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# Analytical Solution for Velocity Field Scattered by Submerged Permeable <br> <br> Breakwaters 

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수중 투과성 방파제에 의해 산란된 유속장의 해석해

2023년 2월

서울대학교 대학원
건설환경공학부 건설환경공학전공
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# Abstract <br> Analytical Solution for Velocity Field Scattered by Submerged Permeable Breakwaters 

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Presented herein are the formulation of the problem and its analytical solution for the velocity field scattered by submerged permeable breakwaters under the linear monochromatic wave.

This study set the problem that a permeable breakwater is submerged in the water with vertically occupying a finite interval under a small amplitude linear wave train. Assuming that the fluid has an infinite depth and the flow is incompressible, inviscid, and irrotational, the potential wave theory can be applied. However, the flow through the permeable plate imposes a nonlinear boundary condition on the plate.

Therefore, a perturbation method was applied to resolve this nonlinear boundary condition, with a small parameter representing the permeability. When the problem
was expanded up to the first order, the leading-order problem represents the velocity potential scattered by the impermeable breakwater, whereas the first-order problem gives the correction to the velocity potential considering the wave scattering by the permeable breakwater.

The reduction method was adopted to simplify the boundary conditions, replacing the spatial potential with the reduced potential. This leads to the homogeneous Riemann-Hilbert problem for the leading-order problem and the nonhomogeneous Riemann-Hilbert problem for the first-order problem. The exact, closed-form expressions of the velocity field for each problem were derived.

As an illustrative example of the application of the obtained velocity field, the reflection and transmission coefficients were calculated in various wave and breakwater conditions. Here, an approximate numerical quadrature method for evaluating finite Hilbert transform using Chebyshev polynomials was used. The results showed that the permeable breakwaters could dissipate more wave energy compared to the impermeable breakwaters. In addition to the evaluation of the wave attenuation efficiency, the velocity field can be utilized in various ways, such as calculating the hydrodynamic wave forces exerted on the breakwater.

Keywords: Permeable breakwater, Submerged floating breakwater, Linear wave, Perturbation method, Analytical solution

Student Number: 2021-23413

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## Notation

| $\operatorname{Im}_{(\cdot)}\{\diamond\}$ | Imaginary part of $\diamond$ with respect to $(\cdot)$ |
| :--- | :--- |
| $\operatorname{Re}_{(\cdot)}\{\diamond\}$ | Real part of $\diamond$ with respect to $(\cdot)$ |
| $A$ | Incident wave amplitude |
| $a_{n}$ | Absolute value of the upper point of the $n^{t h}$ plate |
| $a_{r}^{M}$ | Approximate coefficient |
| $b_{n}$ | Absolute value of the lower point of the $n^{t h}$ plate |
| $D$ | Thickness of a plate |
| $E(\varphi, m)$ | Incomplete elliptic integral of the second kind |
| $F(\varphi, m)$ | Incomplete elliptic integral of the first kind |
| $g$ | Gravitational acceleration |
| $i$ | Space-related imaginary unit $(=\sqrt{-1})$ |
| $I((\cdot) ; \diamond)$ | Finite Hilbert transform of $(\cdot)$ where the singularity is at $\diamond$ |
| $j$ | Time-related imaginary unit $(=\sqrt{-1})$ |
| $k$ | Incident wavenumber |
| $L_{n}$ | Interval that the $n^{t h}$ plate occupies |
| $L_{n}^{\prime}$ | Reflection of $L_{n}$ in the real axis |
| $m$ | Parameter of the elliptic integrals |
| $P$ | Number of the plates |
| $P$ | Pressure |
| $P$ | Atmospheric pressure |


| $p_{M}(t)$ | Approximate polynomial of degree $M$ |
| :---: | :---: |
| $Q_{M}((\cdot) ; \diamond)$ | Approximate quadrature of $(\cdot)$ with $M$-interpolating abscissa where the singularity is at $\diamond$ |
| $R$ | Reflection coefficient |
| $r$ | Distance from a point in the point to the near edge of a plate |
| T | Transmission coefficient |
| $T_{r}(t)$ | Chebyshev polynomials of the first kind |
| $W(z)$ | Reduced potential |
| $w(z)$ | Complex potential |
| $x$ | Abscissa of the Cartesian coordinate system representing the mean free surface |
| $y$ | Ordinate of the Cartesian coordinate system representing the vertical upwards direction |
| $\eta(x, t)$ | Free surface elevation |
| $\Gamma_{n}$ | Closed contour surrounding $L_{n}$ |
| $\kappa$ | Porosity of a plate |
| $\mu$ | Ratio of the absolute value of the top point and the bottom point of a breakwater |
| $\nabla_{2}$ | Two-dimensional gradient operator |
| $\nu$ | Kinematic viscosity of the water |
| $\omega$ | Incident wave angular frequency |
| $\phi(x, y)$ | Spatial velocity potential |
| $\Phi(x, y, t)$ | Velocity potential |
| $\psi(x, y)$ | Stream function |
| $\rho$ | Density of the water |
| $\varepsilon$ | Perturbation parameter |
| $\varphi$ | Argument of the elliptic integrals |


| $(\cdot)^{+\infty}$ | $(\cdot)$ at $x \rightarrow \infty$ |
| :--- | :--- |
| $(\cdot)^{+}$ | $(\cdot)$ on the positive side of the plates |
| $(\cdot)^{-\infty}$ | $(\cdot)$ at $x \rightarrow-\infty$ |
| $(\cdot)^{-}$ | $(\cdot)$ on the negative side of the plates |
| $[\cdot]$ | Floor function of $(\cdot)$ |
| $\overline{(\cdot)}$ | Complex conjugate of $(\cdot)$ |
| $(\cdot)_{0}$ | Leading order term of $(\cdot)$ |
| $(\cdot)_{1}$ | First order term of $(\cdot)$ |
| $(\cdot)_{I}$ | $(\cdot)$ of the incident wave |
| $(\cdot)_{R}$ | $(\cdot)$ of the reflected wave |
| $(\cdot)_{T}$ | $(\cdot)$ of the transmitted wave |

## Chapter 1. Introduction

### 1.1 Motivations

Waves from deep water propagate into the shoreline and affect the coastal area, often causing severe problems. Therefore, various shapes and functions of breakwaters have been investigated and constructed in maritime and offshore areas to dissipate wave energy from the open sea. By their properties, such as installation method, geometry, and permeability, breakwaters can be classified into various kinds.

The breakwaters can be classified into two broad categories by their method of installation: fixed type and floating type. Traditional gravity-based breakwaters were built on the ground and fixed by their own weight or pillar base in the ground. By their shape, they can be classified again as rubble mound type, vertical wall type, or composite type. Although these gravity-based type breakwaters are commonly seen by their effectiveness in breaking waves, there are some disadvantages in that they are typically expensive to build and difficult to remove since they resist the wave by incorporating sufficient mass. Furthermore, these conventional breakwaters are usually impermeable, so they may hinder seawater circulation, leading to ocean environmental problems. Thus, floating or permeable breakwaters have often been considered as the solution to this problem.

In recent decades, there has been a growing body of research that explores
floating breakwaters. Considering floating breakwaters are less reliant on the seabed topography and simply removable when additional sea space is required, they are recognized as adequate alternatives to traditional gravity-type breakwaters (Ji et al., 2017). In addition, they have the advantages of being able to circulate water, making a fishway, and transporting sediment under the breakwater, and may be relatively economical by protecting closer to the water surface where wave action is most noticeable (Isaacson et al., 1998). Thus, these structures may be appropriate to attenuate waves that are not extreme as tsunamis or storm waves, and especially useful to block a significant portion of the wave energy near the surface in the case of deep water waves (Briggs et al., 2002). Therefore, from an engineering perspective, investigating the floating breakwaters is an important problem.

One of the noteworthy works on floating breakwaters is the Rapidly Installed Breakwater System (RIBS) developed by U.S. Army Engineer Research and Development Center (ERDC) Coastal and Hydraulics Laboratory (Briggs et al., 2002). As shown in Fig. (1.1), RIBS has the shape of vertical barriers with a truss structure covered with fabric. Although floating breakwaters can have various shapes, such as a vertical barrier, a box, a pontoon, and so forth, highly inspired by this structure of RIBS, the present study gave attention to vertical-shaped floating breakwaters.

In addition to the vertical shape, floating breakwaters can either emerge to the water surface or submerge into the water. When it comes to the moored floating


Figure 1.1 (a) Rapidly Installed Breakwater System (RIBS) concept (Briggs, 2001); (b) Prototype design of RIBS(XM99), illustrating the structure of truss connected with fabric (Briggs et al., 2002)
breakwaters, not the freely floating breakwaters relying on pontoons or empty boxes, the breakwaters may be slightly submerged in the water. Although there are reports on emerging floating breakwaters or surface-piercing breakwaters, there are few studies on submerged floating breakwaters. Hence, the submerged floating breakwaters are sought with a concentration in this study.

Meanwhile, in certain instances, permeable breakwaters can also resolve the problems that conventional impermeable breakwaters have. For example, the permeable breakwaters can be selected in order to reduce excessive reflected waves by the impermeable barrier (Isaacson et al., 1998). Moreover, Lee and Chwang (2000) maintained that the porous barrier is effective in reducing the hydrodynamic wave forces applied to the breakwater, as well as attenuating wave amplitude. Thus, herein the effect of permeability of the breakwaters is considered necessary since the necessity of porous coastal structures has grown recently, from both an engineering and environmental perspective as prescribed.

In order to examine the use of submerged floating permeable breakwaters, the interaction between the breakwaters and the waves will be essential for designing this type of breakwater. For instance, when the wave velocity field scattered by the breakwater is given, engineers can easily calculate the factors used in designing breakwaters, such as wave force applied on breakwaters, desired reflection and transmission coefficients.

Therefore, from these points of view, this paper focuses on deriving an analytical solution for the velocity field in the two-dimensional wave scattering problem by an arbitrary number of vertical submerged floating permeable breakwaters in infinitedepth water, using linear water-wave theory. Considering that this paper focuses on the submerged case of the permeable breakwaters, hereinafter, the vertical submerged floating permeable breakwaters are denoted as the submerged permeable breakwaters.

### 1.2 Literature review

Dean (1945) first solved the wave scattering problem due to the semi-infinite vertical barrier at a distance below the water surface. Ursell (1948) obtained the velocity field of waves by a finite thin plate oscillating with a small angle, following the work of Havelock (1940). Subsequently, several authors, notably Lewin (1963) and Mei (1966), have contributed to generalizing the problem of wave generation and scattering by any number of vertical breakwaters in deep water. Later, Evans (1970) solved the diffraction problems on a completely submerged rolling plate in close form, determining the velocity potential everywhere in the fluid.

The problem of a permeable barrier has hitherto rarely received attention. Macaskill (1979) first tried to numerically solve the water wave reflection by a permeable barrier, although the permeable barrier was considered an impermeable barrier with numerous gaps. Chwang (1983) showed that the porosity can reduce the hydro-
dynamic force impinging on the wavemakers as well as the wave amplitude. Yu and Chwang (1994) thoroughly investigated the porous-effect parameter considering the friction resistance effect and the inertial effect in the porous medium, incorporating the results derived by Macaskill (1979) and Chwang (1983). Lee and Chwang (2000) applied eigenfunction expansion to convert the boundary value problems into certain dual series relations and solved them with the least square method. While the previous researchers used the domain decomposition method, Gayen and Mondal (2014) utilized a hypersingular integral equation for the discontinuity of the potential across the plate and numerically solved the equation with Chebyshev polynomial.

Fig. (1.2) shows the geometry of vertical breakwaters in previous studies, and Table (1.1) briefly summarises the earlier studies on vertical breakwaters.

In the present study, an analytic solution for the velocity potential of the scattering problem around submerged floating permeable breakwaters is obtained for the first time. A small parameter representing the permeability of each plate is defined, and a perturbation method is used to overcome the nonlinearity of the boundary condition.

### 1.3 Research objective

The research objectives of this study are listed below:
(i) Derive an analytical solution for the velocity potential around the submerged
permeable breakwaters;
(ii) Numerically calculate the analytical expression of the reflection and transmission coefficients.

Table 1.1 Previous studies on vertical breakwaters

| Author | Permeability | Plates \# | Remarks |
| :---: | :---: | :---: | :---: |
| Dean (1945) | X | 1 | Semi-infinite barrier |
| Ursell (1948) | X | 1 | Oscillating, <br> free surface punching plate |
| Lewin (1963) | X | N | Riemann-Hilbert problem |
| Mei (1966) | X | N fixed, <br> 1 rolling | Rolling, <br> free surface punching plate |
| Evans (1970) | X | 1 | Rolling, submerged plate |
| Macaskill (1979) | O | 1 | Set of impermeable plates |
| Lee and Chwang <br> (2000) | O | 1 or 2 | Finite depth, <br> eigenfunction series expansion |
| Gayen and Mondal <br> (2014) | O | 1 | Numerically solved a second kind <br> hypersingular integral equation |



Figure 1.2 Geometry of the breakwaters in (a) Dean (1945); (b) Ursell (1948); (c) Lewin (1963); (d) Mei (1966); (e) Evans (1970); (f) Macaskill (1979); (g) Lee and Chwang (2000); (h) Gayen and Mondal (2014)

## Chapter 2. Formulation

### 2.1 Statement of problem

As prescribed in Chapter 1, we consider a problem involving the interaction of linear monochromatic water waves and thin flat permeable plates. The geometry of the breakwater is illustrated in Fig. (2.1). Here, we take the Cartesian coordinate system $(x, y)$ where the $x$-axis is the mean free surface, and the $y$-axis directs vertically upwards. Under the water surface, $N$ permeable plates occupy the intervals $L_{n}: x=$ $0,-b_{n}<y<-a_{n}$, where $n=1,2,3, \cdots, N$. And on the water surface, a train of waves with a small amplitude $A$ and an angular wave frequency $\omega$ propagates from $x=+\infty$ in the negative $x$-direction.


Figure 2.1 Vertical plates occupy the intervals along the $y$ axis and a regular wave train of small amplitude propagates in the $x$-direction

In order to solve the problem in closed form using potential wave theory, it is assumed that the fluid has infinite depth and is incompressible, inviscid, and irrotational. Also, we assume that the wave amplitude is small and there is no wave breaking. With the first assumption, the viscous effects on the boundary layer of the fluid can be neglected since we are interested in the wave scattering problem, and hydrodynamic forces are essentially due to the pressure gradient.

For an irrotational flow in two-dimension, there exists a velocity potential $\Phi(x, y, t)$, and the negative of the gradient of the potential field becomes the velocity field, $\mathbf{v}=(u, v)$. This velocity potential will also have a frequency of $\omega$ and be harmonic in time. Without loss of generality, the velocity potential can be written as,

$$
\begin{equation*}
\Phi(x, y, t)=\phi(x, y) e^{-j \omega t} \tag{2.1}
\end{equation*}
$$

where $\phi(x, y)$ is the spatial velocity potential, and $j$ is the time-related imaginary unit defined as $j=\sqrt{-1}$.

We take the Euler equations for inviscid flow as the governing equations. Using the total velocity potential $\Phi$, these governing equations appear as,

$$
\begin{align*}
\frac{\partial^{2} \Phi}{\partial x^{2}}+\frac{\partial^{2} \Phi}{\partial y^{2}} & =0  \tag{2.2}\\
\frac{\partial \Phi}{\partial t}+\frac{1}{2} \nabla_{2} \Phi \cdot \nabla_{2} \Phi-g y & =\frac{1}{\rho}\left(P-P_{a}\right) \tag{2.3}
\end{align*}
$$

Here, $\nabla_{2}$ is the two-dimensional gradient operator, $g$ is the gravitational acceleration, $\rho$ is the density of the fluid, $P$ is the pressure, and $P_{a}$ is the atmospheric pressure. These fundamental equations have boundary conditions at the free surface and the bottom. Also, there is no surface tension or external force on the surface. Thus, the pressure at the free surface, $\eta$, is the same as the atmospheric pressure:

$$
\begin{equation*}
\frac{\partial \Phi}{\partial t}+\frac{1}{2} \nabla_{2} \Phi \cdot \nabla_{2} \Phi-g y=0, \quad \text { on } y=\eta . \tag{2.4}
\end{equation*}
$$

Expanding Eq. (2.4) into Taylor series with respect to $y=0$ and ignoring the higher-order terms, we get,

$$
\begin{equation*}
\eta=-\frac{1}{g} \frac{\partial \Phi}{\partial t}, \quad \text { on } y=0 \tag{2.5}
\end{equation*}
$$

which is called linearized dynamic free surface boundary condition.
Next, we require a kinematic boundary condition that the free surface remains the free surface. That is to say, the water particles in the free surface do not change, and this condition can be imposed by using the material derivative as below:

$$
\begin{equation*}
\frac{D}{D t}(y-\eta)=0, \quad \text { on } y=\eta \tag{2.6}
\end{equation*}
$$

where $\frac{D}{D t}=\frac{\partial}{\partial t}+\mathbf{v} \cdot \nabla$.
Again, we obtain the linearized kinematic free surface boundary condition as
below by expanding Eq. (2.6) around $y=0$ and neglecting the small higher-order terms:

$$
\begin{equation*}
\frac{\partial \Phi}{\partial y}=\frac{\partial \eta}{\partial t}, \quad \text { on } y=0 \tag{2.7}
\end{equation*}
$$

Combining Eq. (2.5) and Eq. (2.7) with differentiating Eq. (2.5) with respect to $t$ once, we have the single form of free surface boundary condition in terms of the velocity potential only:

$$
\begin{equation*}
\frac{\partial^{2} \Phi}{\partial t^{2}}+g \frac{\partial \Phi}{\partial y}=0, \quad \text { on } y=0 \tag{2.8}
\end{equation*}
$$

Now, since we defined the velocity potential as Eq. (2.1), we can simplify the governing equation and the combined free surface boundary condition by substituting Eq. (2.1) into Eq. (2.2) and Eq. (2.8):

$$
\begin{align*}
& \frac{\partial^{2} \phi}{\partial x^{2}}+\frac{\partial^{2} \phi}{\partial y^{2}}=0, \quad \text { in the fluid, }  \tag{2.9}\\
& k \phi-\frac{\partial \phi}{\partial y}=0, \text { on } y=0 \tag{2.10}
\end{align*}
$$

Here, $k=\omega^{2} / g$ is known as the wavenumber. The dispersion relation of $\omega$ and $k$ at depth $h$ is originally $\omega^{2}=g k \tanh k h$, but from the infinite-depth condition, the dispersion relation is reduced to $\omega^{2}=g k$.

Linear wave theory allows us to split the potential into the incident and scattered wave potential. The incident wave velocity potential $\Phi_{I}$ is also represented with
the combination of spatial-related potential and the time-harmonic part, as below:

$$
\begin{equation*}
\Phi_{I}(x, y, t)=\phi_{I}(x, y) e^{-j \omega t} \tag{2.11}
\end{equation*}
$$

Since the incident wave elevation is aforementioned as

$$
\begin{equation*}
\eta_{I}(x, t)=-\frac{1}{g} \frac{\partial \Phi_{I}}{\partial t}=A \cos (k x+\omega t) \tag{2.12}
\end{equation*}
$$

the incident spatial velocity potential in Eq. (2.11) would be

$$
\begin{equation*}
\phi_{I}(x, y)=-\frac{j g A}{\omega} e^{k y-j k x} . \tag{2.13}
\end{equation*}
$$

Since the scattering of the wave is due to the presence of the plate, this indicates that the radiation condition for the spatial velocity potential is needed. The scattered wave propagating to the positive side of the $x$-axis will be superposed with the incident wave, and the scattered wave propagating to the negative side of the $x$-axis will travel outwards to $-\infty$. Therefore, the desired behavior at $x \rightarrow \pm \infty$ would be,

$$
\begin{array}{ll}
\phi^{+\infty}(x, y) \sim A^{+\infty} e^{k y+j k x}-\frac{j g A}{\omega} e^{k y-j k x}, & \\
\phi^{-\infty}(x, y) \sim+\infty  \tag{2.15}\\
\sim A^{-\infty} e^{k y-j k x}, & x \rightarrow-\infty .
\end{array}
$$

for some constants $A^{ \pm \infty}$. Superscripts $(\cdot)^{+\infty}$ and $(\cdot)^{-\infty}$ mean $(\cdot)$ at $x \rightarrow \pm \infty$, respec-
tively. $A^{+\infty} e^{k y+j k x}$ in Eq. (2.14) represents the spatial velocity potential generated by the reflected wave propagating to $+x$ direction, and $A^{-\infty} e^{k y-j k x}$ in Eq. (2.15) represents the spatial velocity potential due to the transmitted wave propagating to $-x$ direction.

Also, the fluid velocity will vanish as $y \rightarrow-\infty$. Thus,

$$
\begin{equation*}
\frac{\partial \phi}{\partial x} \sim 0, \frac{\partial \phi}{\partial y} \sim 0, \quad y \rightarrow-\infty \tag{2.16}
\end{equation*}
$$

The velocity components are bounded everywhere, but at the edges of the plates, the velocity may be unbounded and permit a mild, integrable singularity as below:

$$
\begin{equation*}
\frac{\partial \phi}{\partial r}=\mathcal{O}\left(\frac{1}{r^{\lambda}}\right), \quad 0<\lambda<1, \text { near } z=-i a_{n},-i b_{n} \tag{2.17}
\end{equation*}
$$

Here, $r$ is the distance from a point in the fluid to either of these points.

### 2.2 Permeable boundary condition on the plates

As suggested by Taylor (1956), the boundary condition on the permeable breakwaters is obtained with the assumption that the flow through the permeable plates is due to the pressure difference between both sides of the plates. This assumption makes the
boundary condition in Eq. (2.18) have a similar form to Darcy's law:

$$
\begin{equation*}
\frac{\partial \Phi}{\partial x}=-\frac{\kappa}{\rho \nu D}\left(p^{+}-p^{-}\right), \quad \text { on } L_{n}, n=1,2,3, \cdots, N \tag{2.18}
\end{equation*}
$$

Here, $\kappa$ is the porosity of the plates, $\rho$ is the density of the water, $\nu$ is the kinematic viscosity of the water, $D$ is the thickness of the plates, and $p^{+}$and $p^{-}$are the pressure on the positive and the negative side of the plates, respectively.

Meanwhile, for an infinitesimal amplitude, Bernoulli's equation may be linearized, and the pressure can be expressed as,

$$
\begin{equation*}
p(x, y, t)=-\rho \frac{\partial \Phi}{\partial t}(x, y, t)-\rho g y . \tag{2.19}
\end{equation*}
$$

Substituting Eq. (2.19) into Eq. (2.18), the spatial derivative of the total velocity potential is represented by the difference with the time derivative of the total wave on both sides of the plates:

$$
\begin{equation*}
\frac{\partial \Phi}{\partial x}=\frac{\kappa}{\nu D}\left(\frac{\partial \Phi^{+}}{\partial t}-\frac{\partial \Phi^{-}}{\partial t}\right), \quad \text { on } L_{n} \tag{2.20}
\end{equation*}
$$

Superscripts $(\cdot)^{+}$and $(\cdot)^{-}$mean $(\cdot)$ on the positive side and the negative side of the plates, respectively. Here, $\kappa / \nu D$ can be considered as a parameter representing permeability. Since $\Phi(x, y, t)=\phi(x, y) e^{-j w t}$ from Eq. (2.1), we can simplify Eq.
(2.20) as below:

$$
\begin{equation*}
\frac{\partial \phi}{\partial x}=-j \frac{\kappa \omega}{\nu D}\left(\phi^{+}-\phi^{-}\right), \quad \text { on } L_{n} \tag{2.21}
\end{equation*}
$$

In Eq. (2.21), the spatial velocity potential itself is affected by the spatial velocity potential on both sides of the plates. This leads to the nonlinearity of the problem, and the perturbation method is introduced to solve this problem.

### 2.3 Perturbation expansion

To convert the original problem into a perturbation problem, define a small, dimensionless parameter $\varepsilon$ as

$$
\begin{equation*}
\varepsilon=\frac{\kappa \omega}{\nu D k} . \tag{2.22}
\end{equation*}
$$

Introducing the small parameter $\varepsilon$, we may assume a perturbative solution to Eq. (2.21) of the form,

$$
\begin{equation*}
\phi=\phi_{0}+\varepsilon \phi_{1}+\cdots . \tag{2.23}
\end{equation*}
$$

Here in Eq. (2.23), $\phi_{0}$ represents the spatial velocity potential due to the impermeable plates, and $\phi_{1}$ shows the correction when the plate has permeability. Thus, $\phi_{0}$ will explain the tendency of the solution, where $\phi_{1}$ adds some correction on $\phi_{0}$ and explains the effect of permeability.

Although there are higher-order terms in Eq. (2.23), this paper only considers the leading and first-order terms. In order to investigate the effect of the permeability
of the breakwater.

Substituting Eq. (2.23) into Eq. (2.21), we can get the boundary conditions of the leading-order and the first-order spatial velocity potential function on the vertical plates:

$$
\begin{gather*}
\frac{\partial \phi_{0}}{\partial x}=0 \quad \text { on } L_{n}, \quad n=1,2,3, \cdots, N  \tag{2.24}\\
\frac{\partial \phi_{1}}{\partial x}=-j k\left(\phi_{0}^{+}-\phi_{0}^{-}\right), \quad \text { on } L_{n}, \quad n=1,2,3, \cdots, N . \tag{2.25}
\end{gather*}
$$

Also, in view of Eq. (2.14) and Eq. (2.15), we require the behavior of $\phi_{0}$ at $x \rightarrow \pm \infty$ to be,

$$
\begin{array}{ll}
\phi_{0}^{+\infty}(x, y) \sim A_{0}^{+\infty} e^{k y+j k x}-\frac{j g A}{\omega} e^{k y-j k x}, & x \rightarrow+\infty \\
\phi_{0}^{-\infty}(x, y) \sim A_{0}^{-\infty} e^{k y-j k x}, & x \rightarrow-\infty \tag{2.27}
\end{array}
$$

and for $\phi_{1}$,

$$
\begin{array}{ll}
\phi_{1}^{+\infty}(x, y) \sim A_{1}^{+\infty} e^{k y+j k x}, & x \rightarrow+\infty \\
\phi_{1}^{-\infty}(x, y) \sim A_{1}^{-\infty} e^{k y-j k x}, & x \rightarrow-\infty \tag{2.29}
\end{array}
$$

From the equation, it is intuitively seen that the first order solution $\phi_{1}$ does not affect by the incident wave but is generated by the leading order solution.

With these boundary conditions on the plate and at $x \rightarrow \pm \infty$, we can derive the leading order and the first order solution of the spatial velocity potential.

## Chapter 3. Leading-order solution

### 3.1 Leading-order solution for the case of multiple plates

Define the complex potential $w_{0}(z)$ as,

$$
\begin{equation*}
w_{0}(z)=\phi_{0}(x, y)+i \psi_{0}(x, y), \quad \operatorname{Im}_{i}\{z\}<0 \tag{3.1}
\end{equation*}
$$

where $z=x+i y, i=\sqrt{-1}$ and $\psi_{0}(x, y)$ is the two-dimensional stream function. Here, $i$ is independent of $j$ and is used to explain the complex potential $w_{0}(z)$ as a space-related imaginary unit.

In addition, since the breakwater has the form of thin plates, it is convenient to consider the reduced potential $W_{0}(z)$, defined by,

$$
\begin{equation*}
W_{0}(z)=\frac{d w_{0}}{d z}+i k w_{0}, \quad \operatorname{Im}_{i}\{z\}<0 \tag{3.2}
\end{equation*}
$$

This method is called the reduction method, which is effective for simplifying the boundary conditions (Porter, 1972). By the use of the reduced potential, from Eq. (2.10),

$$
\begin{equation*}
\operatorname{Im}_{i}\left\{W_{0}(z)\right\}=0, \quad \text { along the horizontal axis }(\mathrm{y}=0) \tag{3.3}
\end{equation*}
$$

If the imaginary part of $W_{0}(z)$ is zero on $x$-axis, the function $\overline{W_{0}(z)}$ is the
analytic continuation of $W_{0}(z)$ through this interval (Muskhelishvili, 1977). This property is called Schwarz's principle of reflection. Therefore, $W_{0}(z)$ can be defined not only on the lower half-plane but also on the upper half-plane. Now, $W_{0}(z)$ can be continued by Schwarz's reflection principle into $y>0$, where,

$$
\begin{equation*}
W_{0}(\bar{z})=\overline{W_{0}(z)} . \tag{3.4}
\end{equation*}
$$

Since $W_{0}(z)$ is a single-valued function outside the circle $|z|=b_{N}$, it has a Laurent expansion of the form,

$$
\begin{equation*}
W_{0}(z)=\sum_{n=-\infty}^{\infty} c_{n} z^{n} . \tag{3.5}
\end{equation*}
$$

Assume Eq. (2.14) and Eq. (2.15) may be differentiated once with respect to $x$ or $y$. Then it follows that,

$$
\begin{equation*}
W_{0}(z)=\mathcal{O}(1), \quad|z| \rightarrow \infty \tag{3.6}
\end{equation*}
$$

which means that $W_{0}(z)$ is bounded as $z$ goes to infinity.
In addition, from Eq. (2.24), the real part of $W_{0}(z)$ is zero on the plate:

$$
\begin{equation*}
\operatorname{Re}_{i}\left\{W_{0}(z)\right\}=0, \quad \text { on } L_{n}, \quad n=1,2,3, \cdots, N . \tag{3.7}
\end{equation*}
$$

From Eq. (3.4), the equation above can be expanded through the whole complex plane:

$$
\begin{equation*}
\operatorname{Re}_{i}\left\{W_{0}(z)\right\}=0, \quad \text { on } L_{n}+L_{n}^{\prime} \tag{3.8}
\end{equation*}
$$

where $L_{n}^{\prime}$ is the interval $x=0, a_{n}<y<b_{n}$, the reflection of $L_{n}$ with respect to the real axis.

Finally, $W_{0}(z)$ may be unbounded near the ends of $L_{n}, L_{n}^{\prime}$. Thus,

$$
\begin{equation*}
W_{0}(z)=\mathcal{O}\left(\frac{1}{r^{\lambda}}\right), \quad 0<\lambda<1, \text { near } z= \pm i a_{n}, \pm i b_{n} \tag{3.9}
\end{equation*}
$$

The problem of determining $W_{0}(z)$ satisfying Eq. (3.6), Eq. (3.8), and Eq. (3.9) is a typical homogeneous Riemann-Hilbert problem.

The solution of the homogeneous Riemann-Hilbert problem for the plane with cuts distributed along a straight line is given by Muskhelishvili (1977, p. 261):

$$
\begin{equation*}
W_{0}(z)=\frac{C_{0}+\sum_{n=1}^{N} C_{n} z^{2 n}}{\prod_{n=1}^{N} \sqrt{z^{2}+a_{n}^{2}} \sqrt{z^{2}+b_{n}^{2}}} . \tag{3.10}
\end{equation*}
$$

Here, $C_{0}$ and $C_{n}(n=1,2,3, \cdots, N)$ are real constants with respect to i. Substituting Eq. (3.10) into Eq. (3.2), the complex potential $w_{0}(z)$ is found by integrating Eq. (3.2):

$$
\begin{equation*}
w_{0}(z)=e^{-i k z}\left[B_{0}+\int_{-i a_{1}}^{z} e^{i k \zeta} W_{0}(\zeta) d \zeta\right] \tag{3.11}
\end{equation*}
$$

where $B_{0}$ is an arbitrary real constant with respect to $i$.

It remains to determine the constants $B_{0}, C_{0}$, and $C_{n}$. From the first assumption that the circulation around each plate is zero, which implies the potential is single-valued within the fluid, it follows that,

$$
\begin{equation*}
\operatorname{Re}_{i}\left\{\oint_{\Gamma_{n}} e^{i k \zeta} W_{0}(\zeta) d \zeta\right\}=0 \tag{3.12}
\end{equation*}
$$

where $\Gamma_{n}$ is a closed contour surrounding $L_{n}$, and yields $N$ conditions.

Eq. (2.26) and Eq. (2.27), the behavior of the solution away from the plates, provide two remaining conditions, which now complete the solution.

### 3.2 Leading-order solution for the case of a single plate

For an illustrative example, the case of the breakwater consisting of a single plate is sought, and the process of calculating the undetermined constants is presented in detail in this section. When $N=1$, Eq. (3.10) and Eq. (3.11) are simplified to,

$$
\begin{gather*}
W_{0}(z)=\frac{C_{0}+D_{0} z^{2}}{\sqrt{\left(z^{2}+a^{2}\right)\left(z^{2}+b^{2}\right)}}  \tag{3.13}\\
w_{0}(z)=e^{-i k z}\left[B_{0}+\int_{-i a}^{z} e^{i k \zeta} W_{0}(\zeta) d \zeta\right] . \tag{3.14}
\end{gather*}
$$

To determine the unknown coefficients, as in the case of multiple plates, the
radiation boundary conditions prescribed in Eq. (2.26) and Eq. (2.27) will be first imposed onto Eq. (3.14). To do this, the asymptotic behavior of the integral term in Eq. (3.14) is sought.


Figure 3.1 Integration path for $w_{0}(z)$ as $z \rightarrow \pm \infty$

For $z \rightarrow+\infty$, let us modify the integral term in Eq. (3.14) as,

$$
\begin{align*}
& \int_{-i a}^{z} e^{i k \zeta} W_{0}(\zeta) d \zeta \\
& =\int_{-i a}^{0^{+}+i \infty} e^{i k \zeta} W_{0}(\zeta) d \zeta+\int_{i \infty+0+}^{z} e^{i k \zeta}\left(W_{0}(\zeta)-D_{0}\right) d \zeta-\frac{i D_{0}}{k} e^{i k z} \tag{3.15}
\end{align*}
$$

When the second integral in Eq. (3.15) is taken following a large arc, which is noted as $\Gamma_{+}$in Fig. (3.1), this term vanishes to zero by Jordan's lemma, since $W_{0}(z)$ is bounded at infinity and $W_{0}(z)-D_{0} \rightarrow 0$ as $|z| \rightarrow \infty$. Thus, for $z \rightarrow+\infty$ the
complex potential turns into,

$$
\begin{equation*}
w_{0}^{+\infty}(z) \sim e^{-i k z}\left[B_{0}+\int_{-i a}^{i \infty+0+} e^{i k \zeta} W_{0}(\zeta) d \zeta\right]-\frac{i D_{0}}{k} \tag{3.16}
\end{equation*}
$$

Contracting the integral term in Eq. (3.16) onto the $y$-axis from the right, the integration should be separately examined for some interval parts on account of the branch cut $(a, b)$ in the reduced potential:

$$
\begin{align*}
& \int_{-i a}^{0^{+}+i \infty} e^{i k \zeta} W_{0}(\zeta) d \zeta \\
& =\int_{-a}^{\infty} e^{-k u} W_{0}^{+}(i u) i d u \\
& =-\int_{-a}^{\infty} e^{-k u} \operatorname{Im}_{i}\left\{W_{0}^{+}(i u)\right\} d u+i \int_{-a}^{\infty} e^{-k u} \operatorname{Re}_{i}\left\{W_{0}^{+}(i u)\right\} d u \\
& =-\int_{a}^{b} \frac{e^{-k u}\left(C_{0}-D_{0} u^{2}\right)}{\sqrt{\left(u^{2}-a^{2}\right)\left(b^{2}-u^{2}\right)}} d u \\
& +i\left(\int_{-a}^{a} \frac{e^{-k u}\left(C_{0}-D_{0} u^{2}\right)}{\sqrt{\left(a^{2}-u^{2}\right)\left(b^{2}-u^{2}\right)}} d u-\int_{b}^{\infty} \frac{e^{-k u}\left(C_{0}-D_{0} u^{2}\right)}{\sqrt{\left(u^{2}-a^{2}\right)\left(b^{2}-u^{2}\right)}} d u\right) \tag{3.17}
\end{align*}
$$

Although it is conventional to denote the left-sided approach of the arc as + side, in the sense that $W_{0}(z)$ is concerned on the positive $x$ side, $(\cdot)^{+}$is used for representing the right side-approaching from the right side-of the plate.

To simplify the expression, it is convenient to define some functions repre-
senting each interval along the integration path,

$$
\begin{align*}
& a_{1}(k)=\int_{a}^{b} \frac{e^{-k u}}{\sqrt{\left(u^{2}-a^{2}\right)\left(b^{2}-u^{2}\right)}} d u  \tag{3.18}\\
& a_{2}(k)=\int_{-a}^{a} \frac{e^{-k u}}{\sqrt{\left(a^{2}-u^{2}\right)\left(b^{2}-u^{2}\right)}} d u  \tag{3.19}\\
& a_{3}(k)=\int_{b}^{\infty} \frac{e^{-k u}}{\sqrt{\left(u^{2}-a^{2}\right)\left(u^{2}-b^{2}\right)}} d u \tag{3.20}
\end{align*}
$$

and their second derivative with respect to $k$,

$$
\begin{equation*}
a_{i}^{\prime \prime}(k)=\frac{d^{2} a_{i}}{d k^{2}}, \quad i=1,2,3 \tag{3.21}
\end{equation*}
$$

Fig. (3.2-3.3) are figures of constants $a_{i}(k), a_{i}^{\prime \prime}(k), a_{1}(-k)$ and $a_{1}^{\prime \prime}(-k)$ varying with $\mu$ and $k b$. Here, $\mu$ is the ratio of the absolute values of the top point of the breakwater, $a$, and the bottom end of the breakwater, $b$, defined as $\mu=a / b$.

Then, for $z \rightarrow+\infty$,

$$
\begin{align*}
w_{0}^{+\infty}(z) & \sim e^{-i k z}\left[B_{0}-\left(C_{0} a_{1}(k)-D_{0} a_{1}^{\prime \prime}(k)\right)\right. \\
& \left.+i\left(\left(C_{0} a_{2}(k)-D_{0} a_{2}^{\prime \prime}(k)\right)-\left(C_{0} a_{3}(k)-D_{0} a_{3}^{\prime \prime}(k)\right)\right)\right]-\frac{i D_{0}}{k} \tag{3.22}
\end{align*}
$$

Again for brief-expression, some functions are defined for each interval:

$$
\begin{equation*}
\gamma_{0}(k)=C_{0} a_{1}(k)-D_{0} a_{1}^{\prime \prime}(k), \tag{3.23}
\end{equation*}
$$



Figure 3.2 (left) $a_{i}(k)$ and (right) $a_{i}^{\prime \prime}(k)$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (—, $a_{1}(k)$ and $a_{1}^{\prime \prime}(k)$; $\cdot-\cdot$, $a_{2}(k)$ and $a_{2}^{\prime \prime}(k) ; \cdots \cdots, a_{3}(k)$ and $\left.a_{3}^{\prime \prime}(k).\right)$


Figure 3.2 (left) $a_{i}(k)$ and (right) $a_{i}^{\prime \prime}(k)$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (一, $a_{1}(k)$ and $a_{1}^{\prime \prime}(k)$; --•, $a_{2}(k)$ and $a_{2}^{\prime \prime}(k) ; \cdots \cdots, a_{3}(k)$ and $a_{3}^{\prime \prime}(k)$.) (cont.)


Figure 3.3 (left) $a_{1}(-k)$ and (right) $a_{1}^{\prime \prime}(-k)$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$.


Figure 3.3 (left) $a_{1}(-k)$ and (right) $a_{1}^{\prime \prime}(-k)$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$ (cont.).

$$
\begin{align*}
& \alpha_{0}(k)=C_{0} a_{2}(k)-D_{0} a_{2}^{\prime \prime}(k),  \tag{3.24}\\
& \beta_{0}(k)=C_{0} a_{3}(k)-D_{0} a_{3}^{\prime \prime}(k) \tag{3.25}
\end{align*}
$$

Finally, Eq. (3.22) is shortened as below:

$$
\begin{equation*}
w_{0}^{+\infty}(z) \sim e^{-i k z}\left[B_{0}-\gamma_{0}(k)+i\left(\alpha_{0}(k)-\beta_{0}(k)\right)\right]-\frac{i D_{0}}{k} . \tag{3.26}
\end{equation*}
$$

Similarly, for $z \rightarrow-\infty$, the integration path of an integral part in Eq. (3.14) is taken as a union of a vertical upward line contracted to the left side of the $y$-axis and an arc noted as $\Gamma_{-}$. Because of the branch cut on $(a, b)$, the imaginary part of $W_{0}^{-}(i u)$ has a different sign from $W_{0}^{+}(i u)$. Thus, the asymptotic behavior of leading-order complex potential as $z \rightarrow \pm \infty$ is expressed in terms of $\gamma_{0}(k), \alpha_{0}(k)$, and $\beta_{0}(k)$ :

$$
\begin{equation*}
w_{0}^{ \pm \infty}(z) \sim e^{-i k z}\left[B_{0} \mp \gamma_{0}(k)+i\left(\alpha_{0}(k)-\beta_{0}(k)\right)\right]-\frac{i D_{0}}{k} \tag{3.27}
\end{equation*}
$$

Taking the real part of Eq. (3.27) with respect to $i$, the leading-order spatial velocity potential for $z \rightarrow \pm \infty$ can be obtained:

$$
\begin{align*}
\phi_{0}^{ \pm \infty}(x, y) & =\operatorname{Re}_{i}\left\{w_{0}(z)\right\} \\
& \sim e^{k y}\left[\left(B_{0} \mp \gamma_{0}(k)\right) \cos k x+\left(\alpha_{0}(k)-\beta_{0}(k)\right) \sin k x\right] \tag{3.28}
\end{align*}
$$

Substituting this into the radiation boundary conditions-Eq. (2.26) and Eq. (2.27)—and dividing both sides with $e^{k y}$, a pair of equations is obtained:

$$
\begin{align*}
& \left(B_{0}-\gamma_{0}(k)\right) \cos k x+\left(\alpha_{0}(k)-\beta_{0}(k)\right) \sin k x \\
& \quad=\left(A_{0}^{+\infty}-\frac{j g A}{\omega}\right) \cos k x+\left(j A_{0}^{+\infty}-\frac{g A}{\omega}\right) \sin k x \tag{3.29}
\end{align*}
$$

$\left(B_{0}+\gamma_{0}(k)\right) \cos k x+\left(\alpha_{0}(k)-\beta_{0}(k)\right) \sin k x$

$$
\begin{equation*}
=A_{0}^{-\infty} \cos k x-j A_{0}^{-\infty} \sin k x \tag{3.30}
\end{equation*}
$$

which gives a set of simultaneous equations:

$$
\left\{\begin{array}{l}
B_{0}-\gamma_{0}(k)=A_{0}^{+\infty}-\frac{j g A}{\omega}  \tag{3.31a}\\
\alpha_{0}(k)-\beta_{0}(k)=j A_{0}^{+\infty}-\frac{g A}{\omega} \\
B_{0}+\gamma_{0}(k)=A_{0}^{-\infty} \\
\alpha_{0}(k)-\beta_{0}(k)=-j A_{0}^{-\infty}
\end{array}\right.
$$

From the equation set above, the value of $B_{0}$ is found as,

$$
\begin{equation*}
B_{0}=-\frac{j g A}{\omega} \tag{3.32}
\end{equation*}
$$

and $A_{0}^{ \pm \infty}$ as,

$$
\begin{equation*}
A_{0}^{+\infty}=-\gamma_{0}(k)=-j\left(\alpha_{0}(k)-\beta_{0}(k)+\frac{g A}{\omega}\right) \tag{3.33}
\end{equation*}
$$

$$
\begin{equation*}
A_{0}^{-\infty}=\gamma_{0}(k)-\frac{j g A}{\omega}=j\left(\alpha_{0}(k)-\beta_{0}(k)\right) \tag{3.34}
\end{equation*}
$$

The relation of $\alpha_{0}(k), \beta_{0}(k)$, and $\gamma_{0}(k)$ in Eq. (3.33) and Eq. (3.34) can be rearranged with respect to $C_{0}$ and $D_{0}$, and this gives one equation for determining $C_{0}$ and $D_{0}$ :

$$
\begin{align*}
\left(a_{1}(k)-j\left(a_{2}(k)-a_{3}(k)\right)\right) C_{0}-\left(a_{1}^{\prime \prime}(k)-j\left(a_{2}^{\prime \prime}(k)-a_{3}^{\prime \prime}(k)\right)\right. & \left.D_{0}\right) \\
& =\frac{j g A}{\omega} \tag{3.35}
\end{align*}
$$

In addition to the radiation boundary condition, the zero-circulation condition can be applied to find $C_{0}$ and $D_{0}$. Assuming that the circulation around a plate is zero and applying the zero-circulation condition Eq. (3.12) with contracting the path of integration onto $L$,

$$
\begin{align*}
\operatorname{Re}_{i}\left\{\oint_{\Gamma} e^{i k \zeta} W_{0}(\zeta) d \zeta\right\} & =\int_{-b}^{-a} \frac{e^{-k u}\left(C_{0}-D_{0} u^{2}\right)}{\sqrt{\left(u^{2}-a^{2}\right)\left(b^{2}-u^{2}\right)}} d u \\
& =\int_{a}^{b} \frac{e^{k u}\left(C_{0}-D_{0} u^{2}\right)}{\sqrt{\left(u^{2}-a^{2}\right)\left(b^{2}-u^{2}\right)}} d u=0 . \tag{3.36}
\end{align*}
$$

Introducing the functions in Eq. (3.18) and Eq. (3.23), a simplified expression of Eq. (3.36) is given below:

$$
\begin{equation*}
\gamma_{0}(-k)=a_{1}(-k) C_{0}-a_{1}^{\prime \prime}(-k) D_{0}=0 . \tag{3.37}
\end{equation*}
$$

Thus, combining Eq. (3.35) and Eq. (3.37), a system of linear equations in $C_{0}$ and $D_{0}$ is obtained. Especially in a matrix form,

$$
\left[\begin{array}{cc}
a_{1}(k)-j\left(a_{2}(k)-a_{3}(k)\right) & -a_{1}^{\prime \prime}(k)+j\left(a_{2}^{\prime \prime}(k)-a_{3}^{\prime \prime}(k)\right)  \tag{3.38}\\
a_{1}(-k) & -a_{1}^{\prime \prime}(-k)
\end{array}\right]\left[\begin{array}{l}
C_{0} \\
D_{0}
\end{array}\right]=\left[\begin{array}{c}
\frac{j g A}{\omega} \\
0
\end{array}\right] .
$$

Using Cramer's rule, the unknown constants $C_{0}$ and $D_{0}$ can be determined after some algebra:

$$
\begin{align*}
C_{0} & =\frac{j g A}{\omega} \frac{a_{1}^{\prime \prime}(-k)}{\Delta_{123}}  \tag{3.39}\\
D_{0} & =\frac{j g A}{\omega} \frac{a_{1}(-k)}{\Delta_{123}} \tag{3.40}
\end{align*}
$$

where,

$$
\begin{gather*}
\Delta_{123}=\Delta_{11}-j\left(\Delta_{12}-\Delta_{13}\right),  \tag{3.41}\\
\Delta_{1 i}=\left|\begin{array}{cc}
a_{i}(k) & a_{1}(-k) \\
a_{i}^{\prime \prime}(k) & a_{1}^{\prime \prime}(-k)
\end{array}\right|, \quad \text { for } i=1,2,3 . \tag{3.42}
\end{gather*}
$$

Constants $\Delta_{1 i}$ and $C_{0}, D_{0}$ for various $\mu$ and $k b$ conditions are illustrated below in Fig. (3.4-3.5).


Figure $3.4 \Delta_{1 i}$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (一, $\Delta_{11} ; \cdots \cdot \cdot, \Delta_{12} ; \cdots \cdots, \Delta_{13}$.)


Figure $3.4 \Delta_{1 i}$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (一, $\Delta_{11} ; \cdots \cdot \Delta_{12} ; \cdots \cdots, \Delta_{13}$.) (cont.).


Figure $3.4 \Delta_{1 i}$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (-, $\Delta_{11} ; \cdots, \Delta_{12} ; \cdots \cdots, \Delta_{13}$.) (cont.).


Figure 3.5 (left) $C_{0}$ and (right) $D_{0}$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (-, $\operatorname{Re}_{j}\left\{C_{0}\right\}$ and $\operatorname{Re}_{j}\left\{D_{0}\right\} ;--$, $\operatorname{Im}_{j}\left\{C_{0}\right\}$ and $\operatorname{Im}_{j}\left\{D_{0}\right\}$.)


Figure 3.5 (left) $C_{0}$ and (right) $D_{0}$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (-, $\operatorname{Re}_{j}\left\{C_{0}\right\}$ and $\operatorname{Re}_{j}\left\{D_{0}\right\} ;--$, $\operatorname{Im}_{j}\left\{C_{0}\right\}$ and $\operatorname{Im}_{j}\left\{D_{0}\right\}$.) (cont.)

Thus, the leading-order term of the complex potential is fully determined below:

$$
\begin{equation*}
w_{0}(z)=-\frac{j g A}{\omega} e^{-i k z}\left[1-\frac{1}{\Delta_{123}} \int_{-i a}^{z} \frac{e^{i k \zeta}\left(a_{1}^{\prime \prime}(-k)+a_{1}(-k) \zeta^{2}\right)}{\sqrt{\left(\zeta^{2}+a^{2}\right)\left(\zeta^{2}+b^{2}\right)}} d \zeta\right] . \tag{3.43}
\end{equation*}
$$

Taking the real part of Eq. (3.43) with respect to $i$ and multiplying the harmonic term $e^{-j \omega t}$, the leading-order term of the velocity potential can be attained.

### 3.3 Wave scattering by an impermeable plate

An analytical solution can be utilized in a wide variety of ways. One of the usages of the solution is evaluating the wave attenuation efficiency by comparing the reflected and transmitted wave amplitude and that of the incident wave. Especially for the leading-order solution, the leading-order terms of the reflected and transmitted wave indicate the waves scattered by an impermeable plate.

Let the leading-order reflection coefficient $R_{0}$ and the leading-order transmission coefficient $T_{0}$ defined as the ratio of the wave amplitude of the leading-order component of the reflected and transmitted wave at $x \rightarrow \pm \infty$ to the incident wave amplitude.

$$
\begin{equation*}
R_{0}=\frac{A_{R 0}}{A}, T_{0}=\frac{A_{T 0}}{A} \tag{3.44}
\end{equation*}
$$

where $A_{R 0}$ and $A_{T 0}$ denotes the leading-order reflected and transmitted wave ampli-
tude at $x \rightarrow \pm \infty$, respectively. These wave amplitudes are calculated from the water surface elevation of the reflected wave and the transmitted wave.

Substituting the coefficients for the leading-order spatial velocity potential derived as in Eq. (3.33), and Eq. (3.34), the leading-order spatial velocity potentials at $x \rightarrow \pm \infty$ Eq. (2.26) and Eq. (2.27) are rephrased as,

$$
\begin{array}{ll}
\phi_{0}^{+\infty}(x, y) \sim-\gamma_{0}(k) e^{k y+j k x}-\frac{j g A}{\omega} e^{k y-j k x}, & x \rightarrow+\infty, \\
\phi_{0}^{-\infty}(x, y) \sim\left(\gamma_{0}(k)-\frac{j g A}{\omega}\right) e^{k y-j k x}, & x \rightarrow-\infty . \tag{3.46}
\end{array}
$$

In Eq. (3.45), the leading-order spatial velocity potential at $x \rightarrow+\infty$ is the sum of the spatial velocity potential of the leading-order reflected wave and the spatial velocity potential of the incident wave:

$$
\begin{equation*}
\phi_{0}^{+\infty}(x, y) \sim \phi_{R 0}(x, y)+\phi_{I}(x, y) \tag{3.47}
\end{equation*}
$$

From Eq. (3.47) and the linearized dynamic free surface boundary condition, the free surface elevation of leading-order reflected wave, $\eta_{R 0}$, is calculated as below:

$$
\begin{aligned}
\eta_{R 0} & =-\frac{1}{g} \frac{\partial}{\partial t}\left(\Phi_{R 0}(x, 0, t)\right) \\
& =-\frac{1}{g} \frac{\partial}{\partial t}\left(\phi_{R 0}(x, 0) e^{-j \omega t}\right) \\
& =-\frac{1}{g} \frac{\partial}{\partial t}\left(-\gamma_{0}(k) e^{j(k x-\omega t)}\right)
\end{aligned}
$$

$$
\begin{equation*}
=-j \frac{\omega}{g} \gamma_{0}(k) e^{j(k x-\omega t)} \tag{3.48}
\end{equation*}
$$

Since $\gamma_{0}(k)$ is predefined in Eq. (3.23) as,

$$
\begin{align*}
\gamma_{0}(k) & =C_{0} a_{1}(k)-D_{0} a_{1}^{\prime \prime}(k) \\
& =\frac{j g A}{\omega} \frac{1}{\Delta_{123}}\left(a_{1}^{\prime \prime}(-k) a_{1}(k)-a_{1}(-k) a_{1}^{\prime \prime}(k)\right) \\
& =\frac{j g A}{\omega} \frac{\Delta_{11}}{\Delta_{123}} \tag{3.49}
\end{align*}
$$

the leading-order reflected wave elevation finally has the form of,

$$
\begin{equation*}
\eta_{R 0}=-j \frac{\omega}{g} \frac{j g A}{\omega} \frac{\Delta_{11}}{\Delta_{123}} e^{j(k x-\omega t)}=A \frac{\Delta_{11}}{\Delta_{123}} e^{j(k x-\omega t)} \tag{3.50}
\end{equation*}
$$

Thus, from Eq. (3.50), the leading-order reflected wave amplitude is found below:

$$
\begin{equation*}
A_{R 0}=\left|A \frac{\Delta_{11}}{\Delta_{123}}\right|=A \frac{\left|\Delta_{11}\right|}{\sqrt{\Delta_{11}^{2}+\left(\Delta_{12}-\Delta_{13}\right)^{2}}} \tag{3.51}
\end{equation*}
$$

Similarly, the leading-order spatial velocity potential at $x \rightarrow-\infty$ in Eq. (3.46) can be thought of as the spatial velocity potential of the leading-order transmitted wave:

$$
\begin{equation*}
\phi_{0}^{-\infty}(x, y) \sim \phi_{T 0}(x, y) \tag{3.52}
\end{equation*}
$$

From Eq. (3.52) and the linearized dynamic free surface boundary condition,
the leading-order transmitted wave is obtained as,

$$
\begin{align*}
\eta_{T 0} & =-\frac{1}{g} \frac{\partial}{\partial t}\left(\Phi_{T 0}^{+\infty}(x, 0, t)\right) \\
& =-\frac{1}{g} \frac{\partial}{\partial t}\left(\phi_{T 0}^{+\infty}(x, 0) e^{-j \omega t}\right) \\
& =-\frac{1}{g} \frac{\partial}{\partial t}\left(\left(\gamma_{0}(k)-\frac{j g A}{\omega}\right) e^{-j(k x+\omega t)}\right) \\
& =j \frac{\omega}{g}\left(\gamma_{0}(k)-\frac{j g A}{\omega}\right) e^{-j(k x+\omega t)} \\
& =j \frac{\omega}{g}\left(\frac{j g A}{\omega} \frac{\Delta_{11}}{\Delta_{123}}-\frac{j g A}{\omega}\right) e^{-j(k x+\omega t)} \\
& =A \frac{-j\left(\Delta_{12}-\Delta_{13}\right)}{\Delta_{123}} e^{-j(k x+\omega t)} . \tag{3.53}
\end{align*}
$$

Then, the wave amplitude of the leading-order transmitted wave will be,

$$
\begin{align*}
A_{T 0} & =\left|A \frac{-j\left(\Delta_{12}-\Delta_{13}\right)}{\Delta_{123}}\right| \\
& =A \frac{\left|\Delta_{12}-\Delta_{13}\right|}{\sqrt{\Delta_{11}^{2}+\left(\Delta_{12}-\Delta_{13}\right)^{2}}} \tag{3.54}
\end{align*}
$$

Dividing Eq. (3.51) and Eq. (3.54) with the incident wave amplitude $A$, leading-order reflection coefficient and transmission coefficient can be obtained:

$$
\begin{align*}
R_{0} & =\frac{A_{R 0}}{A}=\frac{\left|\Delta_{11}\right|}{\sqrt{\Delta_{11}^{2}+\left(\Delta_{12}-\Delta_{13}\right)^{2}}}  \tag{3.55}\\
T_{0} & =\frac{A_{T 0}}{A}=\frac{\left|\Delta_{12}-\Delta_{13}\right|}{\sqrt{\Delta_{11}^{2}+\left(\Delta_{12}-\Delta_{13}\right)^{2}}} \tag{3.56}
\end{align*}
$$

The reflection and transmission coefficients in Eq. (3.55) and Eq. (3.56) in various wave and breakwater conditions are plotted in Fig. (3.6) and Fig. (3.7).

As in the function of $k b$ for different values of $\mu$, Fig. (3.55) illustrates the leading-order reflection coefficients, and Fig. (3.56) shows the leading-order transmission coefficients. Assuming that $k$ is fixed, larger $k b$ means that the breakwater is more extended into the bottom. Thus, the physical explanation of increasing $R_{0}$ and decreasing $T_{0}$ is obvious. In the view of fixed $b$, small $k b$ indicates the long wavelength. If the incident wave has a very long wavelength, then the wave would not even sense the existence of the plate. Therefore, the reflection is small, and most waves are transmitted.

When $\mu=0$, the plate intersects the surface, and $R_{0}$ keeps increasing as $k b$ becomes larger. However, in contrast, when $\mu$ has a specific value, $R_{0}$ has the maximum value in the intermediate value of $k b$. This is because $\mu$ represents the ratio of $a$ and $b$, not the difference between them. If $b$ becomes extremely large, $a$ would also have a large value since $\mu$ is a constant. This means that the breakwater goes further from the water surface and cannot reflect the surface waves effectively. Therefore, for each $\mu>0, R_{0}$ will have its maximum value on a certain intermediate value of $k b$.


Figure 3.6 Leading-order reflection coefficient $R_{0}$ versus $k b: *, \mu=0.001 ;-$, , $\mu=0.01 ;-, \mu=0.05 ;---\mu=0.1 ;-\cdots, \mu=0.25 ; \cdots \cdots, \mu=0.5$.


Figure 3.7 Leading-order transmission coefficient $T_{0}$ versus $k b: *, \mu=0.001 ; ~ \bullet-$, $\mu=0.01 ;-, \mu=0.05 ;--, \mu=0.1 ;-\cdot-, \mu=0.25 ; \cdots \cdots, \mu=0.5$.

## Chapter 4. First-order solution

### 4.1 First-order solution for the case of a single plate

As proposed in the formulation section, it is assumed that the velocity potential comprises the perturbed sum of the leading-order and the first-order velocity potential, which represent the potential due to the impermeable plate and the permeable plate, respectively. In this chapter, the first-order solution for the case of a single permeable plate is sought.

In a similar way that the complex potential and the reduced potential are introduced to seek the solution for the leading-order problem, the first-order complex potential $w_{1}(z)$ and the first-order reduced potential $W_{1}(z)$ are proposed as in Eq. (4.1) and Eq. (4.2):

$$
\begin{gather*}
w_{1}(z)=\phi_{1}+i \psi_{1}, \quad \operatorname{Im}_{i}\{z\}<0,  \tag{4.1}\\
W_{1}(z)=\frac{d w_{1}(z)}{d z}+i k w_{1}(z), \quad \operatorname{Im}_{i}\{z\}<0 . \tag{4.2}
\end{gather*}
$$

As in the previous chapter, the reduced potential $W_{1}(z)$ can be continued into $\operatorname{Im}_{i}\{z\}>0$ by Schwarz's reflection principle since the imaginary part of $W_{1}(z)$ is zero along the $x$-axis.

Firstly, the boundary condition on $L$ was sought before reflecting to $L^{\prime}$. Using
the notation of Eq. (4.2), the boundary condition of the first-order reduced potential on the permeable plate becomes,

$$
\begin{equation*}
\operatorname{Re}_{i}\left\{W_{1}(z)\right\}=\frac{\partial \phi_{1}}{\partial x}-k \psi_{1}, \quad \text { on } L . \tag{4.3}
\end{equation*}
$$

Unlike in the leading-order problem, the partial derivative of the spatial velocity potential is not zero (see Eq. (2.25)); thus, defining the boundary condition on the plate is necessary. In order to fully describe the boundary condition of the firstorder reduced potential on the plate, the partial derivative of the first-order spatial potential $\phi_{1}$ with respect to $x$ and the first-order stream function must be derived. When the breakwater comprises a single permeable plate, the boundary condition of the first-order spatial velocity potential on the breakwater in Eq. (2.25) becomes,

$$
\begin{equation*}
\frac{\partial \phi_{1}}{\partial x}=-j k\left(\phi_{0}^{+}(0, y)-\phi_{0}^{-}(0, y)\right), \quad \text { on } L \tag{4.4}
\end{equation*}
$$

Taking the real part of the leading-order complex potential, which is presented in Eq. (3.43), the leading-order spatial velocity potential on the plate can be obtained:

$$
\begin{align*}
& \phi_{0}(x, y)=\operatorname{Re}_{i}\left\{w_{0}(z)\right\} \\
& =\operatorname{Re}_{i}\left\{-\frac{j g A}{\omega} e^{-i k z}\left[1-\frac{1}{\Delta_{123}} \int_{-i a}^{z} \frac{e^{i k \zeta}\left(a_{1}^{\prime \prime}(-k)+a_{1}(-k) \zeta^{2}\right)}{\sqrt{\left(\zeta^{2}+a^{2}\right)\left(\zeta^{2}+b^{2}\right)}} d \zeta\right]\right\} \tag{4.5}
\end{align*}
$$

Contracting the integration path onto the right and left sides of the plate, the leading-order spatial velocity potential on both sides of the plates can be calculated as follows:

$$
\begin{align*}
& \phi_{0}^{ \pm}(0, y)=\lim _{x \rightarrow 0 \pm} \phi_{0}(x, y) \\
& =\operatorname{Re}_{i}\left\{-\frac{j g A}{\omega} e^{k y}\left[1-\frac{1}{\Delta_{123}} \int_{-i a}^{i y+0 \pm} \frac{e^{i k \zeta}\left(a_{1}^{\prime \prime}(-k)+a_{1}(-k) \zeta^{2}\right)}{\sqrt{\left(\zeta^{2}+a^{2}\right)\left(\zeta^{2}+b^{2}\right)}} d \zeta\right]\right\} \\
& =-\frac{j g A}{\omega} e^{k y}\left[1 \mp \frac{1}{\Delta_{123}} \int_{-a}^{y} \frac{e^{-k \eta}\left(a_{1}^{\prime \prime}(-k)-a_{1}(-k) \eta^{2}\right)}{\sqrt{\left(\eta^{2}-a^{2}\right)\left(b^{2}-\eta^{2}\right)}} d \eta\right] \\
& =-\frac{j g A}{\omega} e^{k y}\left[1 \mp \frac { 1 } { \Delta _ { 1 2 3 } } \left(\int_{-b}^{y} \frac{e^{-k \eta}\left(a_{1}^{\prime \prime}(-k)-a_{1}(-k) \eta^{2}\right)}{\sqrt{\left(\eta^{2}-a^{2}\right)\left(b^{2}-\eta^{2}\right)}} d \eta\right.\right. \\
& \left.\left.\quad+\int_{-a}^{-b} \frac{e^{-k \eta}\left(a_{1}^{\prime \prime}(-k)-a_{1}(-k) \eta^{2}\right)}{\sqrt{\left(\eta^{2}-a^{2}\right)\left(b^{2}-\eta^{2}\right)}} d \eta\right)\right] \tag{4.6}
\end{align*}
$$

## Defining

$$
\begin{equation*}
\gamma_{0}(-k, y)=-\frac{j g A}{\omega} \frac{1}{\Delta_{123}} \int_{-b}^{y} \frac{e^{-k \eta}\left(a_{1}^{\prime \prime}(-k)-a_{1}(-k) \eta^{2}\right)}{\sqrt{\left(\eta^{2}-a^{2}\right)\left(b^{2}-\eta^{2}\right)}} d \eta \tag{4.7}
\end{equation*}
$$

Eq. (4.6) is shortened to,

$$
\begin{equation*}
\phi_{0}^{ \pm}(0, y)=e^{k y}\left[-\frac{j g A}{\omega} \mp \gamma_{0}(-k, y)\right], \quad \text { on } L . \tag{4.8}
\end{equation*}
$$

Substituting Eq. (4.8) into Eq. (4.4) gives a complete expression of the partial
derivative of the first-order spatial velocity potential with respect to $x$ :

$$
\begin{align*}
\frac{\partial \phi_{1}}{\partial x} & =-j k\left(e^{k y}\left[-\frac{j g A}{\omega}-\gamma_{0}(-k, y)\right]-e^{k y}\left[-\frac{j g A}{\omega}+\gamma_{0}(-k, y)\right]\right) \\
& =j 2 k e^{k y} \gamma_{0}(-k, y) . \tag{4.9}
\end{align*}
$$

To utilize Eq. (4.9) in finding the first-order stream function, the equation on the relation of the first-order spatial velocity potential and the first-order stream function is brought about. Applying the Cauchy-Riemann equation on Eq. (4.1),

$$
\begin{equation*}
\frac{\partial \phi_{1}}{\partial x}=\frac{\partial \psi_{1}}{\partial y} \tag{4.10}
\end{equation*}
$$

Integrating the both sides of Eq. (4.4) with respect to $y$, especially $y \in$ $(-b,-a), \psi_{1}(x, y)$ on the barrier can be derived. Although previous studies (Evans, 1970 , etc.) which dealt with impermeable breakwaters nicely defined $\psi_{1}(0, y)$ so that the constant of integration becomes zero, or just gave the indefinite integral function of $\frac{\partial \psi_{1}}{\partial y}$ with respect to $y$, the arbitrariness in $\phi_{1}$ should be resolved in this study since there is a flow through the permeable breakwater. Thus, this study defined the first-order stream function on $L$ as the sum of the definite integral of $\frac{\partial \psi_{1}}{\partial y}$ from $-b$ to $y$ and the constant of integration, $I$ :

$$
\begin{equation*}
\psi_{1}(0, y)=\int_{-b}^{y}\left(j 2 k e^{k \eta} \gamma_{0}(-k, \eta)\right) d \eta+I \tag{4.11}
\end{equation*}
$$

Using integration by parts and substituting Eq. (4.7) into Eq. (4.11), $\psi_{1}(x, y)$ along the plate is obtained as a function of $y$ :

$$
\begin{align*}
& \psi_{1}(0, y)=\left[j 2 e^{k \eta} \gamma_{0}(-k, \eta)\right]_{-b}^{y}-\int_{-b}^{y} j 2 e^{k \eta} \frac{d\left(\gamma_{0}(-k, \eta)\right)}{d \eta} d \eta+I \\
& =j 2 e^{k y} \gamma_{0}(-k, y)-j 2 e^{-k b} \gamma_{0}(-k,-b) \\
& \quad-\int_{-b}^{y} j 2 e^{k \eta}\left(-\frac{j g A}{\omega} \frac{1}{\Delta_{123}} \frac{e^{-k \eta}\left(a_{1}^{\prime \prime}(-k)-a_{1}(-k) \eta^{2}\right)}{\sqrt{\left(\eta^{2}-a^{2}\right)\left(b^{2}-\eta^{2}\right)}}\right) d \eta+I \\
& =j 2 e^{k y} \gamma_{0}(-k, y) \\
& \quad-\frac{g A}{\omega} \frac{2}{\Delta_{123}} \int_{-b}^{y} \frac{a_{1}^{\prime \prime}(-k)-a_{1}(-k) \eta^{2}}{\sqrt{\left(\eta^{2}-a^{2}\right)\left(b^{2}-\eta^{2}\right)}} d \eta+I, \quad \text { on } L . \tag{4.12}
\end{align*}
$$

$$
\gamma_{0}(-k,-b)=\gamma_{0}(-k)=0 \text { on } L, \text { thus eliminated. Now, substituting Eq. (4.9) }
$$

and Eq. (4.12) into Eq. (4.3), the boundary condition of the reduced potential at the permeable plate can be obtained:

$$
\begin{align*}
& \operatorname{Re}_{i}\left\{W_{1}(z)\right\}=\frac{\partial \phi_{1}}{\partial x}-k \psi_{1} \\
& =j 2 k e^{k y} \gamma_{0}(-k, y)-j 2 k e^{k y} \gamma_{0}(-k, y) \\
& \quad+\frac{g A}{\omega} \frac{2 k}{\Delta_{123}} \int_{-b}^{y} \frac{a_{1}^{\prime \prime}(-k)-a_{1}(-k) \eta^{2}}{\sqrt{\left(\eta^{2}-a^{2}\right)\left(b^{2}-\eta^{2}\right)}} d \eta-k I \\
& =\frac{g A}{\omega} \frac{2 k}{\Delta_{123}} \int_{-b}^{y} \frac{a_{1}^{\prime \prime}(-k)-a_{1}(-k) \eta^{2}}{\sqrt{\left(\eta^{2}-a^{2}\right)\left(b^{2}-\eta^{2}\right)}} d \eta-k I \\
& =f(y), \quad \text { on } L . \tag{4.13}
\end{align*}
$$

And this can be reflected in the upper-half complex plane:

$$
\begin{equation*}
\operatorname{Re}_{i}\left\{W_{1}(z)\right\}=f(-|y|), \quad \text { on } L+L^{\prime} \tag{4.14}
\end{equation*}
$$

Solving for $W_{1}(z)$ which satisfies Eq. (4.13) is a non-homogeneous RiemannHilbert problem, and the solution that vanishes at infinity can be found as,

$$
\begin{equation*}
W_{1}(z)=\frac{C_{1}+D_{1} z^{2}+\frac{2}{\pi} \int_{-b}^{-a} \frac{\sqrt{\left(y^{2}-a^{2}\right)\left(b^{2}-y^{2}\right)} y f(y)}{y^{2}+z^{2}} d y}{\sqrt{\left(z^{2}+a^{2}\right)\left(z^{2}+b^{2}\right)}} \tag{4.15}
\end{equation*}
$$

where $C_{1}, D_{1}$ are some constants (Muskhelishvili, 1977).
Similarly, from the leading order solution, the complex potential $w_{1}(z)$ can be obtained as,

$$
\begin{equation*}
w_{1}(z)=e^{-i k z}\left[B_{1}+\int_{-i a}^{z} e^{i k \zeta} W_{1}(\zeta) d \zeta\right] \tag{4.16}
\end{equation*}
$$

where $B_{1}$ is a constant.

Especially when the integration path is taken to follow the plate, i.e., $y \in$ $(-b,-a)$, the complex potential on the plate is as below:

$$
\begin{aligned}
w_{1}^{ \pm}(i y)= & e^{k y}\left[B_{1}+\int_{-i a}^{+i y+0 \pm} e^{i k \zeta} W_{1}(\zeta) d \zeta\right] \\
= & e^{k y}\left[B_{1}-\int_{-a}^{y} e^{-k u} \operatorname{Im}_{i}\left\{W_{1}^{ \pm}(i u)\right\} d u\right. \\
& \left.+i \int_{-a}^{y} e^{-k u} \operatorname{Re}_{i}\left\{W_{1}^{ \pm}(i u)\right\} d u\right]
\end{aligned}
$$

$$
\begin{equation*}
=e^{k y}\left[B_{1}-\int_{-a}^{y} e^{-k u} \operatorname{Im}_{i}\left\{W_{1}^{ \pm}(i u)\right\} d u+i \int_{-a}^{y} e^{-k u} f(u) d u\right] \tag{4.17}
\end{equation*}
$$

Taking the imaginary part of Eq. (4.17) only, the first-order stream function on the plate is obtained:

$$
\begin{equation*}
\psi_{1}(0, y)=\operatorname{Im}_{i}\left\{w_{1}^{ \pm}(i y)\right\}=e^{k y} \int_{-a}^{y} e^{-k u} f(u) d u, \quad \text { on } L \tag{4.18}
\end{equation*}
$$

Here, $\psi_{1}(0, y)$ in Eq. (4.12) and $\psi_{1}(0, y)$ in Eq. (4.18) should be the same. In other words, the integral of $\phi(0, y)$ along the plate and the imaginary part of $w_{1}(z)$ on the plate must have the same value. Hence, one equation for determining the constant of integration, $I$, is presented as:

$$
\begin{align*}
& j 2 e^{k y} \gamma_{0}(-k, y)-\frac{g A}{\omega} \frac{2}{\Delta_{123}} \int_{-b}^{y} \frac{a_{1}^{\prime \prime}(-k)-a_{1}(-k) \eta^{2}}{\sqrt{\left(\eta^{2}-a^{2}\right)\left(b^{2}-\eta^{2}\right)}} d \eta+I \\
& =e^{k y} \int_{-a}^{y} e^{-k u}\left(\frac{g A}{\omega} \frac{2 k}{\Delta_{123}} \int_{-b}^{u} \frac{a_{1}^{\prime \prime}(-k)-a_{1}(-k) \eta^{2}}{\sqrt{\left(\eta^{2}-a^{2}\right)\left(b^{2}-\eta^{2}\right)}} d \eta-k I\right) d u \tag{4.19}
\end{align*}
$$

When $y \rightarrow-b$,

$$
\begin{aligned}
& 0+0+I \\
& =\frac{g A}{\omega} \frac{2 k}{\Delta_{123}} e^{-k b} \int_{-a}^{-b} e^{-k u} \int_{-b}^{u} \frac{a_{1}^{\prime \prime}(-k)-a_{1}(-k) \eta^{2}}{\sqrt{\left(\eta^{2}-a^{2}\right)\left(b^{2}-\eta^{2}\right)}} d \eta d u \\
& \quad-k I e^{-k b} \int_{-a}^{-b} e^{-k u} d u
\end{aligned}
$$

$$
\begin{array}{r}
=\frac{g A}{\omega} \frac{2 k}{\Delta_{123}} e^{-k b} \int_{-a}^{-b} e^{-k u} \int_{-b}^{u} \frac{a_{1}^{\prime \prime}(-k)-a_{1}(-k) \eta^{2}}{\sqrt{\left(\eta^{2}-a^{2}\right)\left(b^{2}-\eta^{2}\right)}} d \eta d u \\
+\left(1-e^{k(a-b)}\right) I \tag{4.20}
\end{array}
$$

Therefore, rearranging the equation above gives the value of $I$ :

$$
\begin{equation*}
I=\frac{g A}{\omega} \frac{2 k}{\Delta_{123}} e^{-k a} \int_{-a}^{-b} e^{-k u} \int_{-b}^{u} \frac{a_{1}^{\prime \prime}(-k)-a_{1}(-k) \eta^{2}}{\sqrt{\left(\eta^{2}-a^{2}\right)\left(b^{2}-\eta^{2}\right)}} d \eta d u . \tag{4.21}
\end{equation*}
$$

$I$ differs by the wave number $k$ and $\mu$, the ratio of $a$ and $b$. The behavior of $I$ is well presented in Fig. (4.1).

Thus, $f(y)$ is now fully defined by substituting the expression of $I$ in Eq. (4.21) into Eq. (4.13):

$$
\begin{align*}
f(y)= & \frac{g A}{\omega} \frac{2 k}{\Delta_{123}} \int_{-b}^{y} \frac{a_{1}^{\prime \prime}(-k)-a_{1}(-k) \eta^{2}}{\sqrt{\left(\eta^{2}-a^{2}\right)\left(b^{2}-\eta^{2}\right)}} d \eta \\
& -k \frac{g A}{\omega} \frac{2 k}{\Delta_{123}} e^{-k a} \int_{-a}^{-b} e^{-k u} \int_{-b}^{u} \frac{a_{1}^{\prime \prime}(-k)-a_{1}(-k) \eta^{2}}{\sqrt{\left(\eta^{2}-a^{2}\right)\left(b^{2}-\eta^{2}\right)}} d \eta d u \\
= & \frac{g A}{\omega} \frac{2 k}{\Delta_{123}}\left[\int_{-b}^{y} \frac{a_{1}^{\prime \prime}(-k)-a_{1}(-k) \eta^{2}}{\sqrt{\left(\eta^{2}-a^{2}\right)\left(b^{2}-\eta^{2}\right)}} d \eta\right. \\
& \left.\quad-k e^{-k a} \int_{-a}^{-b} e^{-k u} \int_{-b}^{u} \frac{a_{1}^{\prime \prime}(-k)-a_{1}(-k) \eta^{2}}{\sqrt{\left(\eta^{2}-a^{2}\right)\left(b^{2}-\eta^{2}\right)}} d \eta d u\right] \\
= & \frac{g A}{\omega} \frac{2 k}{\Delta_{123}}\left[\int_{-b}^{y} \frac{a_{1}^{\prime \prime}(-k)-a_{1}(-k) \eta^{2}}{\sqrt{\left(\eta^{2}-a^{2}\right)\left(b^{2}-\eta^{2}\right)}} d \eta-k \tilde{I}\right] \\
= & \frac{g A}{\omega} \tilde{f}(y) . \tag{4.22}
\end{align*}
$$



Figure $4.1 I$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (-, $\operatorname{Re}_{j}\{I\} ;--, \operatorname{Im}_{j}\{I\}$.)


Figure $4.1 I$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (,$- \operatorname{Re}_{j}\{I\} ;--, \operatorname{Im}_{j}\{I\}$.) (cont.).


Figure $4.1 I$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (-, $\operatorname{Re}_{j}\{I\} ;--, \operatorname{Im}_{j}\{I\}$.) (cont.).

In Eq. (4.22), it is worth noting that the first integral term can be expressed in elliptic integral form.

$$
\begin{equation*}
\int_{-y}^{b} \frac{a_{1}^{\prime \prime}(-k)-a_{1}(-k) \eta^{2}}{\sqrt{\left(\eta^{2}-a^{2}\right)\left(b^{2}-\eta^{2}\right)}} d \eta=\frac{a_{1}^{\prime \prime}(-k)}{b} F(\varphi, m)-a_{1}(-k) b E(\varphi, m) \tag{4.23}
\end{equation*}
$$

where $F(\varphi, m)$ is the incomplete elliptic integral of the first kind and $E(\varphi, m)$ is the incomplete elliptic integral of the second kind (Byrd and Friedman, 1954, p. 56), and each parameter represents,

$$
\begin{equation*}
\varphi=\sin ^{-1} \sqrt{\frac{b^{2}-y^{2}}{b^{2}-a^{2}}}, \quad m=\frac{b^{2}-a^{2}}{b^{2}} \tag{4.24}
\end{equation*}
$$

Using these continuous elliptic integral functions and the previously calculated value of $I, f(y)$ with various $k b$ and $\mu$ can be plotted as shown in Fig. (4.2).

To define the unknown constants $B_{1}, C_{1}$, and $D_{1}$, the same way used in the leading-order problem can be utilized. First, the radiational boundary conditions will be applied. Since $W_{1}(z)-D_{1} \rightarrow 0$ as $|z| \rightarrow \pm \infty$, for large $z$,

$$
\begin{align*}
w_{1}(z) \sim & e^{-i k z}\left[B_{1}+\int_{-i a}^{i \infty} e^{i k u} W_{1}(\zeta) d \zeta\right. \\
& \left.\quad+\int_{i \infty}^{z} e^{i k \zeta}\left(W_{1}(\zeta)-D_{1}\right) d \zeta-\frac{i D_{1}}{k} e^{i k \zeta}\right] \\
\sim & e^{-i k z}\left[B_{1}+\int_{-i a}^{i \infty} e^{i k \zeta} W_{1}(\zeta) d \zeta-\frac{i D_{1}}{k} e^{i k z}\right] \tag{4.25}
\end{align*}
$$



Figure $4.2 f(y)$ versus $y$ for (top) $k b=0.5$, (center) $k b=1.5$, (bottom) $k b=2.5$. (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$.
$\left(-, \operatorname{Re}_{j}\{f(y)\} ;---\operatorname{Im}_{j}\{f(y)\}\right.$.)


Figure $4.2 f(y)$ versus $y$ for (top) $k b=0.5$, (center) $k b=1.5$, (bottom) $k b=2.5$.
(a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. $\left(-, \operatorname{Re}_{j}\{f(y)\} ;--, \operatorname{Im}_{j}\{f(y)\}\right.$.) (cont.)


Figure $4.2 f(y)$ versus $y$ for (top) $k b=0.5$, (center) $k b=1.5$, (bottom) $k b=2.5$. (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$.
$\left(-, \operatorname{Re}_{j}\{f(y)\} ;--, \operatorname{Im}_{j}\{f(y)\}\right.$.) (cont.)


Figure $4.2 f(y)$ versus $y$ for (top) $k b=0.5$, (center) $k b=1.5$, (bottom) $k b=2.5$. (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$.
$\left(-, \operatorname{Re}_{j}\{f(y)\} ;---\operatorname{Im}_{j}\{f(y)\}.\right)($ cont.)


Figure $4.2 f(y)$ versus $y$ for (top) $k b=0.5$, (center) $k b=1.5$, (bottom) $k b=2.5$. (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$.
$\left(-, \operatorname{Re}_{j}\{f(y)\} ;---\operatorname{Im}_{j}\{f(y)\}\right.$.) (cont.)


Figure $4.2 f(y)$ versus $y$ for (top) $k b=0.5$, (center) $k b=1.5$, (bottom) $k b=2.5$. (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. $\left(-, \operatorname{Re}_{j}\{f(y)\} ;--, \operatorname{Im}_{j}\{f(y)\}\right.$.) (cont.)

As in the previous chapter, the integration path along the $y$-axis is split into three intervals by defining functions as below:

$$
\begin{align*}
& F(u)=\frac{g A}{\omega} \frac{2}{\pi} \int_{-b}^{-a} \frac{\sqrt{\left(y^{2}-a^{2}\right)\left(b^{2}-y^{2}\right)} y \tilde{f}(y)}{y^{2}-u^{2}} d y=\frac{g A}{\omega} \tilde{F}(u),  \tag{4.26}\\
& a_{1}(k, F)=\frac{g A}{\omega} \int_{a}^{b} \frac{e^{-k u} \tilde{F}(u)}{\sqrt{\left(u^{2}-a^{2}\right)\left(b^{2}-u^{2}\right)}} d u=\frac{g A}{\omega} a_{1}(k, \tilde{F}),  \tag{4.27}\\
& a_{2}(k, F)=\frac{g A}{\omega} \int_{-a}^{a} \frac{e^{-k u} \tilde{F}(u)}{\sqrt{\left(a^{2}-u^{2}\right)\left(b^{2}-u^{2}\right)}} d u=\frac{g A}{\omega} a_{2}(k, \tilde{F}),  \tag{4.28}\\
& a_{3}(k, F)=\frac{g A}{\omega} \int_{b}^{\infty} \frac{e^{-k u} \tilde{F}(u)}{\sqrt{\left(u^{2}-a^{2}\right)\left(u^{2}-b^{2}\right)}} d u=\frac{g A}{\omega} a_{3}(k, \tilde{F})  \tag{4.29}\\
& \gamma_{1}(k)=C_{1} a_{1}(k)-D_{1} a_{1}^{\prime \prime}(k)+a_{1}(k, F),  \tag{4.30}\\
& \alpha_{1}(k)=C_{1} a_{2}(k)-D_{1} a_{2}^{\prime \prime}(k)+a_{2}(k, F),  \tag{4.31}\\
& \beta_{1}(k)=C_{1} a_{3}(k)-D_{1} a_{3}^{\prime \prime}(k)+a_{3}(k, F),  \tag{4.32}\\
& \delta_{1}(k)=\frac{g A}{\omega} \int_{a}^{b} e^{-k u} \tilde{f}(u) d u=\frac{g A}{\omega} \tilde{\delta}_{1}(k) . \tag{4.33}
\end{align*}
$$

Then, Eq. (4.16) can be simplified:

$$
\begin{equation*}
w_{1}(z) \sim e^{-i k z}\left[B_{1} \mp \gamma_{1}(k)+i\left(\alpha_{1}(k)-\beta_{1}(k)+\delta_{1}(k)\right)\right]-\frac{i D_{1}}{k} \tag{4.34}
\end{equation*}
$$

Although $\alpha_{1}(k), \beta_{1}(k)$ and $\gamma_{1}(k)$ cannot be presented since they require a numerical calculation of $F(u), \delta_{1}(k)$ can be illustrated since it only needs the value of $f(y) . \delta_{1}(k)$ for various $k b$ is represented as in Fig. (4.3).


Figure $4.3 \delta_{1}(k)$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (—, $\operatorname{Re}_{j}\left\{\delta_{1}(k)\right\} ;---\operatorname{Im}_{j}\left\{\delta_{1}(k)\right\}$.)


Figure $4.3 \delta_{1}(k)$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (一, $\operatorname{Re}_{j}\left\{\delta_{1}(k)\right\} ;---\operatorname{Im}_{j}\left\{\delta_{1}(k)\right\}$.) (cont.)

Around the plate, the zero-circulation condition is enforced:

$$
\begin{align*}
& \operatorname{Re}_{i}\left\{\oint_{\Gamma} e^{i k u} W_{1}(u) d u\right\} \\
& =\int_{-b}^{-a} e^{-k u} \frac{C_{1}-D_{1} u^{2}+\frac{2}{\pi} \int_{-b}^{-a} \frac{\sqrt{\left(y^{2}-a^{2}\right)\left(b^{2}-y^{2}\right) y f(y)}}{y^{2}-u^{2}}}{\sqrt{\left(u^{2}-a^{2}\right)\left(b^{2}-u^{2}\right)}} d y \\
& =0 . \tag{4.35}
\end{align*}
$$

Eq. (4.35) can be shortened using the notations above:

$$
\begin{equation*}
\gamma_{1}(-k)=C_{1} a_{1}(-k)-D_{1} a_{1}^{\prime \prime}(-k)+a_{1}(-k, F)=0 . \tag{4.36}
\end{equation*}
$$

We also defined the radiation boundary conditions at $x \rightarrow \pm \infty$ in Eq. (2.28) and Eq. (2.29). Substituting the first-order complex potential Eq. (4.34) into these radiation boundary conditions gives,

$$
\begin{equation*}
\gamma_{1}(k)=j\left(\alpha_{1}(k)-\beta_{1}(k)+\delta_{1}(k)\right) . \tag{4.37}
\end{equation*}
$$

Combining Eq. (4.36) and Eq. (4.37) gives simultaneous linear equations:

$$
\left\{\begin{array}{l}
\gamma_{1}(k)=j\left(\alpha_{1}(k)-\beta_{1}(k)+\delta_{1}(k)\right)  \tag{4.38}\\
\gamma_{1}(-k)=0
\end{array}\right.
$$

Rewriting in a matrix form,

$$
\begin{gather*}
{\left[\begin{array}{cc}
a_{1}(k)-j\left(a_{2}(k)-a_{3}(k)\right) & -a_{1}^{\prime \prime}(k)+j\left(a_{2}^{\prime \prime}(k)-a_{3}^{\prime \prime}(k)\right) \\
a_{1}(-k) & -a_{1}^{\prime \prime}(-k)
\end{array}\right]\left[\begin{array}{l}
C_{1} \\
D_{1}
\end{array}\right]} \\
=\left[\begin{array}{c}
-a_{1}(k, F)+j\left(a_{2}(k, F)-a_{3}(k, F)+\delta_{1}(k)\right) \\
-a_{1}(-k, F)
\end{array}\right] . \tag{4.39}
\end{gather*}
$$

By Cramer's rule, $B_{1}, C_{1}$, and $D_{1}$ are obtained to be,

$$
\begin{gather*}
B_{1}=0  \tag{4.40}\\
C_{1}=\frac{\Delta_{F 1}^{\prime \prime}-j\left(\Delta_{F 2}^{\prime \prime}-\Delta_{F 3}^{\prime \prime}-a_{1}^{\prime \prime}(-k) \delta_{1}(k)\right)}{\Delta_{123}}  \tag{4.41}\\
D_{1}=\frac{\Delta_{F 1}-j\left(\Delta_{F 2}-\Delta_{F 3}-a_{1}(-k) \delta_{1}(k)\right)}{\Delta_{123}} \tag{4.42}
\end{gather*}
$$

where,

$$
\begin{align*}
\Delta_{F i} & =\left|\begin{array}{cc}
a_{i}(k) & a_{1}(-k) \\
a_{i}(k, F) & a_{1}(-k, F)
\end{array}\right|, \quad i=1,2,3  \tag{4.43}\\
\Delta_{F i}^{\prime \prime} & =\left|\begin{array}{cc}
a_{i}^{\prime \prime}(k) & a_{1}^{\prime \prime}(-k) \\
a_{i}(k, F) & a_{1}(-k, F)
\end{array}\right| . \tag{4.44}
\end{align*}
$$

Now, the first-order solution of the complex potential is fully defined, and the
spatial potential is also obtained from taking the real part of the complex potential:

$$
\begin{equation*}
\phi_{1}(x, y)=\operatorname{Re}_{i}\left\{w_{1}(z)\right\} . \tag{4.45}
\end{equation*}
$$

### 4.2 Wave scattering by a permeable plate

In order to consider the effect of the permeability of the plate, perturbed expressions for a reflection coefficient and a transmitted coefficient are presented in this section.

Using the constants we obtained, the first-order spatial wave potential at $x \rightarrow$ $+\infty$ is calculated as below:

$$
\begin{align*}
\phi_{1}^{+\infty}(x, y) & \sim e^{k y}\left[\left(B_{1}-\gamma_{1}(k)\right) \cos k x+\left(\alpha_{1}(k)-\beta_{1}(k)+\delta_{1}(k)\right) \sin k x\right] \\
& \sim e^{k y}\left[-\gamma_{1}(k) \cos k x-j \gamma_{1}(k) \sin k x\right](\because \text { Eq. }(4.37)) \\
& \sim-\gamma_{1}(k) e^{j k x+k y} \\
& \sim \phi_{R 1}^{+\infty}(x, y) \tag{4.46}
\end{align*}
$$

To distinguish the real and imaginary part of $\gamma_{1}(k)$ with respect to $j$, we rearranged the expression of $\gamma_{1}(k)$.

$$
\gamma_{1}(k)=C_{1} a_{1}(k)-D_{1} a_{1}^{\prime \prime}(k)+a_{1}(k, F)
$$

$$
\begin{align*}
&= \frac{a_{1}(k) \Delta_{F 1}^{\prime \prime}-j\left(a_{1}(k) \Delta_{F 2}^{\prime \prime}-a_{1}(k) \Delta_{F 3}^{\prime \prime}-a_{1}(k) a_{1}^{\prime \prime}(-k) \delta_{1}(k)\right)}{\Delta_{123}} \\
&-\frac{a_{1}^{\prime \prime}(k) \Delta_{F 1}-j\left(a_{1}^{\prime \prime}(k) \Delta_{F 2}-a_{1}^{\prime \prime}(k) \Delta_{F 3}-a_{1}^{\prime \prime}(k) a_{1}(-k) \delta_{1}(k)\right)}{\Delta_{123}} \\
&+a_{1}(k, F) \\
&= \frac{a_{1}(k) \Delta_{F 1}^{\prime \prime}-a_{1}^{\prime \prime}(k) \Delta_{F 1}}{\Delta_{123}}+a_{1}(k, F) \\
&-j \frac{-a_{1}(-k, F)\left(\Delta_{2}-\Delta_{3}\right)-\left(a_{2}(k, F)-a_{3}(k, F)+\delta_{1}(k)\right) \Delta_{11}}{\Delta_{123}} \\
&=-j \frac{1}{\Delta_{123}}\left(a_{1}(k, F)\left(\Delta_{12}-\Delta_{13}\right)-a_{1}(-k, F)\left(\Delta_{2}-\Delta_{3}\right)\right. \\
&=\left.-\frac{j g A}{\omega} \frac{1}{\Delta_{123}}\left(a_{1}(k, F)-a_{3}(k, F)+\delta_{1}(k)\right) \Delta_{11}\right) \\
&=-\frac{j g A}{\omega} \frac{\Lambda}{\Delta_{123}},
\end{align*}
$$

where,

$$
\begin{gather*}
\Delta_{i}=\left|\begin{array}{cc}
a_{i}(k) & a_{1}(k) \\
a_{i}^{\prime \prime}(k) & a_{1}^{\prime \prime}(k)
\end{array}\right|, \quad i=2,3,  \tag{4.48}\\
\Lambda=a_{1}(k, \tilde{F})\left(\Delta_{12}-\Delta_{13}\right)-a_{1}(-k, \tilde{F})\left(\Delta_{2}-\Delta_{3}\right)-\left(a_{2}(k, \tilde{F})-a_{3}(k, \tilde{F})+\tilde{\delta}_{1}(k)\right) \Delta_{11} . \tag{4.49}
\end{gather*}
$$

With this simplified expression, the first-order reflected wave at $x \rightarrow \infty$ below:

$$
\eta_{R 1}=-\frac{1}{g} \frac{\partial}{\partial t}\left(\Phi_{R 1}^{+\infty}(x, 0, t)\right)
$$

$$
\begin{align*}
& =-\frac{1}{g} \frac{\partial}{\partial t}\left(\phi_{R 1}^{+\infty}(x, 0) e^{-j \omega t}\right) \\
& =-\frac{1}{g} \frac{\partial}{\partial t}\left(-\gamma_{1}(k) e^{j(k x-\omega t)}\right) \\
& =-\frac{1}{g}\left(-\gamma_{1}(k)\right)(-j \omega) e^{j(k x-\omega t)} \\
& =-A \frac{\Lambda}{\Delta_{123}} e^{j(k x-\omega t)} \tag{4.50}
\end{align*}
$$

From Eq. (4.50), the first-order reflected wave amplitude is,

$$
\begin{align*}
A_{R 1} & =\left|-A \frac{\Lambda}{\Delta_{123}}\right| \\
& =A \frac{|\Lambda|}{\sqrt{\Delta_{11}^{2}+\left(\Delta_{12}-\Delta_{13}\right)^{2}}} \tag{4.51}
\end{align*}
$$

Likewise, the first-order spatial velocity potential at $x \rightarrow-\infty$ is,

$$
\begin{align*}
\phi_{1}^{-\infty}(x, y) & \sim e^{k y}\left[\left(B_{1}+\gamma_{1}(k)\right) \cos k x+\left(\alpha_{1}(k)-\beta_{1}(k)+\delta_{1}(k)\right) \sin k x\right] \\
& \sim e^{k y}\left[\gamma_{1}(k) \cos k x-j \gamma_{1}(k) \sin k x\right](\because \text { Eq. }(4.37)) \\
& \sim \gamma_{1}(k) e^{-j k x+k y} \\
& \sim \phi_{T 1}^{-\infty}(x, y) \tag{4.52}
\end{align*}
$$

From the result above, the first-order transmitted wave can be obtained.

$$
\eta_{T 1}=-\frac{1}{g} \frac{\partial}{\partial t}\left(\Phi_{T 1}^{+\infty}(x, 0, t)\right)
$$

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$$
\begin{align*}
& =-\frac{1}{g} \frac{\partial}{\partial t}\left(\phi_{T 1}^{+\infty}(x, 0) e^{-j \omega t}\right) \\
& =-\frac{1}{g} \frac{\partial}{\partial t}\left(\gamma_{1}(k) e^{-j(k x+\omega t)}\right) \\
& =-\frac{1}{g} \gamma_{1}(k)(-j \omega) e^{-j(k x+\omega t)} \\
& =A \frac{\Lambda}{\Delta_{123}} e^{-j(k x+\omega t)} \tag{4.53}
\end{align*}
$$

Therefore, the first-order transmitted wave amplitude is calculated below:

$$
\begin{align*}
A_{T 1} & =\left|A \frac{\Lambda}{\Delta_{123}}\right| \\
& =A \frac{|\Lambda|}{\sqrt{\Delta_{11}^{2}+\left(\Delta_{12}-\Delta_{13}\right)^{2}}} . \tag{4.54}
\end{align*}
$$

Consequently, from the perturbed solution form as in Eq. (2.23), the reflected wave and the transmitted wave are expressed in the perturbed series of the form,

$$
\begin{align*}
\eta_{R} & =\eta_{R 0}+\varepsilon \eta_{R 1} \\
& =\frac{A\left(\Delta_{11}-\varepsilon \Lambda\right)}{\Delta_{123}} e^{j(k x-\omega t)} \tag{4.55}
\end{align*}
$$

and,

$$
\begin{align*}
\eta_{T} & =\eta_{T 0}+\varepsilon \eta_{T 1} \\
& =\frac{-A\left(j\left(\Delta_{12}-\Delta_{13}\right)-\varepsilon \Lambda\right)}{\Delta_{123}} e^{-j(k x+\omega t)} . \tag{4.56}
\end{align*}
$$

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Dividing Eq. (4.55) and Eq. (4.56) with the incident wave amplitude $A$, the reflection coefficient and transmission coefficient considering the permeability of the plate are represented below.

$$
\begin{align*}
& R=\frac{A_{R}}{A}=\frac{\left|\Delta_{11}-\varepsilon \Lambda\right|}{\sqrt{\Delta_{11}^{2}+\left(\Delta_{12}-\Delta_{13}\right)^{2}}} .  \tag{4.57}\\
& T=\frac{A_{T}}{A}=\frac{\left|j\left(\Delta_{12}-\Delta_{13}\right)-\varepsilon \Lambda\right|}{\sqrt{\Delta_{11}^{2}+\left(\Delta_{12}-\Delta_{13}\right)^{2}}} . \tag{4.58}
\end{align*}
$$

### 4.3 Numerical approximate integration of $R$ and $T$

Unlike in the case of the leading-order problem, the reflection and transmission coefficients of the first-order solution for the velocity potential cannot be easily evaluated with ordinary sense since it has an improper integral when calculating $F(u)$ in Eq. (4.26). In Eq. (4.26), in the denominator of the integrand, $u^{2}$ can have the same value with $y^{2}$ when $y \in(-b,-a)$ and $u \in(a, b)$. Thus, this integral cannot be calculated in the normal quadrature rule, and it has the same form of finite Hilbert transform—or Cauchy principal value integral—of $\sqrt{\left(y^{2}-a^{2}\right)\left(b^{2}-y^{2}\right)} y f(y)$. This type of singular integral equation can be seen in numerous practical problems in science and engineering disciplines (Keller and Wróbel, 2016; Kaya and Erdogan, 1987). Therefore, an appropriate scheme for the numerical evaluation of this integral is needed.

In this paper, following the uniform approximation methods to finite Hilbert transform by Hasegawa (2004), $F(u)$ was numerically calculated with interpolating numerator with Chebyshev polynomials. First of all, Eq. (4.26) is transformed with subtracting out the singularity, which is the typical procedure for approximating the principal value integral (Kumar, 1980; Davis, 1984; Hasegawa and Torii, 1991; Diethelm, 1999; Hasegawa, 2004; Keller and Wróbel, 2016). Let the Cauchy principal value of $F(u)$ as $\frac{1}{\pi} I\left(g ; u^{2}\right)$, where $g$ represents the numerator of the integrand in $F(u)$.

$$
\begin{align*}
I\left(g ; u^{2}\right) & =2 \text { p.v. } \int_{-b}^{-a} \frac{\sqrt{\left(y^{2}-a^{2}\right)\left(b^{2}-y^{2}\right)} y f(y)}{y^{2}-u^{2}} d y \\
& =\text { p.v. } \int_{b^{2}}^{a^{2}} \frac{\sqrt{\left(\tau-a^{2}\right)\left(b^{2}-\tau\right)} f(-\sqrt{\tau})}{\tau-u^{2}} d \tau \\
& =\text { p.v. } \int_{b^{2}}^{a^{2}} \frac{g(t)}{\tau-u^{2}} d \tau \\
& =\int_{b^{2}}^{a^{2}} \frac{g(\tau)-g\left(u^{2}\right)}{\tau-u^{2}} d \tau+\text { p.v. } \int_{b^{2}}^{a^{2}} \frac{g\left(u^{2}\right)}{\tau-u^{2}} d \tau \\
& =\int_{b^{2}}^{a^{2}} \frac{g(\tau)-g\left(u^{2}\right)}{\tau-u^{2}} d \tau+g\left(u^{2}\right) \ln \left(\frac{a^{2}-u^{2}}{u^{2}-b^{2}}\right) . \tag{4.59}
\end{align*}
$$

Eq. (4.59) may be evaluated by approximating $g(\tau)$ as a polynomial $p_{M}(t)$, which is the finite sum of Chebyshev polynomial of the first kind $T_{r}(t)$ (Hasegawa and Torii, 1991). Here, $\tau$ is the variable, that the interval of $t,[-1,1]$, is mapped to $\left[b^{2}, a^{2}\right]$ by the affine transformation.

$$
\begin{equation*}
\tau=\frac{a^{2}-b^{2}}{2} t+\frac{a^{2}+b^{2}}{2}, \quad t \in[-1,1] . \tag{4.60}
\end{equation*}
$$

When $t$ is given as $t=\cos \theta, T_{r}(t)=\cos r \theta$. Then, the function $g(\tau)$ can be approximated to $p_{M}(t)$ as below (Clenshaw and Curtis, 1960).

$$
\begin{equation*}
p_{M}(t)=\sum_{r=0}^{M}{ }^{\prime \prime} a_{r}^{M} T_{r}(t), \quad-1 \leq t \leq 1, \tag{4.61}
\end{equation*}
$$

where double prime is the symbol of the summation whose first and last terms are halved. Selecting $t_{q}=\cos (\pi q / M), q=0, \cdots, M$ as the $M+1$ interpolating abscissa, the interpolation condition becomes,

$$
\begin{equation*}
g\left(\tau_{q}\right)=p_{M}\left(t_{q}\right)=p_{M}\left(\cos \left(\frac{\pi q}{M}\right)\right), \quad q=0, \cdots, M \tag{4.62}
\end{equation*}
$$

Here, $\tau_{q}=\frac{a^{2}-b^{2}}{2} t_{q}+\frac{a^{2}+b^{2}}{2}$. Then, the coefficients $a_{r}^{M}$ of the approximating polynomial $p_{M}(t)$ in Eq. (4.61) can be calculated as below:

$$
\begin{equation*}
a_{r}^{M}=\frac{2}{M} \sum_{q=0}^{M} g\left(\tau_{q}\right) \cos \left(\frac{\pi q r}{M}\right), \quad r=0, \cdots, M \tag{4.63}
\end{equation*}
$$

Hasegawa and Torii (1994) have shown that the integral term in Eq. (4.59) can be evaluated by substituting $p_{M}(t)$ into $g(\tau)$ and adopting the quadrature rule using Chebyshev expansion in terms of $T_{r}(t)$. Since the interval of integration is not $[-1,1], u^{2}$ should not be directly substituted into the quadrature rule which Hasegawa and Torii (1994) presented. Therefore, $c \in(-1,1)$ is used to map the singular point
$u^{2} \in\left(a^{2}, b^{2}\right)$ into the interval $(-1,1)$, which has the mapping function of,

$$
\begin{equation*}
u^{2}=\frac{b^{2}-a^{2}}{2} c+\frac{a^{2}+b^{2}}{2}, \quad c \in(-1,1) \tag{4.64}
\end{equation*}
$$

Now, for $p_{M}(t)$ defined in Eq. (4.61), the approximated quadrature rule for the integral in Eq. (4.59) is,

$$
\begin{equation*}
\int_{a^{2}}^{b^{2}} \frac{p_{M}(t)-p_{M}(c)}{t-u^{2}} d t=4 \sum_{r=0}^{M-1}{ }^{\prime} A_{r}^{M} T_{r}(c), \quad-1<c<1, \tag{4.65}
\end{equation*}
$$

where $A_{r}^{M}$ is defined as,

$$
\begin{equation*}
A_{r}^{M}=\sum_{m=0}^{[(M-r-1) / 2]} \frac{a_{2 m+r+1}^{M}}{2 m+1}, \quad \leq r \leq M-1, \tag{4.66}
\end{equation*}
$$

with using $\frac{1}{2} a_{M}^{M}$ instead of $a_{M}^{M}$ (Hasegawa, 2004). Here, $[\diamond]$ is the floor function that gives the output of the greatest integer less than or equal to $\diamond$.

In sum, $Q_{M}\left(g ; u^{2}\right)$, which is the approximation to $I\left(g ; u^{2}\right)$ in Eq. (4.59), is given as follows:

$$
\begin{equation*}
Q_{M}\left(g ; u^{2}\right)=4 \sum_{r=0}^{M-1}{ }^{\prime} A_{r}^{M} T_{r}(c)+g\left(u^{2}\right) \ln \left(\frac{a^{2}-u^{2}}{u^{2}-b^{2}}\right) . \tag{4.67}
\end{equation*}
$$

Employing Eq. (4.67) to evaluate $F(u)$ in Eq. (4.26) and sequentially calculating $a_{1}(k, F)$ and $\gamma_{1}(k)$, in Eq. (4.27) and Eq. (4.30), the reflection and transmission
coefficient considering the permeability of the plate can be illustrated.

Before computing the reflection and transmission coefficients, the convergence of $a_{M}^{M}$ was examined. $a_{M}^{M}$, the last coefficient of the approximating polynomial weighs heavily when calculating the upper limit of error. Thus, the reasonable $M$ making the absolute tolerance of $a_{M}^{M}$ as $10^{-6}$ was taken. Therefore, as shown in Fig. (4.4), $M=2^{12}$ was used to determine the number of interpolating abscissa.


Figure $4.4 a_{M}^{M}$ in log scale for $\mu=0.001$ and $k b=1.5$.

Using $M+1=2^{12}+1$ points, $F(u)$ was evaluated. The outline of $F(u)$ can be found in Fig. (4.5). Corresponding $a_{i}(k, F), a_{1}(-k, F), \Delta_{F i}$ and $\Delta_{F i}^{\prime \prime}$ are shown in Fig. (4.8-4.9).


Figure 4.5 $F(u)$ versus $u$ for (top) $k b=0.5$, (center) $k b=1.5$, (bottom) $k b=2.5$. (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. $\left(-, \operatorname{Re}_{j}\{F(u)\} ;---, \operatorname{Im}_{j}\{F(u)\}\right.$.)


Figure 4.5 $F(u)$ versus $u$ for (top) $k b=0.5$, (center) $k b=1.5$, (bottom) $k b=2.5$. (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. $\left(-, \operatorname{Re}_{j}\{F(u)\} ;---\operatorname{Im}_{j}\{F(u)\}\right.$.) (cont.)


Figure 4.5 $F(u)$ versus $u$ for (top) $k b=0.5$, (center) $k b=1.5$, (bottom) $k b=2.5$. (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. $\left(-, \operatorname{Re}_{j}\{F(u)\} ;---\operatorname{Im}_{j}\{F(u)\}\right.$.) (cont.)


Figure 4.5 $F(u)$ versus $u$ for (top) $k b=0.5$, (center) $k b=1.5$, (bottom) $k b=2.5$. (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. $\left(-, \operatorname{Re}_{j}\{F(u)\} ;---\operatorname{Im}_{j}\{F(u)\}\right.$.)(cont.)


Figure 4.5 $F(u)$ versus $u$ for (top) $k b=0.5$, (center) $k b=1.5$, (bottom) $k b=2.5$. (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. $\left(-, \operatorname{Re}_{j}\{F(u)\} ;--, \operatorname{Im}_{j}\{F(u)\}\right.$.) (cont.)


Figure $4.5 F(u)$ versus $y$ for (top) $k b=0.5$, (center) $k b=1.5$, (bottom) $k b=2.5$.
(a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (一, $\operatorname{Re}_{j}\{F(u)\} ;--, \operatorname{Im}_{j}\{F(u)\}$.) (cont.)


Figure 4.6 (left) $\operatorname{Re}_{j}\left\{a_{i}(k, F)\right\}$ and (right) $\operatorname{Im}_{j}\left\{a_{i}(k, F)\right\}$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (—, $a_{1}(k, F)$; $\cdots \cdot, a_{2}(k, F) ; \cdots \cdots, a_{3}(k, F)$.)

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Figure 4.6 (left) $\operatorname{Re}_{j}\left\{a_{i}(k, F)\right\}$ and (right) $\operatorname{Im}_{j}\left\{a_{i}(k, F)\right\}$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (—, $a_{1}(k, F)$; $-\cdots, a_{2}(k, F) ; \cdots \cdots, a_{3}(k, F)$.) (cont.)


Figure 4.7 (left) $\operatorname{Re}_{j}\left\{a_{1}(-k, F)\right\}$ and (right) $\operatorname{Im}_{j}\left\{a_{1}(-k, F)\right\}$ versus $k b$ for (a) $\mu=$ 0.001 , (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$.


Figure 4.7 (left) $\operatorname{Re}_{j}\left\{a_{1}(-k, F)\right\}$ and (right) $\operatorname{Im}_{j}\left\{a_{1}(-k, F)\right\}$ versus $k b$ for (a) $\mu=$ 0.001 , (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$ (cont.).


Figure 4.8 (left) $\operatorname{Re}_{j}\left\{\Delta_{F i}\right\}$ and (right) $\operatorname{Im}_{j}\left\{\Delta_{F i}\right\}$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (—, $\Delta_{F 1} ;---$, $\left.\Delta_{F 2} ;-\cdots, \Delta_{F 3}.\right)$


Figure 4.8 (left) $\operatorname{Re}_{j}\left\{\Delta_{F i}\right\}$ and (right) $\operatorname{Im}_{j}\left\{\Delta_{F i}\right\}$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (-, $\Delta_{F 1}$; ---, $\left.\Delta_{F 2} ;---, \Delta_{F 3}.\right)$ (cont.)


Figure 4.9 (left) $\operatorname{Re}_{j}\left\{\Delta_{F i}^{\prime \prime}\right\}$ and (right) $\operatorname{Im}_{j}\left\{\Delta_{F i}^{\prime \prime}\right\}$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (—, $\Delta_{F 1}^{\prime \prime} ;---$ $\left.\Delta_{F 2}^{\prime \prime} ;-\cdot-, \Delta_{F 3}^{\prime \prime}.\right)$


Figure 4.9 (left) $\operatorname{Re}_{j}\left\{\Delta_{F i}^{\prime \prime}\right\}$ and (right) $\operatorname{Im}_{j}\left\{\Delta_{F i}^{\prime \prime}\right\}$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (-, $\Delta_{F 1}^{\prime \prime} ;--$, $\Delta_{F 2}^{\prime \prime} ;-\cdot-, \Delta_{F 3}^{\prime \prime}$.) (cont.)

Then, from Eq. (4.41) and Eq. (4.42), $C_{1}$ and $D_{1}$ can be calculated.


Figure 4.10 (left) $C_{1}$ and (right) $D_{1}$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (-, $\operatorname{Re}_{j}\left\{C_{1}\right\}$ and $\operatorname{Re}_{j}\left\{D_{1}\right\} ;---$ $\operatorname{Im}_{j}\left\{C_{1}\right\}$ and $\operatorname{Im}_{j}\left\{D_{1}\right\}$.)


Figure 4.10 (left) $C_{1}$ and (right) $D_{1}$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (-, $\operatorname{Re}_{j}\left\{C_{1}\right\}$ and $\operatorname{Re}_{j}\left\{D_{1}\right\} ;---$ $\operatorname{Im}_{j}\left\{C_{1}\right\}$ and $\operatorname{Im}_{j}\left\{D_{1}\right\}$.) (cont.)
$\Lambda$ in Eq. (4.49) determines the magnitude of the first-order term in reflection and transmission coefficients, and critically affects the applicable range of the perturbation method. Hence, $\Delta_{i}$ and $\Lambda$ were also plotted.

(a)

(b)

Figure $4.11 \Delta_{i}$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5 .\left(---, \Delta_{2} ;--\cdot, \Delta_{3}\right.$.)


Figure $4.11 \Delta_{i}$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (---, $\Delta_{2} ;--\cdot, \Delta_{3}$.) (cont.).


Figure $4.11 \Delta_{i}$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (---, $\Delta_{2}$; ---, $\Delta_{3}$.) (cont.).


Figure $4.12 \Lambda$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (一, $\operatorname{Re}_{j}\{\Lambda\} ;---\operatorname{Im}_{j}\{\Lambda\}$.)


Figure $4.12 \Lambda$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (,$- \operatorname{Re}_{j}\{\Lambda\} ;--, \operatorname{Im}_{j}\{\Lambda\}$.) (cont.).


Figure $4.12 \Lambda$ versus $k b$ for (a) $\mu=0.001$, (b) $\mu=0.01$, (c) $\mu=0.05$, (d) $\mu=0.1$, (e) $\mu=0.25$, (f) $\mu=0.5$. (一, $\operatorname{Re}_{j}\{\Lambda\} ;---, \operatorname{Im}_{j}\{\Lambda\}$.) (cont.).

Finally, the reflection and transmission coefficients considering the permeable effect are presented. In Fig. (4.12), it can be found that both the real part and the imaginary part of $\Lambda$ become negative values after crossing certain point. Also, its magnitude diverges when $k b$ grows. This indicates that the range of $k b$ should be limited for the perturbation method to be valid. Thus, the range that $\Lambda$ is positive, and has smaller value than $\Delta_{11}$ in the leading order term is only considered to calculate the reflection and transmission coefficients.

In Fig. (4.13-4.24), it is seen that both the reflection coefficient and the transmission coefficient decrease as the perturbation parameter $\varepsilon$ increases. That is to say, due to the permeability effect, the total wave energy is dissipated while going through the permeable plate. Especially this wave energy decline becomes more evident as $\mu$ is smaller, which means that the more the breakwater stretches to the water surface, the bigger the permeability effect influences the reflection and transmission coefficients. Also, considering the valid range of $k b$, it is found that the perturbation method solution is effective when $k b$ has small value.


Figure 4.13 Reflection coefficient $R$ versus $k b$ for $\mu=0.001$ : -, $\varepsilon=0 ;--$, $\varepsilon=0.25 ; \cdots, \varepsilon=0.5 ; \cdots \cdots, \varepsilon=1$.


Figure 4.14 Transmission coefficient $T$ versus $k b$ for $\mu=0.001:-, \varepsilon=0 ;--$, $\varepsilon=0.25 ; \cdots, \varepsilon=0.5 ; \cdots \cdots, \varepsilon=1$.


Figure 4.15 Reflection coefficient $R$ versus $k b$ for $\mu=0.01$ : -, $\varepsilon=0 ;---\varepsilon=0.25$;
$\cdots \cdot \cdot, \varepsilon=0.5 ; \cdots \cdots, \varepsilon=1$.


Figure 4.16 Transmission coefficient $T$ versus $k b$ for $\mu=0.01:-, \varepsilon=0 ;--$, $\varepsilon=0.25 ; \cdots, \varepsilon=0.5 ; \cdots \cdots, \varepsilon=1$.


Figure 4.17 Reflection coefficient $R$ versus $k b$ for $\mu=0.05$ : -, $\varepsilon=0 ;---\varepsilon=0.25$; $\cdots \cdot \cdot, \varepsilon=0.5 ; \cdots \cdots, \varepsilon=1$.


Figure 4.18 Transmission coefficient $T$ versus $k b$ for $\mu=0.05:-, \varepsilon=0 ;--$, $\varepsilon=0.25 ; \cdots, \varepsilon=0.5 ; \cdots \cdots, \varepsilon=1$.


Figure 4.19 Reflection coefficient $R$ versus $k b$ for $\mu=0.1:-, \varepsilon=0 ;--, \varepsilon=0.25$;
$\cdots-\cdot, \varepsilon=0.5 ; \cdots \cdots, \varepsilon=1$.


Figure 4.20 Transmission coefficient $T$ versus $k b$ for $\mu=0.1:-, \varepsilon=0 ;--$, $\varepsilon=0.25 ; \cdots, \varepsilon=0.5 ; \cdots \cdots, \varepsilon=1$.


Figure 4.21 Reflection coefficient $R$ versus $k b$ for $\mu=0.25$ : -, $\varepsilon=0 ;---, \varepsilon=0.25$;
$-\cdot-\cdot, \varepsilon=0.5 ; \cdots \cdots, \varepsilon=1$.


Figure 4.22 Transmission coefficient $T$ versus $k b$ for $\mu=0.25:-, \varepsilon=0 ;--$, $\varepsilon=0.25 ; \cdots \cdot \varepsilon=0.5 ; \cdots \cdots, \varepsilon=1$.


Figure 4.23 Reflection coefficient $R$ versus $k b$ for $\mu=0.5$ : -, $\varepsilon=0 ;--$, $\varepsilon=0.25$;
$-\cdot-\cdot, \varepsilon=0.5 ; \cdots \cdots, \varepsilon=1$.


Figure 4.24 Transmission coefficient $T$ versus $k b$ for $\mu=0.5:-, \varepsilon=0 ;--$, $\varepsilon=0.25 ; \cdots, \varepsilon=0.5 ; \cdots \cdots, \varepsilon=1$.

## Chapter 5. Conclusion and Future Studies

The analytical solution for the velocity field around submerged permeable breakwaters and its application in computing the reflection and transmission coefficients are presented in this study.

Formulating the two-dimensional problem of wave scattering by a vertical submerged permeable breakwater in infinite-depth water, the potential wave theory is adopted, and mild singularity on the edges of the plate is assumed. Then, a permeable boundary condition on the breakwater is given, which makes the boundary condition nonlinear. Using the perturbation series expansion and the reduction method, homogeneous and nonhomogeneous Riemann-Hilbert problems are defined up to the first order. Then, the exact, closed-form analytical solutions for each problem are derived to obtain the perturbed velocity field.

In order to present the application of the obtained velocity potential, the reflection and transmission coefficients are calculated. From the computed leadingorder reflection and transmission coefficients, the wave attenuation effect is shown to fit well with physical intuition and is consistent with the previous studies which investigated the impermeable vertical breakwaters.

Then, numerical computation of the first-order wave amplitude is presented so that the effect of the permeability of the plate can be qualitatively examined.

Since the singularity along the plate hinders the discrete integration directly using the uniform grid, significant effort into evaluating numerically approximated solutions near the contour was paid. Involving Chebyshev series interpolation and subtracting the singular point, the discontinuity by the plate is approximated, and a numerical approach to the Cauchy principal value integral (finite Hilbert transform) is used to construct a collocation method. This allows for the integration of a singular integral following the discontinuous contour. The reflection and transmission coefficients calculated in the present study appeared that the wave energy dissipation effect is better when using the permeable breakwaters.

In contrast to previous studies that used the eigenfunction expansion method, this study uses the perturbation method to formulate the problem of wave scattering by the submerged floating vertical permeable breakwater. Moreover, the reduction method is adopted to form the Riemann-Hilbert problem so that the exact, closed form of the solution can be derived. Not only presenting the solution, the application of the solution is also considered together by illustrating the reflection coefficient and the transmission coefficient in various wave, breakwater geometry, and permeability conditions. Besides the conditions selected in the present research, one may choose the arbitrary wave, breakwater, and permeability conditions and can obtain the velocity field or calculate the wave attenuation effect.

Despite the effort, there are some points that can be improved by further
research. First, although selecting large enough $M$ does not give much error, applying the automatic quadrature scheme to calculate the singular integral numerically can efficiently handle the truncation error by interpolating integration points or the roundoff error by the machine epsilon. Second, the case of $\mu=0$ is not presented in considering the permeable effect since this needs another approach taking the limit to zero. To compare with the surface-piercing type permeable breakwaters, asymptotic behavior when $\mu \rightarrow 0$ should be considered.

## References

Briggs, M., Ye, W., Demirbilek, Z., and Zhang, J. (2002). "Field and numerical comparisons of the RIBS floating breakwater." Journal of Hydraulic Research, 40(3), 289-301.

Briggs, M. J. (2001). "Analytical and Numerical Models of the RIBS XM99 OceanScalae Prototype." Report No. ERDC/CHL TR 01-19., Coastal and Hydraulics Laboratory (U.S.) Engineer Research and Development Center (U.S.).

Byrd, P. F. and Friedman, M. D. (1954). Handbook of Elliptic Integrals for Engineers and Physicists. Springer Berlin Heidelberg, Berlin, Heidelberg.

Chwang, A. T. (1983). "A porous-wavemaker theory." Journal of Fluid Mechanics, 132, 395-406.

Clenshaw, C. W. and Curtis, A. R. (1960). "A method for numerical integration on an automatic computer." Numerische Mathematik, 2(1), 197-205.

Davis, P. J. (1984). Methods of numerical integration / Philip J. Davis, Philip Robinowitz. Computer science and applied mathematics. Academic, Orlando, 2nd edition.

Dean, W. R. (1945). "On the reflexion of surface waves by a submerged plane barrier." Mathematical Proceedings of the Cambridge Philosophical Society, 41(3), 231238.

Diethelm, K. (1999). "A method for the practical evaluation of the Hilbert transform on the real line." Journal of Computational and Applied Mathematics, 112(1-2), 45-53.

Evans, D. V. (1970). "Diffraction of water waves by a submerged vertical plate." Journal of Fluid Mechanics, 40(3), 433-451.

Gayen, R. and Mondal, A. (2014). "A hypersingular integral equation approach to the porous plate problem." Applied Ocean Research, 46, 70-78.

Hasegawa, T. (2004). "Uniform approximations to finite Hilbert transform and its derivative." Journal of Computational and Applied Mathematics, 163(1), 127-138.

Hasegawa, T. and Torii, T. (1991). "An Automatic Quadrature for Cauchy Principal Value Integrals." Mathematics of Computation, 56(194), 741.

Hasegawa, T. and Torii, T. (1994). "Hilbert and Hadamard transforms by generalized Chebyshev expansion." Journal of Computational and Applied Mathematics, 51(1), 71-83.

Havelock, T. (1940). "XLI. Waves produced by the rolling of a ship." The London,

Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 29(195), 407-414.

Isaacson, M., Premasiri, S., and Yang, G. (1998). "Wave Interactions with Vertical Slotted Barrier." Journal of Waterway, Port, Coastal, and Ocean Engineering, 124(3), 118-126.

Ji, C., Cheng, Y., Yang, K., and Oleg, G. (2017). "Numerical and experimental investigation of hydrodynamic performance of a cylindrical dual pontoon-net floating breakwater." Coastal Engineering, 129, 1-16.

Kaya, A. C. and Erdogan, F. (1987). "On the solution of integral equations with strongly singular kernels." Quarterly of Applied Mathematics, 45(1), 105-122.

Keller, P. and Wróbel, I. (2016). "Computing Cauchy principal value integrals using a standard adaptive quadrature." Journal of Computational and Applied Mathematics, 294, 323-341.

Kumar, S. (1980). "A note on quadrature formulae for cauchy principal value integrals." IMA Journal of Applied Mathematics (Institute of Mathematics and Its Applications), 26(4), 447-451.

Lee, M. M. and Chwang, A. T. (2000). "Scattering and radiation of water waves by permeable barriers." Physics of Fluids, 12(1), 54-65.

Lewin, M. (1963). "The effect of vertical barriers on progressing waves." Journal of Mathematics and Physics, 42(1-4), 287-300.

Macaskill, C. (1979). "Reflexion of water waves by a permeable barrier." Journal of Fluid Mechanics, 95(1), 141-157.

Mei, C. C. (1966). "Radiation and scattering of transient gravity waves by vertical plates." Quarterly Journal of Mechanics and Applied Mathematics, 19(4), 417-440.

Muskhelishvili, N. I. (1977). Singular Integral Equations. Springer Netherlands, Dordrecht.

Porter, D. (1972). "The transmission of surface waves through a gap in a vertical barrier." Mathematical Proceedings of the Cambridge Philosophical Society, 71(2), 411-421.

Taylor, G. I. (1956). "Fluid flow in regions bounded by porous surfaces." Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences, 234(1199), 456-475.

Ursell, F. (1948). "On the waves due to the rolling of a ship." The Quarterly Journal of Mechanics and Applied Mathematics, 1(1), 246-252.

Yu, X. and Chwang, A. T. (1994). "Wave-Induced Oscillation in Harbor with Porous Breakwaters." Journal of Waterway, Port, Coastal, and Ocean Engineering, 120(2), 125-144.

## 국문초록

본 연구는 선형파 조건에서 수중 투과성 방파제에 의해 산란된 유속장에 대한 문제를 정의하고 그에 대한 해석해를 제시한다.

본 연구에서는 문제 상황으로 물에 잠긴 투과성 방파제가 수직하게 유한한 구간만큼 있고, 그 위에 미소진폭파가 오는 상황을 상정하였다. 유체의 수심이 무한 히 깊고 흐름이 비압축성, 비점성, 비회전성을 만족한다면, 포텐셜 이론을 적용할 수 있다. 그러나, 투과성 판을 뚫는 흐름으로 인해 판에서는 비선형적인 경계조건이 주어지고, 이는 문제에 대한 해가 존재할 수 없도록 한다.

따라서, 이러한 비선형적인 경계조건을 해결하기 위해 투과성을 대표하는 작은 매개변수를 설정해 섭동법을 적용하였다. 일차항까지 전개하였을 때, 영차 문제는 불투과성 방파제에 의한 산란된 유속 포텐셜을 나타나개 되고, 일차 문제는 투과성 방파제에 의해 산란된 유속장을 고려해 영차해에 수정 효과를 준다.

이에 더해 경계조건을 더욱 간단히하기 위해 공간 포텐셜에 관한 문제를 축 소 포텐셜(reduced potential)에 관한 문제로 치환하여 축소법(reduction method)을 적용하였다. 이는 영차 문제에 관해서는 동차 리만-힐베르트 문제를 만족시키고, 일차 문제에서는 비동차 리만-힐베르트 문제가 된다. 이렇게 정의한 문제를 풀어 각 차수의 유속장에 대해 완전한 닫힌 형태의 해를 유도하였다.

유도된 유속장 해의 활용 방안 예시로, 다양한 파랑 조건과 방파제 조건에

서의 반사계수와 투과계수를 산정하였다. 이 때, 유한 힐베르트 변환은 체비셰프 다항식을 이용한 근사 수치 구적법으로 계산되었다. 산정 결과 투과성 방파제가 불투과성 방파제보다 파랑 에너지를 더 많이 소산시키는 것을 볼 수 있었다. 본 연 구에서 유도된 유속장은 파고 감쇠 효과를 평가하는 것 외에도 방파제에 가해지는 힘이난 모멘트 등 유체역학적 파력을 산정하는 것과 같이 다양하게 활용될 수 있다.

주요어: 투과성 방파제, 수중 부유식 방파제, 선형파 이론, 섭동법, 해석해 학번: 2021-23413

