이학박사 학위논문

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(정준교환관계와 사교공간의 국소 기하학)

2022년 8월

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이 논문을 이학박사 학위논문으로 제출함

2022년 **4**월

서울대학교 대학원

수리과학부

김 현 문

김 현 문의 이학박사 학위논문을 인준함

2022년 **6**월

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The canonical commutation relations and the local geometry of symplectic spaces

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
to the faculty of the Graduate School of
Seoul National University

by

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August 2022

Abstract

While the Darboux theorem implies there are no local symplectic invariants, many results in quantization suggest there is a necessity to make local choices on symplectic manifolds. We study how representations of the canonical commutation relations arise as a description of local symplectic geometry. As a result, a new family of irreducible representations is obtained. While analytic problems remain, this family unifies known families, extends the parameters describing equivalent representations, and exhibits topologically nontrivial configurations of representations. The unifying framework is provided geometrically, by a partition of the complex Lagrangian Grassmannian induced by complex conjugation.

Key words: Canonical commutation relations, Heisenberg group, irreducible representations, symplectic vector spaces, complex Lagrangian subspaces
Student Number: 2015-30967

Contents

Chapter 1

Introduction

Problem description and historical background

In 1930, Dirac [1] laid out the theoretical framework for quantum mechanics using self-adjoint operators on Hilbert spaces, integrating Heisenberg's matrix mechanics and Schrödinger's wave mechanics. Dirac showed how the noncommutative algebra of operators on a Hilbert space could be interpreted using physical concepts, establishing the rules which are now sometimes referred to as the postulates of quantum mechanics [37]. These rules are stated in the language of abstract Hilbert spaces introduced by Von Neumann [35].

Dirac observed the similarities between the Poisson bracket of smooth functions in Hamiltonian mechanics and the commutator of self-adjoint operators on a separable Hilbert space. From this an analogy between the mathematical structures of classical and quantum theory was built, and from this analogy, concepts in classical mechanics could be associated with concepts in quantum mechanics, providing a means to interpret the mathematics of quantum theory.

Dirac acknowledged that this correspondence could not be applied generally, but the method of classical analogy, also referred to nowadays as *canonical quantization*, is widely used by physicists. For instance, Nobel laureate Steven Weinberg, in a standard text on quantum field theory states the following: *"It seems natural to begin any treatment of the subject today by postulating a Lagrangian and applying to it the rules of canonical quantization.*

This is the approach used in most books on quantum field theory." [33]

However, Dirac's treatment was not entirely rigorous, and a body of mathematical work emerged to rigorously implement Dirac's ideas. Gelfand's rigged Hilbert spaces and Schwartz's distribution theory was developed to treat Dirac's delta function. Inconsistencies in operator ordering were pointed out by Groenwald-van Hove, and different methods were created to bypass these problems. Among them are Kostant-Souriau's geometric quantization, Berezin-Toeplitz quantization, Kontsevich's deformation quantization, Klauder-Daubechies's stochastic path integrals, and Weinstein's approach with groupoids. An account of this history can be found in [32].

Among these approaches, some approaches formulated quantization using the language of symplectic geometry. One reason the author finds this approach interesting is because of the following (albeit subjective) possibility: because symplectic manifolds can 1) describe the laws of classical mechanics in their Hamiltonian formulation 2) can be understood independently of physics as geometric objects, a sufficiently elementary formulation of quantization in the language of symplectic geometry would not only serve as a *description* of quantization, but also a *justification* of it.

A common feature can be observed from the approaches to quantization from the perspective of symplectic geometry. While the Darboux theorem states that there are no local symplectic invariants, quantum structures on symplectic manifolds require making additional local choices. For instance, in Klauder-Daubechies construction [41], an additional compatible complex structure J is necessary, and in Kostant-Souriau geometric quantization, additional data such as the prequantum line bundle with connection and a polarization are required. It is desired that a quantization does not depend on these local choices. *Finding out when and how different methods of quantization are equivalent on symplectic manifolds is an important open problem.*

This work aims to clarify what are the local choices in symplectic geometry that are necessary to describe quantum physics, and how they give rise to quantum structures, and in what sense they do so. Special attention was given so that these choices are independently motivated by mathematics, rather than being imposed by the requirements of physics, following the approach of [7] [40]. We study these questions in the simplified setting of finite dimensional symplectic *vector spaces*, with the aim that the explicitness will make transparent how different approaches (sometimes successfully formulated in more generality) compare.

Our answer is that a choice of transverse pair of complex Lagrangian subspaces, introduced by Hess [31] is a viable candidate for local data that prescribes a quantum description.

The main justification of the claim is the main result of this work. The result is that *transverse pairs of complex Lagrangian subspaces parametrize irreducible representations of the canonical commutation relations.* This result has been published by the author in [38] and we provide more expository comments here. We warn the reader that there are several different notions of *representations* depending on what kind of additional analytic requirements are imposed. The main result holds when we do not impose any additional analytic requirements.

Relation to previous works

Firstly, our construction lifts the positivity restriction for polarizations that appear in Hess's and other works in geometric quantization. We delay imposing the positivity restriction until we have to ask for unitarity. If we ignore the requirement for unitarity, we can obtain *topologically nontrivial configurations of representations of the canonical commutation relations.*

Secondly, our construction behaves differently under symplectic linear transformations from *symplectic spinors* of [34]. To explain this we will describe an unconfirmed speculation that motivates the main result. For a germ of smooth functions $\mathcal{O}_{\mathrm{pt}}$ at a point pt in a symplectic manifold (M, ω), the canonical inclusion of derivations

$$T_{\mathrm{pt}}M = \operatorname{Der}\mathcal{O}_{\mathrm{pt}} \hookrightarrow \operatorname{End}\mathcal{O}_{\mathrm{pt}} \tag{1.1}$$

is a Lie algebra homomorphism. The speculation is that a Lie algebra homomorphism

$$\dot{T}_{\cdot}^{\Gamma_1,\Gamma_2} : \mathfrak{heis}\,(T_{\mathrm{pt}}M, \omega_{\mathrm{pt}}) \to \operatorname{End}\mathcal{O}_{\mathrm{pt}} \tag{1.2}$$

can be a viable replacement of this object, and it is a speculation because the author does not know what are the correct analytic requirements to investigate the direct limits. While there is a canonical inclusion in the classical case, one is forced to make a choice of a transverse pair (Γ_1, Γ_2) from a homogeneous parameter space. A key difference from symplectic spinors is that here symplectic transformations are manifested by *isomorphisms* rather than projective *automorphisms* of the representations (cf. Proposition 4.3.2).

Thirdly, our construction unifies several constructions of families of representations of the Heisenberg group (and Lie algebra). The way different families relate to each other can be understood from the partition of the complex Lagrangian Grassmannians given by complex conjugation.

We can quickly demonstrate the unification in the $\mathbb{R}^2$ case. Here the complex Lagrangian Grassmannian is the complex projective line, and complex conjugation partitions it into the upper hemisphere, equator, and lower hemisphere. Transverse pairs of complex Lagrangian subspaces can be represented by two distinct ordered points on the projective line. The main result result states that we can explicitly construct an irreducible representation of the Heisenberg Lie algebra from *any* such choice of two distinct ordered points. The choices reconstructing the previous representations are summarized in the following table.

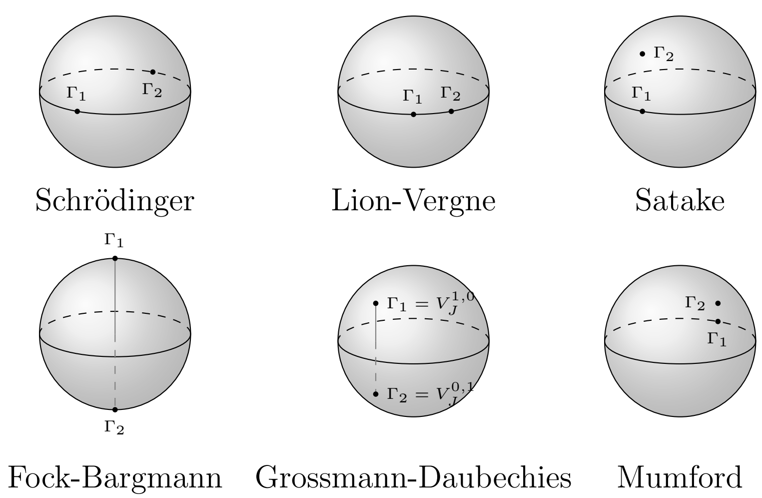

Table 1.1: Pictorial reconstruction dictionary for $V = \mathbb{R}^2$

Additional reasons for transverse pairs

In addition to the main result, transverse pairs of complex Lagrangians are motivated mathematically for the following reasons:

1. They generalize the notion of compatible complex structures
2. They can be naturally associated with Lagrangian subspaces
3. They are canonically obtained from complex Darboux bases
4. They parametrize Poincaré-Birkhoff-Witt isomorphisms prescribing operator ordering rules (cf. Theorem 3.2.12)
5. They are acted on by the real and complex symplectic groups
6. There is an interesting reassembly phenomenon. The Grassmannian of complex Lagrangian subspaces "topologically re-assembles" the Grassmannian of subspaces of *any dimension* in the real symplectic vector space into one homogeneous space. (cf. Theorem 2.7.9)

Summaries of chapters

In Chapter 2, we will review the basic linear algebra of symplectic vector spaces, their complexification, and subspaces. Using these results, we will describe the partition of the complex Lagrangian Grassmannian given by complex conjugation. In fact, we will prove a little bit more, which is the following.

Theorem (2.7.9)**.** *Let (V, ω) be a $2n$-dimensional symplectic vector space, and $\vec{n} := (n_0, n_+, n_-)$ be triples of nonnegative integers such that $n_0 + n_+ + n_- = n$. Then there are partitions of the Grassmannians of k-dimensional subspaces*

$$\mathrm{Gr}(k; V) = \coprod_{\vec{n}: n_0 + 2n_+ = k} \mathrm{Gr}(\vec{n}; V) \quad k = 0, \cdots, 2n \tag{1.3}$$

and a partition of the complex Lagrangian Grassmannian

$$\mathrm{Lag}^{\mathbb{C}}(V) = \coprod_{\vec{n}} \mathrm{Lag}^{\mathbb{C}}(\vec{n}; V) \tag{1.4}$$

such that $\mathrm{Gr}(\vec{n}; V)$ *is homotopic to* $\mathrm{Lag}^{\mathbb{C}}(\vec{n}; V)$.

This theorem describes a "reassembly" phenomenon in which the $2n+1$ Grassmannians $\mathrm{Gr}(0; V), \cdots, \mathrm{Gr}(2n; V)$ split into $\binom{n+2}{2}$-different subsets $\{\mathrm{Gr}(\vec{n}; V)\}_{\vec{n}}$, and each $\mathrm{Gr}(\vec{n}; V)$ can be replaced by a homotopy equivalent $\mathrm{Lag}^{\mathbb{C}}(\vec{n}; V)$ which assemble into *one* homogeneous space $\mathrm{Lag}^{\mathbb{C}}(V)$. So not only does the complex Lagrangian Grassmannian have an interesting partition, there is a sense in which this partition tells us how to think how all subspaces (regardless of the dimension) of V assemble together. For two dimensional symplectic vector spaces, the "assembly" phenomenon was observed independently by M. Hamilton et al, communicated privately to the author.

In Chapter 3, we will review the representation theory of the Heisenberg group and Lie algebra (canonical commutation relations). As suggested by [7], we will view the representations in the context of Equation 1.2. The symmetries are translational symmetries modified by a phase factor, and with this viewpoint, the representations can be understood without referring to their original context in physics by position and momentum operators.

The representation category of the Heisenberg group shares some features with the representation category of finite dimensional representations of finite or compact groups. However, because of the noncompactness of the group and infinite dimensionality of the representations, there are additional conditions (unitarity, topology, convergence, etc) to assume and keep track of, and some subtle differences to keep in mind. We will cite and state relevant results from literature without proof. The results we will review are about exponentiating representations of the Heisenberg Lie algebra into representations of the Heisenberg Lie group, differentiating representations of the Heisenberg Lie group into representations of the Heisenberg Lie algebra, direct integral decompositions (of the Heisenberg group) rather than direct sum decompositions into irreducible representations, and the classification of irreducible unitary representations of the Heisenberg group.

In Chapter 4, we will state our recipe to construct the representations of the Heisenberg group and Lie algebra from pairs of transverse complex Lagrangian subspaces. The key idea comes from the following:

Theorem (4.2.4)**.** *For every transverse pair of complex Lagrangian sub-*

spaces, (Γ_1, Γ_2) *there is a complex valued bilinear form* $(\cdot|\cdot)_{\Gamma_1,\Gamma_2}$ *such that*

$$2(u|v)_{\Gamma_1,\Gamma_2} - 2(v|u)_{\Gamma_1,\Gamma_2} = i\omega(u,v) \quad u,v \in V. \tag{1.5}$$

This generalizes the behavior of antisymmetrization of the hermitian form associated to a compatible complex structure J

$$\frac{1}{2}h_J(u,v) - \frac{1}{2}h_J(v,u) = i\omega(u,v). \tag{1.6}$$

When representations are viewed as Lie group and Lie algebra homomorphisms on vector spaces, we can construct the representations for arbitrary transverse pairs. They are realized as subspaces of the vector space of smooth complex valued functions on V. Real symplectic linear transformations on V act on the space of transverse pairs, and the precomposition operator on functions intertwines the representations whose parameters are in the same orbit of this action.

When representations are viewed analytically, there is a further requirement for them to be realized by unitary or skew-adjoint operators on Hilbert spaces. The previous constructions of Fock-Bargmann, Schrödinger, Satake, Mumford, Lion-Vergne, Grossmann-Daubechies satisfy these requirements, and are all unitarily equivalent if they have the same action of the center. Our construction does not always meet these requirements due to convergence issues. However, the construction produces new parameters that give unitarily equivalent representations.

For representations of the Heisenberg Lie algebra on Hilbert spaces, convergence issues can be circumvented by restricting the domain to a bounded open subset of V. In this case, a polynomial algebra generated by n complex variables is irreducible (as a simple module over the complexified universal enveloping algebra), and is contained as dense subspace of the Hilbert space. In this case, the operators are not always skew-symmetric.

In Chapter 5, we will review the geometry of the parameter spaces, and the reconstruction dictionary that shows how the known families of representations fit together. Then we will proceed to explicitly relate the representations we constructed with the representations of Satake[5], Mumford [12], Lion-Vergne[13], and Grossmann-Daubechies[7][8], as well as the more traditional Schrödinger and Fock-Bargmann representations used by physicists. We will also show there are new parameters that construct equivalent unitary representations.

Chapter 2

Symplectic vector spaces and their complexification

In this chapter we first review the standard notions of symplectic vector spaces, Darboux bases, subspaces of symplectic vector spaces (isotropic, coisotropic, Lagrangian), and compatible complex structures. Notational conventions for block matrix representations of bilinear forms will be set up in the examples. Then we discuss complex Lagrangian subspaces, and end with one of the two main results of this work, asserting the homotopy equivalences between some Grassmannians.

2.1 Symplectic vector spaces

In this section we review the definition of symplectic vector spaces and the fact that finite dimensional symplectic vector spaces are necessarily even dimensional. We also give basic examples, and set up the notation for vectors and matrices we will use for the rest of this work.

Definition 2.1.1 (Symplectic form)**.** *Let V be a real vector space. Then a* symplectic form ω *on V is a real valued bilinear map satisfying the following properties:*

- *(Nondegeneracy) For all nonzero $u \in V$, there exists a $v \in V$ such that $\omega(u, v)$ is nonzero.*

- *(Antisymmetry) For all $u \in V$, $\omega(u,u) = 0$.*

Definition 2.1.2 (Symplectic vector space)**.** *If a vector space V has a symplectic form ω, we will refer to (V,ω) as a* symplectic vector space*. We will only consider finite dimensional symplectic vector spaces. Two symplectic vector spaces (V,ω) and (V',ω') are isomorphic if there exists a linear isomorphism $L : V \xrightarrow{\cong} V'$ such that $\omega'(L\cdot, L\cdot) = \omega(\cdot,\cdot)$.*

Example 2.1.3 ($\mathbb{R}^{2n}$ and the standard symplectic form)**.** *Suppose $u = (q,p)$ and $v = (q',p')$ are elements of $\mathbb{R}^{2n}$ where $q,p,q',p' \in \mathbb{R}^n$. Implicitly identifying n-tuples and $2n$-tuples with column vectors, the standard symplectic form ω_{std} is defined as*

$$\omega_{std}(u,v) := v^t \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} u = (p')^t q - (q')^t p. \tag{2.1}$$

Here 1_n is the $n \times n$ identity matrix and 0 is the $n \times n$ zero matrix. Since

$$\begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} = -\begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}^t \tag{2.2}$$

we can easily check the antisymmetry property:

$$\begin{aligned} \omega_{std}(u,v) + \omega_{std}(v,u) &= v^t \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} u + u^t \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} v && (2.3) \\ &= v^t \left(\begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}^t \right) u && (2.4) \\ &= 0. && (2.5) \end{aligned}$$

Moreover, since

$$\det \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} = 1 \tag{2.6}$$

for every nonzero u let

$$v := \left(\begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}^{-1} \right)^t u. \tag{2.7}$$

Then

$$\omega_{std}(u, v) = u^t u \neq 0. \tag{2.8}$$

and ω_{std} is nondegenerate.

Example 2.1.4 (Skew-symmetric, invertible $2n \times 2n$ matrices)**.** *Suppose M is a $2n \times 2n$ skew-symmetric, invertible real matrix. Then*

$$\omega_M(u, v) := v^t M u \quad u, v \in \mathbb{R}^{2n} \tag{2.9}$$

is a symplectic form on $\mathbb{R}^{2n}$. The arguments from Example 2.1.3 to check that the standard symplectic form is a symplectic form apply directly to show ω_M is a symplectic form.

Remark 2.1.5 (Convention for Gram matrix)**.** *We will follow the convention that the first argument of a bilinear form $B(u, v)$, when written out in matrix form, gets multiplied as a column vector. This implies that the matrix we use to compute a bilinear form $(B)_{\{v_1,\cdots,v_{2n}\}}$ in a particular basis $\{v_1, \cdots, v_{2n}\}$ given by*

$$B\left(\sum_j a_j v_j, \sum_k b_k v_k\right) = \sum_{j,k} B(v_j, v_k) a_j b_k = b^t (B)_{\{v_1,\cdots,v_{2n}\}} a \tag{2.10}$$

is such that

$$((B)_{\{v_1,\cdots,v_{2n}\}})_{jk} = B(v_k, v_j). \tag{2.11}$$

We will refer to this matrix as the Gram matrix*.*

Remark 2.1.6. *Every finite dimensional symplectic vector space is necessarily even dimensional. Suppose (V, ω) is an odd dimensional symplectic vector space. Take any basis $\{v_1, \cdots, v_n\}$. Then consider the matrix M with $M_{jk} := (\omega(v_k, v_j))$. If $u = \sum a_j v_j$ and $v = \sum b_j v_j$ we have*

$$\omega(u, v) = \sum a_j b_k M_{kj} = b^t M a. \tag{2.12}$$

M is skew-symmetric, and therefore

$$\det M = \det M^t = \det(-M) = (-1)^n \det M = -\det M \tag{2.13}$$

So $\det M = 0$ *and* M *is not invertible. Take* $u = \sum a_j v_j$ *in the kernel of* M*. Then for all* $v = \sum b_j v_j$*, we have* $\omega(u, v) = b^t M a = 0$*. So* ω *is not nondegenerate (contradiction).*

2.2 Darboux bases

In this section we review the definition of Darboux bases and some basic examples. The main one that we will use extensively is the Darboux basis given by the column vectors of a symplectic matrix. We end the section by reviewing the existence theorem of Darboux bases in finite dimensional symplectic vector spaces.

Definition 2.2.1 (Darboux basis)**.** *A basis* $\{\mathbf{e}_1, \cdots, \mathbf{e}_n, \mathbf{f}_1, \cdots \mathbf{f}_n\}$ *of a* $2n$ *dimensional symplectic vector space* (V, ω) *is a* Darboux basis *if it satisfies*

$$\omega(\mathbf{e}_j, \mathbf{e}_k) = \omega(\mathbf{f}_j, \mathbf{f}_k) = 0 \quad j, k \in \{1, \cdots, n\}. \tag{2.14}$$

and

$$\omega(\mathbf{e}_j, \mathbf{f}_k) = \delta_{j,k} \tag{2.15}$$

where $\delta_{j,k}$ *is the Kronecker delta. When it is clear from context, we will sometimes denote Darboux bases as simply* $\{\mathbf{e}, \mathbf{f}\}$*.*

Remark 2.2.2. *Every isomorphism of symplectic vector spaces, sends a Darboux basis to a Darboux basis. Conversely, any two Darboux bases of the same cardinality determine an isomorphism of symplectic vector spaces.*

Example 2.2.3 (Standard Darboux basis of $(\mathbb{R}^{2n}, \omega_{std})$)**.** *Let* e_j *be the vector in* $\mathbb{R}^n$ *such that its* j*th component is* 1 *and all other components are zero. Let* $\mathbf{e}_j^{std} := (e_j, 0)$ *and* $\mathbf{f}_j^{std} := (0, e_j)$*. Then* $\{\mathbf{e}^{std}, \mathbf{f}^{std}\}$ *is a Darboux basis of*

$(\mathbb{R}^{2n}, \omega_{std})$. *Indeed,*

$$\omega_{std}(\mathbf{e}_j^{std}, \mathbf{e}_k^{std}) = \begin{pmatrix} e_k^t & 0 \end{pmatrix} \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \begin{pmatrix} e_j \\ 0 \end{pmatrix} = 0 \tag{2.16}$$

$$\omega_{std}(\mathbf{f}_j^{std}, \mathbf{f}_k^{std}) = \begin{pmatrix} 0 & e_k^t \end{pmatrix} \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \begin{pmatrix} 0 \\ e_j \end{pmatrix} = 0 \tag{2.17}$$

$$\omega_{std}(\mathbf{e}_j^{std}, \mathbf{f}_k^{std}) = \begin{pmatrix} 0 & e_k^t \end{pmatrix} \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \begin{pmatrix} e_j \\ 0 \end{pmatrix} = e_k^t e_j = \delta_{j,k}. \tag{2.18}$$

Example 2.2.4 (The column vectors of a symplectic matrix)**.** *A* $2n \times 2n$ *real matrix* S *is* symplectic *if it preserves the standard symplectic form, i.e.*

$$\omega_{std}(S\cdot, S\cdot) = \omega_{std}(\cdot, \cdot). \tag{2.19}$$

Symplectic matrices form a group, which we denote as $\mathrm{Sp}(2n; \mathbb{R})$*. Suppose*

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad A, B, C, D \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \tag{2.20}$$

where $\mathrm{Mat}_{n\times n}(\mathbb{F})$ *denotes the* $n \times n$ *matrices with coefficients in some field* $\mathbb{F}$*. Then the condition that* S *is symplectic is the following*

$$\omega_{std}(Su, Sv) = v^t S^t \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} Su = v^t \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} u \tag{2.21}$$

for all $u, v \in \mathbb{R}^{2n}$*. This is equivalent to*

$$S^t \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} S = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}. \tag{2.22}$$

The left hand side can be expanded as

$$\begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix} \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} C^t A - A^t C & C^t B - A^t D \\ D^t A - B^t C & D^t B - B^t D \end{pmatrix} \tag{2.23}$$

So S is symplectic if and only if its block components satisfy

$$A^tC = C^tA \quad B^tD = D^tB \quad A^tD - C^tB = 1_n. \tag{2.24}$$

Let $\mathbf{e}_j$ be the jth column vector of S and $\mathbf{f}_j$ be the $n+j$th column vector of S for $j = 1, \cdots, n$. Then

$$\begin{aligned}
\omega_{std}(\mathbf{e}_j, \mathbf{e}_k) &= \left(\begin{pmatrix} C^t & A^t \end{pmatrix} \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix}\right)_{kj} = (A^tC - C^tA)_{kj} = 0 \\
\omega_{std}(\mathbf{f}_j, \mathbf{f}_k) &= \left(\begin{pmatrix} D^t & B^t \end{pmatrix} \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \begin{pmatrix} B \\ D \end{pmatrix}\right)_{kj} = (B^tD - D^tB)_{kj} = 0 \\
\omega_{std}(\mathbf{e}_j, \mathbf{f}_k) &= \left(\begin{pmatrix} D^t & B^t \end{pmatrix} \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix}\right)_{kj} = (D^tA - B^tC)_{kj} = \delta_{kj}.
\end{aligned}$$

So the column vectors of any symplectic matrix is a Darboux basis of $\mathbb{R}^{2n}$. Conversely, if the components of any Darboux basis $\{\mathbf{e}_j, \mathbf{f}_j\}_{j=1}^n$ of $(\mathbb{R}^{2n}, \omega_{std})$ are identified as

$$\mathbf{e}_j = \begin{pmatrix} A_{kj} \\ C_{kj} \end{pmatrix} \quad \mathbf{f}_j = \begin{pmatrix} B_{kj} \\ D_{kj} \end{pmatrix} \quad A_{kj}, B_{kj}, C_{kj}, D_{kj} \in \mathbb{R}^n. \tag{2.25}$$

Then the matrix defined by

$$S := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.26}$$

is symplectic.

Definition 2.2.5 (Symplectic linear transformation)**.** *A* symplectic linear transformation *of a symplectic vector space (V, ω) is a linear map $S \in \mathrm{GL}(V)$ such that*

$$\omega(Su, Sv) = \omega(u, v) \quad u, v \in V. \tag{2.27}$$

The set of symplectic linear transformations forms the symplectic group *which we will denote by $\mathrm{Sp}(V, \omega)$.*

Remark 2.2.6 (Symplectic linear transformations and symplectic matrices)**.** *A symplectic linear transformation $S \in \mathrm{Sp}(V, \omega)$ written in matrix form using* *a Darboux basis* $\{\mathbf{e}, \mathbf{f}\}$ *of* V *is a symplectic matrix–i.e.*

$$(S)_{\{\mathbf{e},\mathbf{f}\}} \in \mathrm{Sp}(2n; \mathbb{R}). \tag{2.28}$$

Proposition 2.2.7 (Existence of Darboux basis)**.** *Every symplectic vector space* (V, ω) *has a Darboux basis.*

Proof. The proof is by induction on the dimension of V. If V is 2 dimensional, take any basis $\{u, v\}$ of V. If $\omega(u, v) = 0$, ω is not nondegenerate. So $\omega(u, v) \neq 0$ and $\{u, \omega(u, v)^{-1}v\}$ is a Darboux basis of V. Let $\{v_1, \cdots, v_{2n}\}$ be any basis of V. By nondegeneracy of ω, there exists v_j, v_k such that $\omega(v_j, v_k) \neq 0$. Without loss of generality, let them be v_1 and v_2, and such that $\omega(v_1, v_2) = 1$. Let

$$w_k := v_k - \omega(v_k, v_2)v_1 + \omega(v_k, v_1)v_2 \quad k = 3, \cdots, 2n \tag{2.29}$$

By construction

$$\omega(w_k, v_1) = \omega(w_k, v_2) = 0. \tag{2.30}$$

Then let $W := \mathrm{Span}_{\mathbb{R}}\{w_3, \cdots, w_{2n}\}$. Suppose $\omega|_W$ is not nondegenerate. Then there exists a nonzero vector $w = \sum a_k w_k$ in W such that $\omega(w, v) = 0$ for all $v \in W$. Then

$$\omega(w, v + b_1 v_1 + b_2 v_2) = \omega(w, v) + \sum a_k b_1 \omega(w_k, v_1) + \sum a_k b_2 \omega(w_k, v_2) = 0.$$

Since any vector of V can be expressed as $v + b_1 v_1 + b_2 v_2$ this implies that ω is not nondegenerate (contradiction). Therefore, $\omega|_W$ must be nondegenerate. It is also antisymmetric, so by inductive hypothesis, there exists a Darboux basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_{n-1}, \mathbf{f}_1, \cdots, \mathbf{f}_{n-1}\}$ of W. Then

$$\{\mathbf{e}_1, \cdots, \mathbf{e}_{n-1}, v_1, \mathbf{f}_1, \cdots, \mathbf{f}_{n-1}, v_2\}$$

is a Darboux basis of V. □

Corollary 2.2.8. *Every* $2n$*-dimensional symplectic vector space* (V, ω) *is isomorphic to* $(\mathbb{R}^{2n}, \omega_{std})$.

Proof. Take $\{\mathbf{e}, \mathbf{f}\}$ a Darboux basis of V. Then let $L : V \to \mathbb{R}^{2n}$ be defined as

$$L\mathbf{e}_j := \mathbf{e}_j^{std} \quad L\mathbf{f}_j := \mathbf{f}_j^{std} \quad j = 1, \cdots, n. \tag{2.31}$$

Then by the definition of Darboux basis we have

$$\omega_{std}(L\mathbf{e}_j, L\mathbf{e}_k) = \quad 0 \quad = \omega(\mathbf{e}_j, \mathbf{e}_k) \tag{2.32}$$

$$\omega_{std}(L\mathbf{f}_j, L\mathbf{f}_k) = \quad 0 \quad = \omega(\mathbf{f}_j, \mathbf{f}_k) \tag{2.33}$$

$$\omega_{std}(L\mathbf{e}_j, L\mathbf{f}_k) = \quad \delta_{jk} \quad = \omega(\mathbf{e}_j, \mathbf{f}_k). \tag{2.34}$$

□

2.3 Subspaces of symplectic vector spaces

Unlike the orthogonal complement of an inner product space, a subspace of a symplectic vector space is not necessarily transverse to its symplectic complement. In this section we will review the standard notions of subspaces of symplectic vector spaces (isotropic, coisotropic, Lagrangian), according to how they interact with the symplectic form and end by reviewing a general basis extension theorem.

Definition 2.3.1 (Symplectic subspace)**.** *A subspace $W \subset V$ is a* symplectic subspace *if $(W, \omega|_W)$ is a symplectic vector space.*

Definition 2.3.2 (Symplectic complement)**.** *Let $W \subset V$ be a subspace of a symplectic vector space (V, ω). Then the* symplectic complement of W (in V) *is defined as the subspace*

$$W^\omega := \{v \in V : \omega(v, w) = 0 \text{ for all } w \in W\}. \tag{2.35}$$

Proposition 2.3.3. *Let $W \subset V$ be a subspace of a symplectic vector space (V, ω). Then*

$$\dim_{\mathbb{R}} W + \dim_{\mathbb{R}} W^\omega = \dim_{\mathbb{R}} V. \tag{2.36}$$

Proof. Consider the map $v \mapsto \omega(v, \cdot)|_W$ from V to W^*. The kernel of this map is W^ω and by nondegeneracy of ω, it is surjective. The result follows by the rank-nullity theorem. □

Corollary 2.3.4.

$$(W^\omega)^\omega = W. \tag{2.37}$$

Proof. If $w \in W$, and $v \in W^\omega$, then $\omega(w, v) = 0$ because $v \in W^\omega$. This holds for every $v \in W^\omega$, so $w \in (W^\omega)^\omega$. So $W \subset (W^\omega)^\omega$. By the dimension formula,

$$\dim_{\mathbb{R}}(W^\omega)^\omega = \dim_{\mathbb{R}} V - \dim_{\mathbb{R}} W^\omega = \dim_{\mathbb{R}} W. \tag{2.38}$$

So $W = (W^\omega)^\omega$. □

Example 2.3.5. *$W \cap W^\omega$ may not be 0 and $W + W^\omega$ may not be V. Let (V, ω) be a 6 dimensional symplectic vector space with Darboux basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$. The symplectic complement of*

$$W := \mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_1, \mathbf{f}_1, \mathbf{e}_2\} \tag{2.39}$$

is

$$W^\omega = \mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_2, \mathbf{e}_3, \mathbf{f}_3\} \tag{2.40}$$

so $W \cap W^\omega = \mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_2\}$, and $\mathbf{f}_2 \notin W + W^\omega$. This example is representative.

Proposition 2.3.6. *The following are equivalent:*

(a) W is a symplectic subspace of V.

(b) W^ω is a symplectic subspace of V.

(c) $W \cap W^\omega = 0$.

Proof. (a) $\iff$ (b): Take a Darboux basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_k, \mathbf{f}_1, \cdots, \mathbf{f}_k\}$ of W. If u is a nonzero vector in W^ω, there exists a $v \in V$ such that $\omega(u, v) \neq 0$. Let

$$v' := v - \sum_{j=1}^{k} \omega(v, \mathbf{f}_j)\mathbf{e}_j + \sum_{j=1}^{k} \omega(v, \mathbf{e}_j)\mathbf{f}_j. \tag{2.41}$$

Since $u \in W^\omega$,

$$\omega(u, v') = \omega(u, v) \neq 0. \tag{2.42}$$

Moreover,

$$\omega(v', \mathbf{e}_j) = \omega(v, \mathbf{e}_j) + \omega(v, \mathbf{e}_j)\omega(\mathbf{f}_j, \mathbf{e}_j) = 0 \tag{2.43}$$
$$\omega(v', \mathbf{f}_j) = \omega(v, \mathbf{f}_j) - \omega(v, \mathbf{f}_j)\omega(\mathbf{e}_j, \mathbf{f}_j) = 0. \tag{2.44}$$

So $v' \in W^\omega$. So $\omega|_{W^\omega}$ is nondegenerate.
(a) $\Longrightarrow$ (c) : Suppose $v \in W$. Then take a Darboux basis

$$\{\mathbf{e}_1, \cdots, \mathbf{e}_k, \mathbf{f}_1, \cdots, \mathbf{f}_k\}$$

of W. Then $v = q_1\mathbf{e}_1 + \cdots q_k\mathbf{e}_k + p_1\mathbf{f}_1 + \cdots p_k\mathbf{f}_k$. Then if $v \in W \cap W^\omega$ $\omega(v, \mathbf{e}_j) = \omega(v, \mathbf{f}_j) = 0$ for all $j = 1, \cdots, k$. So $q_j = p_j = 0$ for all $j = 1, \cdots, k$.
(c) $\Longrightarrow$ (a): If w is a nonzero vector in W, by nondegeneracy of ω, there exists a $v \in V$ such that $\omega(w, v) \neq 0$. By the dimension formula, $V \cong W \oplus W^\omega$ so $v = v_W + v_{W^\omega}$ where $v_W \in W$ and $v_{W^\omega} \in W^\omega$. Then $\omega(w, v) = \omega(w, v_W) \neq 0$. So $\omega|_W$ is nondegenerate.

□

Definition 2.3.7 (Isotropic, coisotropic, and Lagrangian subspaces)**.** *A subspace $W \subset V$ of a symplectic vector space is*

- Isotropic *if $W \cap W^\omega = W$, or equivalently, if $\omega|_W = 0$, or $W \subset W^\omega$.*
- Coisotropic *if $W \cap W^\omega = W^\omega$, or equivalently, $W^\omega \subset W$.*
- Lagrangian *if $W \cap W^\omega = W = W^\omega$, or equivalently both isotropic and coisotropic.*

Remark 2.3.8 (Duality between isotropic and coisotropic subspaces)**.** *From the identity $(W^\omega)^\omega = W$ we can see that the symplectic complement exchanges isotropic and coiostropic subspaces. If $W \subset W^\omega$ (W is isotropic), then $(W^\omega)^\omega \subset W^\omega$ (W^ω is coisotropic). Similarly, if $W^\omega \subset W$ (W is coisotropic), then $W^\omega \subset (W^\omega)^\omega$ (W^ω is isotropic).*

Remark 2.3.9 (Dimensions of isotropic, coisotropic, and Lagrangian subspaces)**.** *Suppose (V, ω) is a $2n$-dimensional vector space. Then every isotropic subspace has dimension at most n, every coisotropic subspace has dimension* *at least n, and every Lagrangian subspace has dimension n. This can be seen as follows. Take a basis $\{v_1, \cdots, v_{2n}\}$ of V such that $\{v_1, \cdots, v_k\}$ is a basis of an isotropic subspace W of V. Then if $k > n$ the Gram matrix*

$$(\omega)_{\{v_1,\cdots,v_{2n}\}} = \begin{pmatrix} 0_{k\times k} & -X^t_{k\times(n-k)} \\ X_{(n-k)\times k} & Y_{(n-k)\times(n-k)} \end{pmatrix} \tag{2.45}$$

has linearly dependent columns, and fails to be invertible. This contradicts the nondegeneracy of ω. So an isotropic subspace has dimension at most n, and by the dimension formula, a coisotropic subspace has dimension at least n. A Lagrangian subspace is both isotropic and coisotropic, so has dimension n.

Lemma 2.3.10 (Lagrangian Basis extension)**.** *Let $\mathbf{L}$ be a Lagrangian subspace of a symplectic vector space (V, ω). Then there exists a Darboux basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n, \mathbf{f}_1, \cdots, \mathbf{f}_n\}$ of V such that $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ is a basis of $\mathbf{L}$.*

Proof. Take a basis $\{v_1, \cdots, v_n\}$ of $\mathbf{L}$, and a basis extension $\{v_1, \cdots, v_{2n}\}$ to V. The Gram matrix

$$(\omega)_{\{v_1,\cdots,v_{2n}\}} = \begin{pmatrix} 0 & -X^t_{n\times n} \\ X_{n\times n} & Y_{n\times n} \end{pmatrix} \tag{2.46}$$

is skew-symmetric and nondegenerate, so X is invertible and Y is skew-symmetric. The assertion follows from the matrix identity

$$M^t \begin{pmatrix} 0 & -X^t \\ X & Y \end{pmatrix} M = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \tag{2.47}$$

where

$$M := \begin{pmatrix} 1_n & -\frac{1}{2}X^{-1}Y(X^{-1})^t \\ 0 & (X^{-1})^t \end{pmatrix}. \tag{2.48}$$

□

Proposition 2.3.11. *Let $W \subset V$ be any subspace of a symplectic vector space (V, ω). Then $W \cap W^\omega$ is isotropic, and its symplectic complement is equal to $W + W^\omega$. In particular, $W + W^\omega$ is coisotropic.*

Proof. If $u, v \in W \cap W^\omega$, then $\omega(u,v) = 0$ because $u \in W$ and $v \in W^\omega$. So $W \cap W^\omega$ is isotropic. Suppose $u \in W$, $v \in W^\omega$, and $w \in W \cap W^\omega$. Then

$$\omega(u+v, w) = \omega(u,w) + \omega(v,w) = 0 + 0 = 0. \tag{2.49}$$

So $W + W^\omega \subset (W \cap W^\omega)^\omega$.

$$\begin{aligned} \dim_{\mathbb{R}}(W + W^\omega) &= \dim_{\mathbb{R}} W + \dim_{\mathbb{R}} W^\omega - \dim_{\mathbb{R}}(W \cap W^\omega) && (2.50)\\ &= 2n - \dim_{\mathbb{R}}(W \cap W^\omega) && (2.51)\\ &= \dim_{\mathbb{R}}(W \cap W^\omega)^\omega. && (2.52) \end{aligned}$$

So $W + W^\omega = (W \cap W^\omega)^\omega$. ☐

Example 2.3.12. *Let (V, ω) be a 6-dimensional symplectic vector space with Darboux basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$. Then*

- $\mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_1, \mathbf{e}_2\}$ *is isotropic.*
- $\mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{f}_3\}$ *is coisotropic.*
- $\mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *is Lagrangian.*
- $\mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_1, \mathbf{f}_2, \mathbf{e}_3\}$ *is Lagrangian.*
- $\mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_1, \mathbf{f}_1\}$ *is symplectic.*
- $\mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_1, \mathbf{f}_1, \mathbf{e}_2\}$ *is neither Lagrangian, isotropic, coisotropic, nor symplectic.*

We will see that these are representative examples.

Lemma 2.3.13. *If W is a subspace of (V, ω), define an antisymmetric bilinear form on the quotient space $W/(W \cap W^\omega)$ by*

$$\omega'([u],[v]) := \omega(u,v) \quad u, v \in W. \tag{2.53}$$

Then $(W/(W \cap W^\omega), \omega')$ is a symplectic vector space.

Proof. Suppose u is a vector in W such that $\omega'([u],[v]) = 0$ for all $v \in W$. Then $u \in W^\omega$. So $[u] = 0$. So ω' is nondegenerate. ☐

Theorem 2.3.14 (Linear Relative Darboux theorem)**.** *Let W be a subspace of a symplectic vector space (V, ω). Then there is a Darboux basis of V*

$$\{\mathbf{e}_{W\cap W^\omega}, \mathbf{e}_W, \mathbf{e}_{W^\omega}, \mathbf{f}_{W\cap W^\omega}, \mathbf{f}_W, \mathbf{f}_{W^\omega}\} \tag{2.54}$$

such that

- $\{\mathbf{e}_{W\cap W^\omega}\}$ *is a basis of* $W \cap W^\omega$*, and*
- $\{\mathbf{e}_{W\cap W^\omega}, \mathbf{e}_W, \mathbf{f}_W\}$ *is a basis of* W*, and*
- $\{\mathbf{e}_{W\cap W^\omega}, \mathbf{e}_{W^\omega}, \mathbf{f}_{W^\omega}\}$ *is a basis of* W^ω*.*

If W is coisotropic, then $\{\mathbf{e}_{W^\omega}, \mathbf{f}_{W^\omega}\}$ is empty. If W is isotropic, $\{\mathbf{e}_W, \mathbf{f}_W\}$ is empty. If W is symplectic, then $\{\mathbf{e}_{W\cap W^\omega}, \mathbf{f}_{W\cap W^\omega}\}$ is empty.

Proof. Since $W/(W \cap W^\omega)$ is symplectic, there exists a Darboux basis

$$\{\mathbf{e}_{W/(W\cap W^\omega)}, \mathbf{f}_{W/(W\cap W^\omega)}\}.$$

Similarly, $W^\omega/(W \cap W^\omega)$ is symplectic, so there exists a Darboux basis $\{\mathbf{e}_{W^\omega/(W\cap W^\omega)}, \mathbf{f}_{W^\omega/(W\cap W^\omega)}\}$. Let $\{\mathbf{e}_W, \mathbf{f}_W\}$ and $\{\mathbf{e}_{W^\omega}, \mathbf{f}_{W^\omega}\}$ be vectors in V that are chosen from the cosets defining Darboux bases of $W/(W \cap W^\omega)$ and $W^\omega/(W \cap W^\omega)$. $\mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_W, \mathbf{f}_W, \mathbf{e}_{W^\omega}, \mathbf{f}_{W^\omega}\}$ is a symplectic subspace, and its symplectic complement is a symplectic subspace with $W \cap W^\omega$ as a Lagrangian subspace. By the Lagrangian basis extension, there exists a Darboux basis $\{\mathbf{e}_{W\cap W^\omega}, \mathbf{f}_{W\cap W^\omega}\}$ of $\mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_W, \mathbf{f}_W, \mathbf{e}_{W^\omega}, \mathbf{f}_{W^\omega}\}^\omega$ such that $\{\mathbf{e}_{W\cap W^\omega}\}$ is a basis of $W \cap W^\omega$. Then $\{\mathbf{e}_{W\cap W^\omega}, \mathbf{e}_W, \mathbf{e}_{W^\omega}, \mathbf{f}_{W\cap W^\omega}, \mathbf{f}_W, \mathbf{f}_{W^\omega}\}$ is the desired basis. □

Definition 2.3.15 (Type of a subspace)**.** *Let $\vec{n} := (n_0, n_+, n_-)$ be a triple of nonnegative integers that sum to $n = \frac{1}{2}\dim_{\mathbb{R}} V$. We will say a subspace $W \subset V$ is* of type $\vec{n}$ *if*

$$\dim_{\mathbb{R}} W = n_0 + 2n_+ \quad \dim_{\mathbb{R}}(W \cap W^\omega) = n_0. \tag{2.55}$$

2.4 Compatible complex structures

In this section, we will review the definitions of complex structures, their compatibility with the symplectic form, and some basic examples.

Definition 2.4.1 (Complex structure)**.** *A* complex structure*, or* linear complex structure *on a real vector space V is a linear automorphism $J : V \to V$ such that $J^2 = -\mathrm{Id}_V$. Although they can be identified geometrically, we will reserve the term "complex vector space" for vector spaces over the field of complex numbers, and refer to (V, J) as a vector space with complex structure J.*

Remark 2.4.2. *If a complex structure exists on V, then V is necessarily even dimensional, because* $(\det J)^2 = \det(-\mathrm{Id}_V) = (-1)^{\dim_{\mathbb{R}} V}$*, and* $\det J$ *must be real.*

The minimal polynomial of a complex structure J is x^2+1, which factorizes over the complex numbers as $(x+i)(x-i)$. The characteristic polynomial is of the form $(x+i)^k(x-i)^\ell$ where $k+\ell = 2n$. Since it should have real coefficients, $k = \ell$. So $J^{\mathbb{C}}$, the complex linear extension of J to $V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$, is diagonalizable, with $\pm i$-eigenspaces each with complex dimension n. Let $V_J^{1,0}$ be the $+i$ eigenspace and $V_J^{0,1}$ be the $-i$ eigenspace. The projections to $V_J^{1,0}$ and $V_J^{0,1}$ can be written explicitly

$$\left(\frac{1}{2}(\mathrm{Id}_{V^{\mathbb{C}}} - iJ^{\mathbb{C}})\right)^2 = \frac{1}{2}(\mathrm{Id}_{V^{\mathbb{C}}} - iJ^{\mathbb{C}}) : V^{\mathbb{C}} \to V_J^{1,0} \tag{2.56}$$

$$\left(\frac{1}{2}(\mathrm{Id}_{V^{\mathbb{C}}} + iJ^{\mathbb{C}})\right)^2 = \frac{1}{2}(\mathrm{Id}_{V^{\mathbb{C}}} + iJ^{\mathbb{C}}) : V^{\mathbb{C}} \to V_J^{0,1} \tag{2.57}$$

and it can be checked that

$$J^{\mathbb{C}}\left(\frac{1}{2}(\mathrm{Id}_{V^{\mathbb{C}}} - iJ^{\mathbb{C}})\right) = i\left(\frac{1}{2}(\mathrm{Id}_{V^{\mathbb{C}}} - iJ^{\mathbb{C}})\right) \tag{2.58}$$

$$J^{\mathbb{C}}\left(\frac{1}{2}(\mathrm{Id}_{V^{\mathbb{C}}} + iJ^{\mathbb{C}})\right) = -i\left(\frac{1}{2}(\mathrm{Id}_{V^{\mathbb{C}}} + iJ^{\mathbb{C}})\right). \tag{2.59}$$

It can also be seen that these projections, when restricted to V, give isomorphisms of vector spaces with complex structure.

$$\frac{1}{2}(\mathrm{Id}_{V^{\mathbb{C}}} - iJ^{\mathbb{C}})|_V : (V, J) \cong ((V_J^{1,0})^{\mathbb{R}}, i\cdot) \tag{2.60}$$

$$\frac{1}{2}(\mathrm{Id}_{V^{\mathbb{C}}} + iJ^{\mathbb{C}})|_V : (V, J) \cong ((V_J^{0,1})^{\mathbb{R}}, -i\cdot). \tag{2.61}$$

Here $W^{\mathbb{R}}$ denotes the underlying real vector space of a complex vector space W.

Example 2.4.3 (Standard complex structure on $\mathbb{R}^{2n}$)**.** *Let*

$$J_0 := \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}. \tag{2.62}$$

We can see that there are two ways to identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$. One of the ways is that

$$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto q + ip \tag{2.63}$$

so that

$$J_0 \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} -p \\ q \end{pmatrix} \mapsto -p + iq = i(q + ip). \tag{2.64}$$

Another way is that

$$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto \frac{1}{2}(1_{2n} - iJ_0^{\mathbb{C}}) \begin{pmatrix} q \\ p \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1_n & i \cdot 1_n \\ -i \cdot 1_n & 1_n \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1_n \\ -i \cdot 1_n \end{pmatrix} (q + ip),$$

retaining the information of how $(\mathbb{R}^{2n})_{J_0}^{1,0}$ sits inside of $\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^{2n}$. The coordinates differ by a factor of $1/2$.

Example 2.4.4 (Complex structures in $\mathbb{R}^{2n}$)**.** *Suppose we have a complex structure J on $\mathbb{R}^{2n}$. Then take a complex basis $\{v_1, \cdots, v_n\}$ of $(\mathbb{R}^{2n})_J^{1,0}$. We can view the vectors as elements of $\mathbb{C}^{2n} \cong \mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$, and take complex conjugation componentwise. Since $\{v_1, \cdots, v_n, \overline{v}_1, \cdots, \overline{v}_n\}$ is a complex basis of $(\mathbb{R}^{2n})^{\mathbb{C}}$,*

$$\{\mathrm{Re}\, v_1, \cdots \mathrm{Re}\, v_n, \mathrm{Im}\, v_1, \cdots, \mathrm{Im}\, v_n\} \tag{2.65}$$

is a complex basis of $(\mathbb{R}^{2n})^{\mathbb{C}}$, and a real basis of $\mathbb{R}^{2n}$. In this basis, J takes *the form,*

$$(J)_{\{\mathrm{Re}\, v, \mathrm{Im}\, v\}} = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}. \tag{2.66}$$

So there exists an invertible $2n \times 2n$ matrix X such that

$$J = XJ_0X^{-1}. \tag{2.67}$$

The condition $J_0 = XJ_0X^{-1}$ is equivalent to

$$J_0X = XJ_0, \tag{2.68}$$

which is equivalent to

$$X = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \tag{2.69}$$

in block matrix form. In this case the invertibility of X is equivalent to the invertibility of

$$\frac{1}{2}\begin{pmatrix} 1_n & i \cdot 1_n \\ i \cdot 1_n & 1_n \end{pmatrix} \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} 1_n & -i \cdot 1_n \\ -i \cdot 1_n & 1_n \end{pmatrix} = \begin{pmatrix} A + iB & 0 \\ 0 & A - iB \end{pmatrix}. \tag{2.70}$$

So the set of complex structures on $\mathbb{R}^{2n}$ can be identified with the homogeneous space $\mathrm{GL}(2n; \mathbb{R})/\mathrm{GL}(n; \mathbb{C})$.

Definition 2.4.5 (Compatible complex structure)**.** *A complex structure on a symplectic vector space (V, ω) is* compatible *or ω-compatible if*

- $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$
- $\omega(\cdot, J\cdot)$ *is a positive definite bilinear form on V.*

Remark 2.4.6. $\omega(\cdot, J\cdot)$ *is symmetric. By compatiblity we have*

$$\omega(u, Jv) = \omega(Ju, J^2v) = -\omega(Ju, v) = \omega(v, Ju). \tag{2.71}$$

Example 2.4.7 (Compatible complex structures from Darboux bases)**.** *Let $\{\mathbf{e}_1, \cdots, \mathbf{e}_n, \mathbf{f}_1, \cdots, \mathbf{f}_n\}$ be a Darboux basis of a symplectic vector space (V, ω).*

Then define

$$J\mathbf{e}_j := \mathbf{f}_j \tag{2.72}$$
$$J\mathbf{f}_j := -\mathbf{e}_j \tag{2.73}$$

for $j = 1, \ldots, n$. Then J is an ω-compatible complex structure.

Proposition 2.4.8 (Darboux bases from compatible complex structures)**.** *Let J be an ω-compatible complex structure. Then there exists a Darboux basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n, \mathbf{f}_1, \cdots, \mathbf{f}_n\}$ such that*

$$J\mathbf{e}_j = \mathbf{f}_j \quad J\mathbf{f}_j = -\mathbf{e}_j \quad j = 1, \cdots, n. \tag{2.74}$$

Proof. We can show this by induction on the dimension of V. If V is 2 dimensional, $\{v, Jv\}$ for any (suitably normalized) nonzero $v \in V$ works. Take a $v \in V$ such that $\omega(v, Jv) = 1$. Then let W be the symplectic complement of $\mathrm{Span}_{\mathbb{R}}\{v, Jv\}$. If $w \in W$, by definition

$$\omega(w, Jv) = \omega(w, v) = 0. \tag{2.75}$$

By compatibility,

$$\omega(Jw, v) = \omega(Jw, Jv) = 0. \tag{2.76}$$

So $Jw \in W$. $\{v, Jv\}$ is a symplectic subspace, so W is also a symplectic subspace. Then it can be checked then that $J|_W$ is an $\omega|_W$-compatible complex structure on W. By inductive hypothesis, there exists a Darboux basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_{n-1}, \mathbf{f}_1, \cdots, \mathbf{f}_{n-1}\}$ such that

$$J\mathbf{e}_j = \mathbf{f}_j \quad J\mathbf{f}_j = -\mathbf{e}_j \quad j = 1, \cdots, n-1. \tag{2.77}$$

Then $\{\mathbf{e}_1, \cdots, \mathbf{e}_{n-1}, v, \mathbf{f}_1, \cdots, \mathbf{f}_{n-1}, Jv\}$ is the desired basis. $\square$

Example 2.4.9 (Compatible complex structures on $(\mathbb{R}^{2n}, \omega_{std})$)**.** *Suppose J is a ω-compatible complex structure on $(\mathbb{R}^{2n}, \omega_{std})$. Then by the proposition, there exists a Darboux basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n, \mathbf{f}_1, \cdots, \mathbf{f}_n\}$ such that $J\mathbf{e}_j = \mathbf{f}_j$ and $J\mathbf{f}_j = -\mathbf{e}_j$ for $j = 1, \cdots, n$. Let S be the $2n \times 2n$ real matrix with jth column vector $\mathbf{e}_j$ and $n + j$th column vector $\mathbf{f}_j$. Since $\{\mathbf{e}, \mathbf{f}\}$ is a Darboux basis, S is* *symplectic, and*

$$J = SJ_0S^{-1}. \tag{2.78}$$

The condition $J_0 = SJ_0S^{-1}$ is equivalent to

$$S = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \tag{2.79}$$

in block matrix form. Since S is symplectic $A^tB = B^tA$ and $A^tA+B^tB = 1_n$. This can be identified with the condition that $A+iB$ is unitary, or that S is an orthogonal $2n \times 2n$ matrix. Thus the set of compatible complex structures on $(\mathbb{R}^{2n}, \omega_{std})$ can be identified with the homogeneous space $\mathrm{Sp}(2n;\mathbb{R})/U(n) = \mathrm{Sp}(2n;\mathbb{R})/(\mathrm{Sp}(2n;\mathbb{R}) \cap \mathrm{SO}(2n;\mathbb{R}))$. *We recall that in the following computation for complex structures*

$$\frac{1}{2}\begin{pmatrix} 1_n & i\cdot 1_n \\ i\cdot 1_n & 1_n \end{pmatrix}\begin{pmatrix} A & -B \\ B & A \end{pmatrix}\begin{pmatrix} 1_n & -i\cdot 1_n \\ -i\cdot 1_n & 1_n \end{pmatrix} = \begin{pmatrix} A+iB & 0 \\ 0 & A-iB \end{pmatrix} \tag{2.80}$$

we have

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1_n & i\cdot 1_n \\ i\cdot 1_n & 1_n \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1_n & -i\cdot 1_n \\ -i\cdot 1_n & 1_n \end{pmatrix} \tag{2.81}$$

are complex symplectic matrices (For a definition of complex symmetric matrices, see Remark 3.35).

Example 2.4.10 (Hermitian inner products from compatible complex structures)**.** *If J is an ω-compatible complex structure on (V, ω), then*

$$h_J(u,v) := \omega(u, Jv) + i\omega(u,v) \tag{2.82}$$

is a hermitian inner product on V.

2.5 Complex Lagrangian subspaces

In this section we will review the definition of complex Lagrangian subspaces on the complexification of a (real) symplectic vector space, and some basic examples. From the way complex Lagrangian subspaces interact with the (complexified) symplectic form and complex conjugation, they can be labelled with *types*. We will review splittings (referred to as "standard decompositions" in [10]) of complex Lagrangian subspaces according to their type.

Definition 2.5.1 (Complexification of a symplectic vector space)**.** *Suppose (V,ω) is a symplectic vector space. Let $V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ and $\omega^{\mathbb{C}}$ be the $\mathbb{C}$-bilinear extension of ω to $V^{\mathbb{C}}$. Then we will say $(V^{\mathbb{C}},\omega^{\mathbb{C}})$ is the* complexification *of (V,ω).*

Remark 2.5.2. *The conditions of nondegeneracy and antisymmetry are also well-defined over $\mathbb{C}$. So $(V^{\mathbb{C}},\omega^{\mathbb{C}})$ can be thought of as a a complex symplectic vector space, i.e. a symplectic vector space over the complex numbers.*

Definition 2.5.3 (Complex conjugation)**.** *Let $(V^{\mathbb{C}},\omega^{\mathbb{C}})$ be the complexification of a $2n$-dimensional symplectic vector space (V,ω). Then let $\{\mathbf{e},\mathbf{f}\}$ be a Darboux basis of V. If $v \in V^{\mathbb{C}}$ let*

$$v = q_1\mathbf{e}_1 + \cdots + q_n\mathbf{e}_n + p_1\mathbf{f}_1 + \cdots p_n\mathbf{f}_n \quad q_j, p_j \in \mathbb{C} \quad j = 1, \cdots, n. \tag{2.83}$$

Then let the complex conjugate of v *be*

$$\overline{v} := \overline{q}_1\mathbf{e}_1 + \cdots + \overline{q}_n\mathbf{e}_n + \overline{p}_1\mathbf{f}_1 + \cdots \overline{p}_n\mathbf{f}_n. \tag{2.84}$$

This does not depend on the choice of Darboux basis $\{\mathbf{e},\mathbf{f}\}$ in V. Let $\operatorname{Re} v := \frac{1}{2}(v+\overline{v})$ and $\operatorname{Im} v := \frac{1}{2i}(v-\overline{v})$.

Remark 2.5.4. *A symplectic vector space over the complex numbers does not come with a notion of complex conjugation.*

Definition 2.5.5 (Complex Lagrangian subspace and their splittings)**.** *A complex n-dimensional subspace $\Gamma \subset V^{\mathbb{C}}$ is a* Lagrangian subspace of $V^{\mathbb{C}}$*, or* complex Lagrangian subspace of V *if $\omega^{\mathbb{C}}|_{\Gamma} = 0$. This idea has been referred to as* polarization *(sometimes as distributions of the complexification of the tangent bundle of a symplectic manifold) in geometric quantization.*

Definition 2.5.6 (Type of a complex Lagrangian subspace)**.** *The form*

$$\kappa(u,v) := -i\omega^{\mathbb{C}}(u,\overline{v}) \tag{2.85}$$

is a hermitian form on $V^{\mathbb{C}}$ and all its subspaces. A complex Lagrangian subspace Γ is of type $\vec{n} := (n_0, n_+, n_-)$ *if the zero (respectively, positive, negative) index of inertia of $\kappa|_\Gamma$ is n_0 (respectively, n_+, n_-). Denote by Γ_0 the kernel of $\kappa|_\Gamma$, i.e. the subspace of Γ consisting of all vectors v such that $\kappa|_\Gamma(v, \cdot) = 0$.*

Remark 2.5.7. *A complex Lagrangian subspace of type $(0, n, 0)$ has been referred to by [30] as a* strictly positive polarization *and a complex Lagrangian subspace of type $(k, n-k, 0)$ has been referred to as a* positive polarization.

Example 2.5.8 (Complexification of a Lagrangian subspace)**.** *If $L \subset V$ is a Lagrangian subspace, then $L^{\mathbb{C}} := L \otimes_{\mathbb{R}} \mathbb{C}$ is a Lagrangian subspace of $(V^{\mathbb{C}}, \omega^{\mathbb{C}})$. $L^{\mathbb{C}}$ is a complex Lagrangian subspace of type $(n, 0, 0)$.*

Example 2.5.9 ($\pm i$ eigenspaces of a compatible complex structure J)**.** *If J is an ω-compatible complex structure, let $J^{\mathbb{C}}$ be the $\mathbb{C}$-linear extension of J to $V^{\mathbb{C}}$. Then*

$$\omega^{\mathbb{C}}(J^{\mathbb{C}}\cdot, J^{\mathbb{C}}\cdot) = \omega^{\mathbb{C}}(\cdot, \cdot) \tag{2.86}$$

and we can see that

$$\omega^{\mathbb{C}}((1 \mp iJ^{\mathbb{C}})u, (1 \mp iJ^{\mathbb{C}})v) = 0 \quad u, v \in V^{\mathbb{C}}. \tag{2.87}$$

On the other hand,

$$\mp \frac{i}{2}\omega^{\mathbb{C}}((1 \mp iJ^{\mathbb{C}})u, (1 \pm iJ^{\mathbb{C}})v) = \omega^{\mathbb{C}}(u, J^{\mathbb{C}}v) \mp i\omega(u, v) \quad u, v \in V^{\mathbb{C}} \tag{2.88}$$

When restricted to $u, v \in V$, we recover the hermitian inner product associated to J on the right hand side. $V_J^{1,0}$ is a complex Lagrangian subspace of type $(0, n, 0)$ and $V_J^{0,1}$ is a complex Lagrangian subspace of type $(0, 0, n)$.

Example 2.5.10 (General form)**.** *Let $\{\mathbf{e}^0, \mathbf{e}^+, \mathbf{e}^-, \mathbf{f}^0, \mathbf{f}^+, \mathbf{f}^-\}$ be a Darboux basis of (V, ω). Then the complex span of*

$$\{\mathbf{e}^0, \mathbf{e}^+ - i\mathbf{f}^+, \mathbf{e}^- + i\mathbf{f}^-\} \tag{2.89}$$

is a complex Lagrangian subspace of type (n_0, n_+, n_-). We will see in Theorem 2.6.5 every complex Lagrangian subspace can be constructed in this way.

Example 2.5.11 (Coordinate form)**.** *Suppose* $\{w_1, \cdots, w_n\}$ *is a* $\mathbb{C}$*-basis of a complex Lagrangian subspace* Γ*, and* $\{\mathbf{e}, \mathbf{f}\}$ *a Darboux basis of* (V, ω)*. If*

$$w_j = \sum_{k=1}^{n} (Q_{kj}\mathbf{e}_k + P_{kj}\mathbf{f}_k) \quad j = 1, \cdots, n \tag{2.90}$$

for $w = a_1 w_1 + \cdots + a_n w_n \in \Gamma$*, we have*

$$w = \sum_{j=1}^{n} a_j \left(\sum_{k=1}^{n} (Q_{kj}\mathbf{e}_k + P_{kj}\mathbf{f}_k) \right), \tag{2.91}$$

and obtain the following basis change formula:

$$(w)_{\{\mathbf{e},\mathbf{f}\}} = \begin{pmatrix} Q \\ P \end{pmatrix} (w)_{\{w_j\}} \tag{2.92}$$

where $(w)_{\{w_j\}}$ *is a* $n \times 1$ *column vector with components* a_j*. Therefore we can characterize* Γ *as the complex span of the vectors whose coefficients are given by the column vectors of* $\begin{pmatrix} Q^t & P^t \end{pmatrix}^t$*. The condition for* Γ *to be a complex Lagrangian subspace is equivalent to:*

$$\begin{pmatrix} Q^t & P^t \end{pmatrix} \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix} = P^t Q - Q^t P = 0. \tag{2.93}$$

Definition 2.5.12 (Splitting of a complex Lagrangian subspace)**.** *Suppose* Γ *is a complex Lagrangian subspace of type* $\vec{n}$ *in* $V^{\mathbb{C}}$*. A* (κ_Γ-orthogonal) splitting *of* Γ *is a choice of complex subspaces* $\Gamma_\pm \subset \Gamma$ *such that*

- $\Gamma = \Gamma_0 \oplus \Gamma_+ \oplus \Gamma_-$
- *As hermitian spaces* $(\Gamma_\pm, \kappa|_{\Gamma_\pm}) \cong (\mathbb{C}^{n_\pm}, \pm\langle\cdot,\cdot\rangle_{std})$
- $\kappa|_{\Gamma_+ \times \Gamma_-} = 0.$

We will denote $\Gamma_{\geq 0} := \Gamma_0 \oplus \Gamma_+$ *and* $\Gamma_{\leq 0} := \Gamma_0 \oplus \Gamma_-$*. We will denote* a complex Lagrangian subspace with a splitting (of type $\vec{n}$) *as* $(\Gamma, \Gamma_+, \Gamma_-)$*. We will say two splittings of a complex Lagrangian subspace* $(\Gamma, \Gamma_+, \Gamma_-)$ *and* $(\Gamma, \Gamma_{+'}, \Gamma_{-'})$

are equivalent modulo the kernel *if*

$$\Gamma_\pm \oplus \Gamma_0 = \Gamma_{\pm'} \oplus \Gamma_0$$

as subspaces of Γ*.*

Example 2.5.13 (Splittings from eigenspaces)**.** *A complex Lagrangian subspace* Γ *of* $(\mathbb{R}^{2n}, \omega_{std})$ *can be described by the complex span of the column vectors of*

$$\begin{pmatrix} Q \\ P \end{pmatrix} \quad Q, P \in \mathrm{Mat}_{n\times n}(\mathbb{C}) : Q^t P = P^t Q. \tag{2.94}$$

If $u, v \in \Gamma$*, let*

$$u = \begin{pmatrix} Q \\ P \end{pmatrix} a, \quad v = \begin{pmatrix} Q \\ P \end{pmatrix} b, \quad a, b \in \mathbb{C}^n. \tag{2.95}$$

We can see that

$$\kappa|_\Gamma(u,v) = b^* \begin{pmatrix} Q^* & P^* \end{pmatrix} \begin{pmatrix} 0 & i\cdot 1_n \\ -i\cdot 1_n & 0 \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix} a = b^*(iQ^*P - iP^*Q)a. \tag{2.96}$$

The hermitian matrix $(\kappa|_\Gamma) := iQ^*P - iP^*Q$ *is a self-adjoint operator on* Γ *with respect to the nondegenerate hermitian form* $\langle\cdot,\cdot\rangle_{std}|_\Gamma$ *(the restriction of the standard inner product of* $(\mathbb{R}^{2n})^{\mathbb{C}} = \mathbb{C}^{2n}$*). Let* Γ_+ *be the direct sum of the eigenspaces with positive eigenvalue, and* Γ_- *be the direct sum of the eigenspaces with negative eigenvalue. By the spectral theorem,* $\Gamma_\pm$ *exist, are uniquely defined, and*

$$\Gamma = \Gamma_0 \oplus \Gamma_+ \oplus \Gamma_-. \tag{2.97}$$

Moreover, Γ_+ *and* Γ_- *are orthogonal with respect to* $\langle\cdot,\cdot\rangle_{std}|_\Gamma$*. By the properties of eigenvectors,*

$$\kappa|_{\Gamma_+\times\Gamma_-}(v_+, v_-) = \langle v_-, (\kappa|_\Gamma)v_+\rangle\rangle_{std}|_\Gamma = 0. \tag{2.98}$$

Since every symplectic vector space is symplectomorphic to some $\mathbb{R}^{2n}$*, this example shows that splittings always exist.*

Example 2.5.14 (Other splittings of a complex Lagrangian subspace)**.** *Sup-*

pose Γ *is the complex span of the column vectors of*

$$\frac{1}{\sqrt{2}}\begin{pmatrix} Q \\ P \end{pmatrix}$$

where

$$Q := \begin{pmatrix} \sqrt{2}\cdot 1_{n_0} & 0 & 0 \\ 0 & 1_{n_+} & 0 \\ 0 & 0 & 1_{n_-} \end{pmatrix}, \quad P := \begin{pmatrix} 0 & 0 & 0 \\ 0 & -i\cdot 1_{n_+} & 0 \\ 0 & 0 & i\cdot 1_{n_-} \end{pmatrix}. \tag{2.99}$$

Then we can compute

$$(\kappa|_\Gamma) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1_{n_+} & 0 \\ 0 & 0 & -1_{n_-} \end{pmatrix}. \tag{2.100}$$

We can partition Γ *into* regions

$$\Gamma = \Gamma^0 \sqcup \Gamma^+ \sqcup \Gamma^- \tag{2.101}$$

where

$$\Gamma^\pm := \{v \in \Gamma : \pm\kappa|_\Gamma(v,v) > 0\} \tag{2.102}$$

$$\Gamma^0 := \{v \in \Gamma : \kappa|_\Gamma(v,v) = 0\}. \tag{2.103}$$

Γ^0 *is the* null cone *containing the subspace* $\Gamma_0 \subset \Gamma^0$ *and* $\Gamma_\pm$ *need to be chosen from the various subspaces sitting inside the regions* $\Gamma^\pm \cup \{0\}$.

For instance, if $\vec{n} = (1,1,1)$, *then*

$$(\kappa|_\Gamma) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.104}$$

and

$$\begin{aligned}
\Gamma_0 &\cong \{(z_1, 0, 0) : z_1 \in \mathbb{C}\} && (2.105)\\
\Gamma^0 &\cong \{(z_1, z_2, \pm z_2) : z_1, z_2 \in \mathbb{C}\} && (2.106)\\
\Gamma^{\pm} &\cong \{(z_1, z_2, z_3) : \pm(|z_2|^2 - |z_3|^2) > 0\}. && (2.107)
\end{aligned}$$

So, for instance,

$$\mathrm{Span}_{\mathbb{C}} \begin{pmatrix}1\\0\\0\end{pmatrix} \oplus \mathrm{Span}_{\mathbb{C}} \begin{pmatrix}3\\2\\1\end{pmatrix} \oplus \mathrm{Span}_{\mathbb{C}} \begin{pmatrix}0\\1\\2\end{pmatrix}$$

$$\cong \mathrm{Span}_{\mathbb{C}} \begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix} \oplus \mathrm{Span}_{\mathbb{C}} \begin{pmatrix}3\\ \sqrt{2}\\ 1/\sqrt{2}\\ 0\\ -\sqrt{2}i\\ i/\sqrt{2}\end{pmatrix} \oplus \mathrm{Span}_{\mathbb{C}} \begin{pmatrix}0\\ 1/\sqrt{2}\\ \sqrt{2}\\ 0\\ -i/\sqrt{2}\\ \sqrt{2}i\end{pmatrix} \qquad (2.108)$$

is another splitting of Γ *(or* $(\Gamma, \kappa|_\Gamma)$*). We will see at the end of the chapter that the set of all splittings of a particular type* $\vec{n}$ *is contractible.*

2.6 Real projections

We will review how the images of the splittings of complex Lagrangian subspaces behave under projection to V. A (modulo-the-kernel) equivalence class of splittings is mapped to a real subspace and its symplectic complement. We will end with the statement and proof of the existence of a Darboux basis of V that reconstructs any complex Lagrangian subspace (Theorem 2.6.5). This statement appears as Lemma 5.1 with the proof left as an exercise in [10].

Definition 2.6.1 (Notation for real projection)**.** *For a complex subspace* $W_{\mathbb{C}} \subset V^{\mathbb{C}}$*, let*

$$\mathrm{Re}\, W_{\mathbb{C}} := \{\mathrm{Re}\, w : w \in W^{\mathbb{C}}\}. \qquad (2.109)$$

Lemma 2.6.2.

1. $\operatorname{Re} W_{\mathbb{C}} = (W_{\mathbb{C}} + \overline{W}_{\mathbb{C}}) \cap V$
2. $\operatorname{Re}(W_{\mathbb{C}} + W'_{\mathbb{C}}) = \operatorname{Re} W_{\mathbb{C}} + \operatorname{Re} W'_{\mathbb{C}}$
3. $\operatorname{Re}(W_{\mathbb{C}} \cap W'_{\mathbb{C}}) \subset \operatorname{Re} W_{\mathbb{C}} \cap \operatorname{Re} W'_{\mathbb{C}}$

Proof.

1. $\operatorname{Re} w = \frac{1}{2}(w + \overline{w})$ so $\operatorname{Re} w \in (W_{\mathbb{C}} + \overline{W}_{\mathbb{C}}) \cap V$. If $w \in (W_{\mathbb{C}} + \overline{W}_{\mathbb{C}}) \cap V$, then there exist $u \in W_{\mathbb{C}}$, $v \in \overline{W}_{\mathbb{C}}$ such that $w = u + v$. Since $w \in V$, $w = \operatorname{Re} w = \frac{1}{2}(u + \overline{u}) + \frac{1}{2}(v + \overline{v}) = \operatorname{Re}(u + \overline{v})$. So $w \in \operatorname{Re} W_{\mathbb{C}}$.
2. This follows from $\operatorname{Re}(u + v) = \operatorname{Re} u + \operatorname{Re} v$.
3. If $w \in \operatorname{Re}(W_{\mathbb{C}} \cap W'_{\mathbb{C}})$ there exists a $\tilde{w} \in W_{\mathbb{C}} \cap W'_{\mathbb{C}}$ such that $w = \operatorname{Re} \tilde{w}$. $\operatorname{Re} \tilde{w} \in \operatorname{Re} W_{\mathbb{C}}$ and $\operatorname{Re} \tilde{w} \in \operatorname{Re} W'_{\mathbb{C}}$.

□

Remark 2.6.3. *We can check that the following inclusion is proper*

$$\operatorname{Re}(V_J^{1,0} \cap V_J^{0,1}) = \{0\} \subset \operatorname{Re} V_J^{1,0} \cap \operatorname{Re} V_J^{0,1} = V. \tag{2.110}$$

Lemma 2.6.4.

1. $\operatorname{Re} \Gamma_0 = \Gamma_0 \cap V = \Gamma \cap V$.
2. $\operatorname{Re} \Gamma_0$ *has dimension* n_0, $\operatorname{Re} \Gamma_{\geq 0}$ *has dimension* $n_0 + 2n_+$, *and* $\operatorname{Re} \Gamma_{\leq 0}$ *has dimension* $n_0 + 2n_-$.
3. $(\operatorname{Re} \Gamma_{\geq 0})^{\omega} = \operatorname{Re} \Gamma_{\leq 0}$.

Proof.

1. Suppose $w \in \Gamma$, $v \in \Gamma_0$. Then since v is a 0-eigenvector

$$i\kappa(v, w) = \omega^{\mathbb{C}}(\operatorname{Re} v + i \operatorname{Im} v, \operatorname{Re} w - i \operatorname{Im} w) = 0 \tag{2.111}$$

so the real and imaginary parts vanish

$$\begin{aligned} \omega^{\mathbb{C}}(\operatorname{Re} v, \operatorname{Re} w) + \omega^{\mathbb{C}}(\operatorname{Im} v, \operatorname{Im} w) &= 0 && (2.112)\\ -\omega^{\mathbb{C}}(\operatorname{Re} v, \operatorname{Im} w) + \omega^{\mathbb{C}}(\operatorname{Im} v, \operatorname{Re} w) &= 0. && (2.113) \end{aligned}$$

Since Γ is Lagrangian,
$$\omega^{\mathbb{C}}(v,w)=0 \tag{2.114}$$
so the real and imaginary parts vanish

$$\begin{aligned}\omega^{\mathbb{C}}(\operatorname{Re}v,\operatorname{Re}w)-\omega^{\mathbb{C}}(\operatorname{Im}v,\operatorname{Im}w) &= 0 &(2.115)\\ \omega^{\mathbb{C}}(\operatorname{Re}v,\operatorname{Im}w)+\omega^{\mathbb{C}}(\operatorname{Im}v,\operatorname{Re}w) &= 0 &(2.116)\end{aligned}$$

Therefore
$$\omega^{\mathbb{C}}(\operatorname{Re}v,w)=0 \quad \text{for all } w\in\Gamma. \tag{2.117}$$
Hence $\operatorname{Re}v\in\Gamma^{\omega^{\mathbb{C}}}=\Gamma$. Since $\kappa(\operatorname{Re}v,w)=0$ for all $w\in\Gamma$, $\operatorname{Re}v\in\Gamma_0$. Therefore $\operatorname{Re}\Gamma_0=\Gamma_0\cap V\subset\Gamma\cap V$. If $u\in\Gamma\cap V$, then $u=\overline{u}=\operatorname{Re}u$. Since Γ is Lagrangian

$$\omega^{\mathbb{C}}(v,u)=\omega^{\mathbb{C}}(v,\overline{u})=0 \quad \text{for all } v\in\Gamma. \tag{2.118}$$

Therefore $u\in\Gamma_0\cap V$, and we have $\Gamma_0\cap V=\Gamma\cap V$.

2. $\Gamma_0=(\operatorname{Re}\Gamma_0)^{\mathbb{C}}$ so

$$\dim_{\mathbb{R}}\operatorname{Re}\Gamma_0=\dim_{\mathbb{C}}\Gamma_0=n_0. \tag{2.119}$$

The kernel of the surjective map $\operatorname{Im}:(\Gamma_{\geq 0})^{\mathbb{R}}\to\operatorname{Re}\Gamma_{\geq 0}$ is $V\cap\Gamma_{\geq 0}=\operatorname{Re}\Gamma_0$. So

$$\dim_{\mathbb{R}}\operatorname{Re}\Gamma_{\geq 0}=\dim_{\mathbb{R}}(\Gamma_{\geq 0})^{\mathbb{R}}-\dim_{\mathbb{R}}\operatorname{Re}\Gamma_0=n_0+2n_+. \tag{2.120}$$

3. Suppose $u\in\operatorname{Re}\Gamma_{\geq 0}$ and $v\in\operatorname{Re}\Gamma_{\leq 0}$. Then there exist $\tilde{u}\in\Gamma_{\geq 0}$ and $\tilde{v}\in\Gamma_{\leq 0}$ such that $u=\operatorname{Re}\tilde{u}$ and $v=\operatorname{Re}\tilde{v}$. By the $\kappa|_\Gamma$-orthogonality property of the splitting, we have

$$\kappa(\tilde{u},\tilde{v})=0 \tag{2.121}$$

and since Γ is Lagrangian, we have

$$\omega^{\mathbb{C}}(\tilde{u},\tilde{v})=0. \tag{2.122}$$

Expanding into real and imaginary parts, we get

$$\omega(u,v) = \omega(\operatorname{Re}\tilde{u}, \operatorname{Re}\tilde{v}) = 0. \tag{2.123}$$

Therefore $\operatorname{Re}\Gamma_{\leq 0} \subset (\operatorname{Re}\Gamma_{\geq 0})^{\omega}$. The equality is obtained by the dimension formula.

□

Proposition 2.6.5. *Suppose* $(\Gamma, \Gamma_+, \Gamma_-)$ *is a complex Lagrangian subspace with splitting of type* $\vec{n}$. *Let* $\{v_0\}$ *be a basis of* Γ_0 *such that* $\{v_0\}$ *is a basis of* $\operatorname{Re}\Gamma_0$, *and* $\{v_\pm\}$ *be bases of* $\Gamma_\pm$ *such that* $(\kappa|_{\Gamma_\pm})_{\{v_\pm\}} = \pm 1_{n_\pm}$. *Then there exists* $\{w_0\}$ *such that*

$$\left\{v_0, \frac{1}{\sqrt{2}}\operatorname{Re}v_+, \frac{1}{\sqrt{2}}\operatorname{Re}v_-, w_0, -\frac{1}{\sqrt{2}}\operatorname{Im}v_+, \frac{1}{\sqrt{2}}\operatorname{Im}v_-\right\} \tag{2.124}$$

is a Darboux basis of V.

Proof.

$$\left\{\frac{1}{\sqrt{2}}\operatorname{Re}v_+, \frac{1}{\sqrt{2}}\operatorname{Re}v_-, -\frac{1}{\sqrt{2}}\operatorname{Im}v_+, \frac{1}{\sqrt{2}}\operatorname{Im}v_-\right\} \tag{2.125}$$

is a Darboux basis of its span, which is hence symplectic. The symplectic complement of the span is symplectic, and $\operatorname{Span}_{\mathbb{R}}\{v_0\}$ is a Lagrangian subspace of this space. $\{w_0\}$ is obtained by applying the Lagrangian basis extension. □

Remark 2.6.6. *This shows that every complex Lagrangian subspace is of the form*

$$\operatorname{Span}_{\mathbb{C}}\{\mathbf{e}^0, \mathbf{e}^+ - i\mathbf{f}^+, \mathbf{e}^- + i\mathbf{f}^-\} \tag{2.126}$$

for some Darboux basis $\{\mathbf{e}^0, \mathbf{e}^+, \mathbf{e}^-, \mathbf{f}^0, \mathbf{f}^+, \mathbf{f}^-\}$. *By construction, we have* $\{v_0, \operatorname{Re}v_+, -\operatorname{Im}v_+\}$ *is a basis of* $\operatorname{Re}\Gamma_{\geq 0}$, *and* $\{v_0, \operatorname{Re}v_-, \operatorname{Im}v_-\}$ *is a basis of* $\operatorname{Re}\Gamma_{\leq 0}$.

2.7 The partition of the complex Lagrangian Grassmannian

In this section, we will review how complex conjugation in $V^{\mathbb{C}}$ partitions the complex Lagrangian Grassmannian into $\binom{n+2}{2}$ subsets. Then we will describe each subset as a homogeneous space, using the action of the symplectic group. Moreover, we can partition each Grassmannian of k-dimensional subspaces of V, and show that the of subsets of the partition of the Grassmannians have a bijective correspondence with the subsets in the partition of the complex Lagrangian Grassmannian, in a way that corresponding subsets are homotopic. This describes a "reassembly" phenomenon, in the sense that we can disassemble the $2n+1$ different Grassmannians of V, and–after taking homotopic replacements if each subset–assemble them into *one* homogeneous space.

Definition 2.7.1 (Notation for Grassmannians)**.** *Let (V, ω) be a $2n$ dimensional real symplectic vector space. We will denote by*

$$\mathrm{Gr}\,(k; V) \tag{2.127}$$

the Grassmannian of k-dimensional subspaces of V. We will denote by

$$\mathrm{Gr}\,(\vec{n}; V) \tag{2.128}$$

the Grassmannian of subspaces of $W \subset V$ of dimension $n_0 + 2n_+$ such that $\dim_{\mathbb{R}}(W \cap W^{\omega}) = n_0$. We will denote by

$$\mathrm{Lag}^{\mathbb{C}}(\vec{n}; V) \tag{2.129}$$

the Grassmannian of complex Lagrangian subspaces of $(V^{\mathbb{C}}, \omega^{\mathbb{C}})$ of type $\vec{n}$, and by

$$\mathrm{Lag}^{\mathbb{C}}_{\oplus}(\vec{n}; V) \tag{2.130}$$

the Grassmannian of equivalence classes (modulo the kernel) of complex La-

grangian subspaces with splitting $(\Gamma, \Gamma_+, \Gamma_-)$ *of type* $\vec{n}$ *in* $(V^{\mathbb{C}}, \omega^{\mathbb{C}})$*, and by*

$$\mathrm{Lag}^{\mathbb{C}}(V) \tag{2.131}$$

the Grassmannian of all complex Lagrangian subspaces of $(V^{\mathbb{C}}, \omega^{\mathbb{C}})$*.*

Remark 2.7.2. $\mathrm{Lag}^{\mathbb{C}}(V)$ *has a partition into* $\mathrm{Lag}^{\mathbb{C}}(\vec{n}; V)$*'s.*

$$\mathrm{Lag}^{\mathbb{C}}(V) = \coprod_{\vec{n}} \mathrm{Lag}^{\mathbb{C}}(\vec{n}; V) \tag{2.132}$$

$\mathrm{Gr}(k; V)$ *has a partition into* $\mathrm{Gr}(\vec{n}; V)$*'s.*

$$\mathrm{Gr}(k; V) = \coprod_{\vec{n}:k=n_0+2n_+} \mathrm{Gr}(\vec{n}; V). \tag{2.133}$$

Remark 2.7.3 (Left and right actions of the symplectic group)**.** *Suppose* $\{\mathbf{e}, \mathbf{f}\}$ *and* $\{\mathbf{e}', \mathbf{f}'\}$ *are two Darboux bases which are expressed in terms of some fixed Darboux basis as the column vectors of symplectic matrices* $S_{\{\mathbf{e},\mathbf{f}\}}$ *and* $S_{\{\mathbf{e}',\mathbf{f}'\}}$*. Then the linear map taking* $\{\mathbf{e}, \mathbf{f}\}$ *to* $\{\mathbf{e}', \mathbf{f}'\}$ *can be expressed as both left and right multiplication by some symplectic matrix:*

$$S_{\{\mathbf{e},\mathbf{f}\}} \cdot (S^{-1}_{\{\mathbf{e},\mathbf{f}\}} S_{\{\mathbf{e}',\mathbf{f}'\}}) = S_{\{\mathbf{e}',\mathbf{f}'\}} \tag{2.134}$$
$$(S_{\{\mathbf{e}',\mathbf{f}'\}} S^{-1}_{\{\mathbf{e},\mathbf{f}\}}) \cdot S_{\{\mathbf{e},\mathbf{f}\}} = S_{\{\mathbf{e}',\mathbf{f}'\}}. \tag{2.135}$$

The existence theorems of Darboux bases 2.3.14, 2.6.5 tell us that each W*, (respectively,* $(\Gamma, [(\Gamma_+, \Gamma_-)])$*,* Γ*) can be viewed as equivalence classes of Darboux bases, and if a symplectic linear transformation fixes* W*, (respectively,* $(\Gamma, [(\Gamma_+, \Gamma_-)])$*,* Γ*), it must permute the different Darboux bases in the equivalence class defined by* W *(respectively,* $(\Gamma, [(\Gamma_+, \Gamma_-)])$*,* Γ*). These equivalence classes are defined by a condition of what kind of linear recombinations we allow for the Darboux bases within an equivalence class.*

A right multiplication by a symplectic matrix, rearranges the column vectors of $S_{\{\mathbf{e},\mathbf{f}\}}$ *so it respects the operations of linear combinations of Darboux bases that we use in the proof of theorems 2.3.14, 2.6.5. The same linear recombination rules are applied for two different Darboux bases* $\{\mathbf{e}, \mathbf{f}\}$*,* $\{\mathbf{e}'', \mathbf{f}''\}$ *when* $S_{\{\mathbf{e},\mathbf{f}\}}$ *and* $S_{\{\mathbf{e}'',\mathbf{f}''\}}$ *are multiplied by a symplectic matrix from the right.*

So right multiplication by a symplectic matrix of the form $S^{-1}_{\{\mathbf{e},\mathbf{f}\}}S_{\{\mathbf{e}',\mathbf{f}'\}}$ *preserves the subsets of Darboux bases defined by a condition of linear recombination.*

To see how a left multiplication by a symplectic matrix preserves subsets of Darboux bases defined by a condition of linear recombination, suppose $\{\mathbf{e},\mathbf{f}\}$ *and* $\{\mathbf{e}'',\mathbf{f}''\}$ *satisfies some condition defined by some linear recombination. Then there is a symplectic matrix* S^{right} *such that*

$$S_{\{\mathbf{e}'',\mathbf{f}''\}} = S_{\{\mathbf{e},\mathbf{f}\}}S^{right}. \tag{2.136}$$

So

$$S^{left}S_{\{\mathbf{e}'',\mathbf{f}''\}} = S^{left}S_{\{\mathbf{e},\mathbf{f}\}}S^{right} = S_{\{\mathbf{e}',\mathbf{f}'\}}S^{right} \tag{2.137}$$

So left multiplication by a symplectic matrix also preserves the equivalence class of Darboux bases defined by linear recombination.

Thus the set of symplectic matrices acting on the right

$$\{S^{-1}_{\{\mathbf{e},\mathbf{f}\}}S_{\{\mathbf{e}',\mathbf{f}'\}} : S_{\{\mathbf{e},\mathbf{f}\}} \sim_W S_{\{\mathbf{e}',\mathbf{f}'\}}\} \tag{2.138}$$

and the set of symplectic matrices acting on the left

$$\{S_{\{\mathbf{e}',\mathbf{f}'\}}S^{-1}_{\{\mathbf{e},\mathbf{f}\}} : S_{\{\mathbf{e},\mathbf{f}\}} \sim_W S_{\{\mathbf{e}',\mathbf{f}'\}}\} \tag{2.139}$$

define the right and left stabilizer subgroups of W*. A similar claim can be made for* $(\Gamma, [(\Gamma_+, \Gamma_-)])$ *and* Γ*.*

Therefore, both the left and right actions of $\mathrm{Sp}(V,\omega)$ *on* $\mathrm{Gr}(\vec{n};V)$, $\mathrm{Lag}^{\mathbb{C}}_{\oplus}(\vec{n};V)$, *and* $\mathrm{Lag}^{\mathbb{C}}(\vec{n};V)$ *are well defined, and this action is transitive.*

Now suppose there is a Darboux basis $\{\mathbf{e},\mathbf{f}\}$ *in the equivalence class of* W*, (respectively,* $(\Gamma, [(\Gamma_+, \Gamma_-)])$*,* Γ*), such that*

$$S_{\{\mathbf{e},\mathbf{f}\}} = 1_{2n}. \tag{2.140}$$

Then the left and right stabilizers coincide. When a group acts transitively on a set, the stabilizers at different points are conjugate, hence isomorphic. So all left and right stabilizers are isomorphic.

Proposition 2.7.4 (Right stabilizers)**.** *Let* $N(\vec{n})$ *be the nilpotent group of* *matrices in block form*

$$\begin{pmatrix} 1_{n_0} & E^+ & E^- & Y & F^+ & F^- \\ 0 & 1_{n_+} & 0 & (F^+)^t & 0 & 0 \\ 0 & 0 & 1_{n_-} & (F^-)^t & 0 & 0 \\ 0 & 0 & 0 & 1_{n_0} & 0 & 0 \\ 0 & 0 & 0 & -(E^+)^t & 1_{n_+} & 0 \\ 0 & 0 & 0 & -(E^-)^t & 0 & 1_{n_-} \end{pmatrix} \tag{2.141}$$

where

$$E^\pm, F^\pm \in \mathrm{Mat}_{n_\pm \times n_0}(\mathbb{R}) \tag{2.142}$$

and

$$Y - E^+(F^+)^t - E^-(F^-)^t \tag{2.143}$$

is symmetric (this condition is equivalent to $N(\vec{n})$ being a subgroup of $\mathrm{Sp}(2n;\mathbb{R})$). We can express the stabilizers with respect to the right group action of $\mathrm{Sp}(V,\omega)$ on $\mathrm{Gr}(\vec{n};V)$, $\mathrm{Lag}^{\mathbb{C}}(\vec{n};V)$ and $\mathrm{Lag}^{\mathbb{C}}_{\oplus}(\vec{n};V)$ using $N(\vec{n})$.

1. *If $W \subset V$ is a subspace of type $\vec{n}$, its stabilizer $G^{\mathbb{R},right}_{\vec{n}}(W)$ of the right action of $\mathrm{Sp}(V,\omega)$ on $\mathrm{Gr}(\vec{n};V)$ is isomorphic to the semidirect product*

$$(GL(n_0;\mathbb{R}) \times \mathrm{Sp}(2n_+;\mathbb{R}) \times \mathrm{Sp}(2n_-;\mathbb{R})) \ltimes N(\vec{n}). \tag{2.144}$$

2. *If $\Gamma \subset V^{\mathbb{C}}$ is a complex Lagrangian subspace of type $\vec{n}$, its stabilizer $G^{\mathbb{C},right}_{\vec{n}}(\Gamma)$ of the right action of $\mathrm{Sp}(V,\omega)$ on $\mathrm{Lag}^{\mathbb{C}}(\vec{n};V)$ is isomorphic to the semidirect product*

$$(GL(n_0;\mathbb{R}) \times U(n_+,n_-)) \ltimes N(\vec{n}) \tag{2.145}$$

   where $U(n_+,n_-)$ is the indefinite unitary group.

3. *If $(\Gamma,[(\Gamma_+,\Gamma_-)])$ is a complex Lagrangian subspace with (equivalence class of) splitting of type $\vec{n}$, its stabilizer $G^{\mathbb{C},right}_{\vec{n},\oplus}(\Gamma,[(\Gamma_+,\Gamma_-)])$ of the right action of $\mathrm{Sp}(V,\omega)$ on $\mathrm{Lag}^{\mathbb{C}}_{\oplus}(\vec{n};V)$ is isomorphic to the semidirect product*

$$(GL(n_0;\mathbb{R}) \times U(n_+) \times U(n_-)) \ltimes N(\vec{n}). \tag{2.146}$$

Proof. Denote a fixed Darboux basis of V identifying it with $\mathbb{R}^{2n}$ as

$$\mathrm{fix} := \{\mathbf{e}_1, \cdots, \mathbf{e}_n, \mathbf{f}_1, \cdots, \mathbf{f}_n\}. \tag{2.147}$$

Denote the permuted basis

$$\{\mathbf{e}_j, \mathbf{e}_k, \mathbf{f}_k, \mathbf{e}_\ell, \mathbf{f}_\ell, \mathbf{f}_j\}_{j,k,\ell} \tag{2.148}$$

as $\mathrm{fix}^{\mathbb{R}}_{\vec{n}}$, and the Darboux basis

$$\left\{\mathbf{e}_j, \frac{\mathbf{e}_k - i\mathbf{f}_k}{\sqrt{2}}, \frac{\mathbf{e}_\ell + i\mathbf{f}_\ell}{\sqrt{2}}, \mathbf{f}_j, \frac{-i\mathbf{e}_k + \mathbf{f}_k}{\sqrt{2}}, \frac{i\mathbf{e}_\ell + \mathbf{f}_\ell}{\sqrt{2}}\right\}_{j,k,\ell}$$

as $\mathrm{fix}^{\mathbb{C}}_{\vec{n}}$, where the indices range from

$$\begin{aligned} j &\in \{1, \cdots, n_0\} && (2.149)\\ k &\in \{n_0+1, \cdots, n_0+n_+\} && (2.150)\\ \ell &\in \{n_0+n_++1, \cdots, n\}. && (2.151) \end{aligned}$$

Then denote the change of basis matrices

$$\begin{aligned} M^{\mathbb{R}}_{\vec{n}} &:= \begin{pmatrix} 1_{n_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_{n_+} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n_+} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_{n_0} \\ 0 & 0 & 1_{n_-} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_{n_-} & 0 \end{pmatrix} \\ M^{\mathbb{C}}_{\vec{n}} &:= \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\cdot 1_{n_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_{n_+} & 0 & 0 & -i\cdot 1_{n_+} & 0 \\ 0 & 0 & 1_{n_-} & 0 & 0 & i\cdot 1_{n_-} \\ 0 & 0 & 0 & \sqrt{2}\cdot 1_{n_0} & 0 & 0 \\ 0 & -i\cdot 1_{n_+} & 0 & 0 & 1_{n_+} & 0 \\ 0 & 0 & i\cdot 1_{n_-} & 0 & 0 & 1_{n_-} \end{pmatrix}. \end{aligned}$$

$M^{\mathbb{R}}_{\vec{n}}$ is not necessarily symplectic, but $M^{\mathbb{C}}_{\vec{n}}$ is symplectic. Then a vector in fix basis is expressed in $\mathrm{fix}^{\mathbb{R}}_{\vec{n}}$ basis by multiplying its expression in fix basis by $(M^{\mathbb{R}}_{\vec{n}})^{-1}$ from the left, and a vector in fix basis is expressed in $\mathrm{fix}^{\mathbb{C}}_{\vec{n}}$ basis by multiplying its expression in fix by $(M^{\mathbb{C}}_{\vec{n}})^{-1}$ from the left. A symplectic linear transformation, expressed by a symplectic matrix $(S)_{\mathrm{fix}}$ in the fix basis is expressed in the $\mathrm{fix}^{\mathbb{R}}_{\vec{n}}$ basis by $(S)_{\mathrm{fix}^{\mathbb{R}}_{\vec{n}}} = (M^{\mathbb{R}}_{\vec{n}})^{-1}(S)_{\mathrm{fix}}M^{\mathbb{R}}_{\vec{n}}$, . This can be summarized in the following:

$$\begin{aligned}
(Sv)_{\mathrm{fix}^{\mathbb{R}}_{n}} &= (M^{\mathbb{R}}_{\vec{n}})^{-1}(Sv)_{\mathrm{fix}} && (2.152)\\
&= ((M^{\mathbb{R}}_{\vec{n}})^{-1}(S)_{\mathrm{fix}}M^{\mathbb{R}}_{\vec{n}}) \cdot (M^{\mathbb{R}}_{\vec{n}})^{-1}(v)_{\mathrm{fix}} && (2.153)\\
&= (S)_{\mathrm{fix}^{\mathbb{R}}_{\vec{n}}} \cdot (v)_{\mathrm{fix}^{\mathbb{R}}_{\vec{n}}} && (2.154)
\end{aligned}$$

and similarly for the $\mathrm{fix}^{\mathbb{C}}_{\vec{n}}$ basis.

1. If $W \subset V$ is a subspace of type $\vec{n}$, then the stabilizer $G^{\mathbb{R},right}_{\vec{n}}(W) \subset \mathrm{Sp}(V,\omega)$ consists of symplectic linear transformations S such that, in the $\mathrm{fix}^{\mathbb{R}}_{\vec{n}}$ basis, has block form

$$(S)_{\mathrm{fix}^{\mathbb{R}}_{\vec{n}}} = (M^{\mathbb{R}}_{\vec{n}})^{-1}(S)_{\mathrm{fix}}M^{\mathbb{R}}_{\vec{n}} = \begin{pmatrix} * & * & * & * & * & * \\ 0 & A^+ & B^+ & 0 & 0 & * \\ 0 & C^+ & D^+ & 0 & 0 & * \\ 0 & 0 & 0 & A^- & B^- & * \\ 0 & 0 & 0 & C^- & D^- & * \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}, \quad (2.155)$$

where

$$A^{\pm}, B^{\pm}, C^{\pm}, D^{\pm} \in \mathrm{Mat}_{n_{\pm}\times n_{\pm}}(\mathbb{R}). \quad (2.156)$$

This is because right multiplication by $(S)_{\mathrm{fix}^{\mathbb{R}}_{\vec{n}}}$ must preserve W, W^{ω} and $W \cap W^{\omega}$. In this block form we can see that $G^{\mathbb{R},right}_{\vec{n}}(W)$ is isomoprhic to a semidirect product of the group of matrices of block form

$$\begin{pmatrix} * & 0 & 0 & 0 & 0 & 0 \\ 0 & A^+ & B^+ & 0 & 0 & 0 \\ 0 & C^+ & D^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & A^- & B^- & 0 \\ 0 & 0 & 0 & C^- & D^- & 0 \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}, \tag{2.157}$$

whose image under $M_{\vec{n}}^{\mathbb{R}}(\cdot)(M_{\vec{n}}^{\mathbb{R}})^{-1}$ is symplectic, and the group of matrices of block form

$$\begin{pmatrix} 1_{n_0} & * & * & * & * & * \\ 0 & 1_{n_+} & 0 & 0 & 0 & * \\ 0 & 0 & 1_{n_+} & 0 & 0 & * \\ 0 & 0 & 0 & 1_{n_-} & 0 & * \\ 0 & 0 & 0 & 0 & 1_{n_-} & * \\ 0 & 0 & 0 & 0 & 0 & 1_{n_0} \end{pmatrix}, \tag{2.158}$$

whose image under $M_{\vec{n}}^{\mathbb{R}}(\cdot)(M_{\vec{n}}^{\mathbb{R}})^{-1}$ is symplectic. The former can be identified as $\mathrm{GL}(n_0;\mathbb{R}) \times \mathrm{Sp}(2n_+;\mathbb{R}) \times \mathrm{Sp}(2n_-;\mathbb{R})$ and the latter can be identified as $N(\vec{n})$. The condition for the block forms to have image under $M_{\vec{n}}^{\mathbb{R}}(\cdot)(M_{\vec{n}}^{\mathbb{R}})^{-1}$ to be symplectic is equivalent to the block forms being

$$\begin{pmatrix} X & 0 & 0 & 0 & 0 & 0 \\ 0 & A^+ & B^+ & 0 & 0 & 0 \\ 0 & C^+ & D^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & A^- & B^- & 0 \\ 0 & 0 & 0 & C^- & D^- & 0 \\ 0 & 0 & 0 & 0 & 0 & (X^t)^{-1} \end{pmatrix}, \tag{2.159}$$

and

$$\begin{pmatrix} 1_{n_0} & E^+ & F^+ & E^- & F^- & Y \\ 0 & 1_{n_+} & 0 & 0 & 0 & (F^+)^t \\ 0 & 0 & 1_{n_+} & 0 & 0 & (-E^+)^t \\ 0 & 0 & 0 & 1_{n_-} & 0 & (F^-)^t \\ 0 & 0 & 0 & 0 & 1_{n_-} & (-E^-)^t \\ 0 & 0 & 0 & 0 & 0 & 1_{n_0} \end{pmatrix}. \tag{2.160}$$

2. If $\Gamma \subset V^{\mathbb{C}}$ is a complex Lagrangian subspace of type $\vec{n}$, then the stabilizer $G^{\mathbb{C},right}_{\vec{n}}(\Gamma) \subset \mathrm{Sp}(V,\omega)$ consists of symplectic linear transformations S such that, in the $\mathrm{fix}^{\mathbb{C}}_{\vec{n}}$ basis, has block form

$$(S)_{\mathrm{fix}^{\mathbb{C}}_{\vec{n}}} = (M^{\mathbb{C}}_{\vec{n}})^{-1}(S)_{\mathrm{fix}} M^{\mathbb{C}}_{\vec{n}} = \begin{pmatrix} * & * \\ 0_{n\times n} & * \end{pmatrix}. \tag{2.161}$$

This condition is equivalent to $(S)_{\mathrm{fix}}$ being of block form

$$(S)_{\mathrm{fix}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} * & * & * & * & * & * \\ 0 & A^{++} & A^{+-} & * & B^{++} & B^{+-} \\ 0 & A^{-+} & A^{--} & * & B^{-+} & B^{--} \\ 0 & 0 & 0 & * & 0 & 0 \\ 0 & -B^{++} & B^{+-} & * & A^{++} & -A^{+-} \\ 0 & B^{-+} & -B^{--} & * & -A^{-+} & A^{--} \end{pmatrix}. \tag{2.162}$$

If we compute $(M^{\mathbb{R}}_{\vec{n}})^{-1}(S)_{\mathrm{fix}} M^{\mathbb{R}}_{\vec{n}}$, we get

$$\begin{pmatrix} * & * & * & * & * & * \\ 0 & A^{++} & B^{++} & A^{+-} & B^{+-} & * \\ 0 & -B^{++} & A^{++} & B^{+-} & -A^{+-} & * \\ 0 & A^{-+} & B^{-+} & A^{--} & B^{--} & * \\ 0 & B^{-+} & -A^{-+} & -B^{--} & A^{--} & * \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix} \tag{2.163}$$

from which we can again see that $G^{\mathbb{C},right}_{\vec{n}}(\Gamma)$ is isomorphic to the semidirect product of the group of symplectic matrices of the form

$$(S)_{\text{fix}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} X & 0 & 0 & 0 & 0 & 0 \\ 0 & A^{++} & A^{+-} & 0 & B^{++} & B^{+-} \\ 0 & A^{-+} & A^{--} & 0 & B^{-+} & B^{--} \\ 0 & 0 & 0 & (X^t)^{-1} & 0 & 0 \\ 0 & -B^{++} & B^{+-} & 0 & A^{++} & -A^{+-} \\ 0 & B^{-+} & -B^{--} & 0 & -A^{-+} & A^{--} \end{pmatrix}$$

with $N(\vec{n})$. In this group, the condition that $A^tD - C^tB = 1_n$ is equivalent to

$$U = \begin{pmatrix} A^{++} - iB^{++} & A^{+-} + iB^{+-} \\ A^{-+} - iB^{-+} & A^{--} + iB^{--} \end{pmatrix} \tag{2.164}$$

satisfying

$$\operatorname{Re}\left(U^* \begin{pmatrix} 1_{n_+} & 0 \\ 0 & -1_{n_-} \end{pmatrix} U \right) = \begin{pmatrix} 1_{n_+} & 0 \\ 0 & -1_{n_-} \end{pmatrix}, \tag{2.165}$$

and the condition that $A^tC = C^tA$, which is equivalent to $B^tD = D^tB$, is equivalent to U satisfying

$$\operatorname{Im}\left(U^* \begin{pmatrix} 1_{n_+} & 0 \\ 0 & -1_{n_-} \end{pmatrix} U \right) = 0. \tag{2.166}$$

So $G^{\mathbb{C},right}_{\vec{n}}(\Gamma)$ is isomorphic to

$$(\mathrm{GL}(n_0;\mathbb{R}) \times U(n_+, n_-)) \ltimes N(\vec{n}). \tag{2.167}$$

3. We can repeat the argument above, except now S must preserve the $\kappa|_\Gamma$-orthogonality condition, so

$$A^{+-} = A^{-+} = B^{+-} = B^{-+} = 0. \tag{2.168}$$

Looking at the formula for U, we can see $G^{\mathbb{C},right}_{\vec{n},\oplus}(\Gamma, [(\Gamma_+, \Gamma_-)])$ is iso-

morphic to
$$(\mathrm{GL}(n_0;\mathbb{R}) \times U(n_+) \times U(n_-)) \ltimes N(\vec{n}). \tag{2.169}$$
□

Remark 2.7.5 (The map to the indefinite unitary group)**.** *To see where the identification with the indefinite unitary group comes from, we can look at a stabilizer of the left action. Suppose* Γ *be the complex span of the column vectors of*
$$\frac{1}{\sqrt{2}}\begin{pmatrix} Q \\ P \end{pmatrix}$$
where
$$Q := \begin{pmatrix} 1_{n_+} & 0 \\ 0 & 1_{n_-} \end{pmatrix}, \quad P := \begin{pmatrix} -i \cdot 1_{n_+} & 0 \\ 0 & i \cdot 1_{n_-} \end{pmatrix}. \tag{2.170}$$

If
$$S := \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A^{++} & A^{+-} & B^{++} & B^{+-} \\ A^{-+} & A^{--} & B^{-+} & B^{--} \\ C^{++} & C^{+-} & D^{++} & D^{+-} \\ C^{-+} & C^{--} & D^{-+} & D^{--} \end{pmatrix} \tag{2.171}$$
Then S *fixes* Γ *if and only if there exists an* $n \times n$ *invertible matrix* U *such that*
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} Q \\ P \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} Q \\ P \end{pmatrix} U. \tag{2.172}$$
For the given Q*,* P*, we can explicitly compute this is possible if and only if*
$$U = \begin{pmatrix} A^{++} - iB^{++} & A^{+-} + iB^{+-} \\ A^{-+} - iB^{-+} & A^{--} + iB^{--} \end{pmatrix} \tag{2.173}$$
and
$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A^{++} & A^{+-} & B^{++} & B^{+-} \\ A^{-+} & A^{--} & B^{-+} & B^{--} \\ -B^{++} & B^{+-} & A^{++} & -A^{+-} \\ B^{-+} & -B^{--} & -A^{-+} & A^{--} \end{pmatrix}. \tag{2.174}$$

Remark 2.7.6. *Since the right stabilizers do not depend on the choice of Darboux basis, and hence of the point inside* $\mathrm{Gr}(\vec{n}; V)$*,* $\mathrm{Lag}^{\mathbb{C}}_{\oplus}(\vec{n}; V)$*, and* $\mathrm{Lag}^{\mathbb{C}}(\vec{n}; V)$*, we will denote the right stabilizers by* $G^{\mathbb{R}}_{\vec{n}}$*,* $G^{\mathbb{C}}_{\vec{n},\oplus}$*, and* $G^{\mathbb{C}}_{\vec{n}}$*. Then we obtain the diffeomorphisms with the left coset spaces*

$$\begin{aligned} \mathrm{Lag}^{\mathbb{C}}_{\oplus}(\vec{n}; V) &\cong \mathrm{Sp}(2n; \mathbb{R})/G^{\mathbb{C}}_{\vec{n},\oplus} && (2.175)\\ \mathrm{Lag}^{\mathbb{C}}(\vec{n}; V) &\cong \mathrm{Sp}(2n; \mathbb{R})/G^{\mathbb{C}}_{\vec{n}} && (2.176)\\ \mathrm{Gr}(\vec{n}; V) &\cong \mathrm{Sp}(2n; \mathbb{R})/G^{\mathbb{R}}_{\vec{n}}. && (2.177) \end{aligned}$$

Lemma 2.7.7. $G^{\mathbb{R}}_{\vec{n}}/G^{\mathbb{C}}_{\vec{n},\oplus}$ *and* $G^{\mathbb{C}}_{\vec{n}}/G^{\mathbb{C}}_{\vec{n},\oplus}$ *are contractible.*

Proof. We can compute

$$G^{\mathbb{R}}_{\vec{n}}/G^{\mathbb{C}}_{\vec{n},\oplus} \cong \mathrm{Sp}(2n_+; \mathbb{R})/U(n_+) \times \mathrm{Sp}(2n_-; \mathbb{R})/U(n_-) \tag{2.178}$$

which is a product of Siegel upper half planes, and hence contractible (cf. Example 5.2.1). We can also compute

$$G^{\mathbb{C}}_{\vec{n}}/G^{\mathbb{C}}_{\vec{n},\oplus} \cong U(n_+, n_-)/(U(n_+) \times U(n_-)) \tag{2.179}$$

which is a quotient by the maximal compact subgroup, and is hence contractible by the Cartan-Malcev-Iwasawa theorem. □

Remark 2.7.8. $G^{\mathbb{C}}_{\vec{n}} = G^{\mathbb{C}}_{\vec{n},\oplus}$ *for the coisotropic case* $n_- = 0$ *has been obtained in Proposition 3.3 of [31].*

Theorem 2.7.9. *For all* $\vec{n}$*,* $\mathrm{Gr}(\vec{n}; V)$*,* $\mathrm{Lag}^{\mathbb{C}}(\vec{n}; V)$*, and* $\mathrm{Lag}^{\mathbb{C}}_{\oplus}(\vec{n}; V)$ *are homotopic.*

Proof. The map $(\Gamma, \Gamma_+, \Gamma_-) \mapsto \mathrm{Re}\,\Gamma_{\geq 0}$ from $\mathrm{Lag}^{\mathbb{C}}_{\oplus}(\vec{n}; V)$ to $\mathrm{Gr}(\vec{n}; V)$ has contractible fibers $G^{\mathbb{R}}_{\vec{n}}/G^{\mathbb{C}}_{\vec{n},\oplus}$. The map $(\Gamma, \Gamma_+, \Gamma_-) \mapsto \Gamma$ from $\mathrm{Lag}^{\mathbb{C}}_{\oplus}(\vec{n}; V)$ to $\mathrm{Lag}^{\mathbb{C}}(\vec{n}; V)$ has contractible fibers $G^{\mathbb{C}}_{\vec{n}}/G^{\mathbb{C}}_{\vec{n},\oplus}$. Therefore $\mathrm{Lag}^{\mathbb{C}}(\vec{n}; V)$, $\mathrm{Lag}^{\mathbb{C}}_{\oplus}(\vec{n}; V)$, and $\mathrm{Gr}(\vec{n}; V)$ are homotopic. □

Example 2.7.10 (The complex Lagrangian Grassmanian of $(\mathbb{R}^2, \omega_{std})$)**.** *We* *can identify the Grassmannians of subspaces of dimensions* $1, 2, 0$ *in* $\mathbb{R}^2$.

$$\begin{aligned}
\mathrm{Gr}(1;\mathbb{R}^2) = &\quad \mathrm{Gr}((1,0,0);\mathbb{R}^2) &= \mathbb{P}^1(\mathbb{R}) \qquad (2.180)\\
\mathrm{Gr}(2;\mathbb{R}^2) = &\quad \mathrm{Gr}((0,1,0);\mathbb{R}^2) &= \{\mathbb{R}^2\} \qquad (2.181)\\
\mathrm{Gr}(0;\mathbb{R}^2) = &\quad \mathrm{Gr}((0,0,1);\mathbb{R}^2) &= \{\{0\}\}. \qquad (2.182)
\end{aligned}$$

The stabilizer groups $G^{\mathbb{R}}_{\vec{n}}$ *can be computed*

$$\begin{aligned}
G^{\mathbb{R}}_{(1,0,0)} &= \{\textit{upper triangular matrices of } \mathrm{Sp}(2;\mathbb{R})\} \qquad (2.183)\\
G^{\mathbb{R}}_{(0,1,0)} &= \mathrm{Sp}(2;\mathbb{R}) \qquad (2.184)\\
G^{\mathbb{R}}_{(0,0,1)} &= \mathrm{Sp}(2;\mathbb{R}). \qquad (2.185)
\end{aligned}$$

On the other hand, since every 1*-dimensional subspace of* $\mathbb{C}^2$ *is Lagrangian, the complex Lagrangian Grassmannian of* $\mathbb{R}^2$ *can be identified with the Riemann sphere:*

$$\mathrm{Lag}^{\mathbb{C}}(\mathbb{R}^2) = \mathbb{P}^1(\mathbb{C}). \qquad (2.186)$$

Identifying Γ *with* $[q:p]$, $\kappa|_\Gamma$ *is*

$$-i\omega^{\mathbb{C}}(\cdot,\bar{\cdot})|_\Gamma = -i\begin{pmatrix}\overline{q} & \overline{p}\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}q\\ p\end{pmatrix} = 2\mathrm{Im}\,(q\overline{p}).$$

The complex Lagrangian Grassmannians are partitioned into different types as the equator, upper hemisphere, and lower hemisphere:

$$\begin{aligned}
\mathrm{Lag}^{\mathbb{C}}((1,0,0);\mathbb{R}^2) &= \{[q:p] : \mathrm{Im}(q\overline{p}) = 0\} \qquad (2.187)\\
\mathrm{Lag}^{\mathbb{C}}((0,1,0);\mathbb{R}^2) &= \{[q:p] : \mathrm{Im}(q\overline{p}) > 0\} \qquad (2.188)\\
\mathrm{Lag}^{\mathbb{C}}((0,0,1);\mathbb{R}^2) &= \{[q:p] : \mathrm{Im}(q\overline{p}) < 0\}. \qquad (2.189)
\end{aligned}$$

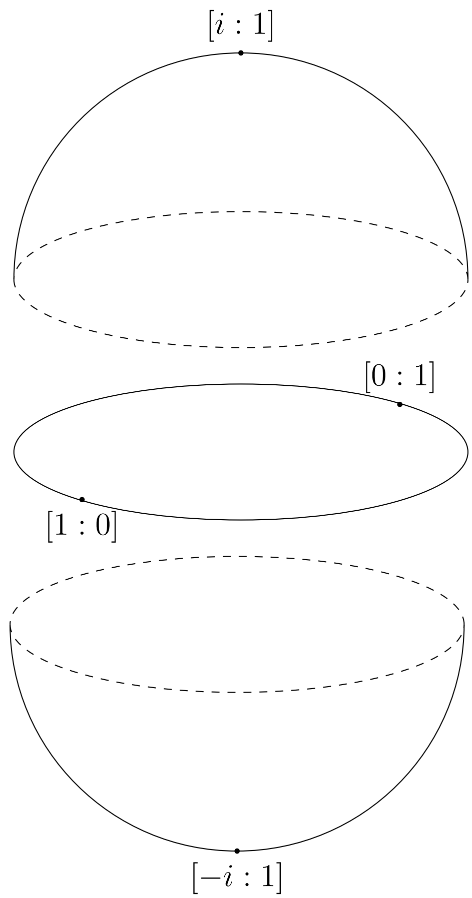

Figure 1: The partition of the complex Lagrangian Grassmannian of $\mathbb{R}^2$

The stabilizer groups $G^{\mathbb{C}}_{\vec{n}} = G^{\mathbb{C}}_{\vec{n},\oplus}$ *can be computed*

$$\begin{aligned} G^{\mathbb{C}}_{(1,0,0)} &= \{\textit{upper triangular matrices of } \mathrm{Sp}(2;\mathbb{R})\} && (2.190)\\ G^{\mathbb{C}}_{(0,1,0)} &= \mathrm{Sp}(2;\mathbb{R}) \cap \mathrm{SO}(2;\mathbb{R}) \cong U(1) && (2.191)\\ G^{\mathbb{C}}_{(0,0,1)} &= \mathrm{Sp}(2;\mathbb{R}) \cap \mathrm{SO}(2;\mathbb{R}) \cong U(1). && (2.192) \end{aligned}$$

Chapter 3

Representation theory of the Heisenberg group

The canonical commutation relations can be expressed using the symmetries of the Heisenberg group and Lie algebra.

In this chapter we will review the definitions for the Heisenberg group and Lie algebra, and cite some relevant, but by no means comprehensive, results in their representation theory without proof. The Heisenberg group is isomorphic to a matrix group, but it is nilpotent, so results on semisimple Lie groups do not apply. Moreover, since the Heisenberg group is not compact, it can (and does) have irreducible infinite dimensional representations. So we need to assume the setup of Hilbert spaces and unitarity, and keep track of topologies of bounded operators and the domains of unbounded operators, the choice of latter possibly being very sensitive about pointwise boundary conditions. Moreover, additional conditions need to be checked to ensure desired properties to hold.

The first property we will review is about the complete reducibility of a group representation into irreducible representations. For finite or compact groups, every (unitary) representation is isomorphic to a direct sum of finite dimensional irreducible representations. The decomposition statement holds for locally compact groups of type I (such as the Heisenberg group), when we consider *(strongly) continuous unitary representations*, and *direct integral decompositions* rather than direct sum decompositions. One subtlety about direct integral decompositions is that a direct integrand of a direct integral decomposition does not necessarily have to be a *subrepresentation*.

The Stone-Von Neumann theorem classifies the irreducible, infinite dimensional, (strongly) continuous unitary representations of the Heisenberg group. Together with the direct integral decomposition, this result helps us have an idea of the category of (strongly) continuous unitary representations of the Heisenberg group.

For representations of the Heisenberg Lie algebra, the Dixmier-Rellich theorem states that direct *sum* decompositions exist for representations satisfying some additional assumptions. These assumptions include the ones induced from the (strong) continuity and unitarity for the representations of the Heisenberg group. At the time of writing, the author is not aware of a treatment of the decomposition or classification (of irreducible, infinite dimensional representations) problem using only concepts from Lie algebras.

Thus, the second property we will review is about the correspondence between representations of the Lie group and Lie algebra. For a (strongly) continuous unitary representation of a locally compact group, the formula for differentiation gives a representation of the Lie algebra on some dense subspaces of the Hilbert space. A representation of a Lie algebra by skew-symmetric operators on a dense subspace of a Hilbert space, exponentiates uniquely into a (strongly) continuous unitary representation of the corresponding Lie group, if in addition, it satisfies the Nelson condition, or the Flato-Simon-Snellman-Sternheimer condition.

In this chapter, endomorphisms will refer to linear operators of vector spaces, with no additional assumptions about their structure. Likewise, the general linear group of a vector space will consist of invertible linear operators of a vector space, with no additional assumptions about preserving any additional structure.

3.1 Translations in symplectic vector spaces

In this section, we will review the definitions of the Heisenberg group and Lie algebra, and compare them with the abelian group of translations of a vector space, and its abelian Lie algebra. This point of view appears in [7], and allows us to think about position, momentum, creation, and annihilation operators in quantum mechanics, as instances of infinitesimal translational symmetries.

Let V be a real inner product space, with an orthonormal basis that identifies it with $\mathbb{R}^n$ with its smooth structure, Euclidean metric and Lebesgue measure. If f is a function on V, then we can denote the translate of f by $-a \in V$ as follows:

$$\tau_a^0 f(v) := f(v+a). \tag{3.1}$$

This can be extended linearly as an endomorphism of $\mathbb{C}^V := \{f : V \to \mathbb{C}\}$ to itself. Then we can check that the operators $\{\tau_a^0\}_{a\in V}$ satisfy

$$\tau_a^0 \tau_b^0 = \tau_{a+b}^0 \quad a, b \in V. \tag{3.2}$$

If f is smooth, then we can differentiate

$$\dot{\tau}_a^0 f(v) := \lim_{t\to 0} \frac{1}{t}(\tau_{ta}^0 f(v) - f(v)) = d_a f(v) \tag{3.3}$$

and obtain the directional derivative of f. We can check that the commutator vanishes in $\mathrm{End}\,(C^\infty(V;\mathbb{C}))$:

$$[\dot{\tau}_a^0, \dot{\tau}_b^0] = 0 \quad a, b \in V. \tag{3.4}$$

We observe that for the Lie group $(V, +)$, $\tau_\cdot^0$ is a representation on $\mathbb{C}^V$. Moreover, $\dot{\tau}_\cdot : a \mapsto \tau_a^0$ is a Lie algebra homomorphism from $(V, 0) \to \mathrm{End}(C^\infty(V;\mathbb{C}))$. The action of the Heisenberg group on some function spaces will retain many properties analogous to the ones we have just observed.

Now let (V, ω) be a symplectic vector space, with a Darboux basis that identifies it with $(\mathbb{R}^{2n}, \omega_{std})$ with its smooth structure. This identification identifies the volume form with the determinant, and so we can also assume (V, ω) has a well-defined Lebesgue measure pulled back from $\mathbb{R}^{2n}$. If f is a function on V, we can compose the translation by $-a \in V$ with the multiplication by $e^{\frac{i}{2}\lambda\omega(v,a)}$, and consider

$$\tau_a^\lambda f(v) := e^{\frac{i}{2}\lambda\omega(v,a)} f(v+a) \quad a \in V, \lambda \in \mathbb{R}. \tag{3.5}$$

We can consider $\{\tau_a^\lambda\}_{a\in V}$ as endomorphisms of $\mathbb{C}^V$, and check they satisfy

$$\tau_a^\lambda \tau_b^\lambda = e^{\frac{i}{2}\lambda\omega(a,b)}\tau_{a+b}^\lambda \quad a,b\in V. \tag{3.6}$$

When $\lambda = 0$ we recover the translation operators $\{\tau_a^0\}_{a\in V}$ on Euclidean spaces. If f is smooth, we can differentiate

$$\dot{\tau}_a^\lambda f(v) := \lim_{t\to 0}\frac{1}{t}(\tau_{ta}^\lambda f(v) - f(v)) = \left(d_a + \frac{i\lambda}{2}\omega(v,a)\right) f(v) \quad a\in V. \tag{3.7}$$

Then we can check that the commutator satisfies

$$[\dot{\tau}_a^\lambda, \dot{\tau}_b^\lambda] = i\lambda\omega(a,b) \tag{3.8}$$

in $\mathrm{End}\,(C^\infty(V;\mathbb{C}))$:

When $\lambda = 1$, we will drop the superscript on τ and $\dot{\tau}$.

Definition 3.1.1 (Heisenberg group)**.** *For (V,ω) a symplectic vector space, let $H(\omega) := \mathbb{R}\times V$ be the* Heisenberg group *or* Heisenberg-Weyl group *with group multiplication*

$$(s,a)\cdot(t,b) := \left(s+t+\frac{1}{2}\omega(a,b), a+b\right) \quad a,b\in V, \quad s,t\in\mathbb{R}. \tag{3.9}$$

$H(\omega)$ has a smooth structure when the smooth structure is pulled back from $(\mathbb{R}^{2n},\omega_{std})$ to (V,ω) by a fixed Darboux basis.

Remark 3.1.2 (The polarized Heisenberg group)**.** *If $a\in V$ is identified with $(q,p)\in\mathbb{R}^{2n}$, the map*

$$(s,a)\mapsto \begin{pmatrix} 1 & p^t & s-\frac{1}{2}p^tq \\ 0 & 1_n & q \\ 0 & 0 & 1 \end{pmatrix} \tag{3.10}$$

is a group isomorphism from $H(\omega)$ to the subgroup of $\mathrm{GL}(\mathbb{R}^{n+2})$ *consisting* *of matrices of the form*

$$\begin{pmatrix} 1 & * & * \\ 0 & 1_n & * \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.11}$$

This subgroup is sometimes referred to as the polarized Heisenberg group *(or just the Heisenberg group). It is not compact, so it does not have any faithful, unitary, finite dimensional representations.*

Definition 3.1.3 (Heisenberg Lie algebra)**.** *For (V, ω) a symplectic vector space, let $\mathfrak{heis}\,\omega := i\mathbb{R} \oplus V$ with Lie bracket*

$$[is + a, it + b] = i\omega(a, b) \quad a, b \in V. \tag{3.12}$$

3.2 Universal enveloping algebras

In this section we will review the definition of the universal enveloping algebra of a Lie algebra and its (formal) completion. In this work, this is viewed as a setup to see what happens when we do all computations formally, without taking into consideration analytic issues. We will see at the end of this section how the set of Poincaré-Birkhoff-Witt isomorphisms given by complex Darboux bases can be identified with the set of transverse pairs of complex Lagrangian subspaces.

Definition 3.2.1 (Universal enveloping algebra)**.** *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. Take the tensor algebra of $\mathfrak{g}$:*

$$T\mathfrak{g} := \bigoplus_{k=0}^{\infty} \mathfrak{g}^{\otimes_{\mathbb{F}} k} \tag{3.13}$$

and consider the two sided ideal I in $T\mathfrak{g}$ generated by

$$\{x \otimes_{\mathbb{F}} y - y \otimes_{\mathbb{F}} x - [x, y] : x, y \in \mathfrak{g}\}. \tag{3.14}$$

Then the universal enveloping algebra *(of $\mathfrak{g}$) is defined as*

$$\mathfrak{U}\mathfrak{g} := T\mathfrak{g}/I. \tag{3.15}$$

Definition 3.2.2 (Symmetric algebra)**.** *If V is a vector space over a field $\mathbb{F}$,* the symmetric algebra of V $\operatorname{Sym} V$ *is the quotient of the tensor algebra of V*

$$TV := \bigoplus_{k=0} V^{\otimes_{\mathbb{F}} k} \tag{3.16}$$

by the two sided ideal generated by

$$\{x \otimes_{\mathbb{F}} y - y \otimes_{\mathbb{F}} x : x, y \in V\}. \tag{3.17}$$

Remark 3.2.3 (Completion of universal enveloping algebras)**.** *The universal enveloping algebra has a Hopf algebra structure, and there is an augmentation (counit) map $\eta : \mathfrak{U}\mathfrak{g} \to \mathbb{F}$. Let I_η be the augmentation ideal. Then the completion of the universal enveloping algebra is given by (Example 1.2 in Appendix A of [20])*

$$\hat{\mathfrak{U}}\mathfrak{g} := \varprojlim \mathfrak{U}\mathfrak{g}/I_\eta^k. \tag{3.18}$$

When $\mathbb{F}$ is a field of characteristic zero, this allows one to write down the formal exponential of elements of $\mathfrak{g}$

$$e^a := \sum_{k=0}^{\infty} \frac{a^k}{k!} \quad a \in \mathfrak{g} \tag{3.19}$$

as an element of $\hat{\mathfrak{U}}\mathfrak{g}$. Because of the noncommutativity of the product, in general

$$e^{a+b} \neq e^a e^b. \tag{3.20}$$

Theorem 3.2.4 (Baker-Campbell-Hausdorff formula)**.** *If $a, b \in \mathfrak{g}$, then there exists a $c \in \mathfrak{g}$ such that*

$$e^a e^b = e^c \tag{3.21}$$

in $\hat{\mathfrak{U}}\mathfrak{g}$. c is given by

$$c = a + \int_0^1 \psi((\exp \operatorname{ad} a)(\exp \operatorname{ad} tb)) b dt \tag{3.22}$$

where $\psi(z)$ *is a formal power series expansion of* $\frac{z\log z}{z-1}$ *around* $z = 1$

$$\psi(1+u) = 1 + \frac{u}{2} - \frac{u^2}{6} + \cdots \tag{3.23}$$

and

$$\operatorname{ad} a : b \mapsto [a,b]. \tag{3.24}$$

The first few terms of c *can be written explicitly as*

$$c = a + b + \frac{1}{2}[a,b] + \frac{1}{12}([a,[a,b]] + [b,[b,a]]) - \frac{1}{24}[b,[a,[a,b]]] + \cdots . \tag{3.25}$$

Example 3.2.5 (Baker-Campbell-Hausdorff formula for the Heisenberg Lie algebra)**.** *For the Heisenberg Lie algebra, one iteration of the Lie bracket takes values in the center, so we have*

$$e^a e^b = e^{\frac{1}{2}[a,b]} e^{a+b} = e^{\frac{i}{2}\omega(a,b)} e^{a+b} \quad a, b \in V \subset \mathfrak{heis}\omega. \tag{3.26}$$

In particular, we recover the same algebraic relation for the τ*'s*

$$\tau_a \tau_b = e^{\frac{i}{2}\omega(a,b)} \tau_{a+b} \tag{3.27}$$

Theorem 3.2.6 (Poincaré-Birkhoff-Witt)**.** *Let* $\{a_1, \cdots, a_d\}$ *be an ordered basis of* $\mathfrak{g}$*. Then* $\{a_1^{r_1} \cdots a_d^{r_d} : r_j \in \mathbb{Z}_{\geq 0}\}$ *is a basis of* $\mathfrak{Ug}$*.*

Remark 3.2.7 (PBW isomorphisms)**.** *Whenever* $\{a_1, \cdots, a_d\}$ *be an ordered basis of* $\mathfrak{g}$*, we have an isomorphism of vector spaces (in fact, coalgebras) between the symmetric algebra of* $\mathfrak{g}$*,* $\operatorname{Sym}\mathfrak{g}$ *and* $\mathfrak{Ug}$*. given by the map*

$$PBW_{\{a_j\}_j} : a_1^{r_1} \cdots a_d^{r_d} \mapsto a_1^{r_1} \cdots a_d^{r_d} \quad r_j \in \mathbb{Z}_{\geq 0}. \tag{3.28}$$

On the left side the product is commutative, but on the right side, it is not.

Remark 3.2.8 (Complexification of the universal enveloping algebra and symmetric algebra)**.** *If* $\mathfrak{g}$ *is a Lie algebra over* $\mathbb{R}$*, then denote by* $\mathfrak{g}^{\mathbb{C}}$ *the complex Lie algebra with underlying vector space* $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ *and with Lie bracket* *extended $\mathbb{C}$-bilinearly. Then there are canonical identifications*

$$\mathfrak{U}(\mathfrak{g}^{\mathbb{C}}) \cong (\mathfrak{U}\mathfrak{g}) \otimes_{\mathbb{R}} \mathbb{C} \tag{3.29}$$

and

$$\mathrm{Sym}(\mathfrak{g}^{\mathbb{C}}) \cong (\mathrm{Sym}\,\mathfrak{g}) \otimes_{\mathbb{R}} \mathbb{C} \tag{3.30}$$

We will denote by

$$\mathfrak{heis}\,\omega^{\mathbb{C}} := (\mathfrak{heis}\,\omega)^{\mathbb{C}}. \tag{3.31}$$

Remark 3.2.9 (Complex Darboux bases and complex symplectic matrices)**.** *The statements from Section 2.2 continue to hold over the complex numbers. For a symplectic vector space over the complex numbers $(V_{\mathbb{C}}, \omega_{\mathbb{C}})$ (not necessarily a complexification of a real symplectic vector space), let* a complex Darboux basis *be defined analogously to 2.2.1–i.e. a basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n, \mathbf{f}_1, \cdots, \mathbf{f}_n\}$ of $V_{\mathbb{C}}$ such that*

$$\omega_{\mathbb{C}}(\mathbf{e}_j, \mathbf{e}_k) = \omega_{\mathbb{C}}(\mathbf{e}_j, \mathbf{e}_k) = 0 \quad j, k \in \{1, \cdots, n\} \tag{3.32}$$

and

$$\omega_{\mathbb{C}}(\mathbf{e}_j, \mathbf{f}_k) = \delta_{j,k} \tag{3.33}$$

where $\delta_{j,k}$ is the Kronecker delta.

A complex symplectic matrix *is a $2n \times 2n$ complex matrix*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.34}$$

where A, B, C, D are $n \times n$ complex matrices, that satisfy

$$A^t C = C^t A, \quad B^t D = D^t B \quad \textit{and} \quad A^t D - C^t B = 1. \tag{3.35}$$

We will denote the set of complex symplectic matrices as $\mathrm{Sp}(2n; \mathbb{C})$*. As in Example 2.2.4 the column vectors of a complex symplectic matrix give a complex Darboux basis of $(\mathbb{C}^{2n} = \mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}, \omega^{\mathbb{C}}_{std})$.*

A (complex) symplectic linear transformation $S \in \mathrm{Sp}(V_{\mathbb{C}}, \omega_{\mathbb{C}})$ *is a com-*

plex linear transformation such that

$$\omega_{\mathbb{C}}(S\cdot, S\cdot) = \omega_{\mathbb{C}}(\cdot,\cdot). \tag{3.36}$$

The matrix form of a (complex) symplectic linear transformation in a complex Darboux basis is a complex symplectic matrix–i.e.

$$(S)_{\{\mathbf{e},\mathbf{f}\}} \in \mathrm{Sp}(2n;\mathbb{C}). \tag{3.37}$$

The proofs of Theorem 2.2.7, Corollary 2.2.8 also hold for symplectic vector spaces over the complex numbers.

Remark 3.2.10 (Action of symplectic linear transformations induced by universal property)**.** *A symplectic linear transformation $S \in \mathrm{Sp}(V^{\mathbb{C}}, \omega^{\mathbb{C}})$ induces an automorphism $(\mathrm{Id}_{i\mathbb{R}}, S)$ of $\mathfrak{heis}\,\omega^{\mathbb{C}}$, and by the universal property, induces automorphisms of both $\mathrm{Sym}(\mathfrak{heis}\,\omega^{\mathbb{C}})$ and $\mathfrak{U}(\mathfrak{heis}\,\omega^{\mathbb{C}})$. We will view these automorphisms only as invertible linear transformations, and denote them again by S.*

Example 3.2.11 (PBW isomorphisms from complex Darboux bases)**.** *Suppose we have a complex Darboux basis $\{\mathbf{e},\mathbf{f}\}$ of $(V^{\mathbb{C}}, \omega^{\mathbb{C}})$. Then the PBW theorem gives an isomorphism as vector spaces*

$$PBW_{\{\mathbf{e},\mathbf{f},i\}} : \mathrm{Sym}(\mathfrak{heis}\,\omega^{\mathbb{C}}) \xrightarrow{\cong} \mathfrak{U}(\mathfrak{heis}\,\omega^{\mathbb{C}}). \tag{3.38}$$

Proposition 3.2.12. *If $S \in \mathrm{Sp}(V^{\mathbb{C}}, \omega^{\mathbb{C}})$, then its action commutes with the PBW isomorphism*

$$PBW_{\{\mathbf{e},\mathbf{f},i\}}(S\cdot) = S \cdot PBW_{\{\mathbf{e},\mathbf{f},i\}}(\cdot) \tag{3.39}$$

if and only if S has block form

$$\begin{pmatrix} X & 0 \\ 0 & (X^t)^{-1} \end{pmatrix} \in \mathrm{GL}(\mathrm{Span}_{\mathbb{C}}\{\mathbf{e}\} \oplus \mathrm{Span}_{\mathbb{C}}\{\mathbf{f}\}). \tag{3.40}$$

Proof. For a fixed complex Darboux basis $\{\mathbf{e},\mathbf{f}\}$, denote $PBW_{\{\mathbf{e},\mathbf{f},i\}}$ by PBW.

Let

$$(S)_{\{\mathbf{e},\mathbf{f}\}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{3.41}$$

Then we can compute the following in $U(\mathfrak{heis}\,\omega^{\mathbb{C}})$:

$$\begin{aligned} S \cdot PBW(\mathbf{e}_j\mathbf{e}_k) - PBW(S \cdot (\mathbf{e}_j\mathbf{e}_k)) &= -i(A^tC)_{jk} && (3.42)\\ S \cdot PBW(\mathbf{f}_j\mathbf{f}_k) - PBW(S \cdot (\mathbf{f}_j\mathbf{f}_k)) &= -i(B^tD)_{jk} && (3.43)\\ S \cdot PBW(\mathbf{e}_j\mathbf{f}_k) - PBW(S \cdot (\mathbf{e}_j\mathbf{f}_k)) &= -i(B^tC)_{jk}. && (3.44)\end{aligned}$$

($\Rightarrow$) Since $(S)_{\{\mathbf{e},\mathbf{f}\}}$ is symplectic, $A^tD - C^tB = 1_n = A^tD$. Therefore A and D are invertible, and $B = C = 0$.
($\Leftarrow$) This is immediate. □

Remark 3.2.13 (PBW isomorphisms from Darboux bases)**.** *Therefore the set of PBW isomorphisms induced by a choice of complex Darboux bases can be identified with*

$$\mathrm{Sp}(2n;\mathbb{C})/\mathrm{GL}(n;\mathbb{C}), \tag{3.45}$$

or as the space of transverse pairs of complex Lagrangian subspaces. We will see in Chapter 4 that this space also parametrizes representations of the Heisenberg Lie algebra.

3.3 Hilbert spaces and unitary operators

In this section we will review Hilbert spaces and the unitary groups of Hilbert spaces. The unitary group is defined as a subgroup of the algebra of bounded operators on a Hilbert space, which has many topologies. We will review how the strong and weak topologies on the algebra of bounded operators coincide on the unitary group.

Definition 3.3.1 (Hilbert space)**.** *A Hilbert space is a vector space $\mathscr{H}$ over the complex numbers, with a nondegenerate Hermitian inner product $\langle \cdot, \cdot \rangle_{\mathscr{H}}$, and complete with respect to it. We will only consider separable Hilbert spaces.*

Definition 3.3.2 (Bounded, unitary, and unbounded operators)**.** *An (unbounded)* operator $\mathbf{A}$ *(or* $(\mathbf{A}, D_{\mathbf{A}})$*) on a Hilbert space* $\mathscr{H}$ *is a linear map from a linear subspace* $D_{\mathbf{A}} \subset \mathscr{H}$ *to* $\mathscr{H}$*.* $D_{\mathbf{A}}$ *is called the* domain *of* $\mathbf{A}$*. We will only consider the operators whose domain is a dense subspace of* $\mathscr{H}$*. A* bounded operator $\mathbf{A}$ *is a linear map from* $\mathscr{H}$ *to* $\mathscr{H}$ *for which there is some constant* $c > 0$ *such that*

$$||\mathbf{A}f||_{\mathscr{H}} \leq c||f||_{\mathscr{H}} \quad \textit{for all } f \in \mathscr{H}. \tag{3.46}$$

A unitary operator $\mathbf{A}$ *is a linear map from* $\mathscr{H}$ *to* $\mathscr{H}$ *such that*

$$\langle \mathbf{A}f, \mathbf{A}g\rangle_{\mathscr{H}} = \langle f, g\rangle_{\mathscr{H}} \quad \textit{for all } f, g \in \mathscr{H}. \tag{3.47}$$

Denote by $\mathcal{B}(\mathscr{H})$ *the space of bounded operators on* $\mathscr{H}$*, and* $U(\mathscr{H})$ *the group of unitary operators on* $\mathscr{H}$*.*

Remark 3.3.3. *A unitary operator is always bounded. A bounded operator is an (unbounded) operator, so "unbounded" means "not necessarily bounded" instead of "not bounded."*

Example 3.3.4. *The differentiation operator* $\frac{d}{dx}$ *on the smooth functions on the interval* $(0, 1)$ *is an unbounded operator on the Hilbert space* $L^2((0, 1))$*.*

Remark 3.3.5. *Unitarity also allows the orthogonal complement of a closed invariant subspace to be closed invariant, so it is a reasonable requirement to have to consider a decomposition theory into irreducible representations. The conditions of unitarity also appears naturally from the requirements of quantum mechanics (Wigner's theorem).*

Definition 3.3.6 (Weak, strong, and norm topologies)**.** *The weak topology is the topology on* $\mathcal{B}(\mathscr{H})$ *induced by the maps*

$$\mathbf{A} \mapsto \langle \mathbf{A}f, g\rangle_{\mathscr{H}} \quad f, g \in \mathscr{H}. \tag{3.48}$$

The strong topology is the topology on $\mathcal{B}(\mathscr{H})$ *induced by the maps*

$$\mathbf{A} \mapsto \mathbf{A}f \quad f \in \mathscr{H} \tag{3.49}$$

or the seminorms

$$\mathbf{A} \mapsto ||\mathbf{A}f||_{\mathscr{H}} \quad f \in \mathscr{H}. \tag{3.50}$$

The norm topology is the topology on $\mathcal{B}(\mathscr{H})$ given by the operator norm

$$||\mathbf{A}|| := \sup_{||f||_{\mathscr{H}} \leq 1} ||\mathbf{A}f||_{\mathscr{H}}. \tag{3.51}$$

Remark 3.3.7 (On $U(\mathscr{H})$ the weak and strong topologies coincide)**.** *For all $f, g \in \mathscr{H}$, and $\mathbf{A}, \mathbf{B} \in \mathcal{B}(\mathscr{H})$ we have by the Cauchy-Schwarz inequality,*

$$|\langle \mathbf{A}f, g\rangle_{\mathscr{H}} - \langle \mathbf{B}f, g\rangle_{\mathscr{H}}| = |\langle (\mathbf{A} - \mathbf{B})f, g\rangle_{\mathscr{H}}| \leq ||\mathbf{A}f - \mathbf{B}f||_{\mathscr{H}} ||g||_{\mathscr{H}} \tag{3.52}$$

So if $\{\mathbf{A}_j\}_j$ is a sequence such that $\mathbf{A}_j \to \mathbf{B}$ in the strong topology, it converges in the weak topology. If $\mathbf{A}_j, \mathbf{B} \in U(\mathscr{H})$, then

$$||\mathbf{A}_j f - \mathbf{B}f||^2_{\mathscr{H}} = 2||f||^2_{\mathscr{H}} - 2\mathrm{Re}\langle \mathbf{A}_j f, \mathbf{B}f\rangle_{\mathscr{H}}. \tag{3.53}$$

So if $\mathbf{A}_j \to \mathbf{B}$ in the weak topology, it converges in the strong topology (by unitarity of $\mathbf{B}$ and continuity of Re*).*

3.4 Unbounded operators and adjoints

In this section we will review unbounded operators on Hilbert spaces and their adjoints. An unbounded operator is self-adjoint if it and its adjoint not only agree on the domain where agreement can be defined, but also when their domains of definition fully coincide. This distinction is important because self-adjointness is necessary for the spectral theorem of unbounded operators, and for exponentiation.

Definition 3.4.1 (Adjoint operators)**.** *Suppose $\mathbf{A}$ is a (possibly unbounded) operator on a (dense subspace $D_{\mathbf{A}}$ of a) Hilbert space $\mathscr{H}$. Let $D_{\mathbf{A}^*}$ be the set of $g \in \mathscr{H}$ such that there is a $h \in \mathscr{H}$ such that*

$$\langle \mathbf{A}f, g\rangle_{\mathscr{H}} = \langle f, h\rangle_{\mathscr{H}} \quad \text{for all } f \in D_{\mathbf{A}}. \tag{3.54}$$

For each such $g \in D_{\mathbf{A}^*}$ define $\mathbf{A}^* g := h$. Then $\mathbf{A}^*$ (or $(\mathbf{A}^*, D_{\mathbf{A}^*})$) is called *the* adjoint *of* $\mathbf{A}$.

Remark 3.4.2. $(\mathbf{A}^*, D_{\mathbf{A}^*})$ *is defined such that for all* $f \in D_{\mathbf{A}}$, $g \in D_{\mathbf{A}^*}$,

$$\langle \mathbf{A}f, g\rangle_{\mathscr{H}} = \langle f, \mathbf{A}^* g\rangle_{\mathscr{H}} \tag{3.55}$$

holds, and that $D_{\mathbf{A}^*}$ *is the maximal domain in which this can happen.*

Definition 3.4.3 (Symmetric and skew-symmetric operators)**.** *A densely defined operator* $(\mathbf{A}, D_{\mathbf{A}})$ *on a Hilbert space* $\mathscr{H}$ *is* symmetric *(respectively,* skew-symmetric*) if* $D_{\mathbf{A}} \subset D_{\mathbf{A}^*}$ *and*

$$\mathbf{A}f = \mathbf{A}^* f \quad f \in D_{\mathbf{A}}. \tag{3.56}$$

(respectively, if $\mathbf{A}f = -\mathbf{A}^* f$ *for all* $f \in D_{\mathbf{A}}$*.) Equivalently,* $\mathbf{A}$ *is symmetric if and only if*

$$\langle \mathbf{A}f, g\rangle_{\mathscr{H}} = \langle f, \mathbf{A}g\rangle_{\mathscr{H}} \quad \textit{for all } f, g \in D_{\mathbf{A}} \tag{3.57}$$

(respectively, if $\langle \mathbf{A}f, g\rangle_{\mathscr{H}} = -\langle f, \mathbf{A}g\rangle_{\mathscr{H}}$ *for all* $f, g \in D_{\mathbf{A}}$*.)*

Definition 3.4.4 (Self-adjoint and skew-adjoint operators)**.** *A densely defined operator* $(\mathbf{A}, D_{\mathbf{A}})$ *on a Hilbert space* $\mathscr{H}$ *is* self-adjoint *(respectively,* skew-adjoint*) if and only if* $\mathbf{A}$ *is symmetric (respectively, skew-symmetric) and* $D_{\mathbf{A}} = D_{\mathbf{A}^*}$.

Remark 3.4.5. *If* $\mathbf{A}$ *is respectively, symmetric, skew-symmetric, self-adjoint, skew-adjoint, then* $i\mathbf{A}$ *is respectively, skew-symmetric, symmetric, skew-adjoint, self-adjoint.*

Definition 3.4.6 (Closed, closable, closure of an operator)**.** *A densely defined operator* $(\mathbf{A}, D_{\mathbf{A}})$ *on a Hilbert space* $\mathscr{H}$ *is* closed *if its graph* $\{(f, \mathbf{A}f) : f \in D_{\mathbf{A}}\} \subset \mathscr{H} \times \mathscr{H}$ *is closed with respect to the inner product*

$$\langle (f, g), (f', g')\rangle_{\mathscr{H}\times\mathscr{H}} := \langle f, f'\rangle_{\mathscr{H}} + \langle g, g'\rangle_{\mathscr{H}}. \tag{3.58}$$

A densely defined operator $(\mathbf{A}', D_{\mathbf{A}'})$ *is an* extension *of* $(\mathbf{A}, D_{\mathbf{A}})$ *if its graph contains the graph of* $(\mathbf{A}, D_{\mathbf{A}})$*. A densely defined operator* $(\mathbf{A}, D_{\mathbf{A}})$ *is* closable *if it has an extension* $(\mathbf{A}', D_{\mathbf{A}'})$ *that is closed. Every closable densely* *defined operator* $(\mathbf{A}, D_{\mathbf{A}})$ *has a smallest closed extension, which is the* closure $(\overline{\mathbf{A}}, D_{\overline{\mathbf{A}}})$

Remark 3.4.7 (Symmetric operators are closable)**.** $\mathbf{A}$ *is closable if and only if* $D_{\mathbf{A}^*}$ *is dense. If* $\mathbf{A}$ *is closable, its closure is* $\mathbf{A}^{**}$*. (Theorem VIII.1(b) [14]) A symmetric operator defined on a dense domain has* $D_{\mathbf{A}} \subset D_{\mathbf{A}^*}$ *so* $D_{\mathbf{A}^*}$ *is dense. Therefore a symmetric operator is always closable.*

Definition 3.4.8 (Essentially self-adjoint operator)**.** *A symmetric operator* $(\mathbf{A}, D_{\mathbf{A}})$ *is* essentially self-adjoint *if its closure is self-adjoint.*

Remark 3.4.9. *An essentially self-adjoint operator has a unique self-adjoint extension. In general, a symmetric operator may have many different self-adjoint extensions or none. ([14], p256-259)*

3.5 Direct integral decompositions of strongly continuous unitary representations

In this section we review the direct integral decomposition of strongly continuous unitary representations of a group of type I (including the Heisenberg group). The direct integral decomposition is induced in two stages–first, by the direct integral decompositions of (representations of) von-Neumann algebras into factorial representations, and second, when the group is of type I, each factorial representation is a direct sum of irreducible representations. And for strongly continuous unitary representations of the Heisenberg group, this is always possible!

Definition 3.5.1 (Representation)**.** *A* representation of a group G *is a group homomorphism from a group* G *to the general linear group of a vector space* V*. A* representation of a Lie algebra $\mathfrak{g}$ *is a Lie algebra homomorphism from* $\mathfrak{g}$ *to the endomorphism algebra of a vector space* V*, with Lie bracket the commutator bracket.*

Remark 3.5.2 (Necessity of infinite dimensional representations)**.** *Since the Heisenberg group is noncompact, it cannot have any faithful, continuous, finite dimensional unitary representations. If so, then the image of* $H(\omega)$

inside $U(\mathscr{H})$ would be a closed subset of a compact set, and would be compact. The Heisenberg group can be realized as a matrix group, so it can have faithful, continuous, finite dimensional, nonunitary representations.

Over a field of characteristic zero, the Heisenberg Lie algebra cannot have any faithful finite dimensional representations. Suppose V is such a representation. Then consider the trace of the image of the defining relations in $\operatorname{End} V$*:*

$$[a, b] = i\omega(a, b). \tag{3.59}$$

On the left hand side, we get zero, while on the right hand side we get $i\omega(a, b)$ times the dimension of the representation (contradiction).

Definition 3.5.3 (Invariant subspaces)**.** *Let $\pi : G \to U(\mathscr{H})$ be a strongly continuous unitary representation. A closed subspace $V \subset \mathscr{H}$ is an* invariant subspace *if*

$$\pi(g)V \subset V \quad \textit{for all } g \in G. \tag{3.60}$$

Definition 3.5.4 (Irreducible representation)**.** *A representation is* irreducible *if it does not contain any nontrivial closed invariant subspaces.*

Example 3.5.5 (Nonexistence of direct sum decomposition into irreducibles)**.** *Let $\pi : \mathbb{R} \to L^2(\mathbb{R})$ be the regular representation*

$$\pi(t)f(x) := f(x - t) \quad f \in L^2(\mathbb{R}). \tag{3.61}$$

$L^2(\mathbb{R})$ has many closed invariant subspaces (Theorem 9.17 of [26]), of the form

$$\{f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset E\} \tag{3.62}$$

where $\hat{f}$ is the Fourier transform of f, and $E \subset \mathbb{R}$ is measurable. So $L^2(\mathbb{R})$ is not irreducible. Suppose there exists an irreducible subrepresentation $V \subset L^2(\mathbb{R})$. $\mathbb{R}$ is abelian, so V is one-dimensional, and $\pi(t)|_V$ acts by multiplication by scalars. So for $f \in V$, $\pi(t)f(x) = f(x - t) = c_t f(x)$. Since π is unitary, c_t has modulus 1 and $|f(x)|$ is constant on the real line. Then f cannot be square integrable unless it is the zero element. Therefore $L^2(\mathbb{R})$ is not irreducible, but does not contain any irreducible subrepresentations. The direct integral decomposition in this case is the direct integral of *one dimensional representations*

$$\pi_\xi(t)\hat{f}(\xi) := e^{it\xi}\hat{f}(\xi) \tag{3.63}$$

which is nothing but the Fourier transform.

Definition 3.5.6 (Von Neumann algebras)**.** *A* Von Neumann algebra *(or* ring of operators *or* weak star algebra*) $\mathcal{M}$ is a unital, self-adjoint subalgebra of some $\mathcal{B}(\mathscr{H})$, closed under the weak operator topology.*

Definition 3.5.7 (Commutant)**.** *The* commutant *of a subset $\mathcal{S}$ of an associative algebra $\mathcal{A}$ is the set of all bounded operators commuting with all elements of $\mathcal{S}$.*

$$\mathcal{S}' := \{\mathbf{A} \in \mathcal{A} : \mathbf{AS} = \mathbf{SA} \textit{ for all } \mathbf{S} \in \mathcal{S}\}. \tag{3.64}$$

Example 3.5.8. *If $\mathcal{S} = \mathcal{S}^*$ then $\mathcal{S}'$ is a von Neumann algebra. In particular, suppose $\pi : G \to U(\mathscr{H})$ be a strongly continuous unitary representation. Since $\pi(g)^* = \pi(g^{-1})$, $\pi(G)'$ and $\pi(G)''$ are von Neumann algebras.*

Theorem 3.5.9 (Von Neumann's double commutant theorem)**.** *Let* $\mathbf{M}$ *be a unital self adjoint subalgebra of $\mathcal{B}(\mathscr{H})$. The following are equivalent*

- $\mathbf{M} = \mathbf{M}''$.
- $\mathbf{M}$ *is weakly closed.*
- $\mathbf{M}$ *is strongly closed.*

Definition 3.5.10 (Factors and factor representations)**.** *A von Neumann algebra is a* factor *if its center consists of scalar multiples of the identity. A unitary representation $\pi : G \to U(\mathscr{H})$ of a locally compact group is* factorial, *or a* factor representation, *or a* primary representation *if the center of $\pi(G)''$ consists of only scalar multiplications of the identity.*

Remark 3.5.11 (Factor representations vs. irreducible representations)**.** *According to the Schur lemma, a (strongly continuous) unitary representation is irreducible if and only if every automorphism (isometric intertwining* *operator) is a scalar multiple of the identity. This is equivalent to the condition that the centralizer* $\pi(G)'$ *consists of scalar multiples of the identity. The center of* $\pi(G)''$ *is* $\pi(G)'' \cap \pi(G)'$*, and a strongly continuous unitary representation is a factor representation if and only if* $\pi(G)'' \cap \pi(G)'$ *consists of scalar multiples of the identity. So every irreducible representation is a factor representation, but a factor representation may not be irreducible. This happens when there are projections to closed invariant subspaces in* $\pi(G)'$ *that are not in* $\pi(G)'' \cap \pi(G)'$*. A factor representation can be a countable multiple of an irreducible representation (type I), or it may not even contain any irreducible subrepresentations at all (types II and III). Every finite dimensional factor representation is of type I.*

Theorem 3.5.12 (Direct integral decomposition of Von Neumann algebras into factors, Theorem VII of [27])**.** *Every von Neumann algebra is unitarily equivalent to a direct integral of factors.*

Remark 3.5.13. *Because of the length involved, we refer the interested reader to the excellent texts [25] [11] for the precise definition of direct integral decomposition of representations and its uniqueness.*

Theorem 3.5.14 (Direct integral decomposition of unitary representations cf. Theorem 7.29 of [28])**.** *Suppose* π *is a strongly continuous unitary representation of a separable locally compact group* G *on a Hilbert space* $\mathscr{H}$*. For every commutative von Neumann subalgebra* $\mathcal{A}$ *in the center of* $\pi(G)''$*, there exists a direct integral decomposition of* π*. If* $\mathcal{A}$ *is the center, then almost every direct integrand is an irreducible representation.*

Definition 3.5.15 (Type I factors and groups of type I)**.** *A factor is* type I *if it is unitarily equivalent to a countable direct sum of copies of a single irreducible representation. A group is* type I *if all its primary representations are unitarily equivalent to countable direct sums of copies of a single irreducible representation.*

Theorem 3.5.16 (cf. Theorem 4.1 in [29])**.** *Every nilpotent group is a group of type I.*

Theorem 3.5.17 (Stone-von Neumann)**.** *Any irreducible infinite dimensional unitary representation of the Heisenberg group, and any integrable* *infinite dimensional irreducible representation of the Heisenberg Lie algebra is unitarily equivalent to a Schrödinger representation.*

3.6 Differentiation and exponentiation of representations on Hilbert spaces

In this section, we will review that we can differentiate a (strongly) continuous, unitary representation of a Lie group on a Hilbert space, and when we can exponentiate a representation of a Lie algebra defined by skew-adjoint operators on a dense subspace of a Hilbert space.

Stone's theorem is the earliest result that tells us when we can differentiate and exponentiate, for a one parameter unitary group.

Theorem 3.6.1 (Stone [18])**.** *Let $\pi : \mathbb{R} \to U(\mathscr{H})$ be a strongly continuous one parameter unitary group. Then there exists a unique (possibly unbounded) self-adjoint operator $\mathbf{A}$ defined on a dense subspace $D_{\mathbf{A}} \subset \mathscr{H}$ such that*

$$\pi(t) = e^{it\mathbf{A}} \tag{3.65}$$

where

$$D_{\mathbf{A}} := \left\{ f \in \mathscr{H} : \lim_{t\to 0} \frac{1}{t}\left(\pi(t)f - f\right) \ \textit{exists} \right\}. \tag{3.66}$$

$f \in D_{\mathbf{A}}$ is equivalent to the condition that $t \mapsto \pi(t)f$ is differentiable.

Conversely, let $\mathbf{A} : D_{\mathbf{A}} \to \mathscr{H}$ be a (possibly unbounded) self-adjoint operator. Then the one-parameter family $a(t) := e^{it\mathbf{A}}$ is a strongly continuous one-parameter unitary group.

The following two theorems tell us we can differentiate a (strongly) continuous unitary representation of a Lie group and obtain a representation of its Lie algebra on a dense subspace:

Theorem 3.6.2 (Gårding [19])**.** *Suppose G is a Lie group and $\pi : G \to U(\mathscr{H})$ is a strongly continuous unitary representation. Then if $d\mu_G$ is the left Haar measure on G, the* Gårding domain

$$D_{\mathfrak{g}} := \left\{ \int_G h(g)\pi(g) f \, d\mu_G(g) : f \in \mathscr{H}, h \in C_c^\infty(G) \right\} \tag{3.67}$$

is dense in $\mathscr{H}$. *Moreover,*

$$d\pi(a(t))f := \lim_{t\to 0} \frac{1}{t}(\pi(a(t)) - 1_{\mathcal{B}(\mathscr{H})})f \tag{3.68}$$

exists for all $f \in D_{\mathfrak{g}}$ *and* $d\pi D_{\mathfrak{g}} \subset D_{\mathfrak{g}}$.

Theorem 3.6.3 (Segal-Mautner(Lemma 5.1 and 5.2 of [21]))**.** *Suppose* G *is a connected Lie group and* $\pi : G \to U(\mathscr{H})$ *is a strongly continuous unitary representation. Then* $d\pi$ *is a Lie algebra homomorphism*

$$d\pi : \mathfrak{g} \to \mathrm{End}(D_{\mathfrak{g}}). \tag{3.69}$$

The following theorems tell us when we can exponentiate a representation of the Lie algebra by skew-symmetric operators.

Definition 3.6.4 (Smooth and analytic vectors)**.** *Let* $\pi : G \to U(\mathscr{H})$ *be a strongly continuous unitary representation. Then* f *is a* smooth vector *(respectively,* analytic vector*) if*

$$g \mapsto \pi(g)f \tag{3.70}$$

is smooth (respectively, analytic). The set of smooth (respectively, analytic) vectors are denoted by $\mathscr{H}^\infty$ *(respectively,* $\mathscr{H}^{an}$*). If* $\rho : \mathfrak{g} \to \mathrm{End}(\mathcal{D})$ *is a Lie algebra representation on a dense subspace of a Hilbert space* $\mathscr{H}$*, a vector* $f \in \mathcal{D}$ *is* analytic *if there is a positive* ϵ *such that*

$$\sum_{k=0}^{\infty} \frac{||\rho(g)f||_{\mathscr{H}}^k \epsilon^k}{k!} < \infty \tag{3.71}$$

Theorem 3.6.5 (Gårding, Nelson (Theorem 3 of [22]), Cartier-Dixmier)**.**

$$\mathscr{H}^{an} \subset D_{\mathfrak{g}} \subset \mathscr{H}^\infty \tag{3.72}$$

and $\mathscr{H}^{an}$ *is dense in* $\mathscr{H}$.

Definition 3.6.6 (Nelson and Flato-Simon-Snellman-Sternheimer conditions)**.** *Suppose* $\mathcal{D}$ *is a dense subspace of a Hilbert space* $\mathscr{H}$ *and* $\rho : \mathfrak{g} \to \mathrm{End}(\mathcal{D})$ *is* *a representation of a real Lie algebra* $\mathfrak{g}$ *by skew-symmetric operators on* $\mathcal{H}$. *Let* $\{a_1, \cdots, a_d\}$ *is a basis of a real Lie algebra* $\mathfrak{g}$.

ρ *satisfies the* Nelson condition *if* $\Delta := \rho(a_1)^2 + \cdots \rho(a_d)^2$ *is essentially self adjoint.* ρ *satisfies the* Flato-Simon-Snellman-Sternheimer condition *if* $\mathcal{D} = \cap_{k=1}^{d} \mathcal{D}_k$, *where* $\mathcal{D}_k$ *is the set of analytic vectors of* $\rho(a_k)$.

Theorem 3.6.7 (Theorem 1 of [23], Theorem 5 of [22])**.** *If* ρ *satisfies either of these conditions, then there exists a unique strongly continuous unitary representation* $\pi : G \to U(\mathscr{H})$ *such that (on their domains of definition),*

$$\operatorname{s}\lim_{t\to 0} \frac{1}{t}(\pi(e^{ta}) - 1_{\mathcal{B}(\mathscr{H})}) = \rho(a). \tag{3.73}$$

One of the conditions necessary to exponentiate is also necessary to obtain a direct *sum* decomposition (via the spectral theorem) of the representation of the Heisenberg Lie algebra.

Theorem 3.6.8 (Dixmier-Rellich cf. [39])**.** *Suppose* $\mathcal{D}$ *is a dense subspace of a Hilbert space* $\mathscr{H}$ *and* $\rho : \mathfrak{heis}\,\omega \to \operatorname{End}(\mathcal{D})$ *is a Lie algebra homomorphism by closed skew-symmetric operators satisfying Nelson's condition. Then* $\mathscr{H}$ *is unitarily equivalent to a direct sum of Schrödinger representations.*

Chapter 4

Construction of representations

In this chapter we will construct representations of the Heisenberg group and Lie algebra parametrized by transverse pairs of complex Lagrangian subspaces. There are four situations that are considered, depending on whether we are looking at representations of the Heisenberg group or Lie algebra, and depending on whether we are looking at representations on vector spaces or on Hilbert spaces. The words *isomorphic*, *irreducible*, *new* representations are different according to each situation.

Let m_{Leb} is the pullback of the Lebesgue measure on $\mathbb{R}^{2n}$ to (V,ω) and λ is a real number. Our claims are the following:

1. Representations of the Heisenberg group on the *vector space* of (Γ_1,Γ_2)-*analytic functions*: There exists a vector space $\mathcal{O}_{\Gamma_1,\Gamma_2}(V) \subset C^\infty(V;\mathbb{C})$ and a group homomorphism

$$T_{\cdot}^{\Gamma_1,\Gamma_2,\lambda} : H(\omega) \to \mathrm{GL}(\mathcal{O}_{\Gamma_1,\Gamma_2}(V)) \tag{4.1}$$

   for an *arbitrary transverse pair* of complex Lagrangian subspaces (Γ_1,Γ_2). Precomposition by $S \in \mathrm{Sp}(V,\omega)$ intertwines $T_{S^{-1}\cdot}^{\Gamma_1,\Gamma_2}$ and $T_{\cdot}^{S\Gamma_1,S\Gamma_2}$.

2. Representations of the Heisenberg Lie algebra on the *vector space* of *polynomial functions*: There exists a Lie algebra homomorphism

$$\dot{T}_{\cdot}^{\Gamma_1,\Gamma_2,\lambda} : \mathfrak{heis}\ \omega \to \mathrm{End}(\mathbb{C}[z_{\Gamma_1,\Gamma_2}]) \tag{4.2}$$

for an *arbitrary transverse pair* of complex Lagrangian subspaces (Γ_1, Γ_2). $\mathbb{C}\,[z_{\Gamma_1,\Gamma_2}]$ is a simple $\mathfrak{U}(\mathfrak{heis}\,\omega)^{\mathbb{C}}$ module. They are all isomorphic.

3. *Strongly continuous unitary representations* of the Heisenberg group on *Hilbert spaces*: Let $\mathfrak{F}^{\lambda}_{\Gamma_1,\Gamma_2}(V)$ be the L^2 completion of

$$\mathcal{O}_{\Gamma_1,\Gamma_2}(V) \cap L^2(V; dm_{\Gamma_1,\Gamma_2}). \tag{4.3}$$

Then

$$T^{\Gamma_1,\Gamma_2,\lambda}_{\cdot} : H(\omega) \to U(\mathfrak{F}^{\lambda}_{\Gamma_1,\Gamma_2}(V)) \tag{4.4}$$

s a strongly continuous group homomorphism. There are (Γ_1, Γ_2) such that $\mathfrak{F}^{\lambda}_{\Gamma_1,\Gamma_2}(V)$ is the zero vector space, and unitarity holds vacuously. When (Γ_1, Γ_2) satisfy a positivity condition, these representations contain $\mathbb{C}[z_{\Gamma_1,\Gamma_2}]$ as a dense subspace, and are isomorphic to a Schrödinger representation by the Stone-von Neumann theorem (hence irreducible). There are new parameters that construct unitarily equivalent representations.

4. Representations of the Heisenberg Lie algebra on *Hilbert spaces by unbounded operators*: Let $\mathcal{U} \subset V$ be an open subset of V. Then we can restrict the relevant objects to $\mathcal{U}$. Then

$$\dot{T}^{\Gamma_1,\Gamma_2,\lambda}_{\cdot} : \mathfrak{heis}\,\omega \to \mathrm{End}(\mathbb{C}[z_{\Gamma_1,\Gamma_2}|_{\mathcal{U}}]). \tag{4.5}$$

is a Lie algebra homomorphism.

Let $\mathfrak{F}^{\lambda}_{\Gamma_1,\Gamma_2}(\mathcal{U})$ be the L^2 completion of

$$\mathcal{O}_{\Gamma_1,\Gamma_2}(\mathcal{U}) \cap L^2(\mathcal{U}; dm_{\Gamma_1,\Gamma_2}|_{\mathcal{U}}). \tag{4.6}$$

If $\mathcal{U}$ is bounded, $\mathbb{C}[z_{\Gamma_1,\Gamma_2}|_{\mathcal{U}}]$ is a dense subspace of $\mathfrak{F}^{\lambda}_{\Gamma_1,\Gamma_2}(\mathcal{U})$, and $\dot{T}^{\Gamma_1,\Gamma_2}_{\cdot}$ is defined for an *arbitrary transverse pair* of complex Lagrangian subspaces (Γ_1, Γ_2). If (Γ_1, Γ_2) satisfy the positivity condition, $\dot{T}^{\Gamma_1,\Gamma_2}_{\cdot}$ is a representation of the Heisenberg Lie algebra by skew-adjoint operators. The author does not know whether there is an established notion of irreducibility for these kinds of representations.

4.1 Transverse pairs of complex Lagrangian subspaces

In this section we will show that the space of transverse pairs of complex Lagrangian subspaces of (V, ω) can be identified with the homogeneous space $\mathrm{Sp}(2n;\mathbb{C})/\mathrm{GL}(n;\mathbb{C})$ using a Darboux basis $\{\mathbf{e}, \mathbf{f}\}$. This identification will be used in the sequel as a coordinate description of the space of transverse pairs of complex Lagrangian subspaces.

Proposition 4.1.1. *Let* (Γ_1, Γ_2) *be a transverse pair of complex Lagrangian subspaces. Then there exists a Darboux basis* $\{\mathbf{e}', \mathbf{f}'\}$ *of* $V^{\mathbb{C}}$ *such that* $\mathbf{e}'_j \in \Gamma_1$*, and* $\mathbf{f}'_j \in \Gamma_2$ *for* $j = 1, \cdots, n$*.*

Proof. Let $\{\mathbf{e}'_1, \cdots, \mathbf{e}'_n\}$ be any basis of Γ_1, and $\{\mathbf{f}''_1, \cdots, \mathbf{f}''_n\}$ be any basis of Γ_2. Then let X be an $n \times n$ matrix with components $(X)_{kj} := \omega^{\mathbb{C}}(\mathbf{e}_j, \mathbf{f}''_k)$. By transversality of the Lagrangian subspaces and nondegeneracy of $\omega^{\mathbb{C}}$, X is an invertible matrix. Let

$$\mathbf{f}'_\ell := \sum_{k=1}^{n} (X^{-1})_{\ell k} \mathbf{f}''_k \quad \ell = 1, \cdots, n \tag{4.7}$$

Then $\mathbf{f}'_\ell \in \Gamma_2$ for $\ell = 1, \cdots, n$, and

$$\omega^{\mathbb{C}}(\mathbf{e}'_j, \mathbf{f}'_\ell) = \omega^{\mathbb{C}} \left(\mathbf{e}'_j, \sum_{k=1}^{n} (X^{-1})_{\ell k} \mathbf{f}''_k \right) = \sum_{k=1}^{n} (X^{-1})_{\ell k} X_{kj} = \delta_{\ell j}, \tag{4.8}$$

so $\{\mathbf{e}', \mathbf{f}'\}$ is the desired Darboux basis. $\square$

Example 4.1.2 (Transverse pairs of complex Lagrangian subspaces associated to matrices in $\mathrm{Sp}(2n;\mathbb{C})$)**.** *Let* $\{\mathbf{e}, \mathbf{f}\}$ *be a fixed Darboux basis of* (V, ω)*. Given a complex symplectic matrix*

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n;\mathbb{C}), \tag{4.9}$$

consider the complex span of n *vectors with coefficients given by the* n *first column vectors of* S *and the complex span of the* n *vectors with coefficients* *given by the n last column vectors of S:*

$$\Gamma_1 \quad := \quad \mathrm{Span}_{\mathbb{C}} \left\{ \sum_{j=1}^{n} (A_{jk}\mathbf{e}_j + C_{jk}\mathbf{f}_j) \right\}_{k=1}^{n} \tag{4.10}$$

$$\Gamma_2 \quad := \quad \mathrm{Span}_{\mathbb{C}} \left\{ \sum_{j=1}^{n} (B_{jk}\mathbf{e}_j + D_{jk}\mathbf{f}_j) \right\}_{k=1}^{n}. \tag{4.11}$$

Since

$$\begin{pmatrix} A^t & C^t \end{pmatrix} \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} B^t & D^t \end{pmatrix} \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \begin{pmatrix} B \\ D \end{pmatrix} = 0 \tag{4.12}$$

and S is of maximal rank, (Γ_1, Γ_2) is a transverse pair of complex Lagrangian subspaces. Conversely, given a pair of transverse Lagrangian subspaces (Γ_1, Γ_2), one can find a complex Darboux basis $\{\mathbf{e}', \mathbf{f}'\}$ of $V^{\mathbb{C}}$ such that

$$\Gamma_1 = \mathrm{Span}_{\mathbb{C}}\{\mathbf{e}_j\}_{j=1}^{n} \quad \Gamma_2 = \mathrm{Span}_{\mathbb{C}}\{\mathbf{f}_j\}_{j=1}^{n}. \tag{4.13}$$

The matrix form of the complex symplectic linear transformation taking $\{\mathbf{e}, \mathbf{f}\}$ to $\{\mathbf{e}', \mathbf{f}'\}$ in $\{\mathbf{e}, \mathbf{f}\}$ basis is a complex symplectic matrix.

Remark 4.1.3 (Equivalence classes of complex symplectic matrices)**.** *Suppose $S, S' \in \mathrm{Sp}(2n;\mathbb{C})$ such that there exists an $n \times n$ invertible matrix X such that*

$$S \begin{pmatrix} X & 0 \\ 0 & (X^t)^{-1} \end{pmatrix} = S'. \tag{4.14}$$

Then the transverse pair defined by S and S' are equal. Conversely, if the transverse pair defined by two complex symplectic matrices S and S' are equal, there exists an $n \times n$ invertible matrix X such that Equation 4.14 holds.

Therefore, a fixed Darboux basis of (V, ω) gives a diffeomorphism between the space of transverse pairs of complex Lagrangian subspaces of (V, ω) with the homogeneous space $\mathrm{Sp}(2n;\mathbb{C})/\mathrm{GL}(n;\mathbb{C})$.

4.2 Bilinear forms

In this section we will introduce complex bilinear forms $(\cdot|\cdot)_{\Gamma_1,\Gamma_2}$ on $V^{\mathbb{C}}$ associated to each transverse pair of complex Lagrangian subspaces (Γ_1, Γ_2). We will extensively use these forms in the rest of the chapter. We will describe them in coordinate form and use this form in the sequel, but we will also review a coordinate invariant description suggested by Y. Karshon.

Definition 4.2.1 (Bilinear forms associated with transverse pairs of Lagrangian subspaces)**.** *Suppose* $\{\mathbf{e},\mathbf{f}\}$ *is a Darboux basis of* (V,ω)*, and a transverse pair of complex Lagrangian subspaces* (Γ_1,Γ_2) *is given by the column vectors of a complex symplectic matrix*

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{4.15}$$

Let $z_{\Gamma_1,\Gamma_2}, \zeta_{\Gamma_1,\Gamma_2} \in \mathrm{Hom}_{\mathbb{C}}(V^{\mathbb{C}};\mathbb{C}^n)$ *be defined as*

$$z_{\Gamma_1,\Gamma_2}(v) \;:=\; \begin{pmatrix} D^t & -B^t \end{pmatrix}(v)_{\{\mathbf{e},\mathbf{f}\}} \quad v \in V^{\mathbb{C}} \tag{4.16}$$

$$\zeta_{\Gamma_1,\Gamma_2}(v) \;:=\; \begin{pmatrix} -C^t & A^t \end{pmatrix}(v)_{\{\mathbf{e},\mathbf{f}\}} \quad v \in V^{\mathbb{C}}. \tag{4.17}$$

$$\mathrm{Hom}_{\mathbb{C}}(V^{\mathbb{C}};\mathbb{C}^n) \cong \mathrm{Hom}_{\mathbb{R}}(V;\mathbb{C}^n) \tag{4.18}$$

as both real and complex vector spaces, we will denote also by $z_{\Gamma_1,\Gamma_2}, \zeta_{\Gamma_1,\Gamma_2}$ *the corresponding elements of* $\mathrm{Hom}_{\mathbb{R}}(V;\mathbb{C}^n)$*.*

Define a $\mathbb{C}$*-bilinear form on* $V^{\mathbb{C}}$ *as*

$$(u|v)_{\Gamma_1,\Gamma_2} := -\frac{i}{2} z_{\Gamma_1,\Gamma_2}(v)^t \zeta_{\Gamma_1,\Gamma_2}(u) \quad u, v \in V^{\mathbb{C}} \tag{4.19}$$

and define a complex valued, $\mathbb{R}$*-bilinear hermitian form on* V *as*

$$h(u,v)_{\Gamma_1,\Gamma_2} := (u|v)_{\Gamma_1,\Gamma_2} - (v|u)_{\Gamma_1,\Gamma_2} \quad u, v \in V. \tag{4.20}$$

Denote the Gram matrix of $(\cdot|\cdot)_{\Gamma_1,\Gamma_2}$ *as*

$$M_{\Gamma_1,\Gamma_2} = -\frac{i}{2} \begin{pmatrix} D \\ -B \end{pmatrix} \begin{pmatrix} -C^t & A^t \end{pmatrix} \tag{4.21}$$

and the Gram matrix of $h(\cdot,\cdot)_{\Gamma_1,\Gamma_2}$ as

$$h_{\Gamma_1,\Gamma_2} = M_{\Gamma_1,\Gamma_2} + M^{\dagger}_{\Gamma_1,\Gamma_2}. \tag{4.22}$$

Remark 4.2.2 (Well-definedness of $(\cdot|\cdot)_{\Gamma_1,\Gamma_2}$ and $h(\cdot,\cdot)_{\Gamma_1,\Gamma_2}$)**.** *Suppose the complex symplectic matrices S and S' correspond to the same transverse pair (Γ_1,Γ_2). Then there exists an invertible $n\times n$ matrix X such that*

$$S\begin{pmatrix} X & 0 \\ 0 & (X^t)^{-1}\end{pmatrix} = S'. \tag{4.23}$$

Then the Gram matrices are

$$\begin{aligned}
M'_{\Gamma_1,\Gamma_2} &= -\frac{i}{2}\begin{pmatrix} D' \\ -B'\end{pmatrix}\begin{pmatrix}-(C')^t & (A')^t\end{pmatrix} && (4.24)\\
&= -\frac{i}{2}\begin{pmatrix} D \\ -B\end{pmatrix}(X^t)^{-1}X\begin{pmatrix}-C^t & A^t\end{pmatrix} && (4.25)\\
&= M_{\Gamma_1,\Gamma_2} && (4.26)
\end{aligned}$$

and the same argument applies for the conjugate term.

Remark 4.2.3 (Basic properties of $(\cdot|\cdot)_{\Gamma_1,\Gamma_2}$)**.** *Since*

$$\begin{pmatrix} D^t & -B^t\end{pmatrix}\begin{pmatrix} B \\ D\end{pmatrix} = \begin{pmatrix}-C^t & A^t\end{pmatrix}\begin{pmatrix} A \\ C\end{pmatrix} = 0 \tag{4.27}$$

$z_{\Gamma_1,\Gamma_2}|_{\Gamma_2} = 0$ *and* $\zeta_{\Gamma_1,\Gamma_2}|_{\Gamma_1} = 0$.

If $a \in V^{\mathbb{C}}$ is fixed, then $(a|v)_{\Gamma_1,\Gamma_2}$ is a complex linear combination of the components of $z_{\Gamma_1,\Gamma_2}(v)$, and $(v|a)_{\Gamma_1,\Gamma_2}$ is a complex linear combination of the components of $\zeta_{\Gamma_1,\Gamma_2}(v)$.

Thus

$$\begin{aligned}
(u+a|v)_{\Gamma_1,\Gamma_2} &= (u|v)_{\Gamma_1,\Gamma_2} \quad a\in\Gamma_1 && (4.28)\\
(u|v+a)_{\Gamma_1,\Gamma_2} &= (u|v)_{\Gamma_1,\Gamma_2} \quad a\in\Gamma_2. && (4.29)
\end{aligned}$$

Finally, if $S\in \mathrm{Sp}(V^{\mathbb{C}},\omega^{\mathbb{C}})$, and acts on (Γ_1,Γ_2) from the left as $(S\Gamma_1, S\Gamma_2)$, *then*

$$(S^{-1}u|S^{-1}v)_{\Gamma_1,\Gamma_2} = (u|v)_{S\Gamma_1,S\Gamma_2}. \tag{4.30}$$

Theorem 4.2.4. *For all transverse pairs of complex Lagrangian subspaces* (Γ_1, Γ_2),

$$(u|v)_{\Gamma_1,\Gamma_2} - (v|u)_{\Gamma_1,\Gamma_2} = \frac{i}{2}\omega^{\mathbb{C}}(u,v) \quad u,v \in V^{\mathbb{C}}. \tag{4.31}$$

Proof. $\mathrm{Sp}(2n;\mathbb{C})$ is closed under matrix transpose. Equivalently, the block components of a complex symplectic matrix satisfy

$$AB^t = BA^t, \quad CD^t = DC^t \quad \text{and} \quad DA^t - CB^t = 1. \tag{4.32}$$

We can check

$$M_{\Gamma_1,\Gamma_2} - M^t_{\Gamma_1,\Gamma_2} = \frac{1}{2}\begin{pmatrix} iDC^t - iCD^t & -iDA^t + iCB^t \\ -iBC^t + iAD^t & iBA^t - iAB^t \end{pmatrix} = \frac{i}{2}\begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}.$$

If $(u)_{\{\mathbf{e}_0,\mathbf{f}_0\}} = a$ and $(v)_{\{\mathbf{e}_0,\mathbf{f}_0\}} = b$, then

$$\begin{aligned} (u|v)_{\Gamma_1,\Gamma_2} - (v|u)_{\Gamma_1,\Gamma_2} &= b^t M_{\Gamma_1,\Gamma_2} a - a^t M_{\Gamma_1,\Gamma_2} b && (4.33)\\ &= b^t (M_{\Gamma_1,\Gamma_2} - M^t_{\Gamma_1,\Gamma_2}) a && (4.34)\\ &= \frac{i}{2}\omega(u,v). && (4.35) \end{aligned}$$

□

Remark 4.2.5 (Coordinate invariant form, suggested by Y. Karshon)**.** *Since* (Γ_1, Γ_2) *are transverse,* $V^{\mathbb{C}} = \Gamma_1 \oplus \Gamma_2$. *Let* $\mathrm{pr}_1^{\Gamma_1,\Gamma_2}$ *be the projection from* $V^{\mathbb{C}}$ *to* Γ_1 along Γ_2 *and* $\mathrm{pr}_2^{\Gamma_1,\Gamma_2}$ *be the projection from* $V^{\mathbb{C}}$ *to* Γ_2 along Γ_1*.*

Then

$$\begin{aligned} (\mathrm{pr}_1^{\Gamma_1,\Gamma_2})_{\{\mathbf{e},\mathbf{f}\}} &= \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} && (4.36)\\ (\mathrm{pr}_2^{\Gamma_1,\Gamma_2})_{\{\mathbf{e},\mathbf{f}\}} &= \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1_n \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}. && (4.37) \end{aligned}$$

Since for symplectic matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix} \tag{4.38}$$

we can compute $(\omega^{\mathbb{C}}(\mathrm{pr}_2^{\Gamma_1,\Gamma_2}\cdot, \mathrm{pr}_1^{\Gamma_1,\Gamma_2}\cdot))_{\{\mathbf{e},\mathbf{f}\}}$ *as*

$$\begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix} \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} D^t & -C^t \\ -B^t & A^t \end{pmatrix}$$

$$= -\begin{pmatrix} D \\ -B \end{pmatrix} \begin{pmatrix} -C^t & A^t \end{pmatrix} = -2iM_{\Gamma_1,\Gamma_2} \tag{4.39}$$

So

$$(u|v)_{\Gamma_1,\Gamma_2} = \frac{i}{2}\omega^{\mathbb{C}}(\mathrm{pr}_2^{\Gamma_1,\Gamma_2}u, \mathrm{pr}_1^{\Gamma_1,\Gamma_2}v) = -\frac{i}{2}\omega^{\mathbb{C}}(\mathrm{pr}_1^{\Gamma_1,\Gamma_2}v, \mathrm{pr}_2^{\Gamma_1,\Gamma_2}u) \tag{4.40}$$

and the proof of Theorem 4.2.4 can be stated as

$$\frac{i}{2}\omega^{\mathbb{C}}(u_2, v_1) - \frac{i}{2}\omega^{\mathbb{C}}(v_2, u_1) = \frac{i}{2}\omega^{\mathbb{C}}(u_1 + u_2, v_1 + v_2) \tag{4.41}$$

when $u = u_1 + u_2$, $v = v_1 + v_2$ *according to the splitting given by* $V^{\mathbb{C}} = \Gamma_1 \oplus \Gamma_2$.

4.3 Construction of representations

In this section, we will construct the representations of the Heisenberg group on the space of (Γ_1, Γ_2)-analytic functions, and the irreducible representations of the Heisenberg Lie algebra on the space of polynomials $\mathbb{C}[z_{\Gamma_1,\Gamma_2}]$, for arbitrary transverse pairs of complex Lagrangian subspaces (Γ_1, Γ_2). The representations in this sections will not assume any structure on the function spaces other than the vector space structure, and the representations themselves are only assumed to preserve the group composition and Lie bracket structures. We will also state some results about how the representations are intertwined by real and complex symplectic linear transformations.

For $a \in V$, let $\dot{T}_a^{\Gamma_1,\Gamma_2,\lambda} \in \operatorname{End} C^\infty(V;\mathbb{C})$ be defined by

$$\dot{T}_a^{\Gamma_1,\Gamma_2,\lambda} f(v) := \left(d_a - 2\lambda(a|v)_{\Gamma_1,\Gamma_2}\right) f(v). \tag{4.42}$$

Then we can check that for $a, b \in V$,

$$\begin{aligned}\left[\dot{T}_a^{\Gamma_1,\Gamma_2,\lambda}, \dot{T}_b^{\Gamma_1,\Gamma_2,\lambda}\right] &= 2\lambda(a|b)_{\Gamma_1,\Gamma_2} - 2\lambda(b|a)_{\Gamma_1,\Gamma_2} &(4.43)\\ &= i\lambda\omega(a,b). &(4.44)\end{aligned}$$

We can extend $\dot{T}_\cdot^{\Gamma_1,\Gamma_2,\lambda}$ from V to $\mathfrak{heis}\ \omega$ by letting

$$\dot{T}_{is+a}^{\Gamma_1,\Gamma_2,\lambda} f(v) := \left(d_a + i\lambda s - 2\lambda(a|v)_{\Gamma_1,\Gamma_2}\right) f(v). \tag{4.45}$$

So $\dot{T}_\cdot^{\Gamma_1,\Gamma_2,\lambda}$ is a Lie algebra homomorphism $\mathfrak{heis}\ \omega \to \operatorname{End} C^\infty(V;\mathbb{C})$.

Theorem 4.3.1. *Let $\mathbb{C}[z_{\Gamma_1,\Gamma_2}]$ denote the polynomial algebra generated by the components of z_{Γ_1,Γ_2}. $\mathbb{C}[z_{\Gamma_1,\Gamma_2}]$ is a simple $\mathfrak{U}(\mathfrak{heis}\ \omega)^{\mathbb{C}}$-module.*

Proof. If $\alpha \in \Gamma_1$, then

$$\begin{aligned}\dot{T}_{\operatorname{Re}\alpha}^{\Gamma_1,\Gamma_2,\lambda} + i\dot{T}_{\operatorname{Im}\alpha}^{\Gamma_1,\Gamma_2,\lambda} &= d_{\operatorname{Re}\alpha} + id_{\operatorname{Im}\alpha} - 2\lambda(\alpha|v)_{\Gamma_1,\Gamma_2}\\ &= d_{\operatorname{Re}\alpha} + id_{\operatorname{Im}\alpha}.\end{aligned}$$

The multiplication term vanishes because for $\alpha \in \Gamma_1$

$$(\alpha|\cdot)_{\Gamma_1,\Gamma_2} = 0. \tag{4.46}$$

If $\beta \in \Gamma_2$, then

$$\begin{aligned}\dot{T}_{\operatorname{Re}\beta}^{\Gamma_1,\Gamma_2,\lambda} + i\dot{T}_{\operatorname{Im}\beta}^{\Gamma_1,\Gamma_2,\lambda} &= d_{\operatorname{Re}\beta} + id_{\operatorname{Im}\beta} - 2\lambda(\beta|v)_{\Gamma_1,\Gamma_2}\\ &= -2\lambda(\beta|v)_{\Gamma_1,\Gamma_2}.\end{aligned}$$

The differentiation term vanishes because z_{Γ_1,Γ_2} is constant in the "Γ_2-direction."

$$(d_{\operatorname{Re}\beta} + id_{\operatorname{Im}\beta})z_{\Gamma_1,\Gamma_2}(v) = \lim_{t\to 0}\frac{1}{t}\left(z_{\Gamma_1,\Gamma_2}(v+\beta) - z_{\Gamma_1,\Gamma_2}(v)\right) = 0. \tag{4.47}$$

In fact, if α (respectively, β) is given by the jth (respectively, $n+j$th) column vector of the complex symplectic matrix S associated to (Γ_1, Γ_2) via a Darboux basis $\{\mathbf{e}, \mathbf{f}\}$, then

$$\begin{aligned} \dot{T}^{\Gamma_1,\Gamma_2,\lambda}_{\operatorname{Re}\alpha} + i\dot{T}^{\Gamma_1,\Gamma_2,\lambda}_{\operatorname{Im}\alpha} &= \frac{\partial}{\partial z^j_{\Gamma_1,\Gamma_2}} && (4.48)\\ \dot{T}^{\Gamma_1,\Gamma_2,\lambda}_{\operatorname{Re}\beta} + i\dot{T}^{\Gamma_1,\Gamma_2,\lambda}_{\operatorname{Im}\beta} &= -2\lambda z^j_{\Gamma_1,\Gamma_2}. && (4.49)\end{aligned}$$

Then every cyclic submodule of $\mathbb{C}[z_{\Gamma_1,\Gamma_2}]$ generated by a nonzero element is $\mathbb{C}[z_{\Gamma_1,\Gamma_2}]$. So $\mathbb{C}[z_{\Gamma_1,\Gamma_2}]$ is a simple $\mathfrak{U}(\mathfrak{heis}\ \omega)^{\mathbb{C}}$ module. $\square$

Proposition 4.3.2. *Suppose $S \in \mathrm{Sp}(V^{\mathbb{C}}, \omega^{\mathbb{C}})$ and denote the precomposition operator*

$$\rho(S)f(z_{\Gamma_1,\Gamma_2}(v)) := f(z_{\Gamma_1,\Gamma_2}(S^{-1}v)) \quad f \in \mathcal{O}_{\Gamma_1,\Gamma_2}(V). \tag{4.50}$$

Then $\rho(S)$ intertwines the representations $\dot{T}^{\Gamma_1,\Gamma_2,\lambda}_{S^{-1}\cdot}$ with $\dot{T}^{\Gamma_1,\Gamma_2,\lambda}_{\cdot}$ in the following sense. For all $\alpha \in V^{\mathbb{C}}$

$$(\dot{T}^{S\Gamma_1,S\Gamma_2,\lambda}_{\operatorname{Re}\alpha} + i\dot{T}^{S\Gamma_1,S\Gamma_2,\lambda}_{\operatorname{Im}\alpha}) \cdot \rho(S) = \rho(S) \cdot (\dot{T}^{\Gamma_1,\Gamma_2,\lambda}_{\operatorname{Re}S^{-1}\alpha} + i\dot{T}^{\Gamma_1,\Gamma_2,\lambda}_{\operatorname{Im}S^{-1}\alpha}) \tag{4.51}$$

in $\operatorname{End}\mathbb{C}[z_{\Gamma_1,\Gamma_2}]$.

Proof. We use the identity

$$(S^{-1}u|S^{-1}v)_{\Gamma_1,\Gamma_2} = (u|v)_{S\Gamma_1,S\Gamma_2} \tag{4.52}$$

and compute directly:

$$\begin{aligned} & \rho(S) \cdot (\dot{T}^{\Gamma_1,\Gamma_2,\lambda}_{\operatorname{Re}S^{-1}\alpha} + i\dot{T}^{\Gamma_1,\Gamma_2,\lambda}_{\operatorname{Im}S^{-1}a})f(z_{\Gamma_1,\Gamma_2}(v))\\ = \;& \rho(S) \cdot \left(d_{\operatorname{Re}S^{-1}\alpha} + id_{\operatorname{Im}S^{-1}\alpha} - 2\lambda(S^{-1}\alpha|v)_{\Gamma_1,\Gamma_2}\right) f(z_{\Gamma_1,\Gamma_2}(v))\\ = \;& \rho(S) \cdot (d_{\operatorname{Re}S^{-1}\alpha} + id_{\operatorname{Im}S^{-1}\alpha})f(z_{\Gamma_1,\Gamma_2}(v))\\ & -2\lambda(S^{-1}\alpha|S^{-1}v)_{\Gamma_1,\Gamma_2} f(z_{\Gamma_1,\Gamma_2}(S^{-1}v))\\ = \;& (d_{\operatorname{Re}\alpha} + id_{\operatorname{Im}\alpha} - 2\lambda(\alpha|v)_{S\Gamma_1,S\Gamma_2})f(z_{\Gamma_1,\Gamma_2}(S^{-1}v))\\ = \;& (\dot{T}^{S\Gamma_1,S\Gamma_2,\lambda}_{\operatorname{Re}\alpha} + i\dot{T}^{S\Gamma_1,S\Gamma_2,\lambda}_{\operatorname{Im}\alpha}) \cdot \rho(S)f(z_{\Gamma_1,\Gamma_2}(v)). \end{aligned}$$

□

For $a \in V$, let $T_a^{\Gamma_1,\Gamma_2,\lambda} \in \mathrm{GL}(\mathbb{C}^V)$ be defined by

$$T_a^{\Gamma_1,\Gamma_2,\lambda} f(v) := f(v+a)e^{-\lambda(a|a+2v)_{\Gamma_1,\Gamma_2}}. \tag{4.53}$$

Remark 4.3.3 (Conjugation by $e^{-\lambda(v|v)_{\Gamma_1,\Gamma_2}}$)**.** *Recall $\tau_\cdot^\lambda$ from Equation 3.5. We note that $T_a^{\Gamma_1,\Gamma_2,\lambda}$ is nothing but a conjugation of τ_a^λ by multiplication operators:*

$$T_a^{\Gamma_1,\Gamma_2,\lambda} = e^{\lambda(v|v)_{\Gamma_1,\Gamma_2}} \cdot \tau_a^\lambda \cdot e^{-\lambda(v|v)_{\Gamma_1,\Gamma_2}}. \tag{4.54}$$

We will return to this observation in the next section.

Then we can check that for $a, b \in V$

$$\begin{aligned} T_a^{\Gamma_1,\Gamma_2,\lambda} T_b^{\Gamma_1,\Gamma_2,\lambda} f(v) &= T_a^{\Gamma_1,\Gamma_2,\lambda} f(v+b)e^{-\lambda(b|b+2v)_{\Gamma_1,\Gamma_2}} \\ &= f(v+a+b)e^{-\lambda(b|b+2v+2a)_{\Gamma_1,\Gamma_2}} e^{-\lambda(a|a+2v)_{\Gamma_1,\Gamma_2}} \\ &= f(v+a+b)e^{-\lambda(a+b|a+b+2v)_{\Gamma_1,\Gamma_2}} e^{\lambda(a|b)_{\Gamma_1,\Gamma_2} - \lambda(b|a)_{\Gamma_1,\Gamma_2}} \\ &= e^{\frac{i}{2}\lambda\omega(a,b)} T_{a+b}^{\Gamma_1,\Gamma_2,\lambda} f(v). \end{aligned}$$

We can extend $T_\cdot^{\Gamma_1,\Gamma_2,\lambda}$ from V to $H(\omega)$ by letting

$$T_{(s,a)}^{\Gamma_1,\Gamma_2,\lambda} f(v) := f(v+a)e^{i\lambda s - \lambda(a|a+2v)_{\Gamma_1,\Gamma_2}}. \tag{4.55}$$

So $T_\cdot^{\Gamma_1,\Gamma_2,\lambda}$ is a group homomorphism $H(\omega) \to \mathrm{GL}(\mathbb{C}^V)$.

Remark 4.3.4 (Differentiation)**.** *If f is a smooth function, pointwise we have*

$$\lim_{t\to 0} \frac{1}{t}(T_{ta}^{\Gamma_1,\Gamma_2,\lambda} f(v) - f(v)) = \dot{T}_a^{\Gamma_1,\Gamma_2,\lambda} f(v). \tag{4.56}$$

We will be refrain from viewing this limit as a limit of operators.

Definition 4.3.5 ((Γ_1, Γ_2)-analytic functions)**.** *Suppose $\mathcal{U} \subset V$ is an open subset. Let $f : \mathcal{U} \to \mathbb{C}$ be a* (Γ_1, Γ_2)-analytic function *on $\mathcal{U}$ if it is a pullback of an analytic function on $z_{\Gamma_1,\Gamma_2}(\mathcal{U}) \subset \mathbb{C}^n$ by z_{Γ_1,Γ_2}. Denote the vector space of (Γ_1, Γ_2)-analytic functions by $\mathcal{O}_{\Gamma_1,\Gamma_2}(\mathcal{U})$.*

Remark 4.3.6 (Restriction of $T_{\cdot}^{\Gamma_1,\Gamma_2,\lambda}$ to $\mathcal{O}_{\Gamma_1,\Gamma_2}(V)$)**.** *Since $(a|v)_{\Gamma_1,\Gamma_2}$ is a linear combination of the components of z_{Γ_1,Γ_2},*

$$e^{-\lambda(a|a+2v)_{\Gamma_1,\Gamma_2}} \in \mathcal{O}_{\Gamma_1,\Gamma_2}(V). \tag{4.57}$$

So Equation 4.53 is well defined for $f \in \mathcal{O}_{\Gamma_1,\Gamma_2}(V)$, and $T_{\cdot}^{\Gamma_1,\Gamma_2,\lambda}$ restricts to a group homomorphism $H(\omega) \to \mathrm{GL}(\mathcal{O}_{\Gamma_1,\Gamma_2}(V))$.

Example 4.3.7 (J-holomorphic functions on V)**.** *Suppose J is a compatible complex structure on (V,ω) with Darboux basis $\{\mathbf{e},\mathbf{f}\}$. A J-holomorphic function $f:\mathcal{U}\to\mathbb{C}$ is a function satisfying the Cauchy-Riemann equation:*

$$d_{Ja}f(v) = id_a f(v) \quad a\in V \quad v\in\mathcal{U}. \tag{4.58}$$

Let $\mathcal{O}_J(\mathcal{U})$ denote the J-holomorphic functions on $\mathcal{U}$.

Let $\begin{pmatrix} A & B \\ C & D\end{pmatrix}$ be a real symplectic matrix such that

$$(J)_{\{\mathbf{e},\mathbf{f}\}} = \begin{pmatrix} A & B \\ C & D\end{pmatrix}\begin{pmatrix} 0 & -1_n \\ 1_n & 0\end{pmatrix}\begin{pmatrix} A & B \\ C & D\end{pmatrix}^{-1}. \tag{4.59}$$

The transverse pair of complex Lagrangian subspaces defined by the complex symplectic matrix

$$S = \begin{pmatrix} A & B \\ C & D\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1_n & -i\cdot 1_n \\ -i\cdot 1_n & 1_n\end{pmatrix} = -\frac{i}{\sqrt{2}}\begin{pmatrix} B+iA & A+iB \\ D+iC & C+iD\end{pmatrix}. \tag{4.60}$$

is $(\Gamma_1,\Gamma_2) = (V_J^{1,0}, V_J^{0,1})$ (Example 5.2.2).

The fact that $z_{V_J^{1,0},V_J^{0,1}}$ is constant in the $V_J^{0,1}$ directions (Remark 4.2.3) implies that every (Γ_1,Γ_2)-analytic function f satisfies

$$\begin{aligned}\left(d_{\mathrm{Re}\left(\begin{smallmatrix} A+iB \\ C+iD\end{smallmatrix}\right)a} + id_{\mathrm{Im}\left(\begin{smallmatrix} A+iB \\ C+iD\end{smallmatrix}\right)a}\right) f(z_{V_J^{1,0},V_J^{0,1}}(v)) &= 0 &(4.61)\\ \left(-d_{\mathrm{Im}\left(\begin{smallmatrix} A+iB \\ C+iD\end{smallmatrix}\right)a} + id_{\mathrm{Re}\left(\begin{smallmatrix} A+iB \\ C+iD\end{smallmatrix}\right)a}\right) f(z_{V_J^{1,0},V_J^{0,1}}(v)) &= 0. &(4.62)\end{aligned}$$

This is equivalent to the Cauchy-Riemann equation, since

$$d_{\binom{A}{C}a} + id_{J\binom{A}{C}a} = d_{\binom{A}{C}a} + id_{\binom{B}{D}a} \tag{4.63}$$

$$-d_{\binom{B}{D}a} + id_{J\binom{B}{D}a} = -d_{\binom{B}{D}a} + id_{\binom{A}{C}a}. \tag{4.64}$$

For J-holomorphic functions, satisfying the Cauchy-Riemann equation is equivalent to analyticity in the complex variables, so the $(V_J^{1,0}, V_J^{0,1})$-analytic functions are the J-holomorphic functions.

$$\mathcal{O}_{V_J^{1,0}, V_J^{1,0}}(\mathcal{U}) = \mathcal{O}_J(\mathcal{U}). \tag{4.65}$$

Proposition 4.3.8. *Suppose $S \in \mathrm{Sp}(V, \omega)$. Then the precomposition operator $\rho(S)$ intertwines $T^{\Gamma_1,\Gamma_2,\lambda}_{S^{-1}\cdot}$ with $T^{S\Gamma_1,S\Gamma_2,\lambda}_{\cdot}$. In other words, for all $a \in V$,*

$$T_a^{S\Gamma_1,S\Gamma_2,\lambda} \cdot \rho(S) = \rho(S) \cdot T^{\Gamma_1,\Gamma_2,\lambda}_{S^{-1}a} \tag{4.66}$$

in $\mathrm{GL}(\mathcal{O}_{\Gamma_1,\Gamma_2}(V))$.

Proof. This is again, by direct computation:

$$\begin{aligned}
& \rho(S) \cdot T^{\Gamma_1,\Gamma_2,\lambda}_{S^{-1}a} f(z_{\Gamma_1,\Gamma_2}(v)) \\
= \;& \rho(S) f(z_{\Gamma_1,\Gamma_2}(v + S^{-1}a)) e^{-\lambda(S^{-1}a|S^{-1}a+2v)_{\Gamma_1,\Gamma_2}} \\
= \;& f(z_{\Gamma_1,\Gamma_2}(S^{-1}(v + a))) e^{-\lambda(S^{-1}a|S^{-1}(a+2v))_{\Gamma_1,\Gamma_2}} \\
= \;& f(z_{\Gamma_1,\Gamma_2}(S^{-1}(v + a))) e^{-\lambda(a|a+2v)_{S\Gamma_1,S\Gamma_2}} \\
= \;& T_a^{S\Gamma_1,S\Gamma_2,\lambda} f(z_{\Gamma_1,\Gamma_2}(S^{-1}(v))).
\end{aligned}$$

□

4.4 Construction of representations on Hilbert spaces

In this section we realize, when possible, the representations constructed in the previous section as representations on Hilbert spaces. For representations of the Heisenberg group, convergence issues do not arise when the transverse pair (Γ_1, Γ_2) is a positive pair. For representations of the Heisenberg Lie algebra, convergence issues can be avoided for all transverse pairs by restricting the domain to bounded open subsets. However, we can only guarantee the representation is by skew-adjoint operators when (Γ_1, Γ_2) is a positive pair.

Definition 4.4.1 (Positive pairs)**.** *We will call a transverse pair of complex Lagrangian subspaces* (Γ_1, Γ_2) *a* positive pair *if for all nonzero* $v \in V$*,*

$$h(v,v)_{\Gamma_1,\Gamma_2} > 0. \tag{4.67}$$

Example 4.4.2 (Positive pairs)**.** *For* $V = \mathbb{R}^2$*, the transverse pair* (Γ_1, Γ_2) *given by the complex symplectic matrix*

$$S = \begin{pmatrix} 1+\varepsilon & -i\varepsilon \\ -i\varepsilon & 1-\varepsilon \end{pmatrix} \in \mathrm{Sp}(2;\mathbb{C}) \quad \varepsilon \in (0,1) \tag{4.68}$$

is a positive pair.

We can compute the Gram matrices

$$M_{\Gamma_1,\Gamma_2} = \frac{1}{2}\begin{pmatrix} \varepsilon(1-\varepsilon) & i\varepsilon^2 - i \\ i\varepsilon^2 & \varepsilon(1+\varepsilon) \end{pmatrix} \tag{4.69}$$

and

$$h_{\Gamma_1,\Gamma_2} = \begin{pmatrix} \varepsilon(1-\varepsilon) & -i/2 \\ i/2 & \varepsilon(1+\varepsilon) \end{pmatrix}. \tag{4.70}$$

So

$$h(v,v)_{\Gamma_1,\Gamma_2} = \varepsilon^2(1-\varepsilon)^2 q^2 + \varepsilon^2(1+\varepsilon)^2 p^2 \quad v = \begin{pmatrix} q \\ p \end{pmatrix} \in \mathbb{R}^2 \tag{4.71}$$

is positive for nonzero v*.*

Take a Darboux basis $\{\mathbf{e}, \mathbf{f}\}$ of (V, ω) and pull back the Lebesgue measure on $\mathbb{R}^{2n}$. Let

$$dm_{\Gamma_1,\Gamma_2,\lambda} := |e^{-\lambda(v|v)_{\Gamma_1,\Gamma_2}}|^2 dm_{Leb} = e^{-\lambda h(v,v)_{\Gamma_1,\Gamma_2}} dm_{Leb}. \tag{4.72}$$

We will construct Hilbert spaces as follows:

$$\mathfrak{F}^{\lambda}_{\Gamma_1,\Gamma_2}(\mathcal{U}) := \overline{\mathcal{O}_{\Gamma_1,\Gamma_2}(\mathcal{U}) \cap L^2(\mathcal{U}; dm_{\Gamma_1,\Gamma_2,\lambda}|_{\mathcal{U}})}^{L^2} \tag{4.73}$$

$$\mathscr{H}^{\lambda}_{\Gamma_1,\Gamma_2}(\mathcal{U}) := \overline{\mathcal{O}_{\Gamma_1,\Gamma_2}(\mathcal{U}) e^{-\lambda(v|v)_{\Gamma_1,\Gamma_2}} \cap L^2(\mathcal{U}; dm_{Leb}|_{\mathcal{U}})}^{L^2} \tag{4.74}$$

Remark 4.4.3 (The zero Hilbert space)**.** *These spaces can be degenerate. For (Γ_1, Γ_2) given by the identity matrix*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.75}$$

we have

$$\mathfrak{F}^{\lambda}_{\Gamma_1,\Gamma_2}(\mathbb{R}^2) = \mathscr{H}^{\lambda}_{\Gamma_1,\Gamma_2}(\mathbb{R}^2) = \{0\} \tag{4.76}$$

because any analytic function of $z_{\Gamma_1,\Gamma_2} = q$ with any nonzero value has a divergent norm. We will consider the unitary group of zero Hilbert spaces as consisting of a single identity element.

Remark 4.4.4 (Multiplication by $e^{\mp\lambda(v|v)_{\Gamma_1,\Gamma_2}}$ are isometries)**.** *From the construction, we can immediately see multiplication by $e^{-\lambda(v|v)_{\Gamma_1,\Gamma_2}}$ is an isometry*

$$e^{-\lambda(v|v)_{\Gamma_1,\Gamma_2}} : \mathfrak{F}^{\lambda}_{\Gamma_1,\Gamma_2}(\mathcal{U}) \xrightarrow{\cong} \mathscr{H}^{\lambda}_{\Gamma_1,\Gamma_2}(\mathcal{U}). \tag{4.77}$$

Moreover, for a unitary operator $\mathbf{A} \in U(\mathscr{H}^{\lambda}_{\Gamma_1,\Gamma_2}(\mathcal{U}))$,

$$e^{\lambda(v|v)_{\Gamma_1,\Gamma_2}} \cdot \mathbf{A} \cdot e^{-\lambda(v|v)_{\Gamma_1,\Gamma_2}} \in U(\mathfrak{F}^{\lambda}_{\Gamma_1,\Gamma_2}(\mathcal{U})). \tag{4.78}$$

We can extend the domain of $\tau^{\lambda}_{\cdot}$ (Equation 3.5) from V to $H(\omega)$ by

$$\tau^{\lambda}_{(s,a)} f(v) := e^{\frac{i}{2}\lambda\omega(v,a) + i\lambda s} f(v+a) \quad a \in V, s \in \mathbb{R}. \tag{4.79}$$

Then $\tau^{\lambda}_{(s,a)}$ are compositions of translations and multiplication by a function with values of modulus 1, so $\tau^{\lambda}_{(s,a)}$ are unitary operators on $\mathscr{H}^{\lambda}_{\Gamma_1,\Gamma_2}(V)$. Then

$$T^{\Gamma_1,\Gamma_2,\lambda}_{(s,a)} = e^{\lambda(v|v)_{\Gamma_1,\Gamma_2}} \cdot \tau^{\lambda}_{(s,a)} \cdot e^{-\lambda(v|v)_{\Gamma_1,\Gamma_2}} \tag{4.80}$$

are unitary operators on $\mathfrak{F}^{\lambda}_{\Gamma_1,\Gamma_2}(V)$.

Theorem 4.4.5. *For any real λ and transverse pair of complex Lagrangian subspaces (Γ_1, Γ_2)*

$$\begin{aligned} \tau_{\cdot}^{\lambda} : H(\omega) &\to U(\mathscr{H}_{\Gamma_1,\Gamma_2}^{\lambda}(V)) \qquad (4.81)\\ T_{\cdot}^{\Gamma_1,\Gamma_2,\lambda} : H(\omega) &\to U(\mathfrak{F}_{\Gamma_1,\Gamma_2}^{\lambda}(V)). \qquad (4.82)\end{aligned}$$

are strongly continuous unitary representations of Heisenberg groups. When (Γ_1, Γ_2) are a positive pair, $\mathbb{C}[z_{\Gamma_1,\Gamma_2}] \subset \mathfrak{F}_{\Gamma_1,\Gamma_2}^{\lambda}(V)$, $\mathbb{C}[z_{\Gamma_1,\Gamma_2}]e^{-\lambda(v|v)_{\Gamma_1,\Gamma_2}} \subset \mathscr{H}_{\Gamma_1,\Gamma_2}^{\lambda}(V)$ as dense subspaces.

Proof. We will prove the strong continuity of $\tau_{\cdot}^{\lambda}$. For $f \in \mathscr{H}_{\Gamma_1,\Gamma_2}^{\lambda}(V)$, $f \in L^2(V; dm_{Leb})$, so there exists a continuous function of compact support f_c such that

$$\|f - f_c\|_2^2 < \epsilon. \qquad (4.83)$$

where

$$|| \cdot ||_2 := || \cdot ||_{L^2(V;dm_{Leb})}. \qquad (4.84)$$

Then $\tau_{(s,a)}^{\lambda} f_c \to f_c$ uniformly. Therefore for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|(s,a)| < \delta$ implies

$$\|\tau_{(s,a)}^{\lambda} f_c - f_c\|_2^2 < \epsilon. \qquad (4.85)$$

Therefore, $|(s,a)| < \delta$ implies

$$\begin{aligned} \|\tau_{(s,a)}^{\lambda} f - f\|_2^2 &= \|\tau_{(s,a)}^{\lambda} f - \tau_{(s,a)}^{\lambda} f_c + \tau_{(s,a)}^{\lambda} f_c - f_c + f_c - f\|_2^2 \\ &\le \|\tau_{(s,a)}^{\lambda} f - \tau_{(s,a)}^{\lambda} f_c\|_2^2 + \|\tau_{(s,a)}^{\lambda} f_c - f_c\|_2^2 + \|f - f_c\|_2^2 \\ &< 3\epsilon. \end{aligned}$$

The same proof goes for $T_{\cdot}^{\Gamma_1,\Gamma_2,\lambda}$ because the space of continuous functions of compact support remains dense in $L^2(V; dm_{\Gamma_1,\Gamma_2})$. □

Corollary 4.4.6. *For bounded domains $\mathcal{U} \subset V$, and any pair of transverse Lagrangian subspaces (Γ_1, Γ_2), $\mathbb{C}[z_{\Gamma_1,\Gamma_2}|_{\mathcal{U}}]e^{-\lambda(v|v)_{\Gamma_1,\Gamma_2}}$ is a dense subspace of $\mathscr{H}_{\Gamma_1,\Gamma_2}^{\lambda}(\mathcal{U})$ on which $\dot{\tau}_{\cdot}^{\lambda}$ is an irreducible representation of $\mathfrak{heis}\,\omega$.*

Chapter 5

Reconstruction of known representations

In this chapter, we will explain how our construction of representation in the previous chapter relates to previously studied families of representations [7][8][2][12][5]. The partition of the complex Lagrangian Grassmannian arising from complex conjugation serves as a natural geometric dictionary, which we will see explicitly for $\mathbb{R}^2$. Because the previous families of representations have been studied from different contexts, we will first explain how they fit with the framework provided by the partition of the complex Lagrangian Grassmannian. Then we will explicitly identify the representations themselves.

Representation	Parameter $S \in \mathrm{Sp}(2n;\mathbb{C})$
Schrödinger	$\begin{pmatrix} 1_n & 0 \\ 0 & 1_n \end{pmatrix}$
Momentum	$\begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$
Fock-Bargmann	$\frac{1}{\sqrt{2}} \begin{pmatrix} 1_n & -i \cdot 1_n \\ -i \cdot 1_n & 1_n \end{pmatrix}$
Grossmann-Daubechies	$-\frac{i}{\sqrt{2}} \begin{pmatrix} B+iA & A+iB \\ D+iC & C+iD \end{pmatrix} \quad : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n;\mathbb{R})$
Mumford	$\begin{pmatrix} 0 & -1_n \\ 1_n & Z \end{pmatrix} \quad : \; Z = Z^t \quad \operatorname{Im} Z > 0$
Satake	$\begin{pmatrix} 1_n & Z \\ 0 & 1_n \end{pmatrix} \quad : \; Z = Z^t \quad \operatorname{Im} Z > 0$
Lion-Vergne	$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \in \mathrm{Sp}(2n;\mathbb{R})$

Table 5.1: Reconstruction dictionary by complex symplectic matrices

5.1 Maximal compact subgroups

By the Cartan-Iwasawa-Malcev theorem, all the topology of a real Lie group is contained in its maximal compact subgroup, and we will briefly review them. Maximal compact subgroups are not unique, and are defined up to conjugacy. We will be interested in a particular maximal compact subgroup of $\mathrm{Sp}(2n;\mathbb{R})$ that is easily written in matrix form.

Example 5.1.1 (Involution of $\mathbb{C}^\times$)**.** $\mathbb{C}^\times$ *is abelian, and not a semisimple Lie* *group. So the Cartan decomposition does not apply. However, the involution*

$$\Theta : z \mapsto (\overline{z})^{-1} \tag{5.1}$$

that can be seen visually, does help provide intuition for the Cartan involution of $\mathfrak{sp}(2n;\mathbb{R})$*, which will help understand the maximal compact subgroup.*

View $\mathbb{C}^\times$ *as a real Lie group with identity element* $1 \in \mathbb{C}^\times$*. Its Lie algebra is abelian*

$$\operatorname{Lie}\mathbb{C}^* = T_1\mathbb{C}^\times \cong \{1\} \times \mathbb{C} \cong (\{1\} \times i\mathbb{R}) \oplus (\{1\} \times \mathbb{R}). \tag{5.2}$$

The splitting comes from the ± 1*-eigenspaces of the involution of* $\operatorname{Lie}\mathbb{C}^\times$

$$\theta : v \mapsto -\overline{v} \tag{5.3}$$

whose exponential is Θ*.*

Exponentiating the $+1$ *eigenspace* $\{1\} \times i\mathbb{R}$ *gives the compact group* $U(1)$ *which is the fixed locus of* Θ*, and exponentiating the* -1 *eigenspace* $\{1\} \times \mathbb{R}$ *gives the noncompact group* $\mathbb{R}^\times$*.*

Example 5.1.2 (Cartan involution of $\mathfrak{sp}(2n;\mathbb{R})$ and the maximal compact subgroup of $\mathrm{Sp}(2n;\mathbb{R})$)**.** *The Lie algebra of the symplectic group* $\mathrm{Sp}(2n;\mathbb{R})$ *is*

$$\mathfrak{sp}(2n;\mathbb{R}) := \left\{ \boldsymbol{\sigma} \in \mathrm{Mat}_{2n\times 2n}(\mathbb{R}) : \boldsymbol{\sigma}^t \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \boldsymbol{\sigma} = 0 \right\}$$

or equivalently,

$$\mathfrak{sp}(2n;\mathbb{R}) := \left\{ \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & -\mathbf{a}^t \end{pmatrix} : \mathbf{b} = \mathbf{b}^t, \mathbf{c} = \mathbf{c}^t, \mathbf{a} \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \right\}. \tag{5.4}$$

The Killing form of $\mathfrak{sp}(2n;\mathbb{R})$ *is*

$$B(\boldsymbol{\sigma}, \boldsymbol{\sigma}') = (2n+2)\operatorname{Tr}(\boldsymbol{\sigma}\boldsymbol{\sigma}') \tag{5.5}$$

and

$$\theta : \boldsymbol{\sigma} \mapsto -\boldsymbol{\sigma}^t \tag{5.6}$$

is a Cartan involution, since

$$
\begin{aligned}
-B(\boldsymbol{\sigma},\theta\boldsymbol{\sigma}) &= (2n+2)\,\mathrm{Tr}\begin{pmatrix}\mathbf{a} & \mathbf{b}\\ \mathbf{c} & -\mathbf{a}^t\end{pmatrix}\begin{pmatrix}\mathbf{a}^t & \mathbf{c}^t\\ \mathbf{b}^t & -\mathbf{a}\end{pmatrix} && (5.7)\\
&= (2n+2)\,\mathrm{Tr}(\mathbf{a}\mathbf{a}^t+\mathbf{a}^t\mathbf{a}+\mathbf{b}\mathbf{b}^t+\mathbf{c}\mathbf{c}^t) && (5.8)\\
&= (2n+2)\sum_{j,k=1}^{n}(2\mathbf{a}_{jk}^2+\mathbf{b}_{jk}^2+\mathbf{c}_{jk}^2) && (5.9)
\end{aligned}
$$

is positive for nonzero $\boldsymbol{\sigma}$*.*

Since

$$\boldsymbol{\sigma}=-\boldsymbol{\sigma}^t \iff 1_{2n}\boldsymbol{\sigma}+\boldsymbol{\sigma}^t 1_{2n}=0 \iff \boldsymbol{\sigma}\in\mathfrak{so}(2n;\mathbb{R}) \tag{5.10}$$

the $+1$*-eigenspace of the Cartan involution* θ *is*

$$\mathfrak{sp}(2n;\mathbb{R})\cap\mathfrak{so}(2n;\mathbb{R}). \tag{5.11}$$

This is the Lie algebra corresponding to the maximal compact subgroup

$$\mathrm{Sp}(2n;\mathbb{R})\cap\mathrm{SO}(2n;\mathbb{R})\subset\mathrm{Sp}(2n;\mathbb{R}). \tag{5.12}$$

Proposition 5.1.3 (Block Iwasawa decomposition)**.** *Let*

$$S=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\mathrm{Sp}(2n;\mathbb{C}) \tag{5.13}$$

such that A^tA+C^tC *is invertible. Then* S *can be decomposed into*

$$S=K\boldsymbol{\alpha}N \quad K,\boldsymbol{\alpha},N\in\mathrm{Sp}(2n;\mathbb{C}) \tag{5.14}$$

where

$$K=\begin{pmatrix}X & -Y\\ Y & X\end{pmatrix}\quad \boldsymbol{\alpha}=\begin{pmatrix}R & 0\\ 0 & (R^{-1})^t\end{pmatrix}\quad N=\begin{pmatrix}1_n & Z\\ 0 & 1_n\end{pmatrix}, \tag{5.15}$$

X, Y, Z are complex $n \times n$ matrices satisfying

$$X^tX + Y^tY = 1 \quad X^tY = Y^tX \quad Z = Z^t \tag{5.16}$$

and R is an invertible $n \times n$ matrix.

Proof. Since $A^tA + C^tC$ is invertible, there exists an orthonormal basis of the span of first n column vectors of S. Pick one such orthonormal basis and write it as n column vectors in the standard basis of $\mathbb{R}^{2n}$ as

$$\begin{pmatrix} X \\ Y \end{pmatrix}. \tag{5.17}$$

Since the column vectors form an orthonormal basis, $X^tX + Y^tY = 1$. Since the n column vectors of $\begin{pmatrix} A^t & C^t \end{pmatrix}^t$ and $\begin{pmatrix} X^t & Y^t \end{pmatrix}^t$ span the same subspace there is an invertible $n \times n$ matrix R such that

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} A \\ C \end{pmatrix} R^{-1}. \tag{5.18}$$

Then since

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} R^{-1} & 0 \\ 0 & R^t \end{pmatrix} = \begin{pmatrix} X & BR^t \\ Y & DR^t \end{pmatrix} \tag{5.19}$$

is symplectic, we have that $X^tY = Y^tX$. Thus

$$\begin{pmatrix} X^t & Y^t \\ -Y^t & X^t \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} R^{-1} & 0 \\ 0 & R^t \end{pmatrix} = \begin{pmatrix} 1 & X^tBR^t + Y^tDR^t \\ 0 & 1 \end{pmatrix} \tag{5.20}$$

is symplectic, and $Z' := X^tBR^t + Y^tDR^t$ is symmetric. Therefore we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & (R^{-1})^t \end{pmatrix} \begin{pmatrix} 1 & R^{-1}Z'(R^{-1})^t \\ 0 & 1 \end{pmatrix} \tag{5.21}$$

as the desired decomposition. □

Remark 5.1.4 (Complexification of maximal compact subgroup)**.** *In the* *decomposition,*

$$K \in \mathrm{Sp}(2n;\mathbb{C}) \cap \mathrm{SO}(2n;\mathbb{C}) \tag{5.22}$$

which is not a compact subgroup of $\mathrm{Sp}(2n;\mathbb{C})$.

5.2 Siegel upper half planes and compatible complex structures

By the Cartan-Malcev-Iwasawa theorem,

$$\mathrm{Sp}(2n;\mathbb{R})/(\mathrm{Sp}(2n;\mathbb{R}) \cap \mathrm{SO}(2n;\mathbb{R})) \tag{5.23}$$

is homeomorphic to a Euclidean space. In this section we will see two different descriptions of this space, which appear in the parametrizations of the Satake-Mumford families and Grossmann-Daubechies families.

Example 5.2.1 (Siegel upper half plane)**.** *The Siegel upper half plane of degree n (denoted $\mathbb{H}_n$) consists of complex symmetric $n \times n$ matrices Z with positive definite imaginary part, and it is contractible.*

$\mathrm{Sp}(2n;\mathbb{R})$ *acts transitively on $\mathbb{H}_n$ by Möbius transformations:*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} .Z \mapsto (AZ+B)(CZ+D)^{-1} \tag{5.24}$$

with the stabilizer at $Z = i \cdot 1_n$ being

$$\mathrm{Sp}(2n;\mathbb{R}) \cap \mathrm{SO}(2n;\mathbb{R}). \tag{5.25}$$

So

$$\mathbb{H}_n \cong \mathrm{Sp}(2n;\mathbb{R})/(\mathrm{Sp}(2n;\mathbb{R}) \cap \mathrm{SO}(2n;\mathbb{R})). \tag{5.26}$$

One way to describe the complex Lagrangian Grassmannian of type $(0,n,0)$ of $\mathbb{R}^{2n}$ is by equivalence classes of the complex $2n \times n$ matrices

$$\begin{pmatrix} Q \\ P \end{pmatrix} \quad : Q^tP = P^tQ, \quad iQ^*P - iP^*Q > 0 \tag{5.27}$$

under the equivalence relation

$$\begin{pmatrix} Q \\ P \end{pmatrix} \sim \begin{pmatrix} Q' \\ P' \end{pmatrix} \iff \exists X \in \mathrm{GL}(n;\mathbb{C}) : \begin{pmatrix} Q \\ P \end{pmatrix} X = \begin{pmatrix} Q' \\ P' \end{pmatrix}. \tag{5.28}$$

Then we can identify

$$\begin{aligned} \mathbb{H}_n &\xrightarrow{\cong} \mathrm{Gr}((0,n,0);\mathbb{R}^{2n}) && (5.29) \\ Z &\mapsto \left[\begin{pmatrix} Z \\ 1_n \end{pmatrix}\right] && (5.30) \end{aligned}$$

and the Möbius action gets mapped to the linear action

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}.Z \mapsto \left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} Z \\ 1_n \end{pmatrix}\right] = \left[\begin{pmatrix} (AZ+B)(CZ+D)^{-1} \\ 1_n \end{pmatrix}\right]. \tag{5.31}$$

Example 5.2.2 (Equivariant embedding of compatible complex structures on $(\mathbb{R}^{2n}, \omega_{std})$)**.** *Let $\mathcal{J}(\mathbb{R}^{2n})$ be the set of compatible complex structures on $(\mathbb{R}^{2n}, \omega_{std})$.*

$$J \mapsto (V_J^{1,0}, V_J^{0,1}) \tag{5.32}$$

is an embedding of $\mathcal{J}(\mathbb{R}^{2n})$ into the space of transverse pairs. We will describe this map in coordinates.

Recalling the projections 2.56, we have

$$\begin{aligned} V_J^{1,0} &= (1-iJ)\Gamma && (5.33) \\ V_J^{0,1} &= (1+iJ)\Gamma' && (5.34) \end{aligned}$$

if Γ (respectively, Γ') is an n-dimensional complex vector spaces transverse to $V_J^{1,0}$ (respectively, $V_J^{0,1}$). We can pick Γ, Γ' so that the coordinate description has a nice form.

Recall that every $J \in \mathcal{J}(\mathbb{R}^{2n})$ can be written as

$$J = SJ_0S^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} J_0 \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} \quad S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n;\mathbb{R}). \tag{5.35}$$

The column vectors of

$$\begin{pmatrix} A \\ C \end{pmatrix}, \begin{pmatrix} B \\ D \end{pmatrix} \tag{5.36}$$

span complex Lagrangian subspaces of type $(n,0,0)$*, so are transverse to both* $V_J^{1,0}$ *and* $V_J^{0,1}$*.*

Thus the column vectors of

$$\frac{1}{\sqrt{2}}(1-iJ)\begin{pmatrix} A \\ C \end{pmatrix} = -\frac{i}{\sqrt{2}}\begin{pmatrix} B+iA \\ D+iC \end{pmatrix} \tag{5.37}$$

span $V_J^{1,0}$*, and the column vectors of*

$$\frac{1}{\sqrt{2}}(1+iJ)\begin{pmatrix} B \\ D \end{pmatrix} = -\frac{i}{\sqrt{2}}\begin{pmatrix} A+iB \\ C+iD \end{pmatrix} \tag{5.38}$$

span $V_J^{0,1}$*.*

The map that can be read off this construction can be described as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto -\frac{i}{\sqrt{2}}\begin{pmatrix} B+iA & A+iB \\ D+iC & C+iD \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1_n & -i\cdot 1_n \\ -i\cdot 1_n & 1_n \end{pmatrix} \tag{5.39}$$

is nothing but the multiplication by the complex symplectic matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1_n & -i\cdot 1_n \\ -i\cdot 1_n & 1_n \end{pmatrix} \tag{5.40}$$

from the right. The matrices in $\mathrm{Sp}(2n;\mathbb{R})\cap\mathrm{SO}(2n;\mathbb{R})$*, satisfy the following identity*

$$\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}\begin{pmatrix} 1_n & -i\cdot 1_n \\ -i\cdot 1_n & 1_n \end{pmatrix} = \begin{pmatrix} 1_n & -i\cdot 1_n \\ -i\cdot 1_n & 1_n \end{pmatrix}\begin{pmatrix} X+iY & 0 \\ 0 & X-iY \end{pmatrix}. \tag{5.41}$$

Noting that $X - iY = (X^t + iY^t)^{-1}$*, the map 5.39 gives an equivariant embedding*

$$\mathrm{Sp}(2n;\mathbb{R})/(\mathrm{Sp}(2n;\mathbb{R})\cap\mathrm{SO}(2n;\mathbb{R})) \hookrightarrow \mathrm{Sp}(2n;\mathbb{C})/\mathrm{GL}(n;\mathbb{C}). \tag{5.42}$$

By Example 2.4.9 and Remark 4.1.3 this can be seen as an equivariant embedding of $\mathcal{J}(\mathbb{R}^{2n})$ into the space of transverse pairs.

5.3 Representations from new parameters

Definition 5.3.1 (Notation for specific complex Lagrangian subspaces)**.** *Let (V, ω) have a Darboux basis $\{\mathbf{e}, \mathbf{f}\}$.*

Denote the complex Lagrangian subspaces spanned by vectors with coefficients the column vectors of, respectively,

$$\begin{pmatrix} 1_n \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1_n \end{pmatrix}, \begin{pmatrix} 1_n \\ -i \cdot 1_n \end{pmatrix}, \begin{pmatrix} 1_n \\ i \cdot 1_n \end{pmatrix}, \begin{pmatrix} Z \\ 1_n \end{pmatrix} \tag{5.43}$$

by, respectively, $\mathbf{L}_1$, $\mathbf{L}_2$, $V_{J_0}^{1,0}$, $V_{J_0}^{0,1}$, Γ_Z.

Given a real symplectic matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n; \mathbb{R}) \tag{5.44}$$

Let the complex Lagrangian subspaces spanned by vectors with coefficients the column vectors of, respectively,

$$\begin{pmatrix} A \\ C \end{pmatrix}, \begin{pmatrix} B \\ D \end{pmatrix}, \begin{pmatrix} A - iB \\ C - iD \end{pmatrix}, \begin{pmatrix} A + iB \\ C + iD \end{pmatrix}, \tag{5.45}$$

by, respectively, L_1, L_2, $V_J^{1,0}$, $V_J^{0,1}$.

Representation	Transverse pair
Schrödinger	$(\mathbf{L}_1, \mathbf{L}_2)$
Fock-Bargmann	$(V_{J_0}^{1,0}, V_{J_0}^{0,1})$
Grossmann-Daubechies	$(V_J^{1,0}, V_J^{0,1})$
Mumford	$(\mathbf{L}_2, J_0\Gamma_Z)$
Satake	$(\mathbf{L}_1, \Gamma_Z)$
Lion-Vergne	(L_1, L_2)

Table 5.2: Reconstruction dictionary by transverse pairs

For $V = \mathbb{R}^2$, its complex Lagrangian Grassmannian can be represented as a point on $\mathbb{CP}^1$ by taking a vector spanning it

$$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto [q : p]. \tag{5.46}$$

Thus, a transverse pair of complex Lagrangian subspaces can be represented as two distinct ordered points on $\mathbb{CP}^1$. The reconstruction dictionary can be represented pictorially as follows:

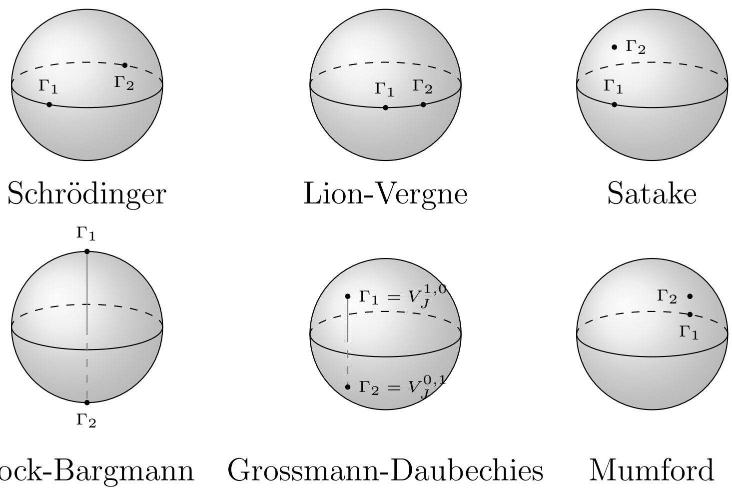

Table 5.3: Pictorial reconstruction dictionary for $V = \mathbb{R}^2$

Remark 5.3.2 (Noncontractibility)**.** *The space of transverse pairs in this case is*

$$(\mathbb{CP}^1 \times \mathbb{CP}^1)\backslash\Delta\mathbb{CP}^1 \tag{5.47}$$

which is homotopic to $\mathbb{CP}^1$*, and hence noncontractible.*

Example 5.3.3 (Unitary representations from new parameters)**.** *Recall from Example 4.4.2 the positive pair* (Γ_1, Γ_2) *given by the complex symplectic matrix*

$$S = \begin{pmatrix} 1+\varepsilon & -i\varepsilon \\ -i\varepsilon & 1-\varepsilon \end{pmatrix} \in \mathrm{Sp}(2;\mathbb{C}) \quad \varepsilon \in (0,1). \tag{5.48}$$

$\Gamma_1 \in \mathrm{Lag}^{\mathbb{C}}((0,1,0);\mathbb{R}^2)$ *and* $\Gamma_2 \in \mathrm{Lag}^{\mathbb{C}}((0,0,1);\mathbb{R}^2)$ *because*

$$\mathrm{Im}((1+\varepsilon)\overline{(-i\varepsilon)}) = \quad \varepsilon(1+\varepsilon) \quad > 0 \tag{5.49}$$

$$\mathrm{Im}((-i\varepsilon)\overline{(1-\varepsilon)}) = \quad -\varepsilon(1-\varepsilon) \quad < 0. \tag{5.50}$$

When $\varepsilon \neq \frac{1}{\sqrt{2}}$, $\Gamma_1 \neq \overline{\Gamma_2}$*, because*

$$\frac{1+\varepsilon}{-i\varepsilon} = \frac{i\varepsilon}{1-\varepsilon} \iff \varepsilon = \frac{1}{\sqrt{2}} \quad (\varepsilon \in (0,1)). \tag{5.51}$$

So $T_{\cdot}^{\Gamma_1,\Gamma_2,\lambda}$*'s are unitary representations from new parameters when* $\varepsilon \neq \frac{1}{\sqrt{2}}$.

In the following, let $\{\mathbf{e}, \mathbf{f}\}$ be a fixed Darboux basis of (V, ω), and denote

$$(v)_{\{\mathbf{e},\mathbf{f}\}} = \begin{pmatrix} q \\ p \end{pmatrix} \in \mathbb{R}^{2n} \tag{5.52}$$

$$q\mathbf{e} := \sum_{j=1}^{n} q_j \mathbf{e}_j \quad p\mathbf{f} := \sum_{j=1}^{n} p_j \mathbf{f}_j \tag{5.53}$$

$$dq := dq_1 \cdots dq_n \quad dp := dp_1 \cdots dp_n. \tag{5.54}$$

5.4 Schrödinger representation

The matrix

$$\begin{pmatrix} 1_n & 0 \\ 0 & 1_n \end{pmatrix} \tag{5.55}$$

gives the transverse pair $(\mathbf{L}_1, \mathbf{L}_2)$, and we can compute:

$$z_{\mathbf{L}_1,\mathbf{L}_2}(v) = q \quad \text{and} \quad z_{\mathbf{L}_1,\mathbf{L}_2}(v) = p \in \mathbb{R}^n. \tag{5.56}$$

Then

$$h_{\mathbf{L}_1,\mathbf{L}_2} = \frac{i}{2} \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \tag{5.57}$$

and

$$e^{-\lambda h(v,v)_{\mathbf{L}_1,\mathbf{L}_2}} \equiv 1 \quad v \in V. \tag{5.58}$$

Let $\{p < 1/2\}$ denote the strip

$$\{v \in V : p(v)_j < 1/2, j = 1, \cdots, n\}. \tag{5.59}$$

Then $\mathfrak{F}^1_{\mathbf{L}_1,\mathbf{L}_2}(\{p < 1/2\})$ is the completion of $\mathcal{O}_{\mathbf{L}_1,\mathbf{L}_2}(\{p < 1/2\})$ by the norm

$$||f||^2_{\{p<1/2\},\mathbf{L}_1,\mathbf{L}_2,1} := \int_{\{p<1/2\}} |f(q)|^2 dq dp = \int_L |f(q)|^2 dq. \tag{5.60}$$

This space is nonzero, because it contains the functions $f(q)e^{-q^t q}$, where f is a polynomial in q. In particular it contains the Hermite functions which describe the energy eigenstates of the quantum harmonic oscillator.

The map $f \mapsto f|_{\mathrm{Re}\,\mathbf{L}_1}$ is an isometry

$$\mathfrak{F}^1_{\mathbf{L}_1,\mathbf{L}_2,1}(\{p < 1/2\}) \xrightarrow{\cong} L^2(\mathrm{Re}\,\mathbf{L}_1, dq) \cong L^2(\mathbb{R}^n, dm_{Leb}). \tag{5.61}$$

Although $T^{\mathbf{L}_1,\mathbf{L}_2,1}_{\cdot}$ is not well defined, $\dot{T}^{\mathbf{L}_1,\mathbf{L}_2,1}_{\cdot}$ is, and we can recover the position and momentum operators.

For instance, the momentum operators are given by:

$$\begin{aligned}
-i\dot{T}^{\mathbf{L}_1,\mathbf{L}_2,1}_{\mathbf{e}_j} &= -id_{\mathbf{e}_j} + i(\mathbf{e}_j|v)_{\mathbf{L}_1,\mathbf{L}_2} && (5.62)\\
&= -i\frac{\partial}{\partial q_j} + \frac{1}{2}\begin{pmatrix} q^t & p^t \end{pmatrix}\begin{pmatrix} 1_n \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1_n \end{pmatrix}\begin{pmatrix} e_j \\ 0 \end{pmatrix} && (5.63)\\
&= -i\frac{\partial}{\partial q_j} && (5.64)\\
&= \hat{p}_j \quad j = 1, \cdots, n. && (5.65)
\end{aligned}$$

The position operators are given by:

$$\begin{aligned}
-i\dot{T}^{\mathbf{L}_1,\mathbf{L}_2,1}_{\mathbf{f}_j} &= -id_{\mathbf{f}_j} + i(\mathbf{f}_j|v)_{\mathbf{L}_1,\mathbf{L}_2} && (5.66)\\
&= \frac{1}{2}\begin{pmatrix} q^t & p^t \end{pmatrix}\begin{pmatrix} 1_n \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1_n \end{pmatrix}\begin{pmatrix} 0 \\ e_j \end{pmatrix} && (5.67)\\
&= q_j && (5.68)\\
&= \hat{q}_j \quad j = 1, \cdots, n. && (5.69)
\end{aligned}$$

It is well known that these operators are self-adjoint, and can be exponentiated.

5.5 Lion-Vergne's family

Let (L_1, L_2) be the transverse pair given by matrices of the form

$$S := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n; \mathbb{R}) \tag{5.70}$$

We will show $\mathfrak{F}^{2\pi}_{L_1,L_2}(\{z_{L_1,L_2} < 1/2\})$ reproduces the representations in p15 of [13] with underlying Hilbert space $L^2(L_1, dz_{L_1,L_2})$.

Let $\{\mathbf{e}', \mathbf{f}'\}$ be the Darboux basis given by the column vectors of S. Then we have for $v \in V$

$$(v)_{\{\mathbf{e}',\mathbf{f}'\}} = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} z_{L_1,L_2}(v) \\ \zeta_{L_1,L_2}(v) \end{pmatrix}. \tag{5.71}$$

So $v = q\mathbf{e} + p\mathbf{f} = z_{L_1,L_2}(v)\mathbf{e}' + \zeta_{L_1,L_2}(v)\mathbf{f}'$.

Let

$$\{\zeta_{L_1,L_2} < 1/2\} := \{v \in V : \zeta_{L_1,L_2}(v)_j < 1/2, j = 1, \cdots, n\}. \tag{5.72}$$

Then $f \mapsto f|_{\operatorname{Re} L_2}$ is an isometry

$$\mathfrak{F}^{2\pi}_{L_1,L_2}(\{\zeta_{L_1,L_2}(v) < 1/2\}) \xrightarrow{\cong} L^2(\operatorname{Re} L_1, dz_{L_1,L_2}). \tag{5.73}$$

We can compute

$$(a\mathbf{e}'|v)_{L_1,L_2} = -\tfrac{i}{2} \begin{pmatrix} D \\ -B \end{pmatrix} \begin{pmatrix} -C^t & A^t \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix} a = 0 \tag{5.74}$$

$$(b\mathbf{f}'|v)_{L_1,L_2} = -\tfrac{i}{2} \begin{pmatrix} D \\ -B \end{pmatrix} \begin{pmatrix} -C^t & A^t \end{pmatrix} \begin{pmatrix} B \\ D \end{pmatrix} b = -\frac{i}{2} b^t z_{L_1,L_2}(v). \tag{5.75}$$

So we have

$$\dot{T}^{L_1,L_2,2\pi}_{a\mathbf{e}'} = d_{a\mathbf{e}'} - 4\pi(a\mathbf{e}'|v)_{L_1,L_2} = -d_{a\mathbf{e}'} \tag{5.76}$$

$$\dot{T}^{L_1,L_2,2\pi}_{b\mathbf{f}'} = d_{b\mathbf{f}'} - 4\pi(b\mathbf{f}'|v)_{L_1,L_2} = 2\pi i b^t z_{L_1,L_2}(v) \tag{5.77}$$

recovering $d\widetilde{W}$ in p15 of [13].

5.6 Fock-Segal-Bargmann space

The transverse pair $(V^{1,0}_{J_0}, V^{0,1}_{J_0})$ corresponds to the matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1_n & -i \cdot 1_n \\ -i \cdot 1_n & 1_n \end{pmatrix}, \tag{5.78}$$

we can compute:

$$z_{V^{1,0}_{J_0}, V^{0,1}_{J_0}}(v) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1_n & i \cdot 1_n \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} = \frac{1}{\sqrt{2}}(q + ip) = z \tag{5.79}$$

$$\zeta_{V^{1,0}_{J_0}, V^{0,1}_{J_0}}(v) = \frac{1}{\sqrt{2}}\begin{pmatrix} i \cdot 1_n & 1_n \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} = \frac{i}{\sqrt{2}}(q - ip) = i\overline{z} \tag{5.80}$$

so $h(\cdot,\cdot)_{V^{1,0}_{J_0}, V^{0,1}_{J_0}}$ agrees with the standard inner product on $\mathbb{C}^n$, when $(\mathbb{C}^n, i)$ is identified with (V, J_0) as complex vector spaces.

Therefore

$$\int_V |f(z_{V^{1,0}_{J_0}, V^{0,1}_{J_0}}(v))|^2 e^{-h(v,v)_{V^{1,0}_{J_0}, V^{0,1}_{J_0}}} dqdp = \int_V |f(z)|^2 e^{-\overline{z}^t z} dqdp. \tag{5.81}$$

$\mathfrak{F}^1_{V^{1,0}_{J_0}, V^{0,1}_{J_0}}(V)$ recovers the Hilbert space in Equation (1.2), p 192 of [2] up to an overall constant factor of $1/\pi^n$.

We can recover the creation and annihilation operators. We first compute for $j = 1, \cdots, n$:

$$\begin{aligned}
\dot{T}_{\mathbf{e}_j}^{V_{J_0}^{1,0}, V_{J_0}^{0,1}, 1} &= d_{\mathbf{e}_j} - (\mathbf{e}_j | v)_{V^{1,0}, V^{0,1}} && (5.82)\\
&= d_{\mathbf{e}_j} + \frac{i}{2} \begin{pmatrix} q^t & p^t \end{pmatrix} \begin{pmatrix} 1_n & -i \cdot 1_n \\ i \cdot 1_n & 1 \end{pmatrix}_n \begin{pmatrix} e_j \\ 0 \end{pmatrix} && (5.83)\\
&= d_{\mathbf{e}_j} + \frac{i}{\sqrt{2}} z_j && (5.84)\\
\dot{T}_{\mathbf{f}_j}^{V_{J_0}^{1,0}, V_{J_0}^{0,1}, 1} &= d_{\mathbf{f}_j} - (\mathbf{f}_j | v)_{V^{1,0}, V^{0,1}} && (5.85)\\
&= d_{\mathbf{f}_j} + \frac{i}{2} \begin{pmatrix} q^t & p^t \end{pmatrix} \begin{pmatrix} 1_n & -i \cdot 1_n \\ i \cdot 1_n & 1_n \end{pmatrix} \begin{pmatrix} 0 \\ e_j \end{pmatrix} && (5.86)\\
&= d_{\mathbf{f}_0} + \frac{1}{\sqrt{2}} z_j. && (5.87)
\end{aligned}$$

Since $\mathcal{O}_{V_{J_0}^{1,0}, V_{J_0}^{0,1}}(V)$ consists of holomorphic functions, $i d_{\mathbf{f}_j} = -d_{\mathbf{e}_j}$. Therefore we obtain

$$\begin{aligned}
-\frac{i}{\sqrt{2}} \left(\dot{T}_{\mathbf{e}_j}^{V_{J_0}^{1,0}, V_{J_0}^{0,1}, 1} + i \dot{T}_{\mathbf{f}_j}^{V_{J_0}^{1,0}, V_{J_0}^{0,1}, 1} \right) &= \quad z_j \quad = \hat{a}^\dagger && (5.88)\\
-\frac{i}{\sqrt{2}} \left(\dot{T}_{\mathbf{e}_j}^{V_{J_0}^{1,0}, V_{J_0}^{0,1}, 1} - i \dot{T}_{\mathbf{f}_j}^{V_{J_0}^{1,0}, V_{J_0}^{0,1}, 1} \right) &= \quad \frac{\partial}{\partial z_j} \quad = \hat{a}_j. && (5.89)
\end{aligned}$$

5.7 Grossmann-Daubechies' family

Let J be a compatible complex structure on (V, ω) given by

$$(J)_{\{\mathbf{e},\mathbf{f}\}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n; \mathbb{R}). \tag{5.90}$$

We will show $\mathscr{H}^1_{V_J^{1,0}, V_J^{0,1}}(V)$ agrees with $\mathscr{H}_J$ in [7] and [8].

By Example 4.3.7 we have (Γ_1, Γ_2)-analytic functions are the J-holomorphic functions:

$$\mathcal{O}_{V_J^{1,0}, V_J^{1,0}}(V) = \mathcal{O}_J(V). \tag{5.91}$$

The Gram matrix of $h(\cdot,\cdot)_{V_J^{1,0},V_J^{0,1}}$ is

$$\frac{1}{2}\begin{pmatrix} CC^t+DD^t & -CA^t-DB^t \\ -AC^t-BD^t & BB^t+AA^t \end{pmatrix}+\frac{1}{2}\begin{pmatrix} 0 & -i\cdot 1_n \\ i\cdot 1_n & 0 \end{pmatrix}. \tag{5.92}$$

On the other hand, the Gram matrix of $1/2(\omega(\cdot,J\cdot)+i\omega(\cdot,\cdot))$ is

$$\frac{1}{2}\begin{pmatrix} CA^t+DB^t & DD^t+CC^t \\ -BB^t-AA^t & -AC^t-BD^t \end{pmatrix}\begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}+-\frac{1}{2}\begin{pmatrix} 0 & -i\cdot 1_n \\ i\cdot 1_n & 0 \end{pmatrix} \tag{5.93}$$

which agrees with the Gram matrix of $h(\cdot,\cdot)_{V_J^{1,0},V_J^{0,1}}$.

Therefore

$$\Omega_J(v):=e^{-\frac{1}{2}\omega(v,Jv)}=e^{-h(v,v)_{V_J^{1,0},V_J^{0,1}}} \tag{5.94}$$

and we recover $\mathscr{H}^1_{V_J^{1,0},V_J^{0,1}}(V)$ as the $\mathscr{H}_J$ defined in p1378.

Finally we recover the group action in p1378:

$$(W(a)f)(v)=e^{i\omega(a,v)}f(v-a)=(\tau_{-a}f)(v). \tag{5.95}$$

5.8 Satake's family

We will show $\mathfrak{F}^{-2\pi}_{\mathbf{L}_1,\Gamma_Z}(V)$ corresponds to the spaces $\mathfrak{F}_z$ that appears in [5]. We first derive the Hilbert space. For the family of matrices

$$\left\{\begin{pmatrix} 1_n & Z \\ 0 & 1_n \end{pmatrix}: Z=Z^t, \operatorname{Im} Z>0\right\} \tag{5.96}$$

We have $z_{\mathbf{L}_1,\Gamma_Z}(v)=q-Zp$, and identify

$$(V,J_0)\xrightarrow{\cong}(\mathbf{L}_1,i):v\mapsto z_{\mathbf{L}_1,\Gamma_Z}(v)\mathbf{e} \tag{5.97}$$

as complex vector spaces. Here $J_0:\mathbf{e}\mapsto\mathbf{f},\mathbf{f}\mapsto-\mathbf{e}$.

Suppose

$$x_Z:=\operatorname{Re} z_{\mathbf{L}_1,\Gamma_Z}\quad y_Z:=\operatorname{Im} z_{\mathbf{L}_1,\Gamma_Z}. \tag{5.98}$$

Then the coordinate change is

$$\begin{pmatrix} x_Z \\ y_Z \end{pmatrix} = \begin{pmatrix} 1_n & -\operatorname{Re} Z \\ 0 & -\operatorname{Im} Z \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \tag{5.99}$$

so the measures are related as follows:

$$\frac{1}{\det \operatorname{Im} Z} dx_Z dy_Z = dq dp. \tag{5.100}$$

Moreover,

$$h(v,v)_{\mathbf{L}_1,\Gamma_Z} = -p^t \operatorname{Im} Z p = -y_Z^t (\operatorname{Im} Z)^{-1} y_Z. \tag{5.101}$$

So $\mathfrak{F}^{-2\pi}_{\mathbf{L}_1,\Gamma_Z}(V)$ is identified with the holomorphic functions on $\mathbf{L}_1$ such that

$$\frac{1}{\det \operatorname{Im} Z} \int_{\mathbf{L}_1} |f(z_{\mathbf{L}_1,\Gamma_Z})|^2 e^{-2\pi y_Z^t (\operatorname{Im} Z)^{-1} y_Z} dx_Z dy_Z < \infty. \tag{5.102}$$

This is the integral that appears in Equation (5), p397. Now we will derive the automorphic factor η. Recall that

$$T^{\mathbf{L}_1,\Gamma_Z,2\pi}_{(s,u)} f(v) = f(v+u) e^{2\pi i s} e^{-2\pi (u|u+2v)_{\mathbf{L}_1,\Gamma_Z}}. \tag{5.103}$$

Let

$$(u)_{\{\mathbf{e},\mathbf{f}\}} = \begin{pmatrix} q \\ p \end{pmatrix} \quad (v)_{\{\mathbf{e},\mathbf{f}\}} = \begin{pmatrix} z \\ 0 \end{pmatrix} \quad (u_Z)_{\{\mathbf{e},\mathbf{f}\}} := \begin{pmatrix} z_{\mathbf{L}_1,\Gamma_Z}(u) \\ 0 \end{pmatrix}. \tag{5.104}$$

Then

$$\begin{aligned} -2\pi(u|u)_{\mathbf{L}_1,\Gamma_Z} &= \quad -\pi i (q^t p - p^t Z p) \quad &= 2\pi i \cdot \frac{1}{2} \omega^{\mathbb{C}}(u, u_Z) \quad &(5.105) \\ -2\pi(u|2v)_{\mathbf{L}_1,\Gamma_z} &= \quad 2\pi i z^t p \quad &= 2\pi i \cdot \omega^{\mathbb{C}}(u,v). \quad &(5.106) \end{aligned}$$

Therefore we have Satake's automorphic factor

$$\begin{aligned} \eta((s,u,\mathrm{Id}),(v,Z)) &= e^{2\pi i \left(s + \frac{1}{2}\omega^{\mathbb{C}}(u,u_Z) + \omega^{\mathbb{C}}(u,v)\right)} &(5.107) \\ &= e^{2\pi i s} e^{-2\pi (u|u+2v)_{\mathbf{L}_1,\Gamma_Z}}. &(5.108) \end{aligned}$$

This is a restriction of the expression that appears in Equation (2), p395.

In terms of the automorphic factor, we get

$$T^{\mathbf{L}_1,\Gamma_Z,-2\pi}_{(s,u)} f(z_{\mathbf{L}_1,\Gamma_Z}(v)) = \eta((s,u,\mathrm{Id}),(v,Z))^{-1} f(z_{\mathbf{L}_1,\Gamma_Z}(v+u)). \tag{5.109}$$

We recover a restriction of the group action defined in Equation (6), p 398.

5.9 Mumford's family

We will show $\mathfrak{F}^{-2\pi}_{\mathbf{L}_2,J_0\Gamma_Z}(V)$ corresponds to $\mathscr{H}^2_\vartheta(\mathbb{C}^n,Z)$ of [12]. Consider the family of matrices

$$\begin{pmatrix} 0 & -1_n \\ 1_n & Z \end{pmatrix} = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \begin{pmatrix} 1_n & Z \\ 0 & 1_n \end{pmatrix} \quad Z = Z^t \tag{5.110}$$

We can compute

$$z_{\mathbf{L}_2,J_0\Gamma_Z}(v) = \begin{pmatrix} Z & 1_n \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} = Zq + p \tag{5.111}$$

$$\zeta_{\mathbf{L}_2,J_0\Gamma_Z}(v) = \begin{pmatrix} -1_n & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} = -q \tag{5.112}$$

In particular, $z_{\mathbf{L}_2,\Gamma_Z}(v)$ agrees with the complex coordinate $\underline{v}$ in p19 of [12].

We can compute

$$h_{\mathbf{L}_2,J_0\Gamma_Z} = \frac{1}{2} \begin{pmatrix} -2\operatorname{Im} Z & -i\cdot 1_n \\ i \cdot 1_n & 0 \end{pmatrix} \tag{5.113}$$

so that $\mathfrak{F}^{-2\pi}_{\mathbf{L}_2,J_0\Gamma_Z}(V)$ is the completion of the functions analytic in $z_{\mathbf{L}_2,J_0\Gamma_Z}(v)$ and such that

$$\int_V |f(z_{\mathbf{L}_2,J_0\Gamma_Z}(v))|^2 e^{-2\pi q^t \operatorname{Im} Z q} dq dp < \infty. \tag{5.114}$$

This agrees with the equation on the top of p20 of [12].

Finally, we can compute the action. If

$$(u)_{\{\mathbf{e},\mathbf{f}\}} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (v)_{\{\mathbf{e},\mathbf{f}\}} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{5.115}$$

then

$$\left(T^{\mathbf{L}_2,J_0\Gamma_Z,-2\pi}_{(s,u)} f\right)\left(z_{\mathbf{L}_2,J_0\Gamma_Z}(v)\right) = e^{-2\pi i s} e^{2\pi i (u|u+2v)_{\mathbf{L}_2,J_0\Gamma_Z}} f(z_{\mathbf{L}_2,J_0\Gamma_Z}(v)) \tag{5.116}$$

where

$$\begin{aligned} 2\pi i (u|u+2v)_{\mathbf{L}_2,J_0\Gamma_Z} &= \pi i \Bigg(\begin{pmatrix} u_1^t & u_2^t \end{pmatrix} \begin{pmatrix} Z & 0 \\ 1_n & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} && (5.117) \\ &\quad +2 \begin{pmatrix} v_1^t & v_2^t \end{pmatrix} \begin{pmatrix} Z & 0 \\ 1_n & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Bigg) && (5.118) \\ &= 2\pi i \left(u_1^t z_{\mathbf{L}_2,J_0\Gamma_Z}(v) + \frac{1}{2} u_1^t z_{\mathbf{L}_2,J_0\Gamma_Z}(u) \right). && (5.119) \end{aligned}$$

We can see that this agrees with the equation on the bottom of p19 of [12].

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국문초록

다르부의 정리에 의해, 사교공간은 국소적으로 불변량을 가지지 않는다는 사실이 잘 알려져 있다. 하지만 많은 양자화 문제의 해법들에 의하면, 사교공간의 국소적인 구조들을 선택해야할 필요성이 제기된다. 이 논문에서 우리는 정준교환관계의 표현이 사교공간의 국소적인 성질을 기술하는 방법으로서 어떻게 나타나는지 연구한다. 그 결과로 정준교환관계의 기약표현들의 새로운 모임을 얻는다. 해석학적인 문제들이 남아있지만, 이 모임은 기존에 알려진 표현들의 모임들을 취합하고, 동형인 표현들의 매개집합을 확장하며, 위상적으로 자명하지 않은 표현들의 배열이 존재함을 보여준다. 서로 다른 표현들의 모임을 취합하는 구조가 기하학적으로 주어진다는 것도 주목할 점이다.

주요어휘: 정준교환관계, 하이젠베르크 군, 기약표현, 사교벡터공간, 복소 라그랑주 공간
학번: 2015-30967

감사의 글

사교기하학과 수학연구의 세계에 입문할 기회를 주시고, 오랫동안 함께 해주신 조철현 지도교수님께 감사드립니다. 논문을 심사해주실 때 피드백을 주신 Otto van Koert, 조철현, 이훈희, 권재훈, 박재석 교수님께 감사드립니다. 연구를 진행하기 어려웠을 때, 연구 내용에 관심을 가져주시고 지지해주신 Maurice de Gosson 교수님께 감사드립니다. 겹선형형식의 좌표불변한 꼴을 제안해주신 Yael Karshon 교수님께 감사드립니다. 긴 대학원 기간동안 함께 한 연구실 동료들, 조교실분들, 수리과학부 행정실 분들에게 감사드립니다. 졸업할 때까지 저를 믿어준 가족과 친구들, 그리고 하늘에 계신 박정식님께 감사드립니다.

힘든 때도 많았지만, 대학원에서 보낸 시간을 감사한 마음으로 기억하겠습니다.