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From giant gravitons to black holes

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Abstract: We study AdS_5 black holes from a recently suggested giant graviton expansion formula for the index of $\mathrm{U}(N)$ maximal super-Yang-Mills theory. We compute the large N entropy at fixed charges and giant graviton numbers n_I by a saddle point analysis, and further maximize it in n_I. This agrees with the dual black hole entropy in the small black hole limit. To get black holes at general sizes, one should note that various giant graviton indices cancel because gauge theory does not suffer from a Hagedorn-like pathology by an infinite baryonic tower. With one assumption on the mechanism of this cancellation, we account for the dual black hole entropy at general sizes. We interpret our results as analytic continuations of the large N free energies of SCFTs, and based on it compute the entropies of $\mathrm{AdS}_{4,7}$ black holes from M5, M2 giant gravitons.

Keywords: AdS-CFT Correspondence, Black Holes in String Theory, D-Branes

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1 Introduction

Understanding the Hilbert space is the most basic task in quantum physics. It has also been a key problem of large N gauge theories and AdS/CFT, often with a focus on the emergent bulk descriptions. For instance, consider 4d maximal super-Yang-Mills theory on $S^3 \times \mathbb{R}$, dual to the type IIB theory on global $AdS_5 \times S^5$. We normalize the energy E to be dimensionless by multiplying the AdS radius. At $E \sim N^0$, the spectrum is described by the gas of gravitons. At $E \sim \sqrt{N g_{\rm YM}^2}$, where $g_{\rm YM}$ is the Yang-Mills coupling, stringy excitations enter. At $E \sim N^1$, a novel finite N effect enters. On the QFT side, this comes from the finite size of the $N \times N$ matrices, imposing trace relations on gauge-invariant operators. In the gravity dual, this is realized as the gravitons polarizing to D3-branes. The branes can stretch either in S^5, called giant gravitons [1], or in AdS_5, called dual giant gravitons [2, 3]. The trace relations are realized either by giant gravitons having a maximal size, or the dual giant gravitons having a maximal number [4, 5]. These descriptions use probe D-brane approaches, whose validity requires that the energy is not too large, say $E \ll N^2$. This is merely a technical limitation, and the concept of giant gravitons may exist at higher energies and provide useful insights. This turned out to be the case in the half-BPS sector [6].

At $E \sim N^2$, semi-classical black hole solutions represent ensembles of states. In this paper, we wish to clarify the giant graviton picture at $E \sim N^2$, studying how black holes emerge from a giant graviton description of the spectral problem. We consider the BPS sector of the maximal super-Yang-Mills theory through the index of [7, 8]. This index has been studied to better understand the dual BPS black holes [9–12]. see [13–15] and references thereof. We shall study the recently suggested reformulation of this index [16–20], called 'giant graviton expansion.' We shall mainly consider the formula by Yosuke Imamura [16].[1]

The giant graviton expansion of the index is given by

$$Z(\Delta_I,\omega_i)=Z_{\rm KK}(\Delta_I,\omega_i)\sum_{n_1,n_2,n_3=0}^{\infty}e^{-N\sum_{I=1}^{3}n_I\Delta_I}Z_{n_1,n_2,n_3}(\Delta_I,\omega_i)\,,\quad I=1,2,3\,,\ i=1,2\,. \tag{1.1}$$

See section 2 for detailed explanations. n_I's are winding numbers of maximal giant gravitons along three different S^3 cycles in S^5. Z_{n_1,n_2,n_3} is 'formally' an index of a $\mathrm{U}(n_1)\times\mathrm{U}(n_2)\times\mathrm{U}(n_3)$ quiver gauge theory, consisting of 4d/2d fields on the D-branes or at their intersections. When $E\sim N^2$, we expect all n_I's are typically at the order of N^1. Our strategy of studying this index is roughly as follows. We first find certain large n_I ($\sim N^1$) saddle points of the integral representation of Z_{n_1,n_2,n_3}. The contour of this integral is complicated and empirically determined only for low n_I's. The contour information is in principle important to decide whether a saddle is relevant for approximating the integral, through the Picard-Lefschetz theory. As often done in practical studies of challenging integrals, we ignore this issue and assume that our saddles are relevant. After Legendre transforming the free energy at large fixed charges q_I and n_I's, one obtains a macroscopic entropy $S(q_I,n_I)$. Further maximizing it in n_I's to find the dominant term Z_{n_1,n_2,n_3}, one would naively find the entropy at fixed q_I. For a reason to be explained, this strategy is correct only in the 'small black hole limit' $\frac{q_I}{N^2}\ll 1$.[2]

To understand why (1.1) is a subtle formula, one should note that n_I's are the numbers of determinant operators in gauge theory, which are morally baryons. Baryons and mesons provide towers of confining spectrum, responsible for fast growth of the high energy density of states. However, since the basic degrees of freedom are gluons, the growth for gauge theories should be much slower. Therefore, for an expansion like (1.1) to correctly capture the gauge theory entropy at high energies, one expects substantial cancellations of different Z_{n_1,n_2,n_3}'s. For instance, if such cancellations do not happen, we shall see that the series (1.1) exhibits very fast growth at large n_I's and cause a Hagedorn-like pathology [21, 22]: due to string and brane states, the canonical partition function becomes ill-defined. This implies that individual $S(q_I,n_I)$ and Z_{n_1,n_2,n_3} lose physical meanings at high energies, while the series (1.1) itself may be physical after cancellations. To address the cancellations rigorously, one should be able to compute the subleading terms in the large N limit. This is beyond the scope of this paper (and the subleading terms depend on the contour choice for Z_{n_1,n_2,n_3}). We shall rather assume a particular mechanism of how the apparently leading contributions

[1]We understand that there is a subtlety in the contour choices in this formula, related to a chemical potential called $a_{\rm loop}$ [16, 19]. We shall comment on it in section 2 when it seems to be relevant.

[2]For the semi-classical black hole solution to be reliable, the normalized charge $\epsilon\equiv\frac{q}{N^2}$ has to be independent of large N. The small black hole limit is defined by ϵ (independent of N) being parametrically smaller than 1. Geometrically, the size of the small black hole r_+ should be much smaller than AdS radius (which we set to 1), but much larger than the 5d Planck length l_P (defined by Newton constant $G\sim l_P^3\sim N^{-2}$) for semiclassical approximation: $N^{-2/3}\ll r_+\ll 1$. r_+ and ϵ are related as $r_+\sim\epsilon^{1/2}$, which for instance can be easily seen from the entropy formula, $S\sim N^2r_+^3\sim\frac{q^{3/2}}{N}$ (see the first line of (2.27)). So ϵ can be chosen to satisfy $N^{-4/3}\ll\epsilon\ll 1$ for large N.

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$e^{S(q_I,n_I)}$ cancel, and then proceed to compute the true entropy that exactly accounts for the dual black holes. As we explain in section 2.2, the mechanism we suggest assumes that the summations over discrete n_I's can be approximated by integrations of these variables, which is valid only if the subleading terms in the large N limit are arranged suitably.

As a byproduct, we find an emergent 2d QFT-like structure in our large N calculation when the three chemical potentials for the $\mathrm{U}(1)^3 \subset \mathrm{SO}(6)$ electric charges are equal. This has to do with the integrand of Z_{n_1,n_2,n_3} reducing to the ratios of theta functions. This concretely justifies a study of [8], which assumed the existence of a hypothetical 2d CFT on the worldvolume of giant gravitons and then proceeded to count small black holes.

Our large N results can be interpreted as an analytic continuation of the maximal super-Yang-Mills index, extending the idea of [17, 20]. After establishing this interpretation on $AdS_5 \times S^5$, we apply it to the large N index on $AdS_4 \times S^7$ based on the expansion of M5-brane giant gravitons [23]. Namely, just assuming the existence of such an expansion and very basic structures, we explain the entropy of the AdS_4 black holes from the large N free energy of 6d SCFTs on M5-branes. Similarly, we find a relation between the entropies of black holes on $AdS_7 \times S^4$ and the large N free energies of 3d SCFTs on M2-branes.

The rest of this paper is organized as follows. In section 2.1, we present a saddle point analysis of Z_{n_1,n_2,n_3}, and show that it accounts for the small black hole entropy. In section 2.2 we explain a possible way in which different Z_{n_1,n_2,n_3} can cancel at general charges. Then assuming this, we compute the true asymptotic large N entropy accounting for the dual black holes. Section 2.3 comments on the similar analysis with three unequal electric charges. In section 3, we make an interpretation of our results from analytic continuations and generalize it to account for the entropies of BPS black holes in $\mathrm{AdS}_{4,7}$. Section 4 concludes with discussions.

2 Giant graviton index and black holes

The index for the $\mathcal{N}=4$ Yang-Mills theory is defined by

$$Z(\Delta_I,\omega_i) = \mathrm{Tr}\left[(-1)^F e^{-\sum_{I=1}^3 \Delta_I Q_I - \sum_{i=1}^2 \omega_i J_i}\right] \tag{2.1}$$

subject to the condition $\sum_{I=1}^3 \Delta_I - \sum_{i=1}^2 \omega_i = 2\pi i\mathbb{Z}$, where Q_I are $\mathrm{U}(1)^3 \subset \mathrm{SO}(6)$ R-charges and J_i are $\mathrm{U}(1)^2 \subset \mathrm{SO}(4)$ angular momenta. See [7, 8] for a unitary matrix integral representation of this index for the $\mathrm{U}(N)$ gauge group. Recently, an alternative expression for this index was proposed. It takes the form of (1.1), where Z_{KK} is the index of low energy gravitons [8]. $Z_{n_1,n_2,n_3}(\Delta_I,\omega_i)$ is given by a $\mathrm{U}(n_1)\times\mathrm{U}(n_2)\times\mathrm{U}(n_3)$ matrix integral of the form [16]

$$Z_{n_1,n_2,n_3} = \oint \prod_{I=1}^{3}\prod_{a=1}^{n_I} du_a^{(I)} \cdot \prod_{I=1}^{3} Z_I^{\mathrm{4d}} \cdot Z_{I,I+1}^{\mathrm{2d}}\,, \quad \text{where} \quad I+3\sim I\,, \tag{2.2}$$

while $Z_{0,0,0}\equiv 1$. The functions Z_I^{4d} and $Z_{I,I+1}^{\mathrm{2d}}$ appearing in the integrand are given as follows. From now on, let us define $\Delta_I \equiv -2\pi i\tau_I = -2\pi i(\tau+z_I)$ with $\sum_{I=1}^3 z_I = 0$, and

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$\omega_1 = -2\pi i(\frac{3\tau}{2} + y - 1)$, $\omega_2 = -2\pi i(\frac{3\tau}{2} - y)$. Then $Z_3^{\rm 4d}$ from the 4d U(n_3) adjoint fields is given by

$$Z_3^{\rm 4d} = \prod_{a,b} \frac{\Gamma(u_{ab}^{(3)} - \tau - z_3; \tau + z_1, \tau + z_2)\Gamma(u_{ab}^{(3)} + \frac{3}{2}\tau \pm y; \tau + z_1, \tau + z_2)}{\Gamma(u_{ab}^{(3)}; \tau + z_1, \tau + z_2)} . \tag{2.3}$$

Here and below, whenever the argument contains $\pm$, corresponding two functions are multiplied. $u_{ab}^{(I)} \equiv u_a^{(I)} - u_b^{(I)}$, and $\Gamma(z; \sigma, \tau)$ is the elliptic Gamma function defined by

$$\Gamma(z; \sigma, \tau) = \prod_{m,n=0}^{\infty} \frac{1 - e^{-2\pi i z} e^{2\pi i((m+1)\sigma + (n+1)\tau)}}{1 - e^{2\pi i z} e^{2\pi i(m\sigma + n\tau)}} . \tag{2.4}$$

Other $Z_I^{\rm 4d}$ are given similarly by permuting the $I = 1, 2, 3$ indices. The integrand from the 2d U(n_1) $\times$ U(n_2) bifundamental fields is given by

$$Z_{1,2}^{\rm 2d} = \prod_{a=1}^{n_1} \prod_{b=1}^{n_2} \frac{\theta\left(\pm(u_{ab}^{(12)} + a_{12}) + \frac{\tau_3}{2} + y, \tau_3\right)}{\theta\left(\pm(u_{ab}^{(12)} + a_{12}) - \tau + \frac{z_3}{2}, \tau_3\right)} , \tag{2.5}$$

where $u_{ab}^{(I,I+1)} \equiv u_a^{(I)} - u_b^{(I+1)}$. $\theta(z, \tau)$ is the q-theta function (with 'q' given by $e^{2\pi i \tau}$) defined by

$$\theta(z, \tau) = \prod_{n=0}^{\infty} (1 - e^{2\pi i z} e^{2\pi i n \tau})(1 - e^{-2\pi i z} e^{2\pi i(n+1)\tau}) . \tag{2.6}$$

Other $Z_{I,I+1}^{\rm 2d}$ are given similarly. The integration contour is complicated, and is related to how the auxiliary parameters $a_{I,I+1}$ are chosen. Only the value of $a_{\rm loop} \equiv a_{12} + a_{23} + a_{31}$ is important. In [16] and [19], two different choices of $a_{\rm loop}$ were made, also with different choices of the integration contour. Both prescriptions are tested till certain low orders. One of $a_{\rm loop} = -\frac{3\tau}{2} \pm y$ was chosen in [16], while $a_{\rm loop} = 0$ was chosen in [19]. The situation might be that both prescriptions work to all orders in n_I's, or one of the two is correct for higher n_I's. Although we have little to say about this issue, we simply note that our saddle point ansatz below works with the choice $a_{\rm loop} = 0$ of [19]. Perhaps with the choice of [16], residue contributions may be more important when the contour crosses poles during its deformation towards the saddle point. (Such an issue may also arise in the original Yang-Mills matrix integral for the index, as commented on in [24].) So we set $a_{I,I+1} = 0$ from now on.

We would like to study the large N behaviors of Z_{n_1,n_2,n_3}. Since n_I's contribute $N n_I$ to the electric charges Q_I, which we want to scale as N^2, we let n_I's to scale linearly in N.

2.1 Large N saddle points and small black holes

In this subsection we shall consider the index at $\Delta_1 = \Delta_2 = \Delta_3$. (We shall comment on the generalization to unequal Δ_I's in section 2.3.) This corresponds to taking the $z_I \to 0$ limit. It was shown [16] that individual Z_{n_1,n_2,n_3} diverges in this limit, while the full index after summing them over remains finite. We are interested in the leading order free energy $\log Z_{n_1,n_2,n_3} \sim N^2$ in this limit. To understand this limit more precisely, we first decompose

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the contributions to $Z_I^{\rm 4d}$ from the N Cartans at $a=b$ and the off-diagonals at $a\neq b$. The former part can be written in the limit as

$$\left[\frac{-e^{2\pi i\tau}\theta\left(\frac{\tau}{2}-y,\tau\right)}{(1-e^{2\pi i z_{I+1,I}})(1-e^{2\pi i z_{I-1,I}})E(\tau)^2}\right]^{n_I}, \tag{2.7}$$

where $z_{I,J}\equiv z_I-z_J$, $E(\tau)\equiv\prod_{n=1}^{\infty}(1-e^{2\pi i n\tau})$. So the Cartan parts diverge in the limit $z_I\to 0$. This accounts for many of the divergences encountered in [16] in this limit. The divergence is linear in N, $\sim N\log\varepsilon$ at small $z_I\sim\varepsilon$. So we ignore this part since we are interested in the leading free energy proportional to N^2. (However, see the later part of this subsection and section 2.2 for important roles of the subleading parts.) The off-diagonal part with $a\neq b$ contains extra divergences in the limit $z_I\to 0$, by the z_I dependent poles pinching the integration contour [16]. Our precise setting of taking the limit is as follows. We are interested in the behaviors of the integrand near the saddle point of our interest, to be presented below. The saddle point is away from the contour, and will not suffer in any sense from the pinching of the z_I dependent poles. So as for this part, we naively take the $z_I\to 0$ limit and simplify the integrand. Using $\Gamma(z+\sigma;\sigma,\tau)=\theta(z,\tau)\Gamma(z;\sigma,\tau)$, $\Gamma(z+\tau;\sigma,\tau)=\theta(z,\sigma)\Gamma(z;\sigma,\tau)$ and $\Gamma(z;\sigma,\tau)=\frac{1}{\Gamma(\sigma+\tau-z;\sigma,\tau)}$, one obtains

$$Z_I^{\rm 4d}\xrightarrow{z_I\to 0}\prod_{1\leq a\neq b\leq n_I}\frac{\theta\left(u_{ab}^{(I)}+\frac{\tau}{2}-y,\tau\right)}{\theta(u_{ab}^{(I)}-\tau,\tau)}\,. \tag{2.8}$$

We have ignored the Cartan part which only makes a subleading N^1 contribution. At the saddle point, all $u_a^{(I)}$'s will be different, so that this function remains finite in the $z_I\to 0$ limit. We realize that the contributions from the 4d fields are given in terms of the theta functions after substantial cancellations. Similarly, $Z_{I,I+1}^{\rm 2d}$ are given in the $z_I\to 0$ limit by

$$Z_{I,I+1}^{\rm 2d}=\prod_{a=1}^{n_I}\prod_{b=1}^{n_{I+1}}\frac{\theta\left(\pm u_{ab}^{(I,I+1)}+\frac{\tau}{2}-y,\tau\right)}{\theta\left(\pm u_{ab}^{(I,I+1)}-\tau,\tau\right)}\,. \tag{2.9}$$

$Z_I^{\rm 4d}$ and $Z_{I,I+1}^{\rm 2d}$ in this limit are invariant under shifting $u_a^{(I)}$ to $u_a^{(I)}+1$ or $u_a^{(I)}+\tau$.

As our large N (and large $n_I\sim N$) saddle point ansatz, we take each set of U(n_I) eigenvalues $u_a^{(I)}$ to be uniformly distributed along the τ-circle,

$$u^{(I)}=x_I\tau\,,\quad 0<x_I<1\,,\quad \rho(x_I)=1\,. \tag{2.10}$$

This is a coarse-grained continuum description of the eigenvalues, which are separated from their nearest neighbor by a distance at order $\frac{1}{N}$. We can typically assume that none of these eigenvalues are at precisely the same values. Therefore, (2.8) and (2.9) do not diverge due to $u_a^{(I)}=u_b^{(J)}$. If such a divergence apparently seems to happen in the continuum description, it should be avoided by integrating over x_I's with a principal-value prescription. It is easy to see that this distribution solves the large n_I saddle point equation. To check this, it is convenient to first S-dualize the integrand using the identity

$$\theta(z,\tau)=e^{-\pi i B(z,\tau)}\theta\left(\frac{z}{\tau},-\frac{1}{\tau}\right)\,,\quad B(z,\tau)\equiv\frac{z^2}{\tau}+z\left(\frac{1}{\tau}-1\right)+\frac{1}{6}\left(\tau+\frac{1}{\tau}\right)-\frac{1}{2}\,. \tag{2.11}$$

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We shall set y to be in the range $0 < y_1 < 1$, where $y \equiv y_1 + y_2\tau$ with real y_1, y_2: this convention can be chosen by a suitable period shift of y. Then, applying the S-dual identities, one obtains

$$Z_I^{\rm 4d} \sim \exp\left[\pi i n_I^2\left(\tfrac{y-y^2}{\tau} - \tfrac{3}{2} + \tfrac{9\tau}{4}\right)\right] \prod_{1\le a\neq b\le n_I} \frac{\theta\left(\frac{u_{ab}^{(I)}-y}{\tau} + \frac{1}{2}, -\frac{1}{\tau}\right)}{\theta\left(\frac{u_{ab}^{(I)}}{\tau}, -\frac{1}{\tau}\right)} \tag{2.12}$$

$$Z_{I,I+1}^{\rm 2d} = \exp\left[2\pi i n_I n_{I+1}\left(\tfrac{y-y^2}{\tau} - \tfrac{3}{2} + \tfrac{9\tau}{4}\right)\right] \prod_{a=1}^{n_I}\prod_{b=1}^{n_{I+1}} \frac{\theta\left(\frac{\pm u_{ab}^{I,I+1}-y}{\tau} + \frac{1}{2}, -\frac{1}{\tau}\right)}{\theta\left(\frac{\pm u_{ab}^{I,I+1}}{\tau}, -\frac{1}{\tau}\right)}\,.$$

Collecting all, the integrand is given by a constant factor

$$\exp\left[\frac{\pi i (n_1+n_2+n_3)^2\left(y-\frac{3\tau}{2}\right)\left(1-y-\frac{3\tau}{2}\right)}{\tau}\right] \tag{2.13}$$

times

$$\tilde{Z}(u^{(I)}) = \prod_{I=1}^{3}\left[\tilde{Z}_I^{\rm 4d}\tilde{Z}_{I,I+1}^{\rm 2d}\right] \equiv \prod_{I=1}^{3}\left[\prod_{1\le a\neq b\le n_I}\frac{\theta\left(\frac{u_{ab}^{(I)}-y}{\tau}+\frac{1}{2},-\frac{1}{\tau}\right)}{\theta\left(\frac{u_{ab}^{(I)}}{\tau},-\frac{1}{\tau}\right)}\cdot\prod_{a=1}^{n_I}\prod_{b=1}^{n_{I+1}}\frac{\theta\left(\frac{\pm u_{ab}^{I,I+1}-y}{\tau}+\frac{1}{2},-\frac{1}{\tau}\right)}{\theta\left(\frac{\pm u_{ab}^{I,I+1}}{\tau},-\frac{1}{\tau}\right)}\right]. \tag{2.14}$$

In order to show that (2.10) is a saddle point, one should show that the force $\frac{\partial}{\partial u_a^{(I)}}\log\tilde{Z}$ vanishes in the large N limit. More precisely, one should show that the leading N^1 order term of the force vanishes. This force is given by

$$-\frac{N}{\tau}\int_0^1 dx'\frac{\partial}{\partial x'}\left[\log\tilde{Z}_I^{\rm 4d}(u(x)-u(x'))+\log\tilde{Z}_{I,I+1}^{\rm 2d}(u(x)-u(x'))+\log\tilde{Z}_{I-1,I}^{\rm 2d}(u(x')-u(x))\right]. \tag{2.15}$$

The expression inside the square bracket of the right hand side is given by a linear combination of the function of the form $\log(1-e^{\pm 2\pi i(x-x')}e^{-\frac{2\pi i(n+\alpha)}{\tau}})$ with $n \in \mathbb{Z} \ge 0$, $\alpha \ge 0$ and $x, x' \in [0,1]$. (α may be either 0 or y.) So all these log functions are periodic in $x' \to x'+1$ shift without crossing the branch cut. Therefore, we integrate the derivative of a periodic function over a circle, which vanishes. The terms with $\alpha = 0$ and $n = 0$ have the branch points on the circle $x' \in [0,1]$, but employing the principal-valued integrals as explained, they also vanish.

One can compute $\log Z_{n_1,n_2,n_3}$ at this saddle point. By evaluating $\log\tilde{Z}$ in the continuum limit, similar to the evaluation of (2.15), one finds $\log\tilde{Z} = 0$. This is because the integral of $\log(1-e^{\pm 2\pi i(x-x')}e^{-\frac{2\pi i(n+\alpha)}{\tau}})$ is zero at $n \ge 0$, $\alpha \ge 0$. So the large N free energy is given by

$$\log Z_{n_1,n_2,n_3} = \frac{\pi i n^2\left(y-\frac{3\tau}{2}\right)\left(1-y-\frac{3\tau}{2}\right)}{\tau}\,, \tag{2.16}$$

where $n \equiv n_1+n_2+n_3$. Note that the leading free energy depends only on one combination of n_I. So there apparently is a large number of degenerate terms if we only consider the

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leading free energy. One can Legendre transform this free energy to obtain the macroscopic entropy at fixed charges q, j conjugate to τ, y, also at fixed n. This amounts to the extremization of

$$S(q,j;\tau,y,n) = \frac{\pi i n^2 y(1-y)}{\tau} - \frac{3\pi i n^2}{2} + \frac{9\pi i n^2}{4}\tau - 2\pi i\tau\cdot(3q-nN) - 2\pi i y\cdot j\,. \tag{2.17}$$

The charges correspond to $q = \frac{Q_1+Q_2+Q_3}{3} + \frac{J_1+J_2}{2}$, $j = J_1 - J_2$. The solution is given by[3]

$$\tau = i\frac{n^2}{2\sqrt{2n^2P - j^2}}\,, \qquad y = \frac{1}{2} - i\frac{j}{2\sqrt{2n^2P - j^2}} \tag{2.18}$$

where $P \equiv 3q - nN - \frac{9n^2}{8}$, and the extremized entropy is given by

$$S(q,j,n) = \frac{\pi}{2}\sqrt{n^2(24q - 8nN - 9n^2) - 4j^2} - \pi i j - \frac{3\pi i n^2}{2}\,. \tag{2.19}$$

The constant imaginary term $-\pi i j$ can be ignored in the discussions below.

Before proceeding, let us comment on the structure of the asymptotic Z_{n_1,n_2,n_3} that we obtained in (2.16). We first investigate the structure of the expansion (1.1) in the grand canonical ensemble with fixed τ, y. Since these parameters are complex, it is helpful to focus on a region which contains the saddle point of the Legendre transformation (2.18).[4] For instance, as for τ, let us take it to be purely imaginary with $\mathrm{Im}(\tau) > 0$. For y, let us freeze $y = \frac{1}{2}$ for simplicity of the discussion. This corresponds to setting $j = 0$ in the microcanonical ensemble, or unrefining the chemical potential y for j in the grand canonical ensemble. Then one finds that the giant graviton expansion (1.1) takes the form of

$$Z \sim \sum_{n_1,n_2,n_3} \Omega(n_I) e^{2\pi i N\tau n} \exp\left[\frac{\pi i n^2}{4\tau}(1-3\tau)^2\right] = \sum_{n_1,n_2,n_3} \Omega(n_I) e^{-Nn\beta} \exp\left[\frac{\pi^2 n^2}{2}\left(\frac{1}{\beta} - \frac{3i}{\pi} - \frac{9\beta}{4\pi^2}\right)\right] \tag{2.20}$$

where $\beta \equiv -2\pi i\tau$ is real and positive. Recall that $n \equiv n_1 + n_2 + n_3$, and $\Omega(n_I)$ come from the subleading contributions to $\log Z_{n_1,n_2,n_3}$ in the large $n \sim N$ expansion. At large n, each $|Z_{n_1,n_2,n_3}|$ grows very fast like e^{an^2} with certain $a > 0$ when $\beta < \beta_c \equiv \frac{2\pi}{3}$. So at high temperatures, unless the subleading factors $\Omega(n_I)$ are given in a manner that various terms substantially cancel, the sum will diverge very badly at large n. One can be more realistic and insert the complex values of $\tau(q)$ as a function of real charge q, at which we know that BPS black hole saddle points exist [14, 25]. Then one finds that $\mathrm{Im}\left[\frac{1}{\tau(q)} - 9\tau(q)\right] < 0$ is always met, again making $|Z_{n_1,n_2,n_3}|$ to grow fast. However, if the expansion (1.1) provides an exact expression for the gauge theory partition function, we expect the series (2.20) to better behave at high temperatures where the system deconfines [25, 27, 28].

Let us elaborate more on why we expect the series (2.20) to behave well for $\beta < \beta_c$. For instance, consider a series of the form

$$\sum_n \Omega(n) e^{\beta_c n} e^{-n\beta} \tag{2.21}$$

[3]During the saddle point analysis, it is essential for τ to be in the upper half-plane. For instance, when $q = \epsilon N^2$ and $n = \alpha N$ with $\epsilon, \alpha \ll 1$, one should take $\alpha < 3\epsilon$.

[4]See [25–27]. Basically, the phases of fugacities should be tuned in the index even in the grand canonical ensemble, to minimize the unwanted boson-fermion 'cancellations' during macroscopic approximations.

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with $\beta_c > 0$, and $\Omega(n)$ does not affect the exponential growth of $e^{\beta_c n}$. The series ceases to converge at $\beta < \beta_c$ outside its radius of convergence. If this series is for a thermal partition function, the divergence is the Hagedorn pathology [21] caused by exponential growth of the density of states at high energy. It happens due to an infinite tower of mesonic states [21], or an infinite tower of string oscillations [22]. The apparent divergence $\sim e^{an^2}$ of (2.20) would make the series worse-behaved than (2.21). Namely, unless cancellations happen, the radius of convergence is zero. One may interpret this divergence (if present) as coming from a much faster asymptotic growth of baryonic states. However, the notion of baryonic states should become ambiguous at large charge q for which $\tau(q)$ enters the deconfining regime. For expressions like (1.1) or (2.20) to remain relevant, $\Omega(n_I)$'s should be arranged so that the apparent asymptotic growth e^{an^2} cancels. The cancellation effects should be more crucial for $\tau(q)$ with larger q, as the system is deeper inside the deconfining regime. If such cancellations are not taken into account, each term in the series (2.19) may over-estimate the microcanonical entropy.

With this caution in mind, let us try to extract the microcanonical entropy from the formula (2.19). We first study the case with $j = 0$. Since n is not a physical charge, we should try to maximize $\text{Re}[S(q, n)]$ as a function of non-negative integer n. $\text{Re}[S(q, n)]$ is positive when

$$0 < n < \frac{4}{9}\left[-N + \sqrt{N^2 + \tfrac{27q}{2}}\right] \equiv n_*(q)\,. \tag{2.22}$$

So to study the macroscopic entropy from this index, one only needs to sum over n till $n_*(q)$. Thus, we consider

$$e^{S(q)} = \oint d\tau e^{-2\pi i\tau\cdot 3q} \sum_{n_1,n_2,n_3} e^{2\pi i N n\tau} Z_{n_1,n_2.n_3} \sim \sum_{n_1,n_2,n_3}^{n\leq n_*(q)} \Omega(n_I)\exp\left[\frac{\pi n}{2}\sqrt{24q - 8nN - 9n^2} - \frac{3\pi i n^2}{2}\right]. \tag{2.23}$$

We study this quantity at large N and large $q \propto N^2$, naively expecting at this moment that the leading contribution comes from certain n at order N^1. We first note that the overall phase factor $e^{-\frac{3\pi i n^2}{2}}$ oscillates between i and 1, depending on whether n is odd or even. So dividing the sum into even/odd n's and naturally expecting that the maximization will not be sensitive to the even/odd nature of n's, this phase does not matter and the dominant contribution to this entropy is given by the maximum of $\text{Re}[S(q, n)]$. This happens at

$$n_0 = \frac{-N + \sqrt{N^2 + 12q}}{3}\,, \tag{2.24}$$

which is in the range $0 < n_0(q) < n_*(q)$. The maximal entropy is given by

$$\text{Re}[S(q)] = \text{Re}[S(q, n_0(q))] = \frac{\pi(-N + \sqrt{N^2 + 12q})}{3\sqrt{6}}\sqrt{N^2 + 18q - N\sqrt{N^2 + 12q}}\,. \tag{2.25}$$

We first study its asymptotic behaviors at $q \ll N^2$ and $q \gg N^2$, which would respectively correspond to the small and large black hole limits. In other words, we first rewrite the entropy as a function of N^2 and $\epsilon \equiv \frac{q}{N^2}$. Then, S is given by N^2 times a function of ϵ.

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Expanding S in small and large ϵ (independent of N), one finds[5]

$$\begin{aligned}
\mathrm{Re}[S(q)] &= \frac{\pi(2q)^{\frac{3}{2}}}{N} - \frac{9\pi q^{\frac{5}{2}}}{\sqrt{2}N^3} + \frac{351\pi q^{\frac{7}{2}}}{8\sqrt{2}N^5} - \frac{8937\pi q^{\frac{9}{2}}}{32\sqrt{2}N^7} + \frac{1048059\pi q^{\frac{11}{2}}}{512\sqrt{2}N^9} + \cdots \quad \text{for } q \ll N^2 \\
\mathrm{Re}[S(q)] &= 2\pi q - \frac{4\pi N q^{\frac{1}{2}}}{3\sqrt{3}} + \cdots \quad \text{for } q \gg N^2 \,.
\end{aligned} \tag{2.26}$$

One can show that this $\mathrm{Re}[S(q)]$ is asymptotically equal to the Bekenstein-Hawking entropy $S_{\mathrm{BH}}(q)$ of the dual black hole when $q \ll N^2$, in which case $\mathrm{Re}[S(q)] \approx \frac{\pi(2q)^{\frac{3}{2}}}{N} \approx S_{\mathrm{BH}}(q)$. Away from the asymptotic limit $q \ll N^2$, $\mathrm{Re}[S(q)] > S_{\mathrm{BH}}(q)$ always holds. In particular, in the two asymptotic limits, S_{BH} is expanded as

$$\begin{aligned}
S_{\mathrm{BH}}(q) &= \frac{\pi(2q)^{\frac{3}{2}}}{N} - \frac{21\pi q^{\frac{5}{2}}}{\sqrt{2}N^3} + \frac{1287\pi q^{\frac{7}{2}}}{8\sqrt{2}N^5} - \frac{46189\pi q^{\frac{9}{2}}}{32\sqrt{2}N^7} + \frac{7243275\pi q^{\frac{11}{2}}}{512\sqrt{2}N^9} + \cdots \quad \text{for } q \ll N^2 \\
S_{\mathrm{BH}}(q) &= \sqrt{3}\pi\left(\frac{N^2q^2}{2}\right)^{\frac{1}{3}} + \cdots \quad \text{for } q \gg N^2 \,.
\end{aligned} \tag{2.27}$$

These expansions are obtained from the Bekenstein-Hawking entropy of the BPS black holes, as explained in appendix A. One finds a small over-estimating deviation $\mathrm{Re}[S(q)] - S_{\mathrm{BH}}(q) \approx \frac{12\pi q^{\frac{5}{2}}}{\sqrt{2}N^3} > 0$ in the small charge expansion, and $\mathrm{Re}[S(q)] \gg S_{\mathrm{BH}}(q)$ in the large charge expansion. As already explained, our interpretation of this over-estimate is that we have been ignoring the possible cancellations of the apparently leading order terms due to nontrivial $\Omega(n_I)$'s in (2.20). The large charge behavior $\mathrm{Re}[S(q)] \sim 2\pi q = 3\beta_c q$ of (2.26) is a Hagedorn growth.

The agreement of the leading entropy $\mathrm{Re}[S(q)] \approx \frac{\pi(2q)^{\frac{3}{2}}}{N}$ with the Bekenstein-Hawking entropy of small black holes might still look a bit miraculous. To better appreciate this, it is first worthwhile to note that small black holes are never dominant saddles in the grand canonical ensemble. Also, they always stay in the confining region in the complex τ space [27]. So it makes sense that they admit a description in terms of D-branes, which are baryonic objects in the confining phase. On the other hand, as q gradually grows, giant gravitons will eventually lose their meaning at high energy. This is because the fundamental high energy degrees of freedom are gluons rather than their bound states. Interestingly, the D3-brane giant graviton approach has been already employed in [8] to account for the entropy of small black holes. The calculation we did with $S(q, n)$ was discussed in [8], in precisely the same computational procedure. The rough idea of [8] is to regard the maximal giant gravitons to be similar to the wrapped D-branes which account for 5d asymptotically flat black holes [29]. Since most of the microscopic accounts for asymptotically flat black holes use branes, and since small AdS_5 black holes are (at least mathematically) identical to the 5d asymptotically flat black holes embedded in large AdS, it is natural that both objects admit similar D-brane-based descriptions. We find that our studies provide precise logical grounds for the calculations of [8].

[5] In these expansions, since we take the expansion parameter $\epsilon \equiv \frac{q}{N^2}$ to be independent of N, the charge q scales like N^2.

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From (2.18), the saddle point value of τ for the Legendre transformation at $j = 0$ is

$$\tau(q,n) = \frac{in}{\sqrt{24q - 8nN - 9n^2}} = \frac{in}{3\sqrt{(n_*(q) - n)(n + n_*(q) + \frac{8N}{9})}} \,. \tag{2.28}$$

In the small black hole limit $q \ll N^2$, n is ranged in $0 < n < n_*(q) \approx \frac{3q}{N}$ and the maximum $n_0(q)$ of $S(q,n)$ is approximately $n_0(q) \approx \frac{2q}{N}$. Around the maximum $n_0(q)$, τ scales like

$$|\tau| \sim \frac{n}{\sqrt{q}} \sim \frac{q^{\frac{1}{2}}}{N} \ll 1 \,. \tag{2.29}$$

So the small black hole limit $q \ll N^2$ corresponds to the 'Cardy limit' $\tau \to i0$ in the 2d-like integrand (2.8), (2.9). We find this to be a concrete realization of the studies made in [8], which assumed the existence of a hypothetical 2d CFT living on the worldvolume of maximal giant gravitons at fixed $n = n_1 + n_2 + n_3$ and used its Cardy formula to account for the small black hole entropy. The 2d CFT was supposed to live on the Hopf fiber circle of the S^5, which is wrapped by the D3-branes. However, there was no logical justification for the existence of such a 2d CFT, since the 4d worldvolume has no scale separation which justifies the 2d reduction. We found from our index in the limit $z_I \to 0$ that the 4d part of the integrand $Z_I^{\rm 4d}$ partly canceled to yield Jacobi theta functions, which are 2 dimensional objects. So what justifies the 2d reduction here is the boson-fermion cancellations in the index. This is much more specific than a reduction based on the scale separation. This 2d description may break down if one studies unprotected quantities beyond the index. For instance, [8] studied the charge relation satisfied by small black holes, $J_1 + J_2 \sim \frac{q^2}{N^2}$. The idea of [8] is as follows. If the 2d CFT exists, small AdS$_5$ black holes are dual to its NS sector. It is related to the CFT in the Ramond sector by a spectral flow. The Ramond sector CFT describes 5 dimensional asymptotically flat black holes [29, 30] satisfying a charge relation $J_1 + J_2 = 0$. So if the 2d description exists universally beyond the index, the spectral flow will connect the charge relation $J_1 + J_2 = 0$ to that of the AdS black holes. The relation obtained from this route does not agree with the charge relation of AdS black holes [8]. We interpret this as the absence of the 2d description beyond the index.

We can generalize the studies to the case with $j \equiv J_1 - J_2 \neq 0$ by keeping $y \neq \frac{1}{2}$. Again we are only able to successfully count small black holes by keeping the leading term

$$\log Z_{n_1,n_2,n_3} \sim \frac{\pi i n^2 y(1-y)}{\tau} \tag{2.30}$$

of (2.16) at small τ. Making a Legendre transformation of this free energy by extremizing

$$S(q,j;n) \sim \frac{\pi i n^2 y(1-y)}{\tau} - 2\pi i\tau(3q - nN) - 2\pi i y j \tag{2.31}$$

and then maximizing $\mathrm{Re}[S(q,j;n)]$ with n, one obtains the entropy given by

$$S(q,j) = \pi\sqrt{\frac{8q^3}{N^2} - j^2} \,. \tag{2.32}$$

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τ, y at these saddles always satisfy $|\tau| \ll 1$, $\text{Re}(y) = \frac{1}{2}$ as long as the charges are away from the closed timelike curve (CTC) bound $j^2 = \frac{8q^3}{N^2}$, staying within the regime that we assumed. This is precisely the entropy of small spinning BPS black holes in $AdS_5 \times S^5$. This can also be regarded as the BMPV black holes [30] embedded in large AdS$_5$. We emphasize that this is the first microscopic counting of small spinning AdS$_5$ black holes at all allowed values of $J_1 \neq J_2$. In fact accounting for black holes at $J_1 \neq J_2$ has been technically tricky from the QFT dual. For instance, in the saddle point approach to the Yang-Mills matrix model, general $J_1 \neq J_2$ was discussed only in the 4d Cardy limit [14]. At general finite charges, [24] found saddle points which cover substantial charge regions for j, but failed to cover the whole parameter space of CTC-free black holes. More precisely, [24] found the saddles when certain inequalities were met, like eq. (2.41) or (2.45) there. In the parametrization of BPS black holes given by [11], these inequalities cover the region $0 < ag, bg < 1$. On the other hand, the CTC-free black holes exist in a bigger region $ag, bg < 1$, $a + b + abg > 0$. This should be due to our limited understanding of the large N matrix model saddles. At least in the small black hole limit, it is amusing that the giant graviton calculation of this paragraph was able to cover the whole CTC-free black holes satisfying $ag, bg \ll 1$, $a + b > 0$.

2.2 Comments on finite size black holes

From the absence of the Hagedorn behavior in gauge theories, we think it is obvious that cancellations of different Z_{n_1,n_2,n_3}'s happen in general. However, computing such cancellations at large N is technically very challenging. This is because the cancellations happen due to relative minus signs, whose precise determination goes beyond the leading order calculation. For instance, we tried to compute such subleading terms in the small τ regime, but found that the precise integral contours for Z_{n_1,n_2,n_3} are needed to compute them. Also, with such a contour dependence, taking $z_I \to 0$ limit is trickier than in the previous subsection.

In this subsection, leaving the full microscopic analysis to the future, we shall make a simple assumption on how these subleading corrections should be arranged. This assumption will allow us to compute the true entropy of this index after the cancellations, which precisely reproduces the dual black hole entropy. The claim is that, once we include the $\frac{1}{N}$ effects to each Z_{n_1,n_2,n_3}'s contribution, the degeneracy at given n will be lifted by small deviations from (2.16) in a way that the sum over n can be replaced by an integral. The microcanonical sum (2.23) over discrete n can be replaced by

$$e^{S(q)} = \oint d\tau e^{-2\pi i\tau\cdot 3q} \sum_{n_1,n_2,n_3} e^{2\pi i N n\tau} Z_{n_1,n_2.n_3} \sim \int_0^{n_*(q)} dn \exp\left[\frac{\pi n}{2}\sqrt{24q-8nN-9n^2}-\frac{3\pi i n^2}{2}\right]. \tag{2.33}$$

The claim asserts that we use the same function $S(q,n)$ but sum over a dense set of n's. Before explaining anything about this claim, we emphasize that we have no derivation of (2.33) except that this formula will give the exact black hole entropy at an arbitrary size.

Let us first explain why this is a nontrivial claim, and in particular why it is related to including $\frac{1}{n_I}$ subleading terms. Our claim is essentially that, once we include the subleading

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corrections at fixed n, Z_{n_1,n_2,n_3} will behave like

$$Z_{n_1,n_2,n_3} \sim \exp\left[S(q,n+\delta_n(n_I))\right] , \qquad S(q,x) \equiv \frac{\pi x}{2}\sqrt{24q-8Nx-9x^2} - \frac{3\pi i x^2}{2} \tag{2.34}$$

with a nontrivial function $\delta_n(n_I) \sim \mathcal{O}(1) \ll n$ of n_I. This makes the distribution of $n+\delta_n(n_I)$ dense over a range of $\mathcal{O}(1)$ width around n. (2.34) is a claim about the $\frac{1}{N}$ subleading corrections of Z_{n_1,n_2,n_3}, since all $\delta_n(n_I)$ dependent terms are subleading. If this happens to all values of n, one would obtain

$$\sum_{n=0}^{n_*(q)} Z_{n_1,n_2,n_3} = \int_0^{n_*(q)} \rho(n)\exp\left[S(q,n)\right] \tag{2.35}$$

where $\rho(n)$ is a suitable distribution determined by $\delta_n(n_I)$'s. Since the total number of summands satisfying $n \leq n_*(q)$ is proportional to n_*^3, one finds $\int_0^{n_*(q)} dn\rho(n) \sim n_*^3 \sim N^3$. So $\log\rho(n)$ is a logarithmic correction to $S(q,n) \sim N^2$. Thus, ignoring it, one obtains (2.33).

If the sum over n is replaced by an integral over n, it is no longer valid to find the dominant contribution by maximizing $\mathrm{Re}[S(q,n)]$. Rather, one should find a saddle point of $S(q,n)$ in the complex n plane. Regarding $S(q,n)$ as a complex function of n, the maxima $n_0(q)$ of $\mathrm{Re}[S(q,n)]$ on the real axis is not a saddle point, due to nontrivial $\mathrm{Im}[S(q,n)] = -\frac{3\pi n^2}{2}$ for continuous n. In summary, part of our claim is about the $\frac{1}{n_I}$ subleading corrections of $\mathrm{Im}[S(q,n)]$, which lift the degeneracy and render substantial cancellations of different Z_{n_1,n_2,n_3}'s.

Given (2.33), one can identify the saddle point on the complex n plane. In fact it is inconvenient to work directly with the last expression of (2.33). Rather, we keep the variables τ, y unintegrated, and consider the multiple integral formula for $e^{S(q,j)}$ given by

$$\begin{aligned} e^{S(q,j)} &\sim \oint d\tau \oint dy \int_0^{n_*(q)} dn e^{\log Z_{n_1,n_2,n_3} - 2\pi i\tau(3q-nN) - 2\pi i y\cdot j} \\ &\sim \oint d\tau \oint dy \int_0^{n_*(q)} dn\ \exp\left[\frac{\pi i n^2\left(y-\frac{3\tau}{2}\right)\left(1-y-\frac{3\tau}{2}\right)}{\tau} - 2\pi i\tau(3q-nN) - 2\pi i y\cdot j\right]. \end{aligned} \tag{2.36}$$

Note that at this stage we reintroduced the refinement with y or j, since the analysis is no more difficult. To find the possible saddle points in the complex n plane, we can simply extremize the 3-dimensional integral (2.36). (Later in this subsection, when numerically discussing the contour deformation, it will be more convenient to use the original 1 dimensional integral (2.33).) It is easy to first extremize in n, since the integrand is Gaussian in n. One finds the saddle point $n_s = -\frac{N\tau^2}{(y-\frac{3\tau}{2})(1-y-\frac{3\tau}{2})}$. Inserting this, the remaining τ, y integral is given by

$$S^{S(q,j)} \sim \oint d\tau \oint dy \exp\left[-\frac{\pi i N^2\tau^3}{\left(\frac{3\tau}{2}-y\right)\left(\frac{3\tau}{2}-1+y\right)} - 2\pi i\tau\cdot 3q - 2\pi i y\cdot j\right]. \tag{2.37}$$

Reintroducing $\omega_1 = -2\pi i\left(\frac{3\tau}{2}+y-1\right)$, $\omega_2 = -2\pi i\left(\frac{3\tau}{2}-y\right)$, the exponent is given by

$$\frac{N^2}{2}\frac{\left(\frac{\omega_1+\omega_2}{3}-\frac{2\pi i}{3}\right)^3}{\omega_1\omega_2} + \omega_1(Q+J_1) + \omega_2(Q+J_2) \tag{2.38}$$

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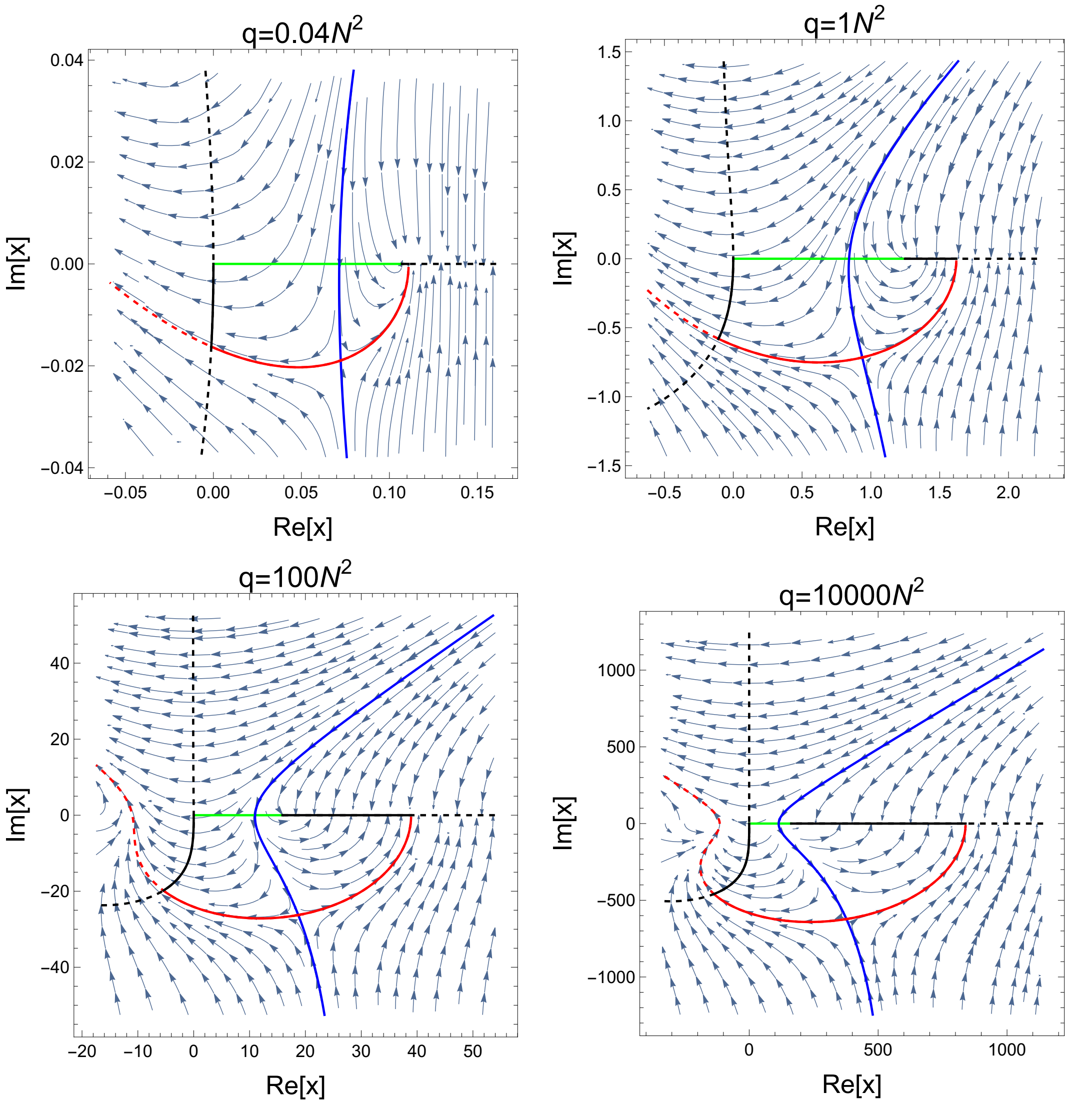

Figure 1. Contour deformations at various q. Green interval is the original integration contour $[0, x_*]$. Blue arrows denote the gradient flow which determines the contour deformation. The deformed contour is the union of two solid black lines C_1, C_3, and the solid red line C_2. The solid blue line is the steepest ascent contour.

where $q \equiv Q + \frac{J_1+J_2}{2}$, $j \equiv J_1 - J_2$. This is precisely the entropy function (at equal electric charges Q_I) for the Bekenstein-Hawking entropy of BPS black holes in $AdS_5 \times S^5$ [33]. Although we discussed the saddle points of the 3-dimensional integral (2.36), the same entropy is obtained with $S(q,n)$ from the last expression of (2.33).

We finally show that the contour can be deformed to pass through this saddle, by extending the standard Picard-Lefschetz theory. Consider the following integral

$$\int_0^{x_*\left(\frac{q}{N^2}\right)} dx e^{N^2 f(x)}\,, \qquad f(x) \equiv \frac{\pi x}{2}\sqrt{24\frac{q}{N^2} - 8x - 9x^2} - \frac{3\pi i x^2}{2}\,, \tag{2.39}$$

where $x = \frac{n}{N}$ and $x_*(\frac{q}{N^2}) = \frac{n_*(q)}{N}$. If the integrand vanishes at the two ends $x = 0, x_*$,

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one can deform the integration contour to the steepest descent contour. The steepest descent contour has maximal $\mathrm{Re}[f(x)]$ at the saddle point, and satisfies the stationary phase condition $\mathrm{Im}[f(x)] = \text{constant}$. The integration on this contour can be approximated at large N by a Gaussian approximation around the saddle point. In our case, the integrand does not vanish at the two ends. Then the steepest descent contour passing through our complex saddle point $x_{\mathrm{s}} \equiv \frac{n_{\mathrm{s}}}{N}$ does not end on $x = 0, x_*$, so we have to slightly extend this standard method. We combine an interval C_2 of the steepest descent contour (solid red line of figure 1) with two more intervals C_1, C_2 of contours satisfying $\mathrm{Re}[f] = \text{constant}$ and ending on $x = 0, x_*$, respectively (solid black). The original contour $C_0 = [0, x_*]$ (green) can be deformed to $C_1 \cup C_2 \cup C_3$. As q decreases, one can see that the complex saddle x_{s} approaches the real maximum $x_0 \equiv \frac{n_0(q)}{N}$ of $\mathrm{Re}[S(q,n)]$. So the complex saddle approach naturally converges the naive analysis with real n of section 2.1 in the small black hole limit.

The dominant term of the integral on C_2 can be computed by the Gaussian approximation around $x = x_{\mathrm{s}}$, yielding a term of the form $\sim e^{N^2 f(x_{\mathrm{s}})}$. Then, denoting the two ends of the interval C_2 by x_1 and x_3, respectively, the integrands on C_1 and C_3 take the form of

$$e^{N^2\mathrm{Re}[f(x_1)]}\int_{C_1} dx e^{iN^2\mathrm{Im}[f(x)]}\,, \qquad e^{N^2\mathrm{Re}[f(x_3)]}\int_{C_3} dx e^{iN^2\mathrm{Im}[f(x)]}\,, \tag{2.40}$$

respectively. Since C_2 is the steepest descent contour, one finds $e^{N^2\mathrm{Re}[f(x_{1,3})]} \ll e^{N^2\mathrm{Re}[f(x_{\mathrm{s}})]}$ and these integrals are bounded as

$$\left|e^{N^2\mathrm{Re}[f(x_{1,3})]}\int_{C_{1,3}} dx e^{iN^2\mathrm{Im}[f(x)]}\right| \le e^{N^2\mathrm{Re}[f(x_{1,3})]}\int_{C_{1,3}} dx\left|e^{iN^2\mathrm{Im}[f(x)]}\right| \ll e^{N^2\mathrm{Re}[f(x_{\mathrm{s}})]}\,. \tag{2.41}$$

Therefore, the contribution from $C_1 \cup C_3$ is subdominant, justifying the approximation using the Gaussian approximation near x_{s}. As illustrated in figure 1, we checked for a wide range of $\frac{q}{N^2}$ that the contour can always be deformed in this way.

2.3 Comments on unequal electric charges

So far, we studied the index with the chemical potentials Δ_I for the three electric charges Q_I unrefined, $\Delta_1 = \Delta_2 = \Delta_3 \equiv -2\pi i\tau$. In this subsection we comment on the generalizations with unequal Δ_I's. Note that in the original Yang-Mills matrix model of [7, 8], taking independent Δ_I was rather straightforward, while introducing the refinement $y \neq \frac{1}{2}$ for two independent angular momenta J_1, J_2 was much trickier [24, 27]. This was basically because independent ω_1, ω_2 for the spacetime charges in QFT could introduce branch points in the matrix model potentials which yield nonzero eigenvalue forces. In fact, as explained in section 2.1, [24] found saddle points for the black holes at $J_1 \neq J_2$ only when certain inequalities are met: see eq. (2.41) or (2.45) of [24]. In the giant graviton index Z_{n_1,n_2,n_3}, since the role of internal and spacetime symmetries are partly exchanged, such as $\Delta_1, \Delta_2 \leftrightarrow \omega_1, \omega_2$ for the giant gravitons with $n_3 \neq 0$, the situation is the other way round. We have seen in section 2.1 that the giant graviton approach sees the black holes at $J_1 \neq J_2$ (i.e. at $y \neq \frac{1}{2}$) rather easily at least when the black hole size is small. This is because $\omega_{1,2}$ are the chemical potentials for internal symmetries in this approach. On the other hand, we find it very difficult to construct saddle points of Z_{n_1,n_2,n_3} when Δ_I's are different.

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More concretely, we have tried to construct the large n_I saddle points by noting that the integrand $Z_I^{\rm 4d}$ resembles the integrand of a Yang-Mills index with $\mathrm{U}(n_I)$ gauge group, except for the tachyon and contour issues [16, 19]. So we tried to use the ansatz of [24] to find the saddle point at unequal Δ_I's. This almost solves the saddle point equations but not quite, due to several branch points in the potential. During the course, however, we could write down a free energy for $\log Z_{n_1,n_2,n_3}$ whose Legendre transformation and maximization in n_I's yield the dual black hole entropy. Although we have a gap in our derivation, we strongly believe that we found the correct answer. So we simply report our findings without any microscopic derivation.

We find that the refined free energy should be given by

$$\log Z_{n_1,n_2,n_3} = \frac{\pi i(n_1\tau_1+n_2\tau_2+n_3\tau_3)^2}{\tau_1\tau_2\tau_3}\left(\tfrac{1}{2}\textstyle\sum_{I=1}^3\tau_I - y\right)\left(\tfrac{1}{2}\textstyle\sum_{I=1}^3\tau_I + y - 1\right) \tag{2.42}$$

and the corresponding entropy function to extremize is

$$S(q_I,j;n_I,\tau_I,y) = \log Z_{n_1,n_2,n_3} - 2\pi i\sum_I \tau_I(q_I - n_I N) - 2\pi i y\cdot j\,. \tag{2.43}$$

Note that $q_I \equiv Q_I + \frac{J_1+J_2}{2}$, $j\equiv J_1 - J_2$. Like the analysis we did for equal τ_I, we should first extremize this in τ_I's, and then maximize the real part with n_I. If we do this calculation, again one generally obtains an entropy $\mathrm{Re}[S(q_I,j)]$ which overestimates the degeneracy unless cancellations of various Z_{n_1,n_2,n_3} are taken into account.

Like the case with equal q_I, the entropy estimated with single Z_{n_1,n_2,n_3} reproduces the black hole entropy in the small black hole limit $q_I, j \ll N^2$. For simplicity, we show this only at $j=0$ ($y=\frac{1}{2}$). The small black hole limit corresponds to $|\tau_I|\ll 1$, in which case one obtains

$$S(q_I;n_I,\tau_I) \approx \frac{\pi i(n_1\tau_1+n_2\tau_2+n_2\tau_3)^2}{4\tau_1\tau_2\tau_3} - 2\pi i\sum_I \tau_I(q_I - n_I N)\,. \tag{2.44}$$

This is a real function for purely imaginary τ_I's. We shall extremize S with τ_I's on this subspace. Since S is real, the next maximization of $\mathrm{Re}[S]$ with n_I is just maximizing S. We can exchange the order of the two extremizations. $S(q_I;n_I,\tau_I)$ is quadratic in n_I, and depends only on $n_1\tau_1+n_2\tau_2+n_3\tau_3$. Therefore, only this linear combination of three n_I is fixed after extremizing with n_I, leaving two parameters unfixed. This is a generalization of sections 2.1 and 2.2 where only $n_1+n_2+n_3\equiv n$ was fixed. After this extremization, one obtains

$$S(q_I;\tau_I) \approx -4\pi i N^2\tau_1\tau_2\tau_3 - 2\pi i\sum_I \tau_I q_I\,. \tag{2.45}$$

This is precisely the entropy function for the small black holes, whose further extremization yields the Bekenstein-Hawking entropy $S(q_I)\approx \frac{\pi}{N}\sqrt{8q_1q_2q_3}$ of small AdS black holes.

Beyond small black holes, we also expect that subleading order terms should render substantial cancellations of different Z_{n_1,n_2,n_3}'s, in order for the index not to exhibit a Hagedorn-like pathology at large n_I's. At complex τ_I, one can only fix $\sum_I n_I\tau_I$ so that one of the three parameters n_I remains unfixed in the leading free energy. Summing over

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them could render cancellations. We also suggest that the concrete mechanism of such cancellation is replacing the sum over discrete n_I's by an integral, as we explained in section 2.2. This allows one to seek a complex saddle point for n_I's in (2.43). Since (2.42) is quadratic in n_I's, one first finds a Gaussian saddle point for $n_1\tau_1 + n_2\tau_2 + n_3\tau_3$, obtaining

$$S(q_I, j; \tau, y) = -\frac{\pi i N^2 \tau_1\tau_2\tau_3}{\left(\frac{1}{2}\sum_I \tau_I - y\right)\left(\frac{1}{2}\sum_I \tau_I + y - 1\right)} - 2\pi i \sum_I \tau_I q_I - 2\pi i y \cdot j\,. \tag{2.46}$$

This is precisely the entropy function of BPS black holes in $AdS_5 \times S^5$ [33], whose further extremization yields the Bekenstein-Hawking entropies of the dual black holes.

3 Analytic continuation and AdS$_{4,7}$ black holes

In this section, we interpret our results in section 2 as the analytic continuation of the maximal super-Yang-Mills index. [17] established such an interpretation for $Z_{0,0,n_3}$. See also [20]. For general Z_{n_1,n_2,n_3}, we find a similar interpretation of its large N free energy.

Let us first review [17] in the language of [16]. The integrand for $Z_{0,0,n_3}$ is simply $Z_3^{\rm 4d}$ of our section 2.1, as the quiver consists only of one adjoint node. This is related to the integrand $Z^{\rm YM}_{\rm int}$ of 4d maximal super-Yang-Mills index $Z^{\rm YM}_{{\rm U}(n_3)}$ in a very simple manner. Let us first note that, when we write the arguments of $Z^{\rm YM}_{\rm int}$ as $Z^{\rm YM}_{\rm int}(\Delta_1, \Delta_2, \Delta_3; \omega_1, \omega_2; u_a)$, the first three denote the ${\rm U}(1)^3 \subset {\rm SO}(6)$ internal rotations while the next two denote the ${\rm U}(1)^2 \subset {\rm SO}(4)$ rotations on the spacetime of the QFT. Then $Z_3^{\rm 4d}$ is given by [16, 17]

$$Z_3^{\rm 4d} = Z^{\rm YM}_{\rm int}(\omega_1, \omega_2, -\Delta_3; \Delta_1, \Delta_2; u_a^{(3)})\,. \tag{3.1}$$

This formula can be understood as follows. Since n_3 D3-branes wrap $S^3 \subset S^5$, the two worldvolume rotation parameters are Δ_1, Δ_2. On the other hand, SO(4) rotations on AdS$_5$ are internal symmetries on D3-branes. So ω_1, ω_2 are their internal rotation parameters. Finally, since the maximal giant gravitons can shrink rather than grow, losing energies, the corresponding transverse scalar is tachyonic. This demands replacing Δ_3 by $-\Delta_3$. In fact, as emphasized in [16, 19], (3.1) has to be defined by analytic continuation since ${\rm Re}(-\Delta_3) < 0$. Suitably choosing the integration contour, one finds that $Z_{0,0,n_3}$ is obtained from $Z^{\rm YM}_{{\rm U}(n_3)}$ by exchanging $\omega_{1,2} \leftrightarrow \Delta_{1,2}$ and replacing $\Delta_3 \to -\Delta_3$ with analytic continuation [17].

One can also get its large $n_3 \sim N$ free energy from analytic continuation. When $\omega_{1,2}$, $-\Delta_3$, $\Delta_{1,2}$ on the right hand side of (3.1) have positive real parts, its large N free energy is given by

$$\log Z_{0,0,n_3} = \log Z^{\rm YM}_{{\rm U}(n_3)}(\omega_1, \omega_2, -\Delta_3; \Delta_1, \Delta_2) \sim \frac{n_3^2}{2}\frac{\omega_1\omega_2(-\Delta_3)}{\Delta_1\Delta_2} \tag{3.2}$$

where the imaginary parts of the chemical potentials are suitably shifted by their periods to satisfy either $\omega_1 + \omega_2 - \Delta_3 - \Delta_1 - \Delta_2 = \pm 2\pi i$. This result can be understood in two different ways. Firstly, it can be understood as derived from various calculations of the Yang-Mills index [14, 15, 24, 31, 32]. Secondly, one can interpret it as the free energy of dual AdS$_5$ black holes [13, 33, 34]. For 4d maximal super-Yang-Mills, both viewpoints are available. Having in mind less explored SCFT's, to be explored later in this section, we emphasize the virtue

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of understanding $\log Z_{\mathrm{SCFT_D}}$ as the free energy of dual black holes in AdS_{D+1}. Once (3.2) is known in the region $\mathrm{Re}(-\Delta_3) > 0$, it is quite immediate to continue it to the physical region $\mathrm{Re}(\Delta_3) > 0$. Namely, we just keep the expression on the right hand side of (3.2). This continuation assumes the absence of the Stokes' phenomena. Let us assume this and proceed. In the parametrization of section 2, we take $\Delta_I = -2\pi i\tau_I$, $\omega_1 = -2\pi i\left(\frac{1}{2}\sum_I \tau_I + y - 1\right)$ and $\omega_2 = -2\pi i\left(\frac{1}{2}\sum_I \tau_I - y\right)$ which satisfy $\Delta_1+\Delta_2+\Delta_3-\omega_1-\omega_2 = -2\pi i$. Then (3.2) is given by

$$\log Z_{0,0,n_3} \sim \frac{\pi i n_3^2 \tau_3 \left(\frac{1}{2}\sum_I \tau_I - y\right)\left(\frac{1}{2}\sum_I \tau_I + y - 1\right)}{\tau_1\tau_2}, \tag{3.3}$$

which is the giant graviton free energy (2.42) or (2.16) at $n_1 = n_2 = 0$. Therefore, our formulae of section 2 can be naturally understood as the analytic continuation of the Yang-Mills index.

In fact one can similarly interpret our general formula (2.42) for Z_{n_1,n_2,n_3}. Expanding the complete square in the numerator, one obtains

$$\begin{aligned}\log Z_{n_1,n_2,n_3} = &\sum_{I=1}^{3} \frac{\pi i n_I^2 \tau_I \left(\frac{1}{2}\sum_J \tau_J - y\right)\left(\frac{1}{2}\sum_J \tau_J + y - 1\right)}{\tau_{I-1}\tau_{I+1}} \\ &+ \sum_{I=1}^{3} \frac{2\pi i n_{I-1}n_{I+1}\left(\frac{1}{2}\sum_J \tau_J - y\right)\left(\frac{1}{2}\sum_J \tau_J + y - 1\right)}{\tau_I}\end{aligned} \tag{3.4}$$

where $I \sim I+3$ is understood. The three terms on the first line are $\log Z_{n_1,0,0}$, $\log Z_{0,n_2,0}$ and $\log Z_{0,0,n_3}$. One can more concretely identify each of them as the saddle point value of the integrand $\log Z_I^{\mathrm{4d}}$. This was derived in section 2.1 at equal τ_I's. At equal τ_I's, the three terms on the second line can also be separately identified as the saddle point values of the integrands $\log Z_{I,I+1}^{\mathrm{2d}}$. So we naturally find a picture of the large N giant graviton free energy, as the sum of three maximal super-Yang-Mills free energies and three 2d free energies at the intersections.

We comment on two features of (2.42) and (3.4). Firstly, the six 4d and 2d contributions factorize in (3.4). This does not always have to be the case, as there is no reason for these degrees of freedom to decouple. We find that it is a rather exceptional property, perhaps for even dimensional QFT's whose free energies can be read off from anomalies. A more fundamental aspect is the n_I dependence through $\sum_{I=1}^3 n_I\Delta_I$. This may be heuristically understood as follows. $\frac{1}{8}$-BPS giant graviton solutions of [35] are given by holomorphic surfaces in $\mathbb{C}^3 \supset S^5$

$$0 = \sum_{n_1,n_2,n_3=0}^{\infty} C_{n_1,n_2,n_3} z_1^{n_1} z_2^{n_2} z_3^{n_3} \tag{3.5}$$

where z_I are the coordinates of $\mathbb{C}^3$. Δ_I can be regarded as $\mathrm{U}(1)^3$ rotation parameters on the moduli space, transforming $z_1^{n_1}z_2^{n_2}z_3^{n_3} \to e^{\sum_I n_I\Delta_I} z_1^{n_1}z_2^{n_2}z_3^{n_3}$. The formula of [16] can be interpreted as an 'equivariant localization' of an integration over the moduli space given by C_{n_1,n_2,n_3}'s (modded out by an overall multiplication of a complex number) [17, 36]. When all Δ_I's assume general values, (3.5) is invariant under the rotation only if a single term is kept on the right hand side. So the moduli are completely lifted. In this case, n_I's label

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a discrete set of points in the moduli space which are invariant under $\mathrm{U}(1)^3$. When Δ_I's assume rational ratios, one may keep multiple terms on the right hand side if the value of $\sum_I n_I\Delta_I$ is the same. In this case, the moduli are partly unlifted. If we can interpret the integration over the unlifted moduli space as the 1-loop zero-mode integral in the large N calculation, only $\sum_I n_I\Delta_I$ would appear in the leading large N free energy since this is the only invariant quantity on the unlifted moduli space. Of course this line of thinking assumes many things, such as the equivariant localization picture of the index, etc. However, we think it is a somewhat natural explanation of the appearance of $\sum_I n_I\Delta_I$.

[17, 20, 23] explored the index of the 6d SCFTs on N M5-branes from M2-brane giant gravitons, and also the index of the 3d SCFTs on N M2-branes from M5-brane giant gravitons. We shall now study the 6d/3d indices accepting the existence of such giant graviton expansions, only assuming the analytic continuation picture and the n_I dependence through $\sum_I n_I\Delta_I$. In the former case, the M2-brane giant gravitons wrap an internal $S^2 \subset S^4$. So analytic continuations of the 3d maximal SCFT index will give the giant graviton index in $AdS_7 \times S^4$. In the latter case, the M5-brane giant gravitons wrap an internal $S^5 \subset S^7$. So the 6d maximal SCFT index will provide the giant graviton index in $AdS_4 \times S^7$. Integrating out the giant graviton numbers, we indeed recover the free energies and entropies of $AdS_{7,4}$ black holes.

***AdS*$_4$ black holes from M5-branes:** we assume the following giant graviton expansion [23] of the 3d index of maximal SCFT living on N M2-branes [37]:

$$Z_{\rm 3d}(\Delta_{1,2,3,4};\omega) = Z_{\rm KK}\sum_{n_1,n_2,n_3,n_4=0}^{\infty} e^{-N\sum_{I=1}^4 \Delta_I n_I} Z_{n_1,n_2,n_3,n_4}(\Delta_I;\omega)\,. \tag{3.6}$$

Δ_I are for the $\mathrm{U}(1)^4 \subset \mathrm{SO}(8)$ R-symmetry and ω is for the $\mathrm{U}(1) \subset \mathrm{SO}(3)$ rotation, satisfying $\sum_I \Delta_I - \omega = 2\pi i\mathbb{Z}$. n_I are the numbers of maximal giant gravitons wrapping four $S^5 \subset S^7$.

When only one n_I is nonzero, say when $n_1 \neq 0$, $\log Z_{n_1,0,0,0}$ is obtain by analytically continuing the free energy $\log Z^{\rm 6d}_{n_1}$ of 6d $(2,0)$ SCFT of A_{n_1-1} type at large n_1. The three spacetime parameters are $\Delta_{2,3,4}$, and the two $\mathrm{U}(1)^2$ internal parameters are $\omega, -\Delta_1$. Either from QFT [14, 38] or gravity [39] considerations, one obtains

$$\log Z_{n_1,0,0,0} = \log Z^{\rm 6d}_{n_1} \equiv -\frac{n_1^3}{24}\frac{\omega^2(-\Delta_1)^2}{\Delta_2\Delta_3\Delta_4} \tag{3.7}$$

at $\sum_I \Delta_I - \omega = \pm 2\pi i$. When all n_I's are nonzero, there would be four different 6d QFT's, and also extra modes supported on the intersection S^3 of two giants and on the intersection S^1 of three giants. We suggest that the net large n_I free energy is given by

$$\log Z_{n_1,n_2,n_3,n_4} = -\frac{\omega^2\left(\sum_{I=1}^4 n_I\Delta_I\right)^3}{24\Delta_1\Delta_2\Delta_3\Delta_4}\,. \tag{3.8}$$

This formula reduces to the expected ones like (3.7) when only one n_I is nonzero. This is the unique expression with the correct limits and the $\sum_I n_I\Delta_I$ dependence.

Expanding the numerator of (3.8), one obtains contributions from the 6d/4d/2d modes:

$$\log Z_{n_1,n_2,n_3,n_4} = \sum_{I=1}^{3} \log Z^{\rm 6d}_{n_I} + \sum_{I<J} \log Z^{\rm 4d}_{n_I,n_J} + \sum_{I<J<K} \log Z^{\rm 2d}_{n_I,n_J,n_K} \,. \tag{3.9}$$

$\log Z^{\rm 6d}_{n_I}$ are given by (3.7) or its permuted versions, and the other terms are given by

$$\log Z^{\rm 4d}_{n_1,n_2} = -\frac{n_1 n_2 \omega^2 \left(n_1\Delta_1 + n_2\Delta_2\right)}{8\Delta_3\Delta_4}\,, \qquad \log Z^{\rm 2d}_{n_1,n_2,n_3} = -\frac{n_1n_2n_3\omega^2}{4\Delta_4}\,. \tag{3.10}$$

We can independently justify them. $\log Z^{\rm 4d}_{n_1,n_2}$ is the free energy of an SCFT at the intersection of n_2 M5-branes on 012345, and n_1 M5-branes on $01236,10$. Its free energy takes the form of $\frac{P(-\Delta_1,-\Delta_2,\omega)}{\Delta_3\Delta_4}$, where P is the cubic anomaly polynomial for $\mathrm{SO}(2)_{45}$, $\mathrm{SO}(2)_{6,10}$, $\mathrm{SO}(3)_{789}$.[6] The $\mathrm{SO}(2)_{45}$-$\mathrm{SO}(3)$-$\mathrm{SO}(3)$ anomaly can be computed by separating the n_2 M5-branes along the 6'th direction, and compactifying the 10'th direction to obtain Witten's $\mathrm{SU}(n_1)^{n_2-1}$ MQCD [40]. ($\mathrm{SO}(2)_{6,10}$ is explicitly broken by the deformations.) This is a linear quiver with n_1 fundamentals attached to each $\mathrm{SU}(n_1)$ node at the end. The large n_I anomaly is given by

$$k_{\Delta_1\omega\omega} \equiv \mathrm{Tr}\left[J_{45}J_{78}J_{78}\right] = \frac{n_{\rm V}}{4} = \frac{(n_1^2-1)(n_2-1)}{4} \approx \frac{n_1^2 n_2}{4}\,, \tag{3.11}$$

where $n_{\rm V}$ is the number of vector multiplets. This yields the following contribution to the anomaly polynomial P (e.g. see eq. (2.34) of [41] and also [42]):

$$\frac{3k_{\Delta_1\omega\omega}(-\Delta_1)\omega^2}{6} = -\frac{n_1^2n_2\Delta_1\omega^2}{8}\,. \tag{3.12}$$

This explains the first term of $\log Z^{\rm 4d}_{n_1,n_2}$ in (3.10). Similarly, its second term is explained from the $\mathrm{SO}(2)_{6,10}$-$\mathrm{SO}(3)^2$ anomaly of the $\mathrm{SU}(n_2)^{n_1-1}$ MQCD. $\log Z^{\rm 2d}_{n_1,n_2,n_3}$ can also be computed from anomalies, but here we just explain a quick check of its coefficient from the entropy of 4d black holes obtained by triply intersecting M5-strings with momentum p. The entropy is given by $2\pi\sqrt{n_1n_2n_3p}$, which is obtained by extremizing $\frac{\pi^2 n_1n_2n_3}{\beta}+p\beta$. This is the Cardy limit of $\log Z^{\rm 2d}_{n_1,n_2,n_3}$ in (3.10), upon taking $\Delta_4=\beta\ll 1$ and $\omega=\sum_I\Delta_I\mp 2\pi i\approx\mp 2\pi i$.

Now we extremize the entropy function given by

$$S(Q_I,J;\Delta_I,\omega,n_I) = \log Z_{n_1,n_2,n_3,n_4} + \sum_{I=1}^{4}\Delta_I(Q_I-Nn_I)+\omega J \tag{3.13}$$

with (3.8). As in section 2, we may either understand it as maximizing $\mathrm{Re}[S]$ with real n_I's for small black holes, or extremizing S with complex n_I's for generic black holes. We present the calculation in the latter viewpoint. Extremizing S in n_I, $\sum_I n_I\Delta_I$ is given by

$$\left(\sum_I n_I\Delta_I\right)^2 = -\frac{8N\Delta_1\Delta_2\Delta_3\Delta_4}{\omega^2} \;\to\; \sum_I n_I\Delta_I = \pm 2\sqrt{2}iN^{\frac{1}{2}}\frac{\sqrt{\Delta_1\Delta_2\Delta_3\Delta_4}}{\omega}\,. \tag{3.14}$$

[6] $-\Delta_1$, $-\Delta_2$ are inserted since they correspond to the tachyonic transverse directions.

We should pick one saddle point solution. To explain this, we employ the convention of [43, 44] on the square-root, which sets $\sqrt{\Delta^4} = -\Delta^2$ when all Δ_I's are equal. Then, among the two solutions of (3.14), one should choose the upper/lower sign at $\sum_{I=1}^4 \Delta_I - \omega = \pm 2\pi i$, respectively. We showed this by a Picard-Lefschetz analysis like that of section 2.2. With this choice, let us first discuss the small black hole limit, in which case $\Delta_I \ll 1$ are small positive numbers and $\omega \approx \mp 2\pi i$. Then (3.14) at equal Δ_I's reduces to

$$\Delta \sum_I n_I \approx \pm 2\sqrt{2} i N^{\frac{1}{2}} \cdot \frac{-\Delta^2}{\mp 2\pi i} = 2\sqrt{2} N^{\frac{3}{2}} \Delta^2 \,, \tag{3.15}$$

yielding real positive $\sum_I n_I$. This ensures that extremizing S with complex n_I is equivalent to maximizing $\mathrm{Re}[S]$ with real n_I in the small black hole limit.

Inserting the solution picked in the previous paragraph into (3.13), one obtains

$$S(Q_I, J; \Delta_I, \omega) = \mp \frac{4\sqrt{2} i N^{\frac{3}{2}}}{3} \frac{\sqrt{\Delta_1\Delta_2\Delta_3\Delta_4}}{\omega} + \sum_{I=1}^{4} \Delta_I Q_I + \omega J \tag{3.16}$$

which perfectly reproduces the entropy function of BPS black holes in $AdS_4 \times S^7$ [43].

***AdS_7* black holes from M2-branes:** we assume the following giant graviton expansion [23] of the index of 6d $(2,0)$ SCFT of A_{N-1} type [37]:

$$Z_{\mathrm{6d}}(\Delta_1, \Delta_2; \omega_1, \omega_2, \omega_3) = Z_{\mathrm{KK}} \sum_{n_1, n_2 = 0}^{\infty} e^{-N \sum_{I=1}^{2} \Delta_I n_I} Z_{n_1, n_2}(\Delta_I; \omega_i) \,. \tag{3.17}$$

Here $\Delta_{1,2}$ are for the $\mathrm{U}(1)^2 \subset \mathrm{SO}(5)$ R-symmetry, and $\omega_{1,2,3}$ are for the $\mathrm{U}(1)^3 \subset \mathrm{SO}(6)$ in the spacetime. They satisfy $\sum_I \Delta_I - \sum_i \omega_i = 2\pi i \mathbb{Z}$. n_I are the numbers of maximal giant gravitons wrapping two different $S^2 \subset S^4$ cycles.

Assuming the analytic continuation picture, $\log Z_{n_1,0}$ at large $n_1 \sim N$ will be given by the 3d free energies on n_1 M2-branes. Taking $\omega_{1,2,3}, -\Delta_1$ as the internal parameters and Δ_2 as the worldvolume parameter, the analytically continued free energy is given by [43–46]

$$\pm \frac{4\sqrt{2} i n_1^{\frac{3}{2}}}{3} \frac{\sqrt{\omega_1\omega_2\omega_3(-\Delta_1)}}{\Delta_2} \,, \quad \sum_{I=1}^{2} \Delta_I - \sum_{i=1}^{3} \omega_i = \pm 2\pi i \,. \tag{3.18}$$

Similar expression is obtained for $\log Z_{0,n_2}$. Now we suggest that the general $\log Z_{n_1,n_2}$ at large $n_I \sim N$ is given by

$$\log Z_{n_1,n_2} \sim \pm \frac{4\sqrt{2} i}{3} \sqrt{\omega_1\omega_2\omega_3} \frac{(-n_1\Delta_1 - n_2\Delta_2)^{\frac{3}{2}}}{-\Delta_1\Delta_2} \,. \tag{3.19}$$

This gives the desired limits when either of n_1, n_2 vanishes, and depends only on $\sum_I n_I \Delta_I$. It defies factorization between 3d-1d degrees of freedom.

Now extremizing the entropy function given by

$$S(Q_I, J_i; \Delta_I, \omega_i, n_I) = \log Z_{n_1,n_2} + \sum_{I=1}^{2} \Delta_I (Q_I - N n_I) + \sum_{i=1}^{3} \omega_i J_i \tag{3.20}$$

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with (3.19), one obtains

$$\pm 2\sqrt{2}i\sqrt{\omega_1\omega_2\omega_3(-n_1\Delta_1 - n_2\Delta_2)} = N\Delta_1\Delta_2 \;\rightarrow\; \sum_{I=1}^{2} n_I\Delta_I = \frac{N^2}{8}\frac{(\Delta_1\Delta_2)^2}{\omega_1\omega_2\omega_3}\,. \tag{3.21}$$

Inserting this back to the entropy function, one obtains

$$S(Q_I, J_i; \Delta_I, \omega_i) = -\frac{N^3}{24}\frac{\Delta_1^2\Delta_2^2}{\omega_1\omega_2\omega_3} + \sum_{I=2}^{2}\Delta_I Q_I + \sum_{i=1}^{3}\omega_i J_i \tag{3.22}$$

which is precisely the entropy function of BPS black holes in $AdS_7 \times S^4$ [39].

Before finishing this exercise on $AdS_7 \times S^4$, we comment on a puzzle that we resolved only partly. In section 2, it was important to have a leading order degeneracy after fixing $\sum_I n_I\Delta_I$ by extremization, for the continuum conjecture and cancellations of Z_{n_I}'s. Here, fixing $n_1\Delta_1 + n_2\Delta_2$ at general complex Δ_I's does not leave any degeneracy. So one may wonder whether the picture of section 2 is valid here. Curiously, we can see that the apparent real maximum n_I^0 of $\mathrm{Re}[S(Q_I, J_i, n_I)]$ is generally subject to severe cancellations already with the leading free energy. To simplify the discussions, let us unrefine Δ_I, ω_i and keep $n \equiv n_1 + n_2$, $q \equiv \frac{Q_1+Q_2}{2} + \frac{1}{3}\sum_i J_i$ only. Then the entropy scales like

$$S(q, n) = N^3 f\left(\tfrac{q}{N^3}, \tfrac{n}{N^2}\right)\,. \tag{3.23}$$

The maximum n_0 satisfies $\mathrm{Re}[f'(\frac{n_0}{N^2})] = 0$. Now considering a neighborhood of n_0, one finds

$$S\left(\tfrac{n_0+\Delta n}{N^2}\right) = N^3 f\left(\tfrac{n_0}{N^2}\right) + Nf'\left(\tfrac{n_0}{N^2}\right)\Delta n + \frac{1}{2N}f''\left(\tfrac{n_0}{N^2}\right)(\Delta n)^2 + \cdots\,. \tag{3.24}$$

Since the real part of the second term vanishes at n_0, the change of $\mathrm{Re}[S]$ is very slow in a wide range of $|\Delta n|$ ($\ll N^{\frac{1}{2}}$). However, $\mathrm{Im}[S]$ generally varies fast in this neighborhood due to the second term $N\mathrm{Im}[f'(\frac{n_0}{N^2})]\Delta n$. This causes cancellations of nearby terms around $n = n_0$, reducing the apparently over-estimated entropy at n_0. This cancellation happens because of the large N scalings $S \sim N^3$ and $n \sim N^2$: similar cancellations do not happen in $\mathrm{AdS}_5/\mathrm{CFT}_4$ with $S \sim N^2$ and $n \sim N$. This provides the expected cancellations of the giant graviton index. We would like to further understand what makes the full continuum conjecture for n_I's possible.

4 Conclusion

In this paper, we studied the microstate counting of BPS black holes in $\mathrm{AdS}_{4,5,7}$ from the viewpoint of giant graviton expansions.

We employed a saddle point approach to compute the large N giant graviton index Z_{n_1,n_2,n_3} in $AdS_5 \times S^5$, when the three electric chemical potentials are unrefined. Further maximizing the corresponding entropy in n_I, we successfully accounted for the microstates of small AdS_5 black holes. To understand the black holes at general sizes, one had to take into account the cancellations of Z_{n_1,n_2,n_3}'s. We conjectured a particular cancellation mechanism which successfully reproduces the entropies of the black holes at arbitrary sizes. We further proposed the large N free energy for Z_{n_1,n_2,n_3} with all chemical potentials refined.

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We interpreted the large N giant graviton free energies as the analytically continued indices of SCFT's. Extending this interpretation to other AdS/CFT examples, we found intriguing connections between the SCFT free energies in various dimensions. We obtained the large N free energies of 6d/3d SCFTs on M5/M2-branes from the analytic continuations of vice versa, after suitably dressing them by certain defect free energies. This suggests that the (BPS) spectrum of quantum gravity admits dual formulations, either in terms of electric or magnetic variables. Presumably, there will be more duality relations of this sort that one can find from giant gravitons and analytic continuation. As a crude but nontrivial exercise, we also tried to account for the large N free energy of the 5d SCFTs dual to the massive IIA theory on $AdS_6 \times S^4/\mathbb{Z}_2$ [43, 47, 48], from analytic continuations of the free energy of D2-brane SCFTs on an orbifold. Although the details need to be clarified, we find that the $N^{\frac{5}{2}}$ scaling of the former free energy is obtained from the $n^{\frac{5}{3}}$ scaling for D2-branes in massive IIA theory.

Perhaps we should also comment on our current understanding of the giant graviton expansion. The formula was constructed from maximal giant gravitons, wrapping the largest (non-topological) $S^3 \subset S^5$. The worldvolume degrees of freedom include tachyons which shrink the giant gravitons. The tachyonic quiver theories on these branes were used in a subtle way [16] to write down the giant graviton index formula. Technically, the tachyonic part of the partition function was first analytically continued. Then one chose the path integral contour for the holonomy zero modes empirically. After these ad hoc steps, the resulting Z_{n_1,n_2,n_3} appears to correctly describe the spectral problem of the Yang-Mills index. However, its physics looks different from that of the original quiver system. Namely, the 'ground states' of Z_{n_1,n_2,n_3} describe the trace relation constraints, subtracting the over-estimated states in the graviton index. We feel that a well-defined QFT cannot show this behavior. So it seems desirable to better understand the relation between the quiver and the formula.

Giant gravitons were originally conceived in [1] as a gravity mechanism of imposing trace relation constraints at $E \sim N^1$. However, in this paper, we found that their worldvolume degrees of freedom are also responsible for enhanced entropies at $E \sim N^2$. Perhaps new BPS states exist thanks to the trace relations. Since crucial roles were played by the open strings connecting various D-branes in our studies, one can construct an ansatz for the BPS cohomologies in terms of open spin chains ending on determinant operators [49]. We expect that such an ansatz will be relevant at least in the small black hole regime, $q \ll N^2$.

In section 2.2, we conjectured a possible mechanism in which various Z_{n_1,n_2,n_3}'s are partly canceled to account for the black hole entropies correctly. A key feature was the discrete sums of n_I's being effectively replaced by integrals. We have no a priori justification for this conjecture, except that the final entropy derived with this assumption is correct. Here we simply want to note that the discrete points labeled by (n_1, n_2, n_3) are special points on the moduli space of $\frac{1}{8}$-BPS giant gravitons [35]. As alluded to in [17, 36], they could be the fixed points of the equivariant localization calculation of an integral over this moduli space. Perhaps it may be helpful to seek such an integral reformulation of this sum to justify our continuum conjecture.

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A BPS black holes and their small black hole limits

In this appendix, we summarize the properties of the BPS black holes of [11, 12] having equal electric charges $Q_1 = Q_2 = Q_3 \equiv Q$ and two angular momenta J_1, J_2. The solutions are labelled by two independent parameters. We shall also explain the small black hole limit. More detailed discussions are given in appendix B of [27].

The metric and the gauge field of the 5d gauged supergravity are given in [11, 12], which are eqs. (B.1) and (B.4) of [27]. Their charges and the entropy are given by eq. (B.6) of [27]. For simplicity, here we explain the case with $J_1 = J_2$ in detail. In this case, the charges and the entropy are given by

$$Q = \frac{N^2}{2}\left[\frac{\mu}{\ell^2} + \frac{\mu^2}{2\ell^4}\right], \quad J_1 = J_2 = \frac{N^2}{2}\left[\frac{3\mu^2}{2\ell^4} + \frac{\mu^3}{\ell^6}\right], \quad S_{\rm BH} = \pi N^2\sqrt{\frac{\mu^3}{\ell^6} + \frac{3\mu^4}{4\ell^8}}\,, \tag{A.1}$$

where ℓ is the radius of the AdS$_5$ (called g^{-1} in [27]). μ is $\mu_1 = \mu_2 = \mu_3$ of [10, 12].

The classical black hole solutions above reliably approximate the quantum gravity when $N^2 \gg 1$ and $\varepsilon \equiv \frac{\mu}{\ell^2} \sim N^0$ is held fixed. The small black hole limit of this paper is defined by ε (a parameter independent of N) being parametrically smaller than 1. More precisely, the large N limit with fixed ε is taken first and the small ε limit is taken later. With large enough N^2, the semi-classical description is still reliable because we can keep the curvature to be small. In this limit, the charges and the entropy are all given by N^2 times a nontrivial series expansion in ε. Similarly, taking $N^2 \gg 1$ and ε to be parametrically larger than 1 yields the large black hole limit. In both limits, $S_{\rm BH}$ as a function of $q \equiv Q + \frac{J_1+J_2}{2}$ can be expanded in q/N^2, which is (2.27).

When $J_1 \neq J_2$, the small black hole limit can be taken similarly. It amounts to taking $N^2 \gg 1$, keeping $\frac{Q}{N^2}$, $\frac{J_1}{N^2}$, and $\frac{J_2}{N^2}$ (independent of N) to be parametrically smaller than 1. More technically, the limit is described by eq. (B.12) of [27].

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