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이학석사 학위논문

Superconformal Structures of Supersymmetric Free
Fermion Vertex Algebras

(초대칭 자유페르미온 꼭지점대수의 초등각대칭 구조)

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Superconformal Structures of Supersymmetric Free Fermion Vertex Algebras

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by

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Abstract

In this paper, we define the shifted superconformal vector of supersymmetric charged free fermion vertex algebras, which is a 1-parameter deformation of the superconformal vector of the SUSY $bc\text{-}\beta\gamma$ system. Moreover, we find the corresponding shifted $N = 2$ superconformal symmetry of SUSY charged free fermion vertex algebras, by using the $N_K = 1$ SUSY vertex algebra formalism. Finally, in order to describe the shifted $N = 2$ superconformal symmetry of the SUSY charged free fermion vertex algebra by $N = 2$ superfields, we construct an $N_K = 2$ SUSY version of the $bc\text{-}\beta\gamma$ system.

Key words: Supersymmetric Vertex Algebras, Superconformal Vertex Algebras, Supersymmetric Free Fermion Vertex Algebras, Superconformal Vectors
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Chapter 1

Introduction

Superconformal symmetries have been studied widely in both mathematics and physics literature, see for example [1, 9, 16, 20]. Especially in the theory of vertex algebras, superconformal vectors give rise to the superconformal structures of vertex algebras (see [13]).

In this paper, we study the superconformal structure and the deformation of superconformal vectors of the supersymmetric charged free fermion vertex algebras. In [4, 14], the non-SUSY charged free fermion vertex algebra appears as a ghost part of the non-SUSY W -algebra. Just as in the non-SUSY case, the SUSY charged free fermion vertex algebra is the ghost part of the SUSY W -algebra, which was studied in [17, 18]. On the other hand, conformal vectors play an important role in the theory of vertex algebras, for example, they induce the notion of energy-momentum fields (see [13]). Also, superconformal vectors can be regarded as supersymmetric analogues of conformal vectors.

Despite some difficulties in finding superconformal vectors in general, the constructions of superconformal vectors were studied in some cases. For example, the superconformal vector of the non-SUSY free fermion vertex algebra was studied in [9], and the Kac-Todorov construction (see [15]) of the non-SUSY affine vertex algebras was given in [13]. In the SUSY case, some basics about the superconformal structures of SUSY vertex algebras and especially superconformal vectors of the SUSY affine vertex algebras and the SUSY bc - $\beta\gamma$ system can be found in [11].

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In the context of BRST cohomology, deformations of the (super)conformal vectors are also important, in addition to the construction of (super)conformal vectors. In [5], the 1-parameter deformation of the conformal vector of the non-SUSY bc - $\beta\gamma$ system was introduced. Also, the modification of the Sugawara construction of the conformal vector of the non-SUSY affine vertex algebra was used in [4, 14], and for the SUSY counterpart, the modified Kac-Todorov superconformal vector of the SUSY affine vertex algebra was presented in [12].

In the present paper, we give a 1-parameter deformation of the superconformal vector of the SUSY bc - $\beta\gamma$ system (Theorem 4.6), which is called the shifted superconformal vector of the SUSY charged free fermion vertex algebras, in order to obtain varying conformal weights of monomial fields of the SUSY charged free fermion vertex algebras. Also we find an $N = 2$ superconformal symmetry (Theorem 4.14), compatible with our shifted superconformal vector.

The organization of this paper is as follows. In Chapter 2, following [11, 13], we briefly review the definitions and basic properties of the vertex algebras and the SUSY vertex algebras. In Chapter 3, we recall the definitions of superconformal vertex algebras in the context of the vertex algebras and the SUSY vertex algebras. In Chapter 4, we state the main results of this work. We define the shifted superconformal vector of the supersymmetric charged free fermion vertex algebras and find the corresponding $N = 2$ superconformal symmetry. At the end of this chapter, we describe the shifted $N = 2$ superconformal structure, using the $N_K = 2$ SUSY vertex algebra formalism. In Appendix A, we present some calculations of lambda brackets of the SUSY charged free fermion vertex algebras. Appendix B contains the charge decomposition of the SUSY charged free fermion vertex algebras.

Chapter 2

Vertex algebras and SUSY vertex algebras

In this chapter, we recollect the definitions of vertex algebras and SUSY vertex algebras. We only consider the $N_K = 1$ and $N_K = 2$ SUSY vertex algebras. The main references are [4], [11] and [13].

2.1 Vertex algebras

Definition 2.1. [13] Let V be a vector superspace. Let z be an even indeterminate. A *field* is an $\text{End}(V)$ -valued formal distribution of the form

$$a(z) = \sum_{j \in \mathbb{Z}} z^{-j-1} a_{(j)}, \quad (2.1)$$

where $a_{(j)}v = 0$ for all but finitely many $j \in \mathbb{Z}$, for each $v \in V$.

Definition 2.2. [4, 13] A *vertex algebra* $(V, |0\rangle, \partial, Y)$ consists of a vector superspace V over \mathbb{C} , an even vector $|0\rangle$, an even endomorphism ∂ , and the *state-field correspondence* Y which is a parity preserving linear map from V to the space of $\text{End}(V)$ -valued fields, such that the following axioms hold :

- (Vacuum)

$$\partial |0\rangle = 0, \quad Y(a, z) |0\rangle |_{z=0} = a, \quad (2.2)$$

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- (Translation covariance)

$$[\partial, Y(a, z)] = \partial_z Y(a, z), \quad (2.3)$$

- (Locality) for any $a, b \in V$, there exists $n \in \mathbb{Z}_{\geq 0}$ such that

$$(z - w)^n [Y(a, z), Y(b, w)] = 0. \quad (2.4)$$

The *normally ordered product* on V is defined by

$$: ab := a_{(-1)}b. \quad (2.5)$$

In this paper, we just denote by ab the normally ordered product and $: a : bc ::$ is denoted by abc .

Definition 2.3. [4] A *Lie conformal algebra* is a $\mathbb{Z}/2\mathbb{Z}$ -graded $\mathbb{C}[\partial]$ -module R with a λ -bracket which is a parity preserving \mathbb{C} -bilinear map :

$$[\lambda] : R \otimes R \rightarrow \mathbb{C}[\lambda] \otimes R, \quad a \otimes b \mapsto [a \lambda b], \quad (2.6)$$

satisfying the following conditions :

- (Sesquilinearity)

$$[\partial a \lambda b] = -\lambda [a \lambda b], \quad [a \lambda \partial b] = (\partial + \lambda)[a \lambda b], \quad (2.7)$$

- (Skew-symmetry)

$$[b \lambda a] = (-1)^{ab+1} [a \lambda b], \quad (2.8)$$

- (Jacobi identity)

$$[a \lambda [b \mu c]] = [[a \lambda b] \lambda + \mu c] + (-1)^{ab} [b \mu [a \lambda c]], \quad (2.9)$$

where we write the λ -brackets as :

$$[a \lambda b] = \sum_{j \in \mathbb{Z}_{\geq 0}} \frac{\lambda^j}{j!} a_{(j)} b. \quad (2.10)$$

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Definition 2.4. [4] A *vertex algebra* is a tuple $(V, \partial, [\lambda], |0\rangle, : :)$ such that :

- $(V, \partial, [\lambda])$ is a Lie conformal algebra,
- $(V, \partial, |0\rangle, : :)$ is a unital differential superalgebra with a derivation ∂ , satisfying the following properties :

(Quasi-commutativity)

$$ab - (-1)^{ab}ba = \int_{-\partial}^0 [a \lambda b] d\lambda, \quad (2.11)$$

(Quasi-associativity)

$$(ab)c - a(bc) = \left(\int_0^\partial d\lambda a \right) [b \lambda c] + (-1)^{ab} \left(\int_0^\partial d\lambda b \right) [a \lambda c], \quad (2.12)$$

- the λ -bracket and the product $: :$ are related by :

(Non-commutative Wick formula)

$$[a \lambda bc] = [a \lambda b]c + (-1)^{ab}b[a \lambda c] + \int_0^\lambda [[a \lambda b] \mu c] d\mu. \quad (2.13)$$

Example 2.5. [4, 14] Let U be a finite-dimensional vector superspace and A be a basis of U . Define two vector superspaces :

$$\varphi_U \simeq \Pi U, \quad \varphi^U \simeq \Pi U^*, \quad (2.14)$$

whose basis elements are denoted by φ_a and φ^a respectively. Here Π denotes the parity reversing functor and U^* is the dual vector space of U . Define the λ -brackets on $R^{ch} = \mathbb{C}[\partial] \otimes (\varphi_U \oplus \varphi^U)$ as :

$$[\varphi_a \lambda \varphi^b] = \delta_{ab}, \quad [\varphi_a \lambda \varphi_b] = [\varphi^a \lambda \varphi^b] = 0, \quad (2.15)$$

for $a, b \in A$. Then R^{ch} is a Lie conformal algebra, and the *charged free fermion vertex algebra* F^{ch} associated to the vector superspace U is the universal enveloping vertex algebra $V(R^{ch})$.

Example 2.6. [6] Let U be the vector superspace with the basis $A = A_{\bar{0}} \cup A_{\bar{1}}$ be a basis of U , where $A_{\bar{0}} = \{\beta, \gamma\}$ is the even part and $A_{\bar{1}} = \{b, c\}$ is the odd part of the basis. Define the λ -brackets on $R^{bc\beta\gamma} = \mathbb{C}[\partial] \otimes U$ as :

$$\begin{aligned} [\beta \lambda \gamma] &= 1, & [\gamma \lambda \beta] &= -1, \\ [b \lambda c] &= 1, & [c \lambda b] &= 1, \end{aligned} \quad (2.16)$$

and other λ -brackets on the basis elements are zeros. Then $R^{bc\beta\gamma}$ is a Lie conformal algebra, and the bc - $\beta\gamma$ system is the universal enveloping vertex algebra $V(R^{bc\beta\gamma})$.

2.2 SUSY vertex algebras

In this section, we recall the definitions of the $N_K = 1$ and $N_K = 2$ SUSY vertex algebras as in [11].

Definition 2.7. [11] Let V be a vector superspace. Let z be an even indeterminate and θ be an odd indeterminate such that $\theta^2 = 0$ and $\theta z = z\theta$. An $N = 1$ superfield is an $\text{End}(V)$ -valued formal distribution of the form

$$a(z, \theta) = \sum_{j \in \mathbb{Z}} z^{-j-1} a_{(j|1)} + \theta \sum_{j \in \mathbb{Z}} z^{-j-1} a_{(j|0)}, \quad (2.1)$$

where $a_{(j|*)}v = 0$ for all but finitely many $j \in \mathbb{Z}$, for each $v \in V$.

Definition 2.8. [11] An $N_K = 1$ supersymmetric vertex algebra is a tuple $(V, |0\rangle, D, Y)$ consisting of a vector superspace V over \mathbb{C} , an even vector $|0\rangle$, an odd endomorphism D , and the state-field correspondence Y which is a parity preserving linear map from V to the space of $\text{End}(V)$ -valued $N = 1$ superfields, such that the following axioms hold :

- (Vacuum)

$$D|0\rangle = 0, \quad Y(a, z, \theta)|0\rangle|_{z=0, \theta=0} = a, \quad (2.2)$$

- (Translation covariance)

$$[D, Y(a, z, \theta)] = (\partial_\theta - \theta\partial_z)Y(a, z, \theta), \quad (2.3)$$

- (Locality) for any $a, b \in V$, there exists $n \in \mathbb{Z}_{\geq 0}$ such that

$$(z - w)^n [Y(a, z, \theta), Y(b, w, \zeta)] = 0. \quad (2.4)$$

The normally ordered product on V is defined by

$$:ab := a_{(-1|1)}b. \quad (2.5)$$

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Remark 2.9. [11] Consider the noncommutative associative superalgebra $\mathcal{H}_{N=1} = \mathbb{C}[\partial, D]$ generated by an even generator ∂ and an odd generator D , with relations :

$$D^2 = \partial, \quad [\partial, D] = 0, \quad (2.6)$$

and consider the noncommutative associative superalgebra $\mathcal{L}_{N=1} = \mathbb{C}[\lambda, \chi]$ generated by an even generator λ and an odd generator χ , with relations :

$$\chi^2 = -\lambda, \quad [\lambda, \chi] = 0. \quad (2.7)$$

Definition 2.10. [11, 18] An $N_K = 1$ *supersymmetric Lie conformal algebra* is a $\mathbb{Z}/2\mathbb{Z}$ -graded $\mathcal{H}_{N=1}$ -module R with a Λ -bracket which is a \mathbb{C} -bilinear map of degree 1 :

$$[\]_{\Lambda} : R \otimes R \rightarrow \mathcal{L}_{N=1} \otimes R, \quad a \otimes b \mapsto [a \]_{\Lambda} b, \quad (2.8)$$

satisfying the following conditions :

- (Sesquilinearity)

$$[Da \]_{\Lambda} b] = \chi[a \]_{\Lambda} b], \quad [a \]_{\Lambda} Db] = (-1)^{a+1}(D + \chi)[a \]_{\Lambda} b], \quad (2.9)$$

where D and χ are subject to the relation $[D, \chi] = 2\lambda$,

- (Skew-symmetry)

$$[b \]_{\Lambda} a] = (-1)^{ab}[a \]_{-\nabla - \Lambda} b], \quad (2.10)$$

where $-\nabla - \Lambda = (-\partial - \lambda, -D - \chi)$ with the relations

$$[D, \chi] = 2\lambda, \quad [\partial, \lambda] = [\partial, \chi] = [D, \lambda] = 0, \quad (2.11)$$

- (Jacobi identity)

$$[a \]_{\Lambda} [b \]_{\Gamma} c] = (-1)^{a+1}[[a \]_{\Lambda} b] \]_{\Lambda + \Gamma} c] + (-1)^{(a+1)(b+1)}[b \]_{\Gamma} [a \]_{\Lambda} c], \quad (2.12)$$

where $\Gamma = (\gamma, \eta)$ and $\Lambda + \Gamma = (\lambda + \gamma, \chi + \eta)$ with relations :

$$[\eta, \eta] = -2\gamma, \quad [\gamma, \gamma] = [\gamma, \eta] = [\lambda, \gamma] = [\lambda, \eta] = [\chi, \gamma] = [\chi, \eta] = 0. \quad (2.13)$$

We write the Λ -brackets as :

$$[a \]_{\Lambda} b] = \sum_{j \in \mathbb{Z}_{\geq 0}} \frac{\lambda^j}{j!} a_{(j|0)} b + \chi \sum_{j \in \mathbb{Z}_{\geq 0}} \frac{\lambda^j}{j!} a_{(j|1)} b. \quad (2.14)$$

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Definition 2.11. [11] An $N_K = 1$ *supersymmetric vertex algebra* is a tuple $(V, D, [\ _\ _], |0\rangle, : :)$ such that :

- $(V, D, [\ _\ _])$ is an $N_K = 1$ SUSY Lie conformal algebra,
- $(V, D, |0\rangle, : :)$ is a unital differential superalgebra with an odd derivation D , satisfying the following properties :
(Quasi-commutativity)

$$ab - (-1)^{ab}ba = \int_{-\nabla}^0 [a \ _\ _ b] d\Lambda, \quad (2.15)$$

(Quasi-associativity)

$$(ab)c - a(bc) = \left(\int_0^\nabla d\Lambda a \right) [b \ _\ _ c] + (-1)^{ab} \left(\int_0^\nabla d\Lambda b \right) [a \ _\ _ c], \quad (2.16)$$

- the Λ -bracket and the product $: :$ are related by :
(Non-commutative Wick formula)

$$[a \ _\ _ bc] = [a \ _\ _ b]c + (-1)^{(a+1)b}b[a \ _\ _ c] + \int_0^\Lambda [[a \ _\ _ b] \ _\ _ c] d\Gamma, \quad (2.17)$$

where the integral $\int_0^\Lambda d\Gamma$ is computed as $\partial_\eta \int_0^\lambda d\gamma$.

Example 2.12. [18] Let U be a finite-dimensional vector superspace and $A = A_{\bar{0}} \cup A_{\bar{1}}$ be a basis of U , where $A_{\bar{0}}$ is the even part and $A_{\bar{1}}$ is the odd part of the basis. Define two vector superspaces :

$$\phi_U \simeq U, \quad \phi_{\bar{U}} \simeq \Pi U^*, \quad (2.18)$$

whose basis elements are denoted by ϕ_a and $\phi_{\bar{a}}$ respectively. Define the Λ -brackets on $R_{N=1}^{ch} = \mathcal{H}_{N=1} \otimes (\phi_U \oplus \phi_{\bar{U}})$ as :

$$[\phi_a \ _\ _ \phi_{\bar{b}}] = \delta_{ab}, \quad [\phi_a \ _\ _ \phi_b] = [\phi_{\bar{a}} \ _\ _ \phi_{\bar{b}}] = 0, \quad (2.19)$$

for $a, b \in A$. Then $R_{N=1}^{ch}$ is an $N_K = 1$ supersymmetric Lie conformal algebra, and the *supersymmetric charged free fermion vertex algebra* $F_{N=1}^{ch}$ associated to the vector superspace U is the universal enveloping supersymmetric vertex algebra $V(R_{N=1}^{ch})$. For the description of the universal enveloping supersymmetric vertex algebras, see Definition 2.13 of [18].

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Definition 2.13. [11] Let V be a vector superspace. Let z be an even indeterminate and θ^1, θ^2 be odd indeterminates such that $[\theta^i, \theta^j] = 0$ and $[\theta^i, z] = 0$. An $N = 2$ *superfield* is an $\text{End}(V)$ -valued formal distribution of the form

$$\begin{aligned} a(z, \theta^1, \theta^2) = & \sum_{j \in \mathbb{Z}} z^{-j-1} a_{(j|11)} + \theta^1 \sum_{j \in \mathbb{Z}} z^{-j-1} a_{(j|01)} \\ & + \theta^2 \sum_{j \in \mathbb{Z}} z^{-j-1} a_{(j|10)} + \theta^1 \theta^2 \sum_{j \in \mathbb{Z}} z^{-j-1} a_{(j|00)}, \end{aligned} \quad (2.20)$$

where $a_{(j|**)}v = 0$ for all but finitely many $j \in \mathbb{Z}$, for each $v \in V$.

Definition 2.14. [11] An $N_K = 2$ *supersymmetric vertex algebra* is a tuple $(V, |0\rangle, D^1, D^2, Y)$ consisting of a vector superspace V over \mathbb{C} , an even vector $|0\rangle$, an odd endomorphisms D^1, D^2 and the *state-field correspondence* Y which is a parity preserving linear map from V to the space of $\text{End}(V)$ -valued $N = 2$ superfields, such that the following axioms hold :

- (Vacuum)

$$D^i |0\rangle = 0, \quad Y(a, z, \theta^1, \theta^2) |0\rangle |_{z=0, \theta^i=0} = a, \quad (2.21)$$

- (Translation covariance)

$$[D^i, Y(a, z, \theta^1, \theta^2)] = (\partial_{\theta^i} - \theta^i \partial_z) Y(a, z, \theta^1, \theta^2), \quad (2.22)$$

- (Locality) for any $a, b \in V$, there exists $n \in \mathbb{Z}_{\geq 0}$ such that

$$(z - w)^n [Y(a, z, \theta^1, \theta^2), Y(b, w, \zeta^1, \zeta^2)] = 0. \quad (2.23)$$

The *normally ordered product* on V is defined by

$$: ab := a_{(-1|11)} b. \quad (2.24)$$

Remark 2.15. [11] Consider the noncommutative associative superalgebra $\mathcal{H}_{N=2} = \mathbb{C}[\partial, D^1, D^2]$ generated by an even generator ∂ and odd generators D^1, D^2 with relations :

$$[D^i, D^j] = 2\delta_{ij}\partial, \quad [\partial, D^i] = 0, \quad (2.25)$$

and the noncommutative associative superalgebra $\mathcal{L}_{N=2} = \mathbb{C}[\lambda, \chi^1, \chi^2]$ generated by an even generator λ and odd generators χ^1, χ^2 with relations :

$$[\chi^i, \chi^j] = -2\delta_{ij}\lambda, \quad [\lambda, \chi^i] = 0. \quad (2.26)$$

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Definition 2.16. [8, 11] An $N_K = 2$ supersymmetric Lie conformal algebra is a $\mathbb{Z}/2\mathbb{Z}$ -graded $\mathcal{H}_{N=2}$ -module R with a Λ -bracket which is a parity preserving \mathbb{C} -bilinear map :

$$[\]_{\Lambda} : R \otimes R \rightarrow \mathcal{L}_{N=2} \otimes R, \quad a \otimes b \mapsto [a \]_{\Lambda} b, \quad (2.27)$$

satisfying the following conditions :

- (Sesquilinearity)

$$[D^i a \]_{\Lambda} b = -\chi^i [a \]_{\Lambda} b, \quad [a \]_{\Lambda} D^i b = (-1)^a (D^i + \chi^i) [a \]_{\Lambda} b, \quad (2.28)$$

where D^i and χ^i are subject to the relation $[D^i, \chi^j] = 2\delta_{ij}\lambda$,

- (Skew-symmetry)

$$[b \]_{\Lambda} a = (-1)^{ab+1} [a \]_{-\nabla-\Lambda} b, \quad (2.29)$$

where $-\nabla - \Lambda = (-\partial - \lambda, -D^1 - \chi^1, -D^2 - \chi^2)$ with the relations

$$[D^i, \chi^j] = 2\delta_{ij}\lambda, \quad [\partial, \lambda] = [\partial, \chi^i] = [D^i, \lambda] = 0, \quad (2.30)$$

- (Jacobi identity)

$$[a \]_{\Lambda} [b \]_{\Gamma} c = [[a \]_{\Lambda} b]_{\Lambda+\Gamma} c + (-1)^{ab} [b \]_{\Gamma} [a \]_{\Lambda} c, \quad (2.31)$$

where $\Gamma = (\gamma, \eta^1, \eta^2)$ and $\Lambda + \Gamma = (\lambda + \gamma, \chi^1 + \eta^1, \chi^2 + \eta^2)$ with relations :

$$\begin{aligned} [\eta^i, \eta^j] &= -2\delta_{ij}\gamma, & [\gamma, \gamma] &= [\gamma, \eta^i] = [\lambda, \gamma] \\ & & &= [\lambda, \eta^i] = [\chi^i, \gamma] = [\chi^i, \eta^j] = 0. \end{aligned} \quad (2.32)$$

We write the Λ -brackets as :

$$\begin{aligned} [a \]_{\Lambda} b &= \sum_{j \geq 0} \frac{\lambda^j}{j!} a_{(j|00)} b - \chi^1 \sum_{j \geq 0} \frac{\lambda^j}{j!} a_{(j|10)} b \\ &\quad - \chi^2 \sum_{j \geq 0} \frac{\lambda^j}{j!} a_{(j|01)} b - \chi^1 \chi^2 \sum_{j \geq 0} \frac{\lambda^j}{j!} a_{(j|11)} b. \end{aligned} \quad (2.33)$$

Definition 2.17. [11] An $N_K = 2$ supersymmetric vertex algebra is a tuple $(V, D^1, D^2, [\]_{\Lambda}, |0\rangle, \cdot, \cdot)$ such that :

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- $(V, D^1, D^2, [\ \Lambda \])$ is an $N_K = 2$ SUSY Lie conformal algebra,
- $(V, D^1, D^2, |0\rangle, : \ :)$ is a unital differential superalgebra with odd derivations D^1 and D^2 satisfying the following properties :
(Quasi-commutativity)

$$ab - (-1)^{ab}ba = \int_{-\nabla}^0 [a \ \Lambda \ b] d\Lambda, \quad (2.34)$$

(Quasi-associativity)

$$(ab)c - a(bc) = \left(\int_0^{\nabla} d\Lambda a \right) [b \ \Lambda \ c] + (-1)^{ab} \left(\int_0^{\nabla} d\Lambda b \right) [a \ \Lambda \ c], \quad (2.35)$$

- the Λ -bracket and the product $: \ :$ are related by :
(Non-commutative Wick formula)

$$[a \ \Lambda \ bc] = [a \ \Lambda \ b]c + (-1)^{ab}b[a \ \Lambda \ c] + \int_0^{\Lambda} [[a \ \Lambda \ b] \ \Gamma \ c] d\Gamma, \quad (2.36)$$

where the integral $\int_0^{\Lambda} d\Gamma$ is $\partial_{\eta^1} \partial_{\eta^2} \int_0^{\lambda} d\gamma$.

Chapter 3

Superconformal vertex algebras

In this chapter, we review the definitions of $N = 1$ and $N = 2$ superconformal vertex algebras in the context of the vertex algebras and the SUSY vertex algebras. Details of the definitions can be found in [11] and [13].

Definition 3.1. [13] Let V be a vertex algebra. An even vector $L \in V$ is called a *conformal vector* if it satisfies the following conditions :

- $Y(L, z)$ is a Virasoro field with central charge c , i.e.

$$[L \lambda L] = (\partial + 2\lambda)L + \frac{c}{12}\lambda^3, \quad (3.1)$$

- $L_{(0)} = \partial$,
- $L_{(1)}$ is diagonalizable on V .

For $v \in V$, we say that v has *conformal weight* $\Delta \in \mathbb{C}$ if :

$$[L \lambda v] = (\partial + \Delta\lambda)v + O(\lambda^2), \quad (3.2)$$

where $O(\lambda^2)$ is a polynomial in λ which has no constant and linear terms. Moreover, if there is no $O(\lambda^2)$ term in (3.2), the vector v is called *primary*.

Definition 3.2. [10, 13] The $N = 1$ *superconformal vertex algebra* is generated by a conformal vector L , and an odd vector G which is called a *super-*

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conformal vector, satisfying the following super-Virasoro relation :

$$\begin{aligned} [L \ \lambda \ L] &= (\partial + 2\lambda)L + \frac{c}{12}\lambda^3, \\ [L \ \lambda \ G] &= (\partial + \frac{3}{2}\lambda)G, \\ [G \ \lambda \ G] &= 2L + \frac{c}{3}\lambda^2. \end{aligned} \tag{3.3}$$

Definition 3.3. [10, 13] The $N = 2$ *superconformal vertex algebra* is generated by a conformal vector L , an even vector J , and two odd vectors G^+ , G^- , satisfying the following λ -bracket relations :

$$\begin{aligned} [L \ \lambda \ L] &= (\partial + 2\lambda)L + \frac{c}{12}\lambda^3, \quad [L \ \lambda \ G^\pm] = (\partial + \frac{3}{2}\lambda)G^\pm, \\ [G^\pm \ \lambda \ G^\pm] &= 0, \quad [G^+ \ \lambda \ G^-] = L + (\frac{1}{2}\partial + \lambda)J + \frac{c}{6}\lambda^2, \\ [L \ \lambda \ J] &= (\partial + \lambda)J, \quad [G^\pm \ \lambda \ J] = \mp G^\pm, \quad [J \ \lambda \ J] = \frac{c}{3}\lambda. \end{aligned} \tag{3.4}$$

Definition 3.4. [11] Let V be an $N_K = 1$ SUSY vertex algebra. A vector $T \in V$ is called an $N = 1$ *superconformal vector* if it satisfies the following conditions :

- $Y(T, z, \theta)$ is a super-Virasoro field with central charge c , i.e.

$$[T \ \Lambda \ T] = (2\partial + 3\lambda + \chi D)T + \frac{c}{3}\lambda^2\chi, \tag{3.5}$$

- $T_{(0|0)} = 2\partial$, $T_{(0|1)} = D$,
- $T_{(1|0)}$ is diagonalizable on V .

For $v \in V$, we say that v has *conformal weight* $\Delta \in \mathbb{C}$ if :

$$[T \ \Lambda \ v] = (2\partial + 2\Delta\lambda + \chi D)v + O(\Lambda^2), \tag{3.6}$$

where $O(\Lambda^2)$ is a polynomial in Λ which has no constant and linear terms. Moreover, if there is no $O(\Lambda^2)$ term in (3.6), the vector v is called *primary*.

Definition 3.5. [10, 11] As an $N_K = 1$ SUSY vertex algebra, the $N = 2$ superconformal vertex algebra is generated by an $N = 1$ superconformal

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vector T and an even vector J satisfying the following $N_K = 1$ Λ -bracket relations :

$$\begin{aligned} [T \underset{\Lambda}{\lrcorner} T] &= (2\partial + 3\lambda + \chi D)T + \frac{c}{3}\lambda^2\chi, \\ [T \underset{\Lambda}{\lrcorner} J] &= (2\partial + 2\lambda + \chi D)J, \\ [J \underset{\Lambda}{\lrcorner} J] &= T + \frac{c}{3}\lambda\chi. \end{aligned} \tag{3.7}$$

Definition 3.6. [11] Let V be an $N_K = 2$ SUSY vertex algebra. A vector $P \in V$ is called an $N = 2$ *superconformal vector* if it satisfies the following conditions :

- $Y(P, z, \theta^1, \theta^2)$ is an $N = 2$ super-Virasoro field with central charge c , i.e.

$$[P \underset{\Lambda}{\lrcorner} P] = (2\partial + 2\lambda + \chi^1 D^1 + \chi^2 D^2)P + \frac{c}{3}\lambda\chi^1\chi^2, \tag{3.8}$$

- $P_{(0|00)} = 2\partial$, $P_{(0|10)} = -D^1$, $P_{(0|01)} = D^2$,
- $P_{(1|00)}$ is diagonalizable on V .

For $v \in V$, we say that v has *conformal weight* $\Delta \in \mathbb{C}$ if :

$$[P \underset{\Lambda}{\lrcorner} v] = (2\partial + 2\Delta\lambda + \chi^1 D^1 + \chi^2 D^2)v + O(\Lambda^2), \tag{3.9}$$

where $O(\Lambda^2)$ is a polynomial in Λ which has no constant and linear terms. Moreover, if there is no $O(\Lambda^2)$ term in (3.9), the vector v is called *primary*.

Chapter 4

Superconformal structures of SUSY charged free fermion vertex algebras

For the construction of superconformal vectors of the SUSY W -algebras in Remark 4.10, a particular deformation of the standard superconformal vector T_{st}^{ch} of the supersymmetric charged free fermion vertex algebra $F_{N=1}^{ch}$ is needed. In this chapter, we prove our main result Theorem 4.6 and find the corresponding $N = 2$ superconformal symmetry of $F_{N=1}^{ch}$ in Theorem 4.14. We also construct an $N_K = 2$ version of the bc - $\beta\gamma$ system, so that Theorem 4.6 and Theorem 4.14 can be unified by the $N_K = 2$ SUSY vertex algebra formalism in Theorem 4.21.

4.1 Standard superconformal vectors of SUSY charged free fermion vertex algebras

In this section, we recall the standard superconformal vectors of the SUSY charged free fermion vertex algebras and the connection with the superconformal vectors of a non-SUSY bc - $\beta\gamma$ system. For the SUSY charged free fermion vertex algebra $F_{N=1}^{ch}$ associated to the vector superspace U , recall that $A = A_0 \cup A_1$ refers to the basis for U .

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Example 4.1. In [14], the *standard conformal vector* L_{st}^{ch} , for the non-SUSY charged free fermion vertex algebra associated to $\mathfrak{osp}(1|2)$, is defined as

$$L_{st}^{ch} = -\frac{1}{2}\varphi^{1/2}\partial\varphi_{1/2} + \frac{1}{2}\partial\varphi^{1/2}\varphi_{1/2} + \partial\varphi_1\varphi^1, \quad (4.1)$$

and we can easily find that the corresponding *standard superconformal vector* of $F^{ch}(\mathfrak{osp}(1|2))$ is

$$G_{st}^{ch} = \varphi^{1/2}\varphi^1 + \partial\varphi_1\varphi_{1/2}, \quad (4.2)$$

i.e. the above two vectors satisfy the following λ -brackets :

$$[L_{st}^{ch} \lambda L_{st}^{ch}] = (\partial + 2\lambda)L_{st}^{ch} + \frac{c}{12}\lambda^3, \quad (4.3)$$

$$[L_{st}^{ch} \lambda G_{st}^{ch}] = \left(\partial + \frac{3}{2}\lambda\right)G_{st}^{ch}, \quad (4.4)$$

$$[G_{st}^{ch} \lambda G_{st}^{ch}] = 2L_{st}^{ch} + \frac{c}{3}\lambda^2, \quad (4.5)$$

with central charge $c = -3$. If we rename the fields $\varphi_{1/2}$, $\varphi^{1/2}$, φ^1 , φ_1 , as b , c , β , γ , respectively with reversing the parity, then the vertex algebra $F^{ch}(\mathfrak{osp}(1|2))$ with L_{st}^{ch} and G_{st}^{ch} is nothing but the well-known bc - $\beta\gamma$ system with the conformal vector and the superconformal vector :

$$L_{st}^{ch} = \frac{1}{2}(-c\partial b + \partial cb + 2\partial\gamma\beta), \quad (4.6)$$

$$G_{st}^{ch} = c\beta + \partial\gamma b, \quad (4.7)$$

with central charge 3. More detailed explanations of bc - $\beta\gamma$ system are in [6]. For details of the physical statements, the reader is referred to [19].

Definition 4.2. For an $N = 1$ superconformal structure of the SUSY charged free fermion vertex algebra $F_{N=1}^{ch}$, we slightly modify the superconformal vector of the SUSY bc - $\beta\gamma$ system in Example 5.12 of [11]. Let T_{st}^{ch} be a *standard superconformal vector* of $F_{N=1}^{ch}$ defined by

$$T_{st}^{ch} = \sum_{a \in A_{\bar{0}}} (\partial\phi_a\phi^{\bar{a}} + D\phi_a D\phi^{\bar{a}}) + \sum_{a \in A_{\bar{1}}} (\phi_a\partial\phi^{\bar{a}} + D\phi_a D\phi^{\bar{a}}). \quad (4.8)$$

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Then the field T_{st}^{ch} satisfies the $N = 1$ superconformal relation

$$[T_{st}^{ch} \wedge T_{st}^{ch}] = (2\partial + 3\lambda + \chi D)T_{st}^{ch} + \frac{c_{st}}{3}\lambda^2\chi, \quad (4.9)$$

where the central charge c_{st} is $3\dim U$. The verification of the above \wedge -bracket follows from Theorem 4.6.

Remark 4.3. [3, 6, 7, 10] Let $F_{N=1}^{ch}$ be the SUSY charged free fermion vertex algebra associated to the vector superspace $U = U_{\bar{0}} \oplus U_{\bar{1}}$. For a basis element a in $A_{\bar{0}}$, if we write each superfields $\phi_a, \phi^{\bar{a}}$ as follows :

$$\begin{aligned} Y(\phi_a, z, \theta) &= Y(\gamma, z) + \theta Y(c, z), \\ Y(\phi^{\bar{a}}, z, \theta) &= Y(b, z) + \theta Y(\beta, z), \end{aligned} \quad (4.10)$$

then we can see that the $A_{\bar{0}}$ -part of T_{st}^{ch} is the $\dim U_{\bar{0}}$ copies of the standard superconformal vector of non-SUSY bc - $\beta\gamma$ system, i.e.

$$Y\left(\sum_{a \in A_{\bar{0}}} (\partial\phi_a\phi^{\bar{a}} + D\phi_a D\phi^{\bar{a}}), z, \theta\right) = Y\left(\sum_{a \in A_{\bar{0}}} (G_{st}^{ch}), z\right) + 2\theta Y\left(\sum_{a \in A_{\bar{0}}} (L_{st}^{ch}), z\right). \quad (4.11)$$

Similarly, if we expand the superfields of $\phi_a, \phi^{\bar{a}}$ as

$$\begin{aligned} Y(\phi^{\bar{a}}, z, \theta) &= Y(\gamma, z) + \theta Y(c, z), \\ Y(\phi_a, z, \theta) &= Y(b, z) + \theta Y(\beta, z), \end{aligned} \quad (4.12)$$

for an odd element a in A , then T_{st}^{ch} is the sum of $\dim U$ copies of G_{st}^{ch} and L_{st}^{ch} , i.e.

$$T_{st}^{ch} = \sum_{a \in A} G_{st}^{ch} + 2\theta \left(\sum_{a \in A} L_{st}^{ch}\right). \quad (4.13)$$

In Remark 4.11, we also get a similar result for our shifted superconformal vectors.

Proof. For any even element a , the superfield expansion of $Y(D\phi_a, z, \theta)$ is obtained as

$$\begin{aligned} Y(D\phi_a, z, \theta) &= (\partial_\theta + \theta\partial_z)Y(\phi_a, z, \theta) \\ &= (\partial_\theta + \theta\partial_z)(Y(\gamma, z) + \theta Y(c, z)) \\ &= Y(c, z) + \theta Y(\partial\gamma, z) \end{aligned} \quad (4.14)$$

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by using the rules in Theorem 4.16 of [11]. Similarly, we have

$$Y(D\phi^{\bar{a}}, z, \theta) = Y(\beta, z) + \theta Y(\partial b, z), \quad (4.15)$$

so that

$$\begin{aligned} Y(D\phi_a D\phi^{\bar{a}}, z, \theta) &= (Y(c, z) + \theta Y(\partial\gamma, z))(Y(\beta, z) + \theta Y(\partial b, z)) \\ &= Y(c\beta, z) + \theta Y(\partial\gamma\beta - c\partial b, z). \end{aligned} \quad (4.16)$$

The negative sign of $c\partial b$ follows from the anti-commutativity of the odd field $Y(c, z)$ and the odd indeterminate θ . The expansion of $Y(\partial\phi_a\phi^{\bar{a}}, z, \theta)$ is also given by

$$\begin{aligned} Y(\partial\phi_a\phi^{\bar{a}}, z, \theta) &= Y(\partial\phi_a, z, \theta)Y(\phi^{\bar{a}}, z, \theta) \\ &= \partial_z Y(\phi_a, z, \theta)Y(\phi^{\bar{a}}, z, \theta) \\ &= (Y(\partial\gamma, z) + \theta Y(\partial c, z))(Y(b, z) + \theta Y(\beta, z)) \\ &= Y(\partial\gamma b, z) + \theta Y(\partial cb + \partial\gamma\beta, z), \end{aligned} \quad (4.17)$$

hence we have

$$\begin{aligned} Y(\partial\phi_a\phi^{\bar{a}} + D\phi_a D\phi^{\bar{a}}, z, \theta) &= Y(c\beta + \partial\gamma b, z) \\ &\quad + 2\theta Y\left(\frac{1}{2}(-c\partial b + \partial cb + 2\partial\gamma\beta), z\right) \\ &= Y(G_{st}^{ch}, z) + 2\theta Y(L_{st}^{ch}, z). \end{aligned} \quad (4.18)$$

For an odd element a , we also get the following superfield expansion

$$\begin{aligned} Y(\phi_a\partial\phi^{\bar{a}}, z, \theta) &= Y(\phi_a, z, \theta)Y(\partial\phi^{\bar{a}}, z, \theta) \\ &= Y(\phi_a, z, \theta)\partial_z Y(\phi^{\bar{a}}, z, \theta) \\ &= (Y(b, z) + \theta Y(\beta, z))(Y(\partial\gamma, z) + \theta Y(\partial c, z)) \\ &= Y(b\partial\gamma, z) + \theta Y(\beta\partial\gamma - b\partial c, z) \\ &= Y(\partial\gamma b, z) + \theta Y(\partial cb + \partial\gamma\beta, z), \\ Y(D\phi_a D\phi^{\bar{a}}, z, \theta) &= (Y(\beta, z) + \theta Y(\partial b, z))(Y(c, z) + \theta Y(\partial\gamma, z)) \\ &= Y(\beta c, z) + \theta Y(\partial bc + \beta\partial\gamma, z) \\ &= Y(c\beta, z) + \theta Y(\partial\gamma\beta - c\partial b, z), \end{aligned} \quad (4.19)$$

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hence the field $Y(\phi_a \partial \phi^{\bar{a}} + D\phi_a D\phi^{\bar{a}}, z, \theta)$ of the $A_{\bar{1}}$ -part has the same superfield expansion as in (4.18). Thus we obtain the result

$$Y(T_{st}^{ch}, z, \theta) = Y\left(\sum_{a \in A} G_{st}^{ch}, z\right) + 2\theta Y\left(\sum_{a \in A} L_{st}^{ch}, z\right). \quad (4.20)$$

□

4.2 Shifted superconformal vectors of SUSY charged free fermion vertex algebras

In Theorem 4.6, we define the shifted superconformal vector T_{sh}^{ch} , which is a deformation of the standard superconformal vector T_{st}^{ch} , by finding the proper ghost term using Ansatz 4.5 that the requiring shifted superconformal vector is a linear combination of three particular quadratic monomial terms, in order to obtain varying conformal weights of each fields ϕ_a and $\phi^{\bar{a}}$.

Lemma 4.4. *Let $F_{N=1}^{ch}$ be the SUSY charged free fermion vertex algebra. For any even element a in A , we have the following Λ -brackets :*

$$[\partial\phi_a\phi^{\bar{a}} \Lambda \partial\phi_a\phi^{\bar{a}}] = \partial(\partial\phi_a\phi^{\bar{a}}) + 2\lambda(\partial\phi_a\phi^{\bar{a}}), \quad (4.21)$$

$$[\partial\phi_a\phi^{\bar{a}} \Lambda \phi_a\partial\phi^{\bar{a}}] = \partial(\phi_a\partial\phi^{\bar{a}}) + 2\lambda(\phi_a\partial\phi^{\bar{a}}) + \lambda^2(\phi_a\phi^{\bar{a}}), \quad (4.22)$$

$$\begin{aligned} [\partial\phi_a\phi^{\bar{a}} \Lambda D\phi_a D\phi^{\bar{a}}] &= \partial(D\phi_a D\phi^{\bar{a}}) + \lambda(D\phi_a D\phi^{\bar{a}}) \\ &\quad + \chi D(D\phi_a D\phi^{\bar{a}}) + \frac{1}{2}\lambda^2\chi - \lambda\chi(D\phi_a\phi^{\bar{a}}), \end{aligned} \quad (4.23)$$

$$[\phi_a\partial\phi^{\bar{a}} \Lambda \partial\phi_a\phi^{\bar{a}}] = -\partial(\partial\phi_a\phi^{\bar{a}}) - 2\lambda(\partial\phi_a\phi^{\bar{a}}) - \lambda^2(\phi_a\phi^{\bar{a}}), \quad (4.24)$$

$$[\phi_a\partial\phi^{\bar{a}} \Lambda \phi_a\partial\phi^{\bar{a}}] = -\partial(\phi_a\partial\phi^{\bar{a}}) - 2\lambda(\phi_a\partial\phi^{\bar{a}}), \quad (4.25)$$

$$\begin{aligned} [\phi_a\partial\phi^{\bar{a}} \Lambda D\phi_a D\phi^{\bar{a}}] &= -\partial(D\phi_a D\phi^{\bar{a}}) - \lambda(D\phi_a D\phi^{\bar{a}}) \\ &\quad - \chi D(D\phi_a D\phi^{\bar{a}}) + \frac{1}{2}\lambda^2\chi - \lambda\chi(\phi_a D\phi^{\bar{a}}), \end{aligned} \quad (4.26)$$

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$$\begin{aligned}
[D\phi_a D\phi^{\bar{a}} \wedge \partial\phi_a \phi^{\bar{a}}] &= \partial(\partial\phi_a \phi^{\bar{a}}) + \lambda(\partial\phi_a \phi^{\bar{a}}) \\
&+ \chi D(\partial\phi_a \phi^{\bar{a}}) + \frac{1}{2}\lambda^2 \chi + \lambda\chi(D\phi_a \phi^{\bar{a}}),
\end{aligned} \tag{4.27}$$

$$\begin{aligned}
[D\phi_a D\phi^{\bar{a}} \wedge \phi_a \partial\phi^{\bar{a}}] &= \partial(\phi_a \partial\phi^{\bar{a}}) + \lambda(\phi_a \partial\phi^{\bar{a}}) \\
&+ \chi D(\phi_a \partial\phi^{\bar{a}}) + \frac{1}{2}\lambda^2 \chi + \lambda\chi(\phi_a D\phi^{\bar{a}}),
\end{aligned} \tag{4.28}$$

$$[D\phi_a D\phi^{\bar{a}} \wedge D\phi_a D\phi^{\bar{a}}] = \partial(D\phi_a D\phi^{\bar{a}}) + 2\lambda(D\phi_a D\phi^{\bar{a}}). \tag{4.29}$$

Proof. For the \wedge -bracket in (4.23), we have :

$$\begin{aligned}
[\partial\phi_a \phi^{\bar{a}} \wedge D\phi_a D\phi^{\bar{a}}] &= [\partial\phi_a \phi^{\bar{a}} \wedge D\phi_a] D\phi^{\bar{a}} + D\phi_a [\partial\phi_a \phi^{\bar{a}} \wedge D\phi^{\bar{a}}] \\
&+ \int_0^\Lambda [[\partial\phi_a \phi^{\bar{a}} \wedge D\phi_a] \wedge D\phi^{\bar{a}}] d\Gamma \\
&= ((D + \chi)\partial\phi_a) D\phi^{\bar{a}} + D\phi_a ((D + \chi)(\partial + \lambda)\phi^{\bar{a}}) \\
&+ \int_0^\Lambda [(D + \chi)\partial\phi_a \wedge D\phi^{\bar{a}}] d\Gamma \\
&= \partial D\phi_a D\phi^{\bar{a}} + \chi\partial\phi_a D\phi^{\bar{a}} + D\phi_a \partial D\phi^{\bar{a}} + \lambda D\phi_a D\phi^{\bar{a}} \\
&- \chi D\phi_a \partial\phi^{\bar{a}} - \lambda\chi D\phi_a \phi^{\bar{a}} + \int_0^\Lambda [(D + \chi)\partial\phi_a \wedge D\phi^{\bar{a}}] d\Gamma \\
&= \partial(D\phi_a D\phi^{\bar{a}}) + \lambda(D\phi_a D\phi^{\bar{a}}) \\
&+ \chi D(D\phi_a D\phi^{\bar{a}}) - \lambda\chi(D\phi_a \phi^{\bar{a}}) + \frac{1}{2}\lambda^2 \chi,
\end{aligned} \tag{4.30}$$

from (A.3) and (A.4) in Appendix A, and the non-commutative Wick formula

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of $N_K = 1$ SUSY LCAs. Indeed, the integral term is computed as follows :

$$\begin{aligned}
\int_0^\Lambda [(D + \chi)\partial\phi_a \Gamma D\phi^{\bar{a}}] d\Gamma &= \int_0^\Lambda (\eta - \chi)(-\gamma)(-1)(D + \eta)[\phi_a \Gamma \phi^{\bar{a}}] d\Gamma \\
&= \int_0^\Lambda (\eta - \chi)\gamma\eta d\Gamma \\
&= \int_0^\Lambda -\gamma^2 - \chi\gamma\eta d\Gamma \\
&= \partial_\eta \left(\int_0^\lambda -\gamma^2 - \chi\gamma\eta d\gamma \right) \\
&= \partial_\eta \left(-\frac{1}{3}\lambda^3 - \frac{1}{2}\chi\lambda^2\eta \right) = \frac{1}{2}\chi\lambda^2.
\end{aligned} \tag{4.31}$$

Similarly, by using the results in Lemma A.1, Lemma A.2 and Lemma A.3, we can compute the other Λ -brackets. \square

Now we will find the shifted superconformal vectors in the SUSY charged free fermion vertex algebras :

Ansatz 4.5. Let T_{sh}^{ch} be the field written as a linear combination of three quadratic terms in Lemma 4.4, i.e.

$$T_{sh}^{ch} = m_1\partial\phi_a\phi^{\bar{a}} + m_2\phi_a\partial\phi^{\bar{a}} + m_3D\phi_aD\phi^{\bar{a}}. \tag{4.32}$$

Then the Λ -bracket of T_{sh}^{ch} with itself follows from Lemma 4.4 :

$$\begin{aligned}
&[T_{sh}^{ch} \Lambda T_{sh}^{ch}] \\
&= \left(\begin{aligned} &(m_1^2 - m_1m_2 + m_1m_3)\partial \\ &+ (2m_1^2 - 2m_1m_2 + m_1m_3)\lambda + (m_1m_3)\chi D \end{aligned} \right) \partial\phi_a\phi^{\bar{a}} \\
&+ \left(\begin{aligned} &(m_1m_2 - m_2^2 + m_2m_3)\partial \\ &+ (2m_1m_2 - 2m_2^2 + m_2m_3)\lambda + (m_2m_3)\chi D \end{aligned} \right) \phi_a\partial\phi^{\bar{a}} \\
&+ \left(\begin{aligned} &(m_1m_3 - m_2m_3 + m_3^2)\partial \\ &+ (m_1m_3 - m_2m_3 + 2m_3^2)\lambda + (m_1m_3 - m_2m_3)\chi D \end{aligned} \right) D\phi_aD\phi^{\bar{a}} \\
&+ (m_1m_3 + m_2m_3)\lambda^2\chi.
\end{aligned} \tag{4.33}$$

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On the other hand,

$$\begin{aligned}
(2\partial + 3\lambda + \chi D)T_{sh}^{ch} + \frac{c_{sh}}{3}\lambda^2\chi &= \left((2m_1)\partial + (3m_1)\lambda + (m_1)\chi D\right)\partial\phi_a\phi^{\bar{a}} \\
&+ \left((2m_2)\partial + (3m_2)\lambda + (m_2)\chi D\right)\phi_a\partial\phi^{\bar{a}} \\
&+ \left((2m_3)\partial + (3m_3)\lambda + (m_3)\chi D\right)D\phi_aD\phi^{\bar{a}} \\
&+ \frac{c_{sh}}{3}\lambda^2\chi.
\end{aligned} \tag{4.34}$$

By comparing the coefficients of each monomial terms, we know that the triple of the form

$$(m_1, m_2, m_3) = (t_a + 1, t_a, 1), \tag{4.35}$$

for any complex number t_a satisfies the lambda bracket equality :

$$[T_{sh}^{ch} \wedge T_{sh}^{ch}] = (2\partial + 3\lambda + \chi D)T_{sh}^{ch} + \frac{c_{sh}}{3}\lambda^2\chi, \tag{4.36}$$

where the central charge is $c_{sh} = 6t_a + 3$. For the odd element $a \in A$, we obtain the result by replacing the fields ϕ_a and $\phi^{\bar{a}}$ with $\phi^{\bar{a}}$ and ϕ_a respectively.

Hence we have shown the following theorem :

Theorem 4.6. *Let T_{sh}^{ch} be the field defined by*

$$\begin{aligned}
T_{sh}^{ch} &= \sum_{a \in A_{\bar{0}}} (t_a + 1)\partial\phi_a\phi^{\bar{a}} + t_a\phi_a\partial\phi^{\bar{a}} + D\phi_aD\phi^{\bar{a}} \\
&+ \sum_{a \in A_{\bar{1}}} t_a\partial\phi_a\phi^{\bar{a}} + (t_a + 1)\phi_a\partial\phi^{\bar{a}} + D\phi_aD\phi^{\bar{a}},
\end{aligned} \tag{4.37}$$

where $(t_a)_{a \in A}$ is a sequence of complex numbers. Then the field T_{sh}^{ch} is a superconformal vector of the SUSY charged free fermion vertex algebra $F_{N=1}^{ch}$ with the central charge

$$c_{sh} = \sum_{a \in A} (6t_a + 3). \tag{4.38}$$

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Definition 4.7. The field T_{sh}^{ch} defined in Theorem 4.6 is called the *shifted superconformal vector* of the supersymmetric charged free fermion vertex algebra $F_{N=1}^{ch}$.

Remark 4.8. If we define the ghost field T_{ghost}^{ch} as

$$T_{ghost}^{ch} = \sum_{a \in A} t_a \partial(\phi_a \phi^{\bar{a}}), \quad (4.39)$$

then the shifted superconformal vector T_{sh}^{ch} can be written as

$$T_{sh}^{ch} = T_{st}^{ch} + T_{ghost}^{ch}. \quad (4.40)$$

Moreover, if the complex numbers t_a are all zeros, then the standard superconformal vector T_{st}^{ch} is recovered.

Now, the conformal weight of each of fields ϕ_a and $\phi^{\bar{a}}$ has one degree of freedom.

Proposition 4.9. *For an even element a in A , the conformal weights of ϕ_a and $\phi^{\bar{a}}$ are respectively $-\frac{1}{2}t_a$ and $\frac{1}{2}t_a + \frac{1}{2}$. For an odd element a in A , the conformal weights are $\frac{1}{2}t_a + \frac{1}{2}$ and $-\frac{1}{2}t_a$ respectively. Moreover, the fields ϕ_a and $\phi^{\bar{a}}$ are primary with respect to T_{sh}^{ch} .*

Proof. From the lambda brackets (A.1), (A.7) and (A.13) in Appendix A, we have

$$\begin{aligned} & [(t_a + 1)\partial\phi_a\phi^{\bar{a}} + t_a\phi_a\partial\phi^{\bar{a}} + D\phi_a D\phi^{\bar{a}} \wedge \phi_a] \\ &= (t_a + 1)[\partial\phi_a\phi^{\bar{a}} \wedge \phi_a] + t_a[\phi_a\partial\phi^{\bar{a}} \wedge \phi_a] + [D\phi_a D\phi^{\bar{a}} \wedge \phi_a] \\ &= (t_a + 1)\partial\phi_a - t_a(\partial + \lambda)\phi_a + (\partial + \chi D)\phi_a \\ &= 2\partial\phi_a - t_a\lambda\phi_a + \chi D\phi_a, \end{aligned} \quad (4.41)$$

for any even element a , and also we have

$$\begin{aligned} & [(t_a + 1)\partial\phi_a\phi^{\bar{a}} + t_a\phi_a\partial\phi^{\bar{a}} + D\phi_a D\phi^{\bar{a}} \wedge \phi^{\bar{a}}] \\ &= (t_a + 1)[\partial\phi_a\phi^{\bar{a}} \wedge \phi^{\bar{a}}] + t_a[\phi_a\partial\phi^{\bar{a}} \wedge \phi^{\bar{a}}] + [D\phi_a D\phi^{\bar{a}} \wedge \phi^{\bar{a}}] \\ &= (t_a + 1)(\partial + \lambda)\phi^{\bar{a}} - t_a\partial\phi^{\bar{a}} + (\partial + \chi D)\phi^{\bar{a}} \\ &= 2\partial\phi^{\bar{a}} + (t_a + 1)\lambda\phi^{\bar{a}} + \chi D\phi^{\bar{a}}, \end{aligned} \quad (4.42)$$

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from the lambda brackets in (A.2), (A.8) and (A.14). Hence, we have the following Λ -brackets :

$$\begin{aligned} [T_{sh}^{ch} \Lambda \phi_a] &= (2\partial - t_a\lambda + \chi D)\phi_a, \\ [T_{sh}^{ch} \Lambda \phi^{\bar{a}}] &= (2\partial + (t_a + 1)\lambda + \chi D)\phi^{\bar{a}}, \end{aligned} \quad (4.43)$$

for any even element a , and

$$\begin{aligned} [T_{sh}^{ch} \Lambda \phi_a] &= (2\partial + (t_a + 1)\lambda + \chi D)\phi_a, \\ [T_{sh}^{ch} \Lambda \phi^{\bar{a}}] &= (2\partial - t_a\lambda + \chi D)\phi^{\bar{a}}, \end{aligned} \quad (4.44)$$

for any odd element a in A . \square

Remark 4.10. Roughly speaking, the SUSY W -algebra is the BRST cohomology of the tensor product of the SUSY affine vertex algebra and the SUSY charged free fermion vertex algebra, i.e. :

$$W(\bar{\mathfrak{g}}, f_{odd}, k) = H(V^k(\bar{\mathfrak{g}}) \otimes F_{N=1}^{ch}, Q), \quad (4.45)$$

where Q is the differential of $V^k(\bar{\mathfrak{g}}) \otimes F_{N=1}^{ch}$ and $\bar{\mathfrak{g}}$ denotes \mathfrak{g} with reversed parity (for details, see Section 4.1 of [18]). For the SUSY affine vertex algebra $V^k(\bar{\mathfrak{g}})$, there exists an $N = 1$ superconformal vector T_{st}^{KT} , the Kac-Todorov superconformal vector which originally came from [15], of the SUSY affine vertex algebra (see Example 5.9 of [11]). Also, for any vector $\bar{v} \in \bar{\mathfrak{g}}$, we know that the vector

$$T_{sh}^{KT} = T_{st}^{KT} + \partial\bar{v}, \quad (4.46)$$

is also an $N = 1$ superconformal vector (see Example 2.13 of [12]). Hence, we propose that the sum of our shifted superconformal vector in Theorem 4.6 and the Kac-Todorov superconformal vector

$$T^W := T_{sh}^{KT} \otimes 1 + 1 \otimes T_{sh}^{ch} \quad (4.47)$$

gives a superconformal vector of the supersymmetric W -algebra.

Remark 4.11. Decomposition of the field T_{sh}^{ch} via non-SUSY superconformal vectors can be done in the same way as Remark 4.3, i.e. the shifted superconformal field T_{sh}^{ch} of the SUSY charged free fermion vertex algebra $F_{N=1}^{ch}$ is the sum of shifted superconformal fields of the non-SUSY bc - $\beta\gamma$ system.

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Proof. For $a \in A_{\bar{0}}$, we have

$$\begin{aligned}
Y(\phi_a \partial \phi^{\bar{a}}, z, \theta) &= Y(\phi_a, z, \theta) Y(\partial \phi^{\bar{a}}, z, \theta) \\
&= Y(\phi_a, z, \theta) \partial_z Y(\phi^{\bar{a}}, z, \theta) \\
&= (Y(\gamma, z) + \theta Y(c, z))(Y(\partial b, z) + \theta Y(\partial \beta, z)) \\
&= Y(\gamma \partial b, z) + \theta Y(c \partial b + \gamma \partial \beta, z),
\end{aligned} \tag{4.48}$$

and, with the equations (4.16) and (4.17), the $A_{\bar{0}}$ -part of the shifted superconformal field T_{sh}^{ch} can be expressed as

$$\begin{aligned}
&(t_a + 1) \partial \phi_a \phi^{\bar{a}} + t_a \phi_a \partial \phi^{\bar{a}} + D \phi_a D \phi^{\bar{a}} \\
&= (t_a) \gamma \partial b + (t_a + 1) \partial \gamma b + c \beta \\
&\quad + 2\theta \left(\frac{1}{2} ((t_a - 1) c \partial b + (t_a + 1) \partial c b + t_a \gamma \partial \beta + (t_a + 2) \partial \gamma \beta) \right).
\end{aligned} \tag{4.49}$$

If we define the ghost parts as

$$L_{ghost}^{ch} = \frac{t_a}{2} (c \partial b + \partial c b + \gamma \partial \beta + \partial \gamma \beta), \tag{4.50}$$

$$G_{ghost}^{ch} = t_a (\gamma \partial b + \partial \gamma b), \tag{4.51}$$

we obtain the shifted superconformal vectors in the bc - $\beta\gamma$ system as

$$\begin{aligned}
L_{sh}^{ch} &= L_{st}^{ch} + L_{ghost}^{ch} \\
&= \frac{1}{2} ((t_a - 1) c \partial b + (t_a + 1) \partial c b + t_a \gamma \partial \beta + (t_a + 2) \partial \gamma \beta),
\end{aligned} \tag{4.52}$$

$$\begin{aligned}
G_{sh}^{ch} &= G_{st}^{ch} + G_{ghost}^{ch} \\
&= (t_a) \gamma \partial b + (t_a + 1) \partial \gamma b + c \beta.
\end{aligned} \tag{4.53}$$

On the other hand, for the odd elements $a \in A_{\bar{1}}$, we have

$$\begin{aligned}
Y(\partial \phi_a \phi^{\bar{a}}, z, \theta) &= Y(\partial \phi_a, z, \theta) Y(\phi^{\bar{a}}, z, \theta) \\
&= \partial_z Y(\phi_a, z, \theta) Y(\phi^{\bar{a}}, z, \theta) \\
&= (Y(\partial b, z) + \theta Y(\partial \beta, z))(Y(\gamma, z) + \theta Y(c, z)) \\
&= Y(\partial b \gamma, z) + \theta Y(\partial \beta \gamma - \partial b c, z),
\end{aligned} \tag{4.54}$$

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and, also with the equations in (4.19), the $A_{\bar{1}}$ -part of the shifted superconformal field T_{sh}^{ch} can be written as

$$\begin{aligned} & (t_a + 1)\partial\phi_a\phi^{\bar{a}} + t_a\phi_a\partial\phi^{\bar{a}} + D\phi_a D\phi^{\bar{a}} \\ &= (t_a)\gamma\partial b + (t_a + 1)\partial\gamma b + c\beta \\ &+ 2\theta\left(\frac{1}{2}((t_a - 1)c\partial b + (t_a + 1)\partial cb + t_a\gamma\partial\beta + (t_a + 2)\partial\gamma\beta)\right), \end{aligned} \tag{4.55}$$

of which the expression is same as the $A_{\bar{0}}$ -parts.

Hence the shifted superconformal field of the SUSY charged free fermion vertex algebra is the sum of $\dim U$ copies of the shifted superconformal field of the non-SUSY bc - $\beta\gamma$ system, i.e.

$$T_{sh}^{ch} = \sum_{a \in A} G_{sh}^{ch} + 2\theta\left(\sum_{a \in A} L_{sh}^{ch}\right). \tag{4.56}$$

□

Remark 4.12. In Remark 4.8, the SUSY vertex algebra case, the field T_{st}^{ch} is a special case of the field T_{sh}^{ch} when $t_a = 0$ for all $a \in A$. Likewise, we can see that the each field of L_{st}^{ch} and G_{st}^{ch} is also a special case of the each field of L_{sh}^{ch} and G_{sh}^{ch} in the non-SUSY bc - $\beta\gamma$ system.

4.3 Shifted $N = 2$ superconformal structures of SUSY charged free fermion vertex algebras

From [3] and [11], we can use the fact that the field

$$J_{st}^{ch} = \sum_{a \in A_{\bar{0}}} D\phi_a\phi^{\bar{a}} + \sum_{a \in A_{\bar{1}}} D\phi^{\bar{a}}\phi_a, \tag{4.57}$$

together with T_{st}^{ch} form an $N = 2$ superconformal vertex algebra structure. For the $N = 2$ superconformal symmetry of $F_{N=1}^{ch}$ compatible with the shifted superconformal vector T_{sh}^{ch} , we find the corresponding ghost term in Theorem 4.14. We also write the shifted $N = 2$ superconformal fields as $N = 2$ superfields in Theorem 4.21.

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Lemma 4.13. *Let $F_{N=1}^{ch}$ be the SUSY charged free fermion vertex algebra. For any even element a in A , we have the following Λ -brackets :*

$$\begin{aligned} [\partial\phi_a\phi^{\bar{a}} \Lambda D\phi_a\phi^{\bar{a}}] &= \partial(D\phi_a\phi^{\bar{a}}) + \lambda(D\phi_a\phi^{\bar{a}}) + \chi D(D\phi_a\phi^{\bar{a}}) \\ &\quad + \chi(D\phi_a D\phi^{\bar{a}}) - \frac{1}{2}\lambda^2, \end{aligned} \quad (4.58)$$

$$\begin{aligned} [\partial\phi_a\phi^{\bar{a}} \Lambda \phi_a D\phi^{\bar{a}}] &= \partial(\phi_a D\phi^{\bar{a}}) + \lambda(\phi_a D\phi^{\bar{a}}) + \chi D(\phi_a D\phi^{\bar{a}}) \\ &\quad - \chi(D\phi_a D\phi^{\bar{a}}) + \lambda\chi(\phi_a\phi^{\bar{a}}) + \frac{1}{2}\lambda^2, \end{aligned} \quad (4.59)$$

$$\begin{aligned} [\phi_a\partial\phi^{\bar{a}} \Lambda D\phi_a\phi^{\bar{a}}] &= -\partial(D\phi_a\phi^{\bar{a}}) - \lambda(D\phi_a\phi^{\bar{a}}) - \chi D(D\phi_a\phi^{\bar{a}}) \\ &\quad - \chi(D\phi_a D\phi^{\bar{a}}) - \lambda\chi(\phi_a\phi^{\bar{a}}) - \frac{1}{2}\lambda^2, \end{aligned} \quad (4.60)$$

$$\begin{aligned} [\phi_a\partial\phi^{\bar{a}} \Lambda \phi_a D\phi^{\bar{a}}] &= -\partial(\phi_a D\phi^{\bar{a}}) - \lambda(\phi_a D\phi^{\bar{a}}) - \chi D(\phi_a D\phi^{\bar{a}}) \\ &\quad + \chi(D\phi_a D\phi^{\bar{a}}) + \frac{1}{2}\lambda^2, \end{aligned} \quad (4.61)$$

$$\begin{aligned} [D\phi_a D\phi^{\bar{a}} \Lambda D\phi_a\phi^{\bar{a}}] &= \partial(D\phi_a\phi^{\bar{a}}) + \lambda(D\phi_a\phi^{\bar{a}}) - \chi(D\phi_a D\phi^{\bar{a}}) \\ &\quad + \frac{1}{2}\lambda^2, \end{aligned} \quad (4.62)$$

$$\begin{aligned} [D\phi_a D\phi^{\bar{a}} \Lambda \phi_a D\phi^{\bar{a}}] &= \partial(\phi_a D\phi^{\bar{a}}) + \lambda(\phi_a D\phi^{\bar{a}}) + \chi(D\phi_a D\phi^{\bar{a}}) \\ &\quad + \frac{1}{2}\lambda^2, \end{aligned} \quad (4.63)$$

$$[D\phi_a\phi^{\bar{a}} \Lambda D\phi_a\phi^{\bar{a}}] = \partial\phi_a\phi^{\bar{a}} + D\phi_a D\phi^{\bar{a}} + \lambda\chi, \quad (4.64)$$

$$[D\phi_a\phi^{\bar{a}} \Lambda \phi_a D\phi^{\bar{a}}] = \phi_a\partial\phi^{\bar{a}} - D\phi_a D\phi^{\bar{a}} + \lambda(\phi_a\phi^{\bar{a}}), \quad (4.65)$$

$$[\phi_a D\phi^{\bar{a}} \Lambda D\phi_a\phi^{\bar{a}}] = -\partial\phi_a\phi^{\bar{a}} - D\phi_a D\phi^{\bar{a}} - \lambda(\phi_a\phi^{\bar{a}}), \quad (4.66)$$

$$[\phi_a D\phi^{\bar{a}} \Lambda \phi_a D\phi^{\bar{a}}] = -\phi_a\partial\phi^{\bar{a}} + D\phi_a D\phi^{\bar{a}} - \lambda\chi. \quad (4.67)$$

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Proof. For the Λ -bracket in (4.58), we have :

$$\begin{aligned}
[\partial\phi_a\phi^{\bar{a}} \ \Lambda \ D\phi_a\phi^{\bar{a}}] &= [\partial\phi_a\phi^{\bar{a}} \ \Lambda \ D\phi_a]\phi^{\bar{a}} + D\phi_a[\partial\phi_a\phi^{\bar{a}} \ \Lambda \ \phi^{\bar{a}}] \\
&\quad + \int_0^\Lambda [[\partial\phi_a\phi^{\bar{a}} \ \Lambda \ D\phi_a] \ \Gamma \ \phi^{\bar{a}}] \ d\Gamma \\
&= ((D + \chi)\partial\phi_a)\phi^{\bar{a}} + D\phi_a(\partial + \lambda)\phi^{\bar{a}} \\
&\quad + \int_0^\Lambda [(D + \chi)\partial\phi_a \ \Gamma \ \phi^{\bar{a}}] \ d\Gamma \\
&= \partial D\phi_a\phi^{\bar{a}} + \chi\partial\phi_a\phi^{\bar{a}} + D\phi_a\partial\phi^{\bar{a}} + \lambda D\phi_a\phi^{\bar{a}} \\
&\quad + \int_0^\Lambda -(\eta - \chi)\gamma \ d\Gamma \\
&= \partial(D\phi_a\phi^{\bar{a}}) + \lambda(D\phi_a\phi^{\bar{a}}) + \chi D(D\phi_a\phi^{\bar{a}}) \\
&\quad + \chi(D\phi_a D\phi^{\bar{a}}) - \frac{1}{2}\lambda^2,
\end{aligned} \tag{4.68}$$

from (A.3) and (A.2) in Lemma A.1. Similarly, we can compute the Λ -brackets in (4.59)–(4.63) by using Lemma A.1, Lemma A.2 and Lemma A.3. The Λ -brackets in (4.64)–(4.67) follows from Lemma A.4. \square

Theorem 4.14. *Let J_{ghost}^{ch} be the field defined as*

$$J_{ghost}^{ch} = \sum_{a \in A_{\bar{0}}} t_a(D\phi_a\phi^{\bar{a}} + \phi_a D\phi^{\bar{a}}) + \sum_{a \in A_{\bar{1}}} t_a(D\phi^{\bar{a}}\phi_a + \phi^{\bar{a}} D\phi_a). \tag{4.69}$$

Then the current field

$$J_{sh}^{ch} = J_{st}^{ch} + J_{ghost}^{ch}, \tag{4.70}$$

and the superconformal field T_{sh}^{ch} generates an $N = 2$ superconformal structure of central charge c_{sh} .

Proof. For the $A_{\bar{0}}$ -parts, by using the Lemma 4.13, we have the following

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Λ -brackets :

$$\begin{aligned}
[T_{st}^{ch} \Lambda J_{st}^{ch}] &= (2\partial + 2\lambda + \chi D)J_{st}^{ch}, \\
[T_{st}^{ch} \Lambda J_{ghost}^{ch}] &= (2\partial + 2\lambda + \chi D)J_{ghost}^{ch} \\
&\quad + \sum_{a \in A_{\bar{0}}} t_a \lambda^2 + \sum_{a \in A_{\bar{0}}} t_a \lambda \chi \phi_a \phi_{\bar{a}}, \\
[T_{ghost}^{ch} \Lambda J_{st}^{ch}] &= - \sum_{a \in A_{\bar{0}}} t_a \lambda^2 - \sum_{a \in A_{\bar{0}}} t_a \lambda \chi \phi_a \phi_{\bar{a}}, \\
[T_{ghost}^{ch} \Lambda J_{ghost}^{ch}] &= 0,
\end{aligned} \tag{4.71}$$

hence, we have

$$[T_{sh}^{ch} \Lambda J_{sh}^{ch}] = (2\partial + 2\lambda + \chi D)J_{sh}^{ch}. \tag{4.72}$$

Also, we can check the following Λ -brackets :

$$\begin{aligned}
[J_{st}^{ch} \Lambda J_{st}^{ch}] &= T_{st}^{ch} + \sum_{a \in A_{\bar{0}}} \lambda \chi, \\
[J_{st}^{ch} \Lambda J_{ghost}^{ch}] &= T_{ghost}^{ch} + \sum_{a \in A_{\bar{0}}} t_a \lambda \phi_a \phi_{\bar{a}} + \sum_{a \in A_{\bar{0}}} t_a \lambda \chi, \\
[J_{ghost}^{ch} \Lambda J_{st}^{ch}] &= - \sum_{a \in A_{\bar{0}}} t_a \lambda \phi_a \phi_{\bar{a}} + \sum_{a \in A_{\bar{0}}} t_a \lambda \chi, \\
[J_{ghost}^{ch} \Lambda J_{ghost}^{ch}] &= 0,
\end{aligned} \tag{4.73}$$

thus we obtain the following result :

$$[J_{sh}^{ch} \Lambda J_{sh}^{ch}] = T_{sh}^{ch} + \sum_{a \in A_{\bar{0}}} (2t_a + 1) \lambda \chi. \tag{4.74}$$

For the $A_{\bar{1}}$ -part, we have the same result by replacing the fields ϕ_a with $\phi_{\bar{a}}$, and vice versa. Hence, the fields T_{sh}^{ch} and J_{sh}^{ch} satisfy the $N = 2$ superconformal symmetry. \square

Remark 4.15. Indeed, we can write (see Example 3.13 in [3]) the $N = 1$ superfields T_{sh}^{ch} and J_{sh}^{ch} as :

$$\begin{aligned}
T_{sh}^{ch}(z, \theta) &= (G_{sh}^{ch,+}(z) + G_{sh}^{ch,-}(z)) + 2\theta L_{sh}^{ch}(z), \\
J_{sh}^{ch}(z, \theta) &= J_{sh}^{ch}(z) + \theta(G_{sh}^{ch,-}(z) - G_{sh}^{ch,+}(z)),
\end{aligned} \tag{4.75}$$

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where

$$L_{sh}^{ch}(z) = \sum_{a \in A} \frac{1}{2} \begin{pmatrix} (t_a - 1)c\partial b + (t_a + 1)\partial cb \\ + t_a \gamma \partial \beta + (t_a + 2)\partial \gamma \beta \end{pmatrix}, \quad (4.76)$$

$$G_{sh}^{ch,+}(z) = \sum_{a \in A} c\beta, \quad (4.77)$$

$$G_{sh}^{ch,-}(z) = \sum_{a \in A} (t_a + 1)\partial \gamma b + (t_a)\gamma \partial b, \quad (4.78)$$

$$J_{sh}^{ch}(z) = \sum_{a \in A} (t_a + 1)cb + t_a \gamma \beta, \quad (4.79)$$

from the equations (4.10), (4.14), (4.15), (4.52) and (4.53). Then we know that the fields L_{sh}^{ch} , $G_{sh}^{ch,+}$, $G_{sh}^{ch,-}$ and J_{sh}^{ch} satisfy the non-SUSY lambda bracket relations of $N = 2$ superconformal vertex algebras.

Remark 4.16. From Definition 4.10 of [11], recall that the lambda brackets of $N_K = 2$ SUSY Lie conformal algebras are given by :

$$\begin{aligned} [u \ \Lambda \ v] &= \sum_{j \geq 0} \frac{\lambda^j}{j!} u_{(j|00)} v - \chi^1 \sum_{j \geq 0} \frac{\lambda^j}{j!} u_{(j|10)} v \\ &\quad - \chi^2 \sum_{j \geq 0} \frac{\lambda^j}{j!} u_{(j|01)} v - \chi^1 \chi^2 \sum_{j \geq 0} \frac{\lambda^j}{j!} u_{(j|11)} v. \end{aligned} \quad (4.80)$$

On the other hand, by sesquilinearity of $N_K = 2$ SUSY LCAs, we have :

$$\begin{aligned} u_{(j|00)} v &= -D^1 D^2 u_{(j|11)} v, \\ u_{(j|10)} v &= D^2 u_{(j|11)} v, \\ u_{(j|01)} v &= -D^1 u_{(j|11)} v, \end{aligned} \quad (4.81)$$

hence we obtain the following equality :

$$\begin{aligned} [u \ \Lambda \ v] &= -[D^1 D^2 u \ \lambda \ v] - \chi^1 [D^2 u \ \lambda \ v] \\ &\quad + \chi^2 [D^1 u \ \lambda \ v] - \chi^1 \chi^2 [u \ \lambda \ v]. \end{aligned} \quad (4.82)$$

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Remark 4.17. Following the Remark 4.15, consider the $N_K = 2$ SUSY vertex algebra structure (see Remark 2.8 in [11]) where the odd endomorphisms are defined by

$$\begin{aligned} D^1 &= (G_{sh}^{ch,+} + G_{sh}^{ch,-})_{(0)} = (G_{st}^{ch,+} + G_{st}^{ch,-})_{(0)}, \\ D^2 &= i(G_{sh}^{ch,+} - G_{sh}^{ch,-})_{(0)} = i(G_{st}^{ch,+} - G_{st}^{ch,-})_{(0)}, \end{aligned} \quad (4.83)$$

and the state-field correspondence is given by

$$\begin{aligned} Y(v, z, \theta^1, \theta^2) &= Y(v, z) + \theta^1 Y(D^1 v, z) \\ &\quad + \theta^2 Y(D^2 v, z) + \theta^2 \theta^1 Y(D^1 D^2 v, z). \end{aligned} \quad (4.84)$$

Let P_{sh}^{ch} be the vector defined as :

$$\begin{aligned} P_{sh}^{ch} &= -i J_{sh(-1)}^{ch} |0\rangle \\ &= -i \sum_{a \in A} (t_a + 1) cb + t_a \gamma \beta. \end{aligned} \quad (4.85)$$

Then the $N = 2$ superfield corresponding to P_{sh}^{ch} is expanded as :

$$\begin{aligned} Y(P_{sh}^{ch}, z, \theta^1, \theta^2) &= -i Y(J_{sh}^{ch}, z) - i \theta^1 Y(D^1 J_{sh}^{ch}, z) \\ &\quad - i \theta^2 Y(D^2 J_{sh}^{ch}, z) - i \theta^2 \theta^1 Y(D^1 D^2 J_{sh}^{ch}, z) \\ &= -i Y(J_{sh}^{ch}, z) - i \theta^1 Y(G_{sh}^{ch,-} - G_{sh}^{ch,+}, z) \\ &\quad - \theta^2 Y(G_{sh}^{ch,+} + G_{sh}^{ch,-}, z) - \theta^2 \theta^1 Y(2L_{sh}^{ch}, z) \\ &= -i Y(J_{sh}^{ch}, z, \theta^1) - \theta^2 Y(T_{sh}^{ch}, z, \theta^1), \end{aligned} \quad (4.86)$$

which follows from the non-SUSY lambda bracket relations of $N = 2$ superconformal vertex algebras.

Proposition 4.18. *The field P_{sh}^{ch} defined in Remark 4.17 satisfies the following lambda bracket :*

$$[P_{sh}^{ch} \wedge P_{sh}^{ch}]_{N_K=2} = (2\partial + 2\lambda + \chi^1 D^1 + \chi^2 D^2) P_{sh}^{ch} + \frac{c_{sh}}{3} \lambda \chi^1 \chi^2. \quad (4.87)$$

Hence, the shifted $N = 2$ superconformal symmetry in Theorem 4.14 is generated by one $N = 2$ superfield P_{sh}^{ch} in the $N_K = 2$ SUSY vertex algebra formalism.

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Proof. Note that the fields L_{sh}^{ch} , $G_{sh}^{ch,+}$, $G_{sh}^{ch,-}$, J_{sh}^{ch} in Remark 4.15 satisfy the non-SUSY lambda bracket relations of $N = 2$ superconformal vertex algebras (see Definition 3.3), and by the definitions of D^1 and D^2 in Remark 4.17, we have :

$$\begin{aligned} D^1 J_{sh}^{ch} &= G_{sh}^{ch,-} - G_{sh}^{ch,+}, \\ iD^2 J_{sh}^{ch} &= G_{sh}^{ch,+} + G_{sh}^{ch,-}, \\ iD^1 D^2 J_{sh}^{ch} &= 2L_{sh}^{ch}. \end{aligned} \tag{4.88}$$

Then, from the equality (4.82) in Remark 4.16, we have :

$$\begin{aligned} [P_{sh}^{ch} \ \Delta \ P_{sh}^{ch}] &= -i[iD^1 D^2 J_{sh}^{ch} \ \lambda \ J_{sh}^{ch}] - i\chi^1[iD^2 J_{sh}^{ch} \ \lambda \ J_{sh}^{ch}] \\ &\quad - \chi^2[D^1 J_{sh}^{ch} \ \lambda \ J_{sh}^{ch}] + \chi^1 \chi^2[J_{sh}^{ch} \ \lambda \ J_{sh}^{ch}] \\ &= -i[2L_{sh}^{ch} \ \lambda \ J_{sh}^{ch}] - i\chi^1[G_{sh}^{ch,+} + G_{sh}^{ch,-} \ \lambda \ J_{sh}^{ch}] \\ &\quad - \chi^2[G_{sh}^{ch,-} - G_{sh}^{ch,+} \ \lambda \ J_{sh}^{ch}] + \chi^1 \chi^2[J_{sh}^{ch} \ \lambda \ J_{sh}^{ch}] \\ &= -i(2\partial + 2\lambda)J_{sh}^{ch} - i\chi^1(G_{sh}^{ch,-} - G_{sh}^{ch,+}) \\ &\quad - \chi^2(G_{sh}^{ch,+} + G_{sh}^{ch,-}) + \chi^1 \chi^2 \frac{C_{sh}}{3} \lambda \\ &= -i(2\partial + 2\lambda)J_{sh}^{ch} - i\chi^1 D^1 J_{sh}^{ch} \\ &\quad - i\chi^2 D^2 J_{sh}^{ch} + \chi^1 \chi^2 \frac{C_{sh}}{3} \lambda \\ &= (2\partial + 2\lambda + \chi^1 D^1 + \chi^2 D^2)(-iJ_{sh}^{ch}) + \frac{C_{sh}}{3} \lambda \chi^1 \chi^2. \end{aligned} \tag{4.89}$$

□

Remark 4.19. Now, using the notations in Remark 4.3, let us define the $N = 2$ superfields Φ_a and $\Phi^{\bar{a}}$ as :

$$\begin{aligned} \Phi_a &= Y(\gamma, z, \theta^1, \theta^2) \\ &= Y(\gamma, z) + \theta^1 Y(D^1 \gamma, z) + \theta^2 Y(D^2 \gamma, z) + \theta^2 \theta^1 Y(D^1 D^2 \gamma, z) \\ &= Y(\gamma, z) + \theta^1 Y(c, z) + \theta^2 Y(ic, z) + \theta^2 \theta^1 Y(i\partial \gamma, z) \\ &= \phi_a + i\theta^2 D^1 \phi_a, \end{aligned} \tag{4.90}$$

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$$\begin{aligned}
\Phi^{\bar{a}} &= Y(b, z, \theta^1, \theta^2) \\
&= Y(b, z) + \theta^1 Y(D^1 b, z) + \theta^2 Y(D^2 b, z) + \theta^2 \theta^1 Y(D^1 D^2 b, z) \\
&= Y(b, z) + \theta^1 Y(\beta, z) + \theta^2 Y(i\beta, z) + \theta^2 \theta^1 Y(i\partial b, z) \\
&= \phi^{\bar{a}} + i\theta^2 D^1 \phi^{\bar{a}},
\end{aligned} \tag{4.91}$$

where the $N_K = 2$ SUSY vertex algebra structure is given by Remark 4.17. Then the Λ -bracket of Φ_a and $\Phi^{\bar{a}}$ is :

$$[\Phi_a \wedge \Phi^{\bar{a}}] = -i\chi^1 + \chi^2. \tag{4.92}$$

We also know that :

$$\begin{aligned}
D^1 \Phi_a &= D^1 \phi_a - i\theta^2 \partial \phi_a, & D^1 \Phi^{\bar{a}} &= D^1 \phi^{\bar{a}} - i\theta^2 \partial \phi^{\bar{a}}, \\
D^2 \Phi_a &= iD^1 \phi_a + \theta^2 \partial \phi_a, & D^2 \Phi^{\bar{a}} &= iD^1 \phi^{\bar{a}} + \theta^2 \partial \phi^{\bar{a}},
\end{aligned} \tag{4.93}$$

so that, for any even element $a \in A_{\bar{0}}$, we have :

$$\begin{aligned}
D^1 \Phi_a \Phi^{\bar{a}} &= D^1 \phi_a \phi^{\bar{a}} - i\theta^2 (\partial \phi_a \phi^{\bar{a}} + D^1 \phi_a D^1 \phi^{\bar{a}}), \\
\Phi_a D^1 \Phi^{\bar{a}} &= \phi_a D^1 \phi^{\bar{a}} - i\theta^2 (\phi_a \partial \phi^{\bar{a}} - D^1 \phi_a D^1 \phi^{\bar{a}}),
\end{aligned} \tag{4.94}$$

and, for any odd element $a \in A_{\bar{1}}$:

$$\begin{aligned}
D^1 \Phi^{\bar{a}} \Phi_a &= D^1 \phi^{\bar{a}} \phi_a - i\theta^2 (\phi_a \partial \phi^{\bar{a}} + D^1 \phi_a D^1 \phi^{\bar{a}}), \\
\Phi^{\bar{a}} D^1 \Phi_a &= \phi^{\bar{a}} D^1 \phi_a - i\theta^2 (\partial \phi_a \phi^{\bar{a}} - D^1 \phi_a D^1 \phi^{\bar{a}}).
\end{aligned} \tag{4.95}$$

Therefore, the $N = 2$ superfield P_{sh}^{ch} in (4.86) can be written as :

$$\begin{aligned}
P_{sh}^{ch} &= -i \left(\begin{aligned} &\sum_{a \in A_{\bar{0}}} (t_a + 1) D^1 \Phi_a \Phi^{\bar{a}} + t_a \Phi_a D^1 \Phi^{\bar{a}} \\ &+ \sum_{a \in A_{\bar{1}}} (t_a + 1) D^1 \Phi^{\bar{a}} \Phi_a + t_a \Phi^{\bar{a}} D^1 \Phi_a \end{aligned} \right) \\
&= -i \left(\begin{aligned} &\sum_{a \in A_{\bar{0}}} (t_a + 1) D^1 \Phi_a \Phi^{\bar{a}} + t_a \Phi_a D^1 \Phi^{\bar{a}} \\ &+ \sum_{a \in A_{\bar{1}}} t_a D^1 \Phi_a \Phi^{\bar{a}} - (t_a + 1) \Phi_a D^1 \Phi^{\bar{a}} \end{aligned} \right).
\end{aligned} \tag{4.96}$$

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Then, by using the $N_K = 2$ lambda bracket (4.92) and the relations :

$$iD^1\Phi_a = D^2\Phi_a, \quad iD^1\Phi^{\bar{a}} = D^2\Phi^{\bar{a}}, \quad (4.97)$$

one can also check directly the field P_{sh}^{ch} is an $N = 2$ superconformal vector (see Theorem 4.21).

Definition 4.20. Let U be a finite-dimensional vector superspace and $A = A_{\bar{0}} \cup A_{\bar{1}}$ be a basis of U , where $A_{\bar{0}}$ is the even part and $A_{\bar{1}}$ is the odd part of the basis. Define two vector superspaces :

$$\Phi_U \simeq U, \quad \Phi^{\bar{U}} \simeq \Pi U^*, \quad (4.98)$$

whose basis elements are denoted by Φ_a and $\Phi^{\bar{a}}$ respectively. Define the Λ -brackets on $R_{N=2}^{bc\beta\gamma} = \mathcal{H}_{N=2} \otimes (\Phi_U \oplus \Phi^{\bar{U}})$ by :

$$[\Phi_a \wedge \Phi^{\bar{b}}] = \delta_{ab}(-i\chi^1 + \chi^2), \quad [\Phi_a \wedge \Phi_b] = [\Phi^{\bar{a}} \wedge \Phi^{\bar{b}}] = 0, \quad (4.99)$$

for $a, b \in A$. Then the $\mathcal{H}_{N=2}$ -module $R_{N=2}^{bc\beta\gamma}$ is an $N_K = 2$ SUSY Lie conformal algebra. Now let $V(R_{N=2}^{bc\beta\gamma})$ be the universal enveloping $N_K = 2$ SUSY vertex algebra of $R_{N=2}^{bc\beta\gamma}$, and let $I(R_{N=2}^{bc\beta\gamma})$ be the ideal generated by elements of the form (4.97). Then we define the $N = 2$ supersymmetric bc - $\beta\gamma$ system associated to the vector superspace U as the quotient $V(R_{N=2}^{bc\beta\gamma})/I(R_{N=2}^{bc\beta\gamma})$.

Theorem 4.21. Let P_{sh}^{ch} be the vector defined by

$$P_{sh}^{ch} = -i \left(\begin{array}{l} \sum_{a \in A_{\bar{0}}} (t_a + 1) D^1 \Phi_a \Phi^{\bar{a}} + t_a \Phi_a D^1 \Phi^{\bar{a}} \\ + \sum_{a \in A_{\bar{1}}} (t_a + 1) D^1 \Phi^{\bar{a}} \Phi_a + t_a \Phi^{\bar{a}} D^1 \Phi_a \end{array} \right), \quad (4.100)$$

where $(t_a)_{a \in A}$ is a sequence of complex numbers. Then the vector P_{sh}^{ch} is an $N = 2$ superconformal vector of the $N = 2$ SUSY bc - $\beta\gamma$ system with the central charge

$$c_{sh} = \sum_{a \in A} (6t_a + 3). \quad (4.101)$$

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Proof. Let a be an even element of A . From the non-commutative Wick formula of $N_K = 2$ SUSY LCAs, we have :

$$[\Phi_a \wedge D^1 \Phi_a \Phi^{\bar{a}}] = (i\chi^1 D^1 - \chi^2 D^1) \Phi_a, \quad (4.102)$$

hence, by skew-symmetry :

$$\begin{aligned} [D^1 \Phi_a \Phi^{\bar{a}} \wedge \Phi_a] &= -[\Phi_a \wedge_{-\nabla-\Lambda} D^1 \Phi_a \Phi^{\bar{a}}] \\ &= (-i(-D^1 - \chi^1)D^1 + (-D^2 - \chi^2)D^1) \Phi_a \\ &= i(2\partial + \chi^1 D^1 + \chi^2 D^2) \Phi_a. \end{aligned} \quad (4.103)$$

Therefore we have :

$$\begin{aligned} [D^1 \Phi_a \Phi^{\bar{a}} \wedge D^1 \Phi_a] &= i(D^1 + \chi^1)(2\partial + \chi^1 D^1 + \chi^2 D^2) \Phi_a \\ &= i(2\partial + \lambda + \chi^1 D^1 + \chi^2 D^2) D^1 \Phi_a - \chi^1 \chi^2 D^1 \Phi_a. \end{aligned} \quad (4.104)$$

Similarly, we obtain :

$$[D^1 \Phi_a \Phi^{\bar{a}} \wedge \Phi^{\bar{a}}] = i(2\partial + \lambda + \chi^1 D^1 + \chi^2 D^2) \Phi^{\bar{a}} + \chi^1 \chi^2 \Phi^{\bar{a}}. \quad (4.105)$$

By the Λ -brackets in (4.104) and (4.105), and from the non-commutative Wick formula, we have :

$$\begin{aligned} [D^1 \Phi_a \Phi^{\bar{a}} \wedge D^1 \Phi_a \Phi^{\bar{a}}] &= [D^1 \Phi_a \Phi^{\bar{a}} \wedge D^1 \Phi_a] \Phi^{\bar{a}} + D^1 \Phi_a [D^1 \Phi_a \Phi^{\bar{a}} \wedge \Phi^{\bar{a}}] \\ &\quad + \int_0^\Lambda [[D^1 \Phi_a \Phi^{\bar{a}} \wedge D^1 \Phi_a] \wedge \Phi^{\bar{a}}] d\Gamma \\ &= i(2\partial + 2\lambda + \chi^1 D^1 + \chi^2 D^2) (D^1 \Phi_a \Phi^{\bar{a}}) - \lambda \chi^1 \chi^2. \end{aligned} \quad (4.106)$$

On the other hand, we can compute :

$$[\Phi_a \Phi^{\bar{a}} \wedge \Phi_a \Phi^{\bar{a}}] = 0. \quad (4.107)$$

From the Λ -brackets in (4.103) and (4.105), we have :

$$\begin{aligned} [D^1 \Phi_a \Phi^{\bar{a}} \wedge \Phi_a \Phi^{\bar{a}}] &= i(2\partial + \lambda + \chi^1 D^1 + \chi^2 D^2) (\Phi_a \Phi^{\bar{a}}) \\ &\quad + \chi^1 \chi^2 (\Phi_a \Phi^{\bar{a}}) - \lambda(-i\chi^1 + \chi^2). \end{aligned} \quad (4.108)$$

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Hence, using sesquilinearity, we find :

$$[D^1\Phi_a\Phi^{\bar{a}} \wedge D^1(\Phi_a\Phi^{\bar{a}})] = i(2\partial + 2\lambda + \chi^1 D^1 + \chi^2 D^2)(D^1(\Phi_a\Phi^{\bar{a}})) \\ + (i\lambda\chi^1 + \lambda\chi^2)(\Phi_a\Phi^{\bar{a}}) - i\lambda^2 - \lambda\chi^1\chi^2, \quad (4.109)$$

and, by skew-symmetry, we have :

$$[D^1(\Phi_a\Phi^{\bar{a}}) \wedge D^1\Phi_a\Phi^{\bar{a}}] = -(i\lambda\chi^1 + \lambda\chi^2)(\Phi_a\Phi^{\bar{a}}) + i\lambda^2 - \lambda\chi^1\chi^2. \quad (4.110)$$

Combining the results in (4.106), (4.107), (4.109) and (4.110), we obtain :

$$[D^1\Phi_a\Phi^{\bar{a}} + t_a D^1(\Phi_a\Phi^{\bar{a}}) \wedge D^1\Phi_a\Phi^{\bar{a}} + t_a D^1(\Phi_a\Phi^{\bar{a}})] \\ = i(2\partial + 2\lambda + \chi^1 D^1 + \chi^2 D^2)(D^1\Phi_a\Phi^{\bar{a}} + t_a D^1(\Phi_a\Phi^{\bar{a}})) \\ - (2t_a + 1)\lambda\chi^1\chi^2, \quad (4.111)$$

hence, we get the result. □

Appendix A

Calculations of lambda brackets

For any even element a in A , we have the following lambda brackets in $F_{N=1}^{ch}$.

Lemma A.1. *Lambda brackets for $\partial\phi_a\phi^{\bar{a}}$:*

$$[\partial\phi_a\phi^{\bar{a}} \wedge \phi_a] = \partial\phi_a, \quad (\text{A.1})$$

$$[\partial\phi_a\phi^{\bar{a}} \wedge \phi^{\bar{a}}] = (\partial + \lambda)\phi^{\bar{a}}, \quad (\text{A.2})$$

$$[\partial\phi_a\phi^{\bar{a}} \wedge D\phi_a] = (D + \chi)\partial\phi_a, \quad (\text{A.3})$$

$$[\partial\phi_a\phi^{\bar{a}} \wedge D\phi^{\bar{a}}] = (D + \chi)(\partial + \lambda)\phi^{\bar{a}}, \quad (\text{A.4})$$

$$[\partial\phi_a\phi^{\bar{a}} \wedge \partial\phi_a] = (\partial + \lambda)\partial\phi_a, \quad (\text{A.5})$$

$$[\partial\phi_a\phi^{\bar{a}} \wedge \partial\phi^{\bar{a}}] = (\partial + \lambda)^2\phi^{\bar{a}}. \quad (\text{A.6})$$

Proof. The \wedge -brackets can be obtained by using the rules of $N_K = 1$ SUSY vertex algebras. For the \wedge -bracket in (A.1) :

$$\begin{aligned} [\phi_a \wedge \partial\phi_a\phi^{\bar{a}}] &= [\phi_a \wedge \partial\phi_a]\phi^{\bar{a}} + (-1)^{(a+1)a}\partial\phi_a[\phi_a \wedge \phi^{\bar{a}}] \\ &\quad + \int_0^\Lambda [[\phi_a \wedge \partial\phi_a] \Gamma \phi^{\bar{a}}] d\Gamma \\ &= \partial\phi_a, \end{aligned}$$

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since, from sesquilinearity, we know that :

$$[\phi_a \wedge \partial\phi_a] = (\partial + \lambda)[\phi_a \wedge \phi_a] = 0.$$

Hence, by skew-symmetry, we get :

$$[\partial\phi_a\phi^{\bar{a}} \wedge \phi_a] = (-1)^{(a+a+1)a}[\phi_a \wedge \partial\phi_a\phi^{\bar{a}}] = \partial\phi_a.$$

Similarly, the other \wedge -brackets also can be obtained by using the rules of \wedge -brackets of $N_K = 1$ SUSY vertex algebras. \square

Lemma A.2. *Lambda brackets for $\phi_a\partial\phi^{\bar{a}}$:*

$$[\phi_a\partial\phi^{\bar{a}} \wedge \phi_a] = -(\partial + \lambda)\phi_a, \quad (\text{A.7})$$

$$[\phi_a\partial\phi^{\bar{a}} \wedge \phi^{\bar{a}}] = -\partial\phi^{\bar{a}}, \quad (\text{A.8})$$

$$[\phi_a\partial\phi^{\bar{a}} \wedge D\phi_a] = -(D + \chi)(\partial + \lambda)\phi_a, \quad (\text{A.9})$$

$$[\phi_a\partial\phi^{\bar{a}} \wedge D\phi^{\bar{a}}] = -(D + \chi)\partial\phi^{\bar{a}}, \quad (\text{A.10})$$

$$[\phi_a\partial\phi^{\bar{a}} \wedge \partial\phi_a] = -(\partial + \lambda)^2\phi_a, \quad (\text{A.11})$$

$$[\phi_a\partial\phi^{\bar{a}} \wedge \partial\phi^{\bar{a}}] = -(\partial + \lambda)\partial\phi^{\bar{a}}. \quad (\text{A.12})$$

Proof. It follows from direct calculations. \square

Lemma A.3. *Lambda brackets for $D\phi_a D\phi^{\bar{a}}$:*

$$[D\phi_a D\phi^{\bar{a}} \wedge \phi_a] = (\partial + \chi D)\phi_a, \quad (\text{A.13})$$

$$[D\phi_a D\phi^{\bar{a}} \wedge \phi^{\bar{a}}] = (\partial + \chi D)\phi^{\bar{a}}, \quad (\text{A.14})$$

$$[D\phi_a D\phi^{\bar{a}} \wedge D\phi_a] = (\partial + \lambda)D\phi_a, \quad (\text{A.15})$$

$$[D\phi_a D\phi^{\bar{a}} \wedge D\phi^{\bar{a}}] = (\partial + \lambda)D\phi^{\bar{a}}, \quad (\text{A.16})$$

$$[D\phi_a D\phi^{\bar{a}} \wedge \partial\phi_a] = (\partial + \lambda)(\partial + \chi D)\phi_a, \quad (\text{A.17})$$

$$[D\phi_a D\phi^{\bar{a}} \wedge \partial\phi^{\bar{a}}] = (\partial + \lambda)(\partial + \chi D)\phi^{\bar{a}}. \quad (\text{A.18})$$

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Proof. The results can be verified by direct computations. \square

Lemma A.4. *Lambda brackets for $D\phi_a\phi^{\bar{a}}$ and $\phi_a D\phi^{\bar{a}}$:*

$$[D\phi_a\phi^{\bar{a}} \wedge \phi_a] = -D\phi_a, \quad (\text{A.19})$$

$$[D\phi_a\phi^{\bar{a}} \wedge \phi^{\bar{a}}] = -(D + \chi)\phi^{\bar{a}}, \quad (\text{A.20})$$

$$[D\phi_a\phi^{\bar{a}} \wedge D\phi_a] = (\partial + \chi D)\phi_a, \quad (\text{A.21})$$

$$[D\phi_a\phi^{\bar{a}} \wedge D\phi^{\bar{a}}] = (\partial + \lambda)\phi^{\bar{a}}, \quad (\text{A.22})$$

$$[\phi_a D\phi^{\bar{a}} \wedge \phi_a] = (D + \chi)\phi_a, \quad (\text{A.23})$$

$$[\phi_a D\phi^{\bar{a}} \wedge \phi^{\bar{a}}] = D\phi^{\bar{a}}, \quad (\text{A.24})$$

$$[\phi_a D\phi^{\bar{a}} \wedge D\phi_a] = -(\partial + \lambda)\phi_a, \quad (\text{A.25})$$

$$[\phi_a D\phi^{\bar{a}} \wedge D\phi^{\bar{a}}] = -(\partial + \chi D)\phi^{\bar{a}}. \quad (\text{A.26})$$

Proof. We also get the results from direct computations. \square

Appendix B

Charge decomposition

Definition B.1. As in Section 5.1 of [12], the eigenvalue of $J_{sh(0|1)}^{ch}$ corresponding to $v \in F_{N=1}^{ch}$ is called the *charge* of v . Then, for any even element a in A , the charges of ϕ_a and $\phi^{\bar{a}}$ are t_a and $-(t_a + 1)$ respectively. For the odd elements, the charges of ϕ_a and $\phi^{\bar{a}}$ are $-(t_a + 1)$ and t_a respectively. Also, if we define the *BRST operator* Q of $F_{N=1}^{ch}$ as follows:

$$Q = \frac{1}{2} \left(T_{sh(0|1)}^{ch} - J_{sh(0|0)}^{ch} \right), \quad (\text{B.1})$$

then we obtain the following result.

Proposition B.2. *The BRST operator Q of $F_{N=1}^{ch}$ satisfies $Q^2 = 0$.*

Proof. If we set $d = \sum_{a \in A} D\phi_a D\phi^{\bar{a}}$, we know that :

$$\begin{aligned} Q &= \frac{1}{2} \left(T_{sh(0|1)}^{ch} - J_{sh(0|0)}^{ch} \right) \\ &= \frac{1}{2} \left(T_{sh(0|1)}^{ch} - D J_{sh(0|1)}^{ch} \right) = d_{(0|1)}, \end{aligned} \quad (\text{B.2})$$

by sesquilinearity of $N_K = 1$ supersymmetric LCAs. Also we have :

$$\begin{aligned} [d \wedge [d \wedge v]] &= [[d \wedge d] \wedge v] + [d \wedge [d \wedge v]] \\ &= [\partial d + 2\lambda d \wedge v] + [d \wedge [d \wedge v]] \\ &= (\lambda - \gamma)[d \wedge v] + [d \wedge [d \wedge v]], \end{aligned} \quad (\text{B.3})$$

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for $v \in F_{N=1}^{ch}$, from the Jacobi identity of $N_K = 1$ supersymmetric LCAs and the equation (4.29) in Lemma 4.4. Hence, by comparing the $\chi\eta$ -terms in (B.3), we have the result :

$$d_{(0|1)}(d_{(0|1)}v) = 0. \quad (\text{B.4})$$

□

Remark B.3. The BRST operator Q is invariant under any values of t_a , that is :

$$Q = \frac{1}{2} \left(T_{st}^{ch}{}_{(0|1)} - J_{st}^{ch}{}_{(0|0)} \right). \quad (\text{B.5})$$

Indeed, if we decompose the fields T_{sh}^{ch} and J_{sh}^{ch} as in (4.75), then the BRST operator Q is the zero mode of the field $G_{sh}^{ch,+}$. On the other hand, the ghost term contributes to the *homotopy operator* H of $F_{N=1}^{ch}$ defined by :

$$H = \frac{1}{2} \left(T_{sh}^{ch}{}_{(0|1)} + J_{sh}^{ch}{}_{(0|0)} \right) = (T_{sh}^{ch} - d)_{(0|1)}, \quad (\text{B.6})$$

which is the zero mode of the field $G_{sh}^{ch,-}$.

Proposition B.4. *Let $v \in F_{N=1}^{ch}$ be a vector of the charge m . Then $Q(v)$ has the charge $m + 1$.*

Proof. From the Lemma 4.13, we have :

$$[d \wedge J_{sh}^{ch}] = (\partial + \lambda)J_{sh}^{ch} - \chi d + \frac{c_{sh}}{6}\lambda^2, \quad (\text{B.7})$$

where $d = \sum_{a \in A} D\phi_a D\phi^{\bar{a}}$. Hence we have the following Λ -bracket :

$$\begin{aligned} [J_{sh}^{ch} \wedge d] &= [d \wedge_{-\Lambda-\nabla} J_{sh}^{ch}] \\ &= -\lambda J_{sh}^{ch} + (\chi + D)d + \frac{c_{sh}}{6}\lambda^2, \end{aligned} \quad (\text{B.8})$$

by skew-symmetry of $N_K = 1$ supersymmetric LCAs, and then :

$$\begin{aligned} [J_{sh}^{ch} \wedge [d \wedge_{\Gamma} v]] &= -[[J_{sh}^{ch} \wedge d] \wedge_{\Lambda+\Gamma} v] + [d \wedge_{\Gamma} [J_{sh}^{ch} \wedge v]] \\ &= [\lambda J_{sh}^{ch} - (\chi + D)d - \frac{c_{sh}}{6} \wedge_{\Lambda+\Gamma} v] + [d \wedge_{\Gamma} [J_{sh}^{ch} \wedge v]] \\ &= \lambda [J_{sh}^{ch} \wedge_{\Lambda+\Gamma} v] - \eta [d \wedge_{\Lambda+\Gamma} v] + [d \wedge_{\Gamma} [J_{sh}^{ch} \wedge v]], \end{aligned} \quad (\text{B.9})$$

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from the Jacobi identity of $N_K = 1$ supersymmetric LCAs. By comparing the coefficients of $\chi\eta$ -terms in (B.9), we obtain the result :

$$\begin{aligned} J_{sh(0|1)}^{ch}Q(v) &= J_{sh(0|1)}^{ch}(d_{(0|1)}v) \\ &= d_{(0|1)}v + d_{(0|1)}(J_{sh(0|1)}^{ch}v) \\ &= Q(v) + Q(mv) = (m+1)Q(v). \end{aligned} \tag{B.10}$$

□

Remark B.5. For the BRST operator Q and the homotopy operator H of $F_{N=1}^{ch}$, defined in Remark B.3, we have the following result from the simple calculations :

$$\begin{aligned} Q(D^i\phi) &= \begin{cases} D^{i+1}\phi, & \text{if } i \in 2\mathbb{Z}_{\geq 0} \\ 0, & \text{if } i \in 2\mathbb{Z}_{\geq 0} + 1, \end{cases} \\ H(D^i\phi) &= \begin{cases} 0, & \text{if } i \in 2\mathbb{Z}_{\geq 0} \\ D^{i+1}\phi, & \text{if } i \in 2\mathbb{Z}_{\geq 0} + 1, \end{cases} \end{aligned} \tag{B.11}$$

where ϕ stands either ϕ_a or $\phi_{\bar{a}}$ for any $a \in A$. Also the homotopy operator H decreases the charge by -1 . Moreover, if the charges of $u, v \in F_{N=1}^{ch}$ are m_u and m_v respectively, then uv has the charge $m_u + (-1)^u m_v$.

Proposition B.6. For $(t_a)_{a \in A} \in \mathbb{Z}^{\dim U}$, the supersymmetric charged free fermion vertex algebra has a \mathbb{Z} -grading :

$$F_{N=1}^{ch} = \bigoplus_{m \in \mathbb{Z}} F_{N=1}^{ch,m}, \tag{B.12}$$

called the charge decomposition, where $F_{N=1}^{ch,m}$ is the subspace whose elements have the charge m .

Proof. Combining the facts in Remark B.5 and Proposition B.4, we obtain the result. □

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국문초록

본 학위 논문에서는, 기존의 초대칭 $bc\text{-}\beta\gamma$ 체계의 초등각대칭 벡터를 확장시킨 형태의 초대칭 자유페르미온 꼭지점대수의 $N = 1$ 초등각대칭 벡터를 정의하고, 이와 함께 쌍을 이루는 $N = 2$ 초등각대칭을 찾았다. 그리고 이 결과들을 $N_K = 2$ 초대칭 꼭지점대수의 언어를 사용하여 나타내었다.

주요어휘: 초대칭 꼭지점대수, 초등각대칭 꼭지점대수, 초대칭 자유페르미온 꼭지점대수, 초등각대칭벡터

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