r}}}} q_v^{-j_2} \epsilon^{\frac{m}{d}}\,\Big)\Big\rceil + 1 \\
&= \Big\lceil\, \frac{d-(d-1)\ell}{\min \mathbf{r}} + \Big\lceil\, \frac{d-(d-1)\ell}{\min \mathbf{s}} - \frac{n}{d}\log_{q_v}\epsilon\,\Big\rceil \\
&\quad + 1 - \frac{m}{d}\log_{q_v}\epsilon - \log_{q_v}\frac{(\delta_r/2)^{\frac{1}{\min \mathbf{r}}}}{q_v^2\ r^{\frac{1}{\max \mathbf{r}}}}\,\Big\rceil + 1 \\
&\leq (\,d-(d-1)\ell\,)\Big(\frac{1}{\min \mathbf{r}} + \frac{1}{\max \mathbf{s}}\Big) - \log_{q_v}\frac{(\delta_r/2)^{\frac{1}{\min \mathbf{r}}}}{q_v^2 r^{\frac{1}{\max \mathbf{r}}}} + 4 - \log_{q_v}\epsilon\ .
\end{aligned}$$

The constants η_A, ℓ, δ_r, and r depend only on the fixed matrix $A \in \mathcal{M}_{m,n}(K_v)$. Hence there exists a constant $c(A) > 0$ depending only on d, $\mathbf{r}$, $\mathbf{s}$ and A such that

$$m - \dim_H \mathbf{Bad}_A(\epsilon) \geq c(A)\ \frac{\epsilon^{\,|\mathbf{r}|}}{\log_{q_v}(1/\epsilon)}\ .$$

This proves Theorem 1.3.2. □

Chapter 5

Weighted singular vectors

5.1 Fractal sutructure and Hausdorff dimension

5.1.1 Fractal structure

A *tree* $\mathcal{T}$ is a connected graph without cycles. If we take a vertex τ_0 and fix it (we call it a *root*), then $\mathcal{T}$ is a *rooted tree*. In this paper, we identify $\mathcal{T}$ with the set of vertices of $\mathcal{T}$. It can be checked directly from the definition of $\mathcal{T}$ that any $\tau \in \mathcal{T}$ can be joined to τ_0 by a unique geodesic edge path. We define the *height* of τ as the length of the geodesic edge path joining τ, τ_0 and denote the set of vertices of height n by $\mathcal{T}_n$. For any $\tau \in \mathcal{T}_n$, there exists a unique $\tau_{n-1} \in \mathcal{T}_{n-1}$ such that τ and τ_{n-1} are adjacent. Then we say τ is a *son* of τ_{n-1} and denote the set of all sons of τ_{n-1} by $\mathcal{T}(\tau_{n-1})$. The *boundary* of $\mathcal{T}$, denoted by $\partial\mathcal{T}$, is the set of all sequences $\{\tau_n\} = \{\tau_n\}_{n\in\mathbb{N}\cup\{0\}}$ where τ_n is a son of τ_{n-1} for all $n \in \mathbb{N}$.

A *fractal structure* on $\mathbb{R}^d$ is a pair $(\mathcal{T}, \beta)$ where $\mathcal{T}$ is a rooted tree and β is a map from $\mathcal{T}$ to the set of nonempty compact subsets of $\mathbb{R}^d$. A *fractal* associated to $(\mathcal{T}, \beta)$ is a set

$$\mathcal{F}(\mathcal{T}, \beta) = \bigcup_{\{\tau_n\}\in\partial\mathcal{T}} \bigcap_{n=0}^{\infty} \beta(\tau_n).$$

A fractal structure $(\mathcal{T}, \beta)$ is said to be *regular* if it satisfies the followings:

- each vertex of $\mathcal{T}$ has at least one son;
- if τ is a son of τ', then $\beta(\tau) \subset \beta(\tau')$;

- for any $\{\tau_n\} \in \partial\mathcal{T}$, $\operatorname{diam}\beta(\tau_n) \to 0$ as $n \to \infty$.

5.1.2 Self-affine structure and lower bound

A *self-affine structure* on $\mathbb{R}^d$ is a fractal structure $(\mathcal{T}, \beta)$ on $\mathbb{R}^d$ such that for $\tau \in \mathcal{T}$ the compact subset $\beta(\tau)$ of $\mathbb{R}^d$ is given by a d-dimensional rectangle with size $L^{(1)}(\tau) \times \cdots \times L^{(d)}(\tau)$. A self-affine structure is *regular* if it is a regular fractal structure.

The following theorem is a generalization of [LSST20, Theorem 2.1] for d-dimensional self-affine structures.

Theorem 5.1.1. *Let $(\mathcal{T}, \beta)$ be a regular self-affine structure on $\mathbb{R}^d$ that associates to sequences $\{\rho_n\}, \{C_n\}, \{L_n^{(j)}\}$ for $j = 1, \ldots, d$ of positive real numbers indexed by $\mathbb{N} \cup \{0\}$ with the following properties:*

1. *The sequence $\{L_n^{(j)}\}$ is decreasing in $n \in \mathbb{N} \cup \{0\}$ for each $j = 1, \ldots, d$.*
2. *There exists $1 \le \ell < d$ such that*
$$L_n^{(1)} = \cdots = L_n^{(\ell)} < L_n^{(\ell+1)} \le \cdots \le L_n^{(d)} \quad \textit{and} \quad L^{(j)}(\tau) = L_n^{(j)}$$
for all $n \in \mathbb{N} \cup \{0\}$, $j = 1, \ldots, d$, and $\tau \in \mathcal{T}_n$;
3. *$C_0 = 1$ and $\#\mathcal{T}(\tau) \ge C_n$ for all $n \in \mathbb{N}$ and $\tau \in \mathcal{T}_{n-1}$;*
4. *$\rho_n \le 1$ for all $n \in \mathbb{N}$ and*
$$\textit{dist}(\beta(\tau), \beta(\kappa)) \ge \rho_{n+1} L_n^{(1)}$$
for all $\tau_n \in \mathcal{T}_n$ and distinct $\tau, \kappa \in \mathcal{T}(\tau_n)$.

We denote by

$$P_n = \prod_{i=0}^{n} C_i,$$
$$D_n = \max\{i \ge n : L_i^{(d)} \ge L_n^{(1)}\},$$
$$s = \sup\left\{ t > 0 : \lim_{n\to\infty} \frac{\log(P_n (L_n^{(1)})^t \rho_{n+1}^t \cdot \prod_{i=n+1}^{D_n} \rho_i^\ell C_i)}{\max\{D_n - n, 1\}} = \infty \right\}.$$

If $s > d - \ell$, then $\dim_H \mathcal{F}(\mathcal{T}, \beta) \ge s$.

Using Theorem 5.1.1, we obtain the following corollary which is a generalization of [LSST20, Corollary 2.3 and Corollary 2.4] for d-dimensional self-affine structures.

Corollary 5.1.2. *With the notations in Theorem 5.1.1, suppose that there exists $k, n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the followings hold:*

(i) $\frac{L_{kn}^{(d)}}{L_{kn-1}^{(d)}} \leq \frac{L_n^{(1)}}{L_{n-1}^{(1)}}$ *and* $L_{kn_0-1}^{(d)} < L_{n_0-1}^{(1)}$,

(ii) $e^{n/k} \leq C_n \leq e^{kn}$,

(iii) $e^{-kn} \leq \rho_n \leq e^{-n/k}$,

(iv) $\rho_n^\ell C_n \prod_{j=\ell+1}^d L_n^{(j)}/L_{n-1}^{(j)} \geq n^{-k}$.

If the limit

$$\lim_{n\to\infty} \frac{\log\left(C_n \prod_{j=\ell+1}^d L_n^{(j)}/L_{n-1}^{(j)}\right)}{-\log\left(L_n^{(1)}/L_{n-1}^{(1)}\right)}$$

exists and is equal to $r > 0$, then $\dim_H \mathcal{F}(\mathcal{T}, \beta) \geq d - \ell + r$.

Proof of Corollary 5.1.2. By the assumptions (iii) and (iv), since the sequence $\{L_n^{(j)}\}$ is decreasing in $n \in \mathbb{N} \cup \{0\}$ for each $j = 1, \ldots, d$, we have

$$\log\left(C_n \prod_{j=\ell+1}^d L_n^{(j)}/L_{n-1}^{(j)}\right) = O(n),$$

which implies that $-\log\left(L_n^{(1)}/L_{n-1}^{(1)}\right) \to \infty$ as $n \to \infty$. Hence, using

$$\frac{\log\left(P_n \prod_{j=\ell+1}^d L_n^{(j)}\right)}{-\log L_n^{(1)}} = \frac{\log\left(C_0 \prod_{j=\ell+1}^d L_0^{(j)}\right) + \sum_{i=1}^n \log\left(C_i \prod_{j=\ell+1}^d L_i^{(j)}/L_{i-1}^{(j)}\right)}{-\log L_0^{(1)} - \sum_{i=1}^n \log\left(L_i^{(1)}/L_{i-1}^{(1)}\right)},$$

it follows that

$$\lim_{n\to\infty} \frac{\log\left(C_n \prod_{j=\ell+1}^d L_n^{(j)}/L_{n-1}^{(j)}\right)}{-\log\left(L_n^{(1)}/L_{n-1}^{(1)}\right)} = \lim_{n\to\infty} \frac{\log\left(P_n \prod_{j=\ell+1}^d L_n^{(j)}\right)}{-\log L_n^{(1)}}.$$

Let us denote by

$$s = d - \ell + r = d - \ell + \lim_{n\to\infty} \frac{\log\left(P_n \prod_{j=\ell+1}^d L_n^{(j)}\right)}{-\log L_n^{(1)}}.$$

By the regularity of the given self-affine structure $(\mathcal{T}, \beta)$, we have that $L_n^{(1)} \to 0$ as $n \to \infty$, which implies

$$s = \sup\left\{ t > 0 : \lim_{n\to\infty} P_n (L_n^{(1)})^t \prod_{j=\ell+1}^{d} \frac{L_n^{(j)}}{L_n^{(1)}} = \infty \right\}. \tag{5.1}$$

We will show that $\dim_H \mathcal{F}(\mathcal{T}, \beta) \geq s$ using the equality (5.1).

Recall that $D_n = \max\left\{ i \geq n : L_i^{(d)} \geq L_n^{(1)} \right\}$. Since the sequence $\{L_n^{(j)}\}$ is decreasing in $n \in \mathbb{N} \cup \{0\}$ for each $j = 1, \dots, d$, we have

$$\begin{aligned}
L_{kn}^{(d)} &= L_{kn_0-1}^{(d)} \prod_{i=kn_0}^{kn} \frac{L_i^{(d)}}{L_{i-1}^{(d)}} \leq L_{kn_0-1}^{(d)} \prod_{i=n_0}^{n} \frac{L_{ki}^{(d)}}{L_{ki-1}^{(d)}} \\
&\leq L_{kn_0-1}^{(d)} \prod_{i=n_0}^{n} \frac{L_i^{(1)}}{L_{i-1}^{(1)}} && \text{by assumption } (i) \\
&= L_{kn_0-1}^{(d)} \frac{L_n^{(1)}}{L_{n_0-1}^{(1)}} \\
&\leq L_n^{(1)} && \text{by assumption } (i).
\end{aligned}$$

Hence we have $D_n \leq kn$.

Given $t > 0$, $\epsilon > 0$, it follows from the assumptions (ii), (iii), and $D_n \leq kn$ that $\rho_{D_n+1}^{\ell} C_{D_n+1} \leq e^{k(D_n+1)} \leq e^{k(kn+1)}$. Since $P_n^{\epsilon} = (\prod_{i=0}^{n} C_i)^{\epsilon} \geq e^{\frac{n(n+1)\epsilon}{2k}}$, we have

$$\rho_{D_n+1}^{\ell} C_{D_n+1} \leq P_n^{\epsilon} \tag{5.2}$$

for all large enough $n \geq 1$. Similarly, it follows from the assumptions (iii), (iv), and $D_n \leq kn$ that

$$\rho_{n+1}^t \prod_{i=n+1}^{D_n+1} \left(\rho_i^\ell C_i \prod_{j=\ell+1}^{d} \frac{L_i^{(j)}}{L_{i-1}^{(j)}} \right) \geq e^{-tk(n+1)} \prod_{i=n+1}^{D_n+1} i^{-k} \geq e^{-tk(n+1)}(kn+1)^{-k(kn-n)}$$
$$\geq e^{-tk(n+1)-k(kn-n)\log(kn+1)}.$$

The inequality $P_n^{-\epsilon} \leq e^{-\frac{n(n+1)\epsilon}{2k}}$ implies that

$$\rho_{n+1}^t \prod_{i=n+1}^{D_n+1} \left(\rho_i^\ell C_i \prod_{j=\ell+1}^{d} \frac{L_i^{(j)}}{L_{i-1}^{(j)}} \right) \geq P_n^{-\epsilon} \tag{5.3}$$

for all large enough $n \geq 1$.

Fix a real number t with $d - \ell < t < s$ and take sufficiently small ϵ such that $d - \ell < t/(1 - 3\epsilon) < s$. By the equality (5.1), we have

$$\lim_{n\to\infty} P_n (L_n^{(1)})^{t/(1-3\epsilon)} \prod_{j=\ell+1}^{d} \frac{L_n^{(j)}}{L_n^{(1)}} \geq 1 \tag{5.4}$$

for all large enough $n \geq 1$.

For all large enough $n \geq 1$ so that the above inequalities (5.2), (5.3), and (5.4) hold, it follows from (5.2) that

$$P_n(L_n^{(1)})^t \rho_{n+1}^t \prod_{i=n+1}^{D_n} \rho_i^\ell C_i \geq P_n^{1-\epsilon}(L_n^{(1)})^t \rho_{n+1}^t \prod_{i=n+1}^{D_n+1} \rho_i^\ell C_i$$
$$= P_n^{1-\epsilon}(L_n^{(1)})^t \rho_{n+1}^t \left(\prod_{j=\ell+1}^{d} \frac{L_n^{(j)}}{L_{D_n+1}^{(j)}} \right) \cdot \prod_{i=n+1}^{D_n+1} \left(\rho_i^\ell C_i \prod_{j=\ell+1}^{d} \frac{L_i^{(j)}}{L_{i-1}^{(j)}} \right)$$
$$\geq P_n^{1-\epsilon}(L_n^{(1)})^t \rho_{n+1}^t \left(\prod_{j=\ell+1}^{d} \frac{L_n^{(j)}}{L_n^{(1)}} \right) \cdot \prod_{i=n+1}^{D_n+1} \left(\rho_i^\ell C_i \prod_{j=\ell+1}^{d} \frac{L_i^{(j)}}{L_{i-1}^{(j)}} \right).$$

Using (5.3) and (5.4), we have

$$\begin{aligned}
P_n(L_n^{(1)})^t \rho_{n+1}^t \prod_{i=n+1}^{D_n} \rho_i^\ell C_i &\geq P_n^{1-\epsilon}(L_n^{(1)})^t \left(\prod_{j=\ell+1}^{d} \frac{L_n^{(j)}}{L_n^{(1)}} \right) P_n^{-\epsilon} \\
&\geq P_n^{1-\epsilon}(L_n^{(1)})^t \left(\prod_{j=\ell+1}^{d} \frac{L_n^{(j)}}{L_n^{(1)}} \right)^{1-3\epsilon} P_n^{-2\epsilon} P_n^{\epsilon} \\
&= \left(P_n (L_n^{(1)})^{t/(1-3\epsilon)} \prod_{j=\ell+1}^{d} \frac{L_n^{(j)}}{L_n^{(1)}} \right)^{1-3\epsilon} P_n^{\epsilon} \\
&\geq P_n^{\epsilon}.
\end{aligned}$$

It follows that for all large enough $n \geq 1$,

$$\log\left(P_n(L_n^{(1)})^t \rho_{n+1}^t \prod_{i=n+1}^{D_n} \rho_i^\ell C_i \right) \geq \epsilon \log P_n \gg \epsilon n^2 \geq \epsilon \frac{n}{k-1}(D_n - n),$$

where the implied constant is independent of n. Hence $\dim_H \mathcal{F}(\mathcal{T}, \beta) \geq t$ by Theorem 5.1.1. Since we choose arbitrary t with $d - \ell < t < s$, it concludes Corollary 5.1.2. □

By *elementary squares* of $\beta(\tau)$ for $\tau \in \mathcal{T}$, we mean closed squares contained in $\beta(\tau)$ whose side length is equal to $L^{(1)}(\tau)$.

Lemma 5.1.3. *For $n \in \mathbb{N} \cup \{0\}$ with $D_n > n$, let $\kappa \in \mathcal{T}_n$ and $\tau \in \mathcal{T}_{i-1}$ where $n + 1 \leq i \leq D_n$. Then for any elementary square S of $\beta(\kappa)$,*

$$\#\{\tau' \in \mathcal{T}(\tau) : \beta(\tau') \cap S \neq \varnothing\} \leq (16d)^d \rho_i^{-\ell}.$$

Proof. Through this proof, we denote the size of a rectange R in $\mathbb{R}^d$ by $l_1(R) \times \cdots \times l_d(R)$.

For a fixed elementary square S of $\beta(\kappa)$, let $R_0 = \beta(\tau) \cap S$ and

$$\mathcal{S} = \{\beta(\tau') \cap S : \tau' \in \mathcal{T}(\tau), \beta(\tau') \cap S \neq \varnothing\}.$$

If $R_0 = \varnothing$, then there is nothing to prove since $\#\{\tau' \in \mathcal{T}(\tau) : \beta(\tau') \cap S \neq \varnothing\} = 0$.

Let $j_i \in \{1, \dots, d-1\}$ be an integer such that $L_i^{(d)} \geq \cdots \geq L_i^{(j_i+1)} \geq$

$L_n^{(1)} > L_i^{(j_i)} \geq \cdots \geq L_i^{(1)}$. Note that $j_i \geq \ell$. Let R_0' be the rectangle with the same center as R_0 such that

$$l_j(R_0') = \begin{cases} 4L_{i-1}^{(j)} & \text{for } j = 1, \dots, j_i \\ 4L_n^{(1)} & \text{for } j = j_i + 1, \dots, d. \end{cases}$$

Similarly, for $R \in \mathcal{S}$ let R' be the rectange with the same center such that

$$l_j(R') = \begin{cases} L_i^{(j)} + \frac{\rho_i}{4\sqrt{d}} L_{i-1}^{(1)} & \text{for } j = 1, \dots, j_i \\ L_n^{(1)} & \text{for } j = j_i + 1, \dots, d. \end{cases}$$

We denote by r_0 (resp. r) the center of R_0' (resp. R'). Here, we note that r_0 and r are contained in both $\beta(\tau)$ and S. For $x \in R'$ and $j = 1, \dots, d$,

$$|x_j - (r_0)_j| \leq |x_j - r_j| + |r_j - (r_0)_j| \leq \frac{1}{2} l_j(R') + \min(L_{i-1}^{(j)}, L_n^{(1)}) \leq \frac{1}{2} l_j(R_0').$$

Thus for all $R \in \mathcal{S}$, $R' \subset R_0'$.

For any distinct $R_1, R_2 \in \mathcal{S}$, let $\tau_1', \tau_2' \in \mathcal{T}(\tau)$ be such that $R_1 = \beta(\tau_1') \cap S$ and $R_2 = \beta(\tau_2') \cap S$, and let r_1, r_2 be the centers of R_1', R_2', respectively. Suppose $\|r_1 - r_2\|_\infty = |(r_1)_j - (r_2)_j| > 0$ for some $j = 1, \dots, j_i$. Then for any $x \in R_1'$ and $y \in R_2'$, we have

$$\begin{aligned} |x_j - y_j| &\geq |(r_1)_j - (r_2)_j| - |x_j - (r_1)_j| - |y_j - (r_2)_j| \\ &\geq \frac{1}{\sqrt{d}} \mathrm{dist}(\beta(\tau_1'), \beta(\tau_2')) + L_i^{(j)} - \frac{1}{2} l_j(R_1') - \frac{1}{2} l_j(R_2') \\ &\geq \left(\frac{\rho_i}{\sqrt{d}} L_{i-1}^{(1)} + L_i^{(j)} \right) - \frac{1}{2} l_j(R_1') - \frac{1}{2} l_j(R_2') \\ &= \frac{3\rho_i}{4\sqrt{d}} L_{i-1}^{(1)} > 0. \end{aligned}$$

Thus $R_1' \cap R_2' = \varnothing$.

Now we suppose $\|r_1 - r_2\|_\infty = |(r_1)_j - (r_2)_j| > 0$ for some $j = j_i + 1, \dots, d$. Observe that

$$L_n^{(1)} \geq l_j(R_1) + l_j(R_2) + \frac{1}{\sqrt{d}} \mathrm{dist}(\beta(\tau_1'), \beta(\tau_2')) > l_j(R_1) + l_j(R_2),$$

which implies that

$$|(r_1)_j - (r_2)_j| = L_n^{(1)} - \frac{1}{2}l_j(R_1) - \frac{1}{2}l_j(R_2) > \frac{1}{2}L_n^{(1)}.$$

Thus, for any fixed $R_1 \in \mathcal{S}$ and $j = j_i + 1, \ldots, d$,

$$\#\{R_2 \in \mathcal{S} \setminus \{R_1\} : \|r_1 - r_2\|_\infty = |(r_1)_j - (r_2)_j| \text{ and } R_1' \cap R_2' \neq \varnothing\} \leq 1.$$

Combining above two arguments, we conclude that every points of R_0' is covered by at most $d - j_i + 1$ rectangles of $\{R' : R \in \mathcal{S}\}$. It follows that

$$\begin{aligned}\left(\frac{\rho_i}{4\sqrt{d}}L_{i-1}^{(1)}\right)^{j_i}\left(L_n^{(1)}\right)^{d-j_i}\#\mathcal{S} &\leq \left(L_i^{(j)} + \frac{\rho_i}{4\sqrt{d}}L_{i-1}^{(1)}\right)^{j_i}\left(L_n^{(1)}\right)^{d-j_i}\#\mathcal{S} \\ &= \operatorname{vol}(R')\#\mathcal{S} \\ &\leq (d - j_i + 1)\operatorname{vol}(R_0') \\ &\leq d4^d\left(L_{i-1}^{(1)}\right)^{j_i}\left(L_n^{(1)}\right)^{d-j_i},\end{aligned}$$

hence, using $j_i \geq \ell$,

$$\#\mathcal{S} \leq d^{1+j_i/2}4^{d+j_i}\rho_i^{-j_i} \leq (16d)^d\rho_i^{-\ell}.$$

This inequality completes the proof. □

Let μ be the unique probability measure on $\mathcal{F}(\mathcal{T}, \beta)$ satisfying the following property: For all $y \in \mathcal{F}(\mathcal{T}, \beta)$ and $n \in \mathbb{N}$,

$$\frac{\mu(\{x \in \mathcal{F}(\mathcal{T}, \beta) : \tau_n(x) = \tau_n(y)\})}{\mu(\{x \in \mathcal{F}(\mathcal{T}, \beta) : \tau_{n-1}(x) = \tau_{n-1}(y)\})} = \frac{1}{\#\mathcal{T}(\tau_{n-1}(y))}, \tag{5.5}$$

where $x = \bigcap_{n\geq 0}\beta(\tau_n(x))$. We remark that for any $n \in \mathbb{N}$ and $\kappa \in \mathcal{T}_n$, it follows from (5.5) that

$$\mu(\beta(\kappa)) \leq \frac{\mu(\mathcal{F}(\mathcal{T}, \beta))}{C_0 \ldots C_n} = \frac{1}{P_n} \tag{5.6}$$

Lemma 5.1.4. *Let $n \in \mathbb{N}$ and $\kappa \in \mathcal{T}_n$. Then for any elementary square S of $\beta(\kappa)$, one has*

$$\mu(S) \leq (16d)^{d(D_n - n)}P_n^{-1}\prod_{i=n+1}^{D_n}\rho_i^{-\ell}C_i^{-1}.$$

Proof. If $D_n = n$, then it follows from (5.6). Assume $D_n > n$. Applying Lemma 5.1.3 for $i = n+1, \dots, D_n$, we have

$$\#\{\tau \in \mathcal{T}_{D_n} : \beta(\tau) \cap S \neq \varnothing\} \leq (16d)^{d(D_n - n)} \prod_{i=n+1}^{D_n} \rho_i^{-\ell}. \tag{5.7}$$

Since $S \cap \mathcal{F}(\mathcal{T}, \beta)$ can be covered by rectangles $\{\beta(\tau) : \tau \in \mathcal{T}_{D_n}, \beta(\tau) \cap S \neq \varnothing\}$, we have

$$\begin{aligned} \mu(S) &\leq \sum_{\substack{\tau \in \mathcal{T}_{D_n} \\ \beta(\tau) \cap S \neq \varnothing}} \mu(\beta(\tau)) \\ &\leq \mu(\beta(\kappa)) \prod_{i=n+1}^{D_n} C_i^{-1} \cdot \#\{\tau \in \mathcal{T}_{D_n} : \beta(\tau) \cap S \neq \varnothing\} \\ &\leq (16d)^{d(D_n - n)} P_n^{-1} \prod_{i=n+1}^{D_n} \rho_i^{-\ell} C_i^{-1}. \end{aligned}$$

In the last inequality, we use (5.6) and (5.7). □

Let U be an open subset of $\mathbb{R}^d$ with $U \cap \mathcal{F}(\mathcal{T}, \beta) \neq \varnothing$. If $U \cap \mathcal{F}(\mathcal{T}, \beta)$ is a single point set, then we denote by $n(U)$ the smallest $n \in \mathbb{N}$ such that $\operatorname{diam}(U) \geq \rho_{n+1} L_n^{(1)}$. In that case, there is a unique $\kappa = \kappa(U) \in \mathcal{T}_{n(U)}$ such that $U \cap \mathcal{F}(\mathcal{T}, \beta) \subset \beta(\kappa)$. If $U \cap \mathcal{F}(\mathcal{T}, \beta)$ contains more than two points, then we denote by $n(U)$ the largest $n \in \mathbb{N}$ such that $U \cap \mathcal{F}(\mathcal{T}, \beta) \subset \beta(\kappa)$ for some $\kappa = \kappa(U) \in \mathcal{T}_n$. We note that $\operatorname{diam}(U) \geq \rho_{n(U)+1} L_{n(U)}^{(1)}$ by the assumption (4) of Theorem 5.1.1.

Lemma 5.1.5. *Let U be an open subset of $\mathbb{R}^d$ with $U \cap \mathcal{F}(\mathcal{T}, \beta) \neq \varnothing$. Let $n = n(U)$ and $\kappa = \kappa(U)$. Then there is a family $\mathcal{S}$ of elementary squares of $\beta(\kappa)$ such that*

1. $\bigcup_{S \in \mathcal{S}} S \supset U \cap \mathcal{F}(\mathcal{T}, \beta)$;
2. $\left(L_n^{(1)}\right)^t \cdot \#\mathcal{S} \leq 2^{d-\ell} \rho_{n+1}^{-t} \operatorname{diam}(U)^t$ *for all* $t \geq d - \ell$.

Proof. If $\operatorname{diam}(U) \leq L_n^{(1)}$, then there exists an elementary square S of $\beta(\kappa)$ such that $S \supset U \cap \mathcal{F}(\mathcal{T}, \beta)$. We set $\mathcal{S} = \{S\}$ so that $\mathcal{S}$ satisfies two conditions.

Now we assume $\operatorname{diam}(U) > L_n^{(1)}$. Then $U \cap \mathcal{F}(\mathcal{T}, \beta)$ can be covered by $\left\lceil \frac{\operatorname{diam}(U)}{L_n^{(1)}} \right\rceil^{d-\ell}$ elementary sqaures. Let $\mathcal{S}$ be the family of these elementary squares. Then

$$\left(L_n^{(1)}\right)^t \cdot \#\mathcal{S} = \left(L_n^{(1)}\right)^t \left\lceil \frac{\operatorname{diam}(U)}{L_n^{(1)}} \right\rceil^{d-\ell} \leq 2^{d-\ell} \left(\frac{\operatorname{diam}(U)}{L_n^{(1)}} \right)^{d-\ell} \left(L_n^{(1)}\right)^t$$
$$\leq 2^{d-\ell} \left(\frac{\operatorname{diam}(U)}{L_n^{(1)}} \right)^t \left(L_n^{(1)}\right)^t \leq 2^{d-\ell} \rho_{n+1}^{-t} \operatorname{diam}(U)^t.$$

□

Proof of theorem 5.1.1. For a real number t such that $d-\ell \leq t < s$, there exists $n_0 = n_0(t)$ such that for all $n \geq n_0$,

$$P_n \left(L_n^{(1)}\right)^t \rho_{n+1}^t \prod_{i=n+1}^{D_n} \rho_i^\ell C_i \geq (16d)^{d\max\{D_n - n, 1\}} \geq (16d)^{d(D_n - n)}. \tag{5.8}$$

Let $\mathcal{U}$ be an open cover of $\mathcal{F}(\mathcal{T},\beta)$. Assume that for all $U \in \mathcal{U}$, $\operatorname{diam}(U)$ is small enough so that $n(U) \geq n_0$. Since $\mathcal{F}(\mathcal{T},\beta)$ is compact, there exists a finite subcover $\mathcal{U}_0$ such that for all $U \in \mathcal{U}_0$, $U \cap \mathcal{F}(\mathcal{T},\beta) \neq \varnothing$.

For $U \in \mathcal{U}_0$, let $\mathcal{S}_U$ be a family of elementary squares given by Lemma 5.1.5. Let $\mathcal{Q} = \bigcup_{U\in\mathcal{U}_0} \mathcal{S}_U$ and $n(S) = n(U)$ for $S \in \mathcal{S}_U$. We note that S may belong to different $\mathcal{S}_U$. However, $n(S)$ is well-difined since a side length of S is $L_{n(U)}^{(1)}$. Then $\mathcal{Q}$ covers $\mathcal{F}(\mathcal{T},\beta)$ and hence

$$\begin{aligned}
\sum_{U\in\mathcal{U}_0} \operatorname{diam}(U)^t &\geq \frac{1}{2^{d-\ell}} \sum_{S\in\mathcal{Q}} \rho_{n(S)+1}^t \left(L_{n(S)}^{(1)}\right)^t && \text{by Lemma 5.1.5} \\
&\geq \frac{1}{2^{d-\ell}} \sum_{S\in\mathcal{Q}} (16d)^{d(D_n-n)} P_{n(S)}^{-1} \prod_{i=n+1}^{D_n} \rho_i^{-\ell} C_i^{-1} && \text{by (5.8)} \\
&\geq \frac{1}{2^{d-\ell}} \sum_{S\in\mathcal{Q}} \mu(S) && \text{by Lemma 5.1.4} \\
&\geq \frac{1}{2^{d-\ell}}.
\end{aligned}$$

Thus we have $\dim_H \mathcal{F}(\mathcal{T},\beta) \geq t$. Since we choose arbitrary t with $d-\ell \leq t < s$, the proof is completed. □

5.2 Counting lattice points in convex sets

In this section, we will generalize the results in [LSST20, §3.2] for $\mathbb{R}^3$ to the general $\mathbb{R}^{d+1}$. In §5.2.1, we first recall the notations and lemmas in [LSST20, §3.1].

5.2.1 Preliminaries for lattice point counting

For a positive integer $D \geq 1$, we write the D-dimensional Euclidean space by $\mathcal{E}_D = \mathbb{R}^D$. For a convex body $K \subset \mathbb{R}^D$ and a lattice $\Lambda \subset \mathbb{R}^D$, let $\lambda_i(K, \Lambda)$ ($i = 1, \dots, D$) be the i-th successive minimum of Λ with respect to K, that is, the infimum of those numbers λ such that $\lambda K \cap \Lambda$ contains i linearly independent vectors. Let $\mathrm{vol}(\cdot)$ be the Lebesgue measure on $\mathbb{R}^D$ and let $\mathrm{cov}(\Lambda)$ be the covolume of a lattice Λ, which is the Lebesgue measure of a fundamental domain of Λ. Denote by

$$\theta(K, \Lambda) := \frac{\mathrm{vol}(K)}{\mathrm{cov}(\Lambda)}.$$

For an affine subspace H of $\mathbb{R}^D$, let $\mathrm{vol}_H(\cdot)$ be the Lebesgue measure on H with respect to the subspace Riemannian structure. We write $\mathrm{vol}_H(S) = \mathrm{vol}_H(S \cap H)$ for a Borel measurable subset S of $\mathbb{R}^D$ by abuse of notation. We say that a subspace H of $\mathbb{R}^D$ is Λ-rational if $H \cap \Lambda$ is a lattice in H, and denote by $\mathrm{cov}_H(\Lambda)$ the covolume of the lattice $H \cap \Lambda$ in H. We also use the same notations for the dual vector space $\mathcal{E}_D^*$ with respect to the standard Euclidean structure.

We use $\|\cdot\|$ for the Euclidean norms on $\mathbb{R}^D$ and $\mathcal{E}_D^*$. For a normed vector space V, denote by $B_r(V)$ (or B_r if $V = \mathbb{R}^D$) the ball of radius r centered at $0 \in V$. We use K-norms on $\mathbb{R}^D$ and $\mathcal{E}_D^*$ defined by

$$\begin{cases} \|\mathbf{v}\|_K = \inf\{r > 0 : \mathbf{v} \in rK\}, & \mathbf{v} \in \mathbb{R}^D, \\ \|\varphi\|_K = \sup_{\mathbf{v}\in K} |\varphi(\mathbf{v})|, & \varphi \in \mathcal{E}_D^*. \end{cases}$$

Recall that $\mathcal{L}_D$ is the space of unimodular lattices in $\mathbb{R}^D$, which can be identified with the homogeneous space $\mathrm{SL}_D(\mathbb{R})/\mathrm{SL}_D(\mathbb{Z})$. For $g \in \mathrm{SL}_D(\mathbb{R})$ let g^* be the adjoint action on $\mathcal{E}_D^*$ defined by $\varphi \mapsto \varphi \circ g$. Then g^* can be represented by the transpose of g with respect to the standard basis $\mathbf{e}_1, \dots, \mathbf{e}_D$ of $\mathbb{R}^D$ and the dual basis $\mathbf{e}_1^*, \dots, \mathbf{e}_D^*$ of $\mathcal{E}_D^*$.

The *dual lattice* of Λ in $\mathbb{R}^D$ is the lattice in $\mathcal{E}_D^*$ defined by

$$\Lambda^* = \{\varphi \in \mathcal{E}_D^* : \varphi(\mathbf{v}) \in \mathbb{Z},\ \forall \mathbf{v} \in \Lambda\}.$$

Let us define the following two sets:

$$\mathcal{K}_\epsilon = \mathcal{K}_\epsilon(D) = \{\Lambda \in \mathcal{L}_D : \|\mathbf{v}\| \geq \epsilon,\ \forall \mathbf{v} \in \Lambda \smallsetminus \{0\}\} = \{\Lambda \in \mathcal{L}_D : \lambda_1(B_1, \Lambda) \geq \epsilon\};$$
$$\mathcal{K}_\epsilon^* = \mathcal{K}_\epsilon^*(D) = \{\Lambda \in \mathcal{L}_D : \|\varphi\| \geq \epsilon,\ \forall \varphi \in \Lambda^* \smallsetminus \{0\}\}.$$

Since $\mathcal{E}_D^*$ can be naturally identified with $\bigwedge_{\mathbb{R}}^{D-1} \mathbb{R}^D$ with the standard Euclidean structure, we have $\Lambda^* = \bigwedge_{\mathbb{Z}}^{D-1} \Lambda$.

A nonzero vector $\mathbf{v} \in \Lambda$ is said to be *primitive* if $(1/n)\mathbf{v} \notin \Lambda$ for all $n \in \mathbb{N}$. The set of primitive vectors in Λ is denoted by $\widehat{\Lambda}$.

We summarize the lemmas in [LSST20, §3.1].

Lemma 5.2.1. *Let $D \geq 2$. For every lattice Λ in $\mathbb{R}^D$ and every bounded centrally symmetric convex subset K of $\mathbb{R}^D$ with $\lambda_d(K, \Lambda) \leq 1$ we have*

$$\#(K \cap \widehat{\Lambda}) = \big(\zeta(D)^{-1} + \eta(K, \Lambda)\big) \cdot \theta(K, \Lambda)$$

where ζ is the Riemann ζ-function and

$$|\eta(K, \Lambda)| \ll_D \lambda_D(K, \Lambda) - \lambda_D(K, \Lambda) \log \lambda_1(K, \Lambda).$$

Lemma 5.2.2. *Let $D \geq 2$. For every lattice Λ in $\mathbb{R}^D$ and every bounded centrally symmetric convex subset K of $\mathbb{R}^D$ with $\lambda_D(K, \Lambda) \leq 1$ we have*

$$\#(K \cap (\Lambda \smallsetminus \{0\})) = (1 + \alpha(K, \Lambda)) \cdot \theta(K, \Lambda)$$

where $|\alpha(K, \Lambda)| \ll_D \lambda_D(K, \Lambda)$

Lemma 5.2.3. *Let K and Λ be as in Lemma 5.2.2. Then*

$$\#(K \cap \Lambda) \asymp_D \theta(K, \Lambda).$$

Lemma 5.2.4. *Let $D \geq 1$. Let Λ be a lattice in $\mathbb{R}^D$ and K be a bounded centrally symmetric convex subset of $\mathbb{R}^D$ with nonempty interior. Then*

$$\#(K^\circ \cap \Lambda) \asymp_D \#(K \cap \Lambda) \asymp_D \#(\overline{K} \cap \Lambda).$$

Lemma 5.2.5. *Let K and Λ be as in Lemma 5.2.2. If $\lambda_i(K,\Lambda) \le s \le s' \le \lambda_{j+1}(K,\Lambda)$ where $1 \le i \le j \le D$, then*

$$\left(\frac{s'}{s}\right)^i \ll_D \frac{\#(s'K \cap \Lambda)}{\#(sK \cap \Lambda)} \ll_D \left(\frac{s'}{s}\right)^j.$$

Lemma 5.2.6. *Let $D \ge 2$. Let K be a bounded centrally symmetric convex subset of $\mathbb{R}^D$ with nonempty interior and let $\varphi \in \mathcal{E}_D^* \smallsetminus \{0\}$. Then*

$$\mathrm{vol}_{H_\varphi}(K) \asymp_D \|\varphi\| \mathrm{vol}(K)/\|\varphi\|_K.$$

We need the following auxiliary lemma.

Lemma 5.2.7. *Given $D \ge 2$ and $r > 0$, let $\Lambda \in \mathcal{K}_r^*(D)$, and let $\mathbf{v}, \mathbf{w} \in \Lambda$ be any nonzero linearly independent vectors. Then there exists a positive constant $c' = c'(D) > 0$ such that $\|\mathbf{v} \wedge \mathbf{w}\| \ge c' r^{D-2}$.*

Proof. Let Λ' be the 2-dimensional sublattice of Λ generated by $\mathbf{v}, \mathbf{w}$. By Minkowski's second theorem, we have

$$\|\mathbf{v} \wedge \mathbf{w}\| \ge \mathrm{cov}(\Lambda') \gg_2 \lambda_1(B_1,\Lambda')\lambda_2(B_1,\Lambda') \ge \lambda_1(B_1,\Lambda)\lambda_2(B_1,\Lambda). \tag{5.9}$$

Agian by Minkowski's second theorem, we have

$$\begin{aligned} 1 \ll_D \lambda_1(B_1,\Lambda)\cdots\lambda_D(B_1,\Lambda) &\le \lambda_1(B_1,\Lambda)\lambda_2(B_1,\Lambda)\lambda_D(B_1,\Lambda)^{D-2} \\ &\le \lambda_1(B_1,\Lambda)\lambda_2(B_1,\Lambda)\frac{1}{r^{D-2}}. \end{aligned} \tag{5.10}$$

The last inequality comes from $\Lambda \in \mathcal{K}_r^*(D)$. The result is following by combining (5.9) and (5.10). □

5.2.2 Lattice point counting in $\mathbb{R}^{d+1}$

For $d \ge 2$ and a $(d+1)$-tuple $\mathbf{r} = (r_1, \dots, r_{d+1})$ of positive real numbers, we estimate the number of lattice points in the set

$$M_{\mathbf{r}} = \{(x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1} : |x_i| \le r_i,\ \forall i = 1, \dots, d+1\}.$$

Let

$$M_{\mathbf{r}}^* = \{\varphi \in \mathcal{E}_{d+1}^* : |x_i^\varphi| \le r_i,\ \forall i = 1, \dots, d+1\},$$

where the element $\varphi \in \mathcal{E}_{d+1}^*$ is represented by $\varphi = \sum_{i=1}^{d+1} x_i^\varphi \mathbf{e}_i^*$.

Lemma 5.2.8. *Let $d \geq 2$. For any real number $c_0 > 1$, there exists positive real number $\tilde{c} < 1$ such that for every lattice Λ in $\mathbb{R}^{d+1}$ and every $(d+1)$-tuple $\mathbf{r}$ of positive real numbers with*

$$\lambda_{d+1}(M_{\mathbf{r}}, \Lambda) \leq \tilde{c} \quad \text{and} \quad -\lambda_{d+1}(M_{\mathbf{r}}, \Lambda) \log \lambda_1(M_{\mathbf{r}}, \Lambda) \leq \tilde{c}$$

one has

$$\frac{1}{c_0 \zeta(d+1)} \theta(M_{\mathbf{r}}, \Lambda) \leq \#(M_{\mathbf{r}} \cap \widehat{\Lambda}) \leq \frac{c_0}{\zeta(d+1)} \theta(M_{\mathbf{r}}, \Lambda).$$

Proof. The proof follows directly from Lemma 5.2.1. □

Now we fix real numbers $s, r_1, \ldots, r_{d+1}$ such that $0 < s < 1/2$, $r_i \geq 1$ for each $i = 1, \ldots, d$, and $r_{d+1} = 1$. Denote by $\mathbf{r} = (r_1, \ldots, r_{d+1})$, $r_M = \max_{1 \leq i \leq d} r_i$, and $r_m = \min_{1 \leq i \leq d} r_i$. Define a norm

$$\|\varphi\|_{\mathbf{r}} = \max\left\{r_i |x_i^{\varphi}| : i = 1, \ldots, d+1\right\}.$$

It follows from the definition that

$$\|\varphi\|_{\mathbf{r}} \leq \|\varphi\|_{M_{\mathbf{r}}} \leq (d+1)\|\varphi\|_{\mathbf{r}}. \tag{5.11}$$

For $q > 0$ let

$$N_q(\mathbf{r}, s) = \left\{\varphi \in \mathcal{E}_{d+1}^* : |x_i^{\varphi}| \leq s,\ \forall i = 1, \ldots, d, \text{ and } \|\varphi\|_{\mathbf{r}} \leq q\right\}.$$

Note that $N_q(\mathbf{r}, s) = M_{\mathbf{r}'}^*$ where $\mathbf{r}' = (r_1', \ldots, r_d', q)$ with $r_i' = \min\{q/r_i, s\}$. For a lattice Λ in $\mathbb{R}^{d+1}$ and $i = 1, \ldots, d+1$, let $q_i(\Lambda, \mathbf{r}, s)$ be the infimum of those positive real number q such that $N_q(\mathbf{r}, s) \cap \Lambda$ contains i linearly independent vectors. We will give an upper bound of the number of

$$\mathcal{S}(\Lambda, \mathbf{r}, s) := \left\{\mathbf{v} \in M_{\mathbf{r}} \cap \widehat{\Lambda} : \varphi(\mathbf{v}) = 0 \text{ for some } \varphi \in N_{(d+1)sr_M}(\mathbf{r}, s) \cap \widehat{\Lambda}^*\right\},$$

where $\widehat{\Lambda}^*$ is the set of primitive vectors in Λ^*.

Lemma 5.2.9. *For $d \geq 2$, let Λ be a unimodular lattice in $\mathbb{R}^{d+1}$ with $q_1(\Lambda, \mathbf{r}, s) \geq s^{-2}$. Then*

1. *if $r_m = r_M$ and $q_{d+1}(\Lambda, \mathbf{r}, s) \leq ds^{-1/2} r_M$, then*

$$\#\mathcal{S}(\Lambda, \mathbf{r}, s) \ll s^{1/2} \cdot \mathrm{vol}(M_{\mathbf{r}});$$

2. if $r_m < r_M$ and $q_{d+1}(\Lambda, \mathbf{r}, s) \log q_{d+1}(\Lambda, \mathbf{r}, s) \leq sr_M$, then

$$\#\mathcal{S}(\Lambda, \mathbf{r}, s) \ll s^2 \cdot \mathrm{vol}(M_{\mathbf{r}}).$$

Proof. For simplicity, we denote by $N_q = N_q(\mathbf{r}, s)$, $q_i = q_i(\Lambda, \mathbf{r}, s)$ and $\mathcal{S} = \mathcal{S}(\Lambda, \mathbf{r}, s)$. If $N_{(d+1)sr_M} \cap \widehat{\Lambda}^*$ is empty then there is nothing to prove. We assume that $N_{(d+1)sr_M} \cap \widehat{\Lambda}^*$ is nonempty. It follows from the definition that

$$\#\mathcal{S} \leq \sum_{\varphi \in N_{(d+1)sr_M} \cap \widehat{\Lambda}^*} \#(H_\varphi \cap M_{\mathbf{r}} \cap \widehat{\Lambda}) \tag{5.12}$$

with the notation $H_\varphi = \ker\varphi$.

We first claim that for every $\varphi \in N_{(d+1)sr_M} \cap \widehat{\Lambda}^*$,

$$\#(H_\varphi \cap M_{\mathbf{r}} \cap \widehat{\Lambda}) \ll \frac{\mathrm{vol}(M_{\mathbf{r}})}{\|\varphi\|_{M_{\mathbf{r}}}} \leq \frac{\mathrm{vol}(M_{\mathbf{r}})}{\|\varphi\|_{\mathbf{r}}}. \tag{5.13}$$

where the second inequality follows from (5.11). If $\#(H_\varphi \cap M_{\mathbf{r}} \cap \widehat{\Lambda}) < d+1$, then it follows from (5.11) that

$$\frac{\mathrm{vol}(M_{\mathbf{r}})}{\|\varphi\|_{M_{\mathbf{r}}}} \geq \frac{2^{(d+1)} r_1 \dots r_{d+1}}{(d+1)\|\varphi\|_{\mathbf{r}}} \geq \frac{2^{(d+1)} r_1 \dots r_{d+1}}{(d+1)^2 sr_M} \gg \#(H_\varphi \cap M_{\mathbf{r}} \cap \widehat{\Lambda}).$$

Otherwise, $H_\varphi \cap M_{\mathbf{r}} \cap \Lambda$ has d linearly independent vectors, hence it follows from Lemma 5.2.3 and Lemma 5.2.6 that

$$\#(H_\varphi \cap M_{\mathbf{r}} \cap \widehat{\Lambda}) \ll \frac{\mathrm{vol}_{H_\varphi}(M_{\mathbf{r}})}{\mathrm{cov}_{H_\varphi}(\Lambda)} \ll \frac{\|\varphi\| \, \mathrm{vol}(M_{\mathbf{r}})}{\mathrm{cov}_{H_\varphi}(\Lambda)\|\varphi\|_{M_{\mathbf{r}}}} \ll \frac{\mathrm{vol}(M_{\mathbf{r}})}{\|\varphi\|_{M_{\mathbf{r}}}},$$

which concludes the claim.

By (5.12) and (5.13), it suffices to estimate

$$\begin{aligned} \eta :=& \sum_{\varphi \in N_{(d+1)sr_M} \cap \widehat{\Lambda}^*} \|\varphi\|_{\mathbf{r}}^{-1} \\ =& \frac{1}{(d+1)sr_M} \#(N_{(d+1)sr_M} \cap \widehat{\Lambda}^*) + \sum_{\varphi \in N_{(d+1)sr_M} \cap \widehat{\Lambda}^*} \int_{\|\varphi\|_{\mathbf{r}}}^{(d+1)sr_M} \frac{1}{q^2} \mathrm{d}q. \end{aligned} \tag{5.14}$$

We denote the first and second terms in the last line by η_1, η_2, respectively.

Observe that

$$\begin{aligned}\eta_2 &= \sum_{\varphi\in N_{(d+1)sr_M}\cap\widehat{\Lambda}^*}\int_{q_1}^{(d+1)sr_M}\frac{\mathbb{1}_q(\|\varphi\|_{\mathbf{r}})}{q^2}\mathrm{d}q \\ &= \int_{q_1}^{(d+1)sr_M}\sum_{\varphi\in N_{(d+1)sr_M}\cap\widehat{\Lambda}^*}\frac{\mathbb{1}_q(\|\varphi\|_{\mathbf{r}})}{q^2}\mathrm{d}q \\ &\le \int_{q_1}^{(d+1)sr_M}\frac{\#(N_q\cap\widehat{\Lambda}^*)}{q^2}\mathrm{d}q.\end{aligned}\tag{5.15}$$

where $\mathbb{1}_q$ denotes the indicator function of the set $\{x\in\mathbb{R} : x\le q\}$.

For $i=2,\dots,d$, if $q_{i-1}\le q<q_i$ then $\#(N_q\cap\widehat{\Lambda}^*)=i\le d$. Thus

$$\int_{q_1}^{q_d}\frac{\#(N_q\cap\widehat{\Lambda}^*)}{q^2}\mathrm{d}q\le\int_{q_1}^{q_d}\frac{d}{q^2}\mathrm{d}q\le\frac{d}{q_1}\ll s^2\le s^{1/2},\tag{5.16}$$

where the third inequality follows from the assumption $q_1\ge s^{-2}$.

Proof of the assertion (1). We claim that $\eta\ll s^{-1/2}$ under the assumption of (1), which concludes the assertion (1). Assume that $r_m=r_M$ and $q_{d+1}\le ds^{-1/2}r_M$. Observe that by definition

$$N_{(d+1)s^{-1/2}r_M}=M^*_{(s,\dots,s,(d+1)s^{-1/2}r_M)}.\tag{5.17}$$

We have an upper bound of η_1 as

$$\eta_1\le\frac{\#(N_{(d+1)s^{-1/2}r_M}\cap\Lambda^*)}{(d+1)sr_M}\ll\frac{\mathrm{vol}(N_{(d+1)s^{-1/2}r_M})}{(d+1)sr_M}\ll s^{d-3/2}\le s^{1/2}.\tag{5.18}$$

The first inequality follows from $s<1/2$, the second inequality follows from Lemma 5.2.3, and the third inequality follows from (5.17).

For an upper bound of η_2, we first compute

$$\begin{aligned}\int_{sr_M}^{(d+1)sr_M}\frac{\#(N_q\cap\widehat{\Lambda}^*)}{q^2}\mathrm{d}q &\le \int_{sr_M}^{(d+1)sr_M}\frac{\#(N_{(d+1)sr_M}\cap\Lambda^*)}{q^2}\mathrm{d}q \\ &\le \frac{\#(N_{(d+1)sr_M}\cap\Lambda^*)}{sr_M} \\ &\ll s^{1/2},\end{aligned}\tag{5.19}$$

where the last inequality can be shown by the same as (5.18).

If $sr_M \le q_d$, then it follows from (5.16) and (5.19) that $\eta_2 \ll s^{1/2}$. Now we suppose that $sr_M > q_d$. For all $q_d < q \le sr_M = sr_m$, observe that

$$N_q = M^*_{(q/r_1,\dots,q/r_{d+1})} = \frac{q}{sr_M} N_{sr_M}.$$

Since $\lambda_d(N_q, \Lambda) = \lambda_d(\frac{q}{sr_M} N_{sr_M}, \Lambda) \le 1 \le sr_M/q$, it follows from Lemma 5.2.5 that

$$\#(N_q \cap \widehat{\Lambda}^*) \le \#\left(\frac{q}{sr_M} N_{sr_M} \cap \Lambda^*\right) \ll \left(\frac{q}{sr_M}\right)^d \#(N_{sr_M} \cap \Lambda^*).$$

By $sr_M \le ds^{-1/2} r_M$ and Lemma 5.2.3, we have

$$\begin{aligned}
\#(N_q \cap \widehat{\Lambda}^*) &\ll \left(\frac{q}{sr_M}\right)^d \#(N_{ds^{-1/2}r_M} \cap \Lambda^*) \\
&\ll \left(\frac{q}{sr_M}\right)^d \mathrm{vol}(N_{ds^{-1/2}r_M}) \\
&\ll \left(\frac{q}{sr_M}\right)^2 s^{d-1/2} r_M \ll \frac{q^2 s^{-1/2}}{r_M}.
\end{aligned}$$

The last line follows from $\frac{q}{sr_M} \le 1$ and $s \le 1$. Thus we have

$$\int_{q_d}^{sr_M} \frac{\#(N_q \cap \widehat{\Lambda}^*)}{q^2} \mathrm{d}q \ll \int_{q_d}^{sr_M} \frac{s^{-1/2}}{r_M} \mathrm{d}q \ll s^{1/2}.$$

It follows that $\eta_2 \ll s^{1/2}$ under the assumption of (1), which concludes the assertion (1). □

Proof of the assertion (2). We will prove that $\eta \ll s^2$ under the assumption of (2). By the assumption, we have $q_{d+1} \ge q_1 \ge s^{-2} \ge 4$ so that $q_{d+1} < sr_M < (d+1)sr_M$ since $q_{d+1} \log q_{d+1} \le sr_M$. Thus $N_{(d+1)sr_M} \cap \Lambda^*$ contains $d+1$ linearly independent vectors. By Lemma 5.2.3, we have

$$\eta_1 \le \frac{\#(N_{(d+1)sr_M} \cap \Lambda^*)}{(d+1)sr_M} \ll \frac{\mathrm{vol}(N_{(d+1)sr_M})}{(d+1)sr_M} \ll s^d \le s^2. \tag{5.20}$$

By (5.15), it suffices to show that

$$\int_{q_1}^{(d+1)sr_M} \frac{\#(N_q \cap \widehat{\Lambda}^*)}{q^2} \mathrm{d}q \ll s^2.$$

We split the domain of integration as $(q_1, q_d) \cup (q_d, q_{d+1}) \cup (q_{d+1}, sr_M) \cup (sr_M, (d+1)sr_M)$ and estimate upper bounds of the integrals.

For each $q \in (sr_M, (d+1)sr_M)$, it follows from Lemma 5.2.3 that $\#(N_q \cap \widehat{\Lambda}^*) \ll \mathrm{vol}(N_q) \ll s^d q$. Thus we have

$$\int_{sr_M}^{(d+1)sr_M} \frac{\#(N_q \cap \widehat{\Lambda}^*)}{q^2} \mathrm{d}q \ll \int_{sr_M}^{(d+1)sr_M} \frac{s^d}{q} \mathrm{d}q = s^d \log(d+1) \ll s^2. \tag{5.21}$$

For each $q \in (q_{d+1}, sr_M)$, it follows from Lemma 5.2.3 that $\#(N_q \cap \widehat{\Lambda}^*) \ll \mathrm{vol}(N_q) \ll s^{d-1} q^2 / r_M$. Thus we have

$$\int_{q_{d+1}}^{sr_M} \frac{\#(N_q \cap \widehat{\Lambda}^*)}{q^2} \mathrm{d}q \ll \int_{q_{d+1}}^{sr_M} \frac{s^{d-1}}{r_M} \mathrm{d}q \le s^d \le s^2. \tag{5.22}$$

By (5.16), the integral over (q_1, q_d) is bounded above by s^2.

Now it remains to show that the integral over (q_d, q_{d+1}) is bounded above by s^2. Let $H = \mathrm{Span}_{\mathbb{R}}(N_{q_d} \cap \Lambda^*)$. We claim that for every $q \in (q_d, q_{d+1})$,

$$\mathrm{vol}_H(N_q) \le \frac{q}{q_{d+1}} \mathrm{vol}_H(N_{q_{d+1}}). \tag{5.23}$$

If H contains $\mathbf{e}_{d+1}^*$, then the claim is easily checked from the definition of N_q. Otherwise, we let $\mathbf{pr}^*$ be the orthogonal projection onto $\mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_1^*, \dots, \mathbf{e}_d^*\}$. Then the volume of $\mathbf{pr}^*(N_q)$ is at most q/q_{d+1} times the volume of $\mathbf{pr}^*(N_{q_{d+1}})$ since $q/r_M < q_{d+1}/r_M < s$. Thus we prove the claim.

For each $q \in (q_d, q_{d+1})$, we have

$$
\begin{aligned}
\#(N_q \cap \widehat{\Lambda}^*) &\ll \frac{\mathrm{vol}_H(N_q)}{\mathrm{cov}_H(\Lambda^*)} && \text{by Lemma 5.2.3} \\
&\leq \frac{q}{q_{d+1}} \frac{\mathrm{vol}_H(N_{q_{d+1}})}{\mathrm{cov}_H(\Lambda^*)} && \text{by (5.23)} \\
&\ll \frac{q}{q_{d+1}} \#(N_{q_{d+1}} \cap H \cap \Lambda^*) && \text{by Lemma 5.2.3} \\
&\ll \frac{q}{q_{d+1}} \#(N^\circ_{q_{d+1}} \cap H \cap \Lambda^*) && \text{by Lemma 5.2.4} \\
&= \frac{q}{q_{d+1}} \#(N^\circ_{q_{d+1}} \cap \Lambda^*) \leq \frac{q}{q_{d+1}} \#(N_{q_{d+1}} \cap \Lambda^*) \\
&\ll \frac{q}{q_{d+1}} \mathrm{vol}(N_{q_{d+1}}) && \text{by Lemma 5.2.3} \\
&\ll s^{d-1} \frac{q_{d+1} q}{r_M}.
\end{aligned}
$$

Therefore, we have

$$
\int_{q_d}^{q_{d+1}} \frac{\#(N_1 \cap \widehat{\Lambda}^*)}{q^2} \mathrm{d}q \ll \int_{q_d}^{q_{d+1}} s^2 \frac{q_{d+1}}{s r_M} \frac{1}{q} \mathrm{d}q \leq s^2 \frac{q_{d+1} \log q_{d+1}}{s r_M} \leq s^2. \tag{5.24}
$$

By combining (5.16), (5.21), (5.22), and (5.24), the proof of (2) is completed. □

This proves Lemma 5.2.9. □

For a weight vector $w = (w_1, \dots, w_d)$ as in the introduction, let $1 \leq \ell \leq d-1$ be the unique integer such that $w_1 = \cdots = w_\ell > w_{\ell+1} \geq \cdots \geq w_d$, and denote by $\xi = \max(1, \frac{d-\ell}{\ell})$. For a fixed lattice $\Lambda \subset \mathbb{R}^{d+1}$ and fixed $\mathbf{r}, s$, we denote $q_i(\Lambda, \mathbf{r}, s)$ by $q_i(\Lambda)$ and $N_q(\mathbf{r}, s)$ by N_q for simplicity. Let us fix a constant $C \geq 1$ which is an implied constant for the conclusion of Lemma 5.2.9 (1) and (2).

Lemma 5.2.10. *Let $d \geq 2$, $s = \epsilon^2$, $\mathbf{r} = (r_1, \dots, r_{d+1}) = (\epsilon e^t, \dots, \epsilon e^t, 1)$, $\Lambda \in \mathcal{K}^*_{\epsilon^2} \cap \mathcal{L}'_{d+1}$, and $a_t = \mathrm{diag}(e^{w_1 t}, \dots, e^{w_d t}, e^{-t})$. Then there exists a positive real number $\tilde{\epsilon} \leq 1$ and $c = c(d) > (d+1)^{1/14}$ such that for all $\epsilon, t > 0$ with $ce^{-w_d t/(2d^3)} < \epsilon < \tilde{\epsilon}$, one has*

$$
\#\mathcal{S}(a_t \Lambda, \mathbf{r}, s) \leq \epsilon^{1/2} \cdot \mathrm{vol}(M_{\mathbf{r}}).
$$

Proof. We will prove the lemma for $\tilde{\epsilon} < 1/C^2$ and the constant c will be determined later. By Lemma 5.2.9 (1), it suffices to show that

$$q_1(a_t\Lambda) \geq s^{-2} \quad \text{and} \quad q_{d+1}(a_t\Lambda) \leq ds^{-1/2}r_d. \tag{5.25}$$

First, note that

$$N_q \cap (a_t\Lambda)^* = N_q \cap a_{-t}^*\Lambda^* = a_{-t}^*(a_t^* N_q \cap \Lambda^*),$$

where a_t^* denotes the transpose of a_t. Hence it is enough to show that $a_t^* N_{s^{-2}}$ has no nonzero lattice point of Λ^* for the first inequality of (5.25). Since $d \geq 2$ and $w_d \leq 1/d$, we have

$$e^{-\frac{t}{7}} < e^{-\frac{w_d t}{2d^3}} < ce^{-\frac{w_d t}{2d^3}} < \epsilon,$$

that is, $s^{-2} < r_1 s$. Thus we have

$$N_{s^{-2}} = M^*_{(s^{-2}/r_1,\ldots,s^{-2}/r_1,s^{-2})} = M^*_{(\epsilon^{-5}e^{-t},\ldots,\epsilon^{-5}e^{-t},\epsilon^{-4})},$$

which implies that

$$a_t^* N_{s^{-2}} = M^*_{(\epsilon^{-5}e^{(w_1-1)t},\ldots,\epsilon^{-5}e^{(w_d-1)t},\epsilon^{-4}e^{-t})}.$$

Since for all $i = 1, \ldots, d$

$$\frac{\epsilon}{(d+1)^{1/14}} > \frac{c}{(d+1)^{1/14}}e^{-\frac{w_d t}{2d^3}} > e^{-\frac{(d-1)w_d t}{7}} \geq e^{\frac{(w_i-1)t}{7}},$$

we have $\epsilon^{-5}e^{(w_i-1)t} < \frac{\epsilon^2}{\sqrt{d+1}}$ for all $i = 1, \ldots, d$. It is clear that $\epsilon^{-4}e^{-t} < \epsilon^{-5}e^{(w_d-1)t} < \frac{\epsilon^2}{\sqrt{d+1}}$. Thus $a_t^* N_{s^{-2}}$ is contained in the interior of $B_{\epsilon^2}(\mathcal{E}^*_{d+1})$. Since $\Lambda \in \mathcal{K}^*_{\epsilon^2}$, there is no lattice point of Λ in $a_t^* N_{s^{-2}}$.

To show the second inequality of (5.25), we will construct a basis for Λ^* of which vectors are contained in $N_{ds^{-1/2}r_d} = N_{de^t}$. Since $de^t > r_d > r_d s$, we have

$$a_t^* N_{de^t} = a_t^* M^*_{(s,\ldots,s,de^t)} = M^*_{(se^{w_1 t},\ldots se^{w_d t},d)}.$$

Let $1/2 < r \leq 1$ be such that $r\mathbf{e}_{d+1} \in \widehat{\Lambda}$ from the assumption $\Lambda \in \mathcal{L}'_{d+1}$. Let $\mathbf{pr} : \mathbb{R}^{d+1} \to \mathbb{R}^d$ be the orthogonal projection onto Span$(\mathbf{e}_1, \ldots, \mathbf{e}_d)$. Note that $\mathbf{pr}(\Lambda)$ is a lattice with covolume $1/r$ in $\mathbb{R}^d$. If $\mathbf{v} \in \Lambda$ satisfies $\|\mathbf{pr}(\mathbf{v})\| =$

$\lambda_1(B_1, \mathbf{pr}(\Lambda))$, then since $\Lambda \in \mathcal{K}^*_{\epsilon^2}$, it follows from Lemma 5.2.7 with $D = d+1$ that

$$\lambda_1(B_1, \mathbf{pr}(\Lambda)) \geq r\lambda_1(B_1, \mathbf{pr}(\Lambda)) = \|\mathbf{v} \wedge r\mathbf{e}_{d+1}\| \geq \bar{c}_1(\epsilon^2)^{d-1} \tag{5.26}$$

for some $\bar{c}_1 = \bar{c}_1(d) < 1$. Since $\mathrm{cov}(\mathbf{pr}(\Lambda)) = 1/r$, it follows from the Minkowski's second theorem and (5.26) that for any $0 < c_1 < \bar{c}_1$

$$c_1^{d-1}(\epsilon^2)^{(d-1)^2}\lambda_d(B_1, \mathbf{pr}(\Lambda)) \leq \lambda_1(B_1, \mathbf{pr}(\Lambda)) \cdots \lambda_d(B_1, \mathbf{pr}(\Lambda)) \ll 1,$$

hence there exists $c_2 = c_2(d) > 1$ such that

$$\lambda_d(B_1, \mathbf{pr}(\Lambda)) \leq c_2(\epsilon^{-2})^{(d-1)^2}. \tag{5.27}$$

Let $\{v^{(i)} : i = 1, \ldots, d\}$ be a Minkowski reduced basis for $\mathbf{pr}(\Lambda)$ such that $\|v^{(i)}\| \leq 2^d\lambda_i(B_1, \mathbf{pr}(\Lambda))$. For each $i = 1, \ldots, d$, let $\mathbf{v}_i \in \Lambda$ be such that $\mathbf{pr}(\mathbf{v}_i) = v^{(i)}$ and $|\mathbf{e}^*_{d+1}(\mathbf{v}_i)| < 1$. Then the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_d, \mathbf{v}_{d+1} = r\mathbf{e}_{d+1}$ form a basis for Λ. Recall that $\mathcal{E}^*_{d+1}$ can be naturally identified with $\bigwedge^d_{\mathbb{R}} \mathbb{R}^{d+1}$ with the standard Euclidean structure. Under this identification, we have $\Lambda^* = \bigwedge^d_{\mathbb{Z}} \Lambda$, hence the vectors $\bigwedge_{j\neq i} \mathbf{v}_j$ for $i = 1, \ldots, d+1$ forms a basis for $\bigwedge^d_{\mathbb{Z}} \Lambda$. We now claim that the vectors $\bigwedge_{j\neq i} \mathbf{v}_j$ for $i = 1, \ldots, d+1$ are contained in $a^*_t N_{de^t}$ via the above identification, which proves that $q_{d+1}(a_t\Lambda) \leq ds^{-1/2}r_d$.

For each $i = 1, \ldots, d+1$, write

$$\bigwedge_{j\neq i} \mathbf{v}_j = \sum_{h=1}^{d+1} \left(x^{(i)}_h \bigwedge_{k\neq h} \mathbf{e}_k \right).$$

Note that $|x^{(d+1)}_{d+1}| = 1/r \leq 2 \leq d$ and $x^{(i)}_{d+1} = 0$ for each $i = 1, \ldots, d$ since $\mathbf{v}_{d+1} = r\mathbf{e}_{d+1}$. By the definition of $\mathbf{v}_i$ and (5.27), since $\epsilon < 1$, we can choose large enough $c_3 = c_3(d) > (d+1)^{d^2/7}$ for each $i = 1, \ldots, d$,

$$\|\mathbf{v}_i\| \leq \sqrt{1 + \|v^{(i)}\|^2} \leq 2^d\sqrt{2}c_2(\epsilon^{-2})^{(d-1)^2} \leq c_3(\epsilon^{-2})^{(d-1)^2}.$$

Thus for each $i = 1, \ldots, d+1$ and $h = 1, \ldots, d$,

$$|x^{(i)}_h| \leq \left\| \bigwedge_{j\neq i} \mathbf{v}_j \right\| \leq \prod_{j\neq i} \|\mathbf{v}_j\| \leq c_3^d(\epsilon^{-2})^{d(d-1)^2}.$$

From the assumption $ce^{-w_d t/(2d^3)} < \epsilon$, it follows that

$$c^{2d^3} e^{-w_d t} < (\epsilon^2)^{d^3} < (\epsilon^2)^{d(d-1)^2+1}.$$

Choosing $c = c_3^{1/2d^2} > (d+1)^{1/14}$, we have

$$|x_h^{(i)}| \leq c_3^d (\epsilon^{-2})^{d(d-1)^2} < \epsilon^2 e^{w_d t} = s e^{w_d t} \leq s e^{w_i t},$$

which concludes the claim. □

Lemma 5.2.11. *Let $d \geq 2$, $\mathbf{r} = (r_1, \ldots, r_{d+1})$, $\bar{b}_t = \operatorname{diag}\left(\bar{b}_{t,1}, \ldots, \bar{b}_{t,d+1}\right)$, and $\Lambda \in \mathcal{K}^*_{\epsilon^2}$, where*

$$r_i = \begin{cases} \epsilon e^{\left(\xi - \frac{1}{\ell}(w_{\ell+1} + \cdots + w_d)\right)t} & \text{if} \quad 1 \leq i \leq \ell, \\ \epsilon e^{(\xi + w_i)t} & \text{if} \quad \ell + 1 \leq i \leq d, \\ 1 & \text{if} \quad i = d+1, \end{cases}$$

and

$$\bar{b}_{t,i} = \begin{cases} e^{\left(\xi w_i - \frac{1}{\ell}(w_{\ell+1} + \cdots + w_d)\right)t} & \text{if} \quad 1 \leq i \leq \ell, \\ e^{(1+\xi) w_i t} & \text{if} \quad \ell + 1 \leq i \leq d, \\ e^{-\xi t} & \text{if} \quad i = d+1. \end{cases}$$

Then there exists a positive real number $\tilde{s} \leq 1$ such that for all $s, t > 0$ with $e^{-\delta t} < \epsilon < s < \tilde{s}$ where $\delta = \frac{1}{18d^2} \min\left(\xi w_d, \xi w_1 - \frac{1}{\ell}(w_{\ell+1} + \cdots + w_d)\right)$, one has

$$\#\mathcal{S}(\bar{b}_t \Lambda, \mathbf{r}, s) \leq s \operatorname{vol}(M_{\mathbf{r}}). \tag{5.28}$$

Proof. Note that $r_m = r_1 < r_M = r_{\ell+1}$. Take $t_0 = t_0(w_1, \ldots, w_d) > 0$ such that for any $t > t_0$ we have

$$e^{\frac{w_d}{20} t} \geq (\xi + \frac{w_d}{2})t. \tag{5.29}$$

Denoting by $c_4 = e^{-\delta t_0}$, then $c_4 \in (0,1)$ depends only on the weights $w_1, \ldots, w_d$, and the inequality (5.29) holds whenever $e^{-\delta t} < c_4$. Let

$$\tilde{s} = \min\left(\frac{1}{C}, c_4, \frac{1}{\sqrt{d+1}}, \left(\frac{\operatorname{vol}(B_1)}{4^{d+1}}\right)^{1/d}\right) \leq 1.$$

By Lemma 5.2.9 (2), it suffices to show that for $e^{-\delta t} < \epsilon < s < \tilde{s}$,

$$q_1(\overline{b}_t\Lambda) \geq s^{-2} \quad \text{and} \quad q_{d+1}(\overline{b}_t\Lambda)\log q_{d+1}(\overline{b}_t\Lambda) \leq s\epsilon e^{(\xi+w_{\ell+1})t}.$$

Since $e^{-\delta t} < \epsilon < s$, it follows from $s^{-3}\epsilon^{-1} < \epsilon^{-4} < e^{4\delta t}$ that $s^{-2}/r_i < s$ for all $i = 1, \ldots, d$, hence

$$\begin{aligned}\overline{b}_t^* N_{s^{-2}} &= \overline{b}_t^* M^*_{(\frac{s^{-2}}{r_1},\ldots,\frac{s^{-2}}{r_d},s^{-2})} \\ &= M^*_{\left(e^{\xi(w_1-1)t}\epsilon^{-1}s^{-2},\ldots,e^{\xi(w_d-1)t}\epsilon^{-1}s^{-2},e^{-\xi t}s^{-2}\right)}.\end{aligned}$$

Since $\tilde{s} \leq \frac{1}{\sqrt{d+1}}$, we have for all $i = 1, \ldots, d$,

$$\frac{s^2\epsilon^3}{\sqrt{d+1}} > \epsilon^6 > e^{-6\delta t} > e^{-(\xi w_1 - \frac{1}{\ell}(w_{\ell+1}+\cdots+w_d))t} \geq e^{\xi(w_i-1)t},$$

and

$$\frac{s^2\epsilon^2}{\sqrt{d+1}} > \epsilon^5 > e^{-5\delta t} > e^{-\xi w_d t} > e^{-\xi t},$$

hence it follows that $\overline{b}_t^* N_{s^{-2}}$ is contained in the interior of $B_{\epsilon^2}(\mathcal{E}^*_{d+1})$. Since $\Lambda \in \mathcal{K}^*_{\epsilon^2}$, there is no lattice point of Λ in $\overline{b}_t^* N_{s^{-2}}$, which concludes $q_1(\overline{b}_t\Lambda) \geq s^{-2}$ as in the proof of the first inequality of (5.25).

Since $\xi = \max(1, \frac{d-\ell}{\ell}) < d$, we have

$$s\epsilon > \epsilon^2 > e^{-\frac{1}{9d^2}\xi w_d t} > e^{-\frac{1}{9d}w_d t}, \tag{5.30}$$

which implies that

$$e^{\frac{w_d}{2}t} = e^{-\frac{w_d}{2}t}e^{w_d t} < e^{-\frac{1}{9d}w_d t}e^{w_d t} < s\epsilon e^{w_d t},$$

hence $e^{(\xi+\frac{w_d}{2})t} < r_d s$. On the other hand, it is clear that $r_\ell s < e^{(\xi+\frac{w_d}{2})t}$, hence $\overline{b}_t^* N_{e^{(\xi+w_d/2)t}}$ is the set of $\varphi = x_1^\varphi \mathbf{e}_1^* + \cdots + x_{d+1}^\varphi \mathbf{e}_{d+1}^* \in \mathcal{E}^*_{d+1}$ such that

$$\begin{cases} |x_i^\varphi| \leq s e^{\left(\xi w_i - \frac{1}{\ell}(w_{\ell+1}+\cdots+w_d)\right)t} & \text{for } 1 \leq i \leq \ell, \\ |x_i^\varphi| \leq \epsilon^{-1}e^{(\xi w_i + \frac{w_d}{2})t} & \text{for } \ell+1 \leq i \leq d, \\ |x_i^\varphi| \leq e^{\frac{1}{2}w_d t} & \text{for } i = d+1. \end{cases}$$

It follows from $\Lambda \in \mathcal{K}^*_{\epsilon^2}$ that $\lambda_1(B_1, \Lambda^*) \geq \epsilon^2$. By Minkowski's second theorem, we have

$$\epsilon^{2d}\lambda_{d+1}(B_1,\Lambda^*) \le \lambda_1(B_1,\Lambda^*)\cdots\lambda_{d+1}(B_1,\Lambda^*) \le \frac{2^{d+1}}{\mathrm{vol}(B_1)},$$

hence $\lambda_{d+1}(B_1,\Lambda^*) \le \frac{2^{d+1}}{\mathrm{vol}(B_1)}\epsilon^{-2d}$. Thus there exists a Minkowski reduced basis $\varphi_1,\dots,\varphi_{d+1}$ of Λ^* such that $\|\varphi_i\| \le \frac{4^{d+1}}{\mathrm{vol}(B_1)}\epsilon^{-2d} \le \epsilon^{-3d}$ for all $i=1,\dots d+1$ since $\epsilon^d < \tilde{s}^d \le \frac{\mathrm{vol}(B_1)}{4^{d+1}}$. Recall that $w_1 = \cdots = w_\ell$, hence it can be easily checked that φ_i's are contained in $\overline{b}_t^* N_{e^{(\xi+w_d/2)t}}$. Thus $q_{d+1}(\overline{b}_t\Lambda) \le e^{(\xi+w_d/2)t}$ so that

$$\begin{aligned} q_{d+1}(\overline{b}_t\Lambda)\log q_{d+1}(\overline{b}_t\Lambda) &\le e^{(\xi+\frac{w_d}{2})t}(\xi+\frac{w_d}{2})t \\ &\le e^{(\xi+\frac{w_d}{2})t}e^{\frac{w_d t}{20}} && \text{by (5.29)} \\ &\le s\epsilon e^{(\xi+w_d)t} && \text{by (5.30)} \\ &\le s\epsilon e^{(\xi+w_{\ell+1})t}. \end{aligned}$$

□

5.3 Lower bound

5.3.1 Construction of the fractal set

For a given weight vector $w=(w_1,\dots,w_d)$, recall that $1 \le \ell \le d-1$ is the unique integer such that $w_1 = \cdots = w_\ell > w_{\ell+1} \ge \cdots \ge w_d$, and $\xi = \max(1,\frac{d-\ell}{\ell})$ (see §5.2.2). We choose a real number $c_0 > 1$ such that

$$\text{(5.31)} \qquad \frac{1}{10} < \left(\frac{2}{c_0}-c_0\right)\frac{1}{\zeta(d+1)} \quad \text{and} \quad \frac{c_0}{\zeta(d+1)} < 1,$$

using $1<\zeta(d+1)<2$. Let $\tilde{c} \le 1$ be a positive real number as in Lemma 5.2.8 with respect to the above c_0, and let $\tilde{\epsilon},\tilde{s} \le 1$ be positive real numbers as in Lemmas 5.2.10 and 5.2.11, respectively. We fix the constants $\epsilon,t,r>0$ with the following properties:

1. $0<\epsilon<r<\frac{1}{10^4 4^d}\min\{\tilde{\epsilon},\tilde{s}\}$;
2. $t \ge 1$ will be chosen large enough so that (5.34), (5.39), (5.41), (5.42), (5.44), (5.45), (5.46), (5.47) hold.

Let $\{\epsilon_n\}$ and $\{t_n\}$ be the sequence defined as follows: for $n \in \mathbb{N}$,

1. $\epsilon_n = \epsilon/n$;
2. $t_n - t_{n-1} = \xi nt$ and $t_0 = 1$.

We will construct the tree $\mathcal{T}$ whose vertices are in the set $\mathbb{Q}^d$ of rational vectors and the map β from $V\mathcal{T}$ to the set of compact subsets in $\mathbb{R}^{d+1}$, inductively. We first set the root of $\mathcal{T}$ to be zero, that is, $\tau_0 = \mathbf{0}$ and define

$$\beta(\tau_0) = \{x \in \mathbb{R}^d : |(\tau_0)_i - x_i| < e^{-w_i t_1}, \forall i = 1, \dots, d\}.$$

For each $\tau \in \mathcal{T}_n$ with $n \geq 1$, let

$$\tilde{\beta}(\tau) = \{x \in \mathbb{R}^d : |\tau_i - x_i| < \epsilon_{n+1} e^{-w_i t_{n+1} - t_n}, \forall i = 1, \dots, d\}.$$

Recall that $a_t = \operatorname{diag}\left(e^{w_1 t}, \dots, e^{w_d t}, e^{-t}\right)$ and $h(x) = \begin{pmatrix} I_d & x \\ 0 & 1 \end{pmatrix}$ for $x \in \mathbb{R}^d$. Denote by

$$b_n = \operatorname{diag}(e^{-\frac{1}{\ell}(w_{\ell+1}+\cdots+w_d)nt}, \dots, e^{-\frac{1}{\ell}(w_{\ell+1}+\cdots+w_d)nt}, e^{w_{\ell+1}nt}, \dots, e^{w_d nt}, 1).$$

Note that the first ℓ terms of b_n are the same.

For each $\kappa \in \mathcal{T}_{n-1}$, we define $\mathcal{T}(\kappa)$ as the set of all $\tau \in \tilde{\beta}(\kappa)$ with the following properties:

$$\begin{aligned} & a_{t_n} h(\tau)\mathbb{Z}^{d+1} \in \mathcal{L}'_{d+1}, \\ & a_{t_n} h(\tau)\mathbb{Z}^{d+1} \in \mathcal{K}^*_{\epsilon_n^2}, \\ & b_n a_{t_n} h(\tau)\mathbb{Z}^{d+1} \in \mathcal{K}^*_r. \end{aligned} \tag{5.32}$$

It follows from the definitions of $\mathcal{T}(\kappa)$ and $\mathcal{L}'_{d+1}$ that $\tau \in \mathbb{Q}^d$ and for any $\tau \in \mathcal{T}(\kappa)$ there exists the unique vector

$$\mathbf{v}(\tau) \in \{r\mathbf{e}_{d+1} : 1/2 < r \leq 1\} \cap a_{t_n} h(\tau)\mathbb{Z}^{d+1}. \tag{5.33}$$

Note that $(d+1)$-th coordinate of $\mathbf{v}(\tau)$ is qe^{-t_n} for some $q \in \mathbb{Z}$ such that $1/2 < qe^{-t_n} \leq 1$. Since $t_n \geq t_{n-1} + 1$, $\mathcal{T}_n$ has empty intersection with $\bigcup_{0 \leq i \leq n-1} \mathcal{T}_i$, which implies that $\mathcal{T}$ is a rooted tree.

For each $\tau \in \mathcal{T}(\kappa)$ with $\kappa \in \mathcal{T}_{n-1}$, define

$$\beta(\tau) = \{x \in \mathbb{R}^d : |\tau_i - x_i| < \epsilon_n e^{-w_i t_{n+1} - t_n}, \forall i = 1, \dots, d\}.$$

Note that for each $\tau \in \mathcal{T}(\kappa)$, it follows from the definitions of $\tilde{\beta}$ and β that $\beta(\tau) \subset \beta(\kappa)$. If follows from Lemma 5.3.1 below that each vertex of $\mathcal{T}$ has sons by choosing $t \geq 1$ large enough so that for any $n \in \mathbb{N}$

$$\frac{1}{100}\epsilon_n^d e^{\xi dnt} \geq 1. \tag{5.34}$$

Hence the pair $(\mathcal{T}, \beta)$ is a regular self-affine structure.

Lemma 5.3.1. *For every $n \in \mathbb{N}$ and $y \in \mathcal{T}_{n-1}$ one has*

$$\frac{1}{100}\epsilon_n^d e^{\xi dnt} \leq \#\mathcal{T}(y) \leq 2^{d+1}\epsilon_n^d e^{\xi dnt}.$$

For fixed $n \in \mathbb{N}$ and $y \in \mathcal{T}_{n-1}$, we let

$$\begin{aligned}
\Lambda &= a_{t_{n-1}} h(y)\mathbb{Z}^{d+1} \in \mathcal{L}'_{d+1} \cap \mathcal{K}^*_{\epsilon^2_{n-1}},\\
\Lambda_1 &= a_{t_n} h(y)\mathbb{Z}^{d+1} = a_{\xi nt}\Lambda,\\
\Lambda_2 &= b_n a_{t_n} h(y)\mathbb{Z}^{d+1} = b_n a_{\xi nt}\Lambda,
\end{aligned}$$

and for $x \in \tilde{\beta}(y)$,

$$\begin{aligned}
\Lambda_1(x) &= a_{t_n} h(x)\mathbb{Z}^{d+1} = a_{t_n} h(x-y) a_{t_n}^{-1}\Lambda_1,\\
\Lambda_2(x) &= b_n a_{t_n} h(x)\mathbb{Z}^{d+1} = b_n a_{t_n} h(x-y) a_{t_n}^{-1} b_n^{-1}\Lambda_2.
\end{aligned}$$

The lattices $\Lambda_1(x)$ and $\Lambda_2(x)$ satisfy $\Lambda_1(x) \in \mathcal{L}'_{d+1} \cap \mathcal{K}^*_{\epsilon^2_n}$ and $\Lambda_2(x) \in \mathcal{K}^*_r$ if and only if $x \in \mathcal{T}(y)$. Hence Lemma 5.3.1 follows from the following lemma.

Lemma 5.3.2. *Let $n \in \mathbb{N}$ and $y \in \mathcal{T}_{n-1}$. Then*

$$\frac{1}{10}\epsilon_n^d e^{\xi dnt} \leq \#\{x \in \tilde{\beta}(y) : \Lambda_1(x) \in \mathcal{L}'_{d+1}\} \leq 2^{d+1}\epsilon_n^d e^{\xi dnt}, \tag{5.35}$$

$$\#\{x \in \tilde{\beta}(y) : \Lambda_1(x) \in \mathcal{L}'_{d+1} \setminus \mathcal{K}^*_{\epsilon^2_n}\} \leq \frac{8}{100}\epsilon_n^d e^{\xi dnt}, \tag{5.36}$$

$$\#\{x \in \tilde{\beta}(y) : \Lambda_2(x) \in \mathcal{L}'_{d+1} \setminus \mathcal{K}^*_r\} \leq \frac{1}{100}\epsilon_n^d e^{\xi dnt}. \tag{5.37}$$

Proof. Let $x \in \tilde{\beta}(y)$ with $\Lambda_1(x) \in \mathcal{L}'_{d+1}$. Then there exists s_x such that $1/2 <$

$s_x \le 1$ and $\Lambda_1(x) \cap \mathbb{R}\mathbf{e}_{d+1} = \{s_x \mathbf{e}_{d+1}\}$. We denote $s_x \mathbf{e}_{d+1}$ by $\mathbf{v}(x)$.

First, we prove (5.35). It can be checked by a direct calculation that the map $x \mapsto a_{t_n} h(y-x) a_{t_n}^{-1} \mathbf{v}(x)$ is a bijection from $\{x \in \tilde{\beta}(y) : \Lambda_1(x) \in \mathcal{L}'_{d+1}\}$ to $M \cap \widehat{\Lambda}_1$ where

$$M = \{(z_1, \dots, z_{d+1}) : \max_{1 \le i \le d} |z_i| \le \epsilon_n e^{\xi nt} |z_{d+1}|,\ 1/2 < |z_{d+1}| \le 1\}.$$

Thus it suffices to estimate $\#(M \cap \widehat{\Lambda}_1)$. Let

$$\begin{aligned} M^{(1)} &= \{(z_1, \dots, z_{d+1}) : \max_{1 \le i \le d} |z_i| \le \frac{1}{2} \epsilon_n e^{\xi nt},\ |z_{d+1}| \le 1\} \\ M^{(2)} &= \{(z_1, \dots, z_{d+1}) : \max_{1 \le i \le d} |z_i| \le \frac{1}{2} \epsilon_n e^{\xi nt},\ |z_{d+1}| \le \frac{1}{2}\}. \end{aligned}$$

Since $M^{(1)} \smallsetminus M^{(2)} \subset M \subset 2M^{(2)}$, we have

$$\#(M^{(1)} \cap \widehat{\Lambda}_1) - \#(M^{(2)} \cap \widehat{\Lambda}_1) \le \#(M \cap \widehat{\Lambda}_1) \le \#(2M^{(2)} \cap \widehat{\Lambda}_1). \tag{5.38}$$

We will use Lemma 5.2.8 to estimate $\#(M^{(i)} \cap \widehat{\Lambda}_1)$ for $i = 1, 2$. Since $\Lambda \in \mathcal{K}^*_{\epsilon_{n-1}^2} \subset \mathcal{K}^*_{\epsilon_n^2}$, it follows from the natural identification $\mathcal{E}_d^{**} = \mathbb{R}^d$ and Minkowski second theorem that there exist contants $C_1, C_2 > 0$ depending only on d such that

$$\lambda_1(B_1, \Lambda) \ge C_1 \epsilon_n^{2d} \text{ and } \lambda_{d+1}(B_1, \Lambda) \le C_2 \epsilon_n^{-2}.$$

Since $\Lambda = a_{\xi nt}^{-1} \Lambda_1$, for $i = 1, 2$, we have

$$\begin{aligned} \lambda_1(M^{(i)}, \Lambda_1) &= \lambda_1(a_{\xi nt}^{-1} M^{(i)}, \Lambda) \ge \lambda_1(a_{\xi nt}^{-1} M^{(1)}, \Lambda) \\ &\ge \lambda_1(B_{(d+1)e^{\xi nt}}, \Lambda) \ge \frac{C_1}{d+1} e^{-\xi nt} \epsilon_n^{2d} \end{aligned}$$

and

$$\begin{aligned} \lambda_{d+1}(M^{(i)}, \Lambda_1) &= \lambda_{d+1}(a_{\xi nt}^{-1} M^{(i)}, \Lambda) \le \lambda_{d+1}(a_{\xi nt}^{-1} M^{(2)}, \Lambda) \\ &\le \lambda_{d+1}(B_{\frac{1}{2}\epsilon_n e^{(1-w_1)\xi nt}}, \Lambda) \le 2C_2 e^{(w_1-1)\xi nt} \epsilon_n^{-3}. \end{aligned}$$

Thus we can choose $t \ge 1$ large enough so that for all $n \in \mathbb{N}$

$$\lambda_{d+1}(M^{(i)}, \Lambda_1) < \widetilde{c} \quad \text{and} \quad -\lambda_{d+1}(M^{(i)}, \Lambda_1) \log \lambda_1(M^{(i)}, \Lambda_1) < \widetilde{c}. \tag{5.39}$$

Using Lemma 5.2.8 and (5.38), we have

$$\left(\frac{2}{c_0} - c_0\right) \frac{1}{\zeta(d+1)} \epsilon_n^d e^{\xi dnt} \leq \#(M \cap \widehat{\Lambda}_1) \leq \frac{c_0}{\zeta(d+1)} 2^{d+1} \epsilon_n^d e^{\xi dnt}.$$

By (5.31), we complete the proof of (5.35).

Next, we prove (5.36) and (5.37). Let $s_1 = \epsilon_n^2$, $s_2 = r$, $a^{(1)} = a_{\xi nt}$, $a^{(2)} = b_n a_{\xi nt}$, and

$$\mathcal{S}_j = \{x \in \tilde{\beta}(y) : \Lambda_i(x) \in \mathcal{L}'_{d+1} \smallsetminus \mathcal{K}^*_{s_j}\} \text{ for } j = 1, 2.$$

Recall that

$$\mathcal{S}(\Lambda, \mathbf{r}, s) = \left\{ \mathbf{v} \in M_{\mathbf{r}} \cap \widehat{\Lambda} : \varphi(\mathbf{v}) = 0 \text{ for some } \varphi \in N_{(d+1)sr_M}(\mathbf{r}, s) \cap \widehat{\Lambda}^* \right\}.$$

We will show that

$$\#\mathcal{S}_j \leq \#\mathcal{S}(\Lambda_j, \mathbf{r}_j, s_j) \quad (j = 1, 2) \tag{5.40}$$

for some $\mathbf{r}_j$ and apply Lemma 5.2.10 and 5.2.11.

Let $z^{(1)}$ and $z^{(2)}$ be vectors in $\mathbb{R}^d$ such that

$$\begin{aligned} z_i^{(1)} &= (y_i - x_i) e^{(w_i+1)t_n} \quad \text{for } 1 \leq i \leq d; \\ z_i^{(2)} &= \begin{cases} (y_i - x_i) e^{-\frac{1}{\ell}(w_{\ell+1} + \cdots + w_d)nt + (w_i+1)t_n} & \text{if } 1 \leq i \leq \ell, \\ (y_i - x_i) e^{w_i nt + (w_i+1)t_n} & \text{if } \ell+1 \leq i \leq d. \end{cases} \end{aligned}$$

Then $h(z^{(j)}) = a^{(j)} a_{t_{n-1}} h(y-x) (a^{(j)} a_{t_{n-1}})^{-1}$ for $j = 1, 2$ and

$$\begin{aligned} |z_i^{(1)}| &\leq \epsilon_n e^{\xi nt} =: r_i^{(1)} \quad \text{for } 1 \leq i \leq d; \\ |z_i^{(2)}| &\leq \begin{cases} \epsilon_n e^{\left(\xi - \frac{1}{\ell}(w_{\ell+1} + \cdots + w_d)\right)nt} =: r_i^{(2)} & \text{if } 1 \leq i \leq \ell, \\ \epsilon_n e^{(\xi + w_i)nt} =: r_i^{(2)} & \text{if } \ell+1 \leq i \leq d. \end{cases} \end{aligned}$$

Since $\mathbf{v}(x) \in \Lambda_1(x) \cap \Lambda_2(x)$, for $j = 1, 2$,

$$\mathbf{w}_j(x) := h(z^{(j)}) \mathbf{v}(x) \in \Lambda_j.$$

For $\mathbf{r}_j = (r_1^{(j)}, \ldots, r_d^{(j)}, 1)$, the map $\mathcal{S}_j \to M_{\mathbf{r}_j} \cap \widehat{\Lambda}_j$ given by $x \mapsto \mathbf{w}_j(x)$ is injective. Hence, in order to show (5.40), we should find $\varphi_j \in N_{(d+1)s_j r_M^{(j)}}(\mathbf{r}_j, s_j) \cap \widehat{\Lambda}_i^*$

such that $\varphi_j(\mathbf{w}_j(x)) = 0$. It follows from the definition of $\mathcal{S}_j$ that for $x \in \mathcal{S}_j$, $a^{(j)} a_{t_{n-1}} h(x)\mathbb{Z}^{d+1} \notin \mathcal{K}^*_{s_j}$. Then there exists $\varphi_j \in \widehat{\Lambda}^*_j$ such that $\|h(z^{(j)})^* \varphi_j\| < s_j$, where $h(z^{(j)})^*$ is the adjoint action defined by $g^*\varphi(\mathbf{v}) = \varphi(g\mathbf{v})$ for all $g \in \mathrm{SL}_{d+1}(\mathbb{R})$, $\varphi \in \mathcal{E}^*_{d+1}$, and $\mathbf{v} \in \mathbb{R}^{d+1}$. It follows from direct calculation that

$$h(z^{(j)})^* \varphi_j = \left(\varphi_j(\mathbf{e}_1), \dots, \varphi_j(\mathbf{e}_d), \sum_{i=1}^{d} z_i^{(j)} \varphi_j(\mathbf{e}_i) + \varphi_j(\mathbf{e}_{d+1}) \right).$$

By choosing $t \geq 1$ large enough so that for all $n \in \mathbb{N}$

$$\epsilon e^{\xi n t} \geq 1, \tag{5.41}$$

it follows from $\|h(z^{(j)})^* \varphi_j\| < s_j$ that

$$\begin{aligned} &|\varphi_j(\mathbf{e}_i)| < s_j \quad \text{for } 1 \leq i \leq d; \\ &|\varphi_j(\mathbf{e}_{d+1})| < s_j + d s_j r_M^{(j)} < (d+1) s_j r_M^{(j)}. \end{aligned}$$

Hence we have $\varphi_j \in N_{(d+1)s_j r_M^{(j)}}$. It follows that

$$\begin{aligned} |\varphi_j(\mathbf{w}_j(x))| &= |h(z^{(j)})^* \varphi_j(h(-z^{(j)})\mathbf{w}_j(x))| = |h(z^{(j)})^* \varphi_j(\mathbf{v}(x))| \\ &\leq |h(z^{(j)})^* \varphi_j(\mathbf{e}_{d+1})| \leq \|h(z^{(j)})^* \varphi_j\| < s_j < 1. \end{aligned}$$

Since $\varphi_j(\mathbf{w}_j(x)) \in \mathbb{Z}$, it follows that $\varphi_j(\mathbf{w}_j(x)) = 0$. This proves (5.40).

We choose $t \geq 1$ large enough so that for all $n \in \mathbb{N}$

$$c e^{-w_d \xi n t/(2d^3)} < \epsilon_n \quad \text{and} \quad e^{-\delta n t} < \epsilon_n. \tag{5.42}$$

Since $\Lambda \in \mathcal{K}^*_{\epsilon^2_{n-1}} \subset \mathcal{K}^*_{\epsilon^2_n}$ and (5.42), it follows from Lemma 5.2.10 and 5.2.11 that

$$\begin{aligned} \mathcal{S}_1 &\leq \#\mathcal{S}(\Lambda_1, \mathbf{r}_1, s_1) \leq \sqrt{\epsilon_n}\, \mathrm{vol}(M_{\mathbf{r}_1}) = 2^{d+1} \sqrt{\epsilon_n} \epsilon_n^d e^{\xi d n t}, \\ \mathcal{S}_2 &\leq \#\mathcal{S}(\Lambda_2, \mathbf{r}_2, s_2) \leq r\, \mathrm{vol}(M_{\mathbf{r}_2}) = 2^{d+1} r \epsilon_n^d e^{\xi d n t}. \end{aligned}$$

By the assumption (1) for ϵ and r, this complete the proof. ☐

The following lemma is d-dimensional version of [LSST20, Lemma 4.1].

Lemma 5.3.3. $\mathcal{F}(\mathcal{T}, \beta) \subset \mathrm{Sing}(w)$.

Proof. This lemma directly follows from the same argument in the proof of [LSST20, Lemma 4.1]. □

5.3.2 The lower bound calculation

In this subsection we complete the proof of main results.

Proposition 5.3.4. *Let $w = (w_1, \dots, w_d) \in \mathbb{R}^d_{>0}$ where $w_1 = \cdots = w_\ell > w_{\ell+1} \geq \cdots \geq w_d > 0$ and $\sum_{i=1}^d w_i = 1$ and let $(\mathcal{T}, \beta)$ be the self-affine strunction on $\mathbb{R}^d$ in the previous section. Then*

$$\dim_H \mathcal{F}(\mathcal{T}, \beta) \geq d - \frac{1}{1 + w_1}.$$

We will prove Proposition 5.3.4 using Corollary 5.1.2. Let $C_n, L_n^{(1)}, \dots, L_n^{(d)}$ be the positive constants defined as follows:

$$C_n = \epsilon_n^d e^{\xi dnt}, \quad L_n^{(i)} = 2\epsilon_n e^{-w_i t_{n+1} - t_n}, \forall i = 1, \dots, d.$$

It can be easily checked that a regular self-affine structure $(\mathcal{T}, \beta)$ satisfies assumptions (1), (2), and (3) of Theorem 5.1.1. For the assumption (4) of Theorem 5.1.1, we need the following lemma.

Lemma 5.3.5. *Let $n \in \mathbb{N}$ be large and $\tau \in \mathcal{T}_{n-1}$. Then*

$$\mathrm{dist}(\beta(x), \beta(y)) \geq L_{n-1}^{(1)} \frac{c' r^{d-1}}{4\sqrt{d}\epsilon_{n-1}} e^{\frac{1}{\ell}(w_{\ell+1} + \cdots + w_d - \xi\ell)nt},$$

where $x, y \in \mathcal{T}(\tau)$ are distinct and c' is the positive constant in Lemma 5.2.7.

Proof. By the construction of $\mathcal{T}$ and the definition of b_n, there are $1/2 \leq s_x, s_y \leq 1$ such that

$$s_x \mathbf{e}_{d+1} \in b_n a_{t_n} h(x) \mathbb{Z}^{d+1}, \quad s_y \mathbf{e}_{d+1} \in b_n a_{t_n} h(y) \mathbb{Z}^{d+1}.$$

Let us denote by

$$\mathbf{v} = b_n a_{t_n} h(y - x)(b_n a_{t_n})^{-1} s_x \mathbf{e}_{d+1} \in b_n a_{t_n} h(y) \mathbb{Z}^{d+1},$$

$$\mathbf{v} \wedge s_y \mathbf{e}_{d+1} = s_x s_y \sum_{i=1}^d u_i \mathbf{e}_i \wedge \mathbf{e}_{d+1}.$$

Observe that

$$u_i = \begin{cases} (y_i - x_i)e^{(w_i+1)t_n - \frac{1}{\ell}(w_{\ell+1}+\cdots+w_d)nt} & \text{for } 1 \le i \le \ell, \\ (y_i - x_i)e^{(w_i+1)t_n + w_i nt} & \text{for } \ell+1 \le i \le d. \end{cases}$$

Since x and y are distinct, the vectors $\mathbf{v}$ and $e_y\mathbf{e}_{d+1}$ are linearly independent, hence it follows from Lemma 5.2.7 that

$$\sqrt{d}\|\mathbf{u}\|_\infty \ge s_x s_y \|\mathbf{u}\| = \|\mathbf{v} \wedge s_y\mathbf{e}_{d+1}\| \ge c' r^{d-1}, \tag{5.43}$$

where $\mathbf{u} = (u_1, \ldots, u_d) \in \mathbb{R}^d$ and $\|\cdot\|_\infty$ denotes the max norm.

Let $x' \in \beta(x)$ and $y' \in \beta(y)$. Suppose that $\|\mathbf{u}\|_\infty = |u_i|$ for some $1 \le i \le \ell$. Then it follows from (5.43) that

$$\begin{aligned} \|y' - x'\| &\ge |y_i' - x_i'| \ge |y_i - x_i| - |x_i - x_i'| - |y_i - y_i'| \\ &\ge e^{-(w_i+1)t_n + \frac{1}{\ell}(w_{\ell+1}+\cdots+w_d)nt}\left(\frac{c' r^{d-1}}{\sqrt{d}} - 2\epsilon_n e^{-\xi w_i(n+1)t - \frac{1}{\ell}(w_{\ell+1}+\cdots+w_d)nt}\right) \\ &\ge e^{-(w_i+1)t_n + \frac{1}{\ell}(w_{\ell+1}+\cdots+w_d)nt}\frac{c' r^{d-1}}{2\sqrt{d}} \\ &\ge L_{n-1}^{(i)}\frac{c' r^{d-1}}{4\sqrt{d}\epsilon_{n-1}} e^{\frac{1}{\ell}(w_{\ell+1}+\cdots+w_d - \xi\ell)nt} \\ &\ge L_{n-1}^{(1)}\frac{c' r^{d-1}}{4\sqrt{d}\epsilon_{n-1}} e^{\frac{1}{\ell}(w_{\ell+1}+\cdots+w_d - \xi\ell)nt}. \end{aligned} \tag{5.44}$$

We choose $t \ge 1$ large enough so that the third line (5.44) holds for all $n \in \mathbb{N}$.

On the other hand, if $\|\mathbf{u}\|_\infty = |u_i|$ for some $\ell+1 \le i \le d$, then we have

$$\begin{aligned}
\|y'-x'\| &\ge |y_i'-x_i'| \ge |y_i-x_i| - |x_i-x_i'| - |y_i-y_i'| \\
&\ge e^{-(w_i+1)t_n - w_i nt}\left(\frac{c'r^{d-1}}{\sqrt{d}} - 2\epsilon_n e^{w_i nt - \xi w_i(n+1)t}\right) \\
&\ge e^{-(w_i+1)t_n - w_i nt}\frac{c'r^{d-1}}{2\sqrt{d}} \qquad (5.45)\\
&\ge L_{n-1}^{(1)}\frac{c'r^{d-1}}{4\sqrt{d}\epsilon_{n-1}} e^{(w_1-w_i)t_n-(\xi+w_i)nt} \\
&\ge L_{n-1}^{(1)}\frac{c'r^{d-1}}{4\sqrt{d}\epsilon_{n-1}} e^{(w_1-w_{\ell+1})t_n-(\xi+w_i)nt} \\
&\ge L_{n-1}^{(1)}\frac{c'r^{d-1}}{4\sqrt{d}\epsilon_{n-1}} e^{\frac{1}{\ell}(w_{\ell+1}+\cdots+w_d-\xi\ell)nt}. \qquad (5.46)
\end{aligned}$$

We choose $t \ge 1$ large enough so that the third line (5.45) and last line (5.46) hold for all $n \in \mathbb{N}$.

This concludes the proof of the lemma. □

We choose $t \ge 1$ large enough so that for all $n \in \mathbb{N}$

$$\rho_n := \frac{c'r^{d-1}}{4\sqrt{d}\epsilon_{n-1}} e^{\frac{1}{\ell}(w_{\ell+1}+\cdots+w_d-\xi\ell)nt} \le 1 \qquad (5.47)$$

since $w_{\ell+1}+\cdots+w_d < \xi\ell$. The assumption (4) of Theorem 5.1.1 follows from Lemma 5.3.5.

Proof of Proposition 5.3.4. We prove the proposition applying Corollary 5.1.2. It can be easily checked that for $k > 4\xi dt$, the assumptions of Corollary 5.1.2 hold. Then we have

$$\begin{aligned}
&\frac{\log(C_n L_n^{(\ell+1)}\cdots L_n^{(d)}/L_{n-1}^{(\ell+1)}\cdots L_{n-1}^{(d)})}{-\log(L_n^{(1)}/L_{n-1}^{(1)})} \\
&= \frac{\xi dnt - \xi(w_{\ell+1}+\cdots+w_d)(n+1)t - \xi(d-\ell)nt + o(n)}{\xi w_1(n+1)t + \xi nt + o(n)} \\
&\to \frac{\ell-(w_{\ell+1}+\cdots+w_d)}{1+w_1} = \ell - \frac{1}{1+w_1} \quad \text{as } n\to\infty
\end{aligned}$$

Hence Corollary 5.1.2 implies

$$\dim_H \mathcal{F}(\mathcal{T},\beta) \geq (d-\ell) + \ell - \frac{1}{1+w_1} = d - \frac{1}{1+w_1}.$$

□

Proof of Theorem 1.4.1. If $w_1 = \cdots = w_d$, then the result follows from [CC16, Theorem 1.1]. If there exists $1 \leq \ell \leq d-1$ such that $w_1 = \cdots = w_\ell > w_{\ell+1} \geq \cdots \geq w_d$, then the result follows from Lemma 5.3.3 and Proposition 5.3.4. □

Proof of Theorem 1.4.2. This theorem directly follows from the same argument in the proof of [LSST20, Theorem 1.5]. □

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국문초록

균질공간에서 군 작용의 동역학을 의미하는 균질동역학은 정수론과 많은 연결관계가 있다. 이러한 연결관계는 지난 수십 년간 광범위하고 집중적으로 연구되었으며 다양한 정수론 결과를 제공하였다.

본 학위 논문에서는 균질동역학과 디오판틴 근사의 관계에 대해 살펴보고 다음과 같은 디오판틴 근사에서의 세가지 대상에 대해 알아볼 것이다: 디리끌레 향상 불가능 아핀형식, 나쁜 근사를 가지는 아핀형식, 가중치를 가지는 특이 벡터.

우선 우리는 약한 L^1 측정을 통해 균질동역학에서의 동등분포 결과를 향상시키고 디오판틴 근사에서의 전이원리를 이용하여 디리끌레 향상 불가능 아핀형식에 대한 국소 편재 체계를 구축한다. 이러한 연구를 바탕으로 디리끌레 향상 불가능 아핀형식의 하우스도르프 측도에 대한 $0-\infty$ 현상을 규명한다.

다음으로 엔트로피 강직성의 효과적인 표현을 건설하는데 이를 이용하여 잘 행동하는 시그마 대수를 건설하고 큰 엔트로피를 가지는 불변측도를 건설함으로써 나쁜 근사를 가지는 아핀형식의 하우스도르프 차원의 효과적인 상계를 얻는다. 뿐만 아니라 나쁜 근사를 가지는 아핀형식이 최대차원을 갖기 위한 필요충분조건으로 평균적 특이성을 보인다. 또한 대역적 함수체 위에서의 디오판틴 근사를 생각하고 비슷한 결과를 얻는다.

마지막으로 가중치를 가지는 특이 벡터의 프랙탈 구조와 관련된 수의 기하학의 격자점 셈을 발전시키고 균질동역학의 투영 성질을 이용하여 가중치를 가지는 특이 벡터의 하우스도르프 차원의 하계를 얻는다.

주요어휘: 균질동역학, 디오판틴 근사, 엔트로피 강직, 수의 기하학, 편재 체계, 대역적 함수체

학번: 2016-23082

감사의 글

먼저 제가 무사히 박사과정을 마칠 수 있도록 가장 큰 도움을 주신 지도 교수님 임선희 교수님께 감사드립니다. 교수님의 가르침 덕분에 연구자로서 나아가야 할 방향을 잡을 수 있었고, 연구의 즐거움 또한 배울 수 있었습니다. 앞으로 그 가르침을 마음 속 깊이 간직하고 나아가는 연구자가 되겠습니다. 다시 한번 감사드립니다.

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