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이학박사 학위논문

A study on structures of digraphs
and graphs in the aspect of
competition in ecosystems

(생태계에서의 경쟁 관점으로 그래프와 유향그래프의
구조 연구)

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서울대학교 대학원

수학교육과

최명호

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A study on structures of digraphs and graphs in the aspect of competition in ecosystems

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
to the faculty of the Graduate School of
Seoul National University

by

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Abstract

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In this thesis, we study m -step competition graphs, $(1, 2)$ -step competition graphs, phylogeny graphs, and competition-common enemy graphs (CCE graphs), which are primary variants of competition graphs. Cohen [11] introduced the notion of competition graph while studying predator-prey concepts in ecological food webs. An ecosystem is a biological community of interacting species and their physical environment. For each species in an ecosystem, there can be m conditions of the good environment by regarding lower and upper bounds on numerous dimensions such as soil, climate, temperature, etc, which may be represented by an m -dimensional rectangle, so-called an ecological niche. An elemental ecological truth is that two species compete if and only if their ecological niches overlap. Biologists often describe competitive relations among species cohabiting in a community by a food web that is a digraph whose vertices are the species and an arc goes from a predator to a prey. In this context, Cohen [11] defined the competition graph of a digraph as follows. The competition graph $C(D)$ of a digraph D is defined to be a simple graph whose vertex set is the same as $V(D)$ and which has an edge joining two distinct vertices u and v if and only if there are arcs (u, w) and (v, w) for some vertex w in D . Since Cohen introduced this definition, its variants such as m -step competition graphs, (i, j) -step competition graphs, phylogeny graphs, CCE graphs, p -competition graphs, and niche

graphs have been introduced and studied.

As part of these studies, we show that the connected triangle-free m -step competition graph on n vertices is a tree and completely characterize the digraphs of order n whose m -step competition graphs are star graphs for positive integers $2 \leq m < n$.

We completely identify $(1, 2)$ -step competition graphs $C_{1,2}(D)$ of orientations D of a complete k -partite graph for some $k \geq 3$ when each partite set of D forms a clique in $C_{1,2}(D)$. In addition, we show that the diameter of each component of $C_{1,2}(D)$ is at most three and provide a sharp upper bound on the domination number of $C_{1,2}(D)$ and give a sufficient condition for $C_{1,2}(D)$ being an interval graph.

On the other hand, we study on phylogeny graphs and CCE graphs of degree-bounded acyclic digraphs. An acyclic digraph in which every vertex has indegree at most i and outdegree at most j is called an (i, j) digraph for some positive integers i and j . If each vertex of a (not necessarily acyclic) digraph D has indegree at most i and outdegree at most j , then D is called an $\langle i, j \rangle$ digraph. We give a sufficient condition on the size of hole of an underlying graph of an $(i, 2)$ digraph D for the phylogeny graph of D being a chordal graph where D is an $(i, 2)$ digraph. Moreover, we go further to completely characterize phylogeny graphs of (i, j) digraphs by listing the forbidden induced subgraphs.

We completely identify the graphs with the least components among the CCE graphs of $(2, 2)$ digraphs containing at most one cycle and exactly two isolated vertices, and their digraphs. Finally, we give a sufficient condition for CCE graphs being interval graphs.

Key words: competition graphs, m -step competition graphs, $(1, 2)$ -step competition graphs, phylogeny graphs, competition-common enemy graph

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Chapter 1

Introduction

1.1 Graph theory terminology and basic concepts

We introduce some basic notions in graph theory, which shall be commonly used in this thesis. For undefined terms, readers may refer to [4].

A *graph* G is defined as an ordered pair (V, E) where V is a set and E is a family of unordered pairs of elements in V . An element of V and an element of E are called a *vertex* and an edge of G , respectively. If $e = \{u, v\}$ is an edge, then we simply write it by uv for convenience when there is no confusion. The set of vertices and the set of edges of a graph G are called the *vertex set* and the *edge set* of G , respectively, and denoted by $V(G)$ and $E(G)$, respectively. Any graph with just one vertex is referred to as *trivial*. All other graphs are *nontrivial*.

Let G be a graph with an edge $e = \{u, v\}$ of G . Then we say that e connects (or joins) u and v , u and v are the *end vertices* of e , and u and v are *adjacent* in G . In addition, each of u and v is said to be *incident* with e , and *vice versa*. If $u = v$, then e is called a *loop*. If $u \neq v$ and there is an edge distinct from e connecting u and v , then $\{u, v\}$ is called a *multiple edge* (or *parallel edge*). A graph is *simple* if it has no loops or parallel edges. The number of vertices and edges in G are called the *order* and *size* of G , respectively. Given a simple graph G , the *complement* \overline{G} of G is defined to be a simple graph obtained by reversing the adjacency of G , i.e., $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv \mid uv \notin E(G)\}$. The *adjacency matrix* of G is a square matrix A of

size $n \times n$ such that its element A_{ij} is one when there is an edge from a vertex v_i to vertex v_j , and zero when there is no edge. The diagonal elements of the matrix are all zero, since edges from a vertex to itself (loops) are not allowed in simple graphs.

Let G be a graph and v be a vertex of G . A vertex of G adjacent to v is called a *neighbor* of v . The set of neighbors of v is called the *neighborhood* of v and denoted by $N_G(v)$. The degree of v is the number of edges incident to v and denoted by $d_G(v)$ (or $\deg_G(v)$). A vertex of degree 0 is called an *isolated* vertex. If no confusion is likely, we sometimes omit the letter G from graph-theoretic symbols and write, for example, $N(v)$, $d(v)$, and $\deg(v)$ instead of $N_G(v)$, $d_G(v)$, and $\deg_G(v)$, respectively.

A *walk* in a graph G is a sequence $W := v_0, e_1, v_1, \dots, v_{k-1}, e_k, v_k$ whose terms are alternately vertices and edges of G (not necessarily distinct) such that v_{i-1} and v_i are the end vertices of e_i for each $1 \leq i \leq k$. We refer to W as a v_0v_k -walk. W is *closed* if $v_0 = v_k$. In a simple graph, W is commonly specified by the sequence v_0, v_1, \dots, v_k of its vertices. The *length* $\ell(W)$ of a walk W is the number of edges belonging to it. The vertices v_0 and v_k are called the *end vertices* of W , v_0 being its *initial vertex* and v_k its *terminal vertex*, the vertices v_1, \dots, v_{k-1} are its *internal vertices*. If there exists a walk starting from a vertex v to a vertex w , then we say that v and w are *connected* by a walk. If any two vertices are connected by a walk in a graph G , then we say G is *connected*. Otherwise, G is said to be *disconnected*. A maximally connected subgraph of G is called a *component* of G .

If the vertices in a walk are distinct, then the walk is called a *path*. A *trail* is a walk without repeated edges. A closed trail whose initial vertex and internal vertices are distinct is called a *cycle*. We denote a path on n vertices by P_n and a cycle on n vertices by C_n . The *length* of a path or a cycle is the number of its edges. A cycle of length 3 is called a *triangle*. A *complete graph* is a simple graph in which any two vertices are adjacent, an *empty graph* one which no two vertices are adjacent (that is, one whose edge set is empty). We denote a complete graph of order n by K_n .

A *digraph* (or *directed graph*) D is defined as an ordered pair $(V(D), A(D))$ where $V(D)$ is a set and $A(D)$ is a family of ordered pairs of elements in $V(D)$. An element of $V(D)$ and an element of $A(D)$ are called a *vertex* and an *arc* (or *directed edge*) of D , respectively. The number of vertices in D is called the *order* of D . A digraph

(resp. graph) is *finite* if both its vertex set and arc set (resp. edge set) are finite. We can associate a graph G on the same vertex set as $V(D)$ simply by replacing each arc (u, v) with an edge uv . This graph is said to be the *underlying graph* of D . A digraph D is called *weakly connected* if the underlying graph of D is connected and a *weak component* of D is a subdigraph of D induced by a component in the underlying graph of D .

Let D be a digraph with an arc (u, v) . Then we say that u and v are the *tail* and the *head* of (u, v) , respectively. In addition, u and v are said to be *adjacent* in D . For convenience, we often use the notation $u \rightarrow v$ for “there is an arc (u, v) in D ”. If $u = v$, then the arc (u, v) is called a *loop*. If there are at least two arcs with the same heads and the same tails, then we call them *parallel edge*. A digraph is *simple* if it has no loops or parallel arcs.

Let D be a digraph and v be a vertex of D . We say a vertex w is an *out-neighbor* or *prey* (resp. *in-neighbor* or *predator*) of v if $(v, w) \in A(D)$ (resp. $(w, v) \in A(D)$). The set of out-neighbors (resp. in-neighbors) of v is called the *out-neighborhood* (resp. *in-neighborhood*) of v in D and denoted by $N_D^+(v)$ (resp. $N_D^-(v)$). The *outdegree* $d_D^+(v)$ is the number of arcs outgoing from v and the *indegree* $d_D^-(v)$ is the number of arcs incoming toward v . We call a vertex with outdegree 0 (resp. indegree 0) a *sink* (resp. *source*) of D .

A *directed walk* in a digraph D is a sequence

$$W := v_0, a_1, v_1, a_2, \dots, v_{k-1}, a_k, v_k$$

whose terms are alternately vertices and arcs of D where v_i is a vertex for each $0 \leq i \leq k$ and a_j is an arc from v_{j-1} to v_j for each $1 \leq j \leq k$. We refer to W as a *directed (v_0, v_k) -walk*. If D has no multiple arcs, W is abbreviated as $v_0 \rightarrow v_1 \rightarrow \dots \rightarrow v_k$. The concepts of directed trails, directed paths, and directed cycles in a digraph are defined analogously to the trails, paths, cycles in a graph, respectively. If D has no directed cycle, then D is said to be *acyclic*.

Two graphs (resp. digraphs) G and H are said to be *isomorphic* if there exist bijections $\theta : V(G) \rightarrow V(H)$ and $\phi : E(G) \rightarrow E(H)$ (resp. $\phi : A(G) \rightarrow A(H)$) such

that for every edge $e \in E(G)$ (resp. arc $a \in A(G)$), e connects vertices u and v in G (resp. $a = (u, v)$) if and only if $\phi(e)$ connects vertices $\theta(u)$ and $\theta(v)$ in H (resp. $\phi(a) = (\theta(u), \theta(v))$). If G and H are isomorphic, then we write $G \cong H$.

Given a graph G (resp. digraph), we call a graph (resp. digraph) H a *subgraph* (resp. *subdigraph*) of G if $V(H) \subset V(G)$, $E(H) \subset E(G)$ (resp. $A(H) \subset A(G)$), and we write $H \subset G$. The subgraph (resp. subdigraph) of G whose vertex set is X and whose edge set (resp. arc set) consists of all edges (resp. arcs) of G which have both ends in X is called the *subgraph* (resp. *subdigraph*) of G *induced by* X and is denoted by $G[X]$. The subgraph (resp. subdigraph) induced by $V(G) - X$ is denoted by $G - X$. For notational convenience, we write $G - v$ instead of $G - \{v\}$ for a vertex v in G . An *induced subgraph* (resp. *induced subdigraph*) is a graph (resp. digraph) by some nonempty subset of $V(G)$. We say that G is *H-free* if no induced subgraph (resp. subdigraph) of G is isomorphic to H . A vertex subset S of $V(G)$ is called a *clique* if the induced subgraph $G[S]$ is complete. A *maximal clique* of a graph G is a clique X of vertices of G , such that there is no clique of G that is a proper superset of X . The size of a maximum clique of a graph G is called a *clique number* and denoted by $\omega(G)$. We call a cycle of length at least 4 as an induced subgraph of G a *hole*. A graph is said to be *chordal* if it does not contain a hole.

In this thesis, we mainly study finite simple graphs and finite digraphs without multiple arcs, and the terms ‘graph’ and ‘digraph’ always means ‘finite simple graph’ and ‘finite digraph without multiple arcs’, respectively.

Mathematical Notation

\mathbb{N}	:	The set of positive integers
\mathbb{Z}	:	The set of integers
$V(G)$:	The vertex set of a graph (or a digraph) G
$E(G)$:	The edge set of a graph (or a digraph) G
$A(D)$:	The arc set of a graph (or a digraph) G
uv in G	:	The edge between a vertex u and a vertex v in a graph G
(u, v) in D	:	The arc from a vertex u and a vertex v in a digraph D
\overline{G}	:	The complement a graph G
$G[X]$:	The subgraph of a graph G induced by a vertex subset X
$G - X$:	The subgraph of a graph G induced by $V(G) - X$
$G - v$:	The subgraph of a graph G induced by $V(G) - \{v\}$
$N_G(v)$:	The neighborhood of a vertex v in a graph G
$d_G(v)$:	The degree of a vertex v in a graph G
$N_D^+(v)$:	The out-neighborhood of a vertex v in a digraph D
$N_D^-(v)$:	The in-neighborhood of a vertex v in a digraph D
P_n	:	A path of length n
C_n	:	A cycle of length n
K_n	:	A complete graph of n vertices
$\omega(G)$:	the clique number of a graph G

1.2 Competition graphs and its variants

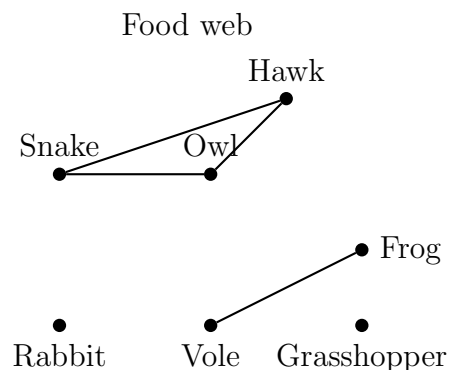
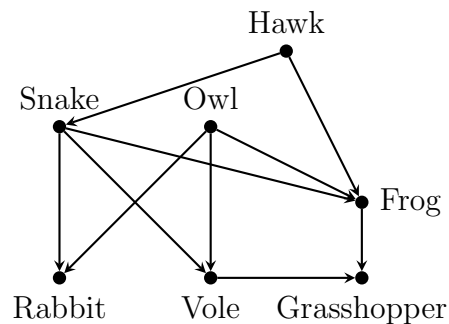
1.2.1 A brief background of competition graphs

An ecosystem is a biological community of interacting species and their physical environment. For each species in an ecosystem, survival is a major issue and there can be conditions of the good environment by regarding lower and upper bounds on numerous dimensions such as soil, climate, temperature, etc. If m different factors of an environment are measured, then m dimensions are needed to describe the environment. Moreover, if the range of each factor is determined, then there is a corresponding region R in m -dimensional Euclidean space such that each point in R lies within the bounds and R is a m -dimensional rectangle with sides parallel to the coordinate axes. Danzer and Grünbaum [16] call such a region a *box*. In addition, this region is frequently called the species' *ecological niche* and the m -dimensional Euclidean space is called *ecological phase space*. An elemental ecological truth is that two species compete if and only if their ecological niches overlap. In this context, Cohen [11] suggested the following question: what number of dimensions is sufficient to describe an ecological phase space only when considering competitive relations among species living together in a biological community? [50]

Biologists often describe competitive relations among species cohabiting in a community by a food web that is a digraph whose vertices are the species and an arc goes from a predator to a prey. Given a food web, we say that two species u and v compete if and only if they have some common prey. To be more precise about competitive relations, Cohen [11] introduced the competition graph of a food web as follows:

Definition 1.1. The *competition graph* $C(D)$ of a digraph D is a simple graph, which has the same vertex set as D and has an edge between two distinct vertices u and v if and only if the arcs (u, x) and (v, x) are in D for some vertex $x \in V(D)$.

Ecological applications of competition graphs can be found in [11, 12]. For a comprehensive introduction to competition graphs, see [17, 42].



Competition Graph

Figure 1.1: A food web and its corresponding digraph and competition graph.

Figure 1.1 shows a simple food web and its competition graph. For example, the Hawk and the Owl compete because they both prey on the Frog. The competition graph of a food web is useful in understanding its structure. Given a graph G , we would like to find a number k and an assignment to each vertex v of a box $B(v)$ in Euclidean k -space such that

$$uv \in E(G) \iff B(u) \cap B(v) \neq \emptyset. \tag{1.1}$$

Following Roberts [48], we call the smallest k satisfying the property in (1.1) the *boxicity* $b(G)$ of G . In addition, he proved that the boxicity of any graph on n vertices

cannot be greater than $\lfloor n/2 \rfloor$. Cozzens [15] showed that computing the boxicity of a graph is a nondeterministic polynomial-hard problem (NP-hard problem for short). Later, this was improved by Yannakakis [60], and finally by Kratochvil [36] who showed that deciding whether the boxicity of a graph is at most 2 itself is an NP-complete problem.

Graphs with boxicity at most 1 are called *interval graphs*. There have been a large number of applications of interval graphs in various fields such as genetics, biology, computer science, and scheduling theory. For those applications, the readers may refer to the book [24] and the paper [49].

Interval graphs admit the elegant structure characterizations, for example, see [22, 23]. One of the significant characterizations was introduced by Lekkerkerker and Boland [40]. To state their result, we need a notion of asteroidal triple. Given a graph G , an *asteroidal triple* (AT for short) is a set of three vertices such that no two of the three vertices are adjacent and, for each pair of these vertices, there is a path in G that does not contain any vertex of the neighborhood of the third.

Theorem 1.2 ([40]). *A graph is an interval graph if and only if it is chordal and AT-free.*

As we have seen previously, interval graphs are related to competition graphs of food webs. The remarkable empirical observation of Cohen [11–13] is that every food web gives rise to a competition graph which is an interval graph, which led to a great deal of research in ecology to determine just why this might be the case. Mathematically, it also has led to a great deal of research on the structure of competition graphs and on the relation between the structure of digraphs and their corresponding competition graphs.

1.2.2 Variants of competition graphs

Many variations of ordinary competition graph have been introduced and studied by many researchers. Analogously to competition graph, the common enemy graph (resource graph) was introduced by Lundgen and Maybee [42].

Definition 1.3. The *common enemy graph* $CE(D)$ of an acyclic digraph D is the graph which has the same vertex set as D and an edge between two distinct vertices u and v if and only if there exists a vertex w in D such that (w, u) and (w, v) are arcs in D .

Scott [56] introduced the competition-common enemy graphs as natural extension of competition and common enemy graphs.

Definition 1.4. The *competition-common enemy graph* $CCE(D)$ of an acyclic digraph D is the graph which has the same vertex set as D and an edge between two distinct vertices u and v if and only if there exist vertices w and x such that (u, w) , (v, w) , (x, u) , and (x, v) are arcs of D .

Since the competition-common enemy graphs of an acyclic digraph D is the intersection of its competition graph and its common enemy graph, it is natural to also consider the union of them, which is a niche graph by Cable [5].

Definition 1.5. The *niche graph* $N(D)$ of an acyclic digraph D is the graph which has the same vertex set as D and an edge between two distinct vertices u and v if and only if there exist vertices w or x such that $\{(u, w), (v, w)\} \subseteq A(D)$ or $\{(x, u), (x, v)\} \subseteq A(D)$.

Kim *et al.* [35] introduced the notion of p -competition graph by changing the condition of the number of the common out-neighbors as follows:

Definition 1.6. Given a positive integer p , the *p -competition graph* $C_p(D)$ of an acyclic digraph D (loops allowed) is the graph which has the same vertex set as D and an edge between two distinct vertices u and v if and only if u and v share p out-neighbors in D .

As another variant related to underlying graph, the notion of phylogeny graphs was introduced by Roberts and Sheng [51].

Definition 1.7. The *phylogeny graph* $P(D)$ of an acyclic digraph D , is the graph which has the same vertex set as D and has an edge between two distinct vertices u

and v if and only if there exists an arc from u to v or an arc from v to u or a common out-neighbor of u and v in D

In studying competition graph and its variants, we frequently assume that a digraph is acyclic. However, the assumption is no longer necessary in more recent study of the variants of competition graphs. Cho *et al.* [7] introduced the notion of m -step competition graph as follows:

Definition 1.8. Given a positive integer m , the m -step competition graph $C^m(D)$ of a digraph D is the graph which has the same vertex set as D and has an edge between two distinct vertices u and v if and only if there exist a directed (u, w) -walk of length m and a directed (v, w) -walk of length m for some w in $V(D)$.

Extending the concept of m -step competition graphs, Factor *et al.* [20] introduced the notion of (i, j) -step competition graph.

Definition 1.9. Given positive integers i and j , the (i, j) -step competition graph $C_{i,j}(D)$ of D , is the graph which has the same vertex set as D and has an edge between two distinct vertices u and v if and only if (i) $d_{D-v}(u, w) \leq i$ and $d_{D-u}(v, w) \leq j$ or (ii) $d_{D-v}(u, w) \leq j$ and $d_{D-u}(v, w) \leq i$.

See the competition graph and its variants of an acyclic digraph D given in Figure 1.2 for an illustration.

In the following subsections, we take a look at some results of m -step competition graph, $(1, 2)$ -step competition graph, CCE graph, and phylogeny graph, which will be mostly dealt with in the body of this thesis. For further information on variants of competition graphs, readers may refer to the survey articles by Kim [33] and Lundgren [42].

1.2.3 m -step competition graphs

Given a digraph D and a positive integer m , we define the m -step digraph D^m of D as follows: $V(D^m) = V(D)$ and there exists an arc (u, v) in D^m if and only if there exists a directed walk of length m from a vertex u to a vertex v . A vertex y (resp. x)

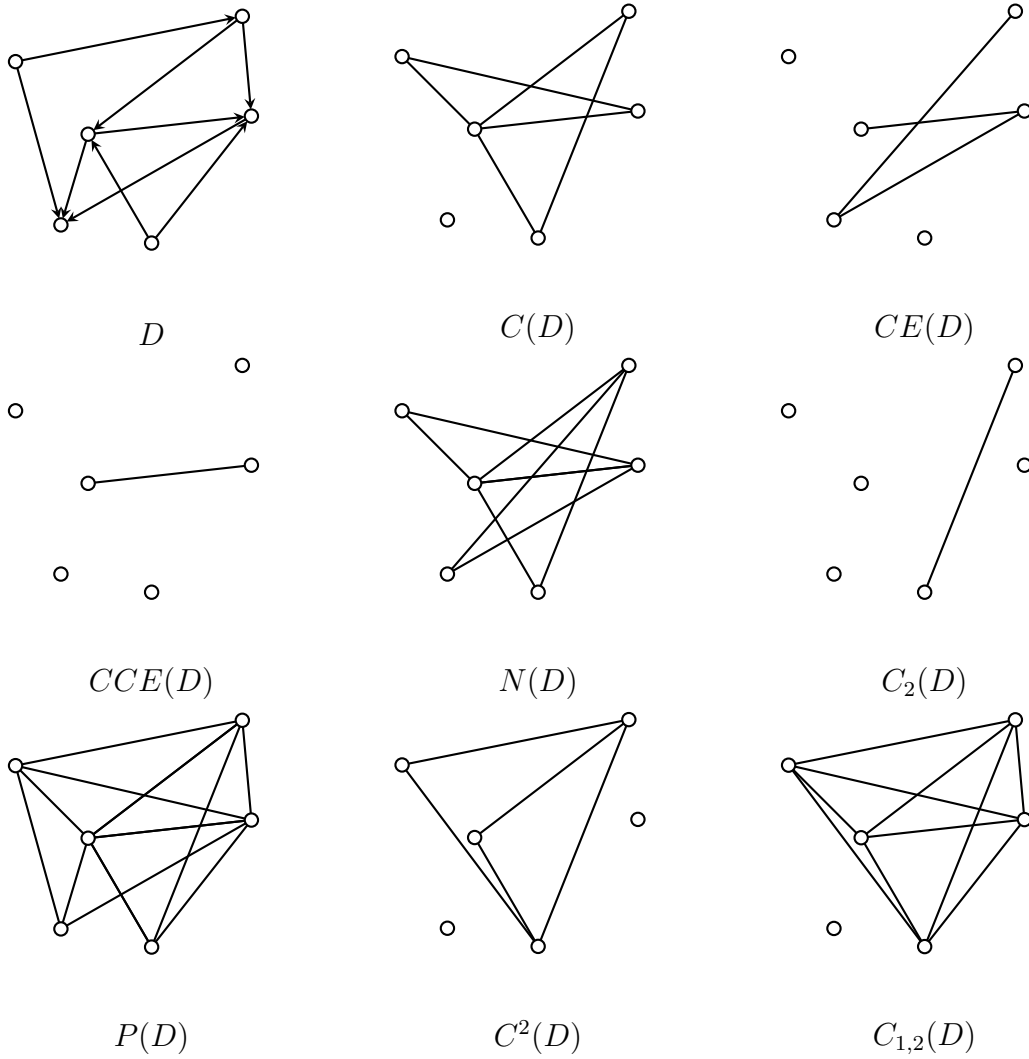


Figure 1.2: The competition graph and its variants of an acyclic digraph D

is an m -step prey (resp. m -step predator) of a vertex x (resp. y) if and only if there exists a directed walk from x to y of length m .

A relationship between the m -step competition graph and the ordinary competition graph was given by Cho *et al.* [7] as follows:

Theorem 1.10 ([7]). *For a digraph D (possibly with loops) and a positive integer m , $C^m(D) = C(D^m)$.*

Since the notion of an m -step competition graph was introduced by Cho *et al.* [7], it has been extensively studied. In 2000, Cho *et al.* [7] posed the following question: For which values of m and n is P_n an m -step competition graph? In 2005, Helleloid [27] partially answered the question and study connected triangle-free m -step competition graphs as follows.

Theorem 1.11 ([27]). *For all positive integers $m \geq n$, the only connected triangle-free m -step competition graph on n vertices is the star graph.*

In 2010, Kuhl *et al.* [37] gave a sufficient condition for $C^m(D) = P_n$. Finally, in 2011, Belmont [3] presented a complete characterization of paths that are m -step competition graphs as follows.

Theorem 1.12 ([3]). *There exists a digraph D such that $C^m(D) = P_n$ if and only if $m \mid n - 1$ or $m \mid n - 2$.*

The structural properties of m -step competition graphs and the matrix sequence $\{C^m(D)\}_{m=1}^{\infty}$ for a digraph D were studied in [7, 8, 19, 28, 43] and [6, 10, 31, 45], respectively. In addition, there is the relation between m -step competition graphs and matrix theory. For the two-element Boolean algebra $\mathcal{B} = \{0, 1\}$, \mathcal{B}_n denotes the set of all $n \times n$ matrices over \mathcal{B} . Under the Boolean operations ($1 + 1 = 1$, $0 + 0 = 0$, $1 + 0 = 1$, $1 \times 1 = 1$, $0 \times 0 = 0$, $1 \times 0 = 0$), matrix addition and multiplication are still well-defined in \mathcal{B}_n . Throughout this thesis, a matrix is Boolean unless otherwise mentioned. Let D be a digraph with vertex set $\{v_1, v_2, \dots, v_n\}$, and $A = (a_{ij})$ be the (Boolean) adjacency matrix of D such that

$$a_{ij} = \begin{cases} 1 & \text{if there is an arc } (v_i, v_j) \text{ in } D, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.13. *The adjacency matrix of $C^m(D)$ for a digraph D of order n is the matrix A_m^* obtained from $A^m(A^T)^m$ by replacing each of diagonal element with 0 where A is the adjacency matrix of D .*

To see why, we take two distinct vertices u and v of D and suppose that the i th row and the j th row are the rows corresponding to u and v , respectively. Then

- u and v are adjacent in $C^m(D)$
- \Leftrightarrow u and v have an m -step common prey in D
- \Leftrightarrow inner product of the i th row and the j th row of A^m is 1
- \Leftrightarrow the (i, j) -entry of A_m^* is 1.

Thus u and v are adjacent in $C^m(D)$ if and only if the (i, j) -entry of A_m^* is 1. Therefore the statements of m -step competition graphs may be restated in terms of matrices, which describe properties of a matrix and give solutions of a particular matrix equation, and so on.

1.2.4 (1, 2)-step competition graphs

An *orientation* of a graph G is a digraph having no directed 2-cycles, no loops, and no multiple arcs whose underlying graph is G . A *tournament* is an orientation of a complete graph.

In 2011, Factor *et al.* [20] characterized the (1, 2)-step competition graphs of tournaments and extended some results to the (i, j) -step competition graphs of tournaments.

Theorem 1.14 ([20]). *A graph G in $n \geq 5$ vertices is the (1, 2)-step competition graph of some strong tournament if and only if G is $K_n, K_n - E(P_2)$, or $K_n - E(P_3)$.*

Theorem 1.15 ([20]). *A graph G on n vertices is the (1, 2) step competition graph of some tournament if and only if G is one of the following graphs:*

- (i) K_n , where $n \neq 2, 3, 4$,
- (ii) $K_{n-1} \cup K_1$, where $n > 1$,

- (iii) $K_n - E(P_3)$ where $n > 2$,
- (iv) $K_n - E(P_2)$ where $n \neq 1, 4$, or
- (v) $K_n - E(K_3)$ where $n \geq 3$.

Then Zhang and Li [62] and Zhang *et al.* [61] studied the $(1, 2)$ -step competition graphs of non-round decomposable pure local tournament and round digraphs, respectively. Recently, Li *et al.* [41] studied the $(1, 2)$ -step competition graph of a hypertournament. On the other hand, Kim *et al.* [34] studied the competition graphs of orientations of complete bipartite graphs. In 2017, Choi *et al.* [9] studied the structure of $(1, 2)$ -step competition graphs of orientations of complete bipartite graphs and obtained the following, which are natural extension of existing results.

Theorem 1.16 ([9]). *Let D be an orientation of complete bipartite graph. Then $C_{1,2}(D)$ has at most one non-trivial component of diameter at most three or consists of exactly two complete components of size at least three.*

In addition, they completely characterized the complete graphs and the disconnected $(1, 2)$ -step competition graph $C_{1,2}(D)$ of an orientation of complete bipartite graph D , which is the disjoint unions of complete graphs, as follows.

Theorem 1.17 ([9]). *The following are true:*

- (i) *For positive integers m and n with $m \geq n$, the disjoint union of the complete graphs K_m and K_n is a $(1, 2)$ -step competition graph of an orientation of a complete bipartite graph if and only if one of the following holds: $n = 1$; $m \geq n \geq 6$; $m \geq 10$ and $n = 5$.*
- (ii) *For a positive integer l , the complete graph K_l is a $(1, 2)$ -step competition graph of an orientation of a complete bipartite graph if and only if $l \geq 12$*

1.2.5 Phylogeny graphs

Pearl [47] introduced the notion of Bayesian network. A *Bayesian network* (also known as a Bayes network) is a probabilistic graphical model that represents a set of

variables and their conditional dependencies via a digraph. “Moral graphs” having arisen from studying Bayesian networks are the same as phylogeny graphs (Definition 1.7). One of the well-known problems, in the context of Bayesian networks, is concerned with the propagation of evidence. It is composed of the assignment of probabilities to the values of the rest of the variables, once the values of some variables are known.

As Cooper [14] showed, this problem is NP-hard. Most remarkable algorithms for this problem are given by Pearl [46], Shachter [57] and by Lauritzen and Spiegelhalter [38]. A step of triangulating a moral graph, adding edges to a moral graph to form a chordal graph, is required in those algorithms (refer to remaining step in Jensen [29]). Thus triangulation of the moral graphs plays a significant role in the process of solving the propagation problem. Even though chordal graphs can be identified in polynomial time, deciding whether or not a graph is moral is NP-complete by Verma and Pearl [59]. In this context, the problem whether phylogeny graph of an acyclic digraph is a chordal or not have been studied.

Steif [58] showed that it might be difficult to figure out the structural properties of acyclic digraphs whose competition graphs are interval. In that regard, Hefner *et al.* [26] studied degree bounded acyclic digraphs having restrictions on the indegree and the outdegree of its vertices to obtain the list of forbidden subdigraphs for acyclic digraphs whose competition graphs are interval. They called an acyclic digraph each vertex of which has indegree at most i and outdegree at most j an (i, j) digraph for positive integers i and j . Hefner *et al.* [26] gave a characterization of $(2, 2)$ digraphs whose competition graphs are interval. The gain of this characterization is a sufficient condition for $(2, 2)$ digraphs having chordal competition graphs.

Recently, research of the phylogeny graphs of degree-bounded have mainly been conducted in two directions: chordality and clique number. Lee *et al.* [39] gave a sufficient condition and a necessary condition for $(2, 2)$ digraphs having chordal phylogeny graphs in terms of its underlying graphs as follows:

Theorem 1.18 ([39]). *Let D be a $(2, 2)$ digraph. If the underlying graph of D contains a hole H of length at least 7, then the subgraph of the phylogeny graph of D induced by $V(H)$ has a hole.*

Theorem 1.19 ([39]). *Let D be a $(2, 2)$ digraph. If the underlying graph is chordal, then the phylogeny graph of D is also chordal.*

Further, Eoh *et al.* [18] studied on chordality of the phylogeny graphs of (i, j) digraphs by extending the result given by Lee *et al.* as follows:

Theorem 1.20 ([18]). *Let $D_{i,j}^*$ be the set of (i, j) digraphs whose underlying graphs are chordal for positive integers i and j . Then the (i, j) phylogeny graph of D is chordal for any $D \in D_{i,j}^*$ if and only if $i \leq 2$ or $j = 1$.*

Then they showed that the phylogeny graph of a $(2, 2)$ digraph is planar if the underlying graph of a $(2, 2)$ digraph is chordal and completely characterized the phylogeny graph of a $(1, i)$ digraph and an $(i, 1)$ digraph, for any positive integer i .

Lee *et al.* [39] and Eoh *et al.* [18] gave an upper bound for the clique number of phylogeny graphs of $(2, 2)$ digraphs and phylogeny graphs of $(2, j)$ digraphs, respectively, as follows:

Theorem 1.21 ([18, 39]). *Let D be a $(2, j)$ digraph for a positive integer j . Then*

$$\omega(P(D)) \leq \begin{cases} j + 2 & \text{if } j \leq 2; \\ j + 3 & \text{otherwise} \end{cases}$$

and the inequalities are tight.

In this context, we study the chordality of phylogeny graphs of $(i, 2)$ digraphs and characterization of its forbidden induced subgraphs in the Chapter 4, which extend the results given by Lee *et al.* [39]. For more information on study on phylogeny graphs, readers may refer to [25, 44, 52–54, 63].

1.2.6 CCE graphs

In 1978, Robert [50] defined $k(G)$, the *competition number* of a graph G , to be the smallest integer k such that $G \cup I_k$ is a competition graph of an acyclic digraph, where I_k is a set of isolated vertices added to G by observing that a competition graph is

obtained by adding sufficiently many isolated vertices to G . Analogously, Scott [56] defined $dk(G)$, the *double competition number*, to be the smallest integer k such that $G \cup I_k$ is a CCE graph of an acyclic digraph, where I_k is a set of isolated vertices added to G by proving that $dk(G)$ is well-defined. In addition, Scott [56] characterize the CCE graphs of acyclic digraphs by computing the double competition numbers as follows.

Theorem 1.22 ([56]). *If G is a path of length at least 2, a cycle of length of at length at least 3, a complete graph of order at least 2 or a nontrivial tree, then $dk(G) = 2$.*

Theorem 1.23 ([56]). *If G is a chordal graph or an interval graph then $dk(G) \leq 2$.*

In this context, we mainly study the CCE graphs of $(2, 2)$ digraphs having exactly two isolated vertices in Chapter 5 and characterize them.

Hefner *et al.* [26] studied competition graphs of (i, j) digraphs, which they called (i, j) *competition graphs*, and gave a characterization of $(2, 2)$ digraphs whose competition graphs are interval graphs as follows.

Theorem 1.24 ([26]). *Let G be a competition graph of a $(2, 2)$ digraph. G is an interval graph if and only if each component is an isolated vertex, a path, or a triangle, and the number of isolated vertices is at least 1.*

In this vein, we give a sufficient condition on the number of components for CCE graphs being interval graphs in Chapter 5. For more results on CCE graphs, readers may refer to [1, 2, 21, 30, 32, 55].

1.3 A preview of the thesis

In Chapter 2, we completely characterize the digraphs of order n whose m -step competition graphs are star graphs for positive integers $2 \leq m < n$. This result in matrix version identifies the solution set to the matrix equation $X^m(X^T)^m = \Lambda_n + I_n$ for positive integers $2 \leq m < n$ where I_n is the identity matrix of order n and Λ_n is a $(0, 1)$ Boolean matrix such that the first row and the first column consist of 1's except $(1, 1)$ -entry and the remaining entries are 0, which is the adjacency matrix of

a star graph of order n . We also derive meaningful properties of the digraphs whose m -step competition graphs are trees. In the process, we extend a result of Helleloid (Theorem 1.11) by showing that for all positive integers $m \geq 2$ and n , the connected triangle-free m -step competition graph on n vertices is a tree.

In Chapter 3, we study $C_{1,2}(D)$ when D is an orientation of a complete k -partite graph for some $k \geq 3$. We completely identify $C_{1,2}(D)$ when each partite set of D forms a clique in $C_{1,2}(D)$. Even if there exists a partite set of D which does not form a clique in $C_{1,2}(D)$, we figure out most of the structure of $C_{1,2}(D)$. Based on these results, we show that the diameter of each component of $C_{1,2}(D)$ is at most three and provide a sharp upper bound on the domination number of $C_{1,2}(D)$. In addition, we list all possible $C_{1,2}(D)$ when D has no vertices of outdegree 0 and $C_{1,2}(D)$ is disconnected. Finally, we give a sufficient condition for $C_{1,2}(D)$ being an interval graph.

In Chapter 4, we give a sufficient condition on the size of hole of an underlying graph of D for $P(D)$ being a chordal graph where D is an $(i, 2)$ digraph. Moreover, we go further to completely characterize (i, j) phylogeny graphs by listing the forbidden induced subgraphs.

In Chapter 5, We characterize the graphs with the least components among the CCE graphs of $(2, 2)$ digraphs containing at most one cycle and exactly two isolated vertices, and their digraphs, which gives a sufficient condition on the number of components for CCE graphs being interval graphs. Further, we completely characterize the graphs having exactly two nontrivial components among the CCE graphs of $(2, 2)$ digraphs with exactly two isolated vertices.

Chapter 2

Digraphs whose m -step competition graphs are trees¹

Recall that given a positive integer m , the m -step competition graph $C^m(D)$ of a digraph D is the graph which has the same vertex set as D and has an edge between two distinct vertices u and v if and only if there exist a directed (u, w) -walk of length m and a directed (v, w) -walk of length m for some w in $V(D)$ (Definition 1.8).

For two vertex-disjoint weakly connected digraphs D_1 and D_2 , it is true that $C^m(D_1 \cup D_2) = C^m(D_1) \cup C^m(D_2)$ for any positive m . In this vein, it is sufficient to consider weakly connected digraphs throughout this chapter. From now on, we assume that any digraph in this chapter is weakly connected unless otherwise mentioned.

We call a complete bipartite graph $K_{1,l}$ for some positive integer l a *star graph*.

In this chapter, we show the following theorem (the definitions of a windmill digraph and an m -conveyor digraph will be given right after the theorem statement).

Theorem 2.1. *For positive integers $2 \leq m < n$, the star graph is an m -step competition graph of a digraph D with n vertices if and only if one of the following holds:*

- (i) D is a windmill digraph;

¹The material in this chapter is from the manuscript ‘Digraphs whose m -step competition graphs are trees’ by Myunggho Choi and Suh-Ryung Kim. The author thanks Prof. Suh-Ryung Kim for allowing him to use its contents for his thesis.

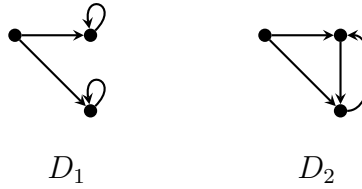


Figure 2.1: The windmill digraphs with three vertices.

(ii) D is an m -conveyor digraph;

(iii) $m = 2$ and D is isomorphic to the digraph given in Figure 2.3.

A *windmill digraph* is defined to be a digraph satisfying the following three conditions:

(W1) D has exactly one source v ;

(W2) $D - v$ is a vertex-disjoint union of directed cycles;

(W3) each vertex except v is a prey of v

(see the windmill digraphs of order 3 in Figure 2.1 for an illustration).

We call a nontrivial directed path or cycle connecting vertices of indegree 2 an *internally secure lane* if each of its interior vertices has indegree 1.

We call a digraph D an m -conveyor digraph for some $m \geq 2$ if D has a vertex v satisfying the following conditions:

(M1) v is the only predator of v ;

(M2) $D - v$ is a vertex-disjoint union of directed cycles;

(M3) each internally secure lane in D has length at most m

(see the 2-conveyor digraphs of order 4 in Figure 2.2 for an illustration).

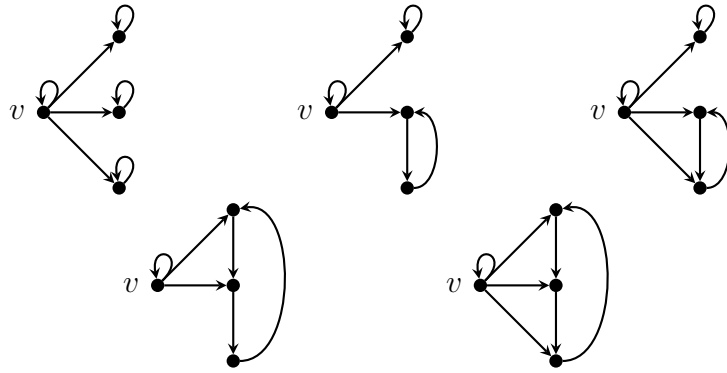


Figure 2.2: The 2-conveyor digraphs of order 4



Figure 2.3: A digraph and its 2-step competition graph

$$\begin{bmatrix} 0 & J & \cdots & J \\ O & \Gamma_{n_1} & O & O \\ \vdots & O & \ddots & O \\ O & O & O & \Gamma_{n_k} \end{bmatrix}, \quad \begin{bmatrix} 1 & Q_{n_1}^{(m)} & \cdots & Q_{n_k}^{(m)} \\ O & \Gamma_{n_1} & O & O \\ \vdots & O & \ddots & O \\ O & O & O & \Gamma_{n_k} \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Figure 2.4: Adjacency matrices of a windmill digraph, a m -conveyor digraph, and the digraph given in Figure 2.3, respectively, where the blocks J and O stand for a matrix of all 1's and a zero matrix, respectively.

The adjacency matrix of a windmill digraph is in the form of the first matrix given in Figure 2.4. Here, Γ_n is the adjacency matrix of a directed cycle of length n , that is,

$$(\Gamma_n)_{ij} = \begin{cases} 1 & \text{if } j = i + 1 \text{ or } (i, j) = (n, 1); \\ 0 & \text{otherwise.} \end{cases}$$

The adjacency matrix of an m -conveyor digraph is in the form of the second matrix given in Figure 2.4. The first row represents v satisfying (M1) and (M2). By (M3), the $(0, 1)$ nonzero matrix $Q_k^{(m)}$ has size $1 \times k$ and satisfies the following properties:

- (F1) the number of consecutive zeros is at most $m - 1$;
- (F2) if the $(1, 1)$ -entry and the $(1, k)$ -entry equal 0, then the number of first consecutive zeros and that of last consecutive zeros add up to at most $m - 1$.

Theorem 2.1 may be restated in terms of matrices. Therefore we have the following corollary restating Theorem 2.1 in terms of matrices:

Corollary 2.2 (Matrix version). *For positive integers $2 \leq m < n$, a square matrix X of order n satisfies $X^m(X^T)^m = \Lambda_n + I_n$ if and only if $P^T X P$ for some permutation matrix P of order n is one of the matrices given in Figure 2.4, where I_n is the identity matrix of order n and Λ_n is the square matrix of order n with the first row and first column of Λ_n consisting of 1's except $(1, 1)$ -entry and the remaining entries being 0.*

We also prove the following result.

Theorem 2.3. *For all positive integers $2 \leq m < n$, the connected triangle-free m -step competition graph on n vertices is a tree.*

Even for a digraph D and an integer $m > |V(D)|$, the same is true as follows.

By Theorems 1.11 and 2.3 we have the following more general result.

Corollary 2.4. *For all positive integers $m \geq 2$ and n , the connected triangle-free m -step competition graph on n vertices is a tree.*

As the rest of this chapter is devoted to proving Theorems 2.1 and 2.3, we may assume from now on that $m \geq 2$ and $m < n$ whenever we are given a digraph of order n whose m -step competition graph is triangle-free.

2.1 The triangle-free m -step competition graphs

In this section, we show that all the connected triangle-free m -step competition graphs are trees.

Let D be a digraph and v be a vertex of D . We denote the m -step prey of v by $N_{D^m}^+(v)$ and the m -step predators of v by $N_{D^m}^-(v)$, respectively. When no confusion is likely, we will just write $N_m^+(v)$ and $N_m^-(v)$. We note that $N_1^+(v) = N^+(v)$ and $N_1^-(v) = N^-(v)$. Technically, we write $N_0^+(v) = N_0^-(v) = \{v\}$. We call $|N_i^-(v)|$ and $|N_i^+(v)|$ the i -step indegree and the i -step outdegree of v , respectively, and denote them by $d_i^-(v)$ and $d_i^+(v)$, respectively. We note that $d_1^+(v) = d^+(v)$ and $d_1^-(v) = d^-(v)$.

We make the following useful observations.

Lemma 2.5. *Let D be a digraph such that $C^m(D)$ is triangle-free. Then the following are true:*

- (1) Any vertex in D has i -step outdegree at least 1 for any positive integer i .
- (2) Any vertex in D has i -step indegree at most 2 for any positive integer $i \leq m$.
- (3) If a directed walk contains at least two vertices and its origin and terminus have indegree 2, then it is a juxtaposition of internally secure lanes.

- (4) For any two internally secure lanes W and W' in D starting at w and w' , respectively, and sharing v as an interior vertex, the (w, v) -section of W and the (w', v) -section of W' coincide.

Proof. Since we have assumed that any digraph has no sinks, part (1) is true.

To prove part (2), suppose, to the contrary, $d_i^-(u) \geq 3$ for some vertex u of D and a positive integer $i \leq m$. Then there exist three distinct i -step predators x, y , and z of u . By part (1), u has an $(m - i)$ -step prey v . Then v is an m -step common prey of x, y , and z . Thus x, y , and z form a triangle in $C^m(D)$, a contradiction. Hence part (2) is true.

Part (3) immediately follows from the definition of internally secure lane and part (2).

To show part (4), let $W = w \rightarrow v_1 \rightarrow \cdots \rightarrow v_l$ and $W' = w' \rightarrow v'_1 \rightarrow \cdots \rightarrow v'_{l'}$ be a pair of internally secure lanes sharing v as an interior vertex for some positive integers l and l' . Then $v = v_k = v'_{k'}$ for some $k \in \{1, \dots, l-1\}$ and $k' \in \{1, \dots, l'-1\}$. By the definition of internally secure lane, $d^-(v_i) = d^-(v'_{i'}) = 1$ for each $1 \leq i \leq k$ and $1 \leq i' \leq k'$. Therefore the (w, v_k) -section of W and the $(w', v'_{k'})$ -section of W' coincide. Thus part (4) is true. \square

Theorem 2.6. *Let G be the m -step competition graph of a digraph D such that G is triangle-free and has the edges as many as the vertices. Then the following are true:*

- (1) *For each vertex u of outdegree at least 2 in D , each prey of u has indegree 2 in D .*
- (2) *Every vertex in D lies on some internally secure lane.*
- (3) *Each internally secure lane of D has length m .*

Proof. We consider a set

$$A = \{(u, \{v, w\}) \mid v \neq w, \{v, w\} \subseteq N_m^-(u)\}.$$

By the definition of m -step competition graph, $|A| \geq |E(G)|$. Thus, by the definition of A and Lemma 2.5(2),

$$|E(G)| \leq |A| = \sum_{v \in V(D)} \binom{d_m^-(u)}{2} \leq \sum_{v \in V(D)} \binom{2}{2} = |V(D)| = |V(G)|.$$

Then, since $|E(G)| = |V(G)|$ by the hypothesis,

$$d_m^-(v) = 2 \tag{2.1}$$

for each vertex v in D . In addition, if u and v are adjacent in G , then

$$|N_m^+(u) \cap N_m^+(v)| = 1,$$

so, for each pair of vertices u and v in D ,

$$|N_m^+(u) \cap N_m^+(v)| \leq 1$$

and

$$|N_m^-(u) \cap N_m^-(v)| \leq 1. \tag{2.2}$$

Suppose for a contradiction that there exist two vertices u and v such that $|N_j^+(u) \cap N_j^+(v)| \geq 2$ for some positive integer $j < m$. Take two distinct vertices w_1 and w_2 in $N_j^+(u) \cap N_j^+(v)$. Then $\{u, v\} \subseteq N_j^-(w_1) \cap N_j^-(w_2)$. Therefore $N_j^-(w_1) = N_j^-(w_2) = \{u, v\}$ by Lemma 2.5(2). Thus $N_m^-(w_1) = N_m^-(w_2)$. Then, since $d_m^-(w_1) = d_m^-(w_2) = 2$ by (2.1), $|N_m^-(w_1) \cap N_m^-(w_2)| = 2$, which contradicts (2.2). Therefore, for each pair of vertices u and v ,

$$|N_i^+(u) \cap N_i^+(v)| \leq 1 \tag{2.3}$$

for any positive integer $i \leq m$.

To show part (1) by contradiction, suppose that there exist a vertex u of outdegree at least 2 and a prey v of u has indegree not equal to 2 in D . Then, by Lemma 2.5(2), v has indegree 1. In addition, since u has outdegree at least 2, u has a prey w other than v . By (2.1), $N_m^-(v) = \{x, y\}$ for some vertices x and y in D . Since u

is the only predator of v , $N_{m-1}^-(u) = \{x, y\}$. Therefore $\{x, y\} \subseteq N_m^-(w)$ and so $\{x, y\} \subseteq N_m^-(v) \cap N_m^-(w)$, which contradicts (2.2). Hence part (1) is true.

To show part (2), take a vertex v in D . If any i -step prey of v has indegree at most 1 for each $1 \leq i \leq m$, then there is an m -step prey of v having m -step indegree 1, which contradicts (2.1). Therefore there exists a j -step prey x of v having indegree at least 2 for some $j \in \{1, \dots, m\}$. Thus x has indegree 2 by Lemma 2.5(2). If v has indegree 2, then a directed (v, x) -walk contains an internally secure lane on which v lies. Suppose that v has indegree not equal to 2. Then v has indegree 1 by Lemma 2.5(2) and (2.1). If each i -step predator of v has indegree at most 1 for each $1 \leq i \leq m-1$, then $d_m^-(v) < 2$, a contradiction to (2.1). Thus there exists a k -step predator y of v having indegree at least 2 for some $k \in \{1, \dots, m-1\}$. Then, by Lemma 2.5(2), each of x and y has indegree 2. Hence we may conclude that every vertex in D lies on some internally secure lane.

To show part (3), take an internally secure lane $W := v_0 \rightarrow v_1 \rightarrow \dots \rightarrow v_j$ for some positive integer j . Then, by the definition of an internally secure lane,

$$N^-(v_0) = \{x, y\}, \quad N^-(v_j) = \{v_{j-1}, w\}$$

for some vertices x, y , and w in D . Then

$$N_{i+1}^-(v_i) \supseteq \{x, y\} \tag{2.4}$$

for each $0 \leq i \leq j$. If $j \geq 2$, then

$$N^-(v_{i+1}) = \{v_i\} \tag{2.5}$$

for each $0 \leq i \leq j-2$ by the definition of internally secure lane and so, by part (1),

$$N^+(v_i) = \{v_{i+1}\} \tag{2.6}$$

for each $0 \leq i \leq j-2$.

To reach a contradiction, suppose $j \neq m$. If $j > m$, then $j \geq 2$ and so, by (2.5),

$N_m^-(v_{j-1}) = \{v_{j-m-1}\}$, which contradicts (2.1). Therefore $j < m$. Then

$$N_{i+1}^-(v_i) = \{x, y\}$$

for each $0 \leq i \leq j$ by (2.4) and Lemma 2.5(2). Since a j -step predator of w is a $(j+1)$ -step predator of v_j , $N_j^-(w) \subseteq \{x, y\}$. Since $N_j^-(v_{j-1}) = \{x, y\}$, $N_j^-(w) \subsetneq \{x, y\}$ by (2.3). Then, since D has no source by (2.1), $N_j^-(w) = \{x\}$ or $\{y\}$. Without loss of generality, we may assume $N_j^-(w) = \{x\}$. Then there exists a directed walk W^* of length j from x to w . If $j = 1$, then $d^-(w) = 1$, w is a prey of x , and $\{v_0, w\} \subseteq N^+(x)$, which contradicts part (1). Therefore $j \geq 2$. Let z be the vertex which x is immediately going toward on W^* . If $z = v_0$, then $w = v_{j-1}$ by (2.6) and we reach a contradiction. Therefore $z \neq v_0$. Thus x has outdegree at least 2. Hence z has a predator x' other than x by part (1). Then attaching the arc (x', z) to the (z, w) -section of W^* results in a directed walk of length j from x' to w . Therefore $\{x, x'\} \subseteq N_j^-(w)$, which contradicts the assumption that $N_j^-(w) = \{x\}$. Thus we may conclude that $j = m$. Since W was arbitrarily chosen from D , part (3) is valid. \square

Given an internally secure lane $W = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_m$, we call v_k the k th interior vertex of W for each $1 \leq k < m$.

Theorem 2.7. *If the m -step competition graph of a digraph is triangle-free and has the edges as many as the vertices, then it has at least m components.*

Proof. Suppose that there exists a digraph D whose m -step competition graph G is triangle-free and has the edges as many as the vertices. Take an internally secure lane W in D (it exists since every vertex in D lies on some internally secure lane by Theorem 2.6(2)). By Theorem 2.6(3), W has length m . Let $W = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_m$ and take v_k for some $k \in \{1, \dots, m-1\}$. Suppose that there exists a vertex w_k adjacent to v_k in G . Then they have an m -step common prey in D and so v_k and w_k have an l -step common prey z that has indegree at least 2 in D for some $l \in \{1, \dots, m\}$. Therefore z has indegree 2 by Lemma 2.5(2). Then there exist a directed (v_k, z) -walk W_1 of length l and a directed (w_k, z) -walk W_2 of length l . Since z has indegree 2, the directed walk W' obtained by concatenating the (v_0, v_k) -section

of W and W_1 contains an internally secure lane. We note that the origin and the terminus of W' have indegree 2. Then, since the length of W' is at most $2m - 1$, W' must be an internally secure lane by Lemma 2.5(3) and Theorem 2.6(3). Thus $l = m - k$. By Theorem 2.6(3) again, W_2 must be a section of an internally secure lane of length l . Thus we may conclude that

(\star) each vertex adjacent to the k th interior vertex of an internally secure lane is the k th interior vertex of an internally secure lane.

Now, for each $1 \leq i \leq m - 1$, we define a vertex set \mathcal{V}_i as follows: $v \in \mathcal{V}_i$ if and only if v is the i th interior vertex of some internally secure lane. Then $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ if $i \neq j$ by Lemma 2.5(4). Moreover, since $v_i \in \mathcal{V}_i$, $\mathcal{V}_i \neq \emptyset$ for each $1 \leq i \leq m - 1$.

Now we choose $j \in \{1, \dots, m - 1\}$. Then take a vertex v in \mathcal{V}_j and let X be the component containing v in G . Take a vertex w in X . Then there exists a shortest path P from v to w . By repeatedly applying (\star) to each vertex on P from the nearest to the farthest from v , we may show that $w \in \mathcal{V}_j$. Therefore $X \subseteq \mathcal{V}_j$. Then, since $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ for $i \neq j$ and $\mathcal{V}_k \neq \emptyset$ for each $1 \leq k \leq m - 1$, G has at least $m - 1$ components each of which is included in \mathcal{V}_j for some $j \in \{1, \dots, m - 1\}$. We note that each vertex in $\bigcup_{i=1}^{m-1} \mathcal{V}_i$ is an interior vertex of an internally secure lane and so, by the definition of internally secure lane, it has indegree 1 for each $1 \leq i \leq m - 1$. Therefore the origin and the terminus of any internally secure lane in D cannot belong to any of the components obtained previously. Hence G has at least m components. \square

Now we are ready to prove Theorem 2.3.

Proof of Theorem 2.3. Suppose, to the contrary, that there exists a digraph D such that $C^m(D)$ is triangle-free and connected but is not a tree. Then, since $C^m(D)$ is connected but is not a tree, $|E(C^m(D))| > |V(C^m(D))| - 1$. By Lemma 2.5(2), $d_m^-(v) = 2$ for each vertex v in D . Therefore $|E(C^m(D))| \leq |V(C^m(D))|$ and so $|E(C^m(D))| = |V(C^m(D))|$. Thus $C^m(D)$ is disconnected by Theorem 2.7 and we reach a contradiction. Hence $C^m(D)$ is a tree. \square

2.2 Digraphs whose m -step competition graphs are trees

In this section, we deduce basic properties of digraphs whose m -step competition graphs are trees.

We call a digraph D with at least three vertices an *m -step tree-inducing digraph* if the m -step competition graph of D is a tree for some integer $m \geq 2$. A digraph is said to be a *tree-inducing digraph* if it is an m -step tree-inducing digraph for some integer $m \geq 2$.

Proposition 2.8. *Let D be an m -step tree-inducing digraph. Then $N_i^+(u) \neq N_i^+(v)$ for any distinct vertices u and v in D and any positive integer $i \leq m$.*

Proof. Suppose $N_i^+(u) = N_i^+(v)$ for some distinct u and v in D and a positive integer $i \leq m$. Then $N_m^+(u) = N_m^+(v)$. Since $N_m^+(u) \neq \emptyset$ by Lemma 2.5(1), u and v are adjacent in $C^m(D)$. Moreover, $N_m^-(w) = \{u, v\}$ for each vertex $w \in N_m^+(u)$ by Lemma 2.5(2). Therefore an edge uw is a component in $C^m(D)$, a contradiction to the connectedness of $C^m(D)$. \square

Proposition 2.9 (Helleloid [27]). *Let D be a digraph with n vertices whose m -step competition graph $C^m(D)$ is a tree. Then there is a one-to-one correspondence between the $n - 1$ pairs of adjacent vertices in $C^m(D)$ and $n - 1$ of n vertices of D ; namely, all but one vertex in D serves as the m -step common prey for exactly one pair of adjacent vertices in $C^m(D)$. The remaining vertex of D can either be the m -step prey of no vertices, of any one vertex, or of any two vertices adjacent in $C^m(D)$.*

Based upon the above proposition, Belmont [3] called the remaining vertex α in D not assigned in a bijection between the edges of $C^m(D)$ and $n - 1$ of the n vertices of D *anomaly*. The author observed that if the remaining vertex of D is the m -step prey of any two vertices adjacent in $C^m(D)$, then the anomaly is not well-defined since there are two vertices with this property and went on to call the one arbitrarily chosen between the two vertices an anomaly. By the definition of an anomaly, it is clear that each tree-inducing digraph has a unique anomaly.

The following proposition gives a necessary and sufficient condition for a vertex of a tree-inducing digraph D being the anomaly, which is actually a restatement of Proposition 2.9.

Proposition 2.10. *Let D be an m -step tree-inducing digraph. Then α in D is the anomaly if and only if α has either at most one m -step predator in D or exactly two m -step predators that have another vertex β as an m -step common prey in D . Furthermore, if the latter of the “if” part is true, then α and β are the only vertices that share two m -step common predators.*

Corollary 2.11. *Let D be an m -step tree-inducing digraph. Then the following are true:*

- (1) *If $|N_m^+(u) \cap N_m^+(v)| \geq 2$ for some u and v in D , then the anomaly is contained in $N_m^+(u) \cap N_m^+(v)$.*
- (2) *If $d_m^-(v) \leq 1$, then v is the anomaly.*

Corollary 2.12. *Let D be an m -step tree-inducing digraph. For the anomaly α of D , exactly one of the following is true:*

- (i) *α has exactly two m -step predators that have a vertex v other than α as an m -step common prey in D , and α and v are the only vertices that share two m -step common predators;*
- (ii) *α has at most one m -step predator and each vertex except α has exactly two m -step predators.*

Theorem 2.13. *Let D be a digraph such that $C^m(D)$ is triangle-free and connected. Then the following are true:*

- (1) *$|\bigcup_{v \in U} N^+(v)| \geq |U|$ for any proper subset U of $V(D)$.*
- (2) *For any vertices u and v in D , $|N_i^+(u) \cap N_i^+(v)| \leq |N_j^+(u) \cap N_j^+(v)|$ for any positive integers i, j with $i \leq j \leq m$.*

(3) For each vertex v in D , $d_i^+(v) \leq d_j^+(v)$ for any positive integers i, j with $i \leq j \leq m$.

Proof. We begin with the proof of the following claim:

Claim. For any nonempty proper subset U of $V(D)$, there exists a vertex $u \in \bigcup_{v \in U} N^+(v)$ such that $|N^-(u) \cap U| = 1$.

To reach a contradiction, suppose that there exists a nonempty proper subset U^* of $V(D)$ with $|N^-(v) \cap U^*| \neq 1$ for each vertex v in $\bigcup_{v \in U^*} N^+(v)$. Since any vertex in $\bigcup_{v \in U^*} N^+(v)$ is a prey of a vertex in U^* , $|N^-(v) \cap U^*| \geq 1$ for each vertex v in $\bigcup_{v \in U^*} N^+(v)$ and so, by Lemma 2.5(2), $|N^-(v) \cap U^*| = 2$. Since U^* is a proper subset of $V(D)$, $V(D) - U^* \neq \emptyset$. Since $U^* \neq \emptyset$ and $C^m(D)$ is connected, there exists a vertex x in $V(D) - U^*$ which is adjacent to a vertex $w \in U^*$ in $C^m(D)$. Then, w and x have an m -step common prey a_m and so there exists a directed (w, a_m) -walk of length m in D . Let a_1 be the vertex outgoing from w on this walk. Then $a_1 \in N^+(w) \subseteq \bigcup_{v \in U^*} N^+(v)$. By the choice of U^* , each vertex of $\bigcup_{v \in U^*} N^+(v)$ has two predators in U^* . Thus there is the other predator y of a_1 that belongs to U^* . Since $x \notin U^*$ and $y \in U^*$, y and x are distinct. Further, y is an m -step predator of a_m and so $\{w, x, y\} \subseteq N_m^-(a_m)$, which is a contradiction to Lemma 2.5(2). Therefore the claim is valid.

We prove part (1) by induction on $|U|$. If $U = \emptyset$, then the inequality trivially holds. Now suppose that $|\bigcup_{v \in U} N^+(v)| \geq |U|$ for any proper vertex subset U of $V(D)$ with $|U| \leq k$ for a nonnegative integer k such that $k \leq |V(D)| - 2$. Take a proper subset W of $V(D)$ with $k + 1$ elements. Then W is nonempty. Suppose, to the contrary, that $|\bigcup_{v \in W} N^+(v)| < |W|$. By the above claim, there exists a vertex $w \in \bigcup_{v \in W} N^+(v)$ such that $|N^-(w) \cap W| = 1$. Then $N^-(w) \cap W = \{x\}$ for some vertex $x \in W$. Since x is the only predator of w in W , $w \notin \bigcup_{v \in W - \{x\}} N^+(v)$. Then, since $w \in \bigcup_{v \in W} N^+(v)$,

$$\left| \bigcup_{v \in W - \{w\}} N^+(v) \right| \leq \left| \bigcup_{v \in W} N^+(v) \right| - 1.$$

By the assumption that $|\bigcup_{v \in W} N^+(v)| < |W|$,

$$\left| \bigcup_{v \in W - \{w\}} N^+(v) \right| < |W| - 1. \quad (2.7)$$

Yet, since $W - \{x\}$ is a proper subset of $V(D)$ with k elements, by the induction hypothesis,

$$\left| \bigcup_{v \in W - \{w\}} N^+(v) \right| \geq |W - \{x\}| = |W| - 1,$$

which contradicts (2.7). Therefore part (1) is true.

To verify part (2), take two vertices u and v of D and fix a positive integer i . We first consider the case $N_i^+(u) \cap N_i^+(v) = V(D)$ and take a vertex w . Then w has at least one predator $z \in N_i^+(u) \cap N_i^+(v)$. Therefore $w \in N_{i+1}^+(u) \cap N_{i+1}^+(v)$. Since w was arbitrarily chosen from D , $N_{i+1}^+(u) \cap N_{i+1}^+(v) = V(D)$.

Now consider the case $N_i^+(u) \cap N_i^+(v) \subsetneq V(D)$.

By part (1),

$$\left| \bigcup_{w \in N_i^+(u) \cap N_i^+(v)} N^+(w) \right| \geq |N_i^+(u) \cap N_i^+(v)|.$$

Then, since

$$\begin{aligned} \bigcup_{w \in N_i^+(u) \cap N_i^+(v)} N^+(w) &\subseteq N_{i+1}^+(u) \cap N_{i+1}^+(v), \\ |N_i^+(u) \cap N_i^+(v)| &\leq |N_{i+1}^+(u) \cap N_{i+1}^+(v)|. \end{aligned}$$

We may repeat this process until we have $|N_i^+(u) \cap N_i^+(v)| \leq |N_j^+(u) \cap N_j^+(v)|$ for any integer j with $i \leq j \leq m$, which will complete the proof of part (2). Part (3) is an immediate consequence of part (2). \square

The inequality in Theorem 2.13(1) is true only for a *proper* subset of $V(D)$ as shown by the following example.

Example 2.14. Fix some integer $m \geq 2$. Let D be the windmill digraph with

$V(D) = \{v_1, v_2, \dots, v_m, w\}$ and $A(D) = \{(v_i, v_{i+1}) \mid 1 \leq i < m\} \cup \{(v_m, v_1)\} \cup \{(w, v_i) \mid 1 \leq i \leq m\}$. Then $C^m(D)$ is a star graph with the center w . However, $|N^+(V(D))| < |V(D)|$ since $w \notin N^+(V(D))$.

Proposition 2.15. *Let D be an m -step tree-inducing digraph with the anomaly α . Suppose $d_i^+(u) \geq l$ for a vertex u in D and positive integers l and $i \leq m$. Then the degree of u is at least $l - 1$ in $C^m(D)$. Especially, if the degree of u equals $l - 1$ in $C^m(D)$, then $d_m^+(u) = l$ and $\alpha \in N_m^+(u)$ in D .*

Proof. Denote by $d(u)$ the degree of a vertex u in $C^m(D)$. Since $d_i^+(u) \geq l$, $d_m^+(u) \geq l$ by Theorem 2.13(3). Then there are at least $l - 1$ vertices in $N_m^+(u)$ each of which serves as the m -step common prey for exactly one pair of adjacent vertices in $C^m(D)$ by Proposition 2.9. Therefore $d(u) \geq l - 1$. To show the ‘‘especially’’ part, suppose, to the contrary, that $d(u) = l - 1$ but $d_m^+(u) \neq l$. Then, by the hypothesis, $d_m^+(u) \geq l + 1$. Thus, by the previous argument, $d(u) \geq l$, a contradiction. Therefore $d_m^+(u) = l$. Yet, $d(u) = l - 1$, so $\alpha \in N_m^+(u)$. \square

Theorem 2.16. *Let D be a tree-inducing digraph without sources. Then each vertex lies on a directed cycle in D .*

Proof. Suppose, to the contrary, that there exists a vertex u which does not lie on any directed cycle in D . Let A, B , and C be subsets of $V(D)$ such that

$$A = \bigcup_{i \geq 1} N_i^+(u); \quad B = \bigcup_{i \geq 1} N_i^-(u); \quad C = V(D) - (A \cup B).$$

By the hypothesis, $N^-(u) \neq \emptyset$, so $B \neq \emptyset$. By Lemma 2.5(1), $N^+(u) \neq \emptyset$, so $A \neq \emptyset$. Since there is no directed cycle containing u , $A \cap B = \emptyset$. If $u \in A$ or $u \in B$, then there exists a closed directed walk containing u and so there exists a directed cycle containing u , which contradicts our assumption. Thus $u \in C$ and so $C \neq \emptyset$. We will claim the following:

$$A \nrightarrow B, \quad A \nrightarrow C, \quad \text{and} \quad C \nrightarrow B \tag{2.8}$$

where $X \nrightarrow Y$ for vertex sets X and Y of D means that there is no arc from a vertex in X to a vertex in Y . Take three vertices $a \in A$, $b \in B$, and $c \in C$.

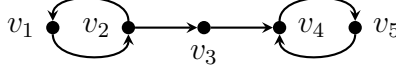


Figure 2.5: A digraph with no sources and no directed cycles containing v_3

If there exists an arc (a, b) , then a directed (u, a) -walk, the arc (a, b) , and a directed (b, u) -walk form a closed directed walk containing u and we reach a contradiction. If there exists an arc (a, c) (resp. an arc (c, b)), then a directed (u, a) -walk and the arc (a, c) form a directed (u, c) -walk (resp. the arc (c, b) and a directed (b, u) -walk form a directed (c, u) -walk), which contradicts the choice of c . Since a , b , and c were arbitrarily chosen from A , B , and C , respectively, the claim is valid.

By choice of the set C ,

$$\{u\} \nrightarrow C, \quad \text{and} \quad C \nrightarrow \{u\} \quad (2.9)$$

Since D is a tree-inducing digraph, by Proposition 2.9, there is a bijection between $E(C^m(D))$ and $V(D) - \{w\}$ where w is the anomaly. Then each of at least $|B|$ vertices in $B \cup \{u\}$ serves as an m -step common prey of a pair of adjacent vertices in $C^m(D)$. Since $u \in C$, no vertex in $A \cup C$ can be an m -step predator of a vertex in $B \cup \{u\}$ by (2.8) and (2.9). Therefore each vertex in $B \cup \{u\}$ has an m -step predator only in B . Consequently, we may conclude that the subgraph H of $C^m(D)$ induced by B has at least $|B|$ edges. Thus H contains a cycle. Then this cycle is contained in $C^m(D)$ and we have reached a contradiction to the hypothesis that $C^m(D)$ is a tree. \square

Remark 2.17. It is likely that, for each vertex of a digraph without sources, there is a directed cycle containing it. However, it is not true. For example, the digraph given in Figure 2.5 has no source and no directed cycle containing the vertex v_3 .

Remark 2.18. For some tree-inducing digraph D with a source, Theorem 2.16 may be false. For example, the vertex w given in Example 2.14 does not lie on any directed cycle in D .

If an m -step tree-inducing digraph D has a loop incident to a vertex on a directed

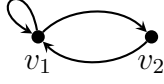


Figure 2.6: A duck digraph with the neck vertex v_1 and the tail vertex v_2

cycle of length 2, then $C^m(D)$ is not a star graph, which will be shown in Lemma 2.27. We call such a configuration a *duck digraph*. That is, a duck digraph is isomorphic to the digraph given in Figure 2.6. We call the vertex with a loop in a duck digraph the *neck vertex* and the other one the *tail vertex*. Given a digraph D , if D contains no subdigraph isomorphic to a duck digraph, we call D a *duck-free digraph*.

Proposition 2.19. *Let D be an m -step tree-inducing digraph such that (a) there exists a vertex u incident to a loop in D and (b) if $m = 2$, then D is duck-free. Then exactly one of the following statements is true.*

- (i) *The vertex u has exactly one predator other than u and $N^+(u) = \{u\}$.*
- (ii) *The vertex u has at least one prey other than u and $N^-(u) = \{u\}$.*

Furthermore, if (i) holds for the vertex u , then the vertex in $N^-(u) - \{u\}$ either is a source or is incident to a loop and (ii) holds for it.

Proof. By the condition (a), $\{u\} \subseteq N^-(u)$ and $\{u\} \subseteq N^+(u)$. If $N^-(u) = N^+(u) = \{u\}$, then u is an isolated vertex in $C^m(D)$, which is a contradiction. Therefore

$$\{u\} \subsetneq N^-(u) \text{ or } \{u\} \subsetneq N^+(u). \quad (2.10)$$

If $N^-(u) = \{u\}$, then the statement (i) does not hold for u and, by (2.10), $\{u\} \subsetneq N^+(u)$ so that (ii) holds for u .

Now suppose that $\{u\} \subsetneq N^-(u)$. Then (ii) does not hold for u . Since $d^-(u) \leq 2$ by Lemma 2.5(2),

$$N^-(u) = \{u, v\}$$

for some vertex v in D . Then

$$u \in N^+(v).$$

Suppose, to the contrary, that there exists a vertex z in $N^-(v) - \{u, v\}$. Then, by using the loop incident to u , we may produce a directed (u, u) -walk, a directed (v, u) -walk, and a directed (z, u) -walk, respectively, of length m . Since $m \geq 2$, $\{u, v, z\} \subseteq N_m^-(u)$, which is a contradiction to Lemma 2.5(2). Hence

$$N^-(v) \subseteq \{u, v\}. \quad (2.11)$$

To reach a contradiction, suppose that $u \in N^-(v)$. Then

$$\{u, v\} \subseteq N^+(u). \quad (2.12)$$

In this case, the subdigraph of D induced by $\{u, v\}$ is a duck digraph. Then, by the condition (b), $m \geq 3$. Suppose that there exists a vertex x in $N^+(v) - \{u, v\}$. By using the loop incident to u , we have $\{u, v, x\} \subseteq N_m^+(v) \cap N_m^+(u)$ for each $m \geq 3$ and so, by Corollary 2.11(1), the anomaly of D is contained in $N_m^+(v) \cap N_m^+(u)$. Therefore $|N_m^+(v) \cap N_m^+(u)| = 2$ by Corollary 2.12(i), which contradicts the fact that $\{u, v, x\} \subseteq N_m^+(v) \cap N_m^+(u)$. Thus

$$u \in N^+(v) \subseteq \{u, v\}. \quad (2.13)$$

Suppose that there exists a vertex y in $N^+(u) - \{u, v\}$. Then, by using the loop incident to u , we have $\{u, v, y\} \subseteq N_m^+(u) \cap N_m^+(v)$ and so we reach a contradiction similarly as above. Therefore $N^+(u) = \{u, v\}$ by (2.12). Thus, by (2.13), $N_m^+(u) = N_m^+(v) = \{u, v\}$, which contradicts Proposition 2.8. Hence $u \notin N^-(v)$ and so, by (2.11),

$$N^-(v) = \emptyset \quad \text{or} \quad N^-(v) = \{v\}. \quad (2.14)$$

Then, by (2.14) and Corollary 2.11(2), v is the anomaly. Now suppose, to the contrary, that $\{u\} \subsetneq N^+(u)$. Take a vertex w in $N^+(u) - \{u\}$. Then $\{u, w\} \subseteq N_m^+(u) \cap N_m^+(v)$. Since v is the anomaly of D , $v \in N_m^+(u) \cap N_m^+(v)$ by Corollary 2.11(1). However, $v \notin N_m^+(u)$ by (2.14), which is a contradiction. Hence $N^+(u) = \{u\}$ and so the statement (i) holds for u . The ‘‘furthermore’’ part is true by (2.14). \square

Corollary 2.20. *Let D be an m -step tree-inducing digraph for an integer $m \geq 3$. Then D is duck-free.*

Proof. Since $m \geq 3$, the condition (b) in Proposition 2.19 is vacuously satisfied. Suppose that there exists a vertex v with a loop in D . Then v satisfies the condition (a) in Proposition 2.19. Thus, by Proposition 2.19, $N^+(v) = \{v\}$ or $N^-(v) = \{v\}$. Thus D is duck-free. \square

Theorem 2.21. *Let D be a duck-free tree-inducing digraph with a loop and without sources. Then there is a vertex u with outdegree at least 2 and $N^-(u) = \{u\}$. Moreover, u is the only one vertex with this property and $d^-(v) = 2$ for each vertex $v \in N^+(u) - \{u\}$.*

Proof. We note that D satisfies the conditions (a) and (b) of Proposition 2.19. Let w be a vertex incident to a loop. If $|N^+(w) - \{w\}| \geq 1$, then Proposition 2.19(ii) holds for w and so we take w as u . Suppose that $|N^+(w) - \{w\}| = 0$. Then Proposition 2.19(i) holds. By the “furthermore” part of Proposition 2.19, the vertex in $N^-(w) - \{w\}$ either is a source or is incident to a loop and Proposition 2.19(ii) holds for it. Since each vertex has indegree at least 1 by the hypothesis, the latter is true and so we take the vertex in $N^-(w) - \{w\}$ as u .

To show the uniqueness, suppose that there exist two vertices x and y each of which has outdegree at least 2 and indegree 1 and is incident to a loop. Therefore $N_m^-(x) = \{x\}$ and $N_m^-(y) = \{y\}$. Then, by Corollary 2.11(2), x and y are anomaly. Therefore $x = y$ by Corollary 2.12(ii). Thus u is the unique vertex with $d^+(u) \geq 2$ and $N^-(u) = \{u\}$.

Suppose, to the contrary, that $d^-(v) \neq 2$ for some vertex $v \in N^+(u) - \{u\}$. Then $d^-(v) \leq 1$ by Lemma 2.5(2). Since D has no source by the hypothesis, $d^-(v) \geq 1$ and so $N^-(v) = \{u\}$. On the other hand, since D is a tree-inducing digraph without sources, there exists a directed cycle C containing v in D by Theorem 2.16. Since $N^-(v) = \{u\}$, u lies on C . Therefore there exists the (v, u) -section of C . However, since $N^-(u) = \{u\}$, there is no directed (v, u) -walk in D and we reach a contradiction. \square

2.3 The digraphs whose m -step competition graphs are star graphs

In this section, we completely characterize the digraphs whose m -step competition graphs are star graphs. The following lemma is easy to check.

Lemma 2.22. *For a digraph D , D is a vertex-disjoint union of directed cycles if and only if each vertex has outdegree 1 in D and any pair of vertices has no common prey in D .*

Theorem 2.23. *An m -step tree-inducing digraph having a source is a windmill digraph.*

Proof. Let D be an m -step tree-inducing digraph having a source v . Then $N_m^-(v) = \emptyset$, so v is the anomaly by Corollary 2.11(2). Therefore v is the only source of D by Proposition 2.9. Thus D satisfies the condition (W1) for being a windmill digraph. In addition,

$$d_m^-(u) = 2 \tag{2.15}$$

for each vertex $u \in V(D) - \{v\}$ by Corollary 2.12(ii).

Fix $u \in V(D) - \{v\}$. Then $d^-(u) \geq 1$ by (2.15). By Lemma 2.5(2), $d^-(u) \leq 2$. Suppose, to the contrary, that $d^-(u) = 1$. Then $N^-(u) = \{x\}$ for some vertex x of D , so $u \notin \bigcup_{v \in V(D) - \{x\}} N^+(v)$. Since $N^-(v) = \emptyset$, $v \notin \bigcup_{v \in V(D) - \{x\}} N^+(v)$ and so $\bigcup_{v \in V(D) - \{x\}} N^+(v) \subseteq (V(D) - \{u, v\})$.

Therefore

$$\left| \bigcup_{v \in V(D) - \{x\}} N^+(v) \right| \leq |V(D) - \{u, v\}| < |V(D) - \{x\}|.$$

Since $V(D) - \{x\}$ is a proper subset of $V(D)$, we reach a contradiction to Theorem 2.13(1). Therefore

$$d^-(u) = 2. \tag{2.16}$$

By Lemma 2.5(1), $d^+(u) \geq 1$. Suppose, to the contrary, that $d^+(u) \geq 2$. Then, by Theorem 2.13(3), $d_{m-1}^+(u) \geq 2$. On the other hand, by (2.16), u is a common prey



Figure 2.7: A digraph and its 2-step competition graph

of two vertices y and y' . Then there exist arcs (y, u) and (y', u) in D . Take a vertex z in $N_{m-1}^+(u)$. Then there exists a directed (u, z) -walk W of length $m - 1$. Therefore $y \rightarrow W$ is a directed (y, z) -walk and $y' \rightarrow W$ is a directed (y', z) -walk both of which have length m . Thus $z \in N_m^+(y) \cap N_m^+(y')$ and so $N_{m-1}^+(u) \subseteq N_m^+(y) \cap N_m^+(y')$. Then, since $d_{m-1}^+(u) \geq 2$, $|N_m^+(y) \cap N_m^+(y')| \geq 2$. Therefore the anomaly v must be contained in $N_m^+(y) \cap N_m^+(y')$ by Corollary 2.11(1), which contracts the fact that $N^-(v) = \emptyset$. Therefore

$$d^+(u) = 1. \quad (2.17)$$

Since u was arbitrarily chosen, (2.16) and (2.17) hold for any vertex in $V(D) - \{v\}$. Now take two distinct vertices x and y in $V(D) - \{v\}$ (they exist since D has at least three vertices by the definition of m -step tree inducing digraph). Then $d^+(x) = d^+(y) = 1$. Therefore, by Proposition 2.8, $N^+(x) \cap N^+(y) = \emptyset$. Thus, by Lemma 2.22, $D - v$ is a vertex-disjoint union of directed cycles and so D satisfies the condition (W2). Hence (2.16) and (2.17) deduce that D satisfies the condition (W3). \square

Lemma 2.24. *Let D be a digraph without sources whose m -step competition graph $C^m(D)$ is a star graph. Then the following are true:*

- (1) *There exist at most two vertices of i -step outdegree at least 2 for each $1 \leq i \leq m$.*
- (2) *If a vertex v has a predator distinct from v , then $d^+(v) \leq 2$.*
- (3) *Each vertex of indegree 2 is a prey of the center of $C^m(D)$.*

Proof. Suppose that there are three vertices x, y , and z having j -step outdegree at least 2 for some $j \in \{1, \dots, m\}$. Since $C^m(D)$ is a star graph, at least two of x, y ,

and z have degree 1 in $C^m(D)$. Without loss of generality, we may assume that y and z have degree 1 in $C^m(D)$. Then the anomaly of D is an m -step common prey of y and z by the “especially” part of Proposition 2.15. Therefore yz is an edge in $C^m(D)$. Then, since y and z have degree 1 in $C^m(D)$, yz is a component in $C^m(D)$, a contradiction. Thus part (1) is true.

To show part (2), suppose that there exists a vertex v that has a predator v' distinct from v . If $d^+(v) \geq 3$, then $d_2^+(v') \geq 3$ and so, by Proposition 2.15, v and v' have degree at least 2 in $C^m(D)$, a contradiction. Therefore $d^+(v) \leq 2$.

Suppose that there exists a vertex of indegree 2 that is a common prey of two vertices x and y . Then x and y are adjacent in $C^m(D)$ by Theorem 2.13(2). Therefore x or y is the center of $C^m(D)$. Thus part (3) is true. \square

We call a directed cycle C in a digraph D an *induced directed cycle* if C is an induced subdigraph of D .

Theorem 2.25. *Let D be a loopless digraph whose m -step competition graph is a star graph. If D has no sources, then $m = 2$ and D is isomorphic to the digraph given in Figure 2.3.*

Proof. Suppose that D has no sources. Then, by Theorem 2.16, D has a directed cycle. We first claim that each directed cycle in D has length at least m . To reach a contradiction, suppose that there exists a directed cycle $C := v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_{l-1} \rightarrow v_0$ of length $l \leq m - 1$. Since D is loopless, $l \geq 2$ and $m \geq 3$.

Suppose that C is not an induced directed cycle. Then, since D is loopless, $l \geq 3$. Moreover, there is an arc (v_i, v_j) for some $i, j \in \{0, 1, \dots, l-1\}$ so that it together with a section of C forms a directed cycle of length at most $l - 1$. Without loss of generality, we may assume that $i = 0$. Then $j \notin \{0, 1\}$ and v_j is a common prey of v_0 and v_{j-1} . Accordingly, v_j is a 2-step common prey of v_{l-1} and v_{j-2} . Therefore v_0v_{j-1} and $v_{l-1}v_{j-2}$ are edges in $C^m(D)$ by Theorem 2.13(2). Thus $C^m(D)$ is not a star graph, a contradiction. Hence C is an induced directed cycle.

Suppose, to the contrary, that no vertex in $V(D) - V(C)$ has a prey in $V(C)$. Then, since C is an induced directed cycle, each vertex on C has exactly one m -step predator in D . Therefore each vertex on C is the anomaly by Corollary 2.11(2). Since

$l \geq 2$, we reach a contradiction to the uniqueness of the anomaly. Therefore there exists a vertex a in $V(D) - V(C)$ that has a prey on C . Without loss of generality, we may assume that v_0 is a prey of a . Therefore v_0 is a common prey of a and v_{l-1} and so, by Theorem 2.13(2), av_{l-1} is an edge in $C^m(D)$. Thus a or v_{l-1} is the center of $C^m(D)$. On the other hand, since a is not source, a has a predator b . To show $b \neq v_{l-2}$, suppose $b = v_{l-2}$. Then $\{a, v_{l-1}\} \subseteq N^+(v_{l-2})$. Therefore $d_2^+(v_{l-3}) \geq 2$ and $d_3^+(v_{l-4}) \geq 2$ (we assume that each subscript of the vertices on C is reduced to modulo l). Thus each of v_{l-2}, v_{l-3} , and v_{l-4} has an m -step outdegree at least 2 by Theorem 2.13(3). Hence $v_{l-2} = v_{l-4}$ by Lemma 2.24(1) and so $l = 2$. Then we can check that $d^+(v_0) \geq 2$, $d_2^+(v_1) \geq 2$, and $d_2^+(a) \geq 2$. Therefore each of v_0, v_1 , and a has an m -step outdegree at least 2 by Theorem 2.13(3), which contradicts Lemma 2.24(1). Thus

$$b \neq v_{l-2}.$$

If b is distinct from v_{l-1} , then v_{l-2} and b are adjacent since v_0 is a 2-step common prey of v_{l-2} and b , a contradiction to the fact that a or v_{l-1} is the center of $C^m(D)$. Therefore $b = v_{l-1}$. Thus v_0 is a 3-step common prey of v_{l-2} and v_{l-3} and so, by Theorem 2.13(2), $v_{l-2}v_{l-3}$ is an edge in $C^m(D)$. Hence v_{l-2} or v_{l-3} is the center of $C^m(D)$. Then, since $v_{l-2} \neq v_{l-1}$, and a or v_{l-1} is the center of $C^m(D)$, $v_{l-1}(= v_{l-3})$ is the center of $C^m(D)$. Therefore $l = 2$. Thus $v_0 \rightarrow v_1 \rightarrow a \rightarrow v_0$ and $a \rightarrow v_0 \rightarrow v_1 \rightarrow v_0$ and so, by Theorem 2.13(2), v_0a is an edge in $C^m(D)$, which contradicts the fact that v_1 is the center of $C^m(D)$.

Hence we have shown that

(*) each directed cycle in D has length at least m .

Since D is loopless and has no sources,

$$d^+(v) \leq 2 \tag{2.18}$$

for each vertex v in D by Lemma 2.24(2). If each vertex has outdegree 1, then each vertex has indegree 1 since D has no sources, and so $C^m(D)$ is edgeless. Therefore

there exists a vertex u of outdegree at least 2. Thus

$$d^+(u) = 2 \tag{2.19}$$

by (2.18). Since D has no sources, there exists a directed walk

$$W := x \rightarrow y \rightarrow u \tag{2.20}$$

in D . Suppose, to the contrary, that $m \geq 3$. Then, by (*), x , y , and u are distinct. Since $d^+(u) = 2$, $d_3^+(x) \geq 2$, and $d_2^+(y) \geq 2$. Therefore each of x, y, u has m -step outdegree at least 2 by Theorem 2.13(3), which contradicts Lemma 2.24(1). Thus $m \leq 2$ and so

$$m = 2.$$

Let c be the center of $C^2(D)$. Since $d_2^+(v) \leq 4$ for each vertex v in D by (2.18), c has degree at most 4 in $C^2(D)$ by Lemma 2.5(2) and so $|V(D)| \leq 5$. Since $|V(D)| > m = 2$, $|V(D)| \in \{3, 4, 5\}$.

Suppose $|V(D)| = 3$. Then, since $u \neq y$, $V(D) = \{u, y, z\}$. By Theorem 2.16, there exists a directed cycle in D . We take a longest directed cycle C of length l . Then, since D is loopless and $|V(D)| = 3$, $l = 2$ or 3. If $l = 2$, then, by Theorem 2.16, D is isomorphic to the digraph given in Figure 2.7 and so $C^2(D)$ has an isolated vertex, a contradiction. Thus $l = 3$. Then $C = u \rightarrow z \rightarrow y \rightarrow u$ or $u \rightarrow y \rightarrow z \rightarrow u$. We note that $N^+(u) = \{y, z\}$ and, by (2.20), $y \rightarrow u$. To show $C = u \rightarrow z \rightarrow y \rightarrow u$, suppose $C = u \rightarrow y \rightarrow z \rightarrow u$. Then u is a common prey of y and z and z is a common prey of u and y , so, by Theorem 2.13(2), yz, uy are edges in $C^2(D)$. Moreover, z is a 2-step common prey of u and z , and so uz is an edge in $C^2(D)$. Thus $C^2(D)$ is a triangle, which contradicts the fact $C^2(D)$ is a star. Therefore $C = u \rightarrow z \rightarrow y \rightarrow u$. Since u has outdegree 2 by (2.19), $u \rightarrow y$. Therefore we obtain a subdigraph isomorphic to the one given in Figure 2.3. Thus y is a common prey of u and z and y is a 2-step common prey of u and y . Hence uz and uy are edges in $C^2(D)$ by Theorem 2.13(2). Now it is easy to check that adding more arcs to the digraph given in Figure 2.3 results in the edge joining y and z in $C^2(D)$. Therefore

we conclude that D is isomorphic to the one given in Figure 2.3.

Now suppose that $|V(D)| = 4$ or 5 . Then c has at least three 2-step prey and so

$$d^+(c) = 2$$

by (2.18). Let u_1 and u_2 be the prey of c . If each of u_1 and u_2 has outdegree 1, then c has at most two 2-step prey, which is impossible. Therefore at least one of them has outdegree 2. Without loss of generality, we may assume that u_1 has outdegree 2. Then c and u_1 are the only vertices of outdegree 2 by Lemma 2.24(1). Hence u_2 has outdegree 1 by (2.18). Moreover, c has at most three 2-step prey and so c has degree at most 3 in $C^2(D)$. Therefore $|V(D)| = 4$. Thus c has degree 3 in $C^2(D)$.

By Lemma 2.5(2), each vertex is a 2-step common prey of at most two vertices. Therefore c has exactly three 2-step prey in D . Let x be one of them. Then, other than c , there is exactly one 2-step predator of x . We denote it by \tilde{x} . Then, for distinct 2-step prey x and y of c , $\tilde{x} \neq \tilde{y}$. Suppose that u_1 has indegree 1. If some prey d of u_1 has indegree 1, then d is a 2-step prey of c and $N_2^-(d) = \{c\}$, which contradicts the existence of \tilde{d} . Therefore each prey of u_1 has indegree 2. Thus, by Lemma 2.24(3), each prey of u_1 is a prey of c . Since D is loopless and u_1 has outdegree 2, $d^+(c) \geq 3$, a contradiction. Thus u_1 has indegree 2 and $|N^-(u_1) - \{c\}| = 1$. Then the vertex in $N^-(u_1) - \{c\}$ is a 2-step common predator of the two prey w and z of u_1 . Now, even if $w \neq z$, $\tilde{w} = \tilde{z}$, a contradiction. Hence the statement is true. \square

Lemma 2.26. *Let D be a windmill digraph or an m -conveyor digraph. Then $C^m(D)$ is a star graph.*

Proof. We suppose that D is a windmill digraph with the source v . Then v and another vertex w have a common prey by (W2) and (W3). Therefore, by (W2), v and w have an m -step common prey for any $m \geq 1$ and so v and w are adjacent in $C^m(D)$. By (W1) and (W2), any two vertices other than v cannot have an m -step common prey for any $m \geq 1$. Thus $C^m(D)$ is a star graph with the center v .

Now we suppose that D is an m -conveyor digraph with the loop v satisfying (M1) and (M2). Thus any vertex w other than v has a unique m -step prey x on a directed

cycle containing w and w is the only m -step predator of x in $V(D) - \{v\}$, and so w is not adjacent to any vertex belonging to $V(D) - \{v\}$ in $C^m(D)$. Since D is a weakly connected digraph, by (M2), each directed cycle in $D - v$ has a vertex that is a prey of v and so there exists an internally secure lane W in D containing x . By (M3), W has length at most m . Since v is incident to a loop by (M1), we may obtain a directed walk of length m from v to x by using the loop incident to v . Hence x is an m -step common prey of v and w and so v and w are adjacent in $C^m(D)$, which implies that $C^m(D)$ is a star graph with the center v . \square

Lemma 2.27. *Let D be a tree-inducing digraph whose m -step competition graph is a star graph. Then D is duck-free.*

Proof. Suppose, to the contrary, that D contains a subdigraph H isomorphic to a duck digraph (see Figure 2.6 for an illustration). Then, by Corollary 2.20, $m = 2$. Let v_1 and v_2 be the neck vertex and the tail vertex, respectively, of H . By the definition of a duck digraph, $\{(v_1, v_1), (v_1, v_2), (v_2, v_1)\} \subseteq A(D)$. It is easy to check that $\{v_1, v_2\} \subseteq N_2^-(v_1)$ and $\{v_1, v_2\} \subseteq N_2^-(v_2)$. By Lemma 2.5(2),

$$N_2^-(v_1) = N_2^-(v_2) = \{v_1, v_2\}.$$

Since a predator of v_1 or v_2 would belong to $N_2^-(v_1)$,

$$N^-(v_1) = \{v_1, v_2\} \quad \text{and} \quad \{v_1\} \subseteq N^-(v_2) \subseteq \{v_1, v_2\}. \quad (2.21)$$

To show $N^+(v_1) = \{v_1, v_2\}$ by contradiction, suppose that there exists a vertex v_3 distinct from v_1 and v_2 in $N^+(v_1)$. Then $\{v_1, v_2, v_3\} \subseteq N_2^+(v_1) \cap N_2^+(v_2)$. Therefore one of v_1, v_2, v_3 is the anomaly of D by Corollary 2.11(1). Thus $|N_2^+(v_1) \cap N_2^+(v_2)| = 2$ by Corollary 2.12(i), which contradicts the fact that $\{v_1, v_2, v_3\} \subseteq N_2^+(v_1) \cap N_2^+(v_2)$. Hence

$$N^+(v_1) = \{v_1, v_2\}. \quad (2.22)$$

If $N^+(v_2) \subseteq \{v_1, v_2\}$, then H is a component of $C^m(D)$ by (2.21) and (2.22), which contradicts the hypothesis that $C^2(D)$ is a star graph with at least three vertices

(a tree-inducing digraph has at least three vertices by definition). Therefore there exists a vertex v_3 in $N^+(v_2) - \{v_1, v_2\}$. If v_3 is incident to a loop, then v_1 , v_2 , and v_3 are 2-step predators of v_3 , which contradicts Lemma 2.5(2). Therefore v_3 is not incident to a loop. Moreover, v_3 has outdegree at least 1 by Lemma 2.5(1). Then, by (2.21), neither v_1 nor v_2 can be a prey of v_3 , so there must be a vertex v_4 in $N^+(v_3) - \{v_1, v_2, v_3\}$. Therefore $\{v_1, v_2, v_4\} \subseteq N_2^+(v_2)$ and $\{v_1, v_2, v_3\} \subseteq N_2^+(v_1)$. Thus each degree of v_1 and v_2 is at least 2 in $C^2(D)$ by Proposition 2.15. Hence $C^2(D)$ is not a star graph, a contradiction. \square

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. To show the “only if” part, suppose that there exists a digraph D with n vertices whose m -step competition graph is a star graph for some $2 \leq m < n$. Then D is duck-free by Lemma 2.27. If D has a source, then D is a windmill digraph by Theorem 2.23. Suppose that D has no sources. If D is loopless, then (iii) is true by Theorem 2.25. Now we suppose that D has a loop. We will show that D is an m -conveyor digraph. Since D has a loop and D is duck-free, there exists a vertex v such that $N^-(v) = \{v\}$ and $d^+(v) \geq 2$ by Theorem 2.21. Since $N^-(v) = \{v\}$, (M1) is satisfied and v is the only m -step predator of v . Then, by Corollary 2.11(2), v is the anomaly.

To reach a contradiction, suppose that there exists a vertex w distinct from v having outdegree at least 2. Then $d_i^+(w) \geq 2$ for each $1 \leq i \leq m$ by Theorem 2.13(3). If w has degree 1 in $C^m(D)$, then $v \in N_m^+(w)$ by the “especially” part of Proposition 2.15, which contradicts the fact that v is the only m -step predator of v . Therefore w has degree at least 2 in $C^m(D)$ and so w is the center of $C^m(D)$. Since D has no sources, w has a predator x . Since $N_{m-1}^+(w) \subseteq N_m^+(x)$, x has at least two m -step prey each of which is not v . Then, since v is the anomaly, x has degree at least 2 by Proposition 2.15, and so x is the center of $C^m(D)$. Thus $x = w$. Since x was arbitrarily taken, $N^-(w) = \{w\}$. Consequently, w is the anomaly by Corollary 2.11(2). Then, since $v \neq w$, we reach a contradiction to the uniqueness of the anomaly. Therefore v is the only vertex of outdegree at least 2 in D and so, by Lemma 2.5(1), each vertex in $V(D) - \{v\}$ has outdegree 1. Thus any pair of vertices in $V(D) - \{v\}$ has no

common prey by Proposition 2.8. Hence $D - v$ is a vertex-disjoint union of directed cycles by Lemma 2.22 and so (M2) is satisfied.

Suppose that there exists an internally secure lane W of length at least $m + 1$. Then the m th interior vertex v' on W has exactly one m -step predator in D . Thus v' is the anomaly in D by Corollary 2.11(2). However, since $N^-(v) = \{v\}$, we obtain $v' \neq v$ and so we reach a contradiction to the uniqueness of the anomaly. Therefore each internally secure lane of D has length at most m and so (M3) is satisfied. Thus D is an m -conveyor digraph. Hence the “only if” part is true.

Now we show the “if” part. If D is a windmill digraph or an m -conveyor digraph, then $C^m(D)$ is a star graph by Lemma 2.26. In addition, it is easy to check that the 2-step competition graph of the digraph given in Figure 2.3 is a star graph. Therefore the “if” part is true and so this completes the proof. \square

Chapter 3

On $(1, 2)$ -step competition graphs of multipartite tournaments¹

Recall that given positive integers i and j , the (i, j) -step competition graph $C_{i,j}(D)$ of D is the graph which has the same vertex set as D and has an edge between two distinct vertices u and v if and only if either $d_{D-v}(u, w) \leq i$ and $d_{D-u}(v, w) \leq j$ or $d_{D-v}(u, w) \leq j$ and $d_{D-u}(v, w) \leq i$. (Definition 1.9).

For a digraph D , we say that vertices u and v in D $(1, 2)$ -*compete* provided there exists a vertex w distinct from u, v and satisfying one of the following:

- there exists an arc (u, w) and a directed (v, w) -path of length 2 not traversing u ;
- there exists a directed (u, w) -path of length 2 not traversing v and an arc (v, w) .

We call w in the above definition a $(1, 2)$ -*step common out-neighbor* of u and v . It is said that two vertices *compete* if they have a common out-neighbor. Thus, $uv \in E(C_{1,2}(D))$ if and only if u and v compete or $(1, 2)$ -compete. In such a case, we say that u and v $\{1, 2\}$ -*compete* in D . If two vertices of a digraph D are adjacent in $C_{1,2}(D)$, then we just say that they are adjacent in the rest of this chapter.

¹The material in this chapter is written based on the manuscript ‘On $(1, 2)$ -step competition graphs of multipartite tournaments’ by Myungho Choi and Suh-Ryung Kim. The author thanks Prof. Suh-Ryung Kim for allowing him to use its contents for his thesis.

We call an orientation of complete k -partite graph for some positive integer k a k -partite tournament. A k -partite tournament for some integer $k \geq 3$ is called a *multipartite tournament* while a 2-partite tournament are called a *bipartite tournament*. A tournament of order n may be regarded as an n -partite tournament for some positive integer n .

Let D be a multipartite tournament. For simplicity, we call a partite set of D *clique* if it forms a clique in $C_{1,2}(D)$. If a partite set of D is not clique, we say that it is *non-clique*. In this thesis, we first completely characterize $C_{1,2}(D)$ when each partite of D is clique (Theorem 3.18, and the “especially” part of Theorem 3.19) and the size of partite sets of D when $C_{1,2}(D)$ is complete (Theorem 3.25). Even if there exists a non-clique partite set, we figure out most of the structure of $C_{1,2}(D)$ (Theorems 3.9 and 3.16). Then we show the diameter of each component of $C_{1,2}(D)$ is at most three (Theorem 3.26) and provide a sharp upper bound on the domination number of $C_{1,2}(D)$ (Theorem 3.27). In addition, we list all possible $C_{1,2}(D)$ when D has no vertices of outdegree 0 and $C_{1,2}(D)$ is disconnected (Theorem 3.30). Finally, we give a sufficient condition for $C_{1,2}(D)$ being an interval graph (Theorem 3.37).

3.1 Preliminaries

Let D be a multipartite tournament. We call a vertex of outdegree 0 in a digraph D a *sink* of D . It is obvious that each non-sink vertex has at least one out-neighbor. For a non-sink vertex u and a vertex v , $u \overset{*}{\rightarrow} v$ means that v is the only out-neighbor of u . We write $u \not\overset{*}{\rightarrow} v$ for the negation of $u \overset{*}{\rightarrow} v$, that is, there is an out-neighbor of u distinct from v . Given a vertex set T of D , we say a vertex v is T -biased if $N^+(v) \cap T \neq \emptyset$ and $N^+(v) \subset T$. See the digraph given in Figure 3.1 for an illustration.

By definition, if two vertices u and v are X -biased for some partite set X of a multipartite tournament, then u and v cannot (1,2)-compete. We may ask if the converse is true. By the way, as long as u and v belong to the same partite set of a multipartite tournament, the answer is yes. For, by the structure of multipartite tournaments, non-sink vertices u and v in the same partite set of a multipartite tournament (1,2)-compete if an out-neighbor of u and an out-neighbor of v belong

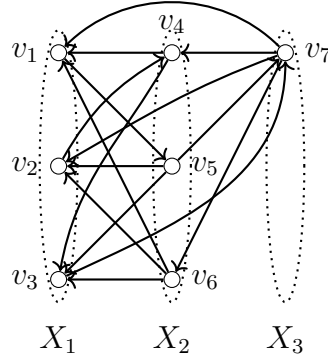


Figure 3.1: $\{v_1, v_2\}$, $\{v_3\}$, and $\{v_4, v_6\}$ are the sets of X_2 -biased vertices, X_3 -biased vertices, and X_1 -biased vertices, respectively.

to distinct partite sets. Thus we have shown the following.

Proposition 3.1. *Let D be a multipartite tournament and u and v be two non-sink vertices belonging to the same partite set in D . Then u and v do not $(1, 2)$ -compete if and only if u and v are X -biased for some partite set X of D*

Proposition 3.2. *Let D be a multipartite tournament and u and v be two non-sink vertices belonging to the same partite set in D . Then the following are equivalent:*

- (i) u and v are not adjacent;
- (ii) $N^+(u) \cap N^+(v) = \emptyset$ and, u and v are X -biased for some partite set X of D

Proof. By Proposition 3.1, $N^+(u) \cap N^+(v) = \emptyset$ and, u and v are X -biased for some partite set X in D if and only if u and v neither compete nor $(1, 2)$ -compete in D , equivalently, u and v are not adjacent. \square

Proposition 3.3. *Let D be a digraph. Suppose $u \xrightarrow{*} v$ for some vertices u and v in D . Then u and v are not adjacent.*

Proof. Since $u \xrightarrow{*} v$, each directed path from u to z must traverse v , so $d_{D-v}(u, z) \neq 1, 2$ for each vertex z in $V(D) \setminus \{u, v\}$. Therefore $uv \notin E(C_{1,2}(D))$. \square

Proposition 3.4. *Let D be a multipartite tournament and u and v be two non-sink vertices with $(u, v) \in A(D)$ belonging to the distinct partite sets in D . Then the following are equivalent:*

- (i) u and v are not adjacent;
- (ii) either $u \xrightarrow{*} v$, or $N^+(u) \cap N^+(v) = \emptyset$ and v is X -biased and u is $X \cup \{v\}$ -biased for some partite set X .

Proof. We first show that (ii) implies (i). If $u \xrightarrow{*} v$ then u and v are not adjacent by Proposition 3.3. Suppose that $u \not\xrightarrow{*} v$, $N^+(u) \cap N^+(v) = \emptyset$, and v is X -biased for some partite set X and u is $X \cup \{v\}$ biased. Then

$$(N^+(u) \cup N^+(v)) \setminus \{u, v\} \subseteq X. \quad (3.1)$$

Suppose, to the contrary, that u and v are adjacent. Then, since $N^+(u) \cap N^+(v) = \emptyset$, u and v must $(1, 2)$ -compete. Let w be a $(1, 2)$ -step common out-neighbor of u and v . Without loss of generality, we assume that there exist a directed (u, w) -path $P = u \rightarrow u' \rightarrow w$ for some vertex u' and an arc (v, w) . Then u' and w are out-neighbor of u and v , respectively, distinct from u and v . Therefore $\{u', w\} \subseteq X$ by (3.1). However, (u', w) is an arc on P and so we reach a contradiction. Thus u and v are not adjacent.

Now we show that (i) implies (ii). Suppose that u and v are not adjacent. Then u and v have no common out-neighbor. If $u \xrightarrow{*} v$, then we are done. Suppose $u \not\xrightarrow{*} v$. Then $N^+(u) \setminus \{v\} \neq \emptyset$. Since $N^+(v) \setminus \{u\} \neq \emptyset$, there exist an out-neighbor x of u and an out-neighbor y of v distinct from v and u , respectively. If x and y belong to different partite sets, then (x, y) or $(y, x) \in A(D)$ and so u and v $(1, 2)$ -compete, which is a contradiction. Therefore x and y belong to the same partite set. Since x and y are arbitrarily chosen, we conclude that v is X -biased and u is $X \cup \{v\}$ biased for some partite set X . \square

The following corollary is immediately true by Propositions 3.2 and 3.4.

Corollary 3.5. *Let D be a multipartite tournament and u, v be two vertices in D . Suppose that $u \not\overset{*}{\rightarrow} v$, $v \not\overset{*}{\rightarrow} u$, and whenever v (resp. u) is X -biased for some partite set X , u is not $X \cup \{v\}$ -biased (resp. v is not $X \cup \{u\}$ -biased). Then u and v are adjacent.*

A *stable set* of a graph is a set of vertices no two of which are adjacent. A stable set in a graph is *maximum* if the graph contains no larger stable set.

As we characterized complete k -partite graphs which can be oriented to become k -partite tournaments whose $(1, 2)$ -step competition graphs are complete, we take a look at multipartite tournaments whose $(1, 2)$ -step competition graphs have maximum stable sets of sizes at least two.

We note that the $(1, 2)$ -step competition graph of a multipartite tournament of order n with a sink constituting a trivial partite set is isomorphic to $K_{n-1} \cup K_1$. In this vein, we only consider a multipartite tournament without a trivial partite set consisting of a sink in the rest of this chapter.

3.2 $C_{1,2}(D)$ with a non-clique partite set of D

In this section, we characterize $(1, 2)$ -step competition graphs of multipartite tournaments with a non-clique partite set.

By our assumption that any multipartite tournament does not have a trivial partite set consisting of a sink, a multipartite tournament with a sink has a non-clique partite set.

We need the following proposition to characterize $(1, 2)$ -step competition graphs of multipartite tournaments with a non-clique partite set.

Given a multipartite tournament D , we say that a vertex set S of D is a $\{1, 2\}$ -stable set if no two vertices in S $\{1, 2\}$ -compete. We note that a vertex set S of D is a $\{1, 2\}$ -stable set if and only if S is a stable set of $C_{1,2}(D)$.

Proposition 3.6. *Let D be a multipartite tournament with a $\{1, 2\}$ -stable set S of size at least two. If S is contained in one partite set X of D , then the following parts are valid:*

(1) $\bigcup_{u \in S} N^+(u)$ is contained in one partite set unless S contains a sink and exactly one non-sink vertex;

(2) if S has at least three vertices, then any pair of vertices in $V(D) \setminus X$ has a common out-neighbor in S ;

Proof. Suppose that S is contained in a partite set X of D . Suppose that S contains a sink. Then there exist at least two non-sink vertices x and y and so, by Proposition 3.2, $N^+(x) \cup N^+(y)$ is contained in a partite set X' of D . Since x and y were arbitrarily chosen from S , each non-sink vertex in S is X' -biased. Therefore the part (1) is true. Now we suppose that S contains no sinks. Since no two vertices in S $\{1, 2\}$ -compete, that is, no two vertices in S are adjacent, the part (1) is true by Proposition 3.2.

To show the part (2), we assume $|S| \geq 3$. Since no two vertices in S $\{1, 2\}$ -compete, each vertex in $V(D) \setminus X$ has at least $|S| - 1$ out-neighbors in S . Since $|S| \geq 3$, $2(|S| - 1) > |S|$ and so, by the Pigeonhole principle, there exists a common out-neighbor in S of each pair of vertices in $V(D) \setminus X$. Therefore the part (2) is true. \square

From now on, when we mention a k -partite tournament D with a non-clique partite set X_1 for some $k \geq 3$, we assume that X_2, \dots, X_k are the remaining partite sets of D . Then

- if D has a sink u , then u is contained in X_1 .

Take a k -partite tournament D with a non-clique partite set X_1 for some $k \geq 3$. We first consider the case where D has a sink u . Then

$$N^-(u) = V(D) \setminus X_1. \quad (3.2)$$

Suppose that a vertex v in X_1 has two out-neighbors w and x belonging to distinct partite sets. Then we take a non-sink vertex y . If $y \in V(D) \setminus X_1$, then u is an out-neighbor of y and so, by Corollary 3.5, v and y are adjacent. If $y \in X_1$, then an

out-neighbor of y and at least one of w or x belong to distinct partite sets and so, by the same corollary, v and y are adjacent. Thus we may conclude that

- I. if D has a sink and a vertex in X_1 has out-neighbors in distinct partite sets, then it is adjacent to all the non-sink vertices (refer to the blocks in the row and column corresponding to X_1^* of Figure 3.2).

Now we consider the case where D has no sinks. Let S be a $\{1,2\}$ -stable set included in X_1 . Suppose that a vertex u in X_1 has two out-neighbors x_1 and x_2 belonging to distinct partite sets. Then $u \notin S$ by Proposition 3.6(1). Moreover, u is adjacent to every vertex in X_1 by Corollary 3.5. Since $|S| \geq 2$, each vertex in $V(D) \setminus X_1$ has at least one out-neighbor in S . Therefore u is adjacent to every vertex in X by Corollary 3.5. Thus we may conclude that

- II. if D has no sink and a vertex in X_1 has out-neighbors in distinct partite sets, then it is a universal vertex (refer to the blocks in the row and column corresponding to X_1^* of Figure 3.2).

By Observations I and II, we have the following proposition.

Proposition 3.7. *Let D be a k -partite tournament D with a non-clique partite set X_1 for some $k \geq 3$. If a vertex in X_1 has out-neighbors in distinct partite sets, then the vertex is adjacent to all the non-sink vertices.*

Hence, in order to characterize $C_{1,2}(D)$ for a multipartite tournament D with a non-clique partite set X_1 , it remains to take a look at a vertex in X_1 with all of its out-neighbors in the same partite set.

If a k -partite tournament D with a non-clique partite set X_1 has a $\{1,2\}$ -stable set S with size at least two in X_1 unless S contains a sink and exactly one non-sink vertex, then each vertex of S is X_j -biased for some $j \in \{2, \dots, k\}$ by Proposition 3.6(1). We may assume $j = 2$, i.e. each vertex in S is X_2 -biased for any k -partite tournament D with a non-clique partite set X_1 and a $\{1,2\}$ -stable set S of size at least two in X_1 .

Theorem 3.8. *Let D be a multipartite tournament with a non-clique partite set X_1 . Then the following parts are valid:*

- (1) For $2 \leq i \leq k$, each X_i -biased vertex belongs to X_1 ;
- (2) $\bigcup_{i=3}^k X_i$ forms a clique in $C_{1,2}(D)$ and each vertex in $\bigcup_{i=3}^k X_i$ is adjacent to each vertex in X_2 in $C_{1,2}(D)$;
- (3) each X_i -biased vertex is adjacent to each X_j -biased vertex and each vertex in X_j for distinct integers $2 \leq i, j \leq k$;
- (4) a X_i -biased vertex x is not adjacent to a vertex y in X_i for some $i \in \{2, \dots, k\}$ if and only if $x \xrightarrow{*} y$ or $y \xrightarrow{*} x$;
- (5) if D has a sink or $\{1, 2\}$ -stable set of size at least three in one partite set, then X_2 forms a clique in $C_{1,2}(D)$.

Proof. We first show the parts (1) and (2). Let S be a $\{1, 2\}$ -stable set with size two in X_1 . Then each vertex in $\bigcup_{i=2}^k X_i$ has an out-neighbor in $S \subseteq X_1$. Therefore no vertex in $\bigcup_{i=2}^k X_i$ is a X_i -biased for any $2 \leq i \leq k$ and so the part (1) is true.

For simplicity, let F_i denote the set of X_i -biased vertices for each $2 \leq i \leq k$. If S contains a sink u , then u is an out-neighbor of each vertex in $\bigcup_{i=3}^k X_i$ and so $\bigcup_{i=3}^k X_i$ forms a clique in $C_{1,2}(D)$. Suppose that S does not contain a sink. Then $S \subseteq F_2$ by the assumption. Therefore the out-neighborhood of each vertex in S is included in X_2 . Thus each vertex in S is an out-neighbor of each vertex in $\bigcup_{i=3}^k X_i$. Hence $\bigcup_{i=3}^k X_i$ forms a clique in $C_{1,2}(D)$. In addition, since each vertex in X_2 has an out-neighbor in $S \subseteq X_1$, each vertex in $\bigcup_{i=3}^k X_i$ is adjacent to each vertex in X_2 in $C_{1,2}(D)$ and so the part (2) is true.

To show the part (3), take two vertices $v \in F_i$ and $x \in X_j \cup F_j$ for distinct i and j in $\{2, \dots, k\}$. If $x \in F_j$, then $\{v, x\} \subseteq X_1$ by the part (1) and so, by Corollary 3.5, v and w are adjacent. Suppose $x \in X_j$. Then x is not an out-neighbor of v . Therefore v is an out-neighbor of x . Moreover, since $v \in F_i$, v has an out-neighbor distinct from x . Thus, by Corollary 3.5, it suffices to show that x has an out-neighbor distinct from v . If S has a sink u , then u is an out-neighbor of x distinct from v and we are done. Therefore we assume that S has no sinks.

Suppose $j \neq 2$. Then, since $S \subseteq F_2$ by the assumption, each vertex in S cannot have $x \in X_j$ as an out-neighbor. Therefore $S \subseteq N^+(x)$. Thus x has at least two out-neighbors in $S \subseteq X_1$ since $|S| \geq 2$. Hence x has an out-neighbor in X_1 distinct from v . Now we suppose $j = 2$. Then $i \neq 2$. Since $|S| \geq 2$, x has at least one out-neighbor x' in S . Since $S \subseteq F_2$ by the assumption, x' is distinct from v . In each case, x has an out-neighbor in X_1 distinct from v . Therefore v and x are adjacent by Corollary 3.5. Since v and x were arbitrarily chosen from F_i and X_j , respectively, the part (3) is true.

The “if” part of the part (4) is true by Proposition 3.3. To show the “only if” part of the part (4), suppose, to the contrary, that there exist a vertex x in F_i and a vertex y in X_i for some $i \in \{2, \dots, k\}$ such that they are not adjacent, $N^+(x) \neq \{y\}$, and $N^+(y) \neq \{x\}$. Then, since x and y are non-sink vertices, $N^+(x) \setminus \{y\} \neq \emptyset$ and $N^+(y) \setminus \{x\} \neq \emptyset$. Since $y \in X_i$, $(N^+(y) \setminus \{x\}) \cap X_j \neq \emptyset$ for some j distinct from i . In addition, since $x \in F_i$, $\emptyset \neq N^+(x) \setminus \{y\} \subset N^+(x) \subset X_i$. Therefore x and y are adjacent by Corollary 3.5, a contradiction.

The part (5) is an immediate consequence of (3.2) and Proposition 3.6(2). \square

Proposition 3.7 and Theorem 3.8 may be summarized as follows.

Theorem 3.9. *Let D be a multipartite tournament with a non-clique partite set X_1 and U be the set of sinks in D (U is possibly vacuous). Then the adjacency matrix of $C_{1,2}(D)$ is in the form given in Figure 3.2.*

By Theorem 3.9, we have the following corollary.

Corollary 3.10. *Let D be a multipartite tournament with a non-clique partite set. Then the following are true:*

- (1) *each component of $C_{1,2}(D)$ has diameter of at most two;*
- (2) *D has no sinks if and only if $C_{1,2}(D)$ is connected;*
- (3) *each stable set of $C_{1,2}(D)$ is contained in at most two partite sets of D .*

$$\begin{array}{c}
U \quad F_2 \quad F_3 \quad \cdots \quad F_k \quad X_1^* \quad X_2 \quad X_3 \quad \cdots \quad X_k \\
\begin{array}{l}
U \\
F_2 \\
F_3 \\
\vdots \\
F_k \\
X_1^* \\
X_2 \\
X_3 \\
\vdots \\
X_k
\end{array}
\begin{pmatrix}
O & O & O & \cdots & O & O & O & O & \cdots & O \\
O & ? & J & \cdots & J & J & ? & J & \cdots & J \\
O & J & ? & \cdots & J & J & J & ? & \cdots & J \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
O & J & J & \cdots & ? & J & J & J & \cdots & ? \\
O & J & J & \cdots & J & J-I & J & J & \cdots & J \\
O & ? & J & \cdots & J & J & M & J & \cdots & J \\
O & J & ? & \cdots & J & J & J & J-I & \cdots & J \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
O & J & J & \cdots & ? & J & J & J & \cdots & J-I
\end{pmatrix}
\end{array}$$

Figure 3.2: The adjacency matrix of $C_{1,2}(D)$ for a multipartite tournament D with a non-clique partite set X_1 where U is the set of sinks in D (U is possibly vacuous); F_i , O , J , and I stand for the set of X_i -biased vertices, a zero matrix, a matrix of all 1's, and an identity matrix, respectively; $X_1^* = X_1 \setminus \left(\bigcup_{i=2}^k F_i \cup U\right)$; M is undetermined, yet, if D has a sink or $\{1, 2\}$ -stable set of size at least three in one partite set, then $M = J - I$; Blocks marked with ? are undetermined.

Proof. By Theorem 3.9, the adjacency matrix of $C_{1,2}(D)$ is in the form of the matrix given in Figure 3.2. Therefore it is easy to see from the matrix given in Figure 3.2 that any pair of vertices of each component is at distance at most two and so the part (1) is true. Moreover, $C_{1,2}(D)$ is connected if and only if $U = \emptyset$, which can be seen from the matrix given in Figure 3.2. By Theorem 3.9, U is the set of sinks in D . Therefore D has no sinks if only if $U = \emptyset$. Thus the part (2) is true.

Now we prove the part (3). Since D has a non-clique partite set, a stable set S of $C_{1,2}(D)$ intersects with at most one partite set among X_2, \dots, X_k by Theorem 3.8(2). Therefore S intersects with at most two partite sets of D . \square

Motivated by Corollary 3.10(3), we may ask the question “Given a multipartite tournament D , what is a biggest set among the $\{1, 2\}$ -stable sets that are not included in any partite set of D ?” Given a positive integer $m \geq 2$ and a tripartite tournament D of order $m + 1$ with $m - 1$ sinks constituting one partite set, $C_{1,2}(D)$ is isomorphic to K_2 with $m - 1$ isolated vertices and so a biggest $\{1, 2\}$ -stable set of D has size m . However, if a multipartite tournament D has no sinks, then the size of such a $\{1, 2\}$ -stable set of D is at most four, which is told by Theorem 3.12.

To justify Theorem 3.12, we need the following lemma.

Lemma 3.11. *Let D be a digraph having a directed cycle C of order l for some $l \in \{3, 4\}$ and X be a subset of $V(D)$. Suppose that each vertex u in $X \setminus V(C)$ has two out-neighbors u_1 and u_2 on C such that both (u_1, u_2) -section and (u_2, u_1) -section of C have length at most 2. Then each vertex in $X \setminus V(C)$ is adjacent to each vertex in X in $C_{1,2}(D)$.*

Proof. We take a vertex u in $X \setminus V(C)$. Let u_1 and u_2 be out-neighbors of u on C satisfying the given condition. Without loss of generality, we may assume that the (u_1, u_2) -section of C has length 2. Then u_2 is a $(1, 2)$ -step out-neighbor of u_1 and u . If $l = 3$, then u_1 is a common out-neighbor of u_2 and u . If $l = 4$, then u_1 is a $(1, 2)$ -step common out-neighbor of u and u_2 . Any vertex on C other than u_2 and u_1 shares u_2 or u_1 as an out-neighbor with u .

Let v be a vertex distinct from u in $X \setminus V(C)$. If u and v do not share a common out-neighbor on C , then $l = 4$ and u_2 is a $(1, 2)$ -step common out-neighbor of u and

v.

□

Theorem 3.12. *Let D be a multipartite tournament of order n with a $\{1, 2\}$ -stable set S which is not included in any partite set of D . Suppose that D has no sinks. Then $|S| \leq 4$. Especially, if $|S| = 4$, then the following are true:*

- (1) *there exist two partite sets X_1 and X_2 of D such that $|S \cap X_1| = |S \cap X_2| = 2$;*
- (2) *$n \geq 5$ and $C_{1,2}(D) \cong K_n - E(K_4)$.*

Proof. Let X_1, \dots, X_k be the partite sets of D and

$$\Lambda = \{i \mid S \cap X_i \neq \emptyset\}.$$

Since there is no partite set of D including S , $|\Lambda| \geq 2$. Suppose $|\Lambda| \geq 4$. We take four vertices in distinct partite sets of D . Then they induce the tournament T of order 4, so there exists a pair of vertices competing in T since T has four vertices and six arcs. Therefore $|\Lambda| \leq 3$. Suppose $|S| \geq 4$. Then there exists a partite set including at least two vertices in S . Therefore D has a non-clique partite set. Thus $|\Lambda| \leq 2$ by Corollary 3.10(3) and so $|\Lambda| = 2$. Hence $|S \cap X_1| = 2$ and $|S \cap X_2| = 2$ by (3) of the same corollary for some partite sets X_1 and X_2 of D , and so $|S| = 4$. Hence we have shown that $|S| \leq 4$. In addition, we have shown that if $|S| = 4$, then there exist two partite sets X_1 and X_2 of D such that $|S \cap X_1| = |S \cap X_2| = 2$ and so (1) of the “especially” part is true.

To show (2) of the “especially” part, suppose $|S| = 4$. As we have shown (1) of the “especially” part, there exist two partite sets X_1 and X_2 of D such that $|S \cap X_1| = |S \cap X_2| = 2$. Then D has a non-clique partite set. In addition, since $k \geq 3$, there exists the partite set X_3 with at least one vertex and so

$$n \geq 5.$$

We note that $S \cap X_1 := \{u_1, u_2\}$ and $S \cap X_2 := \{u_3, u_4\}$ are stable sets of size two in $C_{1,2}(D)$. Therefore neither u_1 nor u_2 is a common out-neighbor of u_3 and u_4 and

vice versa. Without loss of generality, we may assume

$$\{(u_1, u_3), (u_2, u_4), (u_3, u_2), (u_4, u_1)\} \subset A(D).$$

Therefore each of u_1 and u_2 , and each of u_3 and u_4 are X_2 -biased and X_1 -biased, respectively, by Proposition 3.6(1). Now, since D has a non-clique partite set, by Theorem 3.8(4),

$$u_1 \xrightarrow{*} u_3, \quad u_2 \xrightarrow{*} u_4, \quad u_3 \xrightarrow{*} u_2, \quad \text{and} \quad u_4 \xrightarrow{*} u_1.$$

Then the vertices in S form a directed cycle $C := u_1 \rightarrow u_3 \rightarrow u_2 \rightarrow u_4 \rightarrow u_1$ of order 4 in D . Take a vertex x in $V(D) \setminus V(C)$. If $x \in X_1$ or $x \in X_2$, then $\{u_3, u_4\} \subseteq N^+(x)$ or $\{u_1, u_2\} \subseteq N^+(x)$. If $x \notin X_1 \cup X_2$, then $V(C) \subseteq N^+(x)$. Therefore x has two out-neighbors y and z on C such that both (y, z) -section and (z, y) -section of C have length at most 2. Since x was arbitrarily taken from $V(D) \setminus V(C)$, we conclude that each vertex in $V(D) \setminus V(C)$ has two out-neighbors in S satisfying the condition given in Lemma 3.11. Therefore each vertex in $V(D) \setminus V(C)$ is adjacent to each vertex in $V(D)$ in $C_{1,2}(D)$. Then, since $S = \{u_1, u_2, u_3, u_4\}$ is a stable set, $C_{1,2}(D) \cong K_n - E(K_4)$. \square

Corollary 3.13. *Let D be a multipartite tournament. If there exists a $\{1, 2\}$ -stable set of size at least four, then D has a non-clique partite set.*

Proof. Suppose that D has a $\{1, 2\}$ -stable set S of size at least four. If D has a sink, then D has a non-clique partite set. Suppose D has no sinks. If there exists a partite set containing S , then D has a non-clique partite set. If there is no partite set containing S , then $|S| = 4$ and there exists a partite set X of D such that $|S \cap X| = 2$ by Theorem 3.12, and so D has a non-clique partite set. \square

Now we characterize the $(1, 2)$ -step competition graph of a multipartite tournament D if D has a sink or D has no sinks and $C_{1,2}(D)$ has a stable set S of size at least 3 included in one partite set of D . For simplicity, we assign a type to each multipartite tournament D whose $(1, 2)$ -step competition graph in the following way:

- Type I if D has a sink;
- Type II if D has no sinks and $C_{1,2}(D)$ has a stable set S of size at least 3 included in one partite set of D ;

Let D be a k -partite tournament of Type I or II for some $k \geq 3$ and U be the (possibly vacuous) set of sinks in D and F_i denote the set of X_i -biased vertices for each $2 \leq i \leq k$. By Theorem 3.9, except that among the vertices in F_i and that between F_i and X_i for each $2 \leq i \leq k$, the adjacency between two vertices in $C_{1,2}(D)$ is determined. Yet, Proposition 3.2 tells us that

- P1. the vertex set and edge set of the subgraph of $C_{1,2}(D)$ induced by F_i are covered by the set $\{N^-(v) \mid v \in X_i\}$ of cliques for each $2 \leq i \leq k$.

We fix $i \in \{2, \dots, k\}$ and take a vertex v in X_i . Take a vertex $w \in N^-(v) \cap F_i$. If v is the only out-neighbor of a vertex w , then v is not adjacent to w by Proposition 3.3. Now suppose that w has an out-neighbor other than v . Then it is easy to see that a sink u or a vertex in S is a $(1, 2)$ -step common out-neighbor of v and w . Since v was arbitrarily chosen from X_i , we may conclude that for each $2 \leq i \leq k$ and $v \in X_i$,

- P2. if a vertex w in $N^-(v) \cap F_i$ has an out-neighbor other than v , then v and w are adjacent in $C_{1,2}(D)$;
- P3. if v is the only out-neighbor of a vertex w in $N^-(v) \cap F_i$, then v and w are not adjacent in $C_{1,2}(D)$.

A vertex in $N^-(v) \cap F_i$ and a vertex y in X_i distinct from v have a sink u or a vertex in S as a $(1, 2)$ -step common out-neighbor by (3.2) and Proposition 3.6(2), so

- P4. each vertex in $N^-(v) \cap F_i$ and each vertex in X_i distinct from v are adjacent in $C_{1,2}(D)$.

If a graph G is the $(1, 2)$ -step competition graph of a multipartite tournament D of Type I or II, then G satisfies the properties given in Theorem 3.9 and P1-P4. Then we come up with a question,

“Is any graph satisfying these properties the (1, 2)-step competition graph of a multipartite tournament D of Type I or II?”

To answer this question, we consider the set \mathcal{G}_k^* of graphs G satisfying the *good* property stated as follows:

The vertex set of G can be partitioned into $\{X_1, \dots, X_k\}$ for some $k \geq 3$ so that the set U of isolated vertices or a stable set of size at least three is included in X_1 and there exist mutually disjoint subsets F_2, \dots, F_k of $X_1 \setminus U$ such that the adjacency matrix of G is in the form of the matrix given in Figure 3.2 and F_2, \dots, F_k satisfy the following *covering* condition: For each $2 \leq i \leq k$, we may assign an empty set or a clique $K_v \subseteq F_i$ to each vertex $v \in X_i$ so that

- (1) the vertex set and edge set of the subgraph of G induced by F_i are covered by $\mathcal{K}_i := \{K_v \mid v \in X_i\}$ in such a way that, for each vertex v in X_i , a vertex in K_v is adjacent to v if and only if there is another clique in \mathcal{K}_i which covers it; each vertex not in K_v is adjacent to v ;
- (2) each vertex in K_v and each vertex in $X_i \setminus \{v\}$ are adjacent in G ;

Example 3.14. The graph given in Figure 3.3 belongs to \mathcal{G}_3^* . To see why, we denote it by G . We partition the vertex set of G into $X_1 = \{x_1, x_2, x_3\}$, $X_2 = \{x_4\}$, and $X_3 = \{x_5\}$ so that the set $U := \{x_1\}$ of isolated vertices in G is included in X_1 . Now we let $F_2 = \{x_2, x_3\}$ and $F_3 = \emptyset$. Then the adjacency matrix of G is in the form of the matrix given in Figure 3.2 and so the good property is satisfied. Now we assign $\{x_2, x_3\}$ to x_4 and \emptyset to x_5 . Then the covering condition is satisfied. Therefore $G \in \mathcal{G}_3^*$.

Example 3.15. The graph G obtained by deleting the edge x_2x_3 from the one given in Figure 3.3 does not belong to \mathcal{G}_k^* for any $k \geq 3$. To reach a contradiction, suppose that G belongs to \mathcal{G}_k^* . Then $V(G)$ is partitioned into X_1, \dots, X_k for some $k \geq 3$ satisfying the conditions for a graph belonging to \mathcal{G}_k^* . The vertex x_1 is the only isolated vertex in G , so $U = \{x_1\} \subseteq X_1$. By the good property, there exist mutually disjoint subsets F_2, \dots, F_k of $X_1 \setminus U$. Since the size of maximal clique of

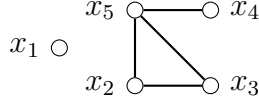


Figure 3.3: A graph belonging to \mathcal{G}_3^*

G is 2, $k = 3$ by the form of the adjacency matrix of G given in Figure 3.2. For the same reason, x_5 belongs to X_2 or X_3 and $|X_2| = |X_3| = 1$. Without loss of generality, $x_5 \in X_2$ and $x_4 \in X_3$. Then, by the form of the matrix in Figure 3.2, $\{x_2, x_3\} = F_2$ or $\{x_2, x_3\} = F_3$. Since x_4 is adjacent to neither x_2 nor x_3 , $\{x_2, x_3\} \neq F_2$. Therefore $\{x_2, x_3\} = F_3$. Then we must assign a clique to x_4 to cover the vertices F_3 by the covering condition(1), which is impossible. Hence we have shown $G \notin \mathcal{G}_k^*$.

Let D be a k -partite tournament of Type I or II for some $k \geq 3$ and v be a vertex in a partite set X of D not containing a sink or a stable set of $C_{1,2}(D)$ with size at least three. Then we take $N^-(v) \cap F$ as K_v for the pure in-neighborhood F of X and so

(\star) the $(1, 2)$ -step competition graph of D belongs to \mathcal{G}_k^* .

Suppose that a graph G belonging to \mathcal{G}_k^* is the $(1, 2)$ -step competition graph of some k -partite tournament D for some integer $k \geq 3$. If D is a multipartite tournament of Type I or II, then the answer to the above question is yes.

Now we answer the proposed question.

Theorem 3.16. *A graph G is the $(1, 2)$ -step competition graph of a k -partite tournament of Type I or II for some $k \geq 3$ if and only if $G \in \mathcal{G}_k^*$*

Proof. The “only if” part is immediately true by (\star).

To show the “if” part, suppose $G \in \mathcal{G}_k^*$. Then, by the good property, $V(G)$ is partitioned into $\{X_1, \dots, X_k\}$ for some $k \geq 3$ so that a stable set of size at least three or the set U of isolated vertices in G is included in X_1 and there exist mutually disjoint subsets F_2, \dots, F_k so that the adjacency matrix of G is in the form of the matrix given in Figure 3.2 and F_2, \dots, F_k satisfy the covering condition.

Given the empty graph with the vertex set $V(G)$, we add the arcs in the following way: We first add the following arc sets whether or not $U = \emptyset$:

$$\{(K_v, v) \mid v \in \bigcup_{i=2}^k X_i, K_v \neq \emptyset\} \quad (3.3)$$

and

$$\bigcup_{i=2}^k \{(v, F_i \setminus \{K_v\}) \mid K_v \subseteq F_i, v \in X_i\} \quad (3.4)$$

where K_v is assigned to v in the covering condition (1);

$$\bigcup_{i=2}^k \{(v, w) \mid v \in \bigcup_{j \neq i} X_j, w \in F_i\}; \quad (3.5)$$

$$\left\{ (v, w) \mid v \in X_1 \setminus \left(U \cup \bigcup_{i=2}^k F_i \right), w \in \bigcup_{i=2}^k X_i \right\}. \quad (3.6)$$

If $U \neq \emptyset$, then we add the arc set

$$\{(v, u) \mid v \in \bigcup_{i=2}^k X_i, u \in U\}. \quad (3.7)$$

Other than those arcs in the above sets, we add arcs arbitrarily oriented to complete a k -partite tournament D with the partite sets X_1, \dots, X_k . By (3.3) and (3.5), F_i is the pure in-neighborhood of X_i for each $2 \leq i \leq k$.

If $U \neq \emptyset$, then D has a sink by (3.7) and so D is of Type I.

Suppose $U = \emptyset$. Then G has a stable set S of size l for some integer $l \geq 3$. By the good property, $S \subseteq \bigcup_{i=2}^k F_i$. Then, by the form of the adjacency matrix of G given in Figure 3.2, there exists a subset F_j including S for some integer $j \in \{2, \dots, k\}$. By permuting partite sets, we may assume

$$S \subseteq F_2. \quad (3.8)$$

Then, since F_2 is the pure in-neighborhood of X_2 , $N^+(s) \subset X_2$ for each vertex s in S by (3.8). Since S is a stable set of G , there is no clique K_v in F_2 containing any pair of vertices of S and so, by (3.3) and (3.4), any pair of vertices of S has no common out-neighbor in X_2 . Therefore S forms a stable set in $C_{1,2}(D)$ by Proposition 3.2. Thus D is of Type II.

The (possibly vacuous) set U is the set of isolated vertices in $C_{1,2}(D)$. To see why, take an isolated vertex x in $C_{1,2}(D)$ if any. Suppose, to the contrary, that x is not a sink in D . If $x \in V(D) \setminus X_1$, then x is not isolated in $C_{1,2}(D)$ by (3.7) or Proposition 3.6(3). Therefore $x \in X_1$. Since x is not a sink, there exists an out-neighbor x' of x in $V(D) \setminus X_1$ and so a sink u in U or a vertex s in S is an out-neighbor of x' . Let y be an out-neighbor in $U \cup S$ of x' . Then, since $k \geq 3$, there exists a vertex z distinct from x' in $V(D) \setminus X_1$ such that y is a common out-neighbor of z and x' . Since there is a (x, y) -directed path of length 2 in D , x is adjacent to z in $C_{1,2}(D)$, which is a contradiction. Thus x is a sink in $C_{1,2}(D)$ and so $x \in U$. Hence the set of isolated vertices in $C_{1,2}(D)$ is included in U . By (3.7), there is no arc outgoing from a vertex in U . Consequently, U is the set of isolated vertices in $C_{1,2}(D)$.

Now we show that G is isomorphic to $C_{1,2}(D)$. We first show that the adjacency matrix of $C_{1,2}(D)$ is in the form of the matrix given Figure 3.2, that is,

- $\left(X_1 \setminus \left(\bigcup_{i=2}^k F_i \cup U \right) \right) \cup \bigcup_{i=2}^k X_i$ forms a clique in $C_{1,2}(D)$;
- The complete multipartite graph with the nonempty sets among $X_1 \setminus \left(\bigcup_{i=2}^k F_i \cup U \right)$, F_2, \dots, F_k as the partite sets is a subgraph of $C_{1,2}(D)$;
- If $F_i \neq \emptyset$ for some $i \in \{2, \dots, k\}$, then the complete bipartite graph with the partite sets F_i and $\bigcup_{j \in \{2, \dots, k\} \setminus \{i\}} X_j$ is a subgraph of $C_{1,2}(D)$.

By (3.6) and Corollary 3.5, $X_1 \setminus \left(\bigcup_{i=2}^k F_i \cup U \right)$ forms a clique in $C_{1,2}(D)$. By (3.7), (3.8), and Proposition 3.6(2),

(#) any pair of vertices in $\bigcup_{i=2}^k X_i$ has a common out-neighbor in U or S .

Therefore $\bigcup_{i=2}^k X_i$ forms a clique in $C_{1,2}(D)$. Moreover, any pair of a vertex in $X_1 \setminus$

$\left(\bigcup_{i=2}^k F_i \cup U\right)$ and a vertex in $\bigcup_{i=2}^k X_i$ has a $(1, 2)$ -step common out-neighbor. Thus $\left(X_1 \setminus \left(\bigcup_{i=2}^k F_i \cup U\right)\right) \cup \bigcup_{i=2}^k X_i$ forms a clique in $C_{1,2}(D)$.

Since K_v were chosen to have $\{K_v \mid v \in X_i\}$ cover F_i for each $i = 2, \dots, n$, the nonempty sets among $X_1 \setminus \left(\bigcup_{i=2}^k F_i \cup U\right)$, F_2, \dots, F_k form the partite sets of complete multipartite graph contained in $C_{1,2}(D)$ by Proposition 3.6(2), (3.3), (3.6), and Corollary 3.5. For the same reason, each sink or each vertex in S is a 2-step out-neighbor of each vertex in F_i if $F_i \neq \emptyset$. Thus, by (\sharp) and Theorem 3.8(3), the complete bipartite graph with the partite sets F_i and $\bigcup_{j \in \{2, \dots, k\} \setminus \{i\}} X_j$ is a subgraph of $C_{1,2}(D)$ if $F_i \neq \emptyset$. Hence we have shown that the adjacency matrix of $C_{1,2}(D)$ is in the form of the matrix given in Figure 3.2.

Now consider the adjacency of G not covered by the matrix given in Figure 3.2. We first show that two vertices in F_i are adjacent in G if and only if they are adjacent in $C_{1,2}(D)$ for each $2 \leq i \leq k$. Suppose that two vertices v and w in F_i are adjacent in G for some $i \in \{2, \dots, k\}$. Then v and w belong to K_x for some $x \in F_i$. By (3.3), v and w are adjacent in $C_{1,2}(D)$. Now suppose that two vertices v and w in F_i are not adjacent in G for some $i \in \{2, \dots, k\}$. Then there is no K_x for any $x \in X_i$ such that $\{v, w\} \subseteq K_x$. Therefore v and w has no common out-neighbor in D by (3.3), (3.4) and (3.5). Thus they are not adjacent in $C_{1,2}(D)$ by Proposition 3.2.

Now we show that a vertex x in F_i and a vertex y in X_i are adjacent in G if and only if they are adjacent in $C_{1,2}(D)$ for each $2 \leq i \leq k$. Suppose that x is adjacent to y in G . Then there exists a vertex z in $X_i \setminus \{y\}$ such that $x \in K_z$ by the covering condition(1). Therefore $(x, z) \in A(D)$ by (3.3). Thus u or a vertex in S is a $(1, 2)$ -step common out-neighbor of x and y by (3.7) and Proposition 3.6(2). Hence x and y are adjacent in $C_{1,2}(D)$. Now we suppose x is not adjacent to y in G . Then $x \in K_y$ and $x \notin K_z$ for any $z \in X_i \setminus \{y\}$ by the covering condition(1). Therefore y is the only out-neighbor of x by (3.3), (3.4), and (3.5). Thus x is not adjacent to y in $C_{1,2}(D)$ by Proposition 3.3. Hence we have shown that G is isomorphic to $C_{1,2}(D)$. \square

3.3 $C_{1,2}(D)$ without a non-clique partite set of D

In this section, we study a multipartite tournament without a non-clique partite sets, that is, a multipartite tournament each of whose partite sets is clique in its $(1, 2)$ -step competition graph. By Corollary 3.13, each $\{1, 2\}$ -stable set of a multipartite tournament without a non-clique partite set has size at most three. In each case of the sizes two and three, we will characterize the $(1, 2)$ -step competition graph of a multipartite tournament each of whose partite sets is clique with a maximum $\{1, 2\}$ -stable set of a given size to come up with Theorems 3.18 and 3.19, whichever applicable.

Lemma 3.17. *Let D be a multipartite tournament without sinks and a non-clique partite set. Suppose D has a set $S = \{u_1, u_2\}$ with $u_1 \xrightarrow{*} u_2$, For any $\{1, 2\}$ -stable set V of size 2 with $V \cap S = \emptyset$, then the following are true:*

- (1) $V = \{v_1, v_2\}$ with $v_1 \xrightarrow{*} v_2$;
- (2) v_1 and u_1 belong to the same partite set of D , and v_2 and u_2 belong to distinct partite sets of D .

Proof. Let X_1 and X_2 be partite sets of D containing u_1 and u_2 , respectively. Then $N^-(u_1) = V(D) \setminus (X_1 \cup \{u_2\})$. Suppose that D has a $\{1, 2\}$ -set $\{v_1, v_2\}$ with $\{v_1, v_2\} \cap S = \emptyset$. If $\{v_1, v_2\} \cap X_1 = \emptyset$, then u_1 is a common out-neighbor of v_1 and v_2 , a contradiction. Therefore $\{v_1, v_2\} \cap X_1 \neq \emptyset$. Then, every partite set of D is clique, not both v_1 and v_2 belong to X_1 . Without loss of generality, we may assume $v_1 \in X_1$ and $v_2 \notin X_1$. Then, v_1 and u_1 belong to the same partite set, so they $\{1, 2\}$ -compete. Since $v_2 \neq u_2$, u_1 is an out-neighbor of v_2 . If v_1 has an out-neighbor distinct from v_2 , then, by Proposition 3.4, v_1 and v_2 $\{1, 2\}$ -compete, a contradiction. Therefore

$$v_1 \xrightarrow{*} v_2$$

since v_1 is not a sink. Then, since $u_1 \xrightarrow{*} u_2$ and $u_2 \neq v_2$, u_1 and v_1 do not compete and so u_1 and v_1 $(1, 2)$ -compete. We also note that u_1 and v_1 are X_2 -biased and the

partite set X containing v_2 . Then, by Proposition 3.2, $X \neq X_2$. Therefore we have shown that the parts are valid. \square

The *complement* of a graph G is a graph \bar{G} on the same vertices such that two distinct vertices of \bar{G} are adjacent if and only if they are not adjacent in G . A tree containing exactly two non-pendant vertices is called a *double-star*. A *caterpillar* is a tree in which all the vertices are within distance 1 of a central path.

Theorem 3.18. *Suppose that a multipartite tournament D has a maximum $\{1, 2\}$ -stable set of size two and every partite set of D is clique. Then the complement of $C_{1,2}(D)$ is one of the following types:*

- A. *a star graph with isolated vertices;*
- B. *a double-star graph with isolated vertices;*
- C. *a disjoint union of at least two star graphs with isolated vertices;*
- D. *a caterpillar which has at least one internal vertex of degree 2 with isolated vertices.*

Proof. Suppose that D has sinks. Since every partite set of D is clique, D has a sink constituting a trivial partite set of D and so $C_{1,2}(D)$ is isomorphic to $K_{|V(D)|-1}$ with an isolated vertex. Therefore the complement $\bar{C}_{1,2}(D)$ of $C_{1,2}(D)$ is of Type A.

Now we assume that D has no sinks. Let $\{u_1, u_2\}$ be a $\{1, 2\}$ -stable set in D , and X_1 and X_2 be partite sets of D containing u_1 and u_2 , respectively. Without loss of generality, we may assume

$$(u_1, u_2) \in A(D).$$

If $C_{1,2}(D) - \{u_1, u_2\}$ is a complete graph, then $\bar{C}_{1,2}(D)$ is of Type A of Type B. Now we suppose that $C_{1,2}(D) - \{u_1, u_2\}$ is not a complete graph. Then there exist two nonadjacent vertices v_1 and v_2 in $C_{1,2}(D) - \{u_1, u_2\}$.

To show $u_1 \xrightarrow{*} u_2$, we suppose that $u_1 \not\xrightarrow{*} u_2$. Then u_2 is not the only out-neighbor of u_1 . Consequently, by Proposition 3.4, there exists a partite set X' such

that $\emptyset \neq N^+(u_1) \setminus \{u_2\} \subseteq X'$ and $\emptyset \neq N^+(u_2) \setminus \{u_1\} \subseteq X'$. Then $X' \neq X_1$ and $X' \neq X_2$. Thus $N^+(u_1) \cap X_2 = \{u_2\}$ and $N^+(u_2) \cap X_1 = \emptyset$.

Now we show that each of v_1 and v_2 has u_1 or u_2 as an out-neighbor. If $v_1 \in X_1$, then u_2 is an out-neighbor of v_1 since $N^+(u_2) \cap X_1 = \emptyset$. If $v_1 \in X_2$, then u_1 is an out-neighbor of v_1 since $N^+(u_1) \cap X_2 = \{u_2\}$. If $v_1 \notin X_1 \cup X_2$, then at least one of u_1 and u_2 is an out-neighbor of v_1 since u_1 and u_2 have no common out-neighbor. Therefore v_1 has u_1 or u_2 as an out-neighbor. By symmetry, v_2 has u_1 or u_2 as an out-neighbor. Thus v_1 and v_2 $\{1, 2\}$ -compete and we reach a contradiction. Hence

$$u_1 \xrightarrow{*} u_2.$$

For simplicity, we call a $\{1, 2\}$ -stable set $\{x_1, x_2\}$ with $x_1 \xrightarrow{*} x_2$ *picky*. Then $\{u_1, u_2\}$ is picky. Therefore $\{v_1, v_2\}$ is also picky by Lemma 3.17(1). Without loss of generality, we may assume

$$v_1 \xrightarrow{*} v_2.$$

Then $v_1 \in X_1$ and $v_2 \notin X_2$ by Lemma 3.17(2). We may assume $v_2 \in X_3$ where X_3 is a partite set of D . Then, since $u_1 \xrightarrow{*} u_2$, any pair of vertices in $V(D) \setminus X_1$ distinct from $\{u_2, v_2\}$ has a common out-neighbor u_1 or v_1 . Moreover, every partite set of D is clique, X_1 forms a clique in $C_{1,2}(D)$. Therefore

(†) any $\{1, 2\}$ -stable set of size 2 distinct from $\{u_2, v_2\}$ intersects with both X_1 and $V(D) \setminus X_1$.

Case 1. u_2 and v_2 $\{1, 2\}$ -compete in D . Then, by (†),

(§) any $\{1, 2\}$ -stable set of size 2 intersects with both X_1 and $V(D) \setminus X_1$.

Let k be the maximum number of disjoint $\{1, 2\}$ -stable sets with size 2 of D .

Subcase 1. $k \geq 3$. Let $\mathcal{S} = \{S_1, S_2, \dots, S_k\}$ where S_i is a $\{1, 2\}$ -stable set with size 2 of D for each $1 \leq i \leq k$ and S_1, \dots, S_k are mutually disjoint. Without loss of generality, we may assume $S_1 = \{u_1, u_2\}$ and $S_2 = \{v_1, v_2\}$. Since $k \geq 3$, $S_3 \in \mathcal{S}$. Then S_3 is picky by Lemma 3.17(1). We denote S_3 by $\{s_{3,1}, s_{3,2}\}$ with $s_{3,1} \xrightarrow{*} s_{3,2}$. Since S_1 is picky and $S_1 \cap S_3 = \emptyset$, Then $s_{3,1} \in X_1$ and $s_{3,2} \notin X_2$ by Lemma 3.17(2).

Since S_2 is also picky and $S_2 \cap S_3 = \emptyset$, $s_{3,2} \notin X_3$ by the same lemma. Therefore we may assume $s_{3,2} \in X_4$ where X_4 is a partite set of D . Inductively, we may let $S_i = \{s_{i,1}, s_{i,2}\}$ so that

$$s_{i,1} \xrightarrow{*} s_{i,2}, \quad s_{i,1} \in X_1, \quad \text{and} \quad s_{i,2} \in X_{i+1} \quad (3.9)$$

where X_{i+1} is a partite set of D for each $1 \leq i \leq k$.

Now take a $\{1, 2\}$ -stable set $\{y_1, y_2\}$ of D with $\{y_1, y_2\} \notin \mathcal{S}$. Then, by the maximality of \mathcal{S} , there exists a pair $\{s_{j,1}, s_{j,2}\}$ in \mathcal{S} such that $\{s_{j,1}, s_{j,2}\} \cap \{y_1, y_2\} \neq \emptyset$ for some $j \in \{1, \dots, k\}$. Without loss of generality, we may assume $y_1 \in X_1$ and $y_2 \in V(D) \setminus X_1$ by (§). We first suppose $\{s_{j,1}, s_{j,2}\} \cap \{y_1, y_2\} = \{s_{j,1}\}$, that is, $y_1 = s_{j,1}$ and $y_2 \neq s_{j,2}$. Then, y_2 has at least $k - 1$ out-neighbors in $T = \{s_{1,1}, s_{2,1}, \dots, s_{k,1}\}$ by (3.9). In addition, for each $1 \leq i \leq k$ except $i = j$, there exists a directed path $P_i := s_{j,1} \rightarrow s_{j,2} \rightarrow s_{i,1}$ which does not traverse y_2 . Therefore $s_{1,1}, s_{2,1}, \dots, s_{k,1}$ except $s_{j,1}$ are 2-step out-neighbors of $s_{j,1}$ obtained by P_1, \dots, P_k . Since $T \supset (N^+(y_2) \cap T) \cup (\{s_{1,1}, s_{2,1}, \dots, s_{k,1}\} \setminus \{s_{j,1}\})$,

$$\begin{aligned} k = |T| &\geq |(N^+(y_2) \cap T)| + |\{s_{1,1}, s_{2,1}, \dots, s_{k,1}\} \setminus \{s_{j,1}\}| \\ &\quad - |(N^+(y_2) \cap T) \cap (\{s_{1,1}, s_{2,1}, \dots, s_{k,1}\} \setminus \{s_{j,1}\})| \\ &\geq 2k - 2 - |(N^+(y_2) \cap T) \cap (\{s_{1,1}, s_{2,1}, \dots, s_{k,1}\} \setminus \{s_{j,1}\})|. \end{aligned}$$

Since $k \geq 3$,

$$|N^+(y_2) \cap T \cap (\{s_{1,1}, s_{2,1}, \dots, s_{k,1}\} \setminus \{s_{j,1}\})| \geq 1.$$

Therefore there exists a $(1, 2)$ -step common out-neighbor of $s_{j,1}$ and y_2 , which belongs to $\{s_{1,1}, s_{2,1}, \dots, s_{k,1}\} \setminus \{s_{j,1}\}$, a contradiction. Thus $\{s_{j,1}, s_{j,2}\} \cap \{y_1, y_2\} = \{s_{j,2}\}$, that is, $y_1 \neq s_{j,1}$ and $y_2 = s_{j,2}$. We will claim that $s_{j,2}$ is the only vertex in $\{s_{1,2}, s_{2,2}, \dots, s_{k,2}\}$ which is not adjacent to y_1 in $C_{1,2}(D)$. Since $k \geq 3$, there exist two vertices $s_{j_1,1}$ and $s_{j_2,1}$ for some $j_1, j_2 \in \{1, \dots, k\} \setminus \{j\}$ and, by (3.9), every vertex except $s_{j_1,2}$ and $s_{j_2,2}$ is an in-neighbor of $s_{j_1,1}$ and $s_{j_2,1}$. Thus $s_{j_1,1}$ and $s_{j_2,1}$ are out-neighbors of $s_{j,2}$ and so $s_{j,2}$ has an out-neighbor distinct from y_1 in X_1 . Since

D has no sinks, y_1 has an out-neighbor. If y_1 has an out-neighbor distinct from $s_{j,2}$, then y_1 and $s_{j,2}$ are adjacent by Corollary 3.5, which is impossible. Therefore $s_{j,2}$ is the out-neighbor of y_1 . Thus y_1 is not adjacent to $s_{j,2}$ by Proposition 3.3. Since $s_{j,1} \xrightarrow{*} s_{j,2}$ by (3.9), $s_{j,1}$ is a common out-neighbor of $s_{1,2}, s_{2,2}, s_{3,2}, \dots, s_{k,2}$ except $s_{j,2}$. Then, since $y_1 \neq s_{j,1}$, $s_{j,2}$ is a $(1, 2)$ -step common out-neighbor of y_1 and $s_{i,2}$ for each $1 \leq i \leq k$ except $i = j$. Thus $s_{j,2}$ is the only vertex in $\{s_{1,2}, s_{2,2}, \dots, s_{k,2}\}$ which is not adjacent to y_1 in $C_{1,2}(D)$.

Since $\{y_1, y_2\}$ and j were arbitrarily chosen, we may conclude that, for every edge except the edges $s_{1,1}s_{1,2}, \dots, s_{k,1}s_{k,2}$, the vertex $s_{j,2}$ for some $j \in \{1, \dots, k\}$ is the only vertex incident to it in $\bar{C}_{1,2}(D)$. This implies that $\bar{C}_{1,2}(D)$ is a disjoint union of k star graphs whose centers are $s_{1,2}, s_{2,2}, \dots, s_{k,2}$ with some isolated vertices and so is of Type C.

Subcase 2. $k = 2$. If $\{u_1, u_2\}$ and $\{v_1, v_2\}$ are the only $\{1, 2\}$ -stable sets in D , then $\bar{C}_{1,2}(D)$ is of Type C. Suppose that there exists a $\{1, 2\}$ -stable set $\{w_1, w_2\}$ in D distinct from $\{u_1, u_2\}$ and $\{v_1, v_2\}$. Then, since $k = 2$, $\{w_1, w_2\} \cap \{u_1, u_2, v_1, v_2\} \neq \emptyset$. Without loss of generality, we may assume $w_1 \in X_1$ and $w_2 \in V(D) \setminus X_1$ by (§). We first suppose $\{w_1, w_2\} \subset \{u_1, u_2, v_1, v_2\}$. Then, either $w_1 = u_1$ and $w_2 = v_2$ or $w_1 = v_1$ and $w_2 = u_2$. We consider the case where $w_1 = u_1$ and $w_2 = v_2$. Then u_1 and v_2 are not adjacent. If $(v_2, u_2) \in A(D)$, then u_2 is a common out-neighbor of u_1 and v_2 , a contradiction. Therefore $(u_2, v_2) \in A(D)$. Thus v_2 is a common out-neighbor of v_1 and u_2 and so v_1 and u_2 are adjacent. In case $w_1 = v_1$ and $w_2 = u_2$, we may conclude that v_2 and u_1 are adjacent by a similar argument. Thus we have shown that

(P1) if $\{w_1, w_2\} \subset \{u_1, u_2, v_1, v_2\}$, then exactly one of $\{u_1, v_2\}$ and $\{u_2, v_1\}$ is a $\{1, 2\}$ -stable set in D .

Now we suppose $\{w_1, w_2\} \not\subset \{u_1, u_2, v_1, v_2\}$. Then $|\{w_1, w_2\} \cap \{u_1, u_2, v_1, v_2\}| = 1$. To the contrary, suppose $\{w_1, w_2\} \cap \{u_1, u_2, v_1, v_2\} = \{w_1\}$. Then $w_1 = u_1$ or $w_1 = v_1$. Without loss of generality, we may assume $w_1 = u_1$. We note that $u_1 \xrightarrow{*} u_2$ and $v_1 \xrightarrow{*} v_2$. Then v_1 is an out-neighbor of u_2 . Since $w_2 \in V(D) \setminus X_1$ and $w_2 \neq v_2$, v_1 is an out-neighbor of w_2 . Then, since $w_2 \neq u_2$, v_1 is a $(1, 2)$ -step common out-neighbor

of u_1 and w_2 , a contradiction. Therefore $\{w_1, w_2\} \cap \{u_1, u_2, v_1, v_2\} = \{w_2\}$. Then $w_2 = u_2$ or $w_2 = v_2$. In addition, $w_1 \neq u_1$ and $w_1 \neq v_1$. Thus $v_1 \in N^+(u_2) \setminus \{w_1\}$ and $u_1 \in N^+(v_2) \setminus \{w_1\}$ (be reminded that $v_1 \in N^+(u_2)$ and $u_1 \in N^+(v_2)$). Since w_1 is a non-sink vertex, $N^+(w_1) \neq \emptyset$. Then, since u_2 and v_2 are distinct, $N^+(w_1) \setminus \{u_2\} \neq \emptyset$ or $N^+(w_1) \setminus \{v_2\} \neq \emptyset$. Therefore w_1 is adjacent to u_2 or v_2 in $C_{1,2}(D)$ by Corollary 3.5. Then, since $w_2 = u_2$ or $w_2 = v_2$, exactly one of $\{w_1, u_2\}$ and $\{w_1, v_2\}$ is a $\{1, 2\}$ -stable set in D . Hence we have shown that

(P2) if $\{w_1, w_2\} \not\subset \{u_1, u_2, v_1, v_2\}$, then $\{w_1, w_2\} \cap \{u_1, u_2, v_1, v_2\} = \{w_2\}$ and exactly one of $\{w_1, u_2\}$ and $\{w_1, v_2\}$ is a $\{1, 2\}$ -stable set in D .

Since $\{w_1, w_2\}$ was arbitrarily chosen, we may conclude that, for every edge e except u_1u_2, v_1v_2 in $\bar{C}_{1,2}(D)$, (i) if the end points of e are contained in $\{u_1, u_2, v_1, v_2\}$, then the subgraph induced by $\{u_1, u_2, v_1, v_2\}$ is an induced path of length 3 by (P1); (ii) otherwise, one end point u of e is contained in $\{u_1, u_2, v_1, v_2\}$ and the other end point v of e must be adjacent to exactly one of u_2 and v_2 , and not adjacent to u_1 and v_1 in $\bar{C}_{1,2}(D)$ by (P2). Hence $\bar{C}_{1,2}(D)$ is of Type C or Type D. Especially, if the latter case holds, then u_1 or v_1 is an internal vertex with degree 2 in $\bar{C}_{1,2}(D)$.

Case 2. u_2 and v_2 do not $\{1, 2\}$ -compete in D . Suppose, to the contrary, that there exists a $\{1, 2\}$ -stable set $\{w_1, w_2\}$ in $V(D) \setminus \{u_2, v_2\}$. Then, by (\dagger), we may assume $w_1 \in X_1$ and $w_2 \in V(D) \setminus X_1$. Since $w_2 \neq v_2$, v_1 is an out-neighbor of w_2 in D . If $w_1 = u_1$, then v_1 is a $(1, 2)$ -step common out-neighbor of w_1 and w_2 since $u_1 \rightarrow u_2 \rightarrow v_1$ is a directed path in D , a contradiction. Therefore $w_1 \neq u_1$. Then, since $w_2 \neq u_2$, $\{w_1, w_2\}$ is a $\{1, 2\}$ -stable set in $V(D) \setminus \{u_1, u_2\}$, so, by Lemma 3.17, $w_1 \xrightarrow{*} w_2$. Therefore w_1 is a common out-neighbor of u_2 and v_2 in D , which is a contradiction to the case assumption. Thus there exists no $\{1, 2\}$ -stable set of size 2 in $V(D) \setminus \{u_2, v_2\}$, that is, $C_{1,2}(D) - \{u_2, v_2\}$ is complete. Then, since u_2 and v_2 do not $\{1, 2\}$ -compete in D by the case assumption, u_2 and v_2 are adjacent in $\bar{C}_{1,2}(D)$ and so they are non-pendant vertices (be reminded that u_1 is adjacent to u_2 and v_1 is adjacent to v_2 in $\bar{C}_{1,2}(D)$). Therefore $\bar{C}_{1,2}(D)$ is of Type B. \square

Theorem 3.19. *Let D be a multipartite tournament of order n with a $\{1, 2\}$ -stable set S which is not included in any partite set of D and t be the number of partite*

sets which intersect with S . Then $t \leq 3$. Especially, if $t = 3$, then $|S| = 3$ and one of the following is true:

- (a) $C_{1,2}(D) \cong K_n - E(K_3)$;
- (b) $n \geq 4$ and $C_{1,2}(D) \cong K_n - (E(K_3) \cup E(K_{1,l}))$ for a positive integer $l \leq n - 3$ where the center of $K_{1,l}$ is a vertex v such that $V(K_3) \cap V(K_{1,l}) = \{v\}$.

Proof. Suppose $t \geq 4$. Then $|S| \geq 4$ and, by Corollary 3.10(3), every partite set of D is clique. Therefore $|S| \leq 3$ by Corollary 3.13, which is a contradiction. Thus $t \leq 3$.

To show the ‘‘especially part’’, suppose $t = 3$. If $|S| \geq 4$, then D has a non-clique partite set, which contradicts Corollary 3.10(3). Therefore $|S| \leq 3$. Since $t = 3$, $|S| = 3$. Let $S = \{u_1, u_2, u_3\}$ and X_1, X_2 , and X_3 be the partite sets of D with $u_i \in X_i$ for each $1 \leq i \leq 3$. Since S is a $\{1, 2\}$ -stable set in D , the vertices in S form a directed cycle C of order 3 in D . Without loss of generality, we may assume

$$\{(u_1, u_2), (u_2, u_3), (u_3, u_1)\} \subset A(D).$$

Case 1. $d^+(u_i) = 1$ for each $1 \leq i \leq 3$. Then each vertex in $V(D) \setminus V(C)$ has at least two out-neighbors in $V(C)$. Therefore each vertex in $V(D) \setminus V(C)$ is adjacent to the vertices in $V(D)$ in $C_{1,2}(D)$ by Lemma 3.11. Thus $C_{1,2}(D)$ contains a subgraph isomorphic to $K_n - E(K_3)$. Then, since S is a stable set in $C_{1,2}(D)$, $C_{1,2}(D) \cong K_n - E(K_3)$.

Case 2. $d^+(u_j) \neq 1$ for some $j \in \{1, 2, 3\}$. Then $d^+(u_j) \geq 2$. Without loss of generality, we may assume $j = 1$. Then, since $(u_1, u_2) \in A(D)$ and $(u_2, u_3) \in A(D)$,

$$\emptyset \neq N^+(u_1) \setminus \{u_2\} \subseteq X_3 \tag{3.10}$$

by Proposition 3.4 and so

$$n \geq 4.$$

We first show that u_1 is the only vertex on C of outdegree at least 2. Suppose that

u_t has outdegree at least 2 for some $t \in \{2, 3\}$. Then

$$N^+(u_t) \setminus \{u_1\} \neq \emptyset$$

and there exists a partite set X such that

$$(N^+(u_1) \setminus \{u_t\} \cup N^+(u_t) \setminus \{u_1\}) \subseteq X$$

by Proposition 3.4. By (3.10), $X = X_3$. Therefore $t \neq 3$ and so $t = 2$. Then $d^+(u_2) \geq 2$. Since $N^+(u_2) = N^+(u_2) \setminus \{u_1\} \subseteq X_3$, there exists a vertex $x \in X_3 \setminus \{u_3\}$ belonging to $N^+(u_2)$. Then $(u_1, x) \in A(D)$ or $(x, u_1) \in A(D)$. If $(u_1, x) \in A(D)$, then x is a common out-neighbor of u_1 and u_2 , a contradiction. Therefore $(x, u_1) \in A(D)$ and so u_1 is a $(1, 2)$ -step common out-neighbor of u_2 and u_3 , a contradiction. Thus u_1 is the only vertex on C of outdegree at least 2. Hence

$$u_2 \overset{*}{\rightarrow} u_3 \quad \text{and} \quad u_3 \overset{*}{\rightarrow} u_1. \tag{3.11}$$

We denote $N^+(u_1) \setminus \{u_2\}$ by N . Then $N \neq \emptyset$. By (3.10) and (3.11), each vertices in $(V(D) \setminus N) \setminus V(C)$ has at least two out-neighbors in $V(C)$. Since the length of C is 3, each vertex in $(V(D) \setminus N) \setminus V(C)$ is adjacent to the vertices in $V(D) \setminus N$ in $C_{1,2}(D)$ by Lemma 3.11.

Now we will show that u_2 is the only vertex nonadjacent to each vertex in N . Take $v \in N$. Then $v \in X_3$. Since $u_2 \overset{*}{\rightarrow} u_3$, v and u_2 have no common out-neighbor in D . In addition, u_1 is the only two-step out-neighbor of u_2 by (3.11), and $v \in N^+(u_1)$. Therefore u_2 is not adjacent to v in $C_{1,2}(D)$. On the other hand, by (3.10), u_2 is a common out-neighbor of u_1 and v and it is a $(1, 2)$ -step common out-neighbor of v and u_3 . Therefore v is adjacent to u_1 and u_3 in $C_{1,2}(D)$. Moreover, since each vertex in $V(D) \setminus V(C)$ has at least one out-neighbor in $V(C)$ by (3.11), v is adjacent to each vertex in $V(D) \setminus V(C)$. Therefore u_2 is the only vertex nonadjacent to v in $C_{1,2}(D)$. Since v was arbitrarily chosen in N , u_2 is the only vertex nonadjacent to each vertex in N . Then, since S is a stable set, $C_{1,2}(D)$ is the graph obtained from the complete graph with the vertex set $V(D)$ by deleting the edges both of whose ends belong to

$V(C)$ and the ones on the star graph with vertex set $N \cup \{u_2\}$ having u_2 as a center. Thus $C_{1,2}(D) \cong K_n - (E(K_3) \cup E(K_{1,l}))$. \square

Remark 3.20. Given a multipartite tournament D without a non-clique partite set, each stable set of $C_{1,2}(D)$ has size at most three by Corollary 3.13. Thus Theorem 3.18 and the “especially” part of Theorem 3.19 completely characterize the $(1, 2)$ -step competition graphs of multipartite tournaments without a non-clique partite sets.

3.4 $C_{1,2}(D)$ as a complete graph

In this section, we characterizes the sizes of partite sets of multipartite tournaments whose $(1, 2)$ -step competition graphs are complete.

If a tournament of order greater than or equal to 5 has minimum outdegree at least two, then, for any pair of vertices u and v , none of u and v is X -biased for any vertex subset X of order 1. Since a tournament of order $k \geq 5$ may be considered as a k -partite tournament, the following is true by Corollary 3.5.

Corollary 3.21. *Let D be a tournament with at least five vertices. If each vertex in D has outdegree at least two, then $C_{1,2}(D)$ is complete.*

A tournament D is *regular* provided all vertices in D have the same out-degree. We say that D is *near regular* provided the largest difference between the out-degrees of any two vertices is 1. It is well-known fact that, for each positive integer n , there exists a regular tournament if n is odd and a near regular tournament when if n is even. Since a regular or near regular tournament with at least five vertices has minimum outdegree at least two, the following is immediately true by Corollary 3.21.

Lemma 3.22. *For $n \geq 5$, there exists a tournament of order n whose $(1, 2)$ -step competition graph is complete.*

Let G be a graph. Two vertices u and v of G are said to be *true twins* if they have the same closed neighborhood. We may introduce an analogous notion for a digraph. Let D be a digraph. Two vertices u and v of D are said to be *true twins* if they have the same open out-neighborhood and open in-neighborhood.

Given a digraph D , if there is a directed path of length 2 from a vertex x to a vertex y in D , we call y a *2-step out-neighbor* of x .

Lemma 3.23. *If two non-sink vertices are true twins in a digraph D , then they are true twins in $C_{1,2}(D)$.*

Proof. Suppose that there exist two non-sink vertices u and v which are true twins in D . Since u is a non-sink vertex, u has an out-neighbor x . Then, since u and v are true twins, x is also an out-neighbor of v . Therefore u and v compete.

Take a vertex $w \neq v$ adjacent to u in $C_{1,2}(D)$. If u and w compete, then v and w also compete since u and v are true twins in D . Suppose that u and w (1, 2)-compete. Then u and w have a (1, 2)-step common out-neighbor y . If y is an out-neighbor of u , then y is also an out-neighbor of v and so v and w (1, 2)-compete. Suppose that y is a 2-step out-neighbor of u . Then there exists a directed path $u \rightarrow z \rightarrow y$ for some vertex z in D . Since u and v are true twins in D , $v \rightarrow z \rightarrow y$ is a directed path and so y is also a 2-step out-neighbor of v . Thus v and w (1, 2)-compete. \square

Lemma 3.24. *Let k be a positive integer with $k \geq 3$; n_1, \dots, n_k be positive integers such that $n_1 \geq \dots \geq n_k$; n'_1, \dots, n'_k be positive integers such that $n'_1 \geq \dots \geq n'_k$, $n'_1 \geq n_1$, $n'_2 \geq n_2, \dots$, and $n'_k \geq n_k$. If D is an orientation of K_{n_1, \dots, n_k} whose (1, 2)-step competition graph is complete, then there exists an orientation D' of $K_{n'_1, \dots, n'_k}$ whose (1, 2)-step competition graph is complete.*

Proof. Suppose that D is an orientation of K_{n_1, \dots, n_k} whose (1, 2)-step competition graph is complete. Let X_1, X_2, \dots, X_k be the partite sets of D satisfying $|X_i| = n_i$ for each $1 \leq i \leq k$. Then we construct an orientation of $K_{n'_1, n_2, \dots, n_k}$ whose (1, 2)-step competition graph is complete in the following way. If $n'_1 = n_1$, then we take D as a desired orientation. Suppose $n'_1 > n_1$. Then we add a new vertex v to X_1 and an arc (v, x) for each out-neighbor x of some vertex u in X_1 to obtain a digraph D_1 so that

$$A(D) \subset A(D_1), \quad \text{and} \quad N_D^+(u) = N_{D_1}^+(u) = N_{D_1}^+(v).$$

Therefore $N_D^-(u) = N_{D_1}^-(u) = N_{D_1}^-(v)$ and so u and v are true twins in D_1 . Since $C_{1,2}(D)$ is complete and $|V(D)| \geq 2$, $N_D^+(u) \neq \emptyset$. Therefore $C_{1,2}(D_1)$ is complete by

Lemma 3.23. We may repeat this process until we obtain a desired orientation $D_{n'_1 - n_1}$. Inductively, we obtain an orientation D_t of $K_{n'_1, \dots, n'_k}$ whose $(1, 2)$ -step competition graph is complete where $t = (n'_1 + \dots + n'_k) - (n_1 + \dots + n_k)$. Therefore the statement is true. \square

The following theorem characterizes the sizes of partite sets of multipartite tournaments whose $(1, 2)$ -step competition graphs are complete.

Theorem 3.25. *Let k be a positive integer with $k \geq 3$ and n_1, n_2, \dots, n_k be positive integers such that $n_1 \geq \dots \geq n_k$. There exists an orientation D of K_{n_1, n_2, \dots, n_k} whose $(1, 2)$ -step competition graph is complete if and only if one of the following holds:*

- (a) $k = 3$, and (i) $n_2 \geq 3$ and $n_3 = 1$ or (ii) $n_3 \geq 2$;
- (b) $k = 4$, and (i) $n_1 \geq 3$ and $n_2 = 1$ or (ii) $n_2 \geq 2$;
- (c) $k \geq 5$.

Proof. We first show the “only if” part. Suppose that there exists an orientation D of K_{n_1, n_2, \dots, n_k} whose $(1, 2)$ -step competition graph is complete. Then, since $k \geq 3$, $|V(D)| \geq 3$ and so each vertex has outdegree at least 1 in D . If there exists a vertex v of outdegree 1 in D , then there exists a vertex nonadjacent to v in $C_{1,2}(D)$ by Proposition 3.3. Therefore

$$d^+(v) \geq 2 \tag{3.12}$$

for each vertex v in D . Thus

$$2|V(D)| \leq |A(D)|.$$

Let X_1, \dots, X_k be the partite sets of D satisfying $|X_i| = n_i$ for each $1 \leq i \leq k$.

Suppose $k = 3$. Then, if $n_2 = 1$, then $|V(D)| = n_1 + 2$ and so $|A(D)| = 2n_1 + 1$, which contradicts $2|V(D)| \leq |A(D)|$. Therefore $n_2 \geq 2$. To show by contradiction, suppose $n_3 = 1$ and $n_2 = 2$. Let $X_2 = \{v_1, v_2\}$, $X_3 = \{v_3\}$. Then each vertex in X_1 is not a common out-neighbor of two vertices in $X_2 \cup X_3$ by (3.12). Therefore each pair of $\{v_1, v_3\}$ and $\{v_2, v_3\}$ has a $(1, 2)$ -step common out-neighbor in D .

Let u be a $(1, 2)$ -step common out-neighbor of v_1 and v_3 . Then $u \in N^+(v_1)$ or $u \in N^+(v_3)$. Suppose $u \in N^+(v_1)$. Then $u \in X_1$ and there exists a (v_3, u) -directed path P of length 2 not traversing v_1 . Thus the interior point on the directed path must be v_2 and so $(v_2, u) \in A(D)$. Hence u has outdegree at most one, a contradiction to (3.12). Therefore $u \in N^+(v_3)$. Then u is a 2-step out-neighbor of v_1 . However, each (v_1, u) -directed path of length 2 must traverse v_3 and so v_1 and v_3 cannot have a $(1, 2)$ -step common out-neighbor, a contradiction. Therefore $u \notin X_1$. By symmetry, any $(1, 2)$ -step common out-neighbor of v_2 and v_3 does not belong to X_1 . Thus $u = v_2$ and v_1 is the only $(1, 2)$ -step common out-neighbor of v_2 and v_3 . Hence v_1 and v_2 must be 2-step out-neighbors of v_2 and v_1 , respectively, and so out-neighbors of v_3 . Therefore $N^+(v_1) \cup N^+(v_2) \subseteq X_1$. Thus v_1 and v_2 do not $(1, 2)$ -compete by Proposition 3.1 and so have a common out-neighbor x . Then $x \in X_1$ and $d^+(x) \leq 1$, which contradicts (3.12). Thus $n_2 \geq 3$ or $n_3 \geq 2$ and so (a) holds.

Suppose $k = 4$ and $n_2 = 1$. Then $|V(D)| = n_1 + 3$ and $|A(D)| = 3n_1 + 3$. By (3.12), $2|V(D)| = 2(n_1 + 3) \leq |A(D)|$ and so $n_1 \geq 3$. Therefore (b) holds. Thus we have shown that the “only if” part is true.

Now we show the “if” part.

Case 1. $k = 3$ or 4 . We consider orientations D_1, D_2, D_3 , and D_4 of $K_{3,3,1}, K_{2,2,2}, K_{3,1,1,1}$, and $K_{2,2,1,1}$, respectively, given in Figure 3.4 whose $(1, 2)$ -step competition graphs are complete. By applying to Lemma 3.24 to D_1, D_2, D_3 , and D_4 , respectively, we may obtain an orientation of K_{n_1, n_2, \dots, n_k} whose $(1, 2)$ competition graph is complete when (a) $k = 3$, and (i) $n_2 \geq 3$ and $n_3 = 1$ or (ii) $n_3 \geq 2$; (b) $k = 4$, and (i) $n_1 \geq 3$ and $n_2 = 1$ or (ii) $n_2 \geq 2$.

Case 2. $k \geq 5$. We obtain a tournament D of order k whose $(1, 2)$ -step competition graph is complete by Lemma 3.22. Then, by applying to Lemma 3.24 to D , we may obtain an orientation of K_{n_1, n_2, \dots, n_k} whose $(1, 2)$ -step competition graph is complete. Therefore the “if” part is true. \square

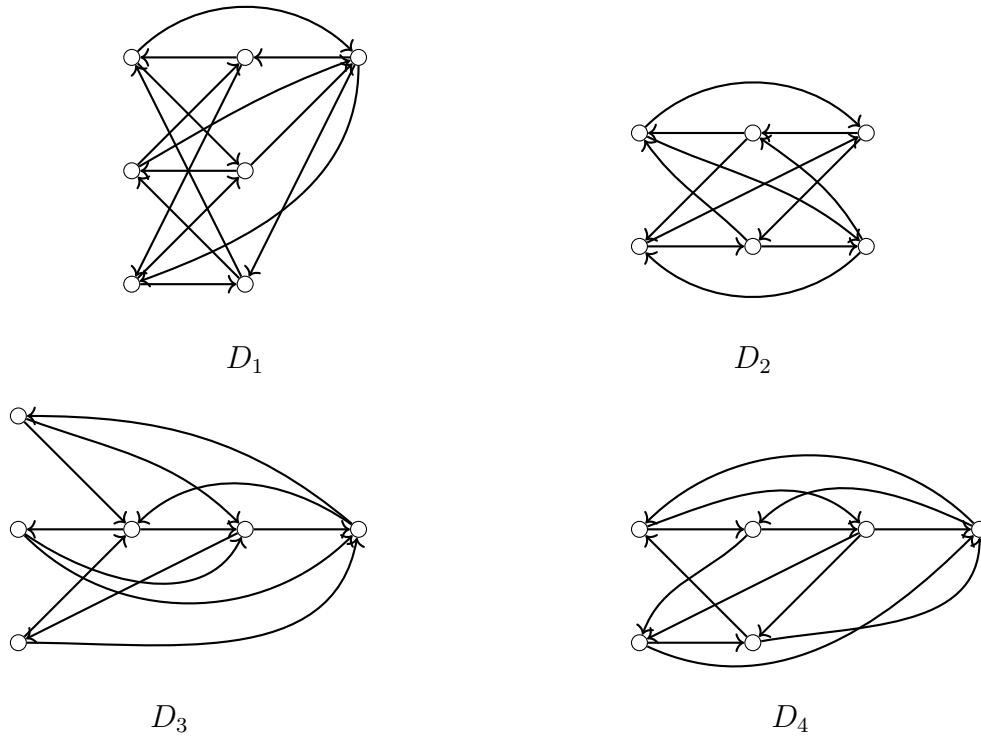


Figure 3.4: D_1 , D_2 , D_3 , and D_4 are orientations of $K_{3,3,1}$, $K_{2,2,2}$, $K_{3,1,1,1}$, and $K_{2,2,1,1}$, respectively, whose (1, 2)-step competition graphs are complete

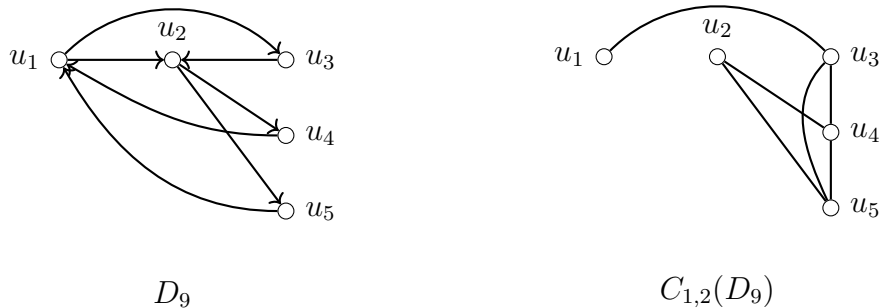


Figure 3.5: A tripartite tournament and its $(1, 2)$ -step competition graph having the diameter three

3.5 Diameters and domination numbers of $C_{1,2}(D)$

See the tripartite tournament D_9 and its $(1, 2)$ -step competition graph $C_{1,2}(D_9)$ given in Figure 3.5. It is easy to check that $C_{1,2}(D_9)$ has a diameter 3. We note that $C_{1,2}(D_9)$ has a maximum stable set $S := \{u_1, u_2\}$ of size 2 and S intersects with two partite sets of D . As a matter of fact, this phenomenon always happens for the $(1, 2)$ -step competition graph of a multipartite tournament.

Theorem 3.26. *Let D be a multipartite tournament. Then each component of $C_{1,2}(D)$ has the diameter at most three. Especially, if there exists a component having the diameter three, then the component itself is $C_{1,2}(D)$, and each maximum stable set of $C_{1,2}(D)$ has size two and intersects with two partite sets of D .*

Proof. We first consider the case where D has a sink. If a sink constitutes a trivial partite set of D , then $C_{1,2}(D)$ is isomorphic to K_{n-1} with an isolated vertex and so the statement is true. If a sink does not constitute a trivial partite set of D , then D has a non-clique partite set and so, by Corollary 3.10(1), the statement is true.

Now we consider the case where D has no sink. If a pair of vertices in the same partite in D is not adjacent, then D has a non-clique partite set and so, by Corollary 3.10(1), the diameter has at most two.

We suppose that each partite set forms a clique. Let S be a maximum stable set in $C_{1,2}(D)$. If $|S| = 1$, then the statement is obviously true. Consider the case where

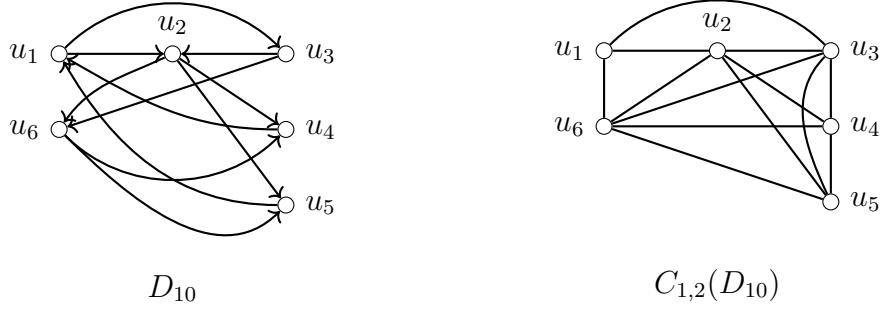


Figure 3.6: A tripartite tournament and its (1, 2)-step competition graph having the diameter two

$|S| \geq 2$. Since each partite set forms a clique, S cannot be included in any partite set of D . Suppose $|S| = 2$. Then each component of $C_{1,2}(D)$ has diameter at most 3. Moreover, if there exists a component having diameter 3, then $C_{1,2}(D)$ is connected. Now suppose $|S| \geq 3$. Since each partite set forms a clique, $|S| = 3$ by Corollary 3.13. Thus it is easy to check that $C_{1,2}(D)$ has a diameter 2 by applying Theorem 3.19. Therefore the statement and the “especially” part are true. \square

The converse of the “especially” part of the above theorem may be false. To see why, consider the tripartite tournament D_{10} given in Figure 3.6, which is obtained from D_9 given in Figure 3.5 by adding a vertex u_6 and arcs (u_2, u_6) , (u_3, u_6) , (u_6, u_4) , and (u_6, u_5) . Then the only pairs of nonadjacent vertices in $C_{1,2}(D_{10})$ are $\{u_1, u_4\}$ and $\{u_1, u_5\}$. Therefore each maximum stable set of $C_{1,2}(D_{10})$ has size two and is intersecting with two partite sets of D . However, $C_{1,2}(D_{10})$ has the diameter two.

A set S of vertices in a graph G is called a *dominating set* if every vertex $v \in V$ is either an element of S or is adjacent to an element of S . The *domination number* $\gamma(G)$ of a graph G equals the minimum cardinality of a dominating set in G .

Given a digraph D , each sink in D is isolated in $C_{1,2}(D)$ and so $m \leq \gamma(C_{1,2}(D))$ where m is the number of sinks in D . The following theorem gives upper bound for the domination number of $C_{1,2}(D)$ when D is a multipartite tournament.

Theorem 3.27. *Let D be a multipartite tournament with m sinks. Then $m \leq \gamma(C_{1,2}(D)) \leq m + 2$ unless $C_{1,2}(D)$ is isomorphic to three isolated vertices.*

Proof. We first suppose that each partite set of D is a clique. Then, by Corollary 3.13, each $\{1, 2\}$ -stable set of D has size at most three. Suppose that D has a $\{1, 2\}$ -stable set of size three. Then $t = 3$ in Theorem 3.19. If $C_{1,2}(D)$ is isomorphic to $K_{|V(D)|} - E(K_3)$, then $C_{1,2}(D)$ has an universal vertex, which implies $\gamma(C_{1,2}(D)) = 1$. If $C_{1,2}(D)$ is isomorphic to $K_{|V(D)|} - (E(K_3) \cup E(K_{1,t}))$, then the center v of $K_{1,t}$ and a vertex u in $K_{1,t} \setminus \{v\}$ forms a dominating set in $C_{1,2}(D)$ and so $\gamma(C_{1,2}(D)) \leq 2$. If D has a maximum $\{1, 2\}$ -stable set of size two, then the complement of $C_{1,2}(D)$ has at least one edge and so, by Theorem 3.18, there exists an edge uv which is incident to a pendent vertex in the complement of $C_{1,2}(D)$, which forms a dominating set in $C_{1,2}(D)$. If D has a maximum $\{1, 2\}$ -stable set of size one, then $C_{1,2}(D)$ is complete.

Now we suppose that $C_{1,2}(D)$ has a non-clique partite set. If D has a sink, then let X_1 be a partite set containing a sink. If D has no sink, then we let X_1 be a non-clique partite set each of whose $\{1, 2\}$ -stable sets is X_2 -biased for some partite set X_2 by Proposition 3.6. Now we take a vertex u in X_3 . If u is adjacent to all vertices except sinks, then $\gamma(C_{1,2}(D)) = m + 1$ since each sink is an isolated vertex in $C_{1,2}(D)$. Suppose that there exists a non-sink vertex v nonadjacent to u . We will show $v \xrightarrow{*} u$ as follows:

Case 1. D has a sink. Then, since X_1 contains a sink, $V(D) \setminus X_1$ forms a clique and so $v \in X_1$. Suppose $v \not\xrightarrow{*} u$. Then, since each sink is an out-neighbor of u , $u \not\xrightarrow{*} v$. Therefore u and v are adjacent by Corollary 3.5, a contradiction. Thus $v \xrightarrow{*} u$.

Case 2. D has no sink. Then the adjacency matrix \mathcal{M} of $C_{1,2}(D)$ is in the form given in Figure 3.2 by Theorem 3.9 in which X_1 was assumed to be a non-clique partite set of D and contain a $\{1, 2\}$ -stable set S with size at least two that is X_2 -biased. Then, since v is not adjacent to u , $v \in F_3$, that is, v is a X_3 -biased vertex by the structure of \mathcal{M} . Therefore $v \xrightarrow{*} u$ or $u \xrightarrow{*} v$ by Theorem 3.8(4). Since $S \subseteq N^+(u)$, $u \not\xrightarrow{*} v$ and so $v \xrightarrow{*} u$.

Thus we have shown that $v \xrightarrow{*} u$. Then u, v together with the sinks form a dominating set in $C_{1,2}(D)$. To see why, we recall that v was arbitrarily chosen from non-sink vertices nonadjacent to u . Therefore u is an out-neighbor of the non-sink

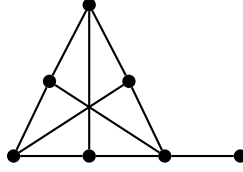


Figure 3.7: A graph with diameter three and domination number three

vertices nonadjacent to u . Thus the set of the non-sink vertices nonadjacent to u forms a clique in $C_{1,2}(D)$. \square

Remark 3.28. The graph G given in Figure 3.7 has diameter three. However, G has domination number three and so G cannot be the $(1, 2)$ -step competition graph of a multipartite tournament by Theorem 3.27.

3.6 Disconnected $(1, 2)$ -step competition graphs

In this section, we list all disconnected $(1, 2)$ -step competition graphs of multipartite tournaments without sinks.

We denote the set of k isolated vertices in a graph by I_k for some positive integer k .

Proposition 3.29. *Let D be a multipartite tournament without a non-clique partite set. Suppose that $C_{1,2}(D)$ has a maximum stable set of size two. If $C_{1,2}(D)$ is disconnected, then $C_{1,2}(D)$ is isomorphic to $K_n \cup I_1$ for some $n \geq 3$.*

Proof. Suppose that $C_{1,2}(D)$ is disconnected. Then $C_{1,2}(D)$ has at least two components. Since $C_{1,2}(D)$ has a maximum stable set of size two, $C_{1,2}(D)$ has exactly two components each of which is complete. Therefore $C_{1,2}(D)$ is isomorphic to $K_n \cup K_m$ for some positive integers n and m with $n + m = |V(D)|$ and $n \geq m$. If $n, m \geq 2$, then the complement of $C_{1,2}(D)$ must have a cycle, which contradicts Theorem 3.18. Therefore $n = 1$ or $m = 1$. Since $n \geq m$, $m = 1$. If $n = 2$, then D must have a sink that forms a trivial partite set, which is the case not to consider (see the last paragraph of section 2). Thus $n \geq 3$. \square

Now we are ready to introduce one of our main results.

Theorem 3.30. *A disconnected graph G is the $(1, 2)$ -step competition graph of a k -partite tournament of order n without sinks for some $k \geq 3$ if and only if G is isomorphic to*

$$\begin{cases} I_3 & n = 3; \\ (K_{n-1} - E(K_2)) \cup I_1 \text{ or } K_{n-1} \cup I_1 & n \geq 4 \end{cases}$$

Proof. To show the “only if” part, suppose that a disconnected graph G is the $(1, 2)$ -step competition graph of a k -partite tournament D of order n without sinks for some $k \geq 3$. If D has a non-clique partite set, then G is connected by Corollary 3.10(2). Therefore every partite set of D is clique. Since G is disconnected, G has a maximum $\{1, 2\}$ -stable set S of size at least two. By Corollary 3.13, $|S| \leq 3$. If $|S| = 3$, then S intersects with three partite sets of D since each partite set of D is clique, and so, by Theorem 3.19, $G \cong K_3 - E(K_3)$ or $G \cong K_n - (E(K_3) \cup E(K_{1, n-3}))$ where $n \geq 4$. Therefore $G \cong I_3$ or $G \cong (K_{n-1} - E(K_2)) \cup I_1$ where $n \geq 4$. If $|S| = 2$, then $G \cong K_{n-1} \cup I_1$ where $n \geq 4$ by Proposition 3.29.

Now we show the “if” part. Let D_{11} be a directed cycle of order 3. Then $C_{1,2}(D_{11}) \cong I_3$. Suppose $n \geq 4$. Let D_{12} be a tripartite tournament with the partite sets $\{u_1\}$, $\{u_2\}$, and $\{u_3, u_4, \dots, u_n\}$ and the arc set

$$A(D_{12}) = \{(u_1, u_3), (u_2, u_1), (u_3, u_2)\} \cup \{(u_i, u_1), (u_2, u_i) \mid 4 \leq i \leq n\}$$

(see the digraph D_{12} given in Figure 3.8 for an illustration). Then u_1 is an isolated vertex in $C_{1,2}(D_{12})$. The vertex u_1 is a common out-neighbor of u_2, u_4, \dots, u_n . In addition, it is a $(1, 2)$ -step common out-neighbor of u_3 and u_i for each $4 \leq i \leq n$. Moreover, u_3 is not adjacent to u_2 by Proposition 3.3. Therefore $C_{1,2}(D_{12}) \cong (K_{n-1} - E(K_2)) \cup I_1$.

Let D_{13} be a tripartite tournament with the partite sets $\{u_1\}$, $\{u_2\}$, $\{u_3, \dots, u_n\}$, and the arc set

$$A(D_{13}) = \{(u_1, u_2), (u_2, u_3), (u_3, u_1)\} \cup \{(u_2, u_i), (u_i, u_1) \mid 4 \leq i \leq n\}$$

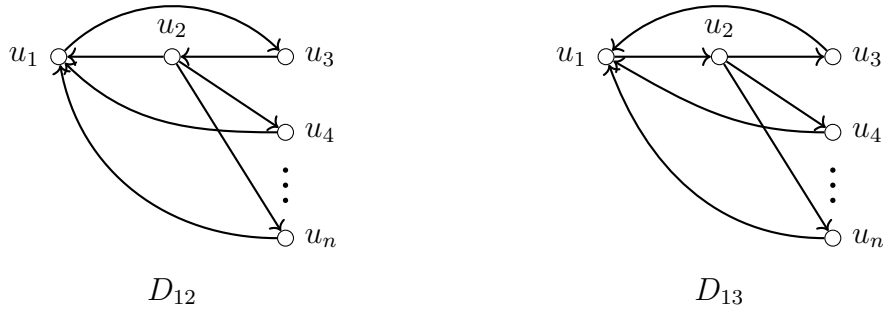


Figure 3.8: Tripartite tournaments in the proof of Theorem 3.30

(see the digraph D_{13} given in Figure 3.8 for an illustration). Then u_1 is an isolated vertex in $C_{1,2}(D_{13})$. The vertex u_1 is a common out-neighbor of u_3, \dots, u_n and a $(1, 2)$ -step common out-neighbor of u_2 and u_i for each $3 \leq i \leq n$. Therefore $C_{1,2}(D_{13}) \cong K_{n-1} \cup I_1$. Since each of D_{11} , D_{12} , and D_{13} has no sink, we have shown that the “if” part is true. \square

Remark 3.31. Let D be a multipartite tournament of order n whose $(1, 2)$ -step competition graph is disconnected. If D has no sink, then $C_{1,2}(D)$ is completely determined by Theorem 3.30. Suppose that D has a sink. If a sink constitutes a trivial partite set, then $C_{1,2}(D) \cong K_{n-1} \cup I_1$. If a sink does not constitute a trivial partite set, then D has a non-clique partite set, so the structure of $C_{1,2}(D)$ is mostly determined by Theorem 3.9.

3.7 Interval $(1, 2)$ -step competition graphs

An *asteroidal triple* of a graph is a set of three vertices such that every pair of vertices are joined by a path outside of the closed neighborhood of the third.

Theorem 3.32. *Let D be a multipartite tournament. If $C_{1,2}(D)$ has an asteroidal triple, then it is contained in a partite set X and $V(D) \setminus X$ forms a clique in $C_{1,2}(D)$.*

Proof. Suppose that $C_{1,2}(D)$ has an asteroidal triple x, y , and z . Then $\{x, y, z\}$ forms a stable set of size three in $C_{1,2}(D)$. Therefore D has a non-clique partite set

by Theorem 3.19 and so, by Theorem 3.9, the adjacency matrix \mathcal{M} of $C_{1,2}(D)$ is in the form given in Figure 3.2. Since $\bigcup_{i=3}^k X_i$ is a clique in $C_{1,2}(D)$, at least two vertices in $\{x, y, z\}$ is contained in a partite set X_1 or X_2 of D . If $\{x, y, z\}$ is contained in X_2 , then X_2 is non-clique. Therefore we may apply Theorem 3.9 to X_2 and so we may assume $\{x, y\} \subset X_1$. Since x and y are not adjacent, $\{x, y\} \subset F_j$ for some $j \in \{2, \dots, k\}$. Suppose, to the contrary, that $z \notin X_1$. Then, since X_j is the only partite set that might have a vertex not adjacent to a vertex in F_j , we have $z \in X_j$. Since x and y have no common out-neighbor and z is not adjacent to x and y , we may assume $x \xrightarrow{*} z$ and $z \xrightarrow{*} y$ by Theorem 3.8(4). By the way, since $\{x, y, z\}$ is an asteroidal triple, y and z are connected by a path P avoiding the neighbourhood of x . Let w be a vertex on P which is adjacent to z . Then w is not adjacent to x and so $w \in F_j$ or $w \in X_j$. Suppose $w \in X_j$. Then, since $x \xrightarrow{*} z$, there is an arc from w to x in D and x is X_j -biased. Therefore w is X_j -biased or $X_j \cup \{x\}$ -biased by Proposition 3.4. Since $w \in X_j$, w is $X_j \cup \{x\}$ -biased. Therefore $w \xrightarrow{*} x$. Then, since $z \xrightarrow{*} y$, w and z has no common out-neighbor. Moreover, w and z are X_1 -biased and so they are not adjacent in $C_{1,2}(D)$ by Proposition 3.2, a contradiction. Therefore $w \notin X_j$ and so $w \in F_j$. Thus there is an arc between w and z . Then, since $z \xrightarrow{*} y$, $(w, z) \in A(D)$ and so z is a common out-neighbor of w and x , a contradiction. Therefore $\{x, y, z\} \subset X_1$. Thus $V(D) \setminus X_1$ forms a clique by the structure of \mathcal{M} . \square

Remark 3.33. By Theorem 3.25, there is a multipartite tournament D whose $(1, 2)$ -competition graph is complete. Such a multipartite tournament satisfies the necessary condition of Theorem 3.32 but its $(1, 2)$ -competition graph does not contain an asteroidal triple. Therefore the converse of Theorem 3.32 is not valid.

Lemma 3.34. *Let D be a multipartite tournament with a non-clique partite set. If two adjacent X_i -biased vertices u and v are not true twins in $C_{1,2}(D)$ for some $i \in \{1, \dots, k\}$, then each vertex in X_i is adjacent to at least one of u and v .*

Proof. Suppose that there are two adjacent X_i -biased vertices u and v which are not true twins in $C_{1,2}(D)$ for some $i \in \{1, \dots, k\}$. To the contrary, suppose there is a vertex w in X_i not adjacent to both u and v . If $u \xrightarrow{*} w$ and $v \xrightarrow{*} w$, then u and v are not true twins in D and they are true twins in $C_{1,2}(D)$ by Lemma 3.23. Therefore

$u \overset{*}{\nrightarrow} w$ or $v \overset{*}{\nrightarrow} w$. Without loss of generality, we may assume $u \overset{*}{\nrightarrow} w$. Then $w \overset{*}{\rightarrow} u$ by Theorem 3.8(4) Therefore $w \overset{*}{\nrightarrow} v$ and so, by the same theorem, $v \overset{*}{\rightarrow} w$. Thus u and v has no common out-neighbor. Hence u and v are not adjacent in $C_{1,2}(D)$ by Proposition 3.2, a contradiction. \square

Proposition 3.35. *Let D be a multipartite tournament. For two adjacent vertices which are not true twins in a non-clique partite set X of D , any vertex not adjacent to the two vertices belongs to X .*

Proof. Suppose that there exist two adjacent vertices u, v which are not true twins in a non-clique partite set X of D and a vertex w not adjacent to any of them. Then, since X is a non-clique, the adjacency matrix \mathcal{M} of $C_{1,2}(D)$ is in the form given in Figure 3.2 by Theorem 3.9 in which $X = X_1$ was assumed. If $w \notin X_1$, then $w \in X_i$ for some $i \in \{2, \dots, k\}$ and so, by the structure of \mathcal{M} , the u and v are X_i -biased, which implies that w is adjacent to one of u and v by Lemma 3.34. Therefore we reach a contradiction and so the statement is true. \square

Given a graph G , we call a vertex of G *universal* if it is adjacent to all other vertices of G .

Theorem 3.36. *Let D be a multipartite tournament. If $C_{1,2}(D)$ has a hole H of length at least five, then the following are true:*

- (1) *there exists a partite set X such that $V(D) \setminus X$ forms a clique in $C_{1,2}(D)$ and every hole L of length at least five is contained in the set of Y -biased vertices included in X for some partite set Y of size at least $|V(L)|$;*
- (2) *$|V(D)| \geq 2|V(H)| + 1$ and if the equality holds, then every vertex not on H is a universal vertex.*

Proof. Suppose that $C_{1,2}(D)$ has a hole $H = v_0v_1 \dots v_{l-1}v_0$ of length $l \geq 5$. Then the complement of H contains a cycle of length at least five, so the complement of $C_{1,2}(D)$ contains a cycle of length at least five. Suppose that each partite set is clique in $C_{1,2}(D)$. Then $C_{1,2}(D)$ has a stable set of size at most three by Theorem 3.19 .

Therefore $C_{1,2}(D)$ is isomorphic to one of the graphs given in Theorem 3.18 or in the “especially part” of Theorem 3.19, none of which contains a cycle of length at least five in its complement. Hence D has a non-clique partite set and so the adjacency matrix \mathcal{M} of $C_{1,2}(D)$ is in the form given in Figure 3.2 by Theorem 3.9 in which X_1 was assumed to be a non-clique partite set of D .

To show the part (1), we first suppose $l \geq 6$. Then there exists an asteroidal triple $\{x, y, z\}$ on H and so, by Theorem 3.32, $\{x, y, z\}$ is contained in a partite set X of D . Since $\{x, y, z\}$ forms a stable set, we may apply Proposition 3.6(2) to claim that if $X = X_2$, then X_1 forms a clique to reach a contradiction. Therefore $\{x, y, z\} \subseteq X_1$. Since $l \geq 6$ by our assumption, each vertex on H can form an asteroidal triple with two vertices on H and so $V(H) \subset X_1$.

We suppose $l = 5$. We first assume that there is no partite set X such that $|V(H) \cap X| \geq 3$. Then, since $l = 5$, there are at least three partite sets intersecting with $V(H)$. Therefore $V(H)$ intersects with two partite sets X_i and X_j for some distinct $i, j \in \{2, \dots, k\}$. By the way, we see from the structure of \mathcal{M} that each vertex in X_i is adjacent to all vertices in X_j . Thus there exist a vertex in $V(H) \cap X_i$ and a vertex in $V(H) \cap X_j$ which are consecutive on H . Without loss of generality, we may assume $v_0 \in X_i \cap V(H)$ and $v_1 \in X_j \cap V(H)$. Then, since v_3 is adjacent to neither v_0 nor v_1 , $v_3 \notin \bigcup_{t=2}^k X_t$. Hence $v_3 \in X_1$ and so $v_3 \in F_i \cup F_j$. However, since the structure of \mathcal{M} shows that each vertex in F_i (resp. F_j) is adjacent to all vertices in X_j (resp. X_i), v_3 is adjacent to v_0 or v_1 , which is impossible. Therefore there exists a partite set X such that $|V(H) \cap X| \geq 3$. Then, since $l = 5$, there exist two consecutive vertices on H belonging to X . Without loss of generality, we may assume $\{v_0, v_1\} \subset X$. Then, by Proposition 3.35, $v_3 \in X$. Suppose, to the contrary, that $v_4 \notin X$ and $v_2 \notin X$. Then $v_4 \in Y$ for a partite set Y distinct from X . Since $\{v_0, v_1, v_3\} \subseteq X$, X is a non-clique. Then $X = X_1$ or X_2 .

We first assume $X = X_1$. Since v_1 is not adjacent to $v_4 \in Y$, we see from the structure of \mathcal{M} that v_1 is Y -biased. Therefore $v_1 \xrightarrow{*} v_4$ or $v_4 \xrightarrow{*} v_1$ by Theorem 3.8(4). If $v_4 \xrightarrow{*} v_1$, then v_4 is a common out-neighbor of v_0 and v_3 and so v_0 and v_3 are adjacent, which is impossible. Therefore $v_1 \xrightarrow{*} v_4$. By the same argument, we may show $v_0 \xrightarrow{*} v_2$. Then, if v_2 or v_4 is an out-neighbor of v_3 , then v_3 is adjacent to v_0 or

v_1 , which is impossible. Thus v_2 and v_4 are in-neighbors of v_3 and so v_2 and v_4 are adjacent, a contradiction. Therefore $v_4 \in X$ or $v_2 \in X$. Then, by Proposition 3.35, $v_2 \in X$ if $v_4 \in X$, or $v_4 \in X$ if $v_2 \in X$, so $V(H) \subseteq X$. Then, since any three vertices on H do not form a triangle, each vertex in $V(D) \setminus X$ has at most 2 in-neighbors in $V(H)$ and so has at least $|V(H)| - 2$ out-neighbors in $V(H)$, that is, $|N^+(w) \cap V(H)| \geq |V(H)| - 2$ for each vertex w in $V(D) \setminus X$. Now, for a pair of vertices u and v in $V(D) \setminus X$,

$$\begin{aligned} |(N^+(u) \cap N^+(v)) \cap V(H)| &\geq |N^+(u) \cap V(H)| + |N^+(v) \cap V(H)| - |V(H)| \\ &\geq 2(|V(H)| - 2) - |V(H)| = |V(H)| - 4 \geq 1. \end{aligned}$$

Therefore any pair of vertices in $V(D) \setminus X$ has a common out-neighbor in $V(H)$ and so $V(D) \setminus X$ forms a clique in $C_{1,2}(D)$.

Now we assume $X = X_2$. Then we may switch X_1 with X_2 to still have the adjacency matrix of $C_{1,2}(D)$ in the form given in Figure 3.2 since Theorem 3.9 is applicable to any non-clique partite set. Thus we may apply the above argument to conclude that $V(D) \setminus X_2$ forms a clique in $C_{1,2}(D)$. However, since X_1 was assumed to be a non-clique partite set, we reach a contradiction. Therefore $X \neq X_2$ and so $X = X_1$. Thus X_1 is the only partite set containing a hole of length at least five. Fix $i \in \{2, \dots, l-2\}$. Then v_0 and v_i are not adjacent, so $\{v_0, v_i\} \subseteq F_j$ for some $j \in \{2, \dots, k\}$ by the structure of \mathcal{M} . Since F_2, \dots, F_k are mutually disjoint, for any $i \in \{2, \dots, l-2\}$, $\{v_0, v_i\} \subseteq F_j$ and so $\{v_0, v_2, v_3, \dots, v_{l-2}\} \subseteq F_j$. Moreover, since v_1 and v_{l-1} are not adjacent to v_3 and v_2 , respectively, $\{v_1, v_{l-1}\} \subseteq F_j$. Thus $V(H) \subseteq F_j$. Hence any pair of adjacent vertices in $V(H)$ must have a common out-neighbor in X_j by Proposition 3.2. Then, since each vertex in X_j can be a common out-neighbor of at most two vertices in $V(H)$, $|E(H)| \leq |X_j|$. Since $|V(H)| = |E(H)|$, $|V(H)| \leq |X_j|$ and so the part (1) is true. Moreover, since D has at least $k \geq 3$ partite set, D has a partite set X_m distinct from X_1 and X_j and so

$$|V(D)| \geq |X_1| + |X_j| + |X_m| \geq |V(H)| + |V(H)| + 1 \geq 2|V(H)| + 1.$$

Therefore the inequality of the part (2) is true. Now suppose the equality holds, that is, $|V(D)| = 2|V(H)| + 1$. Then X_1 , X_j and X_m are the only partite sets of D and $X_1 = V(H)$, $|X_j| = |V(H)|$, $|X_m| = 1$. Therefore every vertex in X_j is a common out-neighbor of exactly two vertices in X_1 . Thus we may label the vertices in X_j as w_0, \dots, w_{l-1} so that

$$w_i \in N^+(v_i) \cap N^+(v_{i+1}), \quad X_1 \setminus \{v_i, v_{i+1}\} \subseteq N^+(w_i)$$

for each $0 \leq i \leq l-1$, identifying v_l with v_0 . In addition, since $V(H) = X_1$ and each vertex on H is X_j -biased, each vertex in X_1 is an out-neighbor of the vertex in X_m . Moreover, since each vertex in X_1 has exactly two out-neighbor in X_j , each vertex in X_1 is adjacent to each vertex in $X_j \cup X_m$ by Corollary 3.5. By the claim (C), $X_j \cup X_m$ forms a clique in $C_{1,2}(D)$. Therefore each vertex in $X_j \cup X_m$ is a universal vertex. Thus the part (2) is true. \square

Theorem 3.37. *Let D be a multipartite tournament such that there exists no partite set X such that $V(D) \setminus X$ forms a clique in $C_{1,2}(D)$. Then $C_{1,2}(D)$ is an interval graph unless $C_{1,2}(D)$ has a hole of length four.*

Proof. We assume that $C_{1,2}(D)$ has no hole of length four. To reach a contradiction, suppose that $C_{1,2}(D)$ is not an interval graph. Then $C_{1,2}(D)$ has a hole or an asteroidal triple by Theorem 1.2. If $C_{1,2}(D)$ has an asteroidal triple, then there exists a partite set X such that $V(D) \setminus X$ forms a clique by Theorem 3.32, which is impossible. Therefore $C_{1,2}(D)$ has a hole H . By our assumption, H has length at least five. Then there exists a partite set X such that $V(D) \setminus X$ forms a clique by Theorem 3.36(1), a contradiction. \square

Chapter 4

The forbidden induced subgraphs of (i, j) phylogeny graphs¹

A graph G is an (i, j) *phylogeny graph* if there is an (i, j) digraph D such that the phylogeny graph of D is isomorphic to G . Throughout this chapter, we assume that variables i and j belong to the set of positive integers unless otherwise stated.

We present two main theorems in this chapter. One of them gives a necessary condition for an $(i, 2)$ digraph having a chordal phylogeny graph as follows:

Theorem 4.1. *Let H be a hole with length l of the underlying graph of an $(i, 2)$ digraph D . If $l \geq 3i + 1$, then the subgraph of the phylogeny graph of D induced by $V(H)$ has a hole. Further, the inequality is tight.*

It extends Theorem 1.18 to $(i, 2)$ phylogeny graphs.

The *join* of two graphs G and H , denoted by $G \vee H$, is the graph formed from disjoint copies of G and H by connecting each vertex of G to each vertex of H .

Based upon the other main theorem in the following, $P_7 \vee I_1$, $C_7 \vee I_1$, $K_{1,4}$, and $K_{3,3}$ are also forbidden induced subgraphs of $(2, 2)$ phylogeny graphs other than K_5 , which extends Theorem 1.21.

¹The material in this chapter is written based on the manuscript ‘The forbidden induced subgraphs of (i, j) phylogeny graphs’ by Myungho Choi and Suh-Ryung Kim. The author thanks Prof. Suh-Ryung Kim for allowing him to use its contents for his thesis.

Theorem 4.2. *The graphs below list forbidden induced subgraphs of the phylogeny graph of an (i, j) digraph with $i, j \geq 2$:*

$$K_{1,j+2}; \quad K_{j+1,j+1}; \quad P_{2j+3} \vee I_1; \quad C_{2j+3} \vee I_1;$$

$$K_{ij+1}, \text{ further, if } i \geq 4 \text{ and } j = 2, K_{\lfloor \frac{3i}{2} \rfloor + 2}.$$

We denote the set of out-neighbors and the set of in-neighbors of a vertex v in a digraph D by $N_D^+(v)$ and $N_D^-(v)$, respectively. In addition, we denote the set of neighbors of a vertex v in a graph G by $N_G(v)$. When no confusion is likely to occur, we omit D or G to just write $N^+(v)$, $N^-(v)$, and $N(v)$.

4.1 A necessary condition for an $(i, 2)$ phylogeny graph being chordal

Let D be an acyclic digraph. Suppose that the underlying graph $U(D)$ of D has a hole $H = v_1v_2 \cdots v_l v_1$ of length l for some $l \geq 5$. Let G be the subgraph of the phylogeny graph $P(D)$ induced by $V(H)$ and D_H be the subdigraph of D induced by $V(H)$. Then, since each vertex has degree 2 in $U(D_H)$, each of v_1, v_2, \dots, v_l has (i) exactly two in-neighbors, or (ii) exactly one out-neighbor and exactly one in neighbor, or (iii) exactly two out-neighbors in D_H . Since D is acyclic, D_H is acyclic and so there exists a vertex in D_H of indegree 2. Let $v_{l_1}, v_{l_2}, \dots, v_{l_k}$ be the vertices in $V(H)$ having two in-neighbors in D_H for an integer $k \geq 1$. We denote the set $\{v_{l_1}, v_{l_2}, \dots, v_{l_k}\}$ by Γ_H . Then any two vertices in Γ_H do not lie consecutively on H and so $1 \leq k \leq \lfloor \frac{l}{2} \rfloor$. Therefore we obtain the cycle C of length $l - k$ in $P(D)$ by deleting $v_{l_1}, v_{l_2}, \dots, v_{l_k}$ from G satisfying the property that each edge of C either is taken care of some vertex in Γ_H or lies on H . We call such a cycle the *cycle obtained from H by Γ_H* . When no confusion is likely to occur, we omit Γ_H in the cycle obtained from H by Γ_H to just write the cycle obtained from H . We note that the length of C is at least $l - \lfloor \frac{l}{2} \rfloor$ and at most $l - 1$. Moreover, by (ii) and (iii), each vertex on C has at least one out-neighbor in $V(H)$. It is easy to check that the number of vertices on C having two out-neighbor in $V(H)$ is equal to $|\Gamma_H|$. In addition, there is no arc in $V(H)$ between

nonconsecutive vertices on C , which implies that each chord of C is a cared edge by a vertex in $V(D) - V(H)$. Hence we immediately have the following lemma.

Lemma 4.3. *Let H be a hole with length $l \geq 5$ in the underlying graph of an acyclic digraph D and C be the cycle obtained from H by Γ_H . Then the following are true:*

- (1) *the length of C is at least $l - \lfloor \frac{l}{2} \rfloor$ and at most $l - 1$;*
- (2) *each vertex on C has at least one out-neighbor in $V(H)$;*
- (3) *the number of vertices on C having two out-neighbors in $V(H)$ is equal to $|\Gamma_H|$;*
- (4) *for each chord uv of C in $P(D)$, uv is a cared edge and each vertex taking care of uv belongs to $V(D) - V(H)$.*

We obtain some useful characteristics on the cycle obtained from a hole in the underlying graph of an $(i, 2)$ digraph as follows.

Proposition 4.4. *Let H be a hole with length $l \geq 5$ in the underlying graph of an $(i, 2)$ digraph D and C be the cycle obtained from H . Suppose that C has a chord uv in $P(D)$. Then the following are true:*

- (1) *there exists exactly one vertex w taking care of uv in D ;*
- (2) *w is the only out-neighbor in $V(D) - V(H)$ of each of u and v ;*
- (3) *for the subgraph induced by the chords of C in $P(D)$, if T is its component containing uv , then w is the common out-neighbor in D of the vertices in T and $V(T)$ forms a clique in $P(D)$.*

Proof. Since uv is a chord of C in $P(D)$, uv is a cared edge and there exists a vertex w taking care of uv , which belongs to $V(D) - V(H)$ by Lemma 4.3(4). Then w is a common out-neighbor of u and v . Since D is an $(i, 2)$ digraph, each of u and v has outdegree at most 2. Then, by Lemma 4.3(2), each of u and v has at most one out-neighbor in $V(D) - V(H)$. Therefore w is the only out-neighbor of each of u and v in $V(D) - V(H)$ and so w is the only vertex taking care of uv in D . Hence parts (1) and (2) are true.

To show part (3), suppose, for the subgraph induced by the chords of C in $P(D)$, T is its component containing uv . Then take a vertex v_1 distinct from u in T . Then there exists a path $P = v_1 \cdots v_t u$ in T and each edge in P is a chord in C . Therefore each edge in P is a cared edge by Lemma 4.3(4). Let y be a vertex taking care of uv_t . Then $y \in V(D) - V(H)$ by Lemma 4.3(4). Since w is the only out-neighbor in $V(D) - V(H)$ of u , $w = y$ and so w is an out-neighbor of v_t . If $t \geq 2$, then, by applying a similar argument for the chord $v_t v_{t-1}$, w takes care of $v_t v_{t-1}$ and w is the only out-neighbor in $V(D) - V(H)$ of v_t by parts (1) and (2). Therefore w is an out-neighbor of v_{t-1} if $t \geq 2$. We repeat this process until we conclude that w is an out-neighbor of v_1 . Therefore part (3) is true. \square

Corollary 4.5. *Let H be a hole with length $l \geq 5$ in the underlying graph of an $(i, 2)$ digraph D and C be the cycle obtained from H . Then, for each vertex on C having two out-neighbors in $V(H)$, it is not incident to any chord of C .*

Proof. Let u be a vertex on C having two out-neighbors in $V(H)$. Suppose, to the contrary, that u is incident a chord uv of C . Then there exists a vertex w in $V(D) - V(H)$ such that w is a common out-neighbor of u and v by Proposition 4.4(2). Thus u has at least three out-neighbors, which contradicts the fact that D is an $(i, 2)$ digraph. \square

To prove one of our main theorems, we need one more result.

Lemma 4.6 ([18]). *Given a graph G and a cycle C of G with length at least four, suppose that a section Q of C forms an induced path of G and contains a path P with length at least two none of whose internal vertices is incident to a chord of C in G . Then P can be extended to a hole H in G so that $V(P) \subsetneq V(H) \subseteq V(C)$ and H contains a vertex on C not on Q .*

Given a vertex subset X of a graph G , a *maximum clique in X* means a clique in $G[X]$ whose size is the maximum among the cliques in $G[X]$.

Theorem 4.7. *Let D be an $(i, 2)$ digraph and C be the cycle of length $l \geq 4$ obtained from a hole in $U(D)$. If a maximum clique in $V(C)$ has size at most $\lfloor \frac{l-1}{2} \rfloor$, then the subgraph of $P(D)$ induced by $V(C)$ has a hole.*

Proof. Suppose that a maximum clique K in $V(C)$ has size at most $\lfloor \frac{l-1}{2} \rfloor$. If C has no chord, then C is a hole and so we are done. Suppose that C has a chord. If a maximum clique has size at least three in $V(C)$, then the clique must contain a chord of C . Otherwise, each chord is a maximum clique itself. Therefore we may assume that $\{u, v\} \subseteq V(K)$ for a chord uv of C . We note that

$$|V(C)| - |V(K)| \geq l - \left\lfloor \frac{l-1}{2} \right\rfloor = \left\lceil \frac{l+1}{2} \right\rceil > \frac{l}{2}.$$

Therefore there exist two consecutive vertices x_1 and x_2 on C each of which does not belong to $V(K)$. Starting from x_1 (resp. x_2), we traverse the (x_1, x_2) -section (resp. the (x_2, x_1) -section) of C that is not the edge x_1x_2 until we first meet a vertex y (resp. z) belonging to $V(K)$. Then the (y, x_1) -section obtained in this way, the edge x_1x_2 , and the (x_2, z) -section obtained in this way form the (y, z) -section Q of C such that y and z are the only vertices belonging to $V(K)$. Since x_1 and x_2 are contained in Q , Q has length at least 3. Let $Q = v_0v_1v_2 \cdots v_t$ where $v_0 = y$ and $v_t = z$ for an integer $t \geq 3$. Then $v_i \notin V(K)$ for each $1 \leq i \leq t-1$. If $v_0 = v_t$, then $V(Q) = V(C)$ and so $V(K) \cap V(C) = \{v_0\}$, which contradicts that the existence of the chord uv . Therefore

$$v_0 \neq v_t.$$

Since $\{v_0, v_t\} \subseteq V(K)$, v_0v_t is an edge in $P(D)$. If v_0v_t is not a chord of C , then $C = v_0v_1 \cdots v_tv_0$ and so, by the choice of Q , $V(K)$ does not contain any chord, which is a contradiction. Therefore v_0v_t is a chord of C .

Let T be the component containing v_0v_t in the induced subgraph by the chords of C . We note that

(\star) any vertex in T cannot be joined to a vertex on $C - T$ by a chord of C .

Let C_1 be the cycle obtained from adding v_0v_t to Q . Suppose $V(T) = \{v_0, v_t\}$. Then C_1 has length at least four and, by (\star), $P_1 := v_1v_0v_t$ is an induced path. Since $V(T) = \{v_0, v_t\}$, v_0 is not incident to any chord of C except v_0v_t and so v_0 is not

incident to any chord of C_1 . Now we suppose

$$V(T) \neq \{v_0, v_t\}.$$

Then $|V(T)| \geq 3$. By the way, since $l \geq 4$, the hole in $U(D)$ containing the vertices on C has length at least 5 by Lemma 4.3(1). Therefore $V(T)$ forms a clique in $P(D)$ by Proposition 4.4(3) and so $|V(K)| \geq 3$ by the maximality. By the choice of Q , K contains a vertex on the (v_0, v_t) -section, say L , of C other than Q . Since v_0v_t is a chord of C , L has length at least two.

Case 1. L has length 2. Let w be the internal vertex on L . Then $V(K) = \{v_0, w, v_t\}$ and so, by the maximality of K , $|V(T)| = 3$. Therefore $T = \{v_0, v_j, v_t\}$ for some $j \in \{1, \dots, t-1\}$. If $l \leq 6$, $\lfloor \frac{l-1}{2} \rfloor < 3$, which contradicts the fact that K has size 3. Therefore $l \geq 7$ and so

$$t \geq 5.$$

If $j = 1$ or 2 , then, by (\star) , $P_2 := v_tv_jv_{j+1}$ is an induced path and v_j is not incident to any chord of C_2 where C_2 is the cycle of length $t-j+1$ obtained from adding v_jv_t to the (v_j, v_t) -section of Q . If $j \geq 3$, then, by (\star) , $P_3 := v_1v_0v_j$ is an induced path and v_0 is not incident to any chord of C_3 where C_3 is the cycle of length $j+1$ obtained from adding v_0v_j to the (v_0, v_j) -section of Q . We note that $t-j+1 \geq 4$ if $j = 1$ or 2 and $j+1 \geq 4$ for $j \geq 3$. Therefore each of C_2 and C_3 has length at least 4.

Case 2. L has length at least 3. Then, for each vertex x on K , v_0x or v_tx is a chord of C . Therefore $V(K) \subseteq V(T)$ and so, by the maximality, $V(K) = V(T)$. Thus, by (\star) , $P_1 = v_1v_0v_t$ is an induced path and v_0 is not incident to any chord of the cycle C_1 .

For each $1 \leq i \leq 3$, by applying Lemma 4.6 to P_i and C_i , we may conclude that P_i can be extended to a hole in $P(D)$ whose vertices are on C_i . Since the cycles C_1, C_2 , and C_3 are contained in $V(C)$, the subgraph of $P(D)$ induced by $V(C)$ has a hole and so the statement is true. \square

Lemma 4.8. *Let G be a graph and C be a cycle of G . Suppose that there exists a*

maximum clique K of size at least four in $V(C)$. If C has length at least five, then, for each pair of vertices in K , there is a path between them consisting of only chords of C .

Proof. Suppose that C has length at least five. Let K^* be the graph obtained from K by deleting edges of C in K . Since $|V(K^*)| = |V(K)|$, there are at least four vertices in $V(K^*)$ and we take four vertices v_1, v_2, v_3, v_4 in $V(K^*)$. Let T be the subgraph induced by $\{v_1, v_2, v_3, v_4\}$ in K^* . Since C has length at least five, at most three edges of C were deleted from the clique on $\{v_1, v_2, v_3, v_4\}$ to obtain T . Even when three edges were deleted, T is isomorphic to a path of length 3. Thus T is connected. Since v_1, v_2, v_3 , and v_4 were arbitrarily chosen from $V(K^*)$, we conclude that K^* is connected and so the statement is true. \square

Lemma 4.9. *Let H be a hole with length $l \geq 5$ in the underlying graph of an $(i, 2)$ digraph D and C be the cycle obtained from H . If $i \geq 3$, then a maximum clique in $V(C)$ has size at most i in $P(D)$.*

Proof. To the contrary, we suppose that $i \geq 3$ and there exists a clique K in $V(C)$ of size at least $i + 1$. Then $|V(K)| \geq 4$. We first assume that C has length at least five. Then, by Lemma 4.8, for each pair of vertices in K , there is a path between them consisting of only chords of C . Therefore, by Proposition 4.4(3), there exists a common out-neighbor w of the vertices in K . Thus w has indegree at least $i + 1$, which contradicts the fact that D is an $(i, 2)$ digraph. Hence C has length at most 4. Then, since $|V(K)| \geq 4$, C has length 4 and so $|V(C)| = |V(K)| = 4$. Thus each vertex on C is incident to a chord. However, since $|\Gamma_H| \geq 1$, there is a vertex on C having two out-neighbors in $V(H)$ by Lemma 4.3(3). Thus the vertex on C is not incident to any chord by Corollary 4.5 and so we reach a contradiction. Hence the statement is true. \square

Theorem 4.10. *Let H be a hole with length $l \geq 5$ in the underlying graph of an $(i, 2)$ digraph D with $i \geq 3$ and C be the cycle of length at least four obtained from H by Γ_H . If*

$$l - |\Gamma_H| \geq 2i + 1 \quad \text{or} \quad l < 3|\Gamma_H|,$$

then the subgraph of the phylogeny graph of D induced by $V(C)$ has a hole.

Proof. By the definition of C , $|V(C)| = l - |\Gamma_H|$. Since $i \geq 3$ by the assumption, each maximum clique in $V(C)$ has size at most i by Lemma 4.9.

We first suppose $l - |\Gamma_H| \geq 2i + 1$. Then $\frac{|V(C)|-1}{2} \geq i$. Since $|V(C)| \leq l - 1$ by Lemma 4.3(1),

$$\frac{|V(C)| - 1}{2} \leq \frac{l - 2}{2} \leq \left\lfloor \frac{l - 1}{2} \right\rfloor.$$

Therefore

$$\left\lfloor \frac{l - 1}{2} \right\rfloor \geq i$$

and so, by Theorem 4.7, the statement is true.

Now we suppose

$$l < 3|\Gamma_H|. \tag{4.1}$$

Let

$$A = \{v \in V(C) \mid |N^+(v) \cap V(H)| = 2\}$$

and

$$B = \{v \in V(C) \mid |N^+(v) \cap V(H)| = 1\}.$$

Take a vertex v on C . Then, by Lemma 4.3(2), $N^+(v) \cap V(H) \neq \emptyset$. Since D is an $(i, 2)$ digraph, $|N^+(v) \cap V(H)| \leq 2$ and so $v \in A \cup B$. Therefore $V(C) = A \sqcup B$ where $A \sqcup B$ represents the disjoint union of sets A and B . Then, since $|A| = |\Gamma_H|$ by Lemma 4.3(3), $|B| = |V(C)| - |A| = l - 2|\Gamma_H|$. Thus $|B| < |A|$ by (4.1). Hence there exist two consecutive vertices u and v on C that belong to A . We take the section $Q := uvw$ of C . Since $\{u, v\} \subseteq A$, neither u nor v is incident to any chord of C by Corollary 4.5. Therefore Q is an induced path of length two. Thus Q can be extended to a hole in $P(D)$ whose vertices are on C by Lemma 4.6. Thus the statement is true. \square

At the end of preparation for the proof of Theorem 4.1, we need a notion of perfect elimination orderings. A *perfect elimination ordering* in a graph is an ordering of the vertices of the graph such that, for each vertex v , v and the neighbors of v that come

after v in the order form a clique. It is well-known that a graph is chordal if and only if it has a perfect elimination ordering. A *simplicial vertex* is one whose neighbors form a clique.

From now on, we use the notation $u \rightarrow v$ (resp. $u \not\rightarrow v$) to represent “ (u, v) is (resp. is not) an arc of a digraph”.

Proof of Theorem 4.1. Suppose $l \geq 3i + 1$. If $i = 2$, then the statement is true by Theorem 1.18. Now we assume $i \geq 3$. Let C be the cycle obtained from H by Γ_H . Since $i \geq 3$, $\frac{l}{2} \geq \frac{3i+1}{2} \geq 5$. Then, since $|V(C)| \geq l - \lfloor \frac{l}{2} \rfloor \geq \frac{l}{2}$ by Lemma 4.3(1), C has length at least five. To the contrary, we suppose, that

$$l - |\Gamma_H| \leq 2i \quad \text{and} \quad l \geq 3|\Gamma_H|.$$

Then $3|\Gamma_H| - |\Gamma_H| \leq l - |\Gamma_H| \leq 2i$ and so we obtain $|\Gamma_H| \leq i$. Since $l \geq 3i + 1$, $l - |\Gamma_H| \geq 2i + 1$, a contradiction. Therefore

$$l - |\Gamma_H| \geq 2i + 1 \quad \text{or} \quad l < 3|\Gamma_H|.$$

Thus the statement is true by Theorem 4.10.

To show the “further” part, we consider an $(i, 2)$ digraph D with the vertex set

$$V(D) = \{u, v_{0,1}, v_{0,2}, v_{0,3}, v_{1,1}, v_{1,2}, v_{1,3}, \dots, v_{i-1,1}, v_{i-1,2}, v_{i-1,3}\}$$

and the arc set

$$A(D) = \{(v_{j,1}, v_{j,2}), (v_{j,2}, v_{j,3}), (v_{j,1}, v_{j-1,3}), (v_{j,2}, u) \mid 0 \leq j \leq i - 1\}$$

(each subscript of the vertices in D is reduced to modulo i and see the digraph given in Figure 4.1 for an illustration). Then $V(D) - \{u\}$ forms a hole H of length $3i$ in $U(D)$. Since $v_{j-1,2} \rightarrow v_{j-1,3}$ and $v_{j,1} \rightarrow v_{j-1,3}$ for each $0 \leq j \leq i - 1$, $C = v_{0,1}v_{0,2}v_{1,1}v_{1,2} \cdots v_{i-1,1}v_{i-1,2}v_{0,1}$ is the cycle obtained from H in $P(D)$. Since u is a common out-neighbor of any pair in $K := \{v_{j,2} \mid 0 \leq j \leq i - 1\}$, K forms a clique in

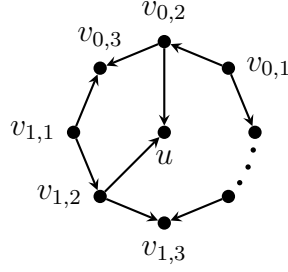


Figure 4.1: The digraph D in the proof for the “further” part of Theorem 4.1

$P(D)$. We can check that for each $0 \leq j \leq i - 1$, in $P(D)$,

$$N(u) = K, \quad N(v_{j,1}) = \{v_{j-1,3}, v_{j-1,2}, v_{j,2}\},$$

$$N(v_{j,2}) = \{u, v_{j,1}, v_{j,3}, v_{j+1,1}\} \sqcup K, \quad \text{and} \quad N(v_{j,3}) = \{v_{j,2}, v_{j+1,1}\}.$$

We note that, for each $0 \leq j \leq i - 1$, $v_{j,3}$ is a simplicial vertex in $P(D) - u$ and $v_{j,1}$ is a simplicial vertex in $P(D) - \{u, v_{0,3}, v_{1,3}, \dots, v_{i-1,3}\}$. Therefore

$$u, v_{0,3}, v_{1,3}, \dots, v_{i-1,3}, v_{0,1}, v_{1,1}, \dots, v_{i-1,1}, v_{0,2}, v_{1,2}, \dots, v_{i-1,2}$$

is a perfect elimination and so $P(D)$ is chordal. Then, since $|V(H)| = 3i$, we conclude that the desired bound $3i + 1$ is achieved by D . \square

4.2 Forbidden subgraphs for phylogeny graphs of degree bounded digraphs

Proposition 4.11. *Let D be an (i, j) digraph and N be a set of neighbors of some vertex in $P(D)$. If any k vertices in N do not form a clique in $P(D)$ for some positive k , then $|N| \leq (k - 1)(j + 1)$.*

Proof. Let u be a vertex such that N is a set of its neighbors in $P(D)$. Suppose that any k vertices in N do not form a clique in $P(D)$ for some positive integer k .

By the definition of phylogeny graph,

$$N(u) = \left(\bigcup_{v \in N^+(u)} N^-(v) - \{u\} \right) \cup N^+(u) \cup N^-(u). \quad (4.2)$$

Take a vertex v in D . Then $N^-(v) \cup \{v\}$ forms a clique in $P(D)$ and so $(N^-(v) \cup \{v\}) \cap N$ is an empty set or forms a clique in $P(D)$. Thus, by our assumption,

$$|N^-(v) \cap N| \leq |(N^-(v) \cup \{v\}) \cap N| \leq k - 1. \quad (4.3)$$

Further, if $v \in N$, then $|N^-(v) \cap N| < |(N^-(v) \cup \{v\}) \cap N|$ and so

$$|N^-(v) \cap N| \leq k - 2. \quad (4.4)$$

We note that $N(u) \cap N = N$ and $u \notin N$. Then, by (4.2), (4.3), and (4.4),

$$\begin{aligned} |N| &\leq \sum_{v \in N^+(u) \cap N} |N^-(v) \cap N| + \sum_{v \in N^+(u) - N} |N^-(v) \cap N| + |N^+(u) \cap N| + |N^-(u) \cap N| \\ &\leq (k - 2) \cdot |N^+(u) \cap N| + (k - 1) \cdot |N^+(u) - N| + |N^+(u) \cap N| + (k - 1) \\ &= (k - 1)(|N^+(u) \cap N| + |N^+(u) - N| + 1) = (k - 1)(|N^+(u)| + 1) \leq (k - 1)(j \boxplus 1). \end{aligned}$$

We say that a graph G is (i, j) *phylogeny-realizable through an (i, j) digraph* if it is the (i, j) phylogeny graph of an (i, j) digraph. (when no confusion is likely to arise, we omit “phylogeny” and “through an (i, j) digraph”)

Proposition 4.12. *If an (i, j) phylogeny graph contains an induced subgraph H isomorphic to $K_{1,l}$ for some positive integer l , then $l \leq j + 1$ and $H \cong K_{1,j+1}$ is realizable.*

Proof. We suppose that an (i, j) phylogeny graph $P(D)$ contains an induced subgraph H isomorphic to $K_{1,j+2}$. Let u be the center of H . Then $V(H) - \{u\}$ is a subset of $N(u)$ such that any two vertices in $V(H) - \{u\}$ do not form a clique in $P(D)$. Therefore $|V(H) - \{u\}| \leq j + 1$ by Proposition 4.11, which is a contradiction.

To show that $H \cong K_{1,j+1}$ is realizable, let D be a digraph with the vertex set

$$V(D) = \{u, v, w_1, \dots, w_j\}$$

and the arc set

$$A(D) = \{(u, v)\} \cup \{(v, w_k) \mid 1 \leq k \leq j\}.$$

Then we can check that D is a $(1, j)$ digraph and $P(D)$ is isomorphic to $K_{1,j+1}$ with the center v and so the statement is true. \square

Lemma 4.13. *If an (i, j) phylogeny graph contains an induced subgraph H isomorphic to $K_{1,j+1}$ with the center v , then $|N^-(v) \cap V(H)| = 1$.*

Proof. Suppose that, for an (i, j) digraph D , $P(D)$ contains an induced subgraph H with the center v isomorphic to $K_{1,j+1}$. Since H is triangle-free, $|N^-(v) \cap V(H)| \leq 1$. To the contrary, suppose that $|N^-(v) \cap V(H)| = 0$. Then, if all the edges in H are cared edges, v has at least $j + 1$ out-neighbors, which contradicts the fact that D is an (i, j) digraph. Therefore H has at least one edge in $U(D)$. Let $|N^+(v) \cap V(H)| = k$ for a positive integer k . Then

$$|N^+(v) - V(H)| \leq j - k \tag{4.5}$$

since v has at most j out-neighbors. Moreover, there are $j + 1 - k$ cared edges incident to v . Let uv be a cared edge. Then, by the definition of cared edges, u and v have a common out-neighbor w . Since H is triangle-free, w belongs to $N^+(v) - V(H)$ and $N^-(w) \cap V(H) = \{u, v\}$. Since the cared edge uv was arbitrarily chosen, we may conclude that the number of cared edges, which equals $j + 1 - k$, is less than or equal to $|N^+(v) - V(H)|$, which contradicts (4.5). \square

Proposition 4.14. *If an (i, j) phylogeny graph contains an induced subgraph H isomorphic to $K_{m,n}$ for some positive integers m and n , then $m \leq j + 1$ and $n \leq j + 1$ where the equalities cannot hold simultaneously, and $H \cong K_{j+1,j}$ is realizable unless $i = 1$.*

Proof. We suppose that there exists an (i, j) phylogeny graph $P(D)$ containing an induced subgraph H isomorphic to $K_{m,n}$ for some positive integers m and n . Take a vertex v in the partite set of H with size m . Let X be the other partite set of H . Then $(\{v\}, X)$ is the bipartition of a subgraph isomorphic to $K_{1,n}$. Thus $n \leq j + 1$ by Proposition 4.12. By symmetry, we conclude $m \leq j + 1$.

To show that either $m < j+1$ or $n < j+1$ by contradiction, suppose $m = n = j+1$. Let v be a vertex in H as before. Then there exists a subgraph H_v in H isomorphic to $K_{1,j+1}$ such that v is the center of H_v . Therefore, by Lemma 4.13, v has one in-neighbor in the subdigraph induced by $V(H_v)$. Then, since H_v is a subgraph of H , v has one in-neighbor in $V(H)$. Since v was arbitrarily chosen from H , each vertex in the subdigraph induced by $V(H)$ has one in-neighbor in $V(H)$. Take a vertex v_1 in $V(H)$. Then there exists an in-neighbor v_2 in $V(H)$. We may repeat this process until we obtain a directed cycle, which contradicts the fact that D is acyclic.

Now we show that $H \cong K_{j+1,j}$ is realizable. We construct an (i, j) digraph whose phylogeny graph contains an induced subgraph isomorphic to $K_{j+1,j}$ for an integer $i \geq 2$. Let D be a digraph with the vertex set

$$V(D) = \{u_1, u_2, \dots, u_{j+1}, v_1, v_2, \dots, v_j\} \cup \{w_{l,m} \mid 1 \leq l, m \leq j\},$$

and the arc set

$$\begin{aligned} A(D) = & \{(u_l, v_l) \mid 1 \leq l \leq j\} \cup \{(v_l, w_{l,m}) \mid 1 \leq l, m \leq j\} \\ & \cup \{(u_l, w_{m,l}) \mid 1 \leq l, m \leq j, l \neq m\} \cup \{(u_{j+1}, w_{l,l}) \mid 1 \leq l \leq j\} \end{aligned}$$

(see the $(2, 2)$ digraph whose phylogeny graph having an induced subgraph isomorphic to $K_{3,2}$ with the bipartition $(\{u_1, u_2, u_3\}, \{v_1, v_2\})$ given in Figure 4.2 for an illustration). Then we can check D is an (i, j) digraph and

$$(\{u_1, u_2, \dots, u_{j+1}\}, \{v_1, v_2, \dots, v_j\})$$

is the bipartition of a subgraph isomorphic to $K_{j+1,j}$ in $P(D)$. □

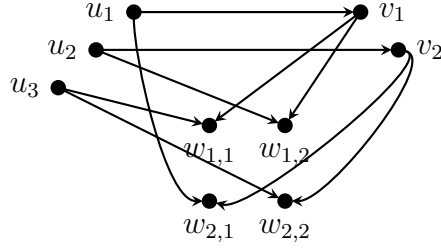


Figure 4.2: A $(2, 2)$ digraph whose phylogeny graph contains $K_{3,2}$ as an induced subgraph

For $i = 1$ or $j = 1$, an (i, j) phylogeny graph is completely characterized by the following theorems. Hereby, we only consider (i, j) phylogeny graphs for $i \geq 2$ and $j \geq 2$.

Theorem 4.15 ([18]). *For a positive integer j , a graph is a $(1, j)$ phylogeny graph if and only if it is a forest with the maximum degree at most $j + 1$.*

Given a graph G , we denote the size of a maximum clique in G by $\omega(G)$. The *clique graph* of a graph G , denoted by $K(G)$, is a simple graph such that (i) every vertex of $K(G)$ represents a maximal clique of G and (ii) two vertices of $K(G)$ are adjacent when they share at least one vertex in common in G .

Theorem 4.16 ([18]). *For a positive integer i , a graph is an $(i, 1)$ phylogeny graph if and only if it is a diamond-free chordal graph with $\omega(G) \leq i + 1$ and its clique graph is a forest.*

Proposition 4.17. *If an (i, j) phylogeny graph contains an induced subgraph H isomorphic to a fan $P_\ell \vee I_1$ or a wheel $C_\ell \vee I_1$ for some positive integers i, j, ℓ with $i, j \geq 2$, then $\ell \leq 2j + 2$, and $H \cong P_{2j+2} \vee I_1$ and $H \cong C_{2j+2} \vee I_1$ are realizable.*

Proof. Suppose there exists an (i, j) digraph D whose phylogeny graph contains an induced subgraph H isomorphic to a fan $P_\ell \vee \{u\}$ or a wheel $C_\ell \vee \{u\}$ in $P(D)$ for some vertex u in D and some positive integer ℓ . Since $j \geq 2$, $2j + 2 \geq 4$ and so the statement is immediately true if $\ell \leq 4$. Suppose $\ell > 4$. Then $V(H) - \{u\} \subseteq N(u)$ and any 3

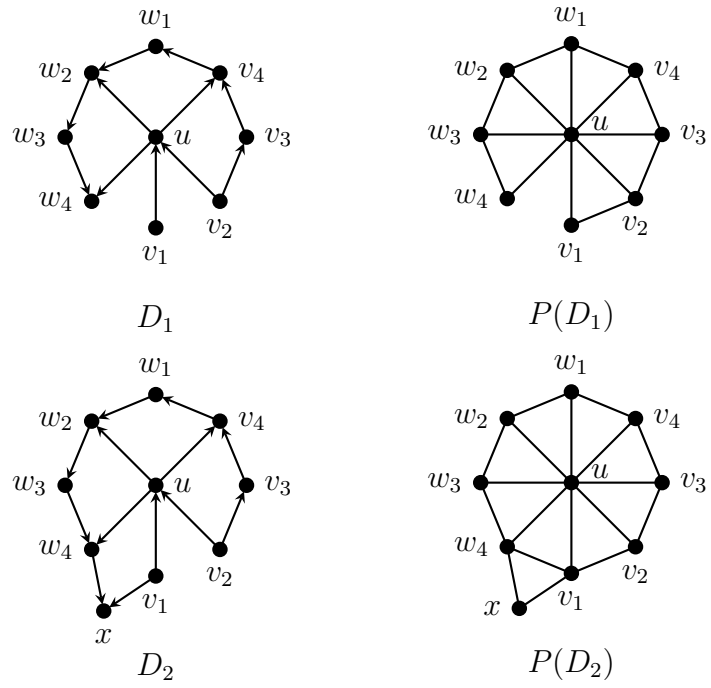


Figure 4.3: The $(2, 3)$ digraphs D_1 and D_2 and the phylogeny graphs $P(D_1)$ and $P(D_2)$ containing $P_8 \vee I_1$ and $C_8 \vee I_1$ as an induced subgraphs, respectively

vertices in $V(H) - \{u\}$ do not form a clique in $P(D)$. Therefore $|V(H) - \{u\}| \leq 2(j+1)$ by Proposition 4.11. Thus $\ell \leq 2(j+1)$.

To show that $H \cong P_{2j+2} \vee I_1$ and $H \cong C_{2j+2} \vee I_1$ are realizable, let D_1 and D_2 be (i, j) digraphs with the vertex sets

$$V(D_1) = \{u, v_1, v_2, v_3, v_4, w_1, \dots, w_{2j-2}\}, \quad V(D_2) = V(D_1) \cup \{x\},$$

the arc sets

$$\begin{aligned} A(D_1) = & \{(u, w_k) \mid k \text{ is an even integer}\} \cup \{(w_k, w_{k+1}) \mid 1 \leq k < 2j - 2\} \\ & \cup \{(v_1, u), (v_2, u), (v_2, v_3), (v_3, v_4), (u, v_4), (v_4, w_1)\} \end{aligned}$$

and $A(D_2) = A(D_1) \cup \{(v_1, x), (w_{2j-2}, x)\}$ (see Figure 4.3 for the digraphs D_1 and D_2 when $i = 2$ and $j = 3$). Then one may check that u is adjacent to $v_1, v_2, v_3, v_4, w_1, \dots, w_{2j-2}$ in both $P(D_1)$ and $P(D_2)$. Moreover, $P := v_1 v_2 v_3 v_4 w_1 w_2 \cdots w_{2j-2}$ is an induced path of length $2j + 1$ in $P(D_1)$ and $C := v_1 v_2 v_3 v_4 w_1 w_2 \cdots w_{2j-2} v_1$ is an induced cycle of length $2j + 2$ in $P(D_2)$, respectively. Thus $P \vee \{u\}$ and $C \vee \{u\}$ are the desired induced subgraphs. Therefore the the statement is true. \square

We call a vertex of indegree 0 in a digraph a *source*.

The following lemma is immediate consequence from the definition of phylogeny graphs.

Lemma 4.18. *Let D be an (i, j) digraph. Suppose that a vertex u has ij neighbors in the phylogeny graph of D and D' is the subdigraph induced by u and these ij neighbors. If u is a source of D' , then the following are true:*

- (1) u has outdegree j in D' ;
- (2) Each out-neighbor v of u has indegree i in D' and $N_D^-(v) \cap N_D^+(u) = \emptyset$;
- (3) $N_D^-(v) \cap N_D^-(w) = \{u\}$ for each pair $\{v, w\}$ of the out-neighbors of u .

Lemma 4.19. *Let D be an acyclic digraph. If u is a source in D and each of its out-neighbors has indegree at least 2, then D has a source distinct from it.*

Proof. Suppose that u is a source in D and each of its out-neighbors has indegree at least 2. Then, since D is acyclic, $D - u$ is acyclic and so there exists a source v in $D - u$. Since each out-neighbor of u has indegree at least 2 in D , v is not an out-neighbor of u in D . Therefore v is a source in D . \square

Now we are ready to extend Theorem 1.21 in Lee *et al.* [39] for an (i, j) digraph.

Theorem 4.20. *Let G be an (i, j) phylogeny graph for positive integers i, j with $i, j \geq 2$. Then $\omega(G) \leq ij$ and the inequality is tight for $i \leq 3$ and $j = 2$.*

Proof. To reach a contradiction, suppose that there is an (i, j) digraph D whose phylogeny graph $P(D)$ contains an induced subgraph H isomorphic to K_{ij+1} . Let D_1 be the subdigraph of D induced by $V(H)$. Since D_1 is acyclic, there is a source u in D_1 . Then $|N_{D_1}^+(u)| = j$ by Lemma 4.18(1). Therefore

$$N_{D_1}^+(u) = N_D^+(u).$$

Let

$$\langle v \rangle^- = N_{D_1}^-(v) - \{u\}$$

for each $v \in N_D^+(u)$. Then, since ij edges in H incident to u , for each $v \in N_D^+(u)$,

$$|\langle v \rangle^-| = i - 1, \quad \langle v \rangle^- \subseteq V(H), \tag{4.6}$$

and

$$\langle v \rangle^- \cap N_D^+(u) = \emptyset \tag{4.7}$$

by Lemma 4.18(2) and

$$\langle v \rangle^- \cap \langle w \rangle^- = \emptyset \tag{4.8}$$

for each pair $\{v, w\}$ of the out-neighbors of u by Lemma 4.18(3). Therefore $N_{D_1}^-(v) = N_D^-(v)$ by (4.6) and so

$$\langle v \rangle^- = N_D^-(v) - \{u\}$$

for each $v \in N_D^+(u)$. We note that $|N_D^+(u)| = j$ and $|\langle v \rangle^-| = i - 1$ for each $v \in N_D^+(u)$

by (4.6). Therefore

$$V(H) = \{u\} \sqcup N_D^+(u) \sqcup \left(\bigsqcup_{v \in N_D^+(u)} \langle v \rangle^- \right). \quad (4.9)$$

Since each out-neighbor of u has indegree i in D_1 by Lemma 4.18(2), D_1 has a source w distinct from u by Lemma 4.19. Then $w \in \bigsqcup_{v \in N_D^+(u)} \langle v \rangle^-$ by (4.9) and so $w \in \langle v_1 \rangle^-$ for some $v_1 \in N_D^+(u)$. Thus, by (4.8),

$$w \not\rightarrow v \quad (4.10)$$

for any $v \in N_D^+(u) - \{v_1\}$. Since w is a source, by Lemma 4.18(1) and (2) applied to w , the out-neighbors of w belong to $V(H)$ and

$$w \not\rightarrow x \quad (4.11)$$

for any $x \in \langle v_1 \rangle^-$. Take v_2 in $N_D^+(u) - \{v_1\}$. Then, by (4.10), $w \not\rightarrow v_2$. Moreover, since w is a source, $v_2 \not\rightarrow w$ and so w and v_2 have a common out-neighbor y_1 in $V(H)$. By (4.11), $y_1 \notin \langle v_1 \rangle^-$. Then, since $u \rightarrow v_2$, and $v_2 \rightarrow y_1$, $u \neq y_1$. If $y_1 \in N_D^+(u)$, then $v_2 \in \langle y_1 \rangle^-$, which contradicts (4.7). Thus $y_1 \notin N_D^+(u)$. Then, since $y_1 \neq u$, $y_1 \in \bigsqcup_{v \in N_D^+(u)} \langle v \rangle^-$ by (4.9). Therefore $y_1 \in \langle v_3 \rangle^-$ for some $v_3 \in N_D^+(u)$. Then $y_1 \rightarrow v_3$. In addition, since $w \rightarrow y_1$, $y_1 \notin \langle v_1 \rangle^-$ by (4.11) and so $v_3 \neq v_1$. We obtain a directed path $P_1 := v_2 \rightarrow y_1 \rightarrow v_3$ whose sequence has two terms v_2 and v_3 belonging to $N_D^+(u) - \{v_1\}$. We note that we only used the fact that v_2 belongs to $N_D^+(u) - \{v_1\}$ to derive the directed path P_1 . Since v_3 also belongs to $N_D^+(u) - \{v_1\}$, we may apply the same argument to obtain a directed path $P_2 := v_3 \rightarrow y_2 \rightarrow v_4$ for some vertices y_2 in $V(H)$ and v_4 in $N_D^+(u) - \{v_1\}$. In the above process, we observe that a directed path $P_1 \rightarrow P_2$ was obtained where $P_a = v_{a+1} \rightarrow y_a \rightarrow v_{a+2}$ for each $a \in \{1, 2\}$, $\{v_2, v_3, v_4\} \subseteq N_D^+(u) - \{v_1\}$, and $\{y_1, y_2\} \subseteq V(H)$. We continue in this way to obtain the directed walk $P := P_1 \rightarrow P_2 \rightarrow \cdots \rightarrow P_{j-1}$. By the way, P contains a closed directed walk. For, v_2, \dots, v_{j+1} belong to $N_D^+(u) - \{v_1\}$ and $|N_D^+(u) - \{v_1\}| = j - 1$ (recall that u has outdegree j) and so $v_l = v_m$ for some

distinct integers $l, m \in \{2, \dots, j + 1\}$. Therefore we reach a contradiction to the fact that D is acyclic. Hence $P(D)$ is K_{ij+1} -free. Consequently, if an (i, j) phylogeny graph contains an induced subgraph isomorphic to K_l for positive integers $i \geq 2$, $j \geq 2$ and l , then $l \leq ij$.

By the digraphs given in Figures 4.4 and 4.5, the inequality is tight when $i \leq 3$ and $j = 2$. Therefore the statement is true. \square

Lemma 4.21. *If there exists an (i, j) phylogeny graph containing an induced subgraph H isomorphic to K_l for a positive integer $l \geq 2$, then, for any positive integer m , there exists an $(i + m, j)$ phylogeny graph containing an induced subgraph isomorphic to K_{l+m} .*

Proof. Suppose that there exists an (i, j) digraph D whose phylogeny graph $P(D)$ contains an induced subgraph H isomorphic to K_l for some positive integer $l \geq 2$. To show the statement, it suffices to construct an $(i + 1, j)$ digraph whose phylogeny graph contains an induced subgraph isomorphic to K_{l+1} .

Let D_1 be the subdigraph induced by $V(H)$ of D . Since D is acyclic, D_1 is acyclic. Then there exists a source u in D_1 . Take a vertex $v \in V(H)$. Then $v = u$ or $v \in N_{D_1}^+(u)$ or u and v have a common out-neighbor, i.e. $v \in N_D^-(x)$ for some $x \in N_D^+(u)$. Since $N_{D_1}^+(u) \subseteq N_D^+(u)$, we have shown

$$V(H) \subseteq \left(\bigcup_{x \in N_D^+(u)} N_D^-(x) \right) \cup N_D^+(u). \quad (4.12)$$

Then, since $|V(H)| = l \geq 2$, there exists a vertex y in $V(H)$ distinct from u . Therefore $y \in \bigcup_{x \in N_D^+(u)} N_D^-(x)$ or $y \in N_D^+(u)$. Thus, in each case, we show $N_D^+(u) \neq \emptyset$. We add a new vertex w and the arc set $\{(w, x) \mid (u, x) \in A(D)\}$ to D . Then the resulting digraph D' is an $(i + 1, j)$ digraph. Moreover, for each v in $V(H)$, v and w have a common out-neighbor or $w \rightarrow v$ by (4.12). Thus $V(H) \cup \{w\}$ forms a clique in $P(D')$. \square

Given an $(i, 2)$ phylogeny graph G for positive integer $i \geq 2$, $\omega(G) \leq 2i$ by Theorem 4.20. Further, if $i \geq 4$, $\omega(G) \leq \frac{3i}{2}$, which is strictly less than $2i$, by the

following theorem.

Theorem 4.22. *Let G be an $(i, 2)$ phylogeny graph for a positive integer $i \geq 4$. Then $\omega(G) \leq \frac{3i}{2} + 1$ and the inequality is tight.*

Proof. Suppose, to the contrary, that there exists an $(i, 2)$ digraph D whose phylogeny graph contains an induced subgraph isomorphic to K_l for some positive integers $i \geq 4$ and $l > \frac{3i}{2} + 1$. It suffices to consider the case where l is the minimum satisfying the inequality and so we may assume

$$l = \left\lfloor \frac{3i}{2} \right\rfloor + 2 = \left\lfloor \frac{i}{2} \right\rfloor + i + 2.$$

If i is even, then there exists an $(i + 1, 2)$ digraph \hat{D} whose phylogeny graph contains an induced subgraph isomorphic to $K_{\lfloor \frac{i+1}{2} \rfloor + (i+1)+2}$ by Lemma 4.21 (note that $(\frac{i}{2} + i + 2) + 1 = \lfloor \frac{i+1}{2} \rfloor + (i + 1) + 2$). By replacing D with \hat{D} , we may assume that i is odd. Therefore D is a $(2k - 1, 2)$ digraph whose phylogeny graph contains an induced subgraph H isomorphic to K_{3k} for some integer $k \geq 3$. We assume that

- (A) D has the smallest number of arcs among the $(i, 2)$ digraphs with the vertex set $V(D)$ whose phylogeny graphs contain a subgraph isomorphic to K_{3k} .

Then

- (B) every vertex not belonging to $V(H)$ has no out-neighbor in D .

Let D_1 be the subdigraph induced by $V(H)$ of D . Then $V(D_1)$ forms a clique in $P(D)$. Moreover, D_1 is acyclic and so D_1 has a source. If a source in D_1 has outdegree at most 1 in D , then it is adjacent to at most $2k - 1$ vertices in H and so it is nonadjacent to some vertex in H in $P(D)$, which is impossible. Therefore each source in D_1 has outdegree 2 in D . Take a source u in D_1 . Then

$$N_D^+(u) = \{v, w\}$$

for some vertices v, w in D . For simplicity, we let

$$[u]^+ = N_{D_1}^+(u), \quad [v]^- = N_D^-(v) \cap V(H), \quad \text{and} \quad [w]^- = N_D^-(w) \cap V(H).$$

Take a vertex $h \in V(H) - \{u\}$. Then h is adjacent to u . Therefore one of the following is true: (i) $h \in [u]^+$; (ii) h is an in-neighbor of exactly one of v and w , that is, h belongs to the symmetric difference $[v]^- \Delta [w]^-$; (iii) h is a common in-neighbor of v and w , i.e. $h \in [v]^- \cap [w]^- - \{u\}$. This together with the fact that $u \in [v]^- \cap [w]^-$ implies that

$$V(H) = ([v]^- \Delta [w]^- - [u]^+) \sqcup [u]^+ \sqcup ([v]^- \cap [w]^-). \quad (4.13)$$

Thus

$$3k = |[v]^- \Delta [w]^- - [u]^+| + |[u]^+| + |[v]^- \cap [w]^-|. \quad (4.14)$$

Hence

$$3k \leq |[v]^- \Delta [w]^-| + |[u]^+| + |[v]^- \cap [w]^-| = |[v]^- \cup [w]^-| + |[u]^+|.$$

Therefore

$$\begin{aligned} |[v]^- - [w]^-| &= |[v]^- \cup [w]^-| - |[w]^-| \geq |[v]^- \cup [w]^-| - (2k - 1) \\ &\geq 3k - |[u]^+| - (2k - 1) = k + 1 - |[u]^+| \end{aligned}$$

and so

$$|[v]^- - [w]^-| \geq k + 1 - |[u]^+|. \quad (4.15)$$

Since there is no distinction between v and w ,

$$|[w]^- - [v]^-| \geq k + 1 - |[u]^+|. \quad (4.16)$$

By our assumption, $k \geq 3$ and $|[u]^+| \leq 2$. Therefore we have the following by (4.15) and (4.16):

$$|[v]^- - [w]^-| \geq 2 \quad \text{and} \quad |[w]^- - [v]^-| \geq 2. \quad (4.17)$$

Suppose $|[u]^+| = 0$. Then $\{v, w\} \cap V(H) = \emptyset$ and so, by the property (B), $N_D^+(v) = N_D^+(w) = \emptyset$. Let D_2 be the subdigraph of D_1 induced by $[v]^- \Delta [w]^-$. Then D_2 is acyclic and so D_2 has a source, namely u' . Without loss of generality, we may assume

$u' \in [w]^- - [v]^-$. Then we obtain the digraph D' by deleting the arc (u', w) and adding the arc (v, w) in D . We can check that D' is a $(2k - 1, 2)$ digraph and the graph induced by $(V(H) - \{u'\}) \cup \{v\}$ of $P(D')$ is isomorphic to K_{3k} . Thus it suffices to consider the case

$$|[u]^+| \geq 1.$$

Claim A. If $v \in [u]^+$ (resp. $w \in [u]^+$) and $v \rightarrow w$ (resp. $w \rightarrow v$), then $N_D^+(v) = \{w\}$ (resp. $N_D^+(w) = \{v\}$).

Proof of Claim A. Suppose that $v \in [u]^+$, $v \rightarrow w$, and $N_D^+(v) \neq \{w\}$. Then v has the other out-neighbor w' distinct from w . By (4.13), $V(H) = [v]^- \cup [w]^- \cup [u]^+$. Let D'' be the digraph obtained from D by deleting the arc (v, w') . The adjacency of two vertices except v in $V(H)$ does not change in $P(D'')$. Now take $a \in V(H) - \{v\}$. Then $a \in [v]^- \cup [w]^- \cup [u]^+$. If $a \in [v]^-$, then $(a, v) \in A(D'')$. If $a \in [w]^-$, then w is a common out-neighbor of a and v in D'' . If $a \in [u]^+$, then $a = w$ and so $(v, a) = (v, w) \in A(D'')$. Thus, in each case, v is adjacent to a in $P(D'')$. Hence $P(D'')$ contains H . Since $D'' \neq D$, we reach a contradiction to the property (A). Thus $N_D^+(v) = \{w\}$. Since there is no distinction between v and w , $N_D^+(w) = \{v\}$ if $w \in [u]^+$ and $w \rightarrow v$. Therefore Claim A is true. \square

Without loss of generality, we may assume

$$v \in [u]^+.$$

Claim B. If there is no arc between v and w , then there exists a vertex v' distinct from v and w such that

$$([v]^- - [w]^-) \sqcup ([w]^- - [v]^- - \{v'\}) \subseteq N_D^-(v').$$

Proof of Claim B. Suppose that there is no arc between v and w . There exists a vertex w_1 in $[w]^- - [v]^-$ by (4.17). Suppose $w_1 \rightarrow a$ for some $a \in [v]^- - [w]^-$. Then $N^+(w_1) = \{a, w\}$. Since $v \in [u]^+ \subseteq V(D_1)$, w_1 and v are adjacent. Then, since $v \not\rightarrow a$ and there is no arc between v and w , w_1 and v have no common out-neighbor and

so $v \rightarrow w_1$. Thus $v \rightarrow w_1 \rightarrow a \rightarrow v$ is a directed cycle, which is impossible. Hence $N_D^+(w_1) \cap ([v]^- - [w]^-) = \emptyset$. Since w_1 was arbitrarily chosen from $[w]^- - [v]^-$, we conclude that

$$N_D^+(z) \cap ([v]^- - [w]^-) = \emptyset \quad (4.18)$$

for each z in $[w]^- - [v]^-$. Take a vertex v_1 in $[v]^- - [w]^-$. If $N_D^+(v_1) = \{v\}$, then v_1 is not adjacent to any vertex in $[w]^- - [v]^-$ by (4.18) (note that $v \not\rightarrow w$). Thus $N_D^+(v_1) = \{v, v'\}$ for some vertex v' distinct from v . Then $v' \neq w$. Take a vertex w_1 in $[w]^- - [v]^- - \{v'\}$. Then v_1 and w_1 are adjacent. By (4.18), $w_1 \not\rightarrow v_1$. Since $w_1 \not\rightarrow v$, $w_1 \rightarrow v'$. Since w_1 was arbitrarily chosen from $[w]^- - [v]^- - \{v'\}$, v' is an out-neighbor for each in $[w]^- - [v]^- - \{v'\}$. Thus

$$\{v_1\} \sqcup ([w]^- - [v]^- - \{v'\}) \subseteq N_D^-(v'). \quad (4.19)$$

Suppose $v_2 \not\rightarrow v'$ for some v_2 in $[v]^- - [w]^- - \{v_1\}$. There exists a vertex w_2 in $[w]^- - [v]^- - \{v'\}$ by (4.17). Then $w_2 \rightarrow w$. Moreover, by (4.19), $w_2 \rightarrow v'$. Thus $N^+(w_2) = \{v', w\}$. Hence v_2 and w_2 have no common prey. Then, since v_2 and w_2 are adjacent and $w_2 \not\rightarrow v_2$ by (4.18), $v_2 \rightarrow w_2$ and so $N_D^+(v_2) = \{v, w_2\}$. If $v' \in [w]^- - [v]^-$, then $v' \not\rightarrow v_2$ by (4.18) and so v' and v_2 are not adjacent, which is impossible. Thus $v' \notin [w]^- - [v]^-$. Then there exists a vertex w_3 in $[w]^- - [v]^- - \{w_2\}$ by (4.17). Since $v' \notin [w]^- - [v]^-$, $w_3 \neq v'$. In addition, $v_2 \not\rightarrow w_3$ and $w_3 \not\rightarrow v_2$. Since v_2 and w_3 are adjacent, $w_3 \rightarrow v_2$ and so, by (4.19), $\{v', w, w_2\} \subseteq N_D^+(w_3)$, which is impossible. Thus there exists an arc from any vertex in $[v]^- - [w]^- - \{v_1\}$ to v' , that is, $[v]^- - [w]^- - \{v_1\} \subseteq N_D^-(v')$. Hence, by (4.19),

$$([v]^- - [w]^-) \sqcup ([w]^- - [v]^- - \{v'\}) \subseteq N_D^-(v'). \quad \square$$

Case 1. $|[u]^+| = 2$. Then $[u]^+ = \{v, w\}$. If $w \rightarrow y$ for some $y \in [v]^-$ and $v \rightarrow z$ for some $z \in [w]^-$, then $w \rightarrow y \rightarrow v \rightarrow z \rightarrow w$ is a closed directed walk, which is impossible. Thus either $w \not\rightarrow y$ for any $y \in [v]^-$ or $v \not\rightarrow z$ for any $z \in [w]^-$. Since

$[u]^+ = \{v, w\}$, we may assume, without loss of generality, that

$$w \not\rightarrow y \tag{4.20}$$

for any $y \in [v]^-$. There exists a vertex v_1 in $[v]^- - [w]^- - \{w\}$ by (4.17). Then $w \not\rightarrow v_1$, $v_1 \not\rightarrow w$, and w and v_1 are adjacent. Therefore there exists a common out-neighbor v^* of w and v_1 .

Subcase 1-1. $w \rightarrow v$. Then, by Claim A, $N^+(w) = \{v\}$ and so $v^* = v$. Thus

$$v \in N_D^+(v_1).$$

Since $w \rightarrow v$,

$$v \not\rightarrow w.$$

If $v \rightarrow z$ for some $z \in [w]^-$, then $v \rightarrow z \rightarrow w \rightarrow v$ is a directed cycle. Thus

$$v \not\rightarrow z \tag{4.21}$$

for any $z \in [w]^-$. There exists a vertex w_1 in $[w]^- - [v]^-$ by (4.17). Then $v \not\rightarrow w_1$, $w_1 \not\rightarrow v$, and v and w_1 are adjacent. Therefore there exists a common out-neighbor w^* of v and w_1 . Thus $w^* \notin [v]^-$ and $w^* \neq w$. Hence $N_D^+(w_1) = \{w, w^*\}$. In addition, $w^* \notin [w]^-$ by (4.21) and so

$$w^* \notin [v]^- \cup [w]^-.$$

Since v_1 and w_1 are adjacent and $w_1 \not\rightarrow v_1$, $v_1 \rightarrow w^*$ or $v_1 \rightarrow w_1$. By the way, there exists a vertex w_2 in $[w]^- - [v]^- - \{w_1\}$ by (4.17). Since $v \not\rightarrow w_2$ and $w_2 \not\rightarrow v$, v and w_2 have a common out-neighbor w^{**} . Then $w^{**} \notin [w]^-$ by (4.21). In addition, $w^* \notin [v]^-$ and $w^{**} \neq w$. Then $N_D^+(w_2) = \{w, w^{**}\}$. Suppose $v_1 \rightarrow w_1$. Then $N_D^+(v_1) = \{v, w_1\}$. Since v_1 and w_2 are adjacent, $w^{**} = w_1$ and so $w^{**} \in [w]^-$, which is impossible. Thus $v_1 \rightarrow w^*$ and so $N_D^+(v_1) = \{v, w^*\}$. Since v_1 and w_2 are adjacent and $w^* \neq w_2$, we conclude $w^* = w^{**}$. Since w_2 was arbitrarily chosen from $[w]^- - [v]^- - \{w_1\}$,

$$[w]^- - [v]^- \subseteq N_D^-(w^*).$$

Moreover, $v_1 \in N_D^-(w^*)$. Since v_1 was arbitrary chosen from $[v]^- - [w]^- - \{w\}$,

$$[v]^- - [w]^- - \{w\} \subseteq N_D^-(w^*).$$

Since $w \rightarrow v$,

$$v \notin [w]^- - [v]^-. \quad (4.22)$$

Then, since $v \in N_D^-(w^*)$,

$$([v]^- - [w]^- - \{w\}) \sqcup ([w]^- - [v]^-) \sqcup \{v\} \subseteq N_D^-(w^*).$$

Let $s = |[v]^- - [w]^- - \{w\}|$ and $t = |[w]^- - [v]^-|$. Since $w \rightarrow v$,

$$w \in [v]^- - [w]^-. \quad (4.23)$$

Then, by (4.15) and (4.16), $s \geq k - 2$ and $t \geq k - 1$. Then, since $|N_D^-(w^*)| \leq 2k - 1$,

$$(s, t) \in \{(k - 2, k - 1), (k - 1, k - 1), (k - 2, k)\}.$$

and so $s + t \leq 2k - 2$. Then, since $[u]^+ = \{v, w\}$,

$$[v]^- \Delta [w]^- - [u]^+ = ([v]^- - [w]^- - \{w\}) \sqcup ([w]^- - [v]^-)$$

by (4.22) and (4.23), and so $|[v]^- \Delta [w]^- - [u]^+| = s + t$. Accordingly, by (4.14),

$$|[v]^- \cap [w]^-| = 3k - (s + t + 2) \geq 3k - 2k = k. \quad (4.24)$$

Thus $|[v]^- \cap [w]^-| \geq k$. Hence

$$2k - 1 \geq |N_D^-(w)| \geq |[w]^- - [v]^-| + |[v]^- \cap [w]^-| \geq k - 1 + k = 2k - 1$$

and so $|N_D^-(w)| = 2k - 1$. Therefore $t = |[w]^- - [v]^-| = k - 1$ and $|[v]^- \cap [w]^-| = k$.

Then $s + t = 2k - 2$ by (4.24) and so $s = |[v]^- - [w]^- - \{w\}| = k - 1$. Further, since

$$\{w\} \sqcup ([v]^- - [w]^- - \{w\}) \sqcup ([v]^- \cap [w]^-) \subseteq N_D^-(v),$$

$$|\{w\}| + |[v]^- - [w]^- - \{w\}| + |[v]^- \cap [w]^-| = 1 + (k-1) + k = 2k \leq |N_D^-(v)|,$$

which is impossible. Therefore Subcase 1-1 cannot happen, i.e. $w \not\rightarrow v$.

Subcase 1-2. $v \rightarrow w$. Then, by Claim A, $N_D^+(v) = \{w\}$. Thus $v \not\rightarrow z$ for any $z \in [w]^-$. Hence, by (4.20), the argument obtained by replacing v with w and adjusting other vertices based upon the replacement in the argument for Subcase 1-1 may be applied to reach a contradiction.

Subcase 1-3. There is no arc between v and w . Then, by Claim B, there exists a vertex v' such that

$$([v]^- - [w]^-) \sqcup ([w]^- - [v]^- - \{v'\}) \subseteq N_D^-(v').$$

Then $v_1 \rightarrow v'$. Since $N_D^+(v_1) = \{v, v'\}$, $v' = v^*$. Since $w \rightarrow v^*$, $v^* \notin [w]^- - [v]^-$ and so

$$([v]^- - [w]^-) \sqcup ([w]^- - [v]^-) \sqcup \{w\} \subseteq N_D^-(v^*).$$

Then, since $|N_D^-(v^*)| \leq 2k-1$, $|[v]^- - [w]^-| = |[w]^- - [v]^-| = k-1$ by (4.15) and (4.16). Hence

$$([v]^- - [w]^-) \sqcup ([w]^- - [v]^-) \sqcup \{w\} = N_D^-(v^*).$$

Then $v \not\rightarrow v^*$. Take a vertex z in $[w]^- - [v]^-$. Therefore $N_D^+(z) = \{v^*, w\}$. Then, since $v \not\rightarrow w$, v and z have no common out-neighbor. Since v is adjacent to z , $z \rightarrow v$ or $v \rightarrow z$. Thus $z \in N_D^+(v)$. Since z was arbitrarily chosen from $[w]^- - [v]^-$, $[w]^- - [v]^- \subseteq N_D^+(v)$ and so, by (4.17), $N_D^+(v) = [w]^- - [v]^-$. Thus v and w have no common out-neighbor. Moreover, since $w \not\rightarrow v$ and $v \not\rightarrow w$, v and w are not adjacent, which is a contradiction.

Case 2. $|[u]^+| \neq 2$. Then

$$[u]^+ = \{v\}.$$

Thus $w \notin V(H)$ and so, by the property (B),

$$N_D^+(w) = \emptyset.$$

Suppose $v \not\rightarrow w$. Then, by Claim B, there exists a vertex v' such that

$$([v]^- - [w]^-) \sqcup ([w]^- - [v]^- - \{v'\}) \subseteq N_D^-(v').$$

Then

$$k + k - 1 \leq |([v]^- - [w]^-)| + |([w]^- - [v]^- - \{v'\})| \leq |N_D^-(v')| \leq 2k - 1$$

by (4.15) and (4.16). Thus $|[v]^- - [w]^-| = k$, $|[w]^- - [v]^- - \{v'\}| = k - 1$,

$$v' \in [w]^- - [v]^- , \tag{4.25}$$

and

$$([v]^- - [w]^-) \sqcup ([w]^- - [v]^- - \{v'\}) = N_D^-(v'). \tag{4.26}$$

Then, since $v \notin [w]^-$, $v \notin N_D^-(v')$ by (4.26) and so $v \not\rightarrow v'$. Since $|[w]^- - [v]^- - \{v'\}| = k - 1 \geq 2$, there are two vertices w_1 and w_2 in $[w]^- - [v]^- - \{v'\}$. Then $N_D^+(w_1) = N_D^+(w_2) = \{w, v'\}$ and so each of w_1 and w_2 shares no out-neighbor with v . Therefore $N_D^+(v) = \{w_1, w_2\}$. Then, since $\{w_1, w_2\} \subseteq [w]^- - [v]^- - \{v'\}$, v and v' have no common out-neighbor by (4.26). In addition, since $v' \in [w]^- - [v]^-$ by (4.25), $v' \not\rightarrow v$. Hence v and v' are not adjacent, which is impossible. Consequently, we have shown

$$v \rightarrow w.$$

Then, by Claim A,

$$N_D^+(v) = \{w\}.$$

Let D_3 be the subdigraph of D induced by $[v]^- \Delta [w]^- - \{v\}$. Since D_3 is acyclic, D_3 has a source, say x . Then $x \in [v]^- - [w]^-$ or $x \in [w]^- - [v]^-$.

Then we claim the following

Claim C. $x \in [v]^- - [w]^-$ and there exists an out-neighbor x^* of x such that $[v]^- \Delta [w]^- - \{v, x^*\} = N_D^-(x^*)$.

Proof of Claim C. Let α denote a vertex between v and w with $x \in [\alpha]^- - [\beta]^-$ and β denote the other vertex. If $N_D^+(x) = \{\alpha\}$, then x cannot be adjacent to any vertex in $[\beta]^- - [\alpha]^- - \{\alpha\}$ since x is a source in D_3 . Thus $N_D^+(x) \neq \{\alpha\}$ and so $N_D^+(x) = \{\alpha, x^*\}$ for some x^* .

To show $N_D^-(x^*) \subseteq [v]^- \Delta [w]^- - \{v, x^*\}$, take $b \in N_D^-(x^*)$. Then $b \in V(H)$ by the property (B). Moreover, $b \neq x^*$. If $b \in [v]^- \cap [w]^-$, then $N_D^+(b) = \{v, w, x^*\}$, which is impossible. Therefore $b \notin [v]^- \cap [w]^-$. By (4.13), $b \in [v]^- \Delta [w]^- - [u]^+$ or $b \in [u]^+$. If $b \in [u]^+$, then $b = v$ and so $N_D^+(b) = N_D^+(v) = \{w\}$, which implies $b \notin N_D^-(x^*)$. Thus $b \in [v]^- \Delta [w]^- - [u]^+$. Then, since $b \neq x^*$, $b \in [v]^- \Delta [w]^- - \{v, x^*\}$ and so

$$N_D^-(x^*) \subseteq [v]^- \Delta [w]^- - \{v, x^*\}. \quad (4.27)$$

Since $x \in [\alpha]^- - [\beta]^-$, $x^* \neq \beta$. Moreover, since x is a source in D_3 and x is adjacent to any vertex in $[\beta]^- - [\alpha]^- - \{\alpha, x^*\}$, x^* is an out-neighbor of any vertex in $[\beta]^- - [\alpha]^- - \{\alpha, x^*\}$. Thus

$$[\beta]^- - [\alpha]^- - \{\alpha, x^*\} \subseteq N_D^-(x^*) \quad (4.28)$$

and

$$N_D^+(z) = \{x^*, \beta\} \quad (4.29)$$

for each z in $[\beta]^- - [\alpha]^- - \{\alpha, x^*\}$.

To show $[\alpha]^- - [\beta]^- - \{\alpha, x^*\} \subseteq N_D^-(x^*)$, we note that $[\alpha]^- - [\beta]^- - \{\alpha, x^*\} = [\alpha]^- - [\beta]^- - \{x^*\}$. Take a vertex y_1 in $[\alpha]^- - [\beta]^- - \{x^*\}$. To the contrary, suppose $y_1 \notin N_D^-(x^*)$, i.e. $y_1 \not\rightarrow x^*$. Then, since $x \rightarrow x^*$, $y_1 \neq x$. Take z_1 in $[\beta]^- - [\alpha]^- - \{\alpha, x^*\}$. Such a vertex exists since $|[\beta]^- - [\alpha]^- - \{\alpha, x^*\}| \geq 1$ by (4.15) and (4.16). Then y_1 and z_1 are adjacent, so $y_1 \rightarrow z_1$ by (4.29). Therefore $N^+(y_1) = \{\alpha, z_1\}$. Since z_1 was arbitrarily chosen from $[\beta]^- - [\alpha]^- - \{\alpha, x^*\}$, $[\beta]^- - [\alpha]^- - \{\alpha, x^*\} \subseteq N^+(y_1)$ and so $[\beta]^- - [\alpha]^- - \{\alpha, x^*\} = \{z_1\}$. Thus $[\beta]^- - [\alpha]^- \subseteq \{\alpha, x^*, z_1\}$. Since $|[\beta]^- - [\alpha]^-| \geq k \geq 3$ by (4.15) and (4.16), $[\beta]^- - [\alpha]^- = \{\alpha, x^*, z_1\}$. Then x^* and y_1 are adjacent. Since

$y_1 \not\rightarrow x^*$ and x^* and y_1 share no out-neighbor, $x^* \rightarrow y_1$. Thus $y_1 \rightarrow z_1 \rightarrow x^* \rightarrow y_1$ is a directed cycles, which is impossible. Hence $y_1 \rightarrow x^*$. Since y_1 was arbitrarily chosen from $[\alpha]^- - [\beta]^- - \{x^*\}$,

$$[\alpha]^- - [\beta]^- - \{\alpha, x^*\} = [\alpha]^- - [\beta]^- - \{x^*\} \subseteq N_D^-(x^*).$$

Then

$$[v]^- \Delta [w]^- - \{\alpha, x^*\} \subseteq N_D^-(x^*)$$

by (4.28). Since $v \in [v]^- \Delta [w]^-$, $\alpha = v$ and $[v]^- \Delta [w]^- - \{v, x^*\} = N_D^-(x^*)$ by (4.27). \square

Since $v \rightarrow w$ and $\{v\} = [u]^+ \subseteq [v]^- \Delta [w]^-$,

$$([v]^- \Delta [w]^-) \sqcup ([v]^- \cap [w]^-) = V(H)$$

by (4.13). Thus, by Claim C,

$$N_D^-(x^*) \sqcup (([v]^- \Delta [w]^-) \cap \{v, x^*\}) \sqcup ([v]^- \cap [w]^-) = V(H). \quad (4.30)$$

Then, since $|N_D^-(x^*)| \leq 2k - 1$ and $|V(H)| = 3k$, we conclude

$$|[v]^- \cap [w]^-| \geq k - 1. \quad (4.31)$$

By the way, since $[u]^+ = \{v\}$ and $v \in [w]^-$, $|[v]^- \cup [w]^-| = 3k$ by (4.13). Then, since $|[u]^+| = 1$,

$$|[v]^- - [w]^-| \geq k \quad \text{and} \quad |[w]^- - [v]^-| \geq k \quad (4.32)$$

by (4.15) and (4.16). Therefore $|[v]^- \Delta [w]^-| \geq 2k$. Since $|[v]^- \cup [w]^-| = 3k$,

$$|[v]^- \cap [w]^-| = |[v]^- \cup [w]^-| - |[v]^- \Delta [w]^-| \leq 3k - 2k = k.$$

If $|[v]^- \cap [w]^-| = k$, then, by (4.32), $|[v]^-| = |[v]^- \cap [w]^-| + |[v]^- - [w]^-| \geq 2k$, which

is impossible. Suppose $|[v]^- \cap [w]^-| = k - 1$. Then $|[v]^- \Delta [w]^-| = 2k + 1$ and so

$$|[v]^- - [w]^-| = k + 1 \quad \text{or} \quad |[w]^- - [v]^-| = k + 1$$

Hence $|[v]^-| = |[v]^- \cap [w]^-| + |[v]^- - [w]^-| \geq 2k$ or $|[w]^-| = |[v]^- \cap [w]^-| + |[w]^- - [v]^-| \geq 2k$, which is impossible. Therefore $|[v]^- \cap [w]^-| \leq k - 2$, which contradicts (4.31). Thus we have shown that there is no $(2k - 1, 2)$ digraph D whose phylogeny graph contains an induced subgraph isomorphic to K_{3k} and so we conclude $\omega(G) \leq \frac{3i}{2} + 1$.

To show that the inequality is tight, we present a $(2k, 2)$ digraph and a $(2k + 1, 2)$ digraph each of whose phylogeny graphs contains K_{3k+1} and K_{3k+2} as an induced subgraph, respectively, for any integer $k \geq 2$. Fix an integer $k \geq 2$. Let D_1 be a $(2k, 2)$ digraph with

$$V(D_1) = \{u, v, w, x_1, x_2, \dots, x_{2k-1}, y_1, \dots, y_{k-1}, z\}$$

and

$$\begin{aligned} A(D_1) = & \{(u, v), (u, w), (v, w), (w, z)\} \cup \bigcup_{i=1}^{2k-1} \{(x_i, v)\} \\ & \cup \bigcup_{i=1}^{k-1} \{(y_i, w), (y_i, z)\} \cup \bigcup_{i=1}^{k-1} \{(x_i, w)\} \cup \bigcup_{i=k}^{2k-1} \{(x_i, z)\} \end{aligned}$$

(see the $(4, 2)$ digraph given in Figure 4.6 for an illustration). In the following, we show that $V(D_1) - \{z\}$ forms a clique of size $3k + 1$ in $P(D_1)$. We note that

$$N^+(u) = \{v, w\}, \quad v \in N^+(u) \cap N^+(x_i), \quad \text{and} \quad w \in N^+(u) \cap N^+(y_j)$$

for each $1 \leq i \leq 2k - 1$ and $1 \leq j \leq k - 1$. Therefore u is adjacent to the vertices in $V(D_1) - \{u, z\}$. We can check that

$$N^+(v) = \{w\}, \quad v \in N^+(x_i), \quad \text{and} \quad w \in N^+(v) \cap N^+(y_j)$$

for each $1 \leq i \leq 2k - 1$ and $1 \leq j \leq k - 1$. Therefore v is adjacent to the vertices in

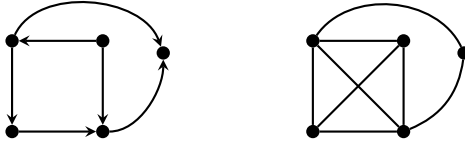


Figure 4.4: A (2,2) digraph and its phylogeny graph

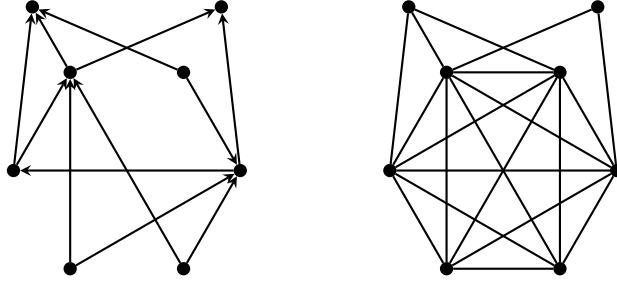


Figure 4.5: A (3,2) digraph and its phylogeny graph

$V(D_1) - \{v, z\}$. Since

$$w \in N^+(x_i) \cap N^+(y_i) \quad \text{and} \quad z \in N^+(w) \cap N^+(x_j)$$

for each $1 \leq i \leq k-1$ and $k \leq j \leq 2k-1$, w is adjacent to the vertices in $V(D_1) - \{w\}$. Take x_i for some $i \in \{1, \dots, 2k-1\}$. Since $\{x_1, \dots, x_{2k-1}\} \subseteq N^-(v)$, $\{x_1, \dots, x_{2k-1}\}$ forms a clique. If $1 \leq i \leq k-1$, then $x_i \rightarrow w$ and so w is a common out-neighbor of x_i and y_j for each $1 \leq j \leq k-1$. If $k \leq i \leq 2k-1$, then $x_i \rightarrow z$ and so z is a common out-neighbor of x_i and y_j for each $1 \leq j \leq k-1$. Therefore x_i is adjacent to the vertices in $\{y_1, \dots, y_{k-1}\}$. Thus x_i is adjacent to the vertices in $V(D_1) - \{x_i, z\}$. Since $\{y_1, \dots, y_{k-1}\} \subseteq N^-(w)$, $\{y_1, \dots, y_{k-1}\}$ forms a clique. Therefore we have shown that $V(D_1) - \{z\}$ forms a clique in $P(D_1)$. Then, by Lemma 4.21, we obtain a $(2k+1, 2)$ digraph whose phylogeny graph contains an induced subgraph isomorphic to K_{3k+2} . Hence we have shown that the inequality is tight. \square

Proof of Theorem 4.2. Propositions 4.12, 4.14, 4.17, and Theorems 4.20 and 4.22

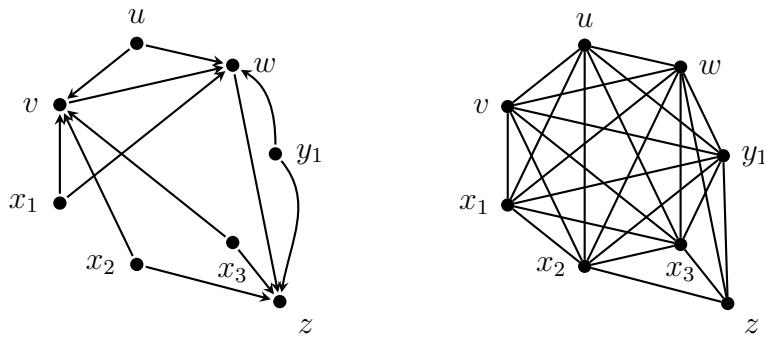


Figure 4.6: A $(4,2)$ digraph and its phylogeny graph

may be summarized in the aspect of forbidden subgraphs in an (i, j) phylogeny graph with $i, j \geq 2$. □

Chapter 5

On CCE graphs of $\langle 2, 2 \rangle$ digraphs¹

The $\langle i, j \rangle$ digraph is a simple digraph satisfying $d^-(x) \leq i$ and $d^+(x) \leq j$ for every vertex x in $V(D)$. By definition, a (i, j) digraph is a $\langle i, j \rangle$ digraph. Given a graph G , we say that G is a $\langle i, j \rangle$ CCE graph if it is the CCE graph of a $\langle i, j \rangle$ digraph.

Proposition 5.1. *The degree of each vertex in a $\langle 2, 2 \rangle$ CCE graph is less than or equal to 2. That is, a $\langle 2, 2 \rangle$ CCE graph has only path components and cycle components, where we identify an isolated vertex with a trivial path.*

Proof. Let D be a $\langle 2, 2 \rangle$ digraph. Take a vertex v of $CCE(D)$. Since D is a $\langle 2, 2 \rangle$ digraph and $CCE(D)$ is a simple graph, the ends of each edge incident to v in $CCE(D)$ have a common prey which is different from a common prey of the ends of another edge incident to v . This implies that v has preys in D at least the number of edges incident to v in $CCE(D)$. Since v has at most 2 prey by $\langle 2, 2 \rangle$ digraph D , the degree of v in $CCE(D)$ is at most 2. \square

The following is an immediate consequence of the definitions of CCE graph and $\langle 2, 2 \rangle$ digraph.

Lemma 5.2. *Let D be a $\langle 2, 2 \rangle$ digraph and u be a vertex which has degree 2 in $CCE(D)$. Then the following are true:*

¹The material in this chapter is written based on the manuscript ‘Interval competition-common enemy graphs of degree-bounded digraphs’ by Myungho Choi, Hojin Chu, and Suh-Ryung Kim. The author thanks the coauthors for allowing him to use its contents for his thesis.

- (i) $d^+(u) = d^-(u) = 2$;
- (ii) two prey (resp. predators) of u have u as the only common predator (resp. prey);
- (iii) $N^+(u) \neq N^+(v)$ and $N^-(u) \neq N^-(v)$ for any vertex v in $V(D) - \{u\}$;
- (iv) each prey (resp. predator) of u is a common prey (resp. predator) of u and one of its neighbors in $CCE(D)$.

Given a $\langle 2, 2 \rangle$ digraph D , $CCE(D) = CCE(D^{\leftarrow})$ where D^{\leftarrow} is the digraph obtained from D by reversing the direction of each arc in D . Thus, given a $\langle 2, 2 \rangle$ CCE graph G and the collection \mathcal{D} of $\langle 2, 2 \rangle$ digraphs each of whose CCE graph is G , if a digraph D chosen arbitrarily from \mathcal{D} has a property α , then D^{\leftarrow} also has the property α since $D^{\leftarrow} \in \mathcal{D}$. Therefore the following proposition is true.

Proposition 5.3. *Let G be a $\langle 2, 2 \rangle$ CCE graph and D be a $\langle 2, 2 \rangle$ digraph satisfying $G = CCE(D)$. Then if α is a property of D , then the statement obtained from α by replacing the term ‘prey’ (resp. ‘predator’) with the term ‘predator’ (resp. ‘prey’) is a property of D .*

Lemma 5.4. *Let G be the CCE graph of a $\langle 2, 2 \rangle$ digraph D . In addition, let u be a vertex of degree 2 in G and two vertices v and w be the prey or the predators of u in D . Then v and w are adjacent in G , or each of v and w has degree at most 1 in G .*

Proof. By Proposition 5.3, it is sufficient to handle the case where v and w are the prey of u in D . Suppose that v and w are the prey of u in D . Assume that one of v and w has degree at least 2 in G . Without loss of generality, we may assume that v has degree at least 2 in G . Then v has degree 2 in G by Proposition 5.1. Since $u \in N^-(v)$, by Lemma 5.2(iv), u is a common predator of v and one of its neighbors in G . Thus, since $N^+(u) = \{v, w\}$, v and w are adjacent in G . □

Definition 5.5. Let D be a digraph.

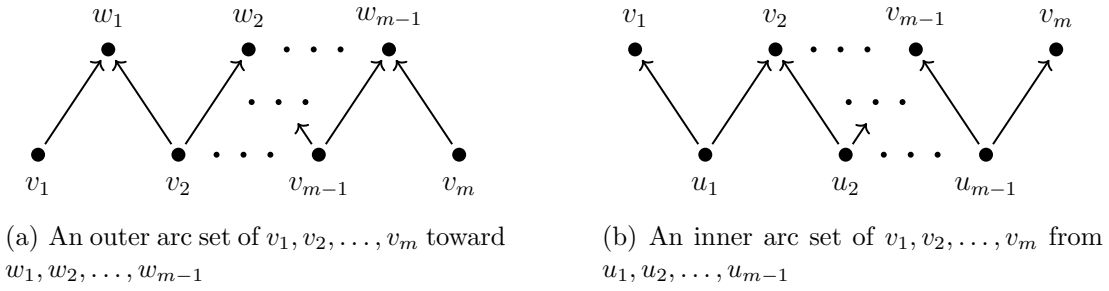


Figure 5.1: An outer arc set and an inner arc set of v_1, v_2, \dots, v_m .

- (i) Given a vertex sequence v_1, v_2, \dots, v_m of D for some integer $m \geq 2$, if an arc set

$$\{(v_1, w_1), (v_2, w_1), (v_2, w_2), \dots, (v_{m-1}, w_{m-1}), (v_m, w_{m-1})\}$$

exists, then we call it an *outer arc set of (the sequence) v_1, v_2, \dots, v_m toward (the sequence) w_1, w_2, \dots, w_{m-1}* (see Figure 5.1(a))

- (ii) Given a vertex sequence v_1, v_2, \dots, v_m of D for some integer $m \geq 2$, if an arc set

$$\{(u_1, v_1), (u_1, v_2), (u_2, v_2), \dots, (u_{m-1}, v_{m-1}), (u_{m-1}, v_m)\}$$

exists, then we call it an *inner arc set of v_1, v_2, \dots, v_m from u_1, u_2, \dots, u_{m-1}* (see Figure 5.1(b)).

We denote a path of length $m - 1$ and a cycle of length m by P_m and C_m , respectively, for a positive integer m . Especially, we denote the path $v_1 v_2 \cdots v_m$ and the cycle $v_1 v_2 \cdots v_m v_1$ by $P_{v,m}$ and $C_{v,m}$, respectively. For a given $C_{v,m}$, we identify v_{m+j} with v_j for any integer j .

By Lemma 5.4 and the definition of outer arc sets and inner arc sets, we have the following proposition and corollaries.

Proposition 5.6. *Let G be the CCE graph of a $\langle 2, 2 \rangle$ digraph D and $P_{u,\ell}$ and $P_{v,m}$ be two nontrivial paths of G . Suppose that there is an arc from u_i to v_j for some positive integers $i \leq \ell$ and $j \leq m$. Then there is either an outer arc set of u_i, \dots, u_a*

toward v_j, \dots, v_b or an inner arc set of v_j, \dots, v_b from u_i, \dots, u_a where the positive integers a and b satisfy one of the following:

$$(a) \ i \leq a \leq \ell; \ j \leq b \leq m; \ |(a-i) - (b-j)| = 1; \ a = \ell \text{ or } b = m;$$

$$(b) \ i \leq a \leq \ell; \ b \leq j; \ |(a-i) - (j-b)| = 1; \ a = \ell \text{ or } b = 1;$$

$$(c) \ a \leq i; \ j \leq b \leq m; \ |(i-a) - (b-j)| = 1; \ a = 1 \text{ or } b = m;$$

$$(d) \ a \leq i; \ b \leq j; \ |(i-a) - (j-b)| = 1; \ a = 1 \text{ or } b = 1.$$

Recall that we use the notation $u \rightarrow v$ (resp. $u \not\rightarrow v$) to represent “ (u, v) is (resp. is not) an arc of a digraph”.

Corollary 5.7. *Let G be the CCE graph of a $\langle 2, 2 \rangle$ digraph D and $P_{u,\ell}$ and $P_{v,m}$ be two nontrivial paths of G . Suppose that $u_1 \rightarrow v_t$, $u_2 \rightarrow v_t$, and $u_2 \rightarrow v_{t+1}$ for some integer $1 \leq t < m$. Then there is an outer arc set of u_1, \dots, u_a toward v_t, \dots, v_b where the integers a and b satisfy*

$$1 < a \leq \ell; \quad t \leq b \leq m; \quad (a-1) - (b-t) = 1; \quad a = \ell \text{ or } b = m.$$

Corollary 5.8. *Let G be the CCE graph of a $\langle 2, 2 \rangle$ digraph D and $P_{u,\ell}$ and $P_{v,m}$ be two nontrivial paths of G . Suppose that $v_t \rightarrow u_1$, $v_t \rightarrow u_1$, and $v_t \rightarrow u_2$ for some integer $1 \leq t < m$. Then there is an inner arc set of u_1, \dots, u_a from v_t, \dots, v_b where the integers a and b satisfy*

$$1 < a \leq \ell; \quad t \leq b \leq m; \quad (a-1) - (b-t) = 1; \quad a = \ell \text{ or } b = m.$$

If the CCE graph of a digraph D contains $P_{v,m}$ or $C_{v,m}$ for an integer $m \geq 3$, then v_i and v_{i+1} have a unique common prey and a unique common predator in D (refer to Figure 5.1) and we denote them by $v_{i,i+1}^-$ and $v_{i,i+1}^+$, respectively. For a given $C_{v,m}$, we identify $v_{m+i,m+j}^-$ and $v_{m+i,m+j}^+$ with $v_{i,j}^-$ and $v_{i,j}^+$, respectively, for any integers i and j .

If $P_{u,\ell} = P_{v,m}$ in Corollary 5.7, then the condition $v_{1,2}^- = v_t$ implies $v_2 \rightarrow v_{t+1}$ and we have the following useful theorem.

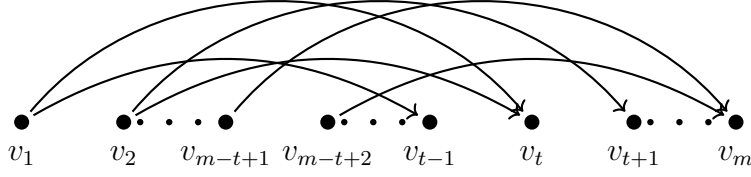


Figure 5.2: The arc set in Theorem 5.9

Given a walk W , we denote by W^{-1} the walk obtained from W by reversing its sequence.

Theorem 5.9. *Let G be the CCE graph of a $\langle 2, 2 \rangle$ digraph D and $P_{v,m}$ be a path in G for some integer $m \geq 3$. If $v_{1,2}^- = v_t$ for some integer $3 \leq t \leq m$, then $v_{i,i+1}^- = v_{t+i-1}$ and $v_{t+i-2,t+i-1}^+ = v_i$ for each integer $1 \leq i \leq m - t + 1$ (see Figure 5.2).*

Proof. Suppose $v_{1,2}^- = v_t$ for some integer $3 \leq t \leq m$. Then $N^-(v_t) = \{v_1, v_2\}$ and so $v_{t-1,t}^+$ is either v_1 or v_2 . To the contrary, suppose $v_{t-1,t}^+ = v_2$. Then $v_2 \rightarrow v_{t-1}$. Since $v_{1,2}^- = v_t$, by applying Corollary 5.7 to $P_{v,m}$ and $P_{v,m}^{-1}$, there is an outer arc set v_1, v_2, \dots, v_a toward v_t, v_{t-1}, \dots, v_b where the integers a and b satisfy the following:

$$1 < a \leq m; \quad 1 \leq b \leq t; \quad (a - 1) - (t - b) = 1; \quad a = m \text{ or } b = 1.$$

If $b = 1$, then $a = t + 1$ and so the outer arc set of v_1, v_2, \dots, v_{t+1} toward v_t, v_{t-1}, \dots, v_1 contains a loop, which is a contradiction. Thus $b \neq 1$. Then $a = m$ and $b = t - m + 2$. Since $t \leq m$, $b \leq 2$ and so $b = 2$. Then $t = m$. Thus the outer arc set of v_1, v_2, \dots, v_m toward v_m, v_{m-1}, \dots, v_2 contains a loop, which is a contradiction. Therefore $v_{t-1,t}^+ = v_1$. Then, if v_{t+1} exists, $v_{t,t+1}^+$ must be v_2 and so, by applying Corollary 5.7 to $P_{v,m}$ and itself, we reach a desired conclusion. \square

By Proposition 5.3, the following corollary immediately follows.

Corollary 5.10. *Let G be the CCE graph of a $\langle 2, 2 \rangle$ digraph D and $P_{v,m}$ be a path in G for some integer $m \geq 3$. If $v_{1,2}^+ = v_t$ for some integer $3 \leq t \leq m$, then $v_{i,i+1}^+ = v_{t+i-1}$ and $v_{t+i-2,t+i-1}^- = v_i$ for each $1 \leq i \leq m - t + 1$.*

Note that $v_{1,2}^+$ or $v_{1,2}^-$ may not be well-defined if $P_{v,2}$ is a component of the CCE graph of a $\langle 2, 2 \rangle$ digraph. For example, a $\langle 2, 2 \rangle$ digraph with the vertex set $\{v_1, v_2, a, b, c\}$ and the arc set $\{(v_1, a), (v_1, b), (v_2, a), (v_2, b), (c, v_1), (c, v_2)\}$ has its CCE graph $P_{v,2}$ with isolated vertices a, b , and c , yet v_1 and v_2 have a and b as common prey. This observation may be generalized as follows.

Proposition 5.11. *Let G be the CCE graph of a $\langle 2, 2 \rangle$ digraph.*

(i) *Given the path $P_{v,m}$ (not necessarily a path component) in G for some integer $m \geq 3$, the sequence*

$$v_{1,2}^- v_{2,3}^- \cdots v_{m-1,m}^-$$

determines the unique outer arc set of $P_{v,m}$ (toward it), while the sequence

$$v_{1,2}^+ v_{2,3}^+ \cdots v_{m-1,m}^+$$

determines the unique inner arc set of $P_{v,m}$ (from it). We denote the outer arc set and the inner arc set of $P_{v,m}$ by $\partial^+(P_{v,m})$ and $\partial^-(P_{v,m})$, respectively. Conversely, if a vertex sequence v_1, v_2, \dots, v_m has both an inner arc set and an outer arc set, then it forms $P_{v,m}$ in CCE graph.

(ii) *Given the cycle $C_{v,m}$ in G for some integer $m \geq 3$, the sequences*

$$v_{1,2}^- v_{2,3}^- \cdots v_{m-1,m}^- v_{m,1}^- \quad \text{and} \quad v_{1,2}^+ v_{2,3}^+ \cdots v_{m-1,m}^+ v_{m,1}^+$$

determine the arc sets

$$\partial^+(P_{v,m}) \cup \{(v_1, v_{m,1}^-), (v_m, v_{m,1}^-)\} \quad \text{and} \quad \partial^-(P_{v,m}) \cup \{(v_{m,1}^+, v_1), (v_{m,1}^+, v_m)\},$$

respectively, which are the unique outer arc set and the unique inner arc set, respectively, of $C_{v,m}$ and we denote them by $\partial^+(C_{v,m})$ and $\partial^-(C_{v,m})$, respectively. Conversely, if a vertex sequence $v_1, v_2, \dots, v_m, v_1$ has both an inner arc set and an outer arc set, then it forms $C_{v,m}$ in CCE graph.

The degree boundedness of $\langle 2, 2 \rangle$ digraph and the previous proposition ensure the following proposition.

Proposition 5.12. *Let G be the CCE graph of a $\langle 2, 2 \rangle$ digraph D . Given the path $P_{v,m}$ (resp. the cycle $C_{v,m}$) in G for some integer $m \geq 3$,*

- (i) $v_{i,i+1}^+ \neq v_{j,j+1}^+$ and $v_{i,i+1}^- \neq v_{j,j+1}^-$ for distinct $1 \leq i, j \leq m-1$ (resp. $1 \leq i, j \leq m$);
- (ii) $N(v_{i,i+1}^+) \subseteq \{v_{i-1,i}^+, v_{i+1,i+2}^+\}$ and $N(v_{i,i+1}^-) \subseteq \{v_{i-1,i}^-, v_{i+1,i+2}^-\}$ for each $2 \leq i \leq m-2$ (resp. $2 \leq i \leq m$).

5.1 CCE graphs of $\langle 2, 2 \rangle$ digraphs

This section is devoted to proving the following theorem which characterizes the $\langle 2, 2 \rangle$ CCE graphs.

Theorem 5.13. *A graph is a $\langle 2, 2 \rangle$ CCE graph if and only if each connected component is a path or a cycle, and the only path component, if exists, is trivial.*

Lemma 5.14. *A nontrivial path is not a $\langle 2, 2 \rangle$ CCE graph.*

Proof. To the contrary, suppose that a nontrivial path $P_{v,m}$ is the CCE graph of a $\langle 2, 2 \rangle$ digraph D for some integer $m > 1$. Any two adjacent vertices of $P_{v,m}$ must have a common prey and a common predator which are distinct in D . Thus $m \geq 4$. Since D is loopless, $v_{1,2}^- = v_t$ for some integer $3 \leq t \leq m$. By Theorem 5.9, $v_{i,i+1}^- = v_{t+i-1}$ and $v_{t+i-2,t+i-1}^+ = v_i$ for each $1 \leq i \leq m-t+1$. Then $v_{t-1,t}^+ = v_1$. Since v_t is already an outneighbor of v_2 , the only neighbor v_2 of v_1 in G cannot be $v_{t-2,t-1}^+$. Thus, by Lemma 5.4, we have $v_{t-2,t-1}^+ = v_m$. Then v_{t-1} is a common prey of v_1 and v_m . If $t < m$, since the only neighbor v_{m-1} of v_m cannot be $v_{m-t,m-t+1}^-$ by Lemma 5.4 again, $v_{m-t,m-t+1}^- = v_1$ and so v_{m-t+1} is a common predator of v_1 and v_m . Suppose $t = m$. That is, $v_{1,2}^- = v_m$. Since v_{t-1} is a common prey of v_1 and v_m , the only neighbor v_{m-1} of v_m cannot be $v_{2,3}^+$. Thus $v_{2,3}^- = v_1$ by Lemma 5.4. Then v_2 is a common predator of v_1 and v_m . Whether $t < m$ or $t = m$, v_1 and v_m have a common

prey and a common predator in D . Thus v_1 and v_m are adjacent in $CCE(D)$, which is a contradiction. \square

Given a positive integer $m \geq 3$, we consider a digraph $D_{v,m}^t$ with the vertex set

$$V(D_{v,m}^t) = \{v_1, v_2, \dots, v_m\}$$

and the arc set

$$A(D_{v,m}^t) = \bigcup_{k=1}^m \{(v_k, v_{k+t}), (v_k, v_{k+t+1})\}$$

for some $t \in \{1, \dots, m-2\}$ (identify v_{m+i} with v_i for each integer i). For each vertex v_i in $D_{v,m}^t$,

$$N^+(v_i) = \{v_{i+t}, v_{i+t+1}\} \quad \text{and} \quad N^-(v_i) = \{v_{i-t-1}, v_{i-t}\}. \quad (5.1)$$

Since $t \in \{1, \dots, m-2\}$, $D_{v,m}^t$ is loopless and so it is a $\langle 2, 2 \rangle$ digraph. Moreover, v_{i+t+1} (resp. v_{i-t}) is a common prey (resp. predator) of v_i and v_{i+1} for each integer $1 \leq i \leq m$. Therefore

$$CCE(D_{v,m}^t) = C_{v,m} \quad (5.2)$$

and

$$v_{i,i+1}^+ = v_{i+t+1} \quad \text{and} \quad v_{i,i+1}^- = v_{i-t} \quad (5.3)$$

in $D_{v,m}^t$ for each integer $1 \leq i \leq m$. Hence we obtain the following proposition.

Lemma 5.15. *A cycle of length at least 3 is a $\langle 2, 2 \rangle$ CCE graph.*

Proposition 5.16. *For positive integers m and n , $P_m \cup P_n$ is a $\langle 2, 2 \rangle$ CCE graph.*

Proof. Fix positive integers m and n . Since an edgeless graph is a $\langle 2, 2 \rangle$ CCE graph, the case $m = n = 1$ is clear. Without loss of generality, we assume $m \geq 2$. We consider the digraph

$$D := D_{v,m+n}^{\overset{m-1}{\curvearrowright}} - (v_1, v_m).$$

Since $D_{v, m+n}^{\widehat{m-1}}$ is a $\langle 2, 2 \rangle$ digraph, D is a $\langle 2, 2 \rangle$ digraph. By (5.3),

$$v_{i, i+1}^- = v_{i+m} \quad \text{and} \quad v_{i, i+1}^+ = v_{i-m+1}$$

in $D_{v, m+n}^{\widehat{m-1}}$ for each integer $1 \leq i \leq m+n$. Especially, $v_{m+n, 1}^- = v_m$ and $v_{m, m+1}^+ = v_1$ in $D_{v, m+n}^{\widehat{m-1}}$ since we identify v_{m+n+j} with v_j for each integer j . Thus removing the arc (v_1, v_m) from $D_{v, m+n}^{\widehat{m-1}}$ deletes the edges $\{v_1, v_{m+n}\}$ and $\{v_m, v_{m+1}\}$ so that the CCE graph of D is the union of paths

$$v_1 v_2 \cdots v_m \quad \text{and} \quad v_{m+1} v_{m+2} \cdots v_{m+n}$$

by (5.2). Hence $CCE(D) \cong P_m \cup P_n$. \square

We denote k path components P_m of a graph by kP_m for positive integers $k \geq 2$ and m . We also denote t isolated vertices by I_t for a positive integer t .

Proposition 5.17. *For positive integers m and n , $2P_m \cup P_n$ is a $\langle 2, 2 \rangle$ CCE graph.*

Proof. Fix positive integers m and n . Since $P_n \cup I_1$ is a $\langle 2, 2 \rangle$ CCE graph by Proposition 5.16, $P_n \cup I_2$ is a $\langle 2, 2 \rangle$ CCE graph. Now we assume $m \geq 2$. We consider the digraph

$$D := D_{v, 2m+n}^{\widehat{m-1}} - (v_1, v_m) - (v_{m+1}, v_{2m}).$$

Since $D_{v, 2m+n}^{\widehat{m-1}}$ is a $\langle 2, 2 \rangle$ digraph, D is a $\langle 2, 2 \rangle$ digraph. Note that by (5.3),

$$v_{i, i+1}^- = v_{i+m} \quad \text{and} \quad v_{i, i+1}^+ = v_{i-m+1}$$

in $D_{v, 2m+n}^{\widehat{m-1}}$ for each integer $1 \leq i \leq 2m+n$. Especially, $v_{2m+n, 1}^- = v_m$, $v_{m, m+1}^+ = v_1$, $v_{m, m+1}^- = v_{2m}$, and $v_{2m, 2m+1}^+ = v_{m+1}$ in $D_{v, 2m+n}^{\widehat{m-1}}$. Thus removing the arcs (v_1, v_m) and (v_{m+1}, v_{2m}) from $D_{v, 2m+n}^{\widehat{m-1}}$ deletes the edges $\{v_1, v_{2m+n}\}$, $\{v_m, v_{m+1}\}$, and $\{v_{2m}, v_{2m+1}\}$ so that the CCE graph of D is the union of paths

$$v_1 v_2 \cdots v_m, \quad v_{m+1} v_{m+2} \cdots v_{2m}, \quad \text{and} \quad v_{2m+1} v_{2m+2} \cdots v_{2m+n}$$

by (5.2). Hence $CCE(D) \cong 2P_m \cup P_n$. \square

Proposition 5.18. *For positive integers l , m and n , $P_l \cup P_m \cup P_n$ is a $\langle 2, 2 \rangle$ CCE graph.*

Proof. Fix positive integers l , m , and n . By Propositions 5.16 and 5.17, it suffices to consider the case $1 < l < m < n$. Suppose

$$1 < l < m < n.$$

Let $D_1 = D_{u, l+m}^{\overset{l-1}{\curvearrowright}}$ and $D_2 = D_{v, m+n}^{\overset{m-l}{\curvearrowright}}$. We consider two digraphs

$$D_3 := D_1 - \partial_{D_1}^-(u_{2l} \cdots u_{l+m}) - (u_{m+1}, u_{l+m})$$

and

$$D_4 := D_2 - \partial_{D_2}^-(v_{m+n-l+1} \cdots v_{m+n}) - (v_{n+l}, v_{m+n})$$

(see Figure 5.3 and, for the notation $\partial_D^-(X)$, refer to Proposition 5.11). Now we obtain a digraph D from digraphs D_3 and D_4 by identifying u_{l+i} with $v_{m+n+1-i}$ for each $1 \leq i \leq m$. Since D_1 and D_2 are $\langle 2, 2 \rangle$ digraphs, D_3 and D_4 are $\langle 2, 2 \rangle$ digraphs. To show that D is a $\langle 2, 2 \rangle$ digraph, it suffices to check the outdegree and indegree of the vertices identified in D_3 and D_4 . By (5.1), we may check the following:

- $d_{D_3}^-(u_{l+1}) = \cdots = d_{D_3}^-(u_{2l-1}) = 2$, $d_{D_3}^-(u_{2l}) = 1$, $d_{D_3}^-(u_{2l+1}) = \cdots = d_{D_3}^-(u_{l+m}) = 0$;
- $d_{D_3}^+(u_{l+1}) = \cdots = d_{D_3}^+(u_m) = 0$, $d_{D_3}^+(u_{m+1}) = 1$, $d_{D_3}^+(u_{m+2}) = \cdots = d_{D_3}^+(u_{l+m}) = 2$;
- $d_{D_4}^-(v_{n+1}) = \cdots = d_{D_4}^-(v_{m+n-l}) = 2$, $d_{D_4}^-(v_{m+n-l+1}) = 1$, $d_{D_4}^-(v_{m+n-l+2}) = \cdots = d_{D_4}^-(v_{m+n}) = 0$;
- $d_{D_4}^+(v_{n+1}) = \cdots = d_{D_4}^+(v_{n+l-1}) = 0$, $d_{D_4}^+(v_{n+l}) = 1$, $d_{D_4}^+(v_{n+l+1}) = \cdots = d_{D_4}^+(u_{m+n}) = 2$.

Thus each vertex identified in D_3 and D_4 has outdegree 2 and indegree 2 in D . Therefore D is a $\langle 2, 2 \rangle$ digraph. By (5.3), we may check the following:

- $A(D_3)$ is the union of an outer arc set of $u_{m+1}, \dots, u_{l+m}, u_1, \dots, u_l$ toward u_1, \dots, u_{2l-1} and an inner arc set of u_1, \dots, u_{2l} from $u_{m+2}, \dots, u_{l+m}, u_1, \dots, u_l$;
- $A(D_4)$ is the union of an outer arc set of $v_{n+l}, \dots, v_{m+n}, v_1, \dots, v_n$ toward v_1, \dots, v_{m+n-l} and an inner arc set of $v_1, \dots, v_{m+n-l+1}$ from $v_{n+l+1}, \dots, v_{m+n}, v_1, \dots, v_n$.

Since we have identified u_{l+i} with $v_{m+n+1-i}$ for each $1 \leq i \leq m$, $CCE(D)$ is the union of paths $u_1 \cdots u_l$, $u_{l+1} \cdots u_{l+m}$, and $v_1 \cdots v_n$. \square

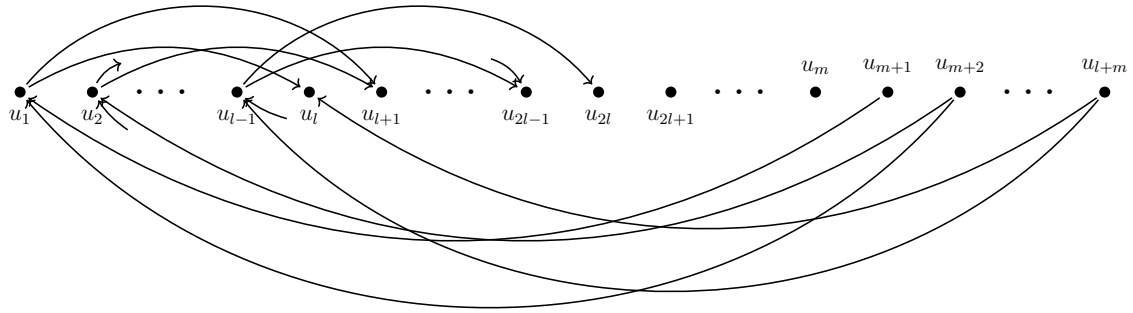
Lemma 5.19. *Let G be a disjoint union of paths. Then G is a $\langle 2, 2 \rangle$ CCE graph if and only if G is not isomorphic to a nontrivial path.*

Proof. The “only if” part follows from Lemma 5.14. Suppose that G is not isomorphic to a nontrivial path. If G is a trivial graph, then it is clear. Assume that G is not a trivial graph. Then G has at least 2 path components. Thus there is a partition \mathcal{P} of the set of path components of G such that each part of \mathcal{P} has size 2 or 3. For each part X , there is a $\langle 2, 2 \rangle$ digraph D_X whose CCE graph is X by Proposition 5.16 if $|X| = 2$ and by Proposition 5.18 if $|X| = 3$. Then $\bigcup_{X \in \mathcal{P}} D_X$ is a $\langle 2, 2 \rangle$ digraph whose CCE graph is G . Thus we have shown the “if” part. \square

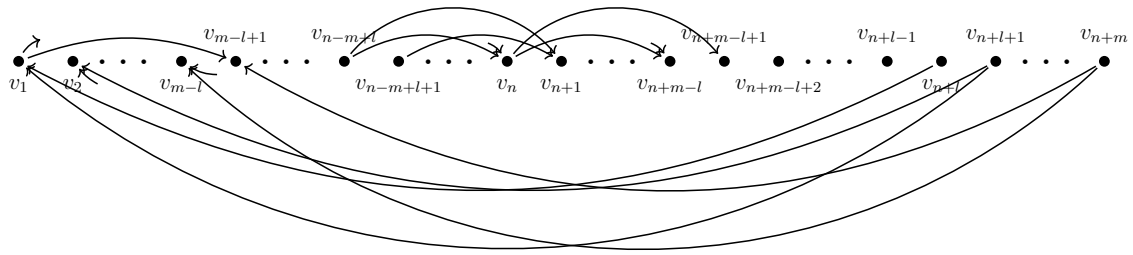
Now we are ready to prove Theorem 5.13.

Proof of Theorem 5.13. For $\langle 2, 2 \rangle$ digraphs D_1 and D_2 , we may check that $CCE(D_1 \cup D_2) = CCE(D_1) \cup CCE(D_2)$. Thus the “if” part follows by Lemmas 5.15 and 5.19.

To show the “only if” part, suppose that G is the CCE graph of a $\langle 2, 2 \rangle$ digraph D . By Proposition 5.1, G is a disjoint union of paths and cycles. To the contrary, suppose that $P_{u,l}$ is the unique path component of G for an integer $l \geq 2$. By Proposition 5.1, each component of G is either a path or a cycle. Since a nontrivial path cannot be the CCE graph of a $\langle 2, 2 \rangle$ digraph by Lemma 5.14, there is a cycle component $C_{v,m}$ in G such that some consecutive vertices of $P_{u,l}$ have a common prey or a common predator on $C_{v,m}$ for some positive integer m . Without loss of generality, we may



(a) D_3



(b) D_4

Figure 5.3: Digraphs in the proof of Proposition 5.18

assume that some consecutive vertices of $P_{u,l}$ have a common predator on $C_{v,m}$. Then $v_j \rightarrow u_i$ for some $1 \leq i \leq l$ and $1 \leq j \leq m$. Thus u_i is either $v_{j-1,j}^-$ or $v_{j,j+1}^-$. By Proposition 5.12, since $P_{u,l}$ is the only path component in G , $l = m$ and there is a positive integer t such that $1 \leq t \leq m$ and $\{u_1, u_l\} = \{v_{t-1,t}^-, v_{t,t+1}^-\}$. Then $v_t = u_{1,l}^+$. By Proposition 5.3, there is a cycle component $C_{w,l}$ in G such that $w_s = u_{1,l}^-$ for some $1 \leq s \leq l$. Thus u_1 and u_l have a common prey v_t and a common predator w_s . Therefore u_1 and u_l are adjacent in G , which is a contradiction that $P_{u,l}$ is a path. \square

5.2 CCE graphs of (2, 2) digraphs

We only consider (2, 2) digraphs with at least three vertices unless otherwise mentioned. Recall that we call a vertex of indegree 0 (resp. outdegree 0) in a digraph D a *source* (resp. *sink*) of D . It is a well-known fact that if a digraph D is an acyclic, then D has a sink and a source. Each sink and each source of a digraph form isolated vertices in its CCE graph. In this context, it is natural to start with a (2, 2) CCE graph including exactly two isolated vertices.

Proposition 5.20. *Let G be a (2, 2) CCE graph with exactly two isolated vertices. If D is a (2, 2) digraph satisfying $CCE(D) = G$, then the following are true:*

- (i) *D has exactly one source x and exactly one sink y which are the two isolated vertices in G .*
- (ii) *x has a prey of indegree 1 and y has a predator of outdegree 1 in D .*
- (iii) *D is weakly connected.*

Proof. Suppose that D is a (2, 2) digraph with $CCE(D) = G$. Part (i) is immediately true by the previous observation. If x (resp. y) has no prey (resp. predator) of indegree 1 (resp. outdegree 1), then $D - x$ (resp. $D - y$) has no a vertex of indegree 0 (resp, outdegree 0), which contradicts the fact that $D - x$ (resp. $D - y$) is an acyclic. Thus part (ii) is true. To show part (iii) by contradiction, suppose that D is not weakly

connected. Then there exist subdigraphs D_1 and D_2 of D such that $D_1 \cup D_2 = D$ and $V(D_1) \cap V(D_2) = \emptyset$. Thus $G = CCE(D) = CCE(D_1) \cup CCE(D_2)$. Since at least one of D_1 and D_2 is a nontrivial acyclic digraph, $CCE(D_1)$ or $CCE(D_2)$ has at least two isolated vertices and so G has at least three isolated vertices, which is impossible. Therefore D is weakly connected. \square

Given a family \mathcal{D} of digraphs, we say that a digraph in \mathcal{D} is *minimal* in \mathcal{D} if there is no proper subdigraph D' of D in \mathcal{D} such that $CCE(D) = CCE(D')$. By the Well-Ordering Axiom, the following lemma is true.

Lemma 5.21. *For a $(2, 2)$ CCE graph G and the set \mathcal{D}_G of $(2, 2)$ digraphs whose CCE graphs are G , there exists a minimal digraph in \mathcal{D}_G .*

Given a $(2, 2)$ CCE graph G , we say that a digraph is a *minimal digraph of G* if D is a minimal digraph among the $(2, 2)$ digraphs whose CCE graphs are G .

It is easy to check that if D is a minimal digraph of a $(2, 2)$ CCE graph G , then D^{\leftarrow} is also a minimal digraph of G . Therefore the following is also true by Proposition 5.3.

Proposition 5.22. *Let G be a $(2, 2)$ CCE graph and D be a $(2, 2)$ minimal digraph of G . Then if α is a property of D , then the statement obtained from α by replacing the term ‘prey’ (resp. ‘predator’) with the term ‘predator’ (resp. ‘prey’) is a property of D .*

Proposition 5.23. *Let D be a minimal digraph of a $(2, 2)$ CCE graph G . Then the following are true:*

- (i) *if a vertex v has exactly one predator (resp. one prey), then v has degree 1 in G and the predator (resp. the prey) of v has the other prey (resp. predator) that is adjacent to v in G .*
- (ii) *if a vertex v has two predators (resp. two prey), then v has degree 2 or the predators (resp. the prey) of v are adjacent in G .*
- (iii) *any two distinct vertices have at most one common prey and at most one common predator.*

Proof. By Proposition 5.22, for showing (i) and (ii), it is sufficient to handle the case where a vertex v has indegree 1 or 2.

To show part (i), suppose that v has indegree 1 in D . Then, since D is a $(2, 2)$ digraph, v has degree at most 1 in G . Suppose that v has degree 0 in G . Then $CCE(D') = G$ for the subdigraph D' with $V(D') = V(D)$ and $A(D') \subsetneq A(D)$ obtained from deleting the incoming arc to v , which contradicts the minimality of D . Therefore v has degree 1 in G . Thus the predator of v has the other prey that is adjacent to v in G .

To verify part (ii), we suppose that v has indegree 2 in D . Let w and x be the predators of v . Assume that v has degree at most 1, and w and x are not adjacent in G . Then deleting any arc of (w, v) and (x, v) does not change the adjacency between w and x . Moreover, since v has degree at most 1 and D is a $(2, 2)$ digraph, we may delete one arc of (w, v) and (x, v) so that the degree of v stays the same in the CCE graph of the resulting digraph D' . Thus $A(D') \subsetneq A(D)$ and $CCE(D') = G$, which contradicts the minimality of D . Hence v has degree 2 or w and x are adjacent in G .

To show part (iii), suppose to the contrary that there are two distinct vertices u_1 and u_2 such that they have at least two common prey or at least two common predators. By Proposition 5.22, we may assume that u_1 and u_2 have at least two common prey v_1 and v_2 . Since D is a $(2, 2)$ digraph,

$$N_D^+(u_1) = N_D^+(u_2) = \{v_1, v_2\}$$

and

$$N_D^-(v_1) = N_D^-(v_2) = \{u_1, u_2\}.$$

Then the pairs that may be affected by deleting the arc (u_1, v_1) from D are that of u_1 and u_2 and that of v_1 and v_2 . Yet, the adjacency of u_1 and u_2 is preserved by the arcs (u_1, v_2) and (u_2, v_2) and that of v_1 and v_2 is preserved by the arcs (u_2, v_1) and (u_2, v_2) . Therefore the CCE graph of the digraph $D - (u_1, v_1)$ is isomorphic to G , which contradicts the fact that D is minimal. \square

Remark 5.24. By Proposition 5.23(iii), two adjacent vertices in the CCE graph G

of a $(2, 2)$ minimal digraph have a unique common predator and a unique common prey. Therefore the notations $v_{1,2}^+$ and $v_{1,2}^-$ may be used for the common predator and the common prey of those two vertices on $P_{v,2}$.

For notational simplicity, given an induced subgraph H of a graph, we may write H to stand for $V(H)$.

Lemma 5.25. *Let G be a $(2, 2)$ CCE graph with exactly two isolated vertices and D be a minimal digraph of G . Then the following are true:*

- (i) *there exists a nontrivial path $P_{v,m}$ in G such that $N^+(x) = \{v_1, v_2\}$ and $N^-(v_1) = \{x\}$ where x is the source of D ;*
- (ii) *there exists a nontrivial path $P_{w,n}$ in G such that $N^-(y) = \{w_1, w_2\}$ and $N^+(w_1) = \{y\}$ where y is the sink of D ;*
- (iii) *$G \cong 2P_1 \cup P_2$ if and only if $N^+(x) = N^-(y)$;*
- (iv) *if $n \geq 3$ and $w_2 \not\rightarrow w_1$, then*
 - (a) *w_1 has a predator w^* of outdegree 1 that is an end vertex of some nontrivial path distinct from $P_{w,n}$;*
 - (b) *if $w_2 \not\rightarrow w^*$, then one predator of w^* has outdegree 1 and the other predator of w^* has outdegree 2.*
- (v) *if $n \geq 3$ and $w_2 \rightarrow w_1$ and G has at least two nontrivial components, then*
 - (a) *$w_n \rightarrow w_{n-1}$ and $A(D)$ contains an outer arc set of w_2, w_3, \dots, w_n toward w_1, w_2, \dots, w_{n-2} ;*
 - (b) *$w_{n-1,n}^+$ is an end vertex of a nontrivial path distinct from $P_{w,n}$;*
 - (c) *the CCE graph of $D - \{w_1, \dots, w_{n-1}, y\}$ is isomorphic to $G - \{w_1, \dots, w_{n-1}, y\}$.*

Proof. By Proposition 5.20(i), D has a unique source x . Since D is weakly connected and x is a source, x has outdegree at least 1. If x has outdegree 1, then x has degree 1 in G by Proposition 5.23(i), which is impossible. Therefore x has outdegree 2. Thus

the prey v_1 and v_2 of x are adjacent in G by Proposition 5.23(ii). Then, since x is the only source in D , v_1 or v_2 has indegree 1 by Proposition 5.20(ii). Without loss of generality, we may assume v_1 has indegree 1. Then v_1 has degree 1 in G and so a path $P_{v,m}$ exists for some integer $m \geq 2$. Hence part (i) is true. Therefore, by Proposition 5.22, D has a unique sink y and $N^-(y) = \{w_1, w_2\}$ and $N^-(w_1) = y$ for a nontrivial path $P_{w,n}$ in G . Thus part (ii) is true.

Now we show part (iii). If $G \cong 2P_1 \cup P_2$, then G has a unique nontrivial path and so $P_{v,2} = P_{w,2}$. Suppose $\{v_1, v_2\} = \{w_1, w_2\}$. To the contrary, assume that v_2 has degree 2 in G . Then $m \geq 3$ and so there exists a common prey v' of v_2 and v_3 . Since v_1 is a predator of y , $y \neq v'$ and so v_2 is not a sink in $D - \{y, v_1\}$. Since $N^-(v_1) = \{x\}$, $N^-(y) = \{v_1, v_2\}$, and $N^+(x) = \{v_1, v_2\}$, $D - \{y, v_1\}$ has no sink, a contradiction. Therefore v_2 has degree 1 in G . Then, since x and y are the isolated vertices, each of v_1 and v_2 has outdegree 1 and indegree 1 by Proposition 5.23. Thus $\{x, v_1, v_2, y\}$ forms a weakly connected component D' in D and so, by Proposition 5.20(iii), $D' = D$. Thus $G \cong 2P_1 \cup P_2$.

To show part (iv), suppose that $n \geq 3$ and $(w_2, w_1) \notin A(D)$. Then w_2 has degree 2 and so $d^+(w_2) = 2$. Accordingly, $d^+(w_2) = 2$ together with $N^+(w_1) = \{y\}$ implies that w_1 is the only sink in the digraph $D - \{y\}$. Now, if each predator of w_1 has outdegree 2, then $D - \{y, w_1\}$, which is acyclic, has no sink since $(w_2, w_1) \notin A(D)$, a contradiction. Therefore at least one predator w^* of w_1 has outdegree 1 in D . Moreover, $d^+(w_{1,2}^+) = 2$. Thus w^* is a unique predator of w_1 having outdegree 1 in D . Since w^* has outdegree 1, w^* has degree 1 by Proposition 5.23(i) and so w^* is an end vertex on some nontrivial path. To the contrary, suppose $w^* \in P_{w,n}$. Then $w^* = w_n$ since $w^* \neq w_1$. Now, since w^* is a predator of w_1 and has outdegree 1, $w_1 = w_{n-1,n}^-$. Thus, by Theorem 5.9(i), $w_{1,2}^+ = w_n$ and so w_n has outdegree 2, a contradiction. Hence $w^* \notin P_{w,n}$. Therefore we have shown that (a) holds.

To show (b), suppose $w_2 \not\rightarrow w^*$. By (a), w^* is an end vertex of some nontrivial path, namely $P_{z,t}$, for some integer $t \geq 2$ with $P_{z,t} \neq P_{w,n}$. Without loss of generality, we may assume $w^* = z_1$. Then, since w_1 is the only prey of z_1 , $N_D^-(w_1) = \{z_1, z_2\}$ and $z_2 = w_{1,2}^+$. Now we consider the digraph $D'' := D - \{w_1, z_1, y\}$. Then D'' is acyclic and the possible sinks of D'' are the predators of some vertex in $\{w_1, z_1, y\}$ that are

w_2, z_2 , and the predators of z_1 . Yet, w_2 is not a sink in D'' since $d_D^+(w_2) = 2$, $w_2 \not\rightarrow z_1$, and $w_2 \not\rightarrow w_1$. Since $z_2 = w_{1,2}^+$, z_2 is not a sink of D'' . Thus one of predator of z_1 must be a sink of D'' . Since $z_{1,2}^+$ is not a sink of D'' , there exists a predator z of z_1 that is a sink of D'' . Then $z \neq w_2$ and $z \neq z_2$. If z has outdegree 2 in D , then w_1 or y is a prey of z and so $z = z_2$ or $z = w_2$, which is impossible. Thus z has outdegree 1 in D and so (b) is true.

To show part (v), suppose $n \geq 3$ and $w_2 \rightarrow w_1$ and G has at least two nontrivial components. Then $w_1 = w_{2,3}^-$. Accordingly, $w_{1,2}^+ = w_3$. Thus, by Corollary 5.10, $A(D)$ contains an arc (w_n, w_{n-1}) and an outer arc set of w_2, w_3, \dots, w_n toward w_1, w_2, \dots, w_{n-2} , and so (a) is true. Hence every vertex on $P_{w,n}$ except w_{n-1} and w_n has indegree 2 which is fulfilled by vertices on $P_{w,n}$. By Lemma 5.2(i), w_{n-1} has two predators. Then the predator of w_{n-1} other than w_n is $w_{n-1,n}^+$ and, by Lemma 5.4, $w_{n-1,n}^+$ is an end vertex of a path in G . If $w_{n-1,n}^+ \in P_{w,n}$, then $w_{n-1,n}^+ = w_1$, which contradicts part (ii). Thus $w_{n-1,n}^+ \notin P_{w,n}$. Suppose that $w_{n-1,n}^+$ is an isolated vertex. Then $w_{n-1,n}^+ = x$ by part (i). Hence the subdigraph D_1 induced by $V(P_{w,n}) \cup \{x, y\}$ is isomorphic to the digraph D_{n+2}^* given in Figure 5.7. Further, it can easily be checked that every vertex in $V(D_1) - \{w_n, x\}$ has indegree 2 in D_1 and every vertex in $V(D_1) - \{w_1, y\}$ has outdegree 2 in D_1 . By part (i), $N^-(w_n) = \{x\}$. By part (ii), $N^+(w_1) = \{y\}$. Thus D_1 is a weak component and so, by Proposition 5.20(iii), $D = D_1$. Therefore $CCE(D_1) = G = P_{w,n} \cup \{x, y\}$, which contradicts the assumption that G has at least two nontrivial components. Hence $w_{n-1,n}^+$ is not isolated and so is an end vertex of a nontrivial path distinct from $P_{w,n}$. Therefore (b) is true.

Now we consider the digraph $D_2 := D - \{w_1, \dots, w_{n-1}, y\}$. As we have shown above, all prey in D of each vertex in $P_{w,n} \cup \{y\}$ lie on $P_{w,n}$ and every vertex on $P_{w,n}$ except w_{n-1} and w_n has indegree 2 in D which is fulfilled by vertices on $P_{w,n}$. Thus the adjacency of any pair of vertices in $V(G) - \{w_1, \dots, w_{n-1}, w_n, y\}$ is preserved in the CCE graph G' of D_2 . We have also shown that the predators of w_{n-1} are w_n and $w_{n-1,n}^+$ in D and w_{n-1} and w_n are not adjacent in G . Hence w_n is isolated in the CCE graph G' of D_2 . Therefore G' is isomorphic to $G - \{w_1, \dots, w_{n-1}, y\}$ and so (c) holds. \square

Proposition 5.26. *Let G be the CCE graph of a $(2, 2)$ digraph D . If there is a cycle*

in G , then there is no arc between its vertices.

Proof. To the contrary, suppose that there is a cycle $C_{v,m}$ in G and an arc between some vertices on $C_{v,m}$ in D . By symmetry, we may assume that there is an arc from v_1 to v_t for some $t \in \{2, \dots, m\}$. Since v_1 is adjacent to v_2 and v_m , v_t is a prey of v_2 or v_m . We may assume that v_t is a prey of v_2 . Then $2 < t \leq m$. By applying Theorem 5.9 to $C_{v,m} - v_1v_m$, we have v_t, \dots, v_m as common prey of v_1 and v_2, \dots, v_{m-t+1} and v_{m-t+2} , respectively. By Theorem 5.9 applied to the path $C_{v,m} - v_1v_2$, v_1 is a common prey of v_{m-t+2} and v_{m-t+3} . By applying the same theorem to $C_{v,m} - v_2v_3, \dots, C_{v,m} - v_{t-1}v_t$ repeatedly, we may obtain an arc set of D

$$A := \bigcup_{k=1}^m \{(v_k, v_{k+t-1}), (v_{k+1}, v_{k+t-1})\}.$$

We consider the subgraph D' of D induced by A . Then it is easy to check that $CCE(D') = C_{v,m}$. Since D is a $(2, 2)$ digraph, D' is a $(2, 2)$ digraph, which contradicts that the CCE graph of a $(2, 2)$ digraph has at least two isolated vertices. \square

Given a vertex set X of a digraph D , we denote by $N^+(X)$ and $N^-(X)$ the sets

$$\{v \in V(D) \mid (x, v) \in A(D), x \in X, v \notin X\} \quad \text{and} \quad \{v \in V(D) \mid (v, x) \in A(D), x \in X, v \notin X\},$$

respectively.

Lemma 5.27. *Let G be the CCE graph of a $(2, 2)$ digraph D . Suppose that G has a cycle C of length m for some $m \geq 3$. Then the following are true:*

(i) $|N^+(C)| = |N^-(C)| = m$;

(ii) $|N^+(C) \cup N^-(C)| \geq m + 3$ and $|N^+(C) \cap N^-(C)| \leq m - 3$;

(iii) each component of G is contained in exactly one of the following:

$$N^+(C) \cap N^-(C); \quad N^+(C) - N^-(C); \quad N^-(C) - N^+(C); \quad V(G) - (N^+(C) \cup N^-(C)).$$

Proof. Let $C := C_{v,m}$. By Proposition 5.26,

$$N^+(C) = \{v_{1,2}^-, v_{2,3}^-, \dots, v_{m,1}^-\} \text{ and } N^-(C) = \{v_{1,2}^+, v_{2,3}^+, \dots, v_{m,1}^+\}.$$

By Proposition 5.12(i), $|N^+(C)| = |N^-(C)| = m$ and so part (i) is true.

To show part (ii), suppose, to the contrary, that $|N^+(C) \cup N^-(C)| \leq m + 2$. Then, since $|N^+(C)| = |N^-(C)| = m$,

$$|N^+(C) \cap N^-(C)| \geq m - 2.$$

Then

$$|N^+(C) - N^-(C)| \leq 2. \tag{5.4}$$

Take a vertex x_1 in $N^+(C) \cap N^-(C)$. Then $x_1 \rightarrow v_j$ and $x_1 \rightarrow v_{j+1}$ for some $j \in \{1, \dots, m\}$. Let $u_i = v_{i,i+1}^-$ for each $i = 1, \dots, m$ ($u_{s+m} = u_s$ and $v_{s+m,t+m}^- = v_{s,t}^-$ for any positive integers s, t). Since $\{u_{j-1}, u_j, u_{j+1}\} \subseteq N^+(C)$, at least one of u_{j-1}, u_j, u_{j+1} belongs to $N^+(C) \cap N^-(C)$ by (5.4). Let x_2 be one of such vertices. Then, since $v_j \rightarrow u_{j-1}$, $v_j \rightarrow u_j$, and $v_{j+1} \rightarrow u_{j+1}$, we obtain a (x_1, x_2) -directed walk W_1 . By similar argument, we obtain a (x_2, x_3) -directed walk W_2 for some $x_3 \in N^+(C) \cap N^-(C)$. By repeating this process, we obtain the directed walk $W := W_1 \rightarrow W_2 \rightarrow \dots \rightarrow W_m$ where W_i is a (x_i, x_{i+1}) -directed walk and $x_i \in N^+(C) \cap N^-(C)$ for each $i = 1, \dots, m$. Then $\{x_1, \dots, x_{m+1}\} \subseteq N^+(C) \cap N^-(C)$. By the way, since $|N^+(C) \cap N^-(C)| \leq m$, $x_k = x_\ell$ for some distinct $k, \ell \in [m+1]$. Thus W contains a closed directed walk, which contradicts the fact that D is acyclic. Hence $|N^+(C) \cup N^-(C)| \geq m+3$. Then, since $|N^+(C)| = |N^-(C)| = m$, $|N^+(C) \cap N^-(C)| \leq m - 3$. Therefore part (ii) is true.

To verify part (iii), suppose that there exists a component T of G such that

$$T \cap (N^+(C) \cup N^-(C)) \neq \emptyset.$$

Without loss of generality, we may assume $T \cap N^+(C) \neq \emptyset$. Then, by Proposition 5.12(ii), $T \subseteq N^+(C)$ and so $T \cap (N^-(C) - N^+(C)) = \emptyset$. If there is an edge uv

in T such that $u \in N^+(C) - N^-(C)$ and $v \in N^+(C) \cap N^-(C)$, then $u \in N^-(C)$ by Proposition 5.12(ii), a contradiction. Therefore every edge of T has end vertices in $N^+(C) - N^-(C)$ or every edge of T has end vertices in $N^+(C) \cap N^-(C)$. Then, since T is a component, $T \subseteq N^+(C) - N^-(C)$ or $T \subseteq N^+(C) \cap N^-(C)$. Hence part (iii) is true. \square

Theorem 5.28. *Let \mathcal{G}_ℓ be the set of graphs having the least components among $(2, 2)$ CCE graphs containing a cycle of length $\ell \geq 3$ and G_ℓ be a graph in \mathcal{G}_ℓ with the least order. Then the following are true:*

- (i) G_ℓ contains at least six isolated vertices;
- (ii) $G_3 \cong C_3 \cup 6P_1$ and $G_4 \cong C_4 \cup 7P_1$;
- (iii) $\ell = 3$ if and only if $G_\ell \cong C_\ell \cup 6P_1$.

Proof. Fix an integer $\ell \geq 3$. For notational convenience, we simply write G for G_ℓ . Let D be a minimal digraph of G . Take a sink x in D . Then x is isolated in $CCE(D)$ and so, by Proposition 5.23(i), x cannot have indegree 1. If x has indegree 0, then $CCE(D - x)$ is a graph having less components than G and $CCE(D - x)$ still has a cycle of length ℓ , which contradicts the choice of G . Thus x has indegree 2. Hence the predators of x are adjacent in G by Proposition 5.23(ii). Suppose that the predators of x lie on a path component P in G . Then the predators have no common prey in $D - x$ by Proposition 5.23(iii) and so they are not adjacent in $CCE(D - x)$. Therefore the component P breaks up into two pieces in $CCE(D - x)$ while one component disappears by deleting x . Thus $CCE(D - x)$ has the same number of components as $CCE(D)$. By the way, $CCE(D - x)$ still has a cycle of length ℓ , which contradicts the choice of G . Therefore the predators of x lie on a cycle component in G . Since x was arbitrarily chosen, we conclude that

- (†) each sink in D has two predators which are consecutive vertices on a cycle.

Thus each predator of a sink has outdegree 2 by Lemma 5.2(i) and so

- (‡) each predator of a sink has a prey distinct from the sink in D .

If D has exactly one sink x , then $D - x$ has no sink by (\ddagger) , which is impossible. Thus D has at least two sinks. By the way, we may show that D has at least three sinks. To show it by contradiction, suppose that x and x' are the only sinks in D . If there is no common predator of x and x' , then $D - \{x, x'\}$ has no sink by (\ddagger) , which is impossible. Thus there exists a common predator y of x and x' . Then y lies on a cycle by (\dagger) . Thus $N_D^-(x) = \{y, y'\}$ and $N_D^-(x') = \{y, y''\}$ where $y'y''$ is a section of C . Since yy' and yy'' are edges of G , y has two predators z_1 and z_2 such that

$$N_D^+(z_1) = \{y, y'\} \quad \text{and} \quad N_D^+(z_2) = \{y, y''\}.$$

By the assumption that x and x' are the only sinks in D , the sinks of $D' := D - \{x, x', y\}$ belong to $N_D^-(x) \cup N_D^-(x') \cup N_D^-(y) - \{x, x', y\} = \{y', y'', z_1, z_2\}$. However, none of these can be a sink of D' . For, it is easy to check that z_1 and z_2 are not sinks of D' . Since each of y' and y'' has degree 2 in G , each of y' and y'' has outdegree 2 by Lemma 5.2(i) and so has a prey not belonging to $\{x, x'\}$. Moreover, $y' \not\rightarrow y$ and $y'' \not\rightarrow y$ by Proposition 5.26. Therefore y' and y'' are not sinks in D' and so D' has no sinks, which is impossible. Thus D has at least three sinks. Hence D has at least three sources by Proposition 5.22. By (\dagger) , D has no vertex of indegree 0 and outdegree 0. Therefore G has at least three sinks and at least three sources. Thus G has at least 6 isolated vertices and so part (i) is true. Let C be a cycle of length ℓ in G . Then

(\diamond) $C \cup 6P_1$ is an induced subgraph of G .

Since $V(C) \cap (N^+(C) \cup N^-(C)) = \emptyset$ by Proposition 5.26, $|V(G)| \geq |V(C)| + |N^+(C) \cup N^-(C)|$. Then, since $|N^+(C) \cup N^-(C)| \geq \ell + 3$ by Lemma 5.27(ii),

$$|V(G)| \geq |V(C)| + |N^+(C) \cup N^-(C)| \geq 2\ell + 3.$$

If $\ell = 3$, then $G \cong C_3 \cup 6P_1$ by (\diamond) , the digraph D_5 , and its CCE graph $CCE(D_5)$ given in Figure 5.4. Suppose $\ell \geq 4$. If $\ell = 4$, then $|V(G)| \geq 11$ and so $G \cong C \cup 7P_1$ by (\diamond) , the digraph D_6 , and its CCE graph $CCE(D_6)$ given in Figure 5.4. Thus part

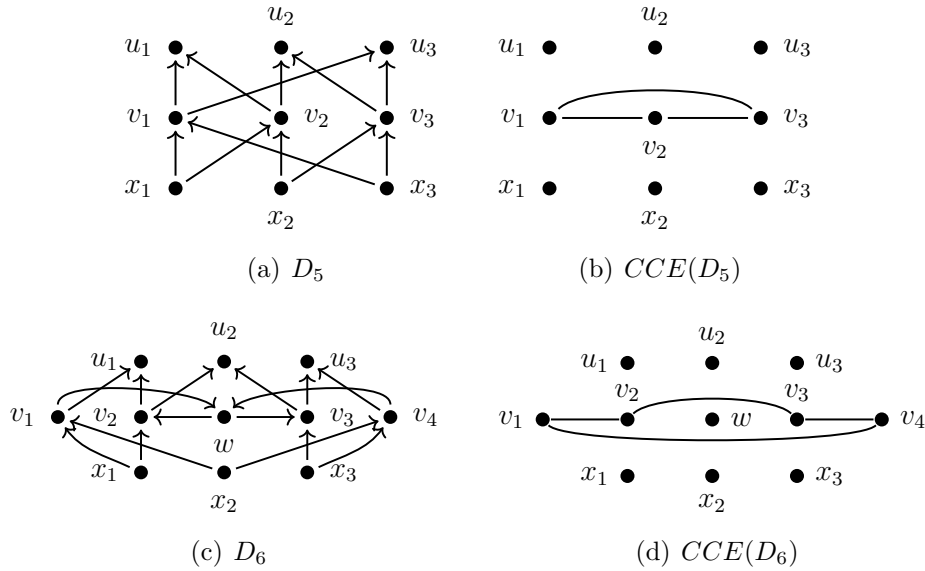


Figure 5.4: Digraphs and its CCE graphs in the proof of Theorem 5.28

(ii) is true. Moreover,

$$|V(G)| \geq 2\ell + 3 > \ell + 6 = |V(C)| + |V(6P_1)|$$

and so G must contain a component not belonging to $C \cup 6P_1$. Hence the “if” part of part (iii) is true and so, by part (ii), the “only if” part is true. Therefore part (iii) is true. \square

Finally, we give a sufficient condition on the number of components for a $(2, 2)$ CCE graph being an interval graph.

Theorem 5.29. *Let G be a $(2, 2)$ CCE graph and t be a number of components of G . If $t \leq 7$, then G is an interval graph. Further, the inequality is tight.*

Proof. Suppose that G is not interval. Then, by Proposition 5.1, G contains a cycle component of length $\ell \geq 4$. Let \mathcal{G}_ℓ be the set of graphs having the least components among the $(2, 2)$ CCE graphs containing a cycle of length ℓ and G_ℓ be a graph

in \mathcal{G}_ℓ with the least order. Then, by (i) and (iii) of Theorem 5.28, G_ℓ contains at least eight components. Thus $t \geq 8$. Therefore we have shown that any $(2, 2)$ CCE graph with at most seven components is interval. Furthermore, by Theorem 5.28(ii), $G_4 \cong C_4 \cup 7P_1$, which has eight components. Since G_4 is not an interval graph, the inequality is tight. \square

Now, we give a characterization of $(2, 2)$ CCE graphs with the least components among $(2, 2)$ CCE graphs containing at most one cycle and exactly two isolated vertices as follows.

Theorem 5.30. *Let G be a graph with the least components among the $(2, 2)$ CCE graphs containing a cycle and exactly two isolated vertices. Then $G \cong C_3 \cup 2P_3 \cup 2P_2 \cup 2P_1$. Further, if D is a minimal digraph of G , then D is isomorphic to D^* or D^* given in Figure 5.6.*

Proof. Let D be a minimal digraph of G . By Proposition 5.20(i), D has a unique source x and a unique sink y , which are the only isolated vertices in G . By (ii) and (iii) of Lemma 5.25, there exist nontrivial paths $P_{v,m}$ and $P_{w,n}$ such that

$$N^+(x) = \{v_1, v_2\}, \quad N^-(v_1) = \{x\}, \quad N^-(y) = \{w_1, w_2\}, \quad N^+(w_1) = \{y\} \quad (5.5)$$

for some integers $m, n \geq 2$. To the contrary, suppose $w_2 \rightarrow w_1$. Then, by Proposition 5.23(ii), w_2 has degree 2 and so $n \geq 3$. Thus, by (c) of Lemma 5.25(v), there exists a CCE graph G' of a $(2, 2)$ digraph such that G' is isomorphic to $G - \{w_1, \dots, w_{n-1}, y\}$. We note that w_n and x are the only isolated vertices in G' , that is, G' contains exactly two isolated vertices and G' has one less nontrivial component than G , which contradicts the choice of G . Therefore

$$w_2 \not\rightarrow w_1. \quad (5.6)$$

Take a cycle C of length ℓ in G for some integer $\ell \geq 3$. Then $x \notin N^+(C)$ and $y \notin N^-(C)$. Since the prey of x are on $P_{v,m}$ and the predators of y are on $P_{w,n}$, $x \notin N^-(C)$ and $y \notin N^+(C)$. Since $|N^+(C)| = |N^-(C)| = \ell$ and $|N^+(C) \cap N^-(C)| \leq \ell - 3$

by (i) and (ii) of Lemma 5.27,

$$N^+(C) - N^-(C) \neq \emptyset \quad \text{and} \quad N^-(C) - N^+(C) \neq \emptyset.$$

Thus there exist components X_1 and X_2 such that $X_1 \cap (N^+(C) - N^-(C)) \neq \emptyset$ and $X_2 \cap (N^-(C) - N^+(C)) \neq \emptyset$. Then, since x and y are the only isolated vertices in G and neither x nor y belongs to any of $N^+(C)$ and $N^-(C)$, it is true that X_1 and X_2 are nontrivial. Moreover, by Lemma 5.27(iii),

$$X_1 \subseteq N^+(C) - N^-(C) \quad \text{and} \quad X_2 \subseteq N^-(C) - N^+(C). \quad (5.7)$$

By Proposition 5.26, $C \neq X_1$ and $C \neq X_2$ and so C, X_1, X_2 are three nontrivial components of G . Let

$$N = N^+(C) \cup N^-(C).$$

We first claim that

$$P_{w,n} \cap N = \emptyset. \quad (5.8)$$

To show the claim by contradiction, suppose $P_{w,n} \cap N \neq \emptyset$. If $P_{w,n} \cap N^-(C) \neq \emptyset$, then, by Lemma 5.27(iii), $P_{w,n} \subseteq N^-(C)$ and so $y \in V(C)$ (recall $N^+(w_1) = \{y\}$), a contradiction. Therefore $P_{w,n} \cap N^-(C) = \emptyset$. Thus $P_{w,n} \cap (N^+(C) - N^-(C)) \neq \emptyset$. Then, by Lemma 5.27(iii),

$$P_{w,n} \subseteq N^+(C) - N^-(C).$$

However, by (5.6) and, Lemma 5.25(iv)(a), there exists a predator w^* of w_1 which has outdegree 1 and is an end vertex on some nontrivial path in G . Then the vertex adjacent to w^* is $w_{1,2}^+$ and so $N_D^-(w_1) = \{w^*, w_{1,2}^+\}$. Since w^* and $w_{1,2}^+$ are on a path, $w_1 \notin N^+(C)$ and so $P_{w,n} \not\subseteq N^+(C)$, a contradiction. Thus (5.8) is valid. Since $P_{w,n}$ is a component containing predators of the sink and $P_{v,w}$ is a component containing prey of the source, by Proposition 5.22, the following is also valid:

$$P_{v,m} \cap N = \emptyset \quad (5.9)$$

Now we claim that

Claim A. If $P_{w,n} = P_{v,m}$, then $n \geq 3$, $x = w_{n-1,n}^+$, and there is a nontrivial path component $P_{z,t}$ for some integer $t \geq 2$ such that $N^+(z_1) = \{w_1\}$, $w_1 = z_{1,2}^-$, $P_{z,t} \neq P_{w,n}$, and $P_{z,t} \cap N = \emptyset$.

Proof of Claim A. Suppose $P_{w,n} = P_{v,m}$. Then, by Lemma 5.25(iii), $N^+(x) \neq N^-(y)$. Thus

$$n \geq 3 \quad \text{and} \quad x = w_{n-1,n}^+$$

Then, by (5.6) and Lemma 5.25(iv)(a), there exists a predator, namely w' , of w_1 having outdegree 1 in D and w' is an end vertex of some nontrivial path component $P_{z,t}$ distinct from $P_{w,n}$. Without loss of generality, we may assume $w' = z_1$. To show $P_{z,t} \cap N = \emptyset$, suppose $P_{z,t} \cap N \neq \emptyset$. Then

$$P_{z,t} \subseteq N^+(C) - N^-(C) \text{ or } P_{z,t} \subseteq N^+(C) \cap N^-(C) \text{ or } P_{z,t} \subseteq N^-(C) - N^+(C)$$

by Lemma 5.27(iii). Since z_1 has outdegree 1, $z_1 \notin N^-(C)$ by Lemma 5.2(iv) and so $z_1 \in N^+(C) - N^-(C)$. Then

$$w_2 \not\rightarrow z_1.$$

Thus there exists one predator of z_1 having outdegree 1 by Lemma 5.25(iv)(b). However, since $z_1 \in N^+(C) - N^-(C)$, each predator of z_1 has outdegree 2 in D , a contradiction. Therefore we have shown that the claim is true. \square

By Claim A, (5.8), and (5.9), whether $P_{v,m} = P_{w,n}$ or not, G has at least two nontrivial paths each of which has no intersection with N . Thus G has at least five nontrivial components.

We may check that the CCE graph of a $(2, 2)$ digraph given in Figure 5.6 is isomorphic to $C_3 \cup 2P_3 \cup 2P_2 \cup 2P_1$. Thus the existence of a $(2, 2)$ digraph given in Figure 5.6 guarantees that G has exactly five nontrivial components. Then G has exactly two nontrivial paths each of which has no intersection with N . Therefore

$$X_1 = N^+(C) \quad \text{and} \quad X_2 = N^-(C), \tag{5.10}$$

so

$$|X_1| = |X_2| = \ell$$

by Lemma 5.27(i).

To show $P_{w,n} \neq P_{v,m}$ by contradiction, we suppose $P_{w,n} = P_{v,m}$. Then, by Claim A, $n \geq 3$, $x = w_{n-1,n}^+$, and there is a nontrivial path component $P_{z,t}$ for some integer $t \geq 2$ such that

$$N^+(z_1) = \{w_1\}, \quad w_1 = z_{1,2}^-, \quad P_{z,t} \neq P_{w,n}, \quad \text{and} \quad P_{z,t} \cap N = \emptyset.$$

Thus the nontrivial components of G are $P_{w,n}, P_{z,t}, X_1, X_2$, and C . Since $N^+(z_1) = \{w_1\}$ and $w_1 = z_{1,2}^-$, $z_2 = w_{1,2}^+$. If $n \leq t$, then $A(D)$ contains an outer arc set of z_1, z_2, \dots, z_n toward w_1, w_2, \dots, w_{n-1} by Corollary 5.7 and so $\{x, z_{n-1}, z_n\} \subseteq N^-(w_{n-1})$, which is impossible. Thus

$$n > t \geq 2$$

and $A(D)$ has an outer arc set of z_1, z_2, \dots, z_t toward w_1, w_2, \dots, w_{t-1} by Corollary 5.7. Then $z_t \rightarrow w_t$. To show $w_2 \not\rightarrow z_1$ by a contradiction, suppose $w_2 \rightarrow z_1$. Then $z_1 = w_{2,3}^-$ and $w_3 = z_{1,2}^+$. Thus $A(D)$ contains an outer arc set of w_2, w_3, \dots, w_{t+1} toward z_1, z_2, \dots, z_{t-1} by Corollary 5.7. Then $w_{t+1} \rightarrow z_t$. By Proposition 5.22 and Claim A, since $P_{w,n}, P_{z,t}, X_1, X_2$, and C are the nontrivial components of G , one of the following holds:

- $N^-(z_1) = \{w_n\}$ and $w_n = z_{1,2}^+$;
- $N^-(z_t) = \{w_n\}$ and $w_n = z_{t-1,t}^+$.

Recall that $z_1 = w_{2,3}^-$ and $w_{t+1} \rightarrow z_t$. Thus $N^-(z_t) = \{w_n\}$, $w_n = z_{t-1,t}^+$, and $t + 1 = n$. Since x is isolated, $N^-(w_n) = \{x\}$ by Proposition 5.23(ii). Thus $V(P_{w,n}) \cup V(P_{z,t}) \cup \{x, y\}$ forms a weak component \tilde{D}_1 (see Figure 5.5(c)) and so D is not weakly connected, which contradicts Proposition 5.20(iii). Hence

$$w_2 \not\rightarrow z_1.$$

Therefore at least one predator of z_1 has outdegree 1 by Lemma 5.25(iv)(b). Let z' be the predator of z_1 having outdegree 1. Thus $z' \neq w_i$ for each $2 \leq i \leq n-1$ and $z' \neq z_i$ for each $1 \leq i \leq t-1$. Further, since $N^+(w_1) = \{y\}$, $z' \neq w_1$. We will claim that $z' \neq w_n$ and $z' \neq z_t$ to show $z' \notin P_{w,n} \cup P_{z,t}$. Since x and y are the only isolated vertices, z' has degree 1. In addition, since z' has outdegree 1, $z' \neq z_{1,2}^+$ and so

$$N^-(z_1) = \{z', z_{1,2}^+\}.$$

Therefore z_1 , which is the only prey of z' , has indegree 2 and so $z' \neq w_n$. Recall that $A(D)$ has an outer arc set of z_1, z_2, \dots, z_t toward w_1, w_2, \dots, w_{t-1} , and $z_t \rightarrow w_t$. Thus $N^+(z_t) = \{w_{t-1}, w_t\}$. Hence $z' \neq z_t$ and so $z' \notin P_{w,n} \cup P_{z,t}$. Then, since $P_{w,n}, P_{z,t}, X_1, X_2$ and C are the components of G , $z' \in X_1$ or $z' \in X_2$. If $z' \in X_2$, then $z_1 \in V(C)$ by (5.7), which is impossible. Thus

$$z' \in X_1$$

and so, by (5.7), z' is a common prey of two consecutive vertices on C . Recall $C = C_{u,\ell}$. Without loss of generality, we may assume $z' = u_{1,2}^-$. Since D is acyclic, $\tilde{D}_2 := D - \{y, w_1, z_1, z'\}$ has a sink. The possible sinks of \tilde{D}_2 are the predators of one of y, w_1, z_1, z' , so w_2, z_2, u_1 , and u_2 are only possible sinks of \tilde{D}_2 . However, none of these can be a sink of \tilde{D}_2 . For, if $w_2 \rightarrow z'$, then $w_2 = u_1$ or $w_2 = u_2$ and so $w_2 \in V(C)$, which is impossible. Thus $w_2 \not\rightarrow z'$. Since w_2 has a prey not belonging to $\{y, w_1, z_1\}$, w_2 is not a sink in \tilde{D}_2 . Since w_2 is a prey of z_2 , z_2 is not a sink in \tilde{D}_2 . Since $N^-(z_1) = \{z', z_{1,2}^+\}$, $u_1 \not\rightarrow z_1$ and $u_2 \not\rightarrow z_1$. We note that each of y and w_1 has two predators distinct from u_1 and u_2 . Now, since u_1 and u_2 have outdegree 2, each of them has a prey not belonging to $\{y, w_1, z_1, z'\}$. Therefore \tilde{D}_2 has no sink and we reach a contradiction. Consequently, we have shown

$$P_{w,n} \neq P_{v,m}. \tag{5.11}$$

Thus $P_{w,n}, P_{v,m}, X_1, X_2$, and C are the nontrivial components of G . To show $n = 2$ by contradiction, suppose $n \geq 3$. Then we may check that the CCE graph of a digraph

$D - y$ is a union of C , X_1 , X_2 , $P_{v,m}$, Q , and two isolated vertices x and w_1 , where Q is a nontrivial path $w_2w_3 \cdots w_n$. We consider the set \mathcal{D} of spanning subdigraphs of $D - y$ whose CCE graphs are isomorphic to

$$C \cup X_1 \cup X_2 \cup P_{v,m} \cup Q \cup \{w_1, x\}$$

and take a minimal digraph D' in \mathcal{D} . Then w_1 is a unique sink of D' . By applying Lemma 5.25(ii) to D' , we may claim that there is a nontrivial path $P_{u,l}$ in $C \cup X_1 \cup X_2 \cup P_{v,m} \cup Q$ such that $N_{D'}^-(w_1) = \{u_1, u_2\}$ and $N_{D'}^+(u_1) = \{w_1\}$. Then $P_{u,l}$ is one of X_1 , X_2 , $P_{v,m}$, and Q . Note that $C \cup X_1 \cup X_2 \cup P_{v,m} \cup Q$ is a graph still satisfying the property that it has the least components among the $(2, 2)$ CCE graphs containing a cycle and exactly two isolated vertices. Thus, by applying (5.8) to D' , we may assert that $P_{u,l}$ is neither X_1 nor X_2 . Moreover, by (5.11) applied to D' , $P_{u,l} \neq P_{v,m}$. Therefore $P_{u,l} = Q$. Since $w_2 \not\rightarrow w_1$ by (5.6), $u_1 \neq w_2$ and so $u_1 = w_n$ and $u_2 = w_{n-1}$. Then $w_1 = w_{n-1,n}^-$ and $w_{n-1} \rightarrow w_1$ in D . By Theorem 5.9 applied to $P_{w,n}^{-1}$, $w_n = w_{1,2}^+$ and so w_n has outdegree 2 in D . Since w_{n-1} has degree 2 in G , it has outdegree 2 in D . Thus each predator of w_1 has outdegree 2 in D , which is a contradiction to Lemma 5.25(iv)(a). Consequently, we have shown

$$n = 2.$$

Since n is the order of the component containing the predator of the sink and m is the order of the component containing the prey of the source,

$$m = 2$$

by Proposition 5.22. Hence $P_{w,2}$, $P_{v,2}$, X_1 , X_2 , and C are the nontrivial components of G . Moreover, $N^+(w_1) = N^+(w_2) = \{y\}$ and $N^-(v_1) = N^-(v_2) = \{x\}$ by Proposition 5.23(ii). Since $X_2 = N^-(C)$, $w_{1,2}^+ \notin X_2$ and so $w_{1,2}^+ \in P_{v,2}$ or $w_{1,2}^+ \in X_1$.

To show that $w_{1,2}^+$ has degree 2 by contradiction, suppose that $w_{1,2}^+$ has degree not equal to 2. Since $w_{1,2}^+ \neq x$ and $w_{1,2}^+ \neq y$, $w_{1,2}^+$ has degree at least 1 and so $w_{1,2}^+$ has degree 1. Then, for the vertex, say w^* , adjacent to $w_{1,2}^+$, either $w^* \rightarrow w_1$

or $w^* \rightarrow w_2$ by Proposition 5.23(iii). Without loss of generality, we may assume $w^* \rightarrow w_1$. Then $w^* \not\rightarrow w_2$. Since each of w_2 and $w_{1,2}^+$ has degree 1, $N_D^-(w_2) = \{w_{1,2}^+\}$ by Proposition 5.23(ii). We consider a digraph $D - \{w_2, y\}$. It is easy to check that the CCE graph of $D - \{w_2, y\}$ is isomorphic to $X_1 \cup X_2 \cup C \cup P_{v,m} \cup \{w_2, x\}$, which contradicts the choice of G . Thus $w_{1,2}^+$ has degree 2. Hence $w_{1,2}^+ \notin P_{v,2}$ and so $w_{1,2}^+ \in X_1$. Therefore X_1 contains a path $z_1 z_2 z_3$ (not necessary be an induced path) such that

$$z_1 \rightarrow w_1, \quad z_2 = w_{1,2}^+, \quad z_3 \rightarrow w_2. \quad (5.12)$$

Since $X_1 = N^+(C)$, there are consecutive vertices, say u_1, u_2, u_3, u_4 , on C such that

$$z_1 = u_{1,2}^-, \quad z_2 = u_{2,3}^-, \quad z_3 = u_{3,4}^- \quad (5.13)$$

by Proposition 5.6. Since u_1 has degree 2, u_1 has a prey not belonging to $\{z_1, z_2\}$. Suppose, to the contrary, that z_1 or z_3 has outdegree 2. Without loss of generality, we may assume that z_3 has outdegree 2. Then z_3 has a prey distinct from w_1 and w_2 and the prey does not belong to $V(C)$ since $X_1 \cap X_2 = \emptyset$ and $X_2 = N^-(C)$. Now we consider the digraph $\tilde{D}_3 := D - \{y, w_1, w_2, z_1, z_2, u_2\}$. Since D is acyclic, \tilde{D}_3 has a sink. The possible sinks of \tilde{D}_3 are $u_1, u_{1,2}^+, u_{2,3}^+, u_3, z_3$ (see Figure 5.5(c)) and we may check that those vertices are not sinks in \tilde{D}_3 . Thus \tilde{D}_3 has no sink, which is a contradiction. Therefore each of z_1 and z_3 has outdegree 1 and so each of them has degree 1 in G . Hence $u_1 = u_4$ and $X_1 = z_1 z_2 z_3 = u_{1,2}^- u_{2,3}^- u_{3,1}^-$. Then, since X_1 is the component containing prey of the vertices on C and X_2 is the component containing predators of the vertices on C , X_1 and X_2 have the same length by Proposition 5.22. Accordingly, $\ell = 3$. Thus $G \cong C_3 \cup 2P_3 \cup 2P_2 \cup 2P_1$.

Note that D was chosen to be a minimal digraph of G . Thus “further” part is true if D is isomorphic to D^* or D^* in Figure 5.6. By applying Proposition 5.22 to the previous argument, X_2 is of the form

$$X_2 = z'_1 z'_2 z'_3$$

with

$$N^-(z'_1) = \{v_1\}, \quad z'_2 = v_{1,2}^-, \quad N^-(z'_3) = \{v_2\}, \quad \text{and} \quad \{z'_1, z'_2, z'_3\} = \{u_{1,2}^+, u_{2,3}^+, u_{3,1}^+\}.$$

Then, together with (5.5), (5.12), and (5.13), we may fix some subdigraph \tilde{D} of D as in Figure 5.6(a) under the isomorphism. Moreover, the remaining arcs of D except the ones in \tilde{D} are determined by $\{z'_1, z'_2, z'_3\} = \{u_{1,2}^+, u_{2,3}^+, u_{3,1}^+\}$. By the way, there are exactly four automorphisms on \tilde{D} . To see why, we consider the two weak components of \tilde{D} . We denote the weak components by F_1 and F_2 as shown in Figure 5.6(a). It is easy to check that each of them has exactly one nonidentity automorphism as follows:

- $f : V(F_1) \rightarrow V(F_1)$ defined by $f(a) = a$ for each $a \in V(D) - \{w_1, w_2, u_{1,2}^-, u_{3,1}^-, u_2, u_3\}$ and

$$(f(w_1), f(w_2), f(u_{1,2}^-), f(u_{3,1}^-), f(u_2), f(u_3)) = (w_2, w_1, u_{3,1}^-, u_{1,2}^-, u_3, u_2),$$

that is, f is the map only switching between w_1 (resp. $u_{1,2}^-$, u_2) and w_2 (resp. $u_{3,1}^-$, u_3);

- $g : V(F_2) \rightarrow V(F_2)$ defined by $g(a) = a$ for each $g \in V(D) - \{z'_1, z'_3, v_1, v_2\}$ and

$$(g(z'_1), g(z'_3), g(v_1), g(v_2)) = (z'_3, z'_1, v_2, v_1),$$

that is, g is the map only switching between z'_1 (resp. v_1) and z'_3 (resp. v_2).

Since F_1 and F_2 are not isomorphic, the automorphisms on \tilde{D} are

$$id_{V(D)}, \quad h_1, \quad h_2, \quad \text{and} \quad h_3$$

where $id_{V(D)}$ is the identity map on $V(D)$ and the rest are

$$h_1 := \begin{cases} f(v) & \text{if } v \in V(F_1) \\ v & \text{if } v \in V(F_2) \end{cases}, \quad h_2 := \begin{cases} v & \text{if } v \in V(F_1) \\ g(v) & \text{if } v \in V(F_2) \end{cases}, \quad h_3 := \begin{cases} f(v) & \text{if } v \in V(F_1) \\ g(v) & \text{if } v \in V(F_2) \end{cases}.$$

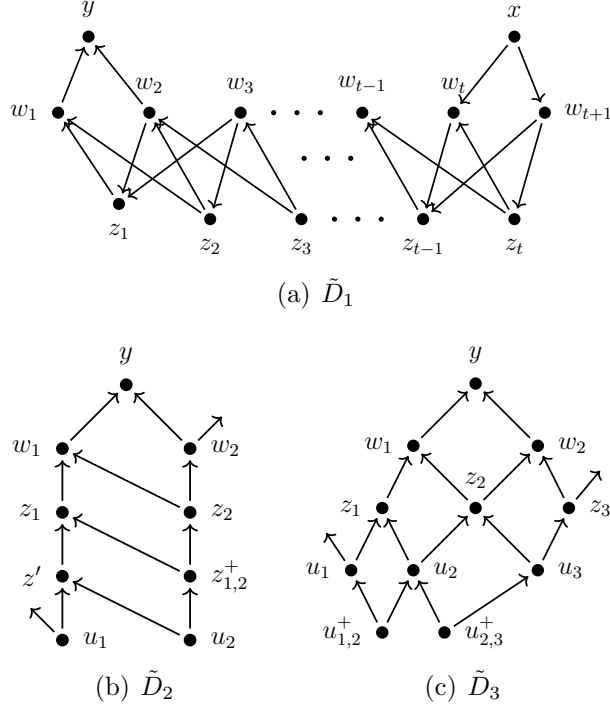


Figure 5.5: Digraphs in the proof of Theorem 5.30

Under the group action of the automorphism group $\{id_{V(D)}, h_1, h_2, h_3\}$ on the set of six digraphs from \tilde{D} obtained by determining a bijection from $\{z'_1, z'_2, z'_3\}$ to $\{u_{1,2}^+, u_{2,3}^+, u_{3,1}^+\}$, it is not difficult to see that there are exactly two orbits:

- one consisting of digraphs determined by $(z'_1, z'_2, z'_3) = (u_{1,2}^+, u_{2,3}^+, u_{3,1}^+)$ or $(z'_1, z'_2, z'_3) = (u_{3,1}^+, u_{2,3}^+, u_{1,2}^+)$, which are isomorphic to D^* ;
- the other one consisting of digraphs determined by $(z'_1, z'_2, z'_3) = (u_{1,2}^+, u_{3,1}^+, u_{2,3}^+)$, $(z'_1, z'_2, z'_3) = (u_{2,3}^+, u_{3,1}^+, u_{1,2}^+)$, $(z'_1, z'_2, z'_3) = (u_{3,1}^+, u_{1,2}^+, u_{2,3}^+)$, or $(z'_1, z'_2, z'_3) = (u_{2,3}^+, u_{1,2}^+, u_{3,1}^+)$, which are isomorphic to D^* .

Therefore D is isomorphic to D^* or D^* given in Figure 5.6. \square

Corollary 5.31. *Let G be a $(2, 2)$ CCE graph with exactly two isolated vertices. If G has at most four nontrivial components, then G has no cycle.*

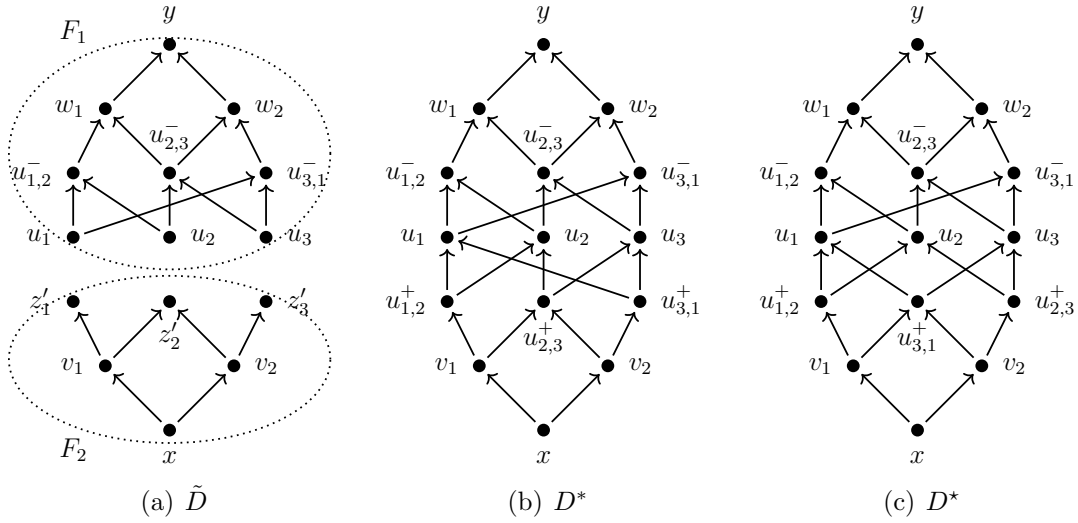


Figure 5.6: A subdigraph \tilde{D} and two digraphs D^* and D^* in the proof of Theorem 5.30

Theorem 5.32. *Let G be a graph with the least components among the $(2, 2)$ CCE graphs containing at least three vertices exactly two of which are isolated. Then $G \cong 2P_1 \cup P_m$ for some positive integer m . Further, if G is isomorphic to $2P_1 \cup P_m$, then a digraph D whose CCE graph is G is isomorphic to: a subdigraph of D_m^* if $m = 1$; D_m^* or $D_m^* - (v_2, v_3)$ if $m = 2$; D_m^* otherwise where D_m^* is the digraph given in Figure 5.7.*

Proof. If G has a cycle, then G contains at least five components by Theorem 5.30. Since each component of G is a path or a cycle by Proposition 5.1, it is sufficient to prove the statement by constructing a $(2, 2)$ digraph whose CCE graph is isomorphic to $2P_1 \cup P_m$ for each m . We consider the digraph D_m^* with $V(D_m^*) = \{v_1, v_2, \dots, v_{m+2}\}$ having an arc (v_1, v_2) and an outer arc set of v_1, \dots, v_{m+1} toward v_3, \dots, v_{m+2} . Then we can check that $CCE(D_m^*)$ is isomorphic to $2P_1 \cup P_m$ and so D_m^* is the desired one.

To show the “further” part, suppose that $G \cong 2P_1 \cup P_m$ for some positive integer m . Let D be a $(2, 2)$ digraph whose CCE graph is G . suppose $m = 1$. Then $|V(D)| = 3$. Since D is acyclic, D is isomorphic to a subdigraph of D_m^* .

Now we suppose $m \geq 2$. Then, by Proposition 5.20(i), D has exactly one source x and exactly one sink y . Therefore x has a prey x' of indegree 1 and y has a predator y' of outdegree 1 by Proposition 5.20(ii). Since x and y are the only two trivial paths in G , x' and y' have degree 1 in G by Lemma 5.2(i) and the vertices other than x and y are on P_m . Let $P_m = P_{x,m}$. Suppose $x' = y'$. Then, without loss of generality, we may assume $x' = y' = x_1$. If $m \geq 3$, then x_2 has degree 2 and so, by Lemma 5.4, x_m is a prey of x_2 and a predator of x_2 , which contradicts that D is acyclic. Thus $m = 2$. Then x_2 must be a prey of x and a predator of y . Therefore D is isomorphic to $D_m^* - (v_2, v_3)$. Now we suppose

$$x' \neq y'.$$

Since G has exactly two vertices of degree 1, $\{x', y'\} = \{x_1, x_m\}$. Without loss of generality, we may assume that $x' = x_1$ and $y' = x_m$. Then, since x_1 has indegree 1 and x_1 is adjacent to x_2 in G , x is a common predator of x_1 and x_2 . Moreover, x_m has outdegree 1 and x_{m-1} is adjacent to x_m in G , y is a common prey of x_{m-1} and x_m . Since D is a $(2, 2)$ digraph,

$$N^+(x) = \{x_1, x_2\}, \quad \text{and} \quad N^-(y) = \{x_{m-1}, x_m\}.$$

Then, since D is acyclic, D is isomorphic to D_m^* or $D_m^* - (v_2, v_3)$ if $m = 2$.

Now we consider the case $m \geq 3$. Since D is a $(2, 2)$ digraph, $x_{1,2}^- \neq y$. Since x is a source, $x_{1,2}^- \neq x$. Thus $x_{1,2}^- = x_t$ for some integer $2 < t \leq m$. Then, by Theorem 5.9,

$$x_{m-t+1} \in N^-(x_m).$$

Suppose, to the contrary, that $t \neq 3$. Then $t \geq 4$ and so $m \geq 4$. We consider the digraph $D' := D - y$. Then we can check that x_m is a sink in D' and $CCE(D')$ is isomorphic to $\{x\} \cup \{x_m\} \cup P_{x,m-1}$. Then, since x_{m-1} has degree 1 in $CCE(D')$, x_m is a common prey of x_{m-1} and x_{m-2} in D' by the above argument. Thus x_m is a common prey of x_{m-1} and x_{m-2} in D . By the way, since $t \geq 4$, $x_{m-t+1} \neq x_{m-1}$ and $x_{m-t+1} \neq x_{m-2}$ and so x_m has at least three predators, a contradiction. Therefore $t = 3$. Then

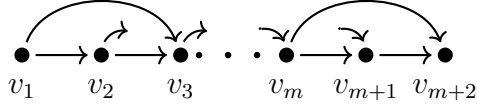


Figure 5.7: The digraph D_m^*

x_i has exactly two prey that are determined for $1 \leq i \leq m - 1$ by Theorem 5.9. Moreover, the arcs incident to each vertex in $\{x, y, x_m\}$ were determined. Then, by letting $x = v_1, y = v_{m+2}$, and $x_i = v_{i+1}$ for each $1 \leq i \leq m$, we can check that D is isomorphic to D_m^* . \square

The above theorem says that a $(2, 2)$ CCE graph G with exactly two isolated vertices has exactly one nontrivial components if and only if G is isomorphic to $P_m \cup 2P_1$ for some integer $m \geq 2$. Extending the result, the following theorem gives a characterization of $(2, 2)$ CCE graphs G having exactly two isolated vertices and two nontrivial components.

Theorem 5.33. *Let G be a $(2, 2)$ CCE graph with exactly two isolated vertices. Then G has exactly two nontrivial components if and only if it is isomorphic to one of the followings:*

- (a) $2P_1 \cup P_2 \cup P_m$ for some $m \geq 2$;
- (b) $2P_1 \cup 2P_m$ for some $m \geq 3$;
- (c) $2P_1 \cup P_m \cup P_{m+1}$ for some $m \geq 3$.

Proof. To show the “only if” part, suppose that G has exactly two nontrivial components. Then there exists a weakly connected minimal digraph D of $G \cup 2P_1$ by Proposition 5.20(iii). By Corollary 5.31, each nontrivial component of G is a nontrivial path. In addition, by Proposition 5.20(i), D has a unique source x and a unique sink y such that $CCE(D) = G \cup \{x, y\}$. Then, by Lemma 5.25(ii), there exists a nontrivial path $P_{w,n}$ such that y is a common prey of w_1 and w_2 , i.e.,

$$N^-(y) = \{w_1, w_2\}. \tag{5.14}$$

By the hypothesis, there is the other nontrivial path $P_{v,m}$ of $G \cup 2P_1$. If $n = 2$ or $m = 2$, then (a) holds. Suppose

$$n \geq 3 \quad \text{and} \quad m \geq 3.$$

To the contrary, suppose $w_2 \rightarrow w_1$. Thus $w_1 = w_{2,3}^-$. By (a) and (b) of Lemma 5.25(v), $w_n \rightarrow w_{n-1}$ and $A(D)$ contains an outer arc set of w_2, w_3, \dots, w_n toward w_1, w_2, \dots, w_{n-2} and $w_{n-1,n}^+$ is an end vertex of a nontrivial path distinct from $P_{w,n}$. Thus $w_{n-1,n}^+ \in P_{v,m}$. Without loss of generality, we may assume that $w_{n-1,n}^+ = v_1$. Then $N^+(v_1) = \{w_{n-1}, w_n\}$ and $N^-(w_{n-1}) = \{w_n, v_1\}$. Thus $w_n = v_{1,2}^-$. Since $N^-(w_{n-1}) = \{w_n, v_1\}$, w_{n-1} cannot be a prey of v_2 . Thus, by Lemma 5.4 (where $u = v_2$), the prey of v_2 other than w_n is an end vertex of a path in G , which is a common prey of v_2 and v_3 . We note that none of y, w_1 , and w_n can be $v_{2,3}^-$ since $N^-(y) = \{w_1, w_2\}$, $N^-(w_1) = \{w_2, w_3\}$, and $N^-(w_n) = \{v_1, v_2\}$. Thus $v_{2,3}^-$ is either v_1 or v_m . If $v_{2,3}^- = v_1$, then w_n and v_1 have a common prey w_{n-1} and a common predator v_2 , which contradicts to the hypothesis that $P_{w,n}$ and $P_{v,m}$ are distinct two paths. Therefore $v_{2,3}^- = v_m$. Then, by applying Theorem 5.9 to the (v_2, v_m) -section of $P_{v,m}$, we may show $\{w_n, v_{m-1}, v_m\} \subseteq N^+(v_2)$, which is impossible. Then

$$w_2 \not\rightarrow w_1,$$

so, by Lemma 5.25(iv)(a), v_1 or v_m is a predator of w_1 and has outdegree 1. Without loss of generality, we may assume that v_1 is a such a vertex. Then $N^+(v_1) = \{w_1\}$,

$$w_1 = v_{1,2}^-, \quad \text{and} \quad v_2 \rightarrow w_2. \tag{5.15}$$

To reach a contradiction, suppose $n > m + 1$. Then, by (5.15) and Corollary 5.7, there is an outer arc set of v_1, \dots, v_m toward w_1, \dots, w_{m-1} and so $v_m = w_{m-1,m}^+$. Moreover, w_{m+1} exists, and $w_{m,m+1}^+$ and v_m are not adjacent. Then, since $w_{m,m+1}^+$ and v_m are predators of w_m , $w_{m,m+1}^+$ is a vertex of degree at most 1 by Lemma 5.4. Hence $w_{m,m+1}^+ = x$ or w_n . If $w_{m,m+1}^+ = x$, then $w_{m+1} = w_n$ by Lemma 5.25(i), which contradicts the case assumption. Thus $w_{m,m+1}^+ = w_n$. Then $w_{n-1} \rightarrow w_m$ or $w_{n-1} \rightarrow w_{m+1}$. If $w_{n-1} \rightarrow w_{m+1}$, then, by Theorem 5.9 applied to the (w_n, w_m) -

section of $P_{w,m}^{-1}$, $w_{n-1} \rightarrow w_m$. Thus $w_{n-1} \rightarrow w_m$ and so, by Theorem 5.9 applied to the (w_n, w_{m-1}) -section of $P_{w,m}^{-1}$, $w_{n-1} \rightarrow w_{m-1}$. Hence $\{v_{m-1}, v_m, w_{n-1}\} \subseteq N^-(w_{m-1})$, a contradiction. Thus

$$n \leq m + 1. \quad (5.16)$$

To show $n \in \{m-1, m, m+1\}$, suppose $n < m-1$. Then, by (5.15) and Corollary 5.7, there is an outer arc set of v_1, \dots, v_{n+1} toward w_1, \dots, w_n . Thus $N^+(x) = \{v_1, v_2\}$ or $\{v_{m-1}, v_m\}$. Then, by Proposition 5.3 and (5.16), $m \leq n + 1$, which contradicts the assumption $n < m - 1$. Therefore $n \in \{m - 1, m, m + 1\}$ and so the ‘‘only if’’ part is true.

Now we show the ‘‘if part’’. First suppose that (a) holds. Then P_2 is a component of G . By Theorem 5.32, $2P_1 \cup P_m$ is the CCE graph of the digraph D_m^* given in Figure 5.7 for each $m \geq 2$. We obtain a digraph D_m^{**} from D_m^* by removing arcs $(v_m, v_{m+2}), (v_{m+1}, v_{m+2})$ and adding two vertices y_1, y_2 and arcs

$$(v_m, y_1), (v_{m+1}, y_1), (v_{m+1}, y_2), (y_1, v_{m+2}), (y_2, v_{m+2}).$$

Then, in $CCE(D_m^{**})$, v_1 and v_{m+2} are the only isolated vertices, $\{v_2, \dots, v_{m+1}\}$ forms a path of length m , and $\{y_1, y_2\}$ forms a path of length 2. Thus $CCE(D_m^{**}) \cong 2P_1 \cup P_2 \cup P_m$.

Second, suppose that (b) holds. Then G is isomorphic to $2P_1 \cup 2P_m$ for some $m \geq 3$. Fix $m \geq 3$. We consider the digraph D_1 with the vertex set $V(D_1) = \{u_1, \dots, u_m, v_1, \dots, v_m, x, y\}$ and the arc set

$$A_1 \cup A_2 \cup \{(u_m, v_{m-1}), (v_m, u_m), (u_1, y), (u_2, y), (x, v_{m-1}), (x, v_m)\}$$

where A_1 (resp. A_2) is the outer arc set of v_1, v_2, \dots, v_m (resp. u_2, u_3, \dots, u_m) toward u_1, u_2, \dots, u_{m-1} (resp. v_1, v_2, \dots, v_{m-2}) (see the digraph D_1 given in Figure 5.8 for an illustration). Obviously, x and y are isolated in $CCE(D_1)$. We can check that $CCE(D_1) = P_{u,m} \cup P_{v,m} \cup \{x, y\}$.

Finally suppose that (c) holds. Then G is isomorphic to $2P_1 \cup P_m \cup P_{m+1}$ for some $m \geq 3$. We consider the digraph D_2 with vertex set $V(D_2) = \{u_1, \dots, u_{m+1}, v_1, \dots, v_m, x, y\}$

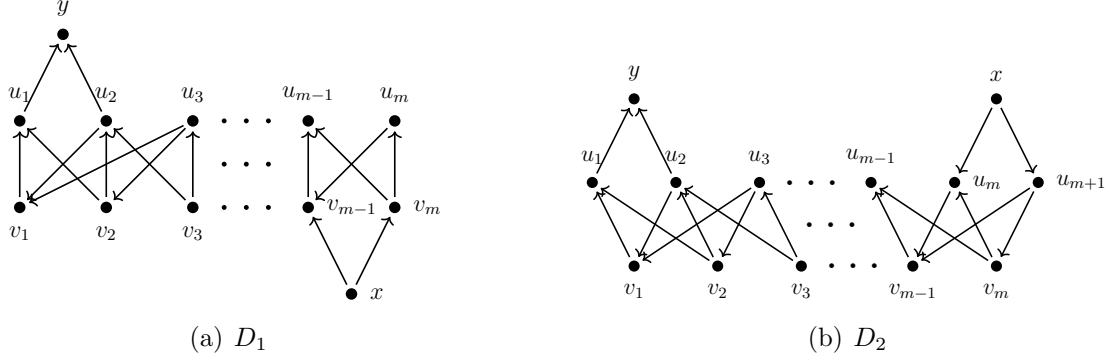


Figure 5.8: The digraphs D_1 and D_2 in the proof of Theorem 5.33

and the arc set

$$A_1 \cup A_2 \cup \{(u_m, v_m), (u_{m+1}, v_m), (u_1, y), (u_2, y), (x, u_m), (x, u_{m+1})\}$$

where A_1 (resp. A_2) is the outer arc set of v_1, v_2, \dots, v_m (resp. u_2, u_3, \dots, u_{m+1}) toward u_1, u_2, \dots, u_{m-1} (resp. v_1, v_2, \dots, v_{m-1}) (see the digraph D_2 given in Figure 5.8 for an illustration). Then in a way similar to the case (b), we can check that $CCE(D_2) = P_{u, m+1} \cup P_{v, m} \cup \{x, y\}$ and this completes the proof of the “if” part. \square

By Theorems 5.30 and 5.32, we completely characterize graphs G having the least components among $(2, 2)$ CCE graphs having at most one cycle component with exactly two isolated vertices. Furthermore, we completely identify CCE graphs G consisting of two nontrivial components with exactly two isolated vertices by Theorem 5.33. Naturally, we come up with a question, “Which $(2, 2)$ CCE graph has two nontrivial components with not two but exactly three isolated vertices?” The following partially answers the question (Theorem 5.36).

Lemma 5.34. *Let $G_1 \cup iP_1$ and $G_2 \cup jP_1$ be $(2, 2)$ CCE graphs for some positive integers i and j . Then $G_1 \cup G_2 \cup (i + j - 1)P_1$ is a $(2, 2)$ CCE graph.*

Proof. Let D_1 and D_2 be $(2, 2)$ digraphs whose CCE graphs are $G_1 \cup iP_1$ and $G_2 \cup jP_1$, respectively. Since D_1 and D_2 are acyclic digraphs, D_1 contains a sink u and D_2

contains a source v . Then we obtain a digraph D from D_1 and D_2 by identifying u with v . Obviously, D is a $(2, 2)$ digraph and $CCE(D) = G_1 \cup G_2 \cup (i + j - 1)P_1$. Therefore the CCE graph of D is $G_1 \cup G_2 \cup (i + j - 1)P_1$. \square

Proposition 5.35. *For each positive integer t , a graph consisting of t path components and $t + 1$ isolated vertices is the CCE graph of a $(2, 2)$ digraph.*

Proof. Fix a positive integer t . Let T_1, T_2, \dots, T_t be the path components. Then, by Theorem 5.32, there exists a digraph D_i whose CCE graph is $T_i \cup 2P_1$ for each $1 \leq i \leq t$. Therefore, by applying Lemma 5.34 to D_1 and D_2 , we obtain a $(2, 2)$ digraph D'_1 whose CCE graph is $T_1 \cup T_2 \cup 3P_1$. Then we apply Lemma 5.34 to D'_1 and D_3 to obtain a $(2, 2)$ digraph D'_2 whose CCE graph is $T_1 \cup T_2 \cup T_3 \cup 4P_1$. We repeat this process until we obtain a digraph D'_{t-1} whose CCE graph is $T_1 \cup T_2 \cup \dots \cup T_t \cup (t + 1)P_1$. \square

Theorem 5.36. *If $P_n \cup P_m \cup iP_1$ is a $(2, 2)$ CCE graph with $3 \leq n$ and $n + 2 \leq m$ for some positive integer i , then $i \geq 3$, further, the inequality is tight.*

Proof. Since every $(2, 2)$ CCE graph contains two isolated vertices, $i \geq 2$. By Theorem 5.33, $P_n \cup P_m \cup 2P_1$ cannot be a $(2, 2)$ CCE graph. Thus $i \neq 2$ and so $i \geq 3$. Now we show “further” part. By Proposition 5.35, there exists a $(2, 2)$ digraph whose CCE digraph is isomorphic to $P_n \cup P_m \cup 3P_3$ and so the inequality is tight. \square

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국문초록

이 논문에서 경쟁그래프의 주요 변이들 중 m -step 경쟁그래프, $(1, 2)$ -step 경쟁그래프, 계통 그래프, 경쟁공적그래프에 대한 연구 결과를 종합했다. Cohen [11]은 먹이사슬에서 포식자-피식자 개념을 연구하면서 경쟁그래프 개념을 고안했다. 생태계는 상호작용하는 종들과 그들의 물리적 환경의 생물학적 체계이다. 생태계의 각 종에 대해서, 토양, 기후, 온도 등과 같은 다양한 차원의 하계 및 상계를 고려하여 좋은 환경을 m 개의 조건들로 나타낼 수 있는데 이를 생태적 지위(ecological niche)라고 한다. 생태학적 기본가정은 두 종이 생태적 지위가 겹치면 경쟁하고(competete), 경쟁하는 두 종은 생태적 지위가 겹친다는 것이다. 흔히 생물학자들은 한 체제에서 서식하는 종들의 경쟁적 관계를 각 종은 꼭짓점으로, 포식자에서 피식자에게는 유향변(arc)을 그어서 먹이사슬로 표현한다. 이러한 맥락에서 Cohen [11]은 다음과 같이 유향그래프의 경쟁그래프를 정의했다. 유향그래프(digraph) D 의 경쟁그래프(competition graph) $C(D)$ 란 $V(D)$ 를 꼭짓점 집합으로 하고 두 꼭짓점 u, v 를 양 끝점으로 갖는 변이 존재한다는 것과 꼭짓점 w 가 존재하여 $(u, w), (v, w)$ 가 모두 D 에서 유향변이 되는 것이 동치인 그래프를 의미한다. Cohen이 경쟁그래프의 정의를 도입한 이후로 그 변이들로 m -step 경쟁그래프(m -step competition graph), (i, j) -step 경쟁그래프((i, j) -step competition graph), 계통그래프(phylogeny graph), 경쟁공적그래프(competition-common enemy graph), p -경쟁그래프(p -competition graph), 그리고 지위그래프(niche graph)가 도입되었고 연구되고 있다.

이 논문의 연구 결과들의 일부는 다음과 같다. 삼각형이 없이 연결된 m -step 경쟁그래프는 트리(tree)임을 보였으며 $2 \leq m < n$ 을 만족하는 정수 m, n 에 대하여 꼭짓점의 개수가 n 개이고 m -step 경쟁그래프가 별그래프(star graph)가 되는 유향그래프를 완벽하게 특징화 하였다.

$k \geq 3$ 이고 방향지어진 완전 k -분할 그래프(oriented complete k -partite graph)의 $(1, 2)$ -step 경쟁그래프 $C_{1,2}(D)$ 에서 각 분할이 완전 부분 그래프를 이룰 때, $C_{1,2}(D)$ 을 모두 특징화 하였다. 또한, $C_{1,2}(D)$ 의 각 성분(component)의 지름(diameter)의 길이가 최대 3이며 $C_{1,2}(D)$ 의 지배수(domination number)에 대한 상계와 최댓값을 구하고 구간그래프(interval graph)가 되기 위한 충분 조건을 구하였다.

차수가 제한된 유향회로를 갖지 않는 유향그래프(degree-bounded acyclic digraph)

의 계통그래프와 경쟁공적그래프에 대해서도 연구하였다. 양의 정수들 i, j 에 대하여 (i, j) 유향그래프란 각 꼭짓점의 내차수는 최대 i , 외차수는 최대 j 인 유향회로 갖지 않는 유향그래프이다. 만약 유향그래프 D 에 각 꼭짓점이 내차수가 최대 i , 외차수가 최대 j 인 경우에 D 를 $\langle i, j \rangle$ 유향그래프라 한다.

D 가 $(i, 2)$ 유향그래프일 때, D 의 계통그래프가 현그래프(chordal graph)가 되기 위한 D 의 방향을 고려하지 않고 얻어지는 그래프(underlying graph)에서 길이가 4 이상인 회로(hole)의 길이에 대한 충분조건을 구하였다. 게다가 (i, j) 유향그래프의 계통그래프에서 나올 수 없는 생성 부분 그래프(forbidden induced subgraph)를 특징화 하였다.

$(2, 2)$ 유향그래프 D 의 경쟁공적그래프 $CCE(D)$ 가 2개의 고립점(isolated vertex)과 최대 1개의 회로를 갖으면서 가장 적은 성분을 갖는 경우일 때의 구조를 규명했다. 마지막으로, $CCE(D)$ 가 구간그래프가 되기 위한 성분의 개수에 대한 충분조건을 구하였다.

주요어휘: 경쟁그래프, m -step 경쟁그래프, $(1, 2)$ -step 경쟁그래프, 계통그래프, 경쟁공적그래프

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