이학석사 학위논문

Cluster Structure in Schubert variety

2023년 2월

서울대학교 대학원
수리과학부
최현재

Cluster Structure in Schubert variety

지도교수 이 승 진

이 논문을 이학석사 학위논문으로 제출함

2022년 10월

서울대학교 대학원
수리과학부
최 현 재

최현재의 이학석사 학위논문을 인준함

2022년 12월

위 원 장	권 재 훈	(인)
부 위 원 장	이 승 진	(인)
위 원	조 철 현	(인)

Cluster Structure in Schubert variety

by

Hyeonjae Choi

A DISSERTATION

Submitted to the faculty of the Graduate School
in partial fulfillment of the requirements
for the degree Master of Science
in the Department of Mathematical Sciences
Seoul National University
February 2023

Abstract

Cluster Structure in Schubert variety

Hyeonjae Choi
Department of Mathematical Sciences
The Graduate School
Seoul National University

In this survey paper, we review the paper [18] which showed how the coordinate ring of (open) Schubert variety in the Grassmannian can be identified with a cluster algebra by using Postnikov's plabic graph. This generalizes a theorem of Scott [17] and proves a conjecture for Schubert varieties which is stated explicitly in [14]. To prove the conjecture, we use a result of Leclerc [10] that coordinate rings of many Richardson varieties in the complete flag variety can be identified with cluster algebra. We also use a construction of Karpman [7] to construct a plabic graph associated with reduced expressions. We generalize the result to skew Schubert varieties using a generalized plabic graph.

Keywords : cluster algebra, Schubert variety.
Student number : 2021–25573

Contents

1 Introduction

Fomin and Zelevinsky introduced the *Cluster algebras* to understand the dual canonical bases for quantum group and total positivity [4]. They are connected to many fields of mathematics such as representation theory of quivers, discrete integrable systems, combinatorics, Riemann surfaces, and Teichmüller theory [12]. They satisfy some properties, including *Laurent phenomenon* [4] and *positivity theorem* [6, 11].

In [17], Scott used plabic graph (or an equivalent object, namely alternating strand diagram) to prove that the coordinate ring of the affine cone over the Grassmannian can be identified with cluster algebra. Some experts expected that there is a natural generalization of plabic graph that gives a cluster structure for (open) Schubert varieties (or more generally, open positroid varieties). In [14], this construction was stated explicitly as a conjecture.

Conjecture 1.1. *Let G be a reduced plabic graph corresponding to an (open) Schubert (or more generally, open positroid) variety. Then the target labeling of the faces of G (which we identify with a collection of Plücker coordinates) together with the dual quiver of G gives rise to a seed for a cluster structure on the coordinate ring of the open Schubert (or positroid) variety.*

Meanwhile, Leclerc [10] shows that when w has a factorization of the form $w = xv$ with $\ell(w) = \ell(x) + \ell(v)$, the coordinate ring of open Richardson variety $\mathcal{R}_{v,w}$ is cluster algebra. Since open Schubert varieties are isomorphic to open Richardson varieties with above property, Leclerc's result implies that their coordinate rings admit cluster algebra. Since Leclerc's cluster quiver is defined in terms of morphisms of modules of the preprojective algebra, his description is far from plabic graph.

In this survey paper, we review that the coordinate ring of a Schubert variety is cluster algebra by relating Leclerc's cluster structure to the plabic graph. We also generalize the result to construct cluster structures in skew Schubert varieties. We use a generalized plabic graph (with boundary vertices which are not cyclically labeled) for skew Schubert variety.

The paper is organized as follows. In Section 2, we give background on cluster algebra, Schubert variety, plabic graph, and reduced expression. In Section 3, we use a particular equivalence class of plabic graph called rectangles seed. We describe the rectangles seed and its dual quiver associated with Schubert variety, bridge graph, and Leclerc's cluster structure. In Section 4, we introduce Theorem 4.1 and Theorem 4.2. We first prove Theorem 4.2 and deduce Theorem 4.1 from Theorem 4.2. This paper is mainly referred to [18].

2 Preliminaries

2.1 Quiver and Cluster Structure

We introduce the definitions of quivers and cluster algebra. The definitions are followed from [3] and [18].

Definition 2.1. A *quiver* is a finite oriented graph, consisting of vertices and directed edges (called arrows). We allow multiple edges, but we disallow loops and oriented 2-cycles.

Some vertices in a quiver are called *frozen* and the remaining vertices are called *mutable*. We always assume that there are no arrows between pairs of frozen vertices. We often express frozen vertices in rectangular boxes.

Definition 2.2. Let k be a mutable vertex in a quiver Q. The *quiver mutation* μ_k transforms Q into a new quiver Q' via a sequence of three steps:

1. For each oriented two-arrow path $i \to k \to j$, add a new arrow $i \to j$.
2. Reverse the direction of all arrows incident to the vertex k.
3. Repeatedly remove oriented 2-cycles until unable to do so.

Example 1. Given a quiver Q, Figure 1 shows $Q' = \mu_k(Q)$ which is obtained from Q by mutating at the vertex k.

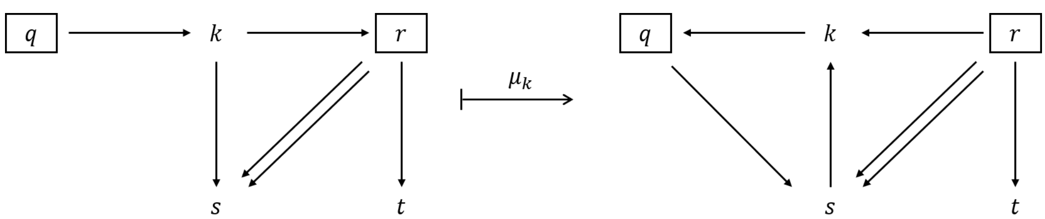


Figure 1: A quiver mutation μ_k, the vertices in square box are frozen.

Remark 2.3. Mutation is an involution, i.e. $\mu_k(\mu_k(Q)) = Q$.

Definition 2.4. Two quivers Q and Q' are called *mutation equivalent* if Q can be transformed into a quiver isomorphic to Q' by a sequence of mutations. The *mutation equivalence class* $[Q]$ of a quiver Q is the set of all quivers (up to isomorphism) which are mutation equivalent to Q.

Definition 2.5. Two quivers Q and Q' are said to have the same *type* if their mutable parts are mutation equivalent. Here the *mutable part* of the quiver refers to the mutable vertices together with all arrows which connect two mutable vertices.

Choose $m \geq n$ positive integers. Let $\mathcal{F}$ be an *ambient field* of rational functions in n independent variables over $\mathbb{C}(x_{n+1}, \ldots, x_m)$.

Definition 2.6. A *labeled seed* in $\mathcal{F}$ is a pair ($\tilde{\mathbf{x}}$,Q) where

- $\tilde{\mathbf{x}} = (x_1, \ldots, x_m)$ is an m-tuple of elements of $\mathcal{F}$ forming a free generating set.
- Q is a quiver on vertices $1, \ldots, n, n+1, \ldots, m$, whose vertices $1, \ldots, n$ are *mutable*, and whose vertices $n+1$,...,m are *frozen*.

We use the following terminology:

- $\tilde{\mathbf{x}}$ is the (labeled) *extended cluster* of the labeled seed ($\tilde{\mathbf{x}}$,Q).
- the n-tuple $\mathbf{x} = (x_1, \ldots, x_n)$ is the (labeled) *cluster* of this seed.
- the elements $x_1, \ldots, x_n$ are its *cluster variables.*
- the remaining elements $x_{n+1}, \ldots, x_m$ of $\tilde{\mathbf{x}}$ are the *frozen variables* (or *coefficient variables*).

Definition 2.7. Let $(\tilde{\mathbf{x}}, Q)$ be a labeled seed in $\mathcal{F}$. The *seed mutation* μ_k in direction k transforms $(\tilde{\mathbf{x}}, Q)$ into the new labeled seed $\mu_k(\tilde{\mathbf{x}}, Q) = (\tilde{\mathbf{x}}', Q')$ defined as follows:

- $Q' = \mu_k(Q)$.

- the extended cluster $\tilde{\mathbf{x}}' = (x'_1, \ldots, x'_m)$ is given by $x'_j = x_j$ for $j \neq k$, whereas $x'_k \in \mathcal{F}$ is determined by the *exchange relation*

$$x_k x'_k = \prod_{k \to r} x_r + \prod_{s \to k} x_s,$$

where the first product is over all arrows $k \to r$ in Q which start at k, and the second product is over all arrows $s \to k$ which end at k.

Remark 2.8. Seed mutation is an involution.

Definition 2.9. Let $\mathbb{T}_n$ denote the *n-regular tree* whose edges are labeled by the number $1, \ldots, n$, so that the n edges incident to each vertex receive different label (see Figure 2).

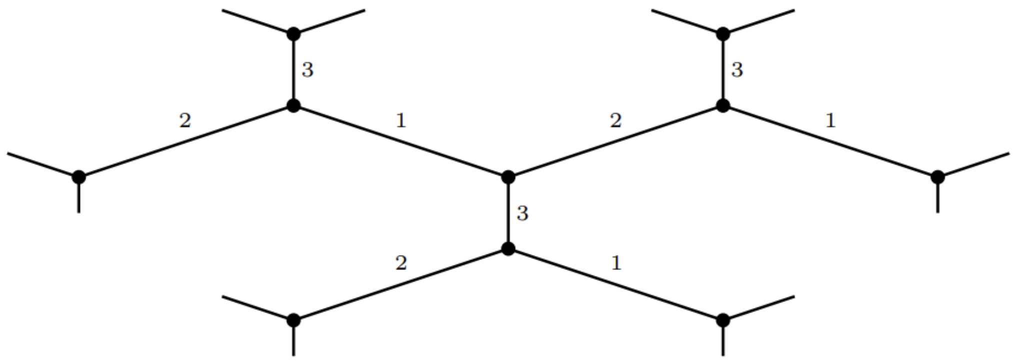


Figure 2: The 3-regular tree $\mathbb{T}_3$.

We write $t \overset{k}{\text{---}} t'$ to indicate that vertices $t, t' \in \mathbb{T}_n$ are joined by an edge labeled by k.

Definition 2.10. A *cluster pattern* is defined by assigning a labeled seed $\Sigma_t = (\tilde{\mathbf{x}}_t, Q_t)$ to every vertex $t \in \mathbb{T}_n$, so that the seeds assigned to the endpoints of any edge $t \overset{k}{\text{---}} t'$ are obtained from each other by the seed mutation in direction k. The components of $\tilde{\mathbf{x}}_t$ are written as $\tilde{\mathbf{x}}_t = (x_{1;t}, \ldots, x_{n;t})$.

Cluster pattern is uniquely determined by an arbitrary seed.

Definition 2.11. Let Σ_t be a cluster pattern, and let

$$\mathcal{X} = \bigcup_{t \in \mathbb{T}_n} \tilde{\mathbf{x}}_t = \{x_{i;t} : t \in \mathbb{T}_n, 1 \leq i \leq n\}$$

the union of clusters of all the seed in the pattern. The elements $x_{i;t} \in \mathcal{X}$ are called *cluster variables*. The *cluster algebra* $\mathcal{A}$ associated with a given pattern is $\mathbb{C}[x_{n+1}^{\pm 1}, \ldots, x_m^{\pm 1}]$-subalgebra of the ambient field $\mathcal{F}$ generated by all cluster variables: $\mathcal{A} = \mathbb{C}[c^{\pm 1}][\mathcal{X}]$. We denote $\mathcal{A} = \mathcal{A}(\tilde{\mathbf{x}}, Q)$, where $(\tilde{\mathbf{x}}, Q)$ is any seed in the underlying cluster pattern. In this generality, $\mathcal{A}$ is called a *cluster algebra from a quiver*, or a *skew-symmetric cluster algebra of geometric type*. We say that $\mathcal{A}$ has rank n because each cluster contains n cluster variables.

Remark 2.12. A common alternative definition is to take $\mathcal{A} = \mathbb{C}[c][\mathcal{X}]$ instead of $\mathcal{A} = \mathbb{C}[c^{\pm 1}][\mathcal{X}]$. With this definition, Scott proved that the coordinate ring of Grassmannian is cluster algebra [17].

2.2 Notation on Schubert Variety

We will give notation about Schubert variety in the Grassmannians [9].

Let GL_n be the general linear group, B the Borel subgroup of lower triangular matrices, B^+ the opposite Borel subgroup of upper triangular matrices, and $W = S_n$ the Weyl group (in this case the symmetric group on n letters). It contains the longest element w_0 with $\ell(w_0) = \binom{n}{2}$. The *complete flag variety* Fl_n is the homogeneous space $B \setminus GL_n$. Precisely, element g of GL_n gives rise to a flag of subspace $\{ V_1 \subset V_2 \subset \cdots \subset V_n\}$, where the span of the top i rows of g is V_i and $V_n = \mathbb{C}^n$. Since the left action of B preserve the flag, we can identify Fl_n with the set of *flags* $\{ V_1 \subset V_2 \subset \cdots \subset V_n\}$ where dim $V_i = i$.

Let $\pi : GL_n \to Fl_n$ denote the natural projection $\pi(g) := Bg$. Let $\dot{w}$ be the matrix representative for w in GL_n. The Bruhat decomposition

$$GL_n = \bigsqcup_{w\in W} B\dot{w}B$$

projects to the Schubert decomposition

$$Fl_n = \bigsqcup_{w\in W} C_w$$

where $C_w = \pi(B\dot{w}B)$ is the *Schubert cell* associated to w, isomorphic to $\mathbb{C}^{\ell(w)}$. We also have the Birkhoff decomposition

$$GL_n = \bigsqcup_{w\in W} B\dot{w}B^+$$

which projects to the opposite Schubert decomposition

$$Fl_n = \bigsqcup_{w\in W} C^w$$

where $C^w = \pi(B\dot{w}B^+)$ is the *opposite Schubert cell* associated to w, isomorphic to $\mathbb{C}^{\ell(w_0)-\ell(w)}$.

The intersection $\mathcal{R}_{v,w} := C^v \cap C_w$ is called an *open Richardson variety* because its closure is a *Richardson variety.* $\mathcal{R}_{v,w}$ is nonempty only if $v \leq w$ in the Bruhat order of W. So we also have

$$Fl_n = \bigsqcup_{v\leq w} \mathcal{R}_{v,w}.$$

Fix $1 < k < n$. The parabolic subgroup $W_K = \langle s_1, \dots, s_{k-1}\rangle \times \langle s_{k+1}, s_{k+2} \dots, s_{n-1}\rangle < W$ gives rise to parabolic subgroup P_K in GL_n, i.e. $P_K = \bigsqcup_{w \in W_K} B\dot{w}B$. W_K contains the longest element w_K with $\ell(w_K) = \binom{k}{2} + \binom{n-k}{2}$.

The *Grassmannian* $Gr_{k,n}$ is the homogeneous space $P_K \setminus GL_n$. We can think $Gr_{k,n}$ as the set of all k-planes in n-dimensional vector space $\mathbb{C}^n$. An element of $Gr_{k,n}$ can be viewed as a full rank $k \times n$ matrix modulo left multiplication by invertible $k \times k$ matrices.

For integers a, b, let $[a, b]$ denote $\{a, a+1, \dots, b-1, b\}$ if $a \leq b$ and the empty set otherwise. Let $[n] := [1, n]$ and $\binom{[n]}{k}$ be the set of all k-element subsets of $[n]$.

Given $V \in Gr_{k,n}$ represented by $k \times n$ matrix A, let $\Delta_I(V)$ be the $k \times k$ minor of A determined by the column set $I \in \binom{[n]}{k}$. The $\Delta_I(V)$ are called *Plücker coordinates* of V. The Plücker coordinates give an embedding $p : Gr_{k,n} \to \mathbb{P}(\bigwedge^r \mathbb{C}^n)$. Choose basis $u_1, \dots, u_k$ in V, we define $p(V) := [u_1 \wedge \cdots \wedge u_k]$. We also have Plücker relation associated to Plücker embedding. Let $I = \{i_1, \dots, i_{k-1}\} \in \binom{[n]}{k-1}$ and $J = \{j_1, \dots, j_{k+1}\} \in \binom{[n]}{k+1}$. The following quadratic relations for $Gr_{k,n}$ are called *Plücker relations*

$$\sum_{\ell=1}^{k+1} (-1)^\ell \Delta_{i_1,\dots,i_{k-1},j_\ell} \Delta_{j_1,\dots,\hat{j_\ell},\dots,j_{k+1}} = 0.$$

Let $W^K = W^K_{\min}$ and $W^K_{\max}$ denote the set of minimal- and maximal-length coset representatives for $W_K \setminus W$. Also let ${}^K W$ (or ${}^K_{\min}W$) denote the set of minimal-length coset representatives for W/W_K. Let $\sigma \in S_n$ be *Grassmannian permutation* if it has at most one descent, and when present, the descent must be in position k, i.e. $\sigma(k) > \sigma(k+1)$.

We have the $\pi_k : Fl_n \to Gr_{k,n}$ where $\pi_k(Bg) = V_k$. $\mathcal{R}_{v,w}$ and $\pi_k(\mathcal{R}_{v,w})$ are isomorphic when $v \in W^K_{\max}$ (or when $w \in W^K_{\min}$). We obtain a stratification

$$Gr_{k,n} = \bigsqcup \pi_k(\mathcal{R}_{v,w})$$

where (v, w) range over all $v \in W^K_{max}, w \in W$, such that $v \leq w$. The strata $\pi_k(\mathcal{R}_{v,w})$ are sometimes called *open positroid varieties* and their closures are called *positroid varieties*. Positroid varieties include Schubert and opposite Schubert varieties in the Grassmannians [8].

Definition 2.13. Let I denote a k-element subset of $[n]$. The *Schubert cell* Ω_I is defined to be Ω_I=$\{V \in Gr_{k,n}$: the lexicographically minimal nonvanishing Plücker coordinate of V is $\Delta_I(V)\}$. The *Schubert variety* X_I is defined to be the closure $\overline{\Omega_I}$ of Ω_I. Meanwhile, the *opposite Schubert cell* Ω^I is defined to be $\Omega^I = \{V \in Gr_{k,n}$: the lexicographically maximal nonvanishing Plücker coordinate of V is $\Delta_I(V)\}$. The *opposite Schubert variety* X^I is defined to be the closure $\overline{\Omega^I}$ of Ω^I.

There is bijection between $v \in W^K_{min}$ (resp. $w \in W^K_{max}$) and k-element subsets of $[n]$, which we denote by $I(v)$ (resp. $I(w)$). When $w \in W^K_{\min}$, $\overline{\pi_k(\mathcal{R}_{e,w})}$ is isomorphic to the *opposite Schubert variety* $X^{I(w)}$, which has dimension $\ell(w)$. Therefore we refer to $\pi_k(\mathcal{R}_{e,w})$ as an *open opposite Schubert variety.* Similarly, when $v \in W^K_{\max}$, $\overline{\pi_k(\mathcal{R}_{v,w_0})}$ is isomorphic to the *Schubert variety* $X_{I(v)}$, which has dimension $\ell(w_0) - \ell(w)$. Therefore we refer to $\pi_k(\mathcal{R}_{v,w_0})$ as an *open Schubert variety.* More generally, if $v \in W^K_{\max}$ and $w \in W$ has a factorization of the form $w = xv$ which is *length-additive*, i.e. where $\ell(w) = \ell(x) + \ell(v)$, then we refer to $\overline{\pi_k(\mathcal{R}_{v,w})}$ (resp. $\pi_k(\mathcal{R}_{v,w})$) as a *skew Schubert variety* (resp. *open skew Schubert variety*).

Let λ be a Young diagram contained in a $k \times (n-k)$ rectangle. We can identify λ with the lattice path $L^{\swarrow}_\lambda$ in the rectangle taking steps west and south from the northeast corner of the rectangle to the southeast corner ("Going southwest"). If we label the steps of the lattice path from 1 to n, then the labels of the south steps give a k-element subset of $[n]$ that we denote by $V^{\swarrow}(\lambda)$. Conversely we can identify each k-element subset I of $[n]$ with a Young diagram in a $k \times (n-k)$ rectangle, which we denote by $\lambda^{\swarrow}(I)$. Hence we can index Schubert and opposite Schubert cells and varieties by Young diagrams, denoting them Ω_λ, Ω^λ, X_λ, and X^λ, respectively. The open Schubert and opposite Schubert varieties are denoted by X°_λ, and $(X^\lambda)^\circ$. The dimension of Ω_λ, X_λ, and X°_λ is $|\lambda|$, the number of boxes of λ, while the codimension of Ω^λ, X^λ, and $(X^\lambda)^\circ$ is $|\lambda|$.

We also associate with a Young diagram λ the *Grassmannian permutation* $\pi^{\swarrow}_\lambda$ of type $(n-k, n)$: in list notation, this permutation is obtained by first reading the labels of the horizontal steps of $L^{\swarrow}_\lambda$, and then read the labels of the vertical steps of $L^{\swarrow}_\lambda$. Any fixed points in position $1, \ldots, n-k$ are “black” and any fixed points in positions $n-k,\ldots,n$ are “white”. Note that $\ell(\pi_\lambda^{\swarrow}) = |\lambda|$.

Remark 2.14. Similarly we can define $L_\lambda^{\nearrow}$ (“Going northeast”) by taking steps east and north from the southeast corner of the rectangle to northeast corner. If we label the path with 1 to n, the labels of north steps give the k-element subset $V^{\nearrow}(\lambda)$. Similarly we can define $\lambda^{\nearrow}(I)$.

2.3 Notion on plabic graphs

We review Postnikov's notion of *plabic graph* [16] which we will use to define cluster structure in Schubert varieties.

Definition 2.15. A *plabic (or planar bicolored) graph* is an undirected graph G drawn inside a disk (considered modulo homotopy) with n *boundary vertices* on the boundary of the disk, labeled $1, \ldots, n$ in clockwise order, as well as some colored *internal vertices*. These internal vertices are strictly inside the disk and are colored in black and white. An internal vertex of degree one adjacent to a boundary vertex is a *lollipop*. We will always assume that no vertices of the same color are adjacent and that each boundary vertex i is adjacent to a single internal vertex.

Figure 3 is an example of a plabic graph.

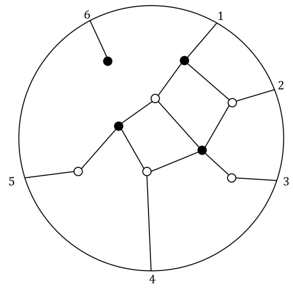


Figure 3: A plabic graph.

Definition 2.16. A *generalized plabic graph* is a plabic graph with boundary vertices are labeled by $1, \ldots, n$ in some order, not necessarily clockwise.

Though the all of following definitions are for plabic graph, we can equally make definitions for generalized plabic graph. Note that we will always assume that a plabic graph G has no isolated components (i.e. every connected components contains at least one boundary vertex). We will also assume that G is *leafless*, i.e. if G has an internal vertex of degree 1, then that vertex must be adjacent to a boundary vertex.

Definition 2.17. There is a natural set of local transformations (moves and reduction) of plabic graphs.

(M1) SQUARE MOVE (Urban renewal). If a plabic graph has a square formed by for trivalent vertices whose colors alternate, then we can switch the colors of these four vertices (and add some degree 2 vertices to preserve the bipartiteness of graph).

(M2) CONTRACTING/EXPANDING A VERTEX. Any degree 2 internal vertex not adjacent to the boundary can be deleted, and the two adjacent vertices merged. This operation can also be reverse.

(M3) MIDDLE VERTEX INSERTION/REMOVAL. We can remove or add degree 2 vertices at will, subject to the condition that the graph remains bipartite.

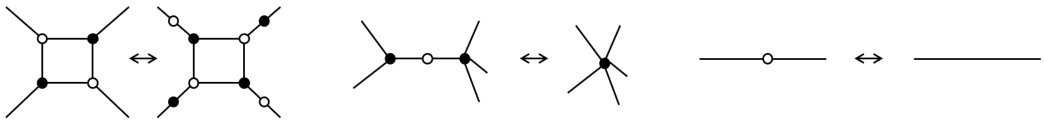


Figure 4: Local transformations of plabic graphs.

(R1) PARALLEL EDGE REDUCTION. If a plabic graph contains two trivalent vertices of different colors connected by a pair of parallel edges, the we can remove these vertices and edges, and glue the remaining pair of edges together.

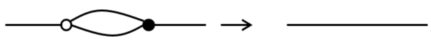


Figure 5: Parallel edge reduction.

Figure 4 and Figure 5 are description of local transformations.

Definition 2.18. Two plabic graphs are called *move-equivalent* if they can be obtained from each other by moves (M1)-(M3). The *move-equivalence class* of a given plabic graph G is the set of all plabic graph which are move-equivalent to G. A leafless plabic graph without isolated components is called *reduced* if there is no graph in its move-equivalence class to which we can apply (R1).

Definition 2.19. A *decorated permutation* $\pi^{:}$ is a permutation $\pi \in S_n$ together with a coloring $i : \pi(i) = i \to \{\text{black, white}\}$.

Definition 2.20. Given a reduced plabic graph G, a *trip* T is a directed path which starts at some boundary vertex i, and follows the "rules of the road": it turns (maximally) right at a black vertex, and (maximally) left at a white vertex. Note that T will also end at a boundary vertex j; we then refer to this trip as $T_{i\to j}$. Setting $\pi(i) = j$ for each such trip, we associate a (decorated) *trip permutation* $\pi_G = (\pi(1), \dots, \pi(n))$ to each reduced plabic graph G, where a fixed point $\pi(i) = i$ is colored white (black) if there is a white (black) lollipop at boundary vertex i. We say that G has *type* π_G.

Example 2. The trip permutation associated to the reduced plabic graph in Figure 6 is (3,4,5,1,2).

Remark 2.21. The trip permutation of a plabic is preserved by the (M1)-(M3), but not by (R1). For reduced plabic graph, any two graphs with the same trip permutation are move-equivalent.

We use the notion of trips to label each face of G by a Plücker coordinate. Note that every trip will partition the faces of a plabic graph into two parts: those on the left of the trip, and those of the right of the trip.

Definition 2.22. Let G be a reduced plabic graph with b boundary vertices. For each one-way trip $T_{i\to j}$ with $i \neq j$, we place label i (resp. j) in every face which is to the left of $T_{i\to j}$. If $i = j$ (that is, i is adjacent to a lollipop), we place the label i in all faces if the lollipop is white and in no faces if the lollipop is black. We then obtain a labeling $\mathcal{F}_{source}(G)$ (resp. $\mathcal{F}_{target}(G)$) of faces of G by subset of $[n]$ which we call the *source* (resp. *target*) *labeling* of G. We identify each k-element subset of $[n]$ with the corresponding Plücker coordinate.

Remark 2.23. All faces of G will be labeled by subset of the same size [16].

We will associate a quiver to each plabic graph and relate quiver mutation to moves on plabic graphs.

Definition 2.24. Let G be a reduced plabic graph. We associate a quiver $Q(G)$ as follows. The vertices of $Q(G)$ are labeled by the faces of G. We say that a vertex of $Q(G)$ is *frozen* if the corresponding face is incident to the

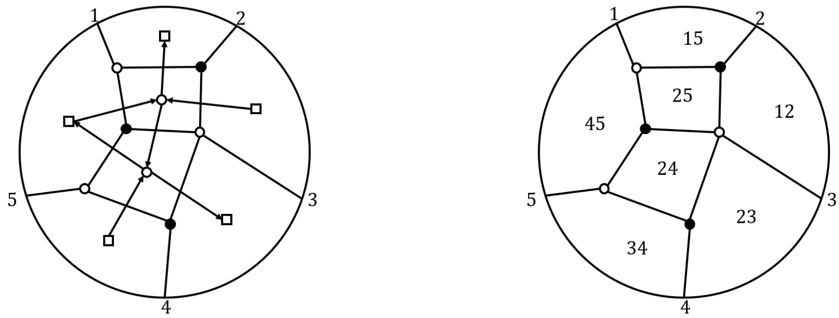


Figure 6: A plabic graph G together with $Q(G)$ and its face labeling $\mathcal{F}_{source}(G)$. $\pi_G = (3, 4, 5, 1, 2)$.

boundary of the disk and is *mutable* otherwise. For each edge e in G which separates two faces, at least one of which is mutable, we introduce an arrow connecting the faces; this arrow is oriented so that it "sees the white endpoint of e to the left and the black endpoint to the right" as it crosses over e. We then remove oriented 2-cycles from the resulting quiver, one by one, to get $Q(G)$.

Example 3. The left-hand side of Figure 6 shows $Q(G)$ and the right-hand side of Figure 6 shows $\mathcal{F}_{source}(G)$.

Definition 2.25. Given a reduced plabic graph G, we let Σ_G^{target} (resp. Σ_G^{source}) be the labeled seed consisting of the quiver $Q(G)$ with vertices labeled by the Plücker coordinates $\mathcal{F}_{target}(G)$ (resp. $\mathcal{F}_{source}(G)$). Given a plabic graph G on n vertices and a permutation $v \in S_n$, we will sometimes use relabeled plabic graph $v^{-1}(G)$ (where the boundary vertices have been modified by applying v^{-1} to them). We will refer to the corresponding seed with the induced target labeling by e.g. $\Sigma_{v^{-1}(G)}^{target}$.

In [17], we can obtain the following lemma.

Lemma 2.26. *If G and G' are related via a square move at a face, then Σ_G^{target} and $\Sigma_{G'}^{target}$ are related via mutation at the corresponding vertex. Similarly for Σ_G^{source} and $\Sigma_{G'}^{source}$.*

We will refer to "mutating" at a nonboundary face of G, meaning that we mutate at the corresponding vertex of quiver $Q(G)$. Note that in general

1	4	6
2	5	7
3		

s_3	s_4	s_5
s_2	s_3	s_4
s_1		

Figure 7: The columnar reading order and the filling with simple transpositions.

the quiver $Q(G)$ admits mutations at vertices which do not correspond to moves of plabic graphs.

Remark 2.27. The trip permutation of $\pi_k(\mathcal{F}_{v,w})$ is $\pi_{v,w} := v^{-1}w$ with all white fixed points lying in $v^{-1}([k])$. One can recover the pair (v, w) from $\pi_{v,w}$ since $v \in W^K_{max}$.

We need some facts on reduced expressions for permutations in ${}^K W$ and W^K.

Lemma 2.28. *[19] Let $x \in^K W$ and let $\lambda := \lambda^{\nearrow}(x([k]))$. Choose a "reading order" for the boxes of λ such that each box is read before the box immediately below it and the box immediately to is right (that is, choose a standard tableaux of shape λ). Fill each box with a simple transposition; the box in row r and column c is filled with s_{k-c+r}. Then reading the fillings of the boxes according to the reading order gives a reduced expression for x (written from right to left).*

Since the elements of W^K are just the inverse of the elements of ${}^K W$, one can also obtain reduced expression for $y \in W^K$ using Lemma 2.28 with $\lambda^{\nearrow}(y^{-1}([k]))$. The only difference is the reduced expression for y is written down from left to right.

Remark 2.29. For simplicity, we will use the *columnar* reading order, which reads the columns of λ from top to bottom, moving left to right. We will call the resulting reduced expressions *columnar expressions.*

Example 4. Figure 7 is an example for $x = (2, 5, 6, 1, 3, 4) \in {}^K W$, and $\lambda^{\nearrow}(x([k])) = (3, 3, 1)$. The left side is the columnar reading order for the boxes $\lambda^{\nearrow}(x([k]))$ and the right side is the filling of $\lambda^{\nearrow}(x([k]))$ with simple transpositions. This reading order produces the reduced expression $\mathbf{x} = s_5s_4s_3s_4s_1s_2s_3$ for $x \in {}^KW$, and the reduced expression $s_3s_2s_1s_4s_3s_5s_4$ for $x^{-1} \in W^K$.

We will be particularly concerned with pairs (v, w) where $v \in W^K_{max}$ and w has a *length-additive factorization* $w = xv$, i.e. $\ell(w) = \ell(x) + \ell(v)$. We will often use reduced expressions for such w that reflect their length-additive factorizations.

Definition 2.30. Let $v \leq w$, with $v \in W^K_{max}$ and $w = xv$ length-additive. Let $v = w_K v'$ be length-additive, where v' is necessarily in W^K_{min}. Then a *standard reduced expression* for w is a reduced expression $\mathbf{w} = \mathbf{x}\mathbf{w}_K\mathbf{v'}$, where $\mathbf{x}$ and $\mathbf{v'}$ are the columnar expressions for x and v', respectively, and $\mathbf{w}_K$ is an arbitrary reduced expression for w_K.

3 The rectangles seed for cluster structure

3.1 The rectangles seed associated to a skew Schubert variety

We will introduce how to associate a pair of permutations with a quiver whose vertices are labeled by Plücker coordinates.

Definition 3.1. (*The rectangles seed* $\Sigma_{v,w}$). Let $v \leq w$ where $v \in W^K_{max}$ and $w = xv$ is a length-additive factorization. Let $\lambda := \lambda^{\nearrow}(x([k]))$. If b is a box of λ, let $\mathrm{Rect}(b)$ be the largest rectangle contained in λ whose lower right corner is b.

We obtain quiver $Q_{v,w}$ as follows: place one vertex in each box of λ. A vertex is mutable if it lies in a box b of the Young diagram and the box immediately southeast of b is also in λ. We add arrows between vertices in adjacent boxes, with all arrows pointing either up or to the left. Finally, in every 2×2 rectangle in λ, we add an arrow from the upper left box to the lower right box. Equivalently, we add an arrow from the vertex in box a to the vertex in box b if

- $\mathrm{Rect}(b)$ is obtained from $\mathrm{Rect}(a)$ by removing a row or column.
- $\mathrm{Rect}(b)$ is obtained from $\mathrm{Rect}(a)$ by adding a hook shape.

We then remove all arrows between two frozen vertices.

To obtain the *rectangle seed* $\Sigma_{v,w}$, we label each vertex of $Q_{v,w}$ with a Plücker coordinate. For b a box of λ, let $J(b) := V^{\nearrow}(\mathrm{Rect}(b))$. The label of the vertex in b is $\Delta_{v^{-1}(J(b))}$. This labeled quiver $\Sigma_{v,w}$ gives a labeled seed, where the Plücker coordinates labeling the vertices give the extended cluster.

Definition 3.2. Let λ be a partition and let b be a box of λ. We say the $\mathrm{Rect}(b)$ is *frozen* for λ or λ-*frozen* if b touches the south or east boundary of λ (either along an edge or at the southeast corner).

Note that the λ-frozen rectangles correspond to the frozen vertices of $\Sigma_{v,w}$.

Figure 8: an example of $\Sigma_{v,w}$ for $k = 3, n = 7, v = w_K$ and $x = wv^{-1} = (3, 5, 7, 1, 2, 4, 6)$.

Example 5. On the left side of Figure 8, λ-frozen rectangles are $\ydiagram{4}$, $\ydiagram{3}$, $\ydiagram{3,3}$, $\ydiagram{2,2}$, $\ydiagram{2,2,2}$, $\ydiagram{1,1}$. On the right side of Figure 8, vertices are replaced by the 3-element subsets $v^{-1}(J(b))$ of $[7]$, which should be interpreted as Plücker coordinates.

3.2 The rectangles seed associated to a bridge graph

We will discuss the generalized plabic graph whose dual quiver (with the target labeling) coincides with $\Sigma_{v,w}$. For this, we introduce a special kind of plabic graph -*bridge graph*- from a pair of permutations [7], and describe how to use this construction to produce the rectangle seed. We first review the notion of *(bounded) affine permutations*. An *affine permutation* of order n is a bijection $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(i+n) = f(i) + n$ for all $i \in \mathbb{Z}$.

Definition 3.3. (The bounded affine permutation associated to a decorated permutation). If σ is a decorated permutation of $[n]$, we define the bounded affine permutation $\tilde{\sigma}$ on $[n]$ as

$$\tilde{\sigma}(i) := \begin{cases} \sigma(i) & \text{if } \sigma(i) > i \text{ or } i \text{ is a black fixed point} \\ \sigma(i) + n & \text{if } \sigma(i) < i \text{ or } i \text{ is a white fixed point} \end{cases}$$

and extend periodically to $\mathbb{Z}$.

Definition 3.4. An $(a\ b)$-*bridge* is a collection of two vertices and three edges inserted at the boundary vertices a and b.

Definition 3.5. Let G be a plabic graph with (bounded affine) trip permutation $\tilde{\sigma}_G$. For a pair of boundary vertices $a < b$, we say that the $(a\ b)$-bridge is called *valid* if $\tilde{\sigma}_G(a) > \tilde{\sigma}_G(b)$, all boundary vertices c between a and b are lollipops, and if a (resp. b) is a lollipop, it is white (resp. black).

To add a bridge to G, choose boundary vertices a,b such that the $(a\ b)$-bridge is valid. Place a white (resp. black) vertex in the middle of the edge adjacent to a (resp. b) and put an edge between two vertices; if a (resp. b) is a lollipop, we use the boundary leaf as the white (resp. black) vertex of the bridge. We then add degree two vertices as necessary to make the resulting graph bipartite.

Definition 3.6. A plabic graph obtained by successively adding valid bridges to a lollipop graph is called *bridge graph*.

Lemma 3.7. *[7] Suppose G is a reduced plabic graph with (bounded affine) trip permutation $\tilde{\sigma}_G$. Let $1 \leq a < b \leq n$ be vertices such that the $(a\ b)$-bridge is valid and let G' be the plabic graph obtained by adding an $(a\ b)$-bridge to G. Then G' is reduced and has trip permutation $\tilde{\sigma}_G \circ (a\ b)$.*

If G' is obtained from G by adding a valid bridge, all faces of G' correspond to faces in G, except for the face bounded by the bridge.

Lemma 3.8. *Suppose G is a reduced plabic graph, $1 \leq a < b \leq n$ vertices such that the $(a\ b)$-bridge is valid, and G' the plabic graph obtained from adding an $(a\ b)$-bridge to G. Then, using the target labeling, the labels of faces in G coincide with the labels of corresponding faces in G'.*

Definition 3.9. Let σ be a decorated permutation of $[n]$. The *Grassmann necklace* of σ is a sequence $\mathcal{J} = (J_1, \ldots, J_n)$ of subsets of $[n]$ where $J_1 := \{i \in [n] : \sigma^{-1}(i) > i \text{ or } i \text{ is a white fixed point}\}$ and

$$J_{i+1} := \begin{cases} (J_i \setminus \{i\}) \cup \{\sigma(i)\} & \text{if } i \in J_i \\ J_i & \textit{else.} \end{cases}$$

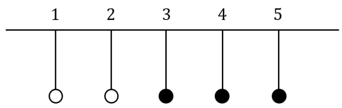
From [15], we obtain if $\sigma_{G'}$ is the trip permutation of G', the boundary face of G' are labeled with the Grassmann necklace of $\sigma_{G'}$. So the label of the face bounded by the $(a\ b)$-bridge is $(a+1)^{st}$ entry of Grassmann necklace of $\sigma_{G'}$.

We will introduce an algorithm for producing a bridge graph with trip permutation vw^{-1} from a pair $(v, \mathbf{w})$, where $v^{-1} \in W^K_{max}$ and $\mathbf{w}$ is a reduced expression for some permutation $w \geq v$.

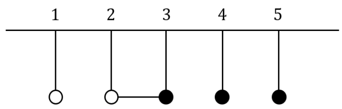
Definition 3.10. Let $w \in W$ with a length-additive factorization $w = xw_K$, where $x \in {}^K W$. Let $\mathbf{x} = s_{i_r} \ldots s_{i_1}$ be the columnar expression for x and let $\mathbf{w}$ be a standard reduced expression for w. We define $B_{w_K, \mathbf{w}}$ to be the bridge graph obtained from the lollipop graph with white lollipops $[k]$ and black $[k+1, n]$ with the bridge sequence $s_{i_1}, s_{i_2}, \ldots, s_{i_r}$.

$B_{w_K, \mathbf{w}}$ is a reduced plabic graph [7]. By Lemma 3.7 $B_{w_K, \mathbf{w}}$ has a (decorated) trip permutation x^{-1} with fixed points in $[k]$ colored white.

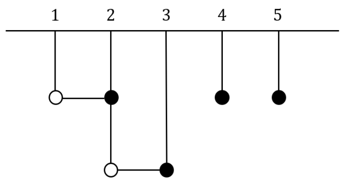
Example 6. Let $k = 2, n = 5, x = (3, 5, 1, 2, 4)$ and $w = xw_K$. The partition $\lambda^{\nearrow}(x([2]))$ corresponding to $x([2]) = \{x(1), x(2)\}$ is $(3, 5)$, and the columnar expression for x is $\mathbf{x} = s_4 s_2 s_3 s_1 s_2$. So the bridge sequence for $B_{w_K,\mathbf{w}}$ is $(2\ 3), (1\ 2), (3\ 4), (2\ 3), (4\ 5)$. To build $B_{w_K,\mathbf{w}}$, we start with the lollipop graph



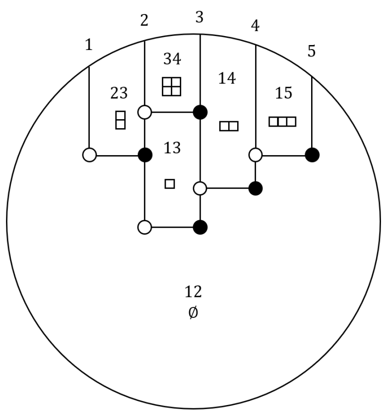
then add the bridge $(2\ 3)$,



the bridge $(1\ 2)$,



and the bridges $(3\ 4), (2\ 3), (4\ 5)$ to obtain the following graph.



Note that the (target) face labels of $B_{w_K,\mathbf{w}}$ correspond to rectangle that fit inside of $\lambda^{\nearrow}(x([2]))$.

The structure of $B_{w_K,\mathbf{w}}$ follows from the structure of its Grassmann necklace.

Lemma 3.11. *Let $x \in {}^K W$.*

1. *The fixed points of x are $[p] \cup [q, n]$ for some $0 \leq p \leq k < q \leq n+1$.*
2. *For $i \in [k]$, $x(i) \geq i$.*

Proof. Note that $x(1) < x(2) < \cdots < x(k)$ and $x(k+1) < \cdots < x(n)$. 2 is clear by increasing condition. For 1, suppose $x(j) = j$ for some $j \in [k]$. Since for $i < j$, $x(i) < x(j)$, we must have that $x([j]) = [j]$. The increasing condition implies that $x(i) = i$ for $i < j$. Similarly if $x(j) = j$ for some $j \in [k+1, n]$, then $x(\ell) = \ell$ for all $\ell > j$. $\square$

Proposition 3.12. *Let $y \in W^K_{min}$ with fixed points $[p] \cup [q, n]$, and let $\lambda := \lambda^{\nearrow}(y^{-1}([k]))$. We color the fixed points of y in $[k]$ white and all others black. Let $\mathcal{J} = (J_1, \ldots, J_n)$ be the Grassmann necklace of y. Then $\lambda^{\nearrow}(J_i) = \emptyset$ for $i \in [p+1] \cup [q, n]$. For other i, $\lambda^{\nearrow}(J_i)$ is a rectangle which is frozen for λ, and $\lambda^{\nearrow}(J_{i+1})$ can be obtained from $\lambda^{\nearrow}(J_i)$ by adding a column to $\lambda^{\nearrow}(J_i)$ if the resulting rectangle fits inside of λ (that is, if $y(i) > k$) or removing a row from $\lambda^{\nearrow}(J_i)$ if it does not (that is, if $y(i) \leq k$). In particular, every λ-frozen rectangle occurs as one of the $\lambda^{\nearrow}(J_i)$.*

Proof. We will proceed with induction of the length of y. If $y = e$, the white fixed points of y are $[k]$, so $J_i = [k]$ for all i, corresponding to the empty set.

Now consider $y \neq e$. By Lemma 3.11, if $i \in [k]$ is not fixed point of y, then $y^{-1}(i) > i$. This together with our choice of decoration implies $y(i) < i$ for all $i \in [k]$.

Suppose the columnar expression for y ends in s_j. Then $z := ys_j$ is an element of W^k corresponding to the partition $\lambda' = \lambda^{\nearrow}(z^{-1}([k]))$, which is λ with the bottom box of the rightmost column removed. Again, we color the fixed points of z in $[k]$ white and the fixed points in $[k+1, n]$ black, and let $\mathcal{I} = (I_1, \ldots, I_n)$ be the Grassmann necklace of z.

Note that $J_r = I_r$ for $r \leq j$ since $y(i) < i, z(i) < i$ for $i \in [k]$ and $y(r) = z(r)$ for $r \neq j, j+1$. Since $\ell(y) > \ell(z), y(j) > y(j+1)$. Note that $y(j) > k \geq y(j+1)$. We can conclude neither j nor $j+1$ are fixed by y by Lemma 3.11. So $J_{j+1} = (I_j \backslash \{j\}) \cup \{y(j)\}$ and $J_{j+1} = (J_{j+1} \backslash \{j+1\} \cup \{y(j+1)\})$.

Note that $z(j+1) > k \geq z(j)$. By induction, $\lambda^{\nearrow}(I_j)$ is a rectangle, so

$I_j = [a] \cup [b, c]$ for $0 \le a \le b, c \le n$. The case is divided by whether j or $j+1$ is fixed by z. If at least one of j and $j+1$ are fixed, it is straightforward, so we prove neither j nor $j+1$ are fixed by z.

Suppose neither j nor $j+1$ are fixed by z, so λ' is obtained from λ by removing a box that is not in the left column or top row. Suppose $I_j = [a] \cup [b, c]$. Since $z(j) \le k$, $\lambda^{\nearrow}(I_{j+1})$ is obtained from $\lambda^{\nearrow}(I_j)$ by removing a row, and we have $I_{j+1} = [a+1] \cup [b+1, c]$. So $j = b$ and $z(j) = a+1$. $\lambda^{\nearrow}(I_{j+1})$ is obtained from $\lambda^{\nearrow}(I_{j+1})$ by adding a column, so $I_{j+2} = [a+1] \cup [b+2.c+1]$. Hence $z(j+1) = c+1$ and $J_{j+1} = [a] \cup [b+1, c+1]$, which means that $\lambda^{\nearrow}(J_{j+1})$ is rectangles obtained from $\lambda^{\nearrow}(J_j)$ by adding a column. This rectangle fits inside of λ because of where we added a box and is also λ-frozen since its lower right corner touches the southeastern boundary of λ. $J_{j+2} = I_{j+2}$ and so $\lambda^{\nearrow}(J_{j+2})$ is obtained from $\lambda^{\nearrow}(J_{j+1})$ by removing a row. Since $I_r = J_r$ for $r \ne j+1$, and all of the rectangles $\lambda^{\nearrow}(I_r)$ are λ-frozen for $r \ne j+1$, the proof is done. $\square$

As a corollary, we obtain the structure of the face labels of the plabic graphs.

Corollary 3.13. *Let $w \in W$ with a length-additive factorization $w = xw_K$, where $x \in {}^K W$. Let $\mathbf{x} = s_{i_r} \dots s_{i_1}$ be the columnar expression for x and $\mathbf{w}$ be a standard expression for w. Let $\lambda := \lambda^{\nearrow}(x([k]))$. Then the set of face labels of $B_{w_K,\mathbf{w}}$ with respect to the target labeling is $\{V^{\nearrow}(\mathrm{Rect}(b)) : b \text{ a box of } \lambda\} \cup \{V^{\nearrow}(\emptyset)\}$. The boundary face labels correspond to the λ-frozen rectangles and the empty set.*

Proof. The bridge sequence of $B_{w_K,\mathbf{w}}$ is $s_{i_r}, \dots, s_{i_1}$ in the columnar expression of x^{-1}. After placing the j^{th} bridge, we obtain a plabic graph with trip permutation $s_{i_1} \dots s_{i_j}$ with white fixed points in $[k]$. Since $s_{i_1} \dots s_{i_j} \in W^K_{min}$, its Grassmann necklace consists of rectangles that are frozen for the partition corresponding $s_{i_1} \dots s_{i_j}$. The face labels of the boundary faces are the Grassmann necklace of the trip permutation, with I_j labeling the face immediately to the left of j. When we add $(j+1)^{th}$ bridge, we introduce a new boundary face and the labels of all other faces stay the same. An old boundary face may be pushed off the boundary by the new face. This occurs when its label is not frozen for the new partition. Further every rectangle that fits into λ is frozen for a partition corresponding to some prefix of $s_{i_1} \dots s_{i_j}$. $\square$

Now we describe the dual quiver of $B_{w_K,\mathbf{w}}$.

Proposition 3.14. *Let w, x, and $\mathbf{w}$ be as in Corollary 3.13, and let $\lambda := \lambda^{\nearrow}(x([k]))$. Let μ, ν be rectangles contained in λ which are not the empty partition. In the dual quiver of $B_{w_K,\mathbf{w}}$, there is an arrow from the face labeled $V^{\nearrow}(\mu)$ to the face labeled $V^{\nearrow}(\nu)$ if*

- *ν is obtained from μ by removing a row or column*
- *ν is obtained from μ by adding a hook shape*

unless both faces are on the boundary, in which case there is no arrow between them. There is also an arrow from the face labeled $V^{\nearrow}(\mu)$, where μ is a single box, to the face labeled $[k]$.

Proof. We will proceed with induction on the number of bridges. We color all boundary vertices of $B_{w_K,\mathbf{w}}$ adjacent to white (black) internal vertices black (white) and add arrows appropriately in the dual quiver. Let $\mathbf{x} = s_{i_r} \dots s_{i_1}$ be the columnar expression for x, so that $s_{i_1}, \dots, s_{i_r}$ is the bridge sequence for $B_{w_K,\mathbf{w}}$. Note that $s_{i_1} = s_k$.

If there is only one single bridge, then $B_{w_K,\mathbf{w}}$ has two faces, one face f labeled with $[k] = V^{\nearrow}(\emptyset)$ and the other face f' labeled with $V^{\nearrow}(\mu)$, where μ is a single box. From the coloring of vertices in a bridge, the dual quiver has one arrow from f' to f.

Now let f' be the new face created by the final bridge $s_{i_r} = (j\ j{+}1)$. Note that j and $j+1$ cannot both be lollipops. Note also that s_{i_r} is preceded by either s_{i_r-1}or s_{i_r+1} in the bridge sequence. If j or $j+1$ is a lollipop, then the face f' shares 2 edges with f, the face labeled with $[k]$. This means there are no edges between these faces in the dual quiver, since 2 shared edges results in an oriented 2-cycle.

We do not have to add additional vertices of degree 2 after placing the bridge to make the graph bipartite. If neither j nor $j{+}1$ are lollipops, from the columnar reading order, there is a s_{j-1} and a s_{j+1} between each occurrence of s_j in the sequence $s_{i_1}, \dots, s_{i_r}$, so j is adjacent to a black internal vertex and

$j+1$ is adjacent to white internal vertex. This means that there is an arrow in the qual quiver between f' and all adjacent faces that are not labeled with $[k]$.

f' is labeled by (the vertical steps of) $\mathrm{Rect}(b)$, where b is the last box of λ in the columnar reading order. From the proof of Corollary 3.13, its right is the face labeled by (the vertical steps of) a partition ν obtained from $\mathrm{Rect}(b)$ by removing a row (since the partition obtained from $\mathrm{Rect}(b)$ by adding a column does not fit in λ). Similarly, its left is the face labeled by (the vertical steps of) a partition ν' obtained from $\mathrm{Rect}(b)$ by removing a column. Below f' is the face labeled by the partition obtained from $\mathrm{Rect}(b)$ by removing a hook shape. Together with the color of vertices in bridges, we complete the proof. □

We will verify the following Lemma 3.15 which will be used in a further section to deduce Theorem 4.2 to Theorem 4.1. When we say "reflect a (generalized) plabic graph in the mirror", we mean the operation shown in Figure 9.

Lemma 3.15. *Let $v \leq w$ where $v \in W^K_{max}$ and $w = xv$ is length-additive and let $\mathbf{w}'$ be a standard reduced expression for xw_K. Consider the following generalized plabic graphs, with the indicated face labeling.*

- *$G_{v,w}$, obtained by applying v^{-1} to the boundary vertices of $B_{w_K,\mathbf{w}'}$, with target labels.*
- *$G^{mir}_{v,w}$, obtained by applying v^{-1} to the boundary vertices of $B_{w_K,\mathbf{w}'}$ and reflecting in the mirror, with source labels.*
- *$H_{v,w}$, obtained by applying w^{-1} to the boundary vertices of $B_{w_K,\mathbf{w}'}$, with source labels.*
- *$H^{mir}_{v,w}$, obtained by applying w^{-1} to the boundary vertices of $B_{w_K,\mathbf{w}'}$ and reflecting in the mirror, with target labels.*

The labeled dual quiver of each of these graphs, with the vertex labeled $v^{-1}([k])$ deleted, is $\Sigma_{v,w}$ (up to reversing all arrows).

Proof. Since reflecting in the mirror reverses all arrows in the dual quiver, $G_{v,w}$ and $H_{v,w}$ have the same (unlabeled) dual quiver as $B_{w_K,\mathbf{w}'}$. By Proposition 3.14, the dual quiver of all of these graphs is $Q_{v,w}$, up to the reversal of all arrows.

Since the face labels of $G_{v,w}$ are obtained from those of $B_{w_K,\mathbf{w}'}$ by applying v^{-1}, the labeled dual quiver of $G_{v,w}$ is $\Sigma_{v,w}$. It suffices to show that the face labels of $G_{v,w}$ agree with the face labels of the 3 other graphs.

The trip permutation of $B_{w_K,\mathbf{w}'}$ is x^{-1}. This implies that applying v^{-1} to a target face label of $B_{w_K,\mathbf{w}'}$ gives the same set as applying $v^{-1}x^{-1} = w^{-1}$ to a source face label of $B_{w_K,\mathbf{w}'}$. Thus the face labels of $H_{v,w}$ are the same as the face labels of $G_{v,w}$.

Reflecting a generalized plabic graph in the mirror reverses all trips and exchanges left and right. As a result, the target labels of $G_{v,w}$ (resp. $H^{mir}_{v,w}$) are the same as the source labels of $G^{mir}_{v,w}$ (resp. $H_{v,w}$). $\square$

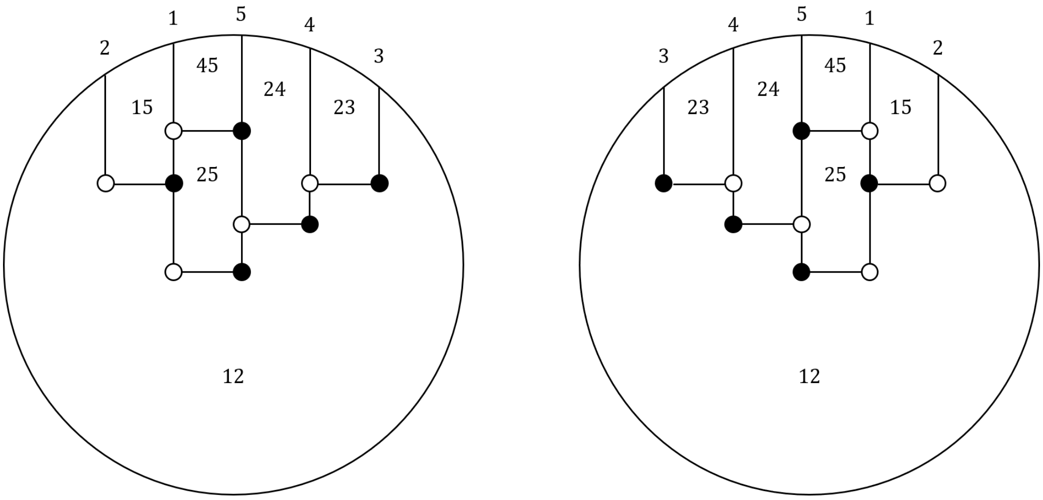


Figure 9: Let $k = 2, n = 5, x = (3, 5, 1, 2, 4)$ and $w = xw_K$ as in Example 6. On the left, we have applied w_K^{-1} to the boundary vertices of $B_{w_K,\mathbf{w}}$ to obtain $G_{w_K,w}$. On the right, we have "reflected $G_{w_K,w}$ in the mirror" to obtain $G^{mir}_{w_K,w}$.

Remark 3.16. We actually interested in the affine cone over $\pi(\mathcal{R}_{v,w})$, we always assume that $\Delta_{v^{-1}([k])}$, the lexicographically minimal nonvanishing Plücker coordinate is equal to 1. So we delete the vertex labeled by $v^{-1}([k])$ in Lemma 3.15.

Remark 3.17. Note that if $v = w_K$, $G^{mir}_{v,w}$ is a "usual" plabic graph(that is, its boundary vertices are $1, \ldots, n$ going clockwise). Similarly if $w = w_0$, $H^{mir}_{v,w}$ is a usual plabic graph.

Remark 3.18. Applying v^{-1} or w^{-1} to the boundary vertices of $B_{w_K,\mathbf{w}'}$ is a mysterious operation. This relabeling takes a plabic graph associated to $\pi_k(\mathcal{R}_{w_K,w_K x^{-1}})$ to one associated to $\pi_k(\mathcal{R}_{v,xv})$. Therefore these positroid varieties are isomorphic.

3.3 The rectangles seed associated to Leclerc's cluster structure

We review the categorical cluster structure on the coordinate of Richardson variety $\mathcal{R}_{v,w}$ in [10]. We are interested in the case of Grassmannians, we restrict our discussion to the construction in type A. In [1], there are more details involving the representation theory of finite-dimensional algebras.

Let Λ be the *preprojective algebra* over $\mathbb{C}$ of type A and rank $n-1$.

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha_1"] & 2 \arrow[l, bend left, "\alpha_1^*"] \arrow[r, bend left, "\alpha_2"] & 3 \arrow[l, bend left, "\alpha_2^*"] \arrow[r, bend left, "\alpha_3"] & \cdots \arrow[l, bend left, "\alpha_3^*"] \arrow[r, bend left, "\alpha_{n-1}"] & n-1 \arrow[l, bend left, "\alpha_{n-1}^*"] \end{tikzcd}$$

It is the finite-dimensional path algebra of the *double quiver* on the vertex set $I = \{1, \ldots, n-1\}$, subject to the relations generated by

$$\sum_i \alpha_i \alpha_i^* - \alpha_i^* \alpha_i = 0.$$

The elements of Λ are linear combinations of paths in the quiver modulo the relations, and multiplication is given by concatenation of paths. Let N be finite-dimensional module N. N is a collection $\{N_i\}_{i \in I}$ of finite-dimensional vector spaces over $\mathbb{C}$ for each vertex $i \in I$, together with a collection of linear maps $\phi_\beta : N_i \to N_j$ for every arrow $\beta : i \to j$ in the quiver. The compositions of these linear maps must satisfy relations induced by the relations on the corresponding.

Let modΛ be the category of finite-dimensional Λ-modules. For any $N \in$ modΛ let $|N|$ be the number of pairwise non-isomorphic indecomposable direct summands of N. We denote add$N$ to be the additive closure of N, i.e. the full subcategory of modΛ whose objects are the Λ-modules isomorphic to a direct sum of direct summands of N Let ind$N$ be the set of indecomposable direct summands of N. For given vertex i, let S_i denote the corresponding simple module and Q_i. The simple module S_i is obtained by placing $\mathbb{C}$ at vertex i and 0's at the remaining vertices of the quiver. In this case $\phi_\beta = 0$ for all arrows β. The injective Λ-module Q_i also has a distinct structure, and we can represent Q_i by its composition factors as follows

$$\begin{array}{cccccccc}
 & & & {\scriptstyle n-i} & & & & \\
 & & {\scriptstyle n-i+1} & & {\scriptstyle n-i-1} & & & \\
 & \iddots & & {\scriptstyle n-i} & & \ddots & & \\
{\scriptstyle n-1} & & \iddots & & \ddots & & \ddots & \\
 & \ddots & & \ddots & & \iddots & & {\scriptstyle 1} \\
 & & \ddots & & {\scriptstyle i} & & \iddots & \\
 & & & {\scriptstyle i+1} & & {\scriptstyle i-1} & & \\
 & & & & {\scriptstyle i} & & &
\end{array}$$

Example 7. When n=6 we obtain the following composition diagrams for the injective modules

$$Q_1 = \begin{smallmatrix} 5 & & & & \\ & 4 & & & \\ & & 3 & & \\ & & & 2 & \\ & & & & 1 \end{smallmatrix} \qquad Q_2 = \begin{smallmatrix} & 4 & & & \\ 5 & & 3 & & \\ & 4 & & 2 & \\ & & 3 & & 1 \\ & & & 2 & \end{smallmatrix} \qquad Q_3 = \begin{smallmatrix} & & 3 & & \\ & 4 & & 2 & \\ 5 & & 3 & & 1 \\ & 4 & & 2 & \\ & & 3 & & \end{smallmatrix}$$

$$Q_4 = \begin{smallmatrix} & & & 2 & \\ & & 3 & & 1 \\ & 4 & & 2 & \\ 5 & & 3 & & \\ & 4 & & & \end{smallmatrix} \qquad Q_5 = \begin{smallmatrix} & & & & 1 \\ & & & 2 & \\ & & 3 & & \\ & 4 & & & \\ 5 & & & & \end{smallmatrix}$$

These numbers can be interpreted as basis vectors or as composition factors. For this example, the module Q_2 is an 8-dimensional Λ-module with dimensional vector $(d_1, d_2, d_3, d_4, d_5) = (1, 2, 2, 2, 1)$.

In general, for every occurrence of $j \in I$ above we obtain the corresponding one-dimensional vector space $V_j \cong \mathbb{C}$ at vertex j of the quiver. Moreover, whenever we see a configuration ${}^{j+1}{}_{j}$ or ${}_{j}{}^{j-1}$ then the linear map between the corresponding spaces $V_{j+1} \to V_j$ or $V_{j-1} \to V_j$ is the identity. The top (resp. socle) of N is a direct sum of simple modules S_i such that the corresponding entry i in the associated composition factor diagram lies at the top (resp. bottom). In the other words, there are no $i-1$ and no $i+1$ appearing directly above (resp. below) i.

For every $i \in I$ and $s_i \in W$ (where W is the symmetric group on n letters) we define $\mathcal{E}_i = \mathcal{E}_{s_i}$ and $\mathcal{E}_i^\dagger = \mathcal{E}_{s_i}^\dagger$ on the category modΛ. Given $N \in$ modΛ let $\mathcal{E}_i(N)$ be the kernel of a surjection $N \twoheadrightarrow S_i^a$ where a is the multiplicity of S_i in the top of N. Similarly, let $\mathcal{E}_i^\dagger(N)$ be the cokernel of an injection $S_i^b \hookrightarrow N$ where b is the multiplicity of S_i in the socle of N. The diagram for $\mathcal{E}_i(N)$ (resp. $\mathcal{E}_i^\dagger(N)$) is obtained from that of N by removing all entries i appearing in the top (resp. bottom). Moreover, for every $w \in W$ we can extend the definition to $\mathcal{E}_w, \mathcal{E}_w^\dagger$ by composing the functors associated to the simple reflections in a reduced expression for w. Given $w \in W$, consider $\mathcal{C}_w = \mathcal{E}_{w^{-1}w_0}(\mathrm{mod}\Lambda)$ and $\mathcal{C}^w = \mathcal{E}^{\dagger}_{w^{-1}w_0}(\mathrm{mod}\Lambda)$, two subcategories of $\mathrm{mod}\Lambda$ associated to w. Now we summarize the main theorem of [10].

Theorem 3.19. *[10] For every $v, w \in W$ with $v \leq w$, the subcategory $\mathcal{C}_{v,w} := \mathcal{C}^v \cap \mathcal{C}_w$ has a cluster structure. Moreover, $\mathcal{C}_{v,w}$ induces a cluster subalgebra in the coordinate ring $\mathbb{C}[\mathcal{R}_{v,w}]$, where the cardinality of the extended cluster is equal to $dim\mathcal{R}_{v,w}$.*

The following definitions are from [10]. Let $\mathcal{B}$ be a subcategory of $\mathrm{mod}\Lambda$ closed under extensions, direct sums, and direct summands. We say T is *$\mathcal{B}$-cluster-tilting* if

$$(X \in \mathcal{B} \text{ and } \mathrm{Ext}^1_{\Lambda}(T, X) = 0) \iff (X \in \mathrm{add}T)$$

We say that T is *basic* if its indecomposable direct summands are pairwise non-isomorphic. Note that the every $\mathcal{B}$-cluster-tilting module T is *rigid*, i.e. $\mathrm{Ext}^1_{\Lambda}(T, T) = 0$. T is $\mathcal{B}$-cluster-tilting if and only if it is a maximal rigid module in $\mathcal{B}$, i.e. the number of pairwise nonisomorphic indecomposable direct summands of T is maximal among rigid modules in $\mathcal{B}$. Theorem 3.19 says that $\mathcal{C}_{v,w}$ is a Frobenius category that admits a cluster-tilting object. Let a basic cluster-tilting module T be given. Then we associate the *endomorphism quiver* Γ_T as follows. The vertices of Γ_T are in bijection with indecomposable direct summands T_i of T. The number of arrows $T_i \to T_j$ in Γ_T corresponds to the dimension of the space of irreducible morphism $T_i \to T_j$ in $\mathrm{add}T$, that is, morphisms that cannot be factored nontrivially in $\mathrm{add}T$.

Given a basic cluster-tilting module $T \in \mathcal{C}_{v,w}$, there is a notion of mutation of T at an indecomposable summand T_i of T, provided that T_i is not projective-injective in $\mathcal{C}_{v,w}$ The *mutation* of T at T_i is a new cluster-tilting module $\mu_{T_i}(T) := T/T_i \oplus T_i'$, obtained by replacing T_i by a unique different indecomposable module $T_i' \in \mathcal{C}_{v,w}$. T_i' is defined by the two short exact sequences

$$0 \to T_i' \to B \xrightarrow{g} T_i \to 0 \qquad 0 \to T_i \xrightarrow{f} B' \to T_i' \to 0$$

where g and f are minimal right and left $\mathrm{add}(T/T_i)$-approximations of T_i. Thus, B is a direct sum of $T_j \in \mathrm{ind}T$. Thus, B is a direct sum of $T_j \in \mathrm{ind}T$

for every arrow $T_j \to T_i$ in Γ_T, and B' is a direct sum of $T_j \in \mathrm{ind}T$ for every arrow $T_i \to T_j$ in Γ_T.

There is a cluster character $\varphi : \mathrm{obj}\mathcal{C}_{v,w} \to \mathbb{C}[\mathcal{R}_{v,w}]$ that maps module $N \in \mathcal{C}_{v,w}$ to functions $\varphi_N \in \mathbb{C}[\mathcal{R}_{v,w}]$. φ satisfies several properties. For every $N, N' \in \mathcal{C}_{v,w}$, we have

$$\varphi_{N\oplus N'} = \varphi_N \varphi_{N'}$$

Moreover, for every mutation μ_{T_i} of cluster-tilting module T, we obtain an exchange relation in $\mathbb{C}[\mathcal{R}_{v,w}]$

$$\varphi_{T_i}\varphi_{T_i'} = \varphi_B + \varphi_{B'}$$

where B and B' come from the short exact sequences above. In this way, the cluster character φ induces a cluster algebra structure in $\mathbb{C}[\mathcal{R}_{v,w}]$ from a categorical cluster structure in $\mathcal{C}_{v,w}$. We now give a more explicit description of Theorem 3.19.

Definition 3.20. Given $v \leq w$ in W and a reduced expression $\mathbf{w} = s_{i_t} \cdots s_{i_1}$ for w, we construct a set of modules $\{U_j\}$ which will give rise to a cluster in $\mathbb{C}[\mathcal{R}_{v,w}]$. Let $\mathbf{v}$ be the reduced subexpression for v in $\mathbf{v}$ that is "rightmost" in $\mathbf{w}$, called the *positive distinguished subexpression* for v in $\mathbf{w}$. Set $w_{(j)} = s_{i_j} \cdots s_{i_2} s_{i_1}$ for $1 \leq j \leq t$, and let $w_{(j)}^{-1} := (w_{(j)})^{-1}$. Let $v_{(j)}$ be the product of all simple reflections in $w_{(j)}$ that are part of $\mathbf{v}$. Define $J \subseteq \{1, \ldots, t\}$ to be the collection of indices j such that the corresponding reflection s_{i_j} in the expression $\mathbf{w}$ is *not* a part of $\mathbf{v}$.

For every $j \in J$ we construct a module U_j from the injective module Q_{i_j}. For $N \in \mathrm{mod}\Lambda$ let $\mathrm{Soc}_{s_i}(N)$ be the direct sum of all submodules of N isomorphic to the simple module S_i. Given a reduced word $z = s_{i_r} \cdots s_{i_2} s_{i_1}$ in W, there is a unique sequence

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_r \subseteq N$$

of submodules of N such that $N_p/N_{p-1} = \mathrm{Soc}_{s_{i_p}}(N/N_{p-1})$. Define $\mathrm{Soc}_z(N) = N_r$. For every $j \in J$, let $V_j = \mathrm{Soc}_{w_{(j)}^{-1}}(Q_{i_j})$ and $U_j = \mathcal{E}^{\dagger}_{v_{(j)}^{-1}} V_j$.

Example 8 gives a construction of a module U_j.

The following theorem describes the cluster algebra structure in the coordinate ring of $\mathcal{R}_{v,w}$ and its additive categorification provided by $\mathcal{C}_{v,w}$.

Theorem 3.21. *[10] Each pair $(v, \mathbf{w})$ as in Definition 3.20 gives a cluster-tilting module $U_{v,\mathbf{w}} := \bigoplus_{j \in J} U_j$ in $\mathcal{C}_{v,w}$, that corresponds via the cluster character φ to a seed in $\mathbb{C}[\mathcal{R}_{v,w}]$ as follows.*

1. *The cluster variables in $\mathbb{C}[\mathcal{R}_{v,w}]$ are the irreducible factors of $\varphi_{U_j} = \Delta_{v_{(j)}^{-1}([i_j]), w_{(j)}^{-1}([i_j])}$ for $j \in J$; they correspond to the indecomposable summands of U_j.*

2. *The frozen variables are irreducible factors of $\prod_{i \in I} \Delta_{v^{-1}([i]), w^{-1}([i])}$; they correspond to the indecomposable summands of $\bigoplus_{i \in I} \mathcal{E}^{\dagger}_{v^{-1}} \mathcal{E}_{w^{-1} w_0}(Q_i)$ (which are the projective-injective objects).*

3. *The extend cluster is the set of cluster and frozen variables, which has cardinality $dim \mathcal{R}_{v,w} = \ell(w) - \ell(v) = |U_{v,\mathbf{w}}|$.*

4. *The quiver associated to the seed is the endomorphism quiver $\Gamma_{U_{v,\mathbf{w}}}$ of the cluster-tilting module. Moreover, the quiver has no loops and no 2-cycles, and the mutation of $U_{v,\mathbf{w}}$ induces mutation on the quiver $\Gamma_{U_{v,\mathbf{w}}}$.*

5. *The cluster algebra $\tilde{\mathcal{R}}_{v,w}$ generated by all cluster variables is a subalgebra of $\mathbb{C}[\mathcal{R}_{v,w}]$; when w can be factored as $w = xv$ with $\ell(w) = \ell(x) + \ell(v)$, the cluster algebra $\tilde{\mathcal{R}}_{v,w}$ is equal to $\mathbb{C}[\mathcal{R}_{v,w}]$.*

Example 8. Let $n = 7$ and consider a pair (v, w) corresponding to a cell in $Gr_{3,7}$, where $v = w_K s_3$ and w is given by the reduced expression

$$\mathbf{w} = s_5 s_6 s_4 s_5 s_2 s_3 s_4 s_1 s_2 s_3 \mathbf{s_1 s_2 s_1 s_4 s_5 s_4 s_6 s_5 s_4 s_3} = s_{i_{20}} \cdots s_{i_2} s_{i_1}.$$

The positive distinguished subexpression for v in $\mathbf{w}$ is indicated in bold, and corresponds to the last ten transpositions at the end of $\mathbf{w}$. The remaining transpositions determine the index set $J = \{11, 12, \ldots, 20\}$, and for each $j \in J$ we obtain a summand U_j of the cluster-tilting module $U_{v,\mathbf{w}}$. We have $v_{(j)} = v$ for all $j \in J$. First compute U_{14}. Recall that

$$Q_4 = \begin{array}{cccccc} & & & 3 & & \\ & & 4 & & 2 & \\ & 5 & & 3 & & 1 \\ 6 & & 4 & & 2 & \\ & 5 & & 3 & & \\ & & 4 & & & \end{array}$$

We build up the composition diagram of $V_{14} = \mathrm{Soc}_{w^{-1}_{(14)}}(Q_4)$ by adding composition factors from the diagram of Q_4, working from the bottom up. We will illustrate below. We add composition factors in the order specified by the reduced expression $w^{-1}_{(14)} = \underline{s_3}s_4\underline{s_5s_6s_4s_5}s_4\underline{s_1s_2}s_1\underline{s_3}s_2s_1\underline{s_4}$ (reading right to left). The underline s_i's indicate when an i is added.

$$\begin{smallmatrix}4\end{smallmatrix} \to \begin{smallmatrix} & 3\\ 4 & \end{smallmatrix} \to \begin{smallmatrix} & & 2\\ & 3 & \\ 4 & & \end{smallmatrix} \to \begin{smallmatrix} & & & 1\\ & & 2 & \\ & 3 & & \\ 4 & & & \end{smallmatrix} \to \begin{smallmatrix} & & & & 1\\ & & & 2 & \\ 5 & & 3 & & \\ & 4 & & & \end{smallmatrix} \to$$

$$\begin{smallmatrix} & & & & 1\\ & 4 & & 2 & \\ 5 & & 3 & & \\ & 4 & & & \end{smallmatrix} \to \begin{smallmatrix} & & & & & 1\\ & & 4 & & 2 & \\ 6 & 5 & & 3 & & \\ & & 4 & & & \end{smallmatrix} \to \begin{smallmatrix} & 5 & & & & 1\\ & & 4 & & 2 & \\ 6 & 5 & & 3 & & \\ & & 4 & & & \end{smallmatrix} \to \begin{smallmatrix} & 5 & & 3 & & 1\\ & & 4 & & 2 & \\ 6 & 5 & & 3 & & \\ & & 4 & & & \end{smallmatrix} = V_{14}$$

We remove composition factors from the diagram of V_{14} to get the composition diagram of $U_{14} = \mathcal{E}^{\dagger}_{v^{-1}}V_{14}$. We remove these factors from the bottom up, in the order specified by reduced expression $v^{-1} = \underline{s_3}s_4s_5\underline{s_6}s_4\underline{s_5s_4}s_1s_2s_1$ right to left. The underlined s_i's indicate when an i is removed.

$$V_{14} = \begin{smallmatrix} & 5 & & 3 & & 1\\ & & 4 & & 2 & \\ 6 & 5 & & 3 & & \\ & & 4 & & & \end{smallmatrix} \to \begin{smallmatrix} & 5 & & 3 & & 1\\ 6 & & 4 & & 2 & \\ & 5 & & 3 & & \end{smallmatrix} \to \begin{smallmatrix} & 5 & & 3 & & 1\\ 6 & & 4 & & 2 & \\ & & & 3 & & \end{smallmatrix} \to \begin{smallmatrix} 5 & & 3 & & 1\\ & 4 & & 2 & \\ & & 3 & & \end{smallmatrix} \to \begin{smallmatrix} 5 & & 3 & & 1\\ & 4 & & 2 & \end{smallmatrix} = U_{14}$$

Similarly, we can obtain the following set of modules:

$$U_{11} = \begin{smallmatrix} 6 & & & & & \\ & 5 & & 3 & & 1\\ & & 4 & & 2 & \end{smallmatrix} \quad U_{12} = \begin{smallmatrix} 6 & & & & \\ & 5 & & 3 & \\ & & 4 & & 2 \end{smallmatrix} \quad U_{13} = \begin{smallmatrix} 6 & & \\ & 5 & \\ & & 4 \end{smallmatrix} \quad U_{15} = \begin{smallmatrix} & 5 & & 3 & & \\ 6 & & 4 & & 2 & \\ & 5 & & 3 & & 1\\ & & 4 & & 2 & \end{smallmatrix}$$

$$U_{16} = \begin{smallmatrix} & 5 & & 3 & \\ 6 & & 4 & & \\ & 5 & & 3 & \\ & & 4 & & 2 \end{smallmatrix} \quad U_{17} = \begin{smallmatrix} & 3 & & 1\\ 4 & & 2 & \end{smallmatrix} \quad U_{18} = \begin{smallmatrix} & & 3 & & \\ & 4 & & 2 & \\ 5 & & 3 & & 1\\ & 4 & & 2 & \end{smallmatrix} \quad U_{19} = \begin{smallmatrix} & 1\\ 2 & \end{smallmatrix} \quad U_{20} = \begin{smallmatrix} & & 2 & \\ & 3 & & 1\\ 4 & & 2 & \end{smallmatrix}$$

The projective-injective objects in $\mathcal{C}_{v,w}$ are $U_{13}, U_{15}, U_{16}, U_{18}, U_{19}, U_{20}$. The endomorphism quiver $\Gamma_{U_{v,\mathbf{w}}}$ is given below.

$$\begin{tikzcd}
\begin{smallmatrix} 6 & & & & & \\ & 5 & & 3 & & 1\\ & & 4 & & 2 & \end{smallmatrix} \arrow[dr] & \begin{smallmatrix} 5 & & 3 & & 1\\ & 4 & & 2 & \end{smallmatrix} \arrow[l] \arrow[dr] & \begin{smallmatrix} & 3 & & 1\\ 4 & & 2 & \end{smallmatrix} \arrow[l] \arrow[dr] & \begin{smallmatrix} & 1\\ 2 & \end{smallmatrix} \arrow[l] \\
\begin{smallmatrix} 6 & & & & \\ & 5 & & 3 & \\ & & 4 & & 2 \end{smallmatrix} \arrow[u] \arrow[dr] & \begin{smallmatrix} & 5 & & 3 & & \\ 6 & & 4 & & 2 & \\ & 5 & & 3 & & 1\\ & & 4 & & 2 & \end{smallmatrix} \arrow[l] \arrow[u] & \begin{smallmatrix} & & 3 & & \\ & 4 & & 2 & \\ 5 & & 3 & & 1\\ & 4 & & 2 & \end{smallmatrix} \arrow[l] \arrow[u] & \begin{smallmatrix} & & 2 & \\ & 3 & & 1\\ 4 & & 2 & \end{smallmatrix} \arrow[l] \arrow[u] \\
\begin{smallmatrix} 6 & & \\ & 5 & \\ & & 4 \end{smallmatrix} \arrow[u] & \begin{smallmatrix} & 5 & & 3 & \\ 6 & & 4 & & \\ & 5 & & 3 & \\ & & 4 & & 2 \end{smallmatrix} \arrow[l] \arrow[u] & &
\end{tikzcd}$$

In general, it is difficult to construct the endomorphism quiver $\Gamma_{U_{v,\mathbf{w}}}$ because it is difficult to determine whether a given morphism is irreducible in $\mathrm{add}U_{v,\mathbf{w}}$.

For example, there is a nonzero morphism $f : U_{15} \to U_{11}$ with image ${}^{5}{}_{4}{}^{3}{}_{2}$ but it factors through U_{12}. Thus f does not induce an arrow in $\Gamma_{U_{v,\mathbf{w}}}$.

We want to find an explicit description of the seed associated to a pair $(v, \mathbf{w})$, where $v \in W^K_{max}$, $w = xv$ is a length-additive factorization, and $\mathbf{w} = \mathbf{xv}$ is a standard expression for w. First, we will determine how to interpret the cluster variables coming from Theorem 3.21 as functions on the Grassmannian. Since each generalized minor from Theorem 3.19 is a minor of a unipotent matrix, we can restrict that matrix to rows $v^{-1}[k]$ and then identify the minor with a Plücker coordinate of the resulting $k \times n$ matrix.

Remark 3.22. Let $J \subseteq [n]$ with $|J| = \ell$. If we project an $n \times n$ unipotent matrix g to the Grassmannian element represented by the span of rows $v^{-1}[\ell]$ of g, the generalized minor $\Delta_{v^{-1}[\ell],J}$ of g equals the following Plücker coordinate of $Gr_{k,n}$:

- If $\ell < k$ and $|J \cup v^{-1}([k] \setminus [\ell])| = k$ then $\Delta_{v^{-1}[\ell],J} = \Delta_{J \cup v^{-1}([k]\setminus[\ell])}$
- If $\ell = k$ then $\Delta_{v^{-1},J} = \Delta_J$
- If $\ell > k$ and $|J \setminus v^{-1}([\ell] \setminus [k])| = k$ then $\Delta_{v^{-1}[\ell],J} = \Delta_{J \setminus v^{-1}([k]\setminus[\ell])}$.

We use Remark 3.22 to show that Leclerc's cluster variables in the seed corresponding $(v, \mathbf{w})$ coincide with those obtained from the rectangle seed defined before.

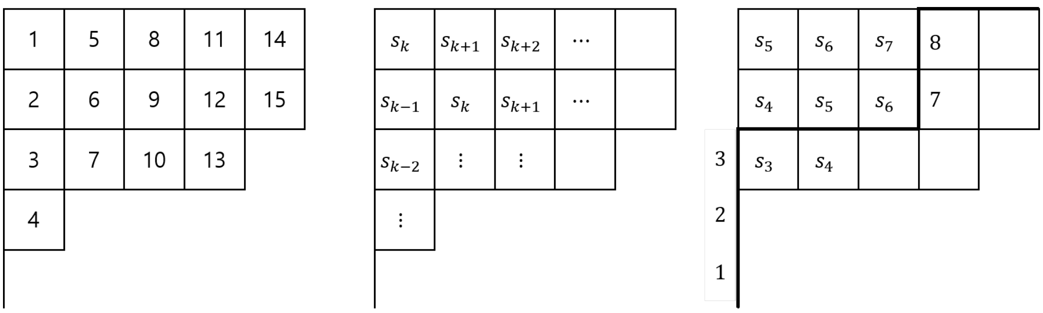


Figure 10:

Lemma 3.23. *Choose a Young diagram contained in a $k \times (n-k)$ rectangle, and label its boxes by simple reflections as in the right of Figure 10. Choose* *a reading order for the boxes as in the left of the Figure 10. Choose any box b and let s_ℓ be its label. Let w_b be the word obtained by reading boxes in order up through b and recording the corresponding simple reflections. For example if b is the box indicated by the bold s_6 in the right of Figure 10, then $w_b = (s_5s_4s_3s_2)(s_6s_5s_4)(s_7s_6)$. Also let $J(b) := V^{\nearrow}(Rect(b))$. In the right of Figure 10 $J(b) = \{1,2,3,7,8\}$. Then for any b and ℓ as above, let $J = w_b[\ell]$ then*

- *If $\ell < k$, then $J(b) = J \cup ([k] \setminus [\ell]) = J \cup \{\ell+1, \ell+2, \dots, k\}$*
- *If $\ell = k$, then $J(b) = J$*
- *If $\ell > k$, then $J(b) = J \setminus ([k] \setminus [\ell]) = J \cup \{k+1, k+2, \dots, \ell\}$.*

Proof. Since the proofs of the three cases are analogous, we prove the first one when $\ell < k$. Let box b be in row r and column c so that its label is $s_\ell = s_{k-r+c}$. We know that $r > c$. Then $J(b) = \{1,2,\dots,k-r\} \cup \{k-r+c+1, k-r+c+2, \dots, k+c\}$, and $J(b) \setminus \{\ell+1, \ell+2, \dots, k\} = J(b) \setminus \{k-r+c+1, k-r+c+2, \dots, k\} = \{1,2,\dots,k-r\} \cup \{k+1, k+2, \dots, k+c\}$. We need to show that $w_b\{1,2,\dots,k-r+c\} = J(b) \setminus \{k-r+c+1, k-r+c+2, \dots, k\} = \{1,2,\dots,k-r\} \cup \{k+1, k+2, \dots, k+c\}$.

Let the labels of the simple generators in the bottom boxes of columns $1, 2, \dots, c-1$ be $i_1, i_2, \dots, i_{c-1}$, respectively. We write $i_c = k-r+c$. Then we have that

$$w_b = (s_k s_{k-1} \cdots s_{i_1})(s_{k+1} s_k \cdots s_{i_2}) \cdots (s_{k+c-2} \cdots s_{i_{c-1}})(s_{k+c-1} \cdots s_{i_c}).$$

Note that $1 \le i_1 < i_2 \cdots < i_{c-1} < i_c = k-r+c$ so that $i_s \le k-r+s$ for all $1 \le s \le c$. Note that for $a < b$ the product $s_b s_{b-1} \cdots s_a$ is equal to the cycle $(b+1, b, b-1, \dots, a+1, a)$ (in cycle notation). Then we see that:

- for $1 \le i_1 - 1, w_b(j) = j \in \{1,2,\dots,k-r\}$
- for $j \in \{i_1, i_2, \dots, i_c\}, w_b(j) \in \{k+1, k+2, \dots, k+c\}$
- for $i_1 < j < i_2, w_b(j) = j-1$
- for $i_2 < j < i_3, w_b(j) = j-2$

- $\vdots$
- for $i_{c-1} < j < i_c, w_b(j) = j - (c-1)$.

So for $i_{s-1} < j < i_s$, we have that $w_b(j) = j - (s-1) < i_s - (s-1) \leq k - r + s - (s-1) = k - r + 1$, and so $w_b(j) \leq k - r$. This shows that for each $j \in \{1, 2, \ldots, k - r + c\}$, $w_b(j) \in \{1, 2, \ldots, k - r\} \cup \{k+1, k+2, \ldots, k+c\}$ and so $w_b\{1, 2, \ldots, k-r+c\} = \{1, 2, \ldots, k-r\} \cup \{k+1, k+2, \ldots, k+c\}$. $\square$

Corollary 3.24. *Consider a skew Schubert variety $\pi_k(\mathcal{R}_{v,w}) \subset Gr_{k,n}$, where $v \leq w, v \in W^K_{max}$, and with $w = xv$ length-additive. Consider the seed for $\mathcal{R}_{v,w}$ given by Theorem 3.21 which is associated to a standard (columnar) reduced expression $\mathbf{w} = \mathbf{xv}$. When we project the cluster variables to $\pi_k(\mathcal{R}_{v,w})$, we obtain the set of Plücker coordinates from the rectangles seed. In other words, they are indexed by boxes b in $\lambda^{\nearrow}(x([k]))$, and are equal to the Plücker coordinates $\Delta_{v^{-1}(J(b))}$ in the Grassmannian.*

Proof. Let $\mathbf{x}$ be the columnar expression for x and $\mathbf{w}$ be a standard reduced expression for w. Let b be a box in $\lambda^{\nearrow}(x([k]))$, and let s_ℓ, w_b, and $J(b)$ be as defined in Lemma 3.23. $w_b = x^{-1}_{(i)}$ for some $1 \leq i \leq \ell(x)$, so $v^{-1}w_b = w^{-1}_{(j)}$ for some j. Applying v^{-1} to Lemma 3.23 implies that the generalized minor $\Delta_{v^{-1}([\ell]), w_j^{-1}([\ell])}$ equals the Plücker coordinate $\Delta_{v^{-1}(J(b))}$ in the Grassmannian (Remark 3.22). Each of those Plücker coordinates is irreducible in $\mathbb{C}[\widehat{\pi_k(\mathcal{R}_{v,w})}]$. $\square$

Lemma 3.25. *Let x be a Grassmannian permutation of type (k, n). Let b be a λ-frozen box of $\lambda = \lambda^{\nearrow}(x([k]))$, and let s_ℓ and w_b be as defined in Lemma 3.23. Then $w_b([\ell]) = x^{-1}([\ell])$. Thus $\Delta_{v^{-1}(J(b))}$ is frozen in the rectangles seed.*

Proof. The boxes in columns to the right of the column of b are filled with s_i such that $i > \ell$. So $x^{-1} = w_b u$, where u is a permutations that fixes $[\ell]$ pointwise, so $w_b([\ell]) = x^{-1}([\ell])$. $\Delta_{v^{-1}(J(b))}$ is the projection of $\Delta_{v^{-1}([\ell]), v^{-1}x^{-1}([\ell])}$ to the Grassmannian, which is frozen by Theorem 3.21. $\square$

We will describe the endomorphism quiver $\Gamma_{U_{v,\mathbf{w}}}$. We need to analyze morphism between indecomposable summand of $U_{v,\mathbf{w}}$. Let $R(x)$ be a Young diagram filled with simple reflections which gives a reduced expression for x. For $b_i \in R(x)$, U_i denotes the associated summand of $U_{v,\mathbf{w}}$. Recall that $\mathrm{Rect}(b_i)$is the maximal rectangle in D whose southeast corner is b_i.

Theorem 3.26. *Consider (v, w) where $v \in W^K_{max}$ and $w = xv$ is length-additive. Let $\mathbf{w} = \mathbf{x}\mathbf{v} = \mathbf{x}\mathbf{w_k}\mathbf{v}'$ be a standard reduced expression for w. For any pair of modules $U_i, U_j \in \mathrm{ind}U_{v,\mathbf{w}}$ there exists an irreducible morphism $U_i \to U_j$ in $\mathrm{add}U_{v,\mathbf{w}}$ if and only if one of the following conditions holds:*

1. *$\mathrm{Rect}(b_j)$ is obtained from $\mathrm{Rect}(b_i)$ by removing a row*
2. *$\mathrm{Rect}(b_j)$ is obtained from $\mathrm{Rect}(b_i)$ by removing a column*
3. *$\mathrm{Rect}(b_j)$ is obtained from $\mathrm{Rect}(b_i)$ by adding a hook shape.*

Moreover, There exists at most one irreducible morphism between U_i and U_j.

By Theorem 3.26, we have the following observation.

Remark 3.27. Let $f : U_i \to U_j$ be a homomorphism and suppose that N is an indecomposable direct summand of $\mathrm{im}f$. Since $\mathrm{im}f$ is a submodule of U_j and is isomorphic to a quotient of U_i, the composition diagram for N embeds into those for U_i, U_j. Moreover N is *closed under predecessors* in U_i: for all vertices x and $y \in I$ in the composition diagrams for N and U_i, respectively, such that y lies immediately above x in U_i (i.e. ${}^{y}{}_{x}$ or ${}_{x}{}^{y}$) we have that y is also in the composition diagram for N. Similarly N is *closed under successors* in U_j: for all vertices $x, y \in I$ in the diagrams for N, U_j such that y lies immediately below x in U_j: for all vertices $x, y \in I$ in the diagrams N, U_j such that y lies immediately below x in U_j (i.e. ${}^{x}{}_{y}$ or ${}_{y}{}^{x}$), we have that Y is also in the diagram for N.

Conversely, for any N that is closed under predecessors in U_i and closed under successors in U_j we get a morphism $f : U_i \to U_j$ with image N.

We will prove Theorem 3.26 in two steps. First, we will prove the case $v' = e$, i.e. $v = w_K$.

Proposition 3.28. *Theorem 3.26 is true when $v' = e$, i.e. $v = w_K$.*

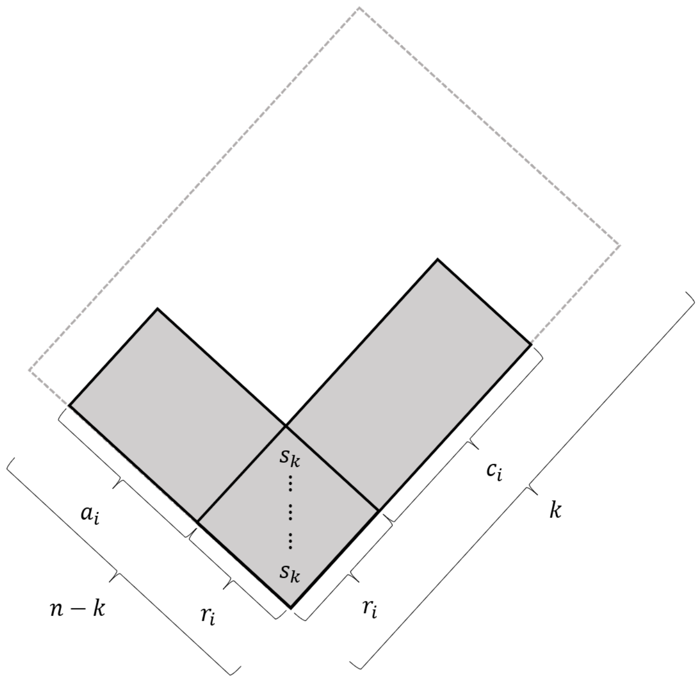


Figure 11: An indecomposable module U_i.

Proof. Note that all indecomposable summands of $U = U_{v,\mathbf{w}}$ are of the form given in Figure 11. Moreover, we have $S_k = \mathrm{Soc} U_i = \mathrm{Soc} U_j$ and either $c_i + r_i = k$ or $a_i + r_i = n - k$ for any $U_i \in \mathrm{ind} U$. We can rephrase the statement of the theorem in terms of a_i, c_i, r_i that define a given summand of U. (1a) and (1b) correspond to case (1) of the theorem depending on if b_i is above or below the main diagonal. Similarly, we have the following correspondence.

(1a) $r_i = r_j + 1, a_i = a_j$, and $c_i + r_i = c_j + r_j = k$

(1b) $r_i = r_j, c_i = c_j - 1$, and $a_i + r_i = a_j + r_j = n - k$

(2a) $r_i = r_j, a_i = a_j - 1$, and $c_i + r_i = c_j + r_j = k$

(2b) $r_i = r_j + 1, c_i = c_j$, and $a_i + r_i = a_j + r_j = n - k$

(3a) $r_i = r_j - 1, a_i = a_j + 1$, and $c_i + r_i = c_j + r_j = k$

(3b) $r_i = r_j - 1, c_i = c_j - 1$, and $a_i + r_i = a_j + r_j = n - k$.

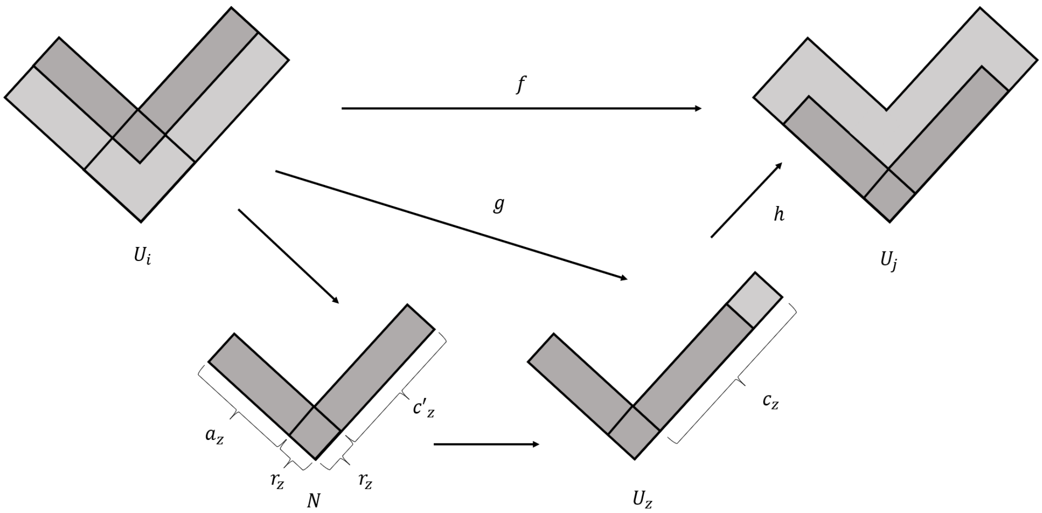


Figure 12: Morphism $f : U_i \to U_j$ with image N.

Given $U_i \in \mathrm{add}U$ defined by a_i, r_i, c_i, a module U_z defined by a_z, r_z, c_z is also in add$u$ if $r_z \leq a_z$ and either $a_z = a_i, c_z \geq b_i$ or $c_z = c_i, a_z \geq a_i$. Every module in add$U$ corresponds to a unique box in $R(x)$. Given a box $b_i \in R(x)$ associated to the module U_i, all the boxes $b_z \in D$ above and the left of b_i are also in $R(x)$. The module U_z with the above properties is precisely the one coming from such a box $b_z \in R(x)$. Thus, $U_z \in \mathrm{add}U$.

Let $f : U_i \to U_j$ be a nonzero nonidentity morphism in modΛ. Since U_j has a one-dimensional socle it follows that im$f$, which is a submodule of U_j. Let $N = \mathrm{im}f$. It is closed under predecessors in U_i and closed under successors in U_j. The socle of N is also S_k, so we obtain the configuration described in Figure 12. Here $r_z \leq r_i, r_j, r_z + c'_z \leq r_j + c_j$, and $r_z + a_z \leq r_j + a_j$. Conversely, for every such N as in the Figure 12 we obtain a nonzero morphism $U_i \to U_j$.

First, we consider the case $r_i + c_i = k$ and $r_j + c_j = k$. Note that N is not necessarily in add$U$. We construct a module $U_z \in \mathrm{ind}\Lambda$ defined by a_z, c_z, r_z satisfying $a_z = a_i$ and $c_z + r_z = k$. Since $r_z \leq r_i$ and $c_z \geq c_i$ it follows that $U_z \in \mathrm{ind}U$. We obtain maps $g : U_i \to U_z$ and $h : U_z \to U_j$ such that $f = hg$. This implies that f is reducible in add$U$ unless $g = 1$ or $h = 1$.

Since we are interested in irreducible morphisms f, suppose that $h = 1$. Thus $U_z = U_j$ and $f = g$. If $r_z = r_i$, then $U_i = U_z$ and $g = f = 1$ contrary to the assumption that f is not identity morphism. For $r_z < r_i$, consider a module U_t defined by $a_t = a_i, r_t = r_z + 1$ and c_t such that $c_t + r_t = k$. In particular $c_t = c_z - 1$. Since $r_t \leq r_i$ and $c_t > c_i$, it follows that $U_t \in \text{add}U$. In this case, f factors through U_t. That is, there exist maps ρ, π as below

$$\begin{tikzcd} U_i \arrow[rr, "f=g"] \arrow[dr, "\rho"'] & & U_z \\ & U_t \arrow[ur, "\pi"'] & \end{tikzcd}$$

such that $f = \pi\rho$. $\pi \neq 1$ implies that $c_t \neq c_z$. Since we are interested in irreducible morphisms f, we consider the case $\rho = 1$ and $f = \pi$. If $f = \pi$ then we have $U_i = U_t$ and $U_j = U_z$. Then $a_i = a_j, r_i = r_j + 1$ and $c_i + r_i = c_j + r_j = k$, which agrees with case (1a). By construction of U_i and U_j such f is indeed irreducible in add$U$.

Now, we consider the case $g = 1$. Thus, $h = f$ and $U_z = U_i$. Let U_q be the module defined by $a_q = a_z + 1,\ r_q = r_z,\ c_q = c_z$, provided that $a_z + r_z < n - k$. Since $r_q = r_z$ and $c_q > a_z$, $U_q \in \text{add}U$. If $a_q + r_q \leq a_j + r_j$, f factors through U_q. There exist morphism σ, δ as below

$$\begin{tikzcd} U_z \arrow[rr, "f=h"] \arrow[dr, "\sigma"'] & & U_j \\ & U_q \arrow[ur, "\delta"'] & \end{tikzcd}$$

Where $f = \delta\sigma$. Since we are interested in irreducible morphisms, we consider the case $\delta = 1$. Note that $\sigma \neq q$ as $a_z < a_q$. In the case $\delta = 1$ we have $f = \sigma$ is injective, and $U_z = U_i, U_q = U_j$. Therefore, $r_i = r_j, a_i = a_j - 1$, and $r_i + c_i = r_j + c_j = k$, which agrees with case (2a). Since f is injective, it is irreducible in add$U$.

Now, we consider the case $f = h$ and $a_z + r_z = a_j + r_j$. We observe that $r_z \neq r_j$; otherwise $U_i = U_z = U_j$ and f are the identity maps. Let U_p be the module defined by $r_p = r_z + 1, a_p = a_z - 1, c_p = c_z - 1$, provided that a_z, c_z are both nonzero. Since $r_p \leq r_j$ and $a_p \geq a_j$, $U_p \in \text{add}U$. Thus we see that $f = h$ factors through U_p

$$\begin{tikzcd} U_z \arrow[rr, "f=h"] \arrow[dr, "\epsilon"'] & & U_z \\ & U_t \arrow[ur, "\theta"'] & \end{tikzcd}$$

where $f = \theta\epsilon$. Note that $\epsilon \neq q$ by construction, therefore we consider the case $\theta = 1$. Thus, $f = \epsilon$ and $U_z = U_i, U_j = U_p$, where $r_i = r_j - 1, a_i = a_i + 1$, and $c_i + r_i = c_j + r_j = k$. This agrees with case (3a). Since f is injective, it is irreducible in add$U$.

Suppose that $f = h$ and $a_z + r_z = a_j + r_j$ as above, but $a_z = 0$ or $c_z = 0$. If $a_z = 0$ then $r_z = a_j + r_j$. Since f maps U_z to U_j injectively, $r_z \leq r_j$ which implies that $a_j = 0$. We obtain $U_z = U_i = U_j$ and f is the identity morphism which is contradiction. On the other hand if $c_z = 0$ then $r_z = k$. Since $r_z \leq r_j \leq k$, we obtain $r_j = k$. Since $a_j + r_j \leq k$, $a_j = 0$ which implies a contradiction.

This completes the proof when $c_i + r_i = c_j + r_j = k$. Similarly, we can prove when $a_i + r_i = a_j + r_j = n - k$. Therefore, it remains to consider the situation when $r_i + c_i = k$ and $a_j + r_j = n - k$ $r_i + a_i < n - k$ and $c_j + r_j < k$ and vice versa. We will show that every morphism in this case is reducible. Suppose $f : U_i \to U_j$ where $r_i + c_i = k$ and $a_j + r_j = n - k$ while $r_i + a_i < n - k$ and $c_j + r_j < k$. Let $U_u \in$ add$U$ be the module defined bu $r_u = r_i, a_u + r_u = n - k$ and $c_u + r_u = k$. U_u is different from both U_i and U_j. Since $r_u = r_i$ and $a_u > a_i$, $U_u \in$ add$U$. We obtained that f factors through U_u. Therefore, f is reducible in add$U$. This shows that such f does not yield any new irreducible module. Similarly, we can prove the other case. □

Second we will relate morphism between summands of $U_{v,\mathbf{w}}$, and morphisms between summands of $U_{w_K,\mathbf{xw_K}}$ where $w = xv$ and $v \in W^K_{max}$.

Lemma 3.29. *Let $w = xv$, where $v \in W^K_{max}$ and $\ell(w) = \ell(x) + \ell(v)$. Denote the cluster-tilting modules coming from standard reduced expressions for the pairs (w_K, xw_K) and (v, w) by U, U' respectively. Let $U_i, U_j \in$ ind$U$ and let $U'_i, U'_j \in$ indU' be the corresponding summands of U'. Then there exists a bijection between irreducible morphisms $U_i \to U_j$ in* add$U$ *and irreducible morphisms $U'_i \to U'_j$ in* addU'.

Proof. There are equivalences of categories $\mathcal{C}_x \xrightarrow{\sim} \mathcal{C}_{v,w}$ and $\mathcal{C}_x \xrightarrow{\sim} \mathcal{C}_{w_k,xw_K}$ [2]. In particular, the category $\mathcal{C}_{v,w}$ and $\mathcal{C}_{w_K,xw_K}$ are equivalent. By [10], this equivalence identifies the two cluster-tilting modules U and U'. This implies that there is a bijection between irreducible morphisms $U_i \to U_j$ in add$U$ and irreducible morphisms $U'_i \to U'_j$ in addU'. □

Together Proposition 3.28 and Lemma 3.29 prove Theorem 3.26. With Lemma 3.23 we have the following theorem.

Theorem 3.30. *Let $w = xv$ be a length-additive factorization and $v \in W^K_{max}$. For a standard reduced expression $\mathbf{w}$ of w, the labeled quiver $\Gamma_{U_{v,\mathbf{w}}}$ coincides with $Q_{v,w}$.*

Proof. By Definition 3.1 and Theorem 3.26, the quivers coincide. By the construction of $\Delta_{v^{-1}(J(b_j))}$ and Lemma 3.23, the labels of the vertices coincide as well. $\square$

4 Main Theorem

4.1 Main Theorem

In this section, there are two main theorems. Theorem 4.2 is generalized one, so we first prove Theorem 4.2 and then deduce Theorem 4.1 from Theorem 4.2.

Theorem 4.1. *[18] Consider the open Schubert variety X_λ° of $Gr_{k,n}$. Let G be a reduced plabic graph (with boundary vertices labeled clockwise from 1 to n) with the trip permutation $\pi_\lambda^{\swarrow}$. Construct the dual quiver of G and label its vertices by the Plücker coordinates given by the target labeling of G, to obtain a labeled seed Σ_G^{target}. Then the coordinate ring $\mathbb{C}[\hat{X}_\lambda^\circ]$ of the (affine cone over) X_λ° coincides with the cluster algebra $\mathcal{A}(\Sigma_G^{target})$.*

Theorem 4.2. *[18] Consider the skew Schubert variety $\pi_k(\mathcal{R}_{v,w})$, where $w \in W_{max}^K$ and w has a length-additive factorization $w = xv$. Let G be a reduced plabic graph (with boundary vertices labeled clockwise from 1 to n) with trip permutation $vw^{-1} = x^{-1}$, and such that boundary lollipops are white if and only if they are in $[k]$. Applying v^{-1} to the boundary vertices of G, obtaining the labeled graph $v^{-1}(G)$, and apply the target labeling to obtain the labeled seed $\Sigma_{v^{-1}(G)}^{target}$. Then the coordinate ring $\mathbb{C}[\widehat{\pi_k(\mathcal{R}_{v,w})}]$ of the (affine cone over) the skew Schubert variety $\pi_k(\mathcal{R}_{v,w})$ coincides with the cluster algebra $\mathcal{A}(\Sigma_{v^{-1}(G)}^{target})$.*

4.2 The proof of Theorem 4.2

Let $v \le w$ be permutations where $v \in W^K_{max}$ and $w = xv$ is a length-additive factorization. Let $\mathbf{w}'$ be a standard reduced expression for $w' := xw_K$ and let $G_{v,w}$ be the graph obtained from the bridge graph $B_{w_K,\mathbf{w}'}$ by applying v^{-1} to the boundary vertices. We label the faces of $G_{v,w}$ using the target labeling and let $Q_{v,w}$ be the labeled dual quiver of $G_{v,w}$ with the vertex labeled $v^{-1}([k])$ removed. We have shown that $Q_{v,w}$ is the rectangle seed and that $Q_{v,w}$ agrees with $\Gamma_{U_v,\mathbf{w}}$.

Let G be a plabic graph obtained from $G_{v,w}$ by a sequence of moves (M1)-(M3). The boundary faces of G have the same labels as the boundary faces of $G_{v,w}$. Let Q be the dual quiver of G, with the vertex labeled $v^{-1}([k])$ removed. A square move at a face of a plabic graph changes the dual quiver via mutation at the corresponding vertex. We can obtain Q from $Q_{v,w}$ by a sequence of mutations. On the other hand, let U and Γ be the one such that the same sequence of mutations performed to the corresponding cluster-tilting module $U_{v,\mathbf{w}}$ and its labeled quiver $\Gamma_{U_v,\mathbf{w}}$. Now, labeling Q with target labels, we claim that $Q = \Gamma_U$. Since two quivers are equal if we ignore the labels, we suffice to show that the labeling coincides. We first show that the face labels of G have the following property.

Definition 4.3. Let $I, J \in \binom{[n]}{k}$. We say I and J are *weakly separated* if for all $a, b \in I \setminus J$ and $c, d \in J \setminus I$ with $a < b$ and $c < d$, we never have that $a < c < b < d$ or $c < a < d < b$.

Proposition 4.4. *Let $v \le w$ be permutations where $v \in W^K_{max}$ and $w = xv$ is a length-additive factorization. Let G be a reduced graph that can be obtained from $G_{v,w}$ by a sequence of moves (M1)-(M3). If $I, J \in \mathcal{F}_{target}(G)$, then I and J are weakly separated.*

Proof. Since $H^{mir}_{v,w}$ is the graph obtained from $B_{w_K,\mathbf{w}'}$ by reflecting in the mirror and applying w^{-1} to the boundary vertices, there is one-to-one correspondence between faces of $G_{v,w}$ and faces of $H^{mir}_{v,w}$, and the target labels of corresponding faces in each graph agree. Performing a sequence of moves to corresponding faces of $G_{v,w}$ and $H^{mir}_{v,w}$ will result in two graphs with the same target face labels. So instead of considering the plabic graph G, we will consider the plabic graph H obtained by performing an analogous sequence of moves to $H^{mir}_{v,w}$.

First, we consider the case when $w = w_0$. Then H^{mir}_{v,w_0} is a normal plabic graph with boundary vertices labeled $1, \ldots, n$ going clockwise. It follows from [15] that $\mathcal{F}_{target}(H)$ consists of pairwise weakly separated sets.

Now, suppose $w < w_0$. Note that $H^{mir}_{v,w}$ can be obtained from H^{mir}_{v,w_0}. $H^{mir}_{v,w}$ is a subgraph of H^{mir}_{v,w_0}, whose boundary labels are inherited from the trips of H^{mir}_{v,w_0}. Therefore, we can perform a sequence of moves to this subgraph to obtain H as a subgraph of a reduced plabic graph. The weak separation of target labels H follows from [15]. □

This property is important because of following lemma, which will ensure that square moves on $G_{v,w}$ correspond to valid 3-term Plücker relations.

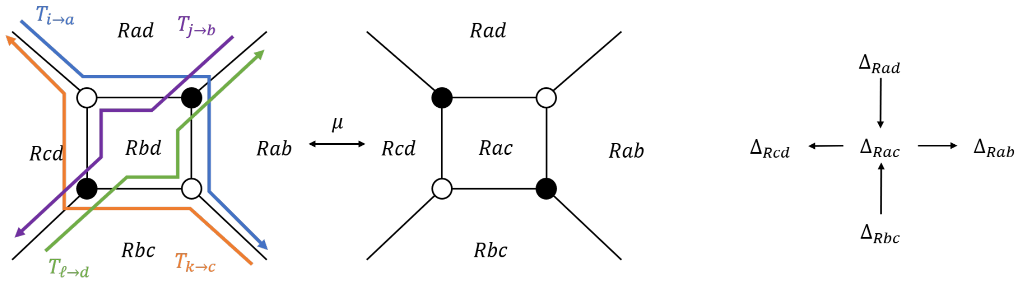


Figure 13: Plabic graph G' and G respectively, and the labeled quiver $Q(G)$.

Lemma 4.5. *Let G be a generalized plabic graph such that the element of $\mathcal{F}_{target}(G)$ are pairwise weakly separated, and let f be a square face of G whose vertices are all of degree 3. Suppose the trips coming into the vertices of f are $T_{i\to a}, T_{j\to b}, T_{k\to c}, T_{l\to d}$ reading clockwise around f (see Figure 13). Then a, b, c, d are cyclically ordered.*

Proof. The target labels of faces around f are Rab, Rbc, Rcd, Rad, where R is some $(k-2)$-element subset of $[n]$ and $Rab := R \cup \{a, b\}$. Since Rad and Rbc are weakly separated, either a, b, c, d or a, c, b, d is cyclically ordered. Similarly, since Rab and Rcd are weakly separated, a, b, c, d are cyclically ordered. □

We now show that if G is a generalized plabic graph move-equivalent to $G_{v,w}$, square moves on G agree with the categorical mutation of modules in $\mathcal{C}_{v,w}$. Together with Theorem 3.30, we complete the proof of Theorem 4.2.

Lemma 4.6. *Let G be a reduced plabic graph that is move-equivalent to $G_{v,w}$. Suppose that the (target) labeled quiver $Q(G) = \Gamma_U$, for some cluster-tilting module $U \in \mathcal{C}_{v,w}$. If G' is obtained from G by performing a square move at some face F of G, then*

$$Q(G') = \Gamma_{U'}$$

as labeled quivers, where U' denotes the mutation of U at the corresponding indecomposable summand U_F of U.

Proof. The label of the square face F and its surrounding faces are given in Figure 13. R is a $(k-2)$-element subset of $[n]$ and $Rac = R \cup \{a, c\}$. F has label Rac in G and after the mutation it has label Rbd. The target face labels of G are pairwise weakly separated by Proposition 4.4, and so a, b, c, d are cyclically ordered by Lemma 4.5. Now, consider the local configuration in Γ_U around the vertex Δ_{Rac} corresponding to the summand U_F of U. Then $U' = U/U_F \oplus U'_F$, where U'_F is defined by the two short exact sequences as follows.

$$0 \to U'_F \to U_{Rbc} \oplus U_{Rad} \to U_F \to 0 \quad 0 \to U_F \to U_{Rab} \oplus U_{Rcd} \to U'_F \to 0$$

Where we identify summand of U with the labels of the corresponding faces in G. Then cluster-character map φ yields the relation

$$\varphi_{U_F}\varphi_{U'_F} = \varphi_{U_{Rbc}}\varphi_{U_{Rad}} + \varphi_{U_{Rab}}\varphi_{U_{Rcd}}$$

Note that if one of the faces adjacent to F has label $v^{-1}([k])$ then the associated module $U_{v^{-1}([k])}$ is the zero module and $\varphi_{U_{v^{-1}([k])}} = \Delta_{v^{-1}([k])} = 1$ by Remark 3.16. Hence, in this case, the above relation still holds. Since the two labeled quiver $Q(G)$ and Γ_U coincide, each function $\varphi_{U_E} \in \mathbb{C}[\mathcal{R}_{w_k,w}]$, where E is a face in G, is a Plücker coordinate coming from the label of the face. Therefore we have the following.

$$\varphi_{U_F} = \varphi_{U_{Rac}} = \Delta_{Rac} \quad \varphi_{U_{Rab}} = \Delta_{Rab} \quad \varphi_{U_{Rbc}} = \Delta_{Rbc}$$
$$\varphi_{U_{Rcd}} = \Delta_{Rcd} \quad \varphi_{U_{Rad}} = \Delta_{Rad}$$

Therefore the relation above will be

$$\Delta_{Rac}\varphi_{U'_F} = \Delta_{Rbc}\Delta_{Rad} + \Delta_{Rab}\Delta Rcd$$

which is a 3-term Plücker relation in the corresponding skew Schubert variety. Thus we conclude that $\varphi_{U'_F} = \Delta_{Rbc}$. This shows that $Q(G')$ and $\Gamma_{U'}$ agree. $\square$

Remark 4.7. Note that all graphs in Lemma 3.15 give rise to the same labeled seed (up to reversing all arrows in the quiver, which does not affect mutation). And a sequence of moves on any one can be translated to a sequence of moves on any other that effects the dual quiver in the same way. Hence any reduced plabic graph move-equivalent to a graph in lemma give rise to a seed for $\pi_k(\mathcal{R}_{v,w})$.

4.3 Deduce Theorem 4.1 from Theorem 4.2

We will show how to deduce Theorem 4.1 from Theorem 4.2.

Proof. Recall that for $v \in W^K_{max}$, $\pi_k(\mathcal{R}_{v,w_0}) = X^\circ_\lambda$, where $V^{\swarrow}(\lambda) = v^{-1}([k])$. The decorated permutation corresponding to $\pi_k(\mathcal{R}_{v,w_0})$ is $v^{-1}w_0$.

Recall also that we can obtain v^{-1} in list notation from λ by labeling the southeast border of λ with $1, \ldots, n$ going southwest and first reading the labels of vertical steps and then reading the labels of vertical steps going northeast and then reading the labels of the horizontal steps going northeast. For $v^{-1}w_0$, we reverse order in which we read the border of λ, first reading the labels of horizontal steps going southwest and then reading the labels of the vertical steps going southwest. So $v{-}1w_0$ is equal to the permutation $\pi^{\swarrow}_\lambda$.

Let $x := w_0v^{-1}$ and then factorization $w_0 = xv$ is length-additive. Let $\mathbf{w}'$ be a standard reduced expression for $w' := xw_K$. If we take $B_{w_K,\mathbf{w}'}$, apply w_0^{-1} to the boundary vertices, and "reflect in the mirror", we obtain a graph H^{mir}_{v,w_0} which has trip permutation $\pi^{\swarrow}_\lambda$ and whose boundary vertices are labeled with $1, \ldots, n$ clockwise. Therefore, by Theorem 3.30 and Lemma 3.15, we obtain a seed for the coordinate ring of (the affine cone over) X°_λ. If G is any reduced plabic graph move-equivalent to H^{mir}_{v,w_0}, then the (target) labeled dual quiver $Q(G)$ gives a seed. $\square$

4.4 Application related to the coordinate rings

Combining main theorem with some known property [13] and [14], we obtain the following corollary.

Corollary 4.8. *Let $v \leq w$, where $v \in W^K_{max}$ and $w = xv$ is length-additive. Then the cluster algebra $\mathbb{C}[\widehat{\pi_k(\mathcal{R}_{v,w}})]$ is locally acyclic, and thus is finitely generated, integrally closed, locally a complete intersection, and equal to its own upper cluster algebra.*

Combining with [5], we can find that the quiver giving rise to the cluster structures for Schubert and skew Schubert varieties admit *green-to-red sequences* which implies that the cluster algebra have *Enough Global Monomials* [6]. Therefore, we obtain the following corollary.

Corollary 4.9. *Let $v \leq w$, where $v \in W^K_{max}$ and $w = xv$ is length-additive. Then the cluster algebra $\mathbb{C}[\widehat{\pi_k(\mathcal{R}_{v,w}})]$ has a canonical basis of theta functions, parametrized by the lattice of g-vectors.*

References

[1] I. Assem, D. Simson, A. Skowronski, et al. *Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory*. Cambridge University Press, 2006.

[2] P. Baumann, J. Kamnitzer, and P. Tingley. Affine mirković-vilonen polytopes. *Publications mathématiques de l'IHÉS*, 120:113–205, 2014.

[3] S. Fomin, L. Williams, and A. Zelevinsky. Introduction to cluster algebras. chapters 1-3, preprint (2016). *arXiv preprint arXiv:1608.05735*, 2016.

[4] S. Fomin and A. Zelevinsky. Cluster algebras i: foundations. *Journal of the American Mathematical Society*, 15(2):497–529, 2002.

[5] N. Ford and K. Serhiyenko. Green-to-red sequences for positroids. *Journal of Combinatorial Theory, Series A*, 159:164–182, 2018.

[6] M. Gross, P. Hacking, S. Keel, and M. Kontsevich. Canonical bases for cluster algebras. *Journal of the American Mathematical Society*, 31(2):497–608, 2018.

[7] R. Karpman. Bridge graphs and deodhar parametrizations for positroid varieties. *Journal of Combinatorial Theory, Series A*, 142:113–146, 2016.

[8] A. Knutson, T. Lam, and D. E. Speyer. Positroid varieties: juggling and geometry. *Compositio Mathematica*, 149(10):1710–1752, 2013.

[9] V. Lakshmibai and J. Brown. The grassmannian variety. In *Developments in Mathematics*, volume 42. Springer, 2015.

[10] B. Leclerc. Cluster structures on strata of flag varieties. *Advances in Mathematics*, 300:190–228, 2016.

[11] K. Lee and R. Schiffler. Positivity for cluster algebras. *Annals of Mathematics*, pages 73–125, 2015.

[12] R. J. Marsh. Lecture notes on cluster algebras. *AMC*, 10:12, 2014.

[13] G. Muller. Locally acyclic cluster algebras. *Advances in Mathematics*, 233(1):207–247, 2013.

[14] G. Muller and D. Speyer. Cluster algebras of grassmannians are locally acyclic. *Proceedings of the American Mathematical Society*, 144(8):3267–3281, 2016.

[15] S. Oh, A. Postnikov, and D. E. Speyer. Weak separation and plabic graphs. *Proceedings of the London Mathematical Society*, 110(3):721–754, 2015.

[16] A. Postnikov. Total positivity, grassmannians, and networks. *arXiv preprint math/0609764*, 2006.

[17] J. S. Scott. Grassmannians and cluster algebras. *Proceedings of the London Mathematical Society*, 92(2):345–380, 2006.

[18] K. Serhiyenko, M. Sherman-Bennett, and L. Williams. Cluster structures in schubert varieties in the grassmannian. *Proceedings of the London Mathematical Society*, 119(6):1694–1744, 2019.

[19] J. R. Stembridge. On the fully commutative elements of coxeter groups. *Journal of Algebraic Combinatorics*, 5(4):353–385, 1996.

국문초록

이 논문에서는 어떻게 그라스만 다양체 안에 있는 슈베르츠 다양체의 좌표환이 Postnikov의 플래빅 그래프를 이용하여 클러스터 대수랑 동일시할 수 있는지 보여준다. 이는 Scott의 정리를 일반화 한것이고 슈베르츠 다양체에 대한 추측을 증명한 것이다. 이 추측을 증명하기 위해 리차드슨 다양체의 좌표 환이 클러스터 대수랑 동일시 될 수 있다는 Leclerc의 결과를 이용한다. Karpman의 설계를 이용하여 축약된 표현과 관련이 있는 플래빅 그래프를 만든다. 마지막으로 일반화된 플래빅 그래프를 이용하여 꼬인 슈베르츠 다양체로 결과를 일반화시킨다.

주요어 : 클러스터 대수, 슈베르츠 다양체
학번 : 2021–25573