M.S. THESIS

Conformal Correlators in Momentum Space

등각장론 상관함수의 운동량 공간 표현

BY

YU SEUNG-YEON

AUGUST 2023

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이 논문을 이학석사 학위논문으로 제출함

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Abstract

For a conformal field theory in an arbitrary dimension, we study the general solution of the conformal Ward identities for scalar n-point function in momentum space. As discovered by Bzowski, McFadden and Skenderis in 2019, the solution is expressed as an integral over $(n-1)$-simplices in momentum space, which we will refer to as a *simplex integral*. The n vertices of the simplex correspond to the n operator insertions. The momenta running between vertices, subject to momentum conservation at each vertex, become the integration variables. The integrand of the integral involves an arbitrary function of *momentum-space cross ratios*. We prove the conformal invariance of the simplex integral using a recursive structure most clearly visible when the function of cross ratios is a monomial in the cross ratios. As an application, we derive the simplex representation of n-point contact Witten diagrams in a holographic conformal field theory.

keywords: Conformal Field Theory, Simplex Integral

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Chapter 1

INTRODUCTION

For half a century, there has been much discussion about position-space n-point functions in a conformal field theory (CFT) and the form of the general n-point functions in position space has been known for 50 years [1]. However, momentum-space n-point functions had been not much discussed for a long time, despite the usefulness for applications, which include various areas like cosmology, condensed matter, anomalies and the bootstrap program. One of the most remarkable applications is the connection with the recent study of inflationary correlators, where the de-Sitter isometries act on late-time slices as conformal transformations. So once we know the general form of CFT n-point functions in momentum space, we can then seek to bootstrap cosmological correlators by supplying additional physical input (e,g., no collinear singularities, appropriate flat-space limit, etc. For a sample of works, see [2–8].).

Yet, recently, several researches have yielded rich and crucial properties of momentum-space n-point functions and most noteworthy is the finding of the expression for the general momentum-space n-point functions by Bzowski, McFadden and Skenderis [9, 10]. Moreover, they found the useful structure of the simplex integrals. This paper will review and follow the discussion of them.

In this thesis, the main goals can be divided into two parts. First, we give the general form of n-point functions of CFT in momentum space as a *simplex integral*, featuring an arbitrary function of momentum-space cross ratios constructed from the integration variables. These cross ratios is the analogue of the position-space cross ratios but they enter only via the integration variables. Second, as an application, we derive the simplex representation for the simplest possible holographic correlators, contact Witten diagrams in Ads/CFT, using two independent methods: one using the star-mesh transformation from electrical circuit theory and the other by a recursive application of the convolution theorem. And We also discuss how do we find the specific form of the function $\hat{f}(\hat{\boldsymbol{u}})$ of the momentum-space cross ratios for this specific example.

The layout of this thesis is as follows. we first briefly review the basic concept of CFT, including conformal algebra, conformal correlation function and conformal block in chapter 2. In chapter 3, we introduce a concept of basic integral transform and give the expression for conformal Ward identities in momentum space. Chapter 4 is the main part of this paper. We present the general scalar n-point function in momentum space as a *simplex integral*. We also introduce a special case of the simplex integral, *mesh integral*. Actually, whenever the functions of the cross ratios $\hat{f}$ is a monomial in the cross ratios, the simplex integral reduces to the mesh integral. We show that the mesh integral has a remarkable recursive structure and prove the conformal invariance of the mesh and simplex integral. In chapter 5, we derive the simplex representation for the simplest possible holographic correlators. We conclude in chapter 6.

Chapter 2

CONFORMAL CORRELATION FUNCTIONS

2.1 Conformal Algebra

To make our discussion self-contained, we present a brief review of the conformal algebra in $d > 2$ dimensions. The conformal group consists of transformations of spacetime which leave the metric tensor invariant up to a scale:

$$g'_{\mu\nu}(x') = e^{2\lambda(x)} g_{\mu\nu}(x). \tag{2.1}$$

For simplicity, we assume that the spacetime is Euclidean. In $d > 2$, the Euclidean conformal group is $SO(d+1,1)$. The finite transformations are known to be

$$\begin{aligned}
x'^{\mu} &= x^{\mu} + a^{\mu}, && \text{(Translation)} \\
x'^{\mu} &= (1+\alpha)x^{\mu}, && \text{(Dilation)} \\
x'^{\mu} &= M^{\mu}{}_{\nu} x^{\nu}, && \text{(Lorentz rotation)} \\
x'^{\mu} &= \frac{x^{\mu} - b^{\mu}x^2}{1 - 2b\cdot x + b^2 x^2}. && \text{(Special conformal transformation)}
\end{aligned} \tag{2.2}$$

Taking the infinitesimal transformations, we can read off the generators of the conformal algebra realized as differential operators,

$$
\begin{aligned}
P_\nu &= -i\partial_\nu, && \text{(Translation)} \\
D &= -ix^\mu\partial_\mu, && \text{(Dilation)} \\
L_{\rho\nu} &= -i(x_\nu\partial_\rho - x_\rho\partial_\nu), && \text{(Lorentz rotation)} \\
K_\rho &= -i(2x_\rho x^\mu\partial_\mu - x^2\partial_\rho). && \text{(Special conformal transformation)}
\end{aligned}
\tag{2.3}
$$

The conformal algebra is realized as Lie algebra among differential operators,

$$
\begin{aligned}
[D, P_\mu] &= iP_\mu, \\
[D, K_\mu] &= -iK_\mu, \\
[K_\mu, P_\nu] &= 2i(g_{\mu\nu}D - L_{\mu\nu}), \\
[K_\rho, L_{\mu\nu}] &= i(g_{\rho\mu}K_\nu - g_{\rho\nu}K_\nu), \\
[P_\rho, L_{\mu\nu}] &= i(g_{\rho\mu}P_\nu - g_{\rho\nu}P_\nu), \\
[L_{\mu\nu}, L_{\rho\sigma}] &= i(g_{\nu\rho}L_{\mu\sigma} + g_{\mu\sigma}L_{\nu\rho} - g_{\mu\rho}L_{\nu\sigma} - g_{\nu\sigma}L_{\mu\rho}).
\end{aligned}
\tag{2.4}
$$

Physicists are interested in the role of conformal symmetry in a QFT. Let F be the function relating the new field Φ' evaluated at the transformed coordinate x' to the old field Φ at x:

$$\Phi'(x') = F(\Phi(x))\,. \tag{2.5}$$

For example, the action of a scale transformation on the field $\Phi(x)$ defines the scaling dimension Δ of Φ:

$$\Phi(\lambda x) = \lambda^{-\Delta}\Phi(x)\,. \tag{2.6}$$

An infinitesimal transformation can be written as

$$
\begin{aligned}
x'^\mu &= x^\mu + \epsilon^a \frac{\delta x^\mu}{\delta\epsilon^a}\,, \\
\Phi'(x') &= \Phi(x) + \epsilon^a \frac{\delta F}{\delta\epsilon^a}\,.
\end{aligned}
\tag{2.7}
$$

The generator G_a of an infinitesimal transformation acting on a field is defined as

$$\delta_\epsilon\Phi(x) \equiv \Phi'(x) - \Phi(x) \equiv -i\epsilon^a G_a\Phi(x)\,, \tag{2.8}$$

so that

$$iG_a\Phi = \frac{\delta x^\mu}{\delta\epsilon^a}\partial_\mu\Phi - \frac{\delta F}{\delta\epsilon^a}\,. \tag{2.9}$$

Let T_a be a matrix representation of an infinitesimal transformation,

$$\Phi'(x') = (1 - i\epsilon^a T_a)\Phi(x)\,. \tag{2.10}$$

Then, we can summarize the transformation rules for the field $\Phi(x)$ as

$$\begin{aligned}
P_\mu\Phi(x) &= -i\partial_\mu\Phi(x), \\
D\Phi(x) &= -i(x^\nu\partial_\nu + \Delta)\Phi(x), \\
L_{\mu\nu}\Phi(x) &= i(x_\mu\partial_\nu - x_\nu\partial_\mu)\Phi(x) + S_{\mu\nu}\Phi(x), \\
K_\mu\Phi(x) &= (\kappa_\mu - i2x_\mu\Delta - x^\nu S_{\mu\nu} - 2ix_\mu x^\nu\partial_\nu + ix^2\partial_\mu)\Phi(x),
\end{aligned} \tag{2.11}$$

where Δ is the scaling dimension and $S_{\mu\nu}$ is the matrix representation of the Lorentz rotation.

2.2 Conformal Correlation Function

One of the most basic observables in a conformal field theory (CFT) is the correlation function of local operators. The constraints imposed by the conformal symmetry on the conformal correlators are expressed by the Ward identities.

Let us denote the n-point correlation function by $\langle\phi_1(x_1)\cdots\phi_n(x_n)\rangle$. For simplicity, we assume that the operators ϕ_i are scalars. The translation and rotation symmetry can be made manifest by appropriate vector or tensor notations. The dilatation Ward identity is slightly less trivial:

$$\sum_{j=1}^{n}\left(x_j^\alpha\frac{\partial}{\partial x_j^\alpha} + \Delta_j\right)\langle\phi_1(x_1)\cdots\phi_n(x_n)\rangle = 0, \tag{2.12}$$

and the special conformal Ward identity,

$$\sum_{j=1}^{n}\left(-x_j^2\frac{\partial}{\partial x_j^\kappa} + 2x_j^\kappa x_j^\alpha\frac{\partial}{\partial x_j^\alpha} + 2\Delta_j x_j^\kappa\right)\langle\phi_1(x_1)\cdots\phi_n(x_n)\rangle = 0, \tag{2.13}$$

where Δ_j is the scaling dimension of the field ϕ_j.

It is well known to what extent conformal invariance restricts n-point correlation functions. For $n \leq 3$, the constraints are so strong that the correlation functions are determined up to constants.

Up to 4-point The only possibility for the one-point functions is a constant,

$$\langle \phi_1(x_1) \rangle = C\,. \tag{2.14}$$

The dilatation Ward identity dictates that the value of C can be non-zero only for the identity operator with $\Delta = 0$.

Translation, rotation and scale invariance fix the two-point functions of scalar operators up to a constant as

$$\langle \phi_1(x_1)\phi_2(x_2) \rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}. \tag{2.15}$$

Then, special conformal symmetry imposes an orthogonality condition,

$$\langle \phi_1(x_1)\phi_2(x_2) \rangle = \frac{C_{12}\delta_{\Delta_1\Delta_2}}{|x_1 - x_2|^{2\Delta_1}}. \tag{2.16}$$

For the three-point functions, translation, dilatation and special conformal transformation together demand that

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \rangle = \frac{f_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{31}^{\Delta_3+\Delta_1-\Delta_2}}, \tag{2.17}$$

where f_{123} is a constant and $x_{ij} \equiv x_i - x_j$. It is a straightforward exercise to verify that the 2-point function (2.15) and the 3-point function (2.17) satisfy the conformal Ward identities (2.12) and (2.13).

The four-point functions, unlike the two- and three-point functions, cannot be fully fixed by conformal invariance. They have an arbitrary dependence on two independent *conformal cross-ratio*:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) \rangle = g(u,v) \prod_{i<J}^{4} x_{ij}^{\Delta/3-\Delta_i-\Delta_j}, \tag{2.18}$$

where u and v are defined by

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}, \tag{2.19}$$

$g(u, v)$ is an arbitrary function, and $\Delta = \sum_{i=1}^{4} \Delta_i$.

General n-point function The higher point functions allow for more conformal cross-ratio. The general form of a scalar n-point function is known to be

$$\langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle = \prod_{1 \le i < j \le n} x_{ij}^{2\alpha_{ij}} f(\boldsymbol{u}), \quad x_{ij} \equiv |x_i - x_j|. \tag{2.20}$$

The parameters α_{ij} are related to the scaling dimensions by

$$\Delta_m = -\sum_{j=1}^{n} \alpha_{mj}, \qquad m = 1, 2, \ldots, n. \tag{2.21}$$

These conditions do not determine α_{ij} uniquely. To avoid ambiguity, it is convenient to make two choices,

$$\alpha_{ji} = \alpha_{ij} \quad \text{and} \quad \alpha_{ii} = 0. \tag{2.22}$$

We denote the full set of parameters collectively by $\boldsymbol{\alpha} = \{\alpha_{ij}\}_{1 \le i < j \le n}$.

The conformal cross-ratio are defined by

$$u_{[pqrs]} = \frac{x_{pr}^2 x_{qs}^2}{x_{pq}^2 x_{rs}^2}, \tag{2.23}$$

where $p, q, r, s = 1, 2, \ldots, n$ are distinct numbers. This construction considerably over-counts the number of cross ratios. At most $n(n-3)/2$ are independent. In sufficiently high spacetime dimension, all $n(n-3)/2$ are independent. In low spacetime dimensions there can be non-trivial relations between the cross ratios. We will ignore this subtlety.

A brief explanation of the counting of the cross ratios is as follows. The number of distinct x_{ij}'s is $n(n-1)/2$. Define the monomial $m(\boldsymbol{x}; \boldsymbol{a})$ by

$$m(x_i; a_{ij}) = \prod_{1 \le i < j \le n} x_{ij}^{a_{ij}}. \tag{2.24}$$

For this monomial to be conformally invariant, the following relation is required:

$$\sum_{j=1}^{i-1} a_{ji} + \sum_{j=i+1}^{n} a_{ij} = 0 \quad \text{for all} \quad i = 1, \ldots, n. \tag{2.25}$$

This gives n constraints on the monomial. So the number of independent cross ratios is reduced to

$$\frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}\,. \tag{2.26}$$

There are various ways of choosing an independent set of u. We will use

$$u_{2a} = u_{[123a]} = \frac{x_{2a}^2 x_{13}^2}{x_{1a}^2 x_{23}^2}, \quad u_{3a} = u_{[132a]} = \frac{x_{3a}^2 x_{12}^2}{x_{1a}^2 x_{23}^2}, \quad u_{ab} = u_{[2a3b]} = \frac{x_{ab}^2 x_{23}^2}{x_{2a}^2 x_{3b}^2}, \tag{2.27}$$

where $a, b = 4, 5, ..., n$ and $a < b$, so we have the right number of ratios:

$$2(n-3) + \frac{(n-3)(n-4)}{2} = \frac{n(n-3)}{2}\,.$$

We denote these independent cross ratios collectively by a vector $\boldsymbol{u}$.

Chapter 3

INTEGRAL TRANSFORMS

3.1 Fourier Transform

We follow high energy physicists' standard convention for Fourier transform:

$$f(x) = \int \frac{d^d p}{(2\pi)^d} e^{ip\cdot x} \tilde{f}(p) \quad \Longleftrightarrow \quad \tilde{f}(p) = \int d^d x e^{-ip\cdot x} f(x)\,. \tag{3.1}$$

Applying the same convention to an n-point correlation function is straightforward,

$$\begin{aligned} & \langle \phi_1(x_1)\cdots\phi_n(x_n)\rangle = \int [dp]_n\, e^{i(p_1x_1+\ldots+p_nx_n)} \langle \phi_1(p_1)\cdots\phi_n(p_n)\rangle \\ \Longleftrightarrow \quad & \langle \phi_1(p_1)\cdots\phi_n(p_n)\rangle = \int [dx]_n\, e^{-(ip_1x_1+\ldots+p_nx_n)} \langle \phi_1(x_1)\cdots\phi_n(x_n)\rangle \ . \end{aligned} \tag{3.2}$$

To avoid clutter, we chose to suppress the dot for inner products ($p\cdot x \to px$) and introduced short-hand notations,

$$[dp]_n = \prod_{i=1}^{n} \frac{d^d p_i}{(2\pi)^d}\,, \quad [dx]_n = \prod_{i=1}^{n} d^d x_i\,. \tag{3.3}$$

A minor but important feature of the Fourier transform of the correlation functions is that translation symmetry implies momentum conservation. To account for this, we adopt the double-bracket notation to denote the reduced correlation function

$$\langle \phi_1(p_1)\cdots\phi_n(p_n)\rangle = (2\pi)^d \delta\left(\sum_{i=1}^{n} p_i\right) \langle\!\langle \phi_1(p_1)\cdots\phi_n(p_n)\rangle\!\rangle\,. \tag{3.4}$$

In the double-bracket, one of the momenta, say p_n, is considered as a dependent variable,

$$p_n = -\sum_{i=1}^{n-1} p_i\,. \tag{3.5}$$

To see the origin of the momentum-conserving delta-function explicitly, we change the variables as

$$x_i = y_i + y_n \quad (1 \le i \le n-1)\,, \quad x_n = y_n\,, \tag{3.6}$$

and use the translation invariance to set

$$\begin{aligned} \langle \phi_1(x_1)\cdots\phi_n(x_n)\rangle &= \langle \phi_1(y_1+y_n)\cdots\phi_{n-1}(y_{n-1}+y_n)\phi_n(y_n)\rangle \\ &= \langle \phi_1(y_1)\cdots\phi_{n-1}(y_{n-1})\phi_n(0)\rangle\ . \end{aligned} \tag{3.7}$$

It is now clear that the integration over y_n produces the delta-function, whereas the integration over $(y_1, \ldots, y_{n-1})$ produces $\langle\!\langle \phi_1(p_1)\cdots\phi_n(p_n)\rangle\!\rangle$.

As simple examples, we show explicitly the Fourier transformation of 1-point and 2-point functions. The 1-point function is simply a constant, say $\langle\phi_1(x_1)\rangle = C$. The Fourier transformation gives $C \times (2\pi)^d \delta(p_1)$.

For the 2-point function (2.16), we should compute the integral,

$$\begin{aligned} \langle \phi_1(p_1)\phi_2(p_2)\rangle &= \int [dp]_2\, e^{ip_1x_1+p_2x_2}\, \langle \phi_1(x_1)\phi_2(x_2)\rangle \\ &= \int [dp]_2\, e^{ip_1x_1+p_2x_2}\, \frac{C_{12}\delta_{\Delta_1\Delta_2}}{|x_1-x_2|^{2\Delta_1}}. \end{aligned} \tag{3.8}$$

For convenience, we set $x_2 = 0$. Then using

$$\int d^dx e^{-ip\cdot x}\frac{1}{x^{2\Delta}} = \frac{\pi^{d/2}2^{d-2\Delta}\Gamma(\frac{d-2\Delta}{2})}{\Gamma(\Delta)}p^{2\Delta-d},$$

where the integral converges for $0 < 2\Delta < d$, we get

$$\langle\!\langle \phi_1(p_1)\phi_2(p_2)\rangle\!\rangle = \frac{C_{12}\pi^{d/2}2^{d-2\Delta}\Gamma(\frac{d-2\Delta}{2})}{\Gamma(\Delta)}p_1^{2\Delta-d} = c_2\, p_1^{2\Delta-d}, \tag{3.9}$$

where $\Delta_1 = \Delta_2 = \Delta$ and we defined the overall constant as c_2. We extracted the Dirac delta function associated with momentum conservation and used the double-bracket notation to denote the reduced correlation function as

$$\langle \phi_1(p_1)\phi_2(p_2)\rangle = (2\pi)^d\delta(p_1+p_2)\langle\!\langle \phi_1(p_1)\phi_2(p_2)\rangle\!\rangle. \tag{3.10}$$

3.2 Conformal Invariance in Momentum Space

We should import the conformal Ward identities to momentum space. Again we focus on the n-point scalar correlators. As discussed above, translation symmetry is incorporated by momentum conservation and the double-bracket notation in (3.4). Rotation symmetry requires that the correlation function be built from invariant scalar products. It remains to spell out the dilatation and special conformal Ward identities in momentum space. We will discuss both the full correlation functions and the reduced correlation functions in the sense of (3.4).

The most convenient (although less rigorous) way to find the momentum-space expressions for the conformal Ward identity operators is to use the familiar rules:

$$x^\alpha \to +i\frac{\partial}{\partial p_\alpha}\,, \quad \frac{\partial}{\partial x^\alpha} \to +ip_\alpha\,. \tag{3.11}$$

These rules are familiar from quantum mechanics. They can be rigorously justified provided that the "wave-functions" behave mildly. We will not dwell on this mathematical point any further.

Dilatation The dilatation Ward identity in the position space is given by (2.12):

$$D = \Delta_t + \sum_{j=1}^{n} x_j^\alpha \frac{\partial}{\partial x_j^\alpha}\,, \quad \Delta_t = \sum_{j=1}^{n} \Delta_i\,. \tag{3.12}$$

Applying the rules (3.11) and switching the ordering between p and $\partial/\partial p$, we find

$$D = \Delta_t - nd - \sum_{j=1}^{n} p_j^\alpha \frac{\partial}{\partial p_j^\alpha}\,. \tag{3.13}$$

In other words, we get the following dilatation Ward identity in momentum space,

$$\left(\Delta_t - nd - \sum_{j=1}^{n} p_j^\alpha \frac{\partial}{\partial p_j^\alpha}\right) \langle \phi_1(p_1)\cdots\phi_n(p_n)\rangle = 0. \tag{3.14}$$

In terms of the reduced correlator, the Ward identity becomes

$$\left(\Delta_t - (n-1)d - \sum_{j=1}^{n-1} p_j^\alpha \frac{\partial}{\partial p_j^\alpha}\right) \langle\!\langle \phi_1(p_1)\cdots\phi_n(p_n)\rangle\!\rangle = 0\,. \tag{3.15}$$

Special conformal transformation We can follow a similar procedure to find the expression for the special conformal Ward identity in momentum space. The Ward identity in the position space is given by (2.13):

$$K^\kappa = \sum_{j=1}^{n} K_j^\kappa \,, \quad K_j^\kappa = -i \left(-x_j^2 \frac{\partial}{\partial x_j^\kappa} + 2x_j^\kappa x_j^\alpha \frac{\partial}{\partial x_j^\alpha} + 2\Delta_j x_j^\kappa \right) . \tag{3.16}$$

Applying the rules (3.11) and pushing the derivatives to the right, we obtain

$$\begin{aligned} K_j^\kappa &= K^\kappa(\Delta_j, p_j) \,, \\ K^\kappa(\Delta, p) &= p^\kappa \frac{\partial^2}{\partial p^\alpha \partial p_\alpha} - 2p_\alpha \frac{\partial^2}{\partial p_\alpha \partial p_\kappa} + 2(\Delta - d) \frac{\partial}{\partial p_\kappa} \,. \end{aligned} \tag{3.17}$$

Using these notations, we can succinctly summarize the special conformal Ward identities as

$$K^\kappa(\boldsymbol{\Delta}) \left\langle \phi_1(p_1) \cdots \phi_n(p_n) \right\rangle = 0, \tag{3.18}$$

$$\tilde{K}^\kappa(\boldsymbol{\Delta}) \langle\!\langle \phi_1(p_1) \cdots \phi_n(p_n) \rangle\!\rangle = 0, \tag{3.19}$$

where the corresponding operators are

$$K^\kappa(\boldsymbol{\Delta}) = \sum_{j=1}^{n} K^\kappa(\Delta_j; p_j) \,, \quad \tilde{K}^\kappa(\boldsymbol{\Delta}) = \sum_{j=1}^{n-1} K^\kappa(\Delta_j; p_j) \,, \tag{3.20}$$

with $K^\kappa(\Delta; p)$ defined in (3.17). In (3.20), $\boldsymbol{\Delta}$ denotes collectively the scaling dimensions of all the operators.

3.3 Mellin-Barnes Transform

Given a function $f(x)$, defined on the positive real axis $\mathbb{R}_+$, its Mellin transform is defined by

$$\varphi(s) = (\mathcal{M}f)(s) = \int_0^\infty x^{s-1} f(x) dx. \tag{3.21}$$

This relation can be inverted by a line integral

$$f(x) = (\mathcal{M}^{-1}\varphi)(x) = \frac{1}{2\pi i} \int_\gamma x^{-s} \varphi(s) ds, \tag{3.22}$$

by an appropriate choice of the contour γ in the complex plane starting at $c - i\infty$ and ending at $c+i\infty$, with $\text{Re}(s) = c > 0$. If the integrand of the transform (3.22) involves products and ratios of Gamma functions, it is called a Mellin-Barnes integral.

Let us give a simple example. The famous integral representation of the gamma function,

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx, \qquad \text{Re}(s) > 0, \tag{3.23}$$

shows that $\Gamma(s)$ is nothing but the Mellin transform of e^{-x}. The inversion formula reads

$$e^{-x} = \int_{c-i\infty}^{c+i\infty} x^{-s}\Gamma(s) ds, \qquad c > 0, \tag{3.24}$$

which provides perhaps the simplest Mellin-Barnes integral. The equivalence between (3.23) and (3.24) can be proved easily by using Cauchy's residue theorem. Let us compute the right-hand side of (3.24) with the rectangular contour given in Figure 3.1. The vertices of the contour are located at $c \pm iR$ $(c > 0)$ and $-(N + 1/2) \pm iR$, where N is a positive integer. The poles of $\Gamma(s)$ inside this contour are at $s = -m$ with residues $(-1)^m/m!$ for $m \in \{0, \ldots, N\}$, respectively.

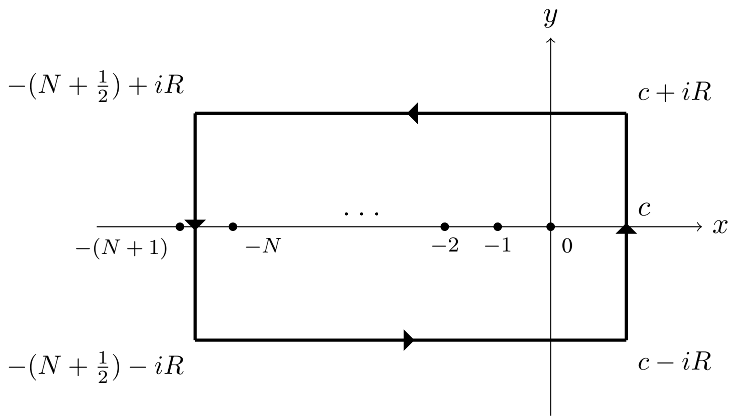

Figure 3.1: Contour for an inverse Mellin transform.Rectangular contour with the vertices $c \pm iR$, $-(N + \frac{1}{2}) \pm iR$.

Now, Cauchy's residue theorem implies that

$$\frac{1}{2\pi i}\int_{\mathcal{R}} x^{-s}\Gamma(s)ds = \sum_{m=0}^{N}\frac{(-1)^m}{m!}x^m\,. \tag{3.25}$$

The next step is to take R and N to infinity. Stirling's approximation implies that the integral on the contour minus the line joining $c-iR$ and $c+iR$ tends to zero. To sum up, we have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\Gamma(s)ds = \lim_{N\to\infty}\sum_{m=0}^{N}\frac{(-1)^m}{m!}x^m = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!}x^m = e^{-x}\,. \tag{3.26}$$

The Mellin-Barnes transform enters the study of CFT through the fact that the position-space correlators can be expressed via the Mellin-Barnes transform [11–13]

$$\langle\phi_1(x_1)\cdots\phi_n(x_n)\rangle = \frac{1}{(2\pi i)^N}\int_{c_{ij}-i\infty}^{c_{ij}+i\infty}[d\gamma_{ij}]\mathcal{M}_n(\gamma_{ij})\prod_{i<j}^{n}x_{ij}^{-2\gamma_{ij}}\Gamma(\gamma_{ij}). \tag{3.27}$$

The integral measure $[d\gamma_{ij}]$ is linear in each γ_{ij} but subject to constraints,

$$\sum_j \gamma_{ij} = \Delta_i\,,\quad \Delta_{ii} = 0\,. \tag{3.28}$$

The number of independent solutions to these constraints is $n(n-3)/2$. The reader may notice that $(-\gamma_{ij})$ satisfy the same conditions as α_{ij} as in (5.73) and (2.22). We will use this similarity in the next chapter, where we will reexamine the conformal invariance of the correlation functions in the momentum space. In the meanwhile, it is convenient to distinguish $(-\gamma_{ij})$ from α_{ij}.

Here, we show that the (3.27) satisfies the conformal Ward identities in the position space. Translation symmetry is manifest since the integral in (3.27) depends only on the relative coordinates x_{ij}. For the dilatation Ward identity (2.12), we observe

$$\begin{aligned}&\sum_{j=1}^{n}\left(x_j^\alpha\frac{\partial}{\partial x_j^\alpha}+\Delta_j\right)\langle\phi_1(x_1)\cdots\phi_n(x_n)\rangle\\ &\quad=\sum_{j=1}^{n}(-\gamma_{ij}+\Delta_j)\langle\phi_1(x_1)\cdots\phi_n(x_n)\rangle = 0\,.\end{aligned} \tag{3.29}$$

Similarly, for the special conformal Ward identity (2.13),

$$\begin{aligned}
&\sum_{j=1}^{n}\left(-x_j^2\frac{\partial}{\partial x_j^\kappa}+2x_j^\kappa x_j^\alpha\frac{\partial}{\partial x_j^\alpha}+2\Delta_j x_j^\kappa\right)\langle\phi_1(x_1)\cdots\phi_n(x_n)\rangle \\
&\quad=\sum_{j=1}^{n}\left(-\gamma_{ij}-\gamma_{ij}+2\Delta_j\right)\langle\phi_1(x_1)\cdots\phi_n(x_n)\rangle=0\,.
\end{aligned} \tag{3.30}$$

Thus we have confirmed that (3.27) is compatible with conformal invariance.

Chapter 4

SIMPLEX AND MESH INTEGRALS

4.1 Simplex Integrals

We are ready to present one of the most important results of the thesis: the simplex integral representation of a scalar n-point function in momentum space, first introduced by Bzowski, McFadden and Skenderis [9, 10].

In [9], the authors showed that the general scalar n-point function in momentum space can be expressed as a *simplex integral*:

$$\langle \phi_1(\boldsymbol{p}_1) \cdots \phi_n(\boldsymbol{p}_n) \rangle = \prod_{1 \leq i < j \leq n} \int \frac{d^d \boldsymbol{q}_{ij}}{(2\pi)^d} \frac{\hat{f}(\hat{\boldsymbol{u}})}{q_{ij}^{2\alpha_{ij}+d}} \prod_{k=1}^{n} (2\pi)^d \delta(\boldsymbol{p}_k + \sum_{l=1}^{n} \boldsymbol{q}_{lk}) \, . \quad (4.1)$$

To define this integral, we introduce an oriented $(n-1)$-simplex. The momenta p_i are assigned to the vertices of the simplex. The integral is defined over the "internal" momenta q_{ij} assigned to the edge from vertex i to j. The orientation is defined such that $q_{ij} = -q_{ji}$ and $q_{ii} = 0$. We thus have $n(n-1)/2$ integration variables which we choose to be q_{ij} with $i < j$. The product of delta functions imposes "momentum conservation" at each vertex as required by the translation invariance of the conformal correlator.

Remarkably, the parameters α_{ij} in (4.1) are the same as the ones that appeared in the position space representation (2.20). We will elaborate on this point later in this chapter. The function $\hat{f}$ is an arbitrary function of momentum-space cross ratios.

$$\hat{u}_{[pqrs]} = \frac{q_{pq}^2 q_{rs}^2}{q_{pr}^2 q_{qs}^2}. \tag{4.2}$$

Just as in position space, only $n(n-3)/2$ of these cross ratios are independent. We will choose the set

$$\hat{u}_{2a} = \hat{u}_{[123a]} = \frac{q_{12}^2 q_{3a}^2}{q_{2a}^2 q_{13}^2}, \quad \hat{u}_{3a} = \hat{u}_{[132a]} = \frac{q_{13}^2 q_{2a}^2}{q_{3a}^2 q_{12}^2}, \quad \hat{u}_{ab} = \hat{u}_{[2a3b]} = \frac{q_{2a}^2 q_{3b}^2}{q_{ab}^2 q_{23}^2}, \tag{4.3}$$

where $a, b = 4, 5, \ldots, n$ and $a < b$. We will denote the set of indices enumerating the independent cross ratios by $\mathcal{U}$, while the ratios themselves will be written collectively as $\hat{\boldsymbol{u}}$, thus $\hat{\boldsymbol{u}} = \{\hat{\boldsymbol{u}}_I\}_{I \in \mathcal{U}}$.

In momentum space, the cross ratios are subject to integration inside the simplex integral (4.1). Putting aside one delta function for overall momentum conservation, we will employ the double-bracket notation

$$\langle \phi_1(\boldsymbol{p}_1) \cdots \phi_n(\boldsymbol{p}_n) \rangle = (2\pi)^d \delta\left(\sum_{i=1}^n \boldsymbol{p}_i\right) \langle\!\langle \phi_1(\boldsymbol{p}_1) \cdots \phi_n(\boldsymbol{p}_n) \rangle\!\rangle. \tag{4.4}$$

Overall, we have $n(n-1)/2$ integrals and $(n-1)$ delta functions. We can now perform the integrals over the variables q_{in} for $i = 1, 2, \ldots, n-1$ in (4.1) to remove the remaining delta functions. This leaves us with $(n-1)(n-2)/2$ integrals still to perform,

$$\langle\!\langle \phi_1(\boldsymbol{p}_1) \cdots \phi_n(\boldsymbol{p}_n) \rangle\!\rangle = \prod_{1 \le i < j \le n-1} \int \frac{d^d \boldsymbol{q}_{ij}}{(2\pi)^d} \frac{\hat{f}(\hat{\boldsymbol{u}})}{\text{Den}_n(\boldsymbol{\alpha})}, \tag{4.5}$$

where the denominator reads

$$\text{Den}_n(\boldsymbol{\alpha}) = \prod_{1 \le i < j \le n-1} q_{ij}^{2\alpha_{ij}+d} \times \prod_{m=1}^{n-1} |\boldsymbol{l}_m - \boldsymbol{p}_m|^{2\alpha_{mn}+d} \tag{4.6}$$

and $\boldsymbol{l}_m$ depends only on the remaining internal momenta,

$$\boldsymbol{l}_m = -\boldsymbol{q}_{mn} + \boldsymbol{p}_m = \sum_{j=1}^{n-1} \boldsymbol{q}_{mj} = -\sum_{j=1}^{m-1} \boldsymbol{q}_{jm} + \sum_{j=m+1}^{n-1} \boldsymbol{q}_{mj}. \tag{4.7}$$

Notice that we have eliminated the momentum $\boldsymbol{p}_n$ and hence all the remaining momenta are independent. All sums and products now extend only up to $n-1$. We will refer to the expression (4.5) as the *reduced simplex integral*.

1-, 2- and 3-point function As a warm-up exercise, let us examine how the simplex integrals works for the 1-, 2- and 3-point functions. In these cases, there are no cross ratios and the function $\hat{f}_n$ can be replaced by constant c_n.

As we discussed in the previous chapter, the 1-point function can be non-vanishing only for the identity operator. The position-space correlator is a constant. Its Fourier transform is proportional to the delta function $\delta(\boldsymbol{p}_1)$. We may say, trivially, the delta function is defined on the vertex of a zero-dimensional simplex. There is no edge to carry any internal momentum.

For a 2-point function, the relevant simplex is a line interval. The simplex integral (4.1) becomes

$$\langle \phi_1(\boldsymbol{p}_1)\phi_2(\boldsymbol{p}_2)\rangle = \int \frac{d^d\boldsymbol{q}_{12}}{(2\pi)^d}\frac{c_2}{q_{12}^{2\alpha_{12}+d}}(2\pi)^{2d}\delta(\boldsymbol{p}_1-\boldsymbol{q}_{12})\delta(\boldsymbol{p}_2+\boldsymbol{q}_{12}). \tag{4.8}$$

Pulling out the momentum conserving delta function, we have

$$\langle \phi_1(\boldsymbol{p}_1)\phi_2(\boldsymbol{p}_2)\rangle = (2\pi)^d\delta(\boldsymbol{p}_1+\boldsymbol{p}_2)\int \frac{d^d\boldsymbol{q}_{12}}{(2\pi)^d}\frac{c_2}{q_{12}^{2\alpha_{12}+d}}(2\pi)^{d}\delta(\boldsymbol{p}_1-\boldsymbol{q}_{12}). \tag{4.9}$$

Performing the integral explicitly, we obtain

$$\langle\!\langle \phi_1(\boldsymbol{p}_1)\phi_2(\boldsymbol{p}_2)\rangle\!\rangle = c_2\, p_1^{-2\alpha_{12}-d}, \tag{4.10}$$

which agrees with (3.9) and (5.73).

The simplex for a 3-point function is a triangle; see Figure 4.2. After pulling out the overall delta function and imposing momentum conservation, one internal momentum remains and give a non-trivial reduced integral. Amusingly, the reduced integral resembles (but differs from) a loop integral in Feynman diagram calculation:

$$\langle\!\langle \phi_1(\boldsymbol{p}_1)\phi_2(\boldsymbol{p}_2)\phi_3(\boldsymbol{p}_3)\rangle\!\rangle = \int \frac{d^d\boldsymbol{q}}{(2\pi)^d}\frac{1}{|\boldsymbol{q}|^{2\alpha_{12}+d}|\boldsymbol{q}-\boldsymbol{p}_1|^{2\alpha_{13}+d}|\boldsymbol{q}+\boldsymbol{p}_2|^{2\alpha_{23}+d}}. \tag{4.11}$$

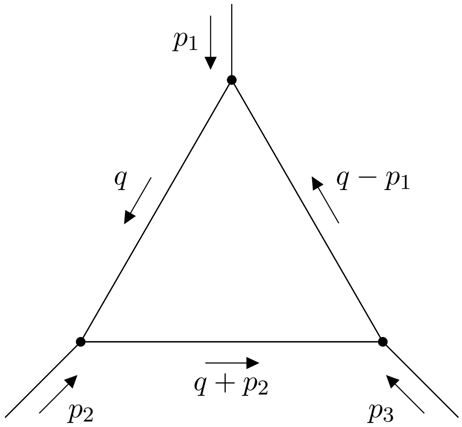

Figure 4.1: The 3-point function as a 2-simplex (triangle) integral.

The α_{ij} parameters, subject to conditions (5.73) and (2.22), are given by

$$\begin{aligned} 2\alpha_{12} &= 2\Delta_3 - \Delta_t = -\Delta_1 - \Delta_2 + \Delta_3\,, \\ 2\alpha_{13} &= 2\Delta_2 - \Delta_t = -\Delta_1 - \Delta_3 + \Delta_2\,, \\ 2\alpha_{23} &= 2\Delta_1 - \Delta_t = -\Delta_2 - \Delta_3 + \Delta_1\,. \end{aligned} \tag{4.12}$$

At this stage, it is not clear how to relate this loop triangle representation of the 3-point function to the other famous integral representation (known as a triple-K integral) whose integrand is the product of three modified Bessel K functions. We will reveal the connection in section 5.1.

4-point function The cross ratios make the 4-point and higher point functions much more complicated than the lower point functions. The reduced integral (4.5) for a 4-point function is

$$\langle\!\langle \phi_1(\boldsymbol{p}_1)\phi_2(\boldsymbol{p}_2)\phi_3(\boldsymbol{p}_3)\phi_4(\boldsymbol{p}_4)\rangle\!\rangle = \int \frac{d^d\boldsymbol{q}_1}{(2\pi)^d}\frac{d^d\boldsymbol{q}_2}{(2\pi)^d}\frac{d^d\boldsymbol{q}_3}{(2\pi)^d}\frac{\hat{f}(\hat{u},\hat{v})}{\mathrm{Den}_4(\boldsymbol{q}_j,\boldsymbol{p}_k)}, \tag{4.13}$$

where the denominator is given by

$$\begin{aligned} \mathrm{Den}_4(\boldsymbol{q}_j,\boldsymbol{p}_k) &= q_3^{2\alpha_{12}+d} q_2^{2\alpha_{13}+d} q_1^{2\alpha_{23}+d} \\ &\quad \times |\boldsymbol{p}_1+\boldsymbol{q}_2-\boldsymbol{q}_3|^{2\alpha_{14}+d} |\boldsymbol{p}_2+\boldsymbol{q}_3-\boldsymbol{q}_1|^{2\alpha_{24}+d} |\boldsymbol{p}_3+\boldsymbol{q}_1-\boldsymbol{q}_2|^{2\alpha_{34}+d}. \end{aligned} \tag{4.14}$$

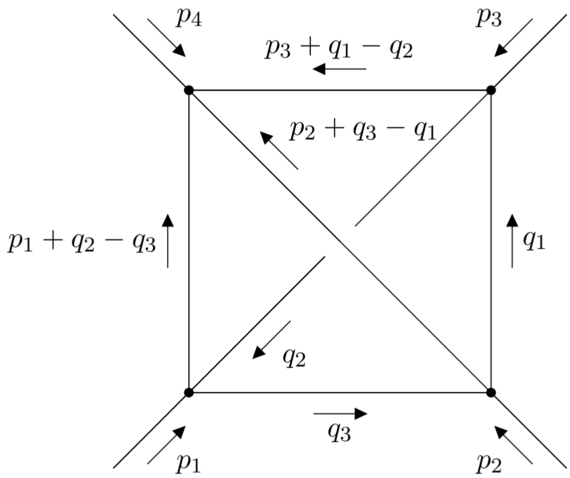

Figure 4.2: The 4-point function as a 3-simplex (tetrahedron) integral.

We simplified the labels somewhat by setting

$$\boldsymbol{q}_i = \frac{1}{2}\epsilon_{ijk}\boldsymbol{q}_{jk}, \qquad (i,j,k = 1,2,3)\,. \tag{4.15}$$

The arbitrary function $\hat{f}(\hat{u},\hat{v})$ is a function of two independent variables,

$$\hat{u} = \frac{q_1^2|\boldsymbol{p}_1 + \boldsymbol{q}_2 - \boldsymbol{q}_3|^2}{q_2^2|\boldsymbol{p}_2 + \boldsymbol{q}_3 - \boldsymbol{q}_1|^2}, \qquad \hat{v} = \frac{q_2^2|\boldsymbol{p}_2 + \boldsymbol{q}_3 - \boldsymbol{q}_1|^2}{q_3^2|\boldsymbol{p}_3 + \boldsymbol{q}_1 - \boldsymbol{q}_2|^2}, \tag{4.16}$$

whose role is analogous to that of the position-space cross ratios u and v. However, they depend on the momenta $\boldsymbol{q}_j$ which are subject to the integration in (4.13).

The above examples illustrate a formal similarity between the simplex integrals and Feynman loop integrals. The main source of the similarity is the momentum conservation at each vertex. But, there are also crucial differences. Most importantly, the Feynman calculus requires sum over many allowed graphs, while there is only one simplex integral for a given n-point function.

4.2 Mesh Integrals

In this section, we take the first step to decompose a general simplex integral into something simpler called *mesh integral*. The mesh integral is a special case of the simplex integral where the position-space expression is a single monomial. It is unlikely that a mesh integral is naturally produced by a simplex integral of a correlator in a CFT. The reason why we study the mesh integral is that we can express an arbitrary simplex integral as a (typically infinite) linear combination of mesh integrals. Moreover, the mesh integrals exhibit a recursive structure, which allows us to build n-point functions in terms of $(n-1)$-point functions.

An n-point mesh integral requires the α_{ij} parameters in the defining formula:

$$M_n(\boldsymbol{\alpha}; \boldsymbol{p}_1, \dots, \boldsymbol{p}_n) = \prod_{1 \le i < j \le n} C_{ij} \int \frac{d^d q_{ij}}{(2\pi)^d} \frac{1}{q_{ij}^{2\alpha_{ij}+d}} \prod_{k=1}^{n} (2\pi)^d \delta(p_k + \sum_{l=1}^{n} q_{lk}), \tag{4.17}$$

where the coefficient

$$C_{ij} = \frac{\pi^{d/2} 2^{d+2\alpha_{ij}}}{\Gamma(-\alpha_{ij})} \Gamma\left(\frac{d}{2} + \alpha_{ij}\right) \tag{4.18}$$

is included for convenience. As above, there are $n(n-1)/2$ integration variables q_{ij} with $i < j$ (and we extend q_{ij} to any i, j by $q_{ij} = -q_{ji}$). So, a mesh integral is a simplex integral with $\hat{f} = 1$. For $n = 1$, it is convenient to define M_1 by

$$M_1(p) = (2\pi)^d \delta(p) \,. \tag{4.19}$$

Just as for simplex integral, we define the *reduced mesh integrals* $\widetilde{M}_n$ by pulling out the momentum conserving delta function,

$$M_n(\boldsymbol{\alpha}; \boldsymbol{p}_1, \dots, \boldsymbol{p}_n) = (2\pi)^d \delta\left(\sum_{i=1}^{n} p_i\right) \widetilde{M}_n(\boldsymbol{\alpha}; \boldsymbol{p}_1, \dots, \boldsymbol{p}_n). \tag{4.20}$$

Up to the factors of C_{ij}, the reduced mesh integrals are given by (4.5) with $\hat{f} = 1$, namely

$$\widetilde{M}_n(\boldsymbol{\alpha}; \boldsymbol{p}_1, \dots, \boldsymbol{p}_n) = \prod_{1 \le i < j \le n-1} C_{ij} \int \frac{d^d q_{ij}}{(2\pi)^d} \frac{1}{\mathrm{Den}_n(\boldsymbol{\alpha})}, \tag{4.21}$$

where the denominator is given by (4.6).

Recursion The mesh integral (4.17) exhibits a remarkable recursive structure. To show this, we pull out factors containing q_{ij} and p_n and rename $q_{in} \mapsto q_i$. Then, the mesh integral can be written recursively as

$$\begin{aligned}
&\widetilde{M}_n(\boldsymbol{\alpha};\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n) \\
&=\prod_{i=i}^{n-1} C_{in}\int \frac{d^dq_i}{(2\pi)^d}\frac{M_{n-1}(\boldsymbol{\alpha};p_1-q_1,\ldots,p_{n-1}-q_{n-1})}{q_1^{2\alpha_{1n}+d}q_2^{2\alpha_{2n}+d}\cdots q_{n-1}^{2\alpha_{n-1,n}+d}}(2\pi)^d\delta(p_n+\sum_{j=1}^{n-1}q_j).
\end{aligned} \tag{4.22}$$

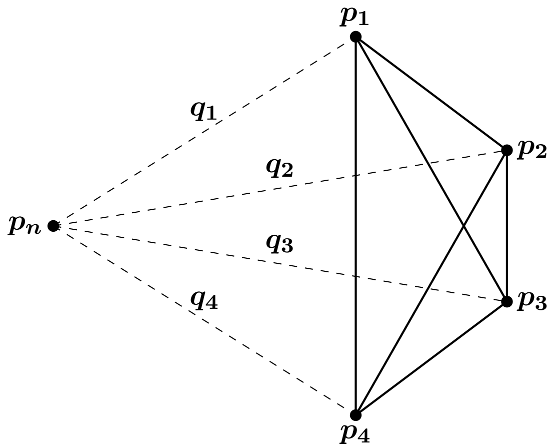

Figure 4.3: The decomposition of the 5-point mesh M_5. The solid internal lines on the right-hand side of the figure represent the 4-point mesh M_4 evaluated with ingoing momenta $\boldsymbol{p}_j-\boldsymbol{q}_j$.

Fourier transform Our next task is to show that the mesh integral (4.17) is the Fourier transform of the position-space conformal n-point function (2.20) when f is a *monomial* in the cross ratios. The n-point function (2.20) takes the form

$$F_n(\boldsymbol{\alpha};x_1,\ldots,x_n)=\prod_{1\le i<j\le n} x_{ij}^{2\alpha_{ij}}, \tag{4.23}$$

where the α_{ij} are still a solution of (5.73). We wish to show that the Fourier transform of (4.23) is equal to (4.17), namely

$$\begin{aligned}
\mathcal{F}[F_n](\boldsymbol{\alpha};\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n)&=\int [dx]_n e^{-i\sum_{j=1}^n x_j\cdot p_j}F_n(\boldsymbol{\alpha};x_1,\ldots,x_n) \\
&=M_n(\boldsymbol{\alpha};p_1,\ldots,p_n).
\end{aligned} \tag{4.24}$$

We can check this explicitly for the 2-point function. The Fourier transform of F_2 is simply,

$$\mathcal{F}[x_{12}^{2\alpha_{12}}] = (2\pi)^d \delta(p_1 + p_2) \frac{C_{12}}{p_1^{2\alpha_{12}+d}}, \tag{4.25}$$

where C_{ij} is given in (4.18). This result matches the corresponding mesh integral, which is, from (4.17),

$$\begin{aligned} M_2(\alpha_{12}; p_1, p_2) &= C_{12} \int \frac{d^d q_{12}}{(2\pi)^d} \frac{1}{q_{12}^{2\alpha_{12}+d}} \delta(p_1 - q_{12})(2\pi)^d \delta(p_2 + q_{12}) \\ &= (2\pi)^d \delta(p_1 + p_2) \frac{C_{12}}{q_{12}^{2\alpha_{12}+d}}. \end{aligned} \tag{4.26}$$

We proceed by induction and use the recursive structure of (4.22). If we assume the statement (4.24) holds true up to the level of the $(n-1)$-point function, we can write F_n as

$$F_n(\boldsymbol{\alpha}; x_1, \ldots, x_n) = x_{1n}^{2\alpha_{1n}} x_{2n}^{2\alpha_{2n}} \cdots x_{n-1,n}^{2\alpha_{n-1,n}} \times F_{n-1}(\boldsymbol{\alpha}; x_1, \ldots, x_{n-1}). \tag{4.27}$$

Using the Fourier transform of a single power function as in (4.25) and denoting convolution by $*$, we find

$$\begin{aligned} &\mathcal{F}[F_n] \\ &= \mathcal{F}[x_{1n}^{2\alpha_{1n}} x_{2n}^{2\alpha_{2n}} \cdots x_{n-1,n}^{2\alpha_{n-1,n}}] * \mathcal{F}[F_{n-1}] \\ &= \left[\frac{(2\pi)^d \delta(\sum_{j=1}^n p_j) \prod_{i=1}^{n-1} C_{in}}{q_1^{2\alpha_{1n}+d} q_2^{2\alpha_{2n}+d} \cdots q_{n-1}^{2\alpha_{n-1,n}+d}} \right] * \left[M_{n-1}(\boldsymbol{\alpha}; p_1, \ldots, p_{n-1})(2\pi)^d \delta(p_n) \right] \\ &= \prod_{i=1}^{n-1} C_{in} \int \frac{d^d q_i}{(2\pi)^d} \frac{M_{n-1}(\boldsymbol{\alpha}; p_1 - q_1, \ldots, p_{n-1} - q_{n-1})}{q_1^{2\alpha_{1n}+d} q_2^{2\alpha_{2n}+d} \cdots q_{n-1}^{2\alpha_{n-1,n}+d}} (2\pi)^d \delta\left(p_n + \sum_{j=1}^{n-1} q_j \right) \\ &= M_n(\boldsymbol{\alpha}; \boldsymbol{p}_1, \ldots, \boldsymbol{p}_n) \,. \end{aligned} \tag{4.28}$$

In the last step, we used the mesh recursion relation (4.22).

4.3 Proof of Conformal Invariance

In this section, we will show that the simplex integrals (4.1) are conformally invariant. Since we showed that the mesh integrals (4.17) are direct Fourier transform of the position-space n-point function (2.20) when f is a monomial in the cross ratios, the conformal invariance of them is already apparent. Even so, we will prove the conformal invariance of the mesh integrals directly using the momentum-space conformal Ward identities, which we derived in chapter 3. As we will see, the conformal invariance of the simplex integrals follows directly from the that of the mesh integrals.

In section 4.3.1, we will first show that the mesh integrals are the solutions of the momentum-space conformal Ward identities. For the special conformal Ward identity, this can be shown in two ways. One is to use the recursive structure of mesh integrals, and the other is to show that the action of the corresponding differential on the mesh integrals yields a total derivative. In section 4.3.2, we then prove the conformal invariance of the simplex integrals. This can be shown in three ways. The first is to use a Mellin-Barnes transformation relating the simplex integrals to the mesh integrals and will be discussed in the first paragraph of section 4.3.2. The other two proceeds by showing that the action of the special conformal Ward identity operator on the simplex integrals yields a total derivative. This can be shown either indirectly or directly, as will be discussed in the second paragraph of section 4.3.2.

4.3.1 Conformal invariance of the mesh integrals

The dilatation Ward identity is a matter of dimension counting. It is easy to verify that the mesh integral (4.17) has the dimension $\Delta_t - nd$. At each edge, an integration increases the dimension by d while a propagator decreases the dimension by $2\alpha_{ij} + d$, resulting in a net decrease by $2\alpha_{ij}$. At each vertex, the momentum-conserving delta functions decreases the dimension by d. Overall, we have

$$-\sum_{1\leq i<j\leq n}\alpha_{ij}-nd=\sum_{i,j=1}^{n}\alpha_{ij}-nd=\Delta_t-nd\,. \tag{4.29}$$

In the last step, we used the sum of all relations in (5.73).

Let us move on to show that the mesh integrals (4.17) also satisfy the special conformal Ward identity (3.18). Recall that the special conformal Ward identity is

$$K^{\kappa}(\boldsymbol{\Delta})\,\langle\phi_1(p_1)\cdots\phi_n(p_n)\rangle=0,$$

where the special conformal differential operator $K^{\kappa}(\boldsymbol{\Delta})$ is given by

$$\begin{aligned}K^{\kappa}(\boldsymbol{\Delta})&=\sum_{j=1}^{n}K^{\kappa}(\Delta_j;p_j)\\ K^{\kappa}(\Delta_j;p_j)&=\left(p_j^{\kappa}\frac{\partial^2}{\partial p_j^{\alpha}\partial p_j^{\alpha}}-2p_j^{\alpha}\frac{\partial^2}{\partial p_j^{\alpha}\partial p_j^{\kappa}}+2(\Delta_j-d)\frac{\partial}{\partial p_j^{\kappa}}\right).\end{aligned} \tag{4.30}$$

We denote the action of the special conformal operator on the n-point mesh integral as

$$\mathcal{E}^{(n)\kappa}(\boldsymbol{\Delta};p_1,\ldots,p_n)=K^{\kappa}(\boldsymbol{\Delta})M_n(\boldsymbol{\alpha};p_1,\ldots,p_n). \tag{4.31}$$

This expression admits a recursive structure similar to (4.22). To see this, let us write the scaling dimensions of the n-point function as $\Delta_m^{(n)}$ and those of the $(n-1)$-point function as $\Delta_m^{(n-1)}$. From (5.73), these are related by

$$\Delta_n^{(n)}=-\sum_{j=1}^{n-1}\alpha_{jn},\qquad \Delta_m^{n}=\Delta_m^{(n-1)}-\alpha_{mn},\quad m=1,\ldots,n-1. \tag{4.32}$$

Thus, given $n-1$ parameters α_{mn} and a set of $\Delta_m^{(n-1)}$ satisfying (5.73) at $(n-1)$ points, we can construct a solution of (5.73) at n points.

To proceed by induction, we first consider the 1-point mesh integral. We defined the mesh integral as $M_1(p_1)=(2\pi)^d\delta(p_1)$, and it is almost trivial to show that

$$K^{\kappa}(0;p_1)M_1(p_1)=0, \tag{4.33}$$

where $\Delta_1 = 0$ is enforced by dilatation invariance. The key element of the induction is an integration by parts identity, a detailed derivation of which can be found in appendix A.1 of [9]. The result is

$$\begin{aligned}
&\mathcal{E}^{(n)\kappa}(\boldsymbol{\Delta}^{(n)}; p_1, \ldots, p_n) \\
&\quad = \prod_{i=1}^{n-1} C_{in} \int \frac{d^d q_i}{(2\pi)^d} \frac{1}{q_1^{2\alpha_{1n}+d} q_2^{2\alpha_{2n}+d} \cdots q_{n-1}^{2\alpha_{n-1,n}+d}} \\
&\quad \times (2\pi)^d \delta\left(p_n + \sum_{j=1}^{n-1} q_j\right) \mathcal{E}^{(n-1)\kappa}(\boldsymbol{\Delta}^{(n-1)}; p_1, \ldots, p_{n-1}).
\end{aligned} \tag{4.34}$$

This identity clearly ensures that, if the $(n-1)$-point mesh integral satisfies the $(n-1)$-point special conformal Ward identity, then the n-point mesh integral satisfies the n-point special conformal Ward identity, completing the desired proof.

4.3.2 Conformal invariance of the simplex integrals

Having proved the conformal invariance of the mesh integral, we now turn to the simplex integral. There exist a few different methods to prove the conformal invariance of the simplex integrals. We will explain two of them discussed in [10].

The first method uses the Mellin-Barnes representation of conformal correlators. A key point is that the Mellin-Barnes transformation decomposes an arbitrary simplex integral as a *linear* combination of mesh integrals. The linear combination runs over continuous parameters, so there can be order of limits problems. Apart from this subtlety, the proof is conceptually simple and technically straightforward.

The second method examines a direct action of the special conformal Ward identity operator on the simplex integrals. Using the conformal invariance of the mesh integrals, we can show that the action of the special conformal Ward identity operator gives a total derivative inside the momentum integral. Under a mild assumption on the convergence of the integral, the integral vanishes and proves the conformal invariance of the simplex integral.

Proof by Mellin-Barnes transform

Mesh integrals are special cases of simplex integrals where $\hat{f}$ is a monomial in the cross ratios. In general, we can express the monomial $\hat{f}$ as

$$\hat{f} = \prod_{I \in \mathcal{U}} \hat{\mathbf{u}}_I^{\gamma_I}, \tag{4.35}$$

where we have $N = n(n-3)/2$ independent cross ratios $\hat{u}_{I_1}, \ldots, \hat{u}_{I_N}$, and γ_I is a set of N exponents. The γ exponents can be absorbed by the α exponents in the definition of the simplex integral such that

$$\frac{\prod_{I \in \mathcal{U}} \hat{\mathbf{u}}_I^{\gamma_I}}{\mathrm{Den}_n(\boldsymbol{\alpha})} = \frac{1}{\mathrm{Den}_n(\boldsymbol{\alpha}_{I_1 \ldots I_N}^{(\gamma_1 \ldots \gamma_N)})}, \tag{4.36}$$

The shifted parameters $\boldsymbol{\alpha}_{I_1 \ldots I_N}^{(\gamma_1 \ldots \gamma_N)} = \{\alpha_{ij, I_1 \ldots I_N}^{(\gamma_1 \ldots \gamma_N)})\}_{1 \leq i < j \leq n}$ are given by

$$\begin{aligned} \boldsymbol{\alpha}_{I_1 \ldots I_N}^{(\gamma_1 \ldots \gamma_N)} &= \alpha_{ij} + \sum_{m=1}^{N} \gamma_m S_{ij, I_m}\,, \\ S_{ij,[pqrs]} &= \delta_{ip}\delta_{jr} + \delta_{iq}\delta_{js} - \delta_{ip}\delta_{jq} - \delta_{ir}\delta_{js}\,. \end{aligned} \tag{4.37}$$

This shift procedure shows that the simplex integral with monomial $\hat{f}$ satisfies the same conformal Ward identities as the simplex integral with $\hat{f} = 1$.

Having established conformal invariance for an arbitrary monomial $\hat{f}$, we can try to proceed further. Since the conformal generators act linearly on the correlators, conformal invariance will continue to hold when $\hat{f}$ is an arbitrary linear combination of monomials. Turning from discrete sums to continuous integrals, we can argue for conformal invariance of the simplex integral when $\hat{f}$ is given by a multiple Mellin-Barnes transform as

$$\hat{f}(\hat{\boldsymbol{u}}) = \frac{1}{(2\pi i)^N} \int_{c_1 - i\infty}^{c_1 + i\infty} ds_1 \ldots \int_{c_N - i\infty}^{c_N + i\infty} ds_N \hat{u}_{I_1}^{s_1} \ldots \hat{u}_{I_N}^{s_N} \hat{F}(s_1, \ldots, s_N), \tag{4.38}$$

for appropriate choice of contour $c_1, \ldots, c_N$ for which the integral converges. As usual, this argument holds provided that switching the order of integration does not cause a convergence issue.

As a final ingredient of the proof, recall that the position-space correlators have a Mellin representation,

$$\langle\phi_1(x_1)\cdots\phi_n(x_n)\rangle = \frac{1}{(2\pi i)^N}\int_{c_{ij}-i\infty}^{c_{ij}+i\infty}[d\gamma_{ij}]\mathcal{M}_n(\gamma_{ij})\prod_{i<j}^{n}x_{ij}^{-2\gamma_{ij}}\Gamma(\gamma_{ij})\,. \quad (4.39)$$

We can align γ_{ij} with the shifted α_{ij} as in (4.37) by writing

$$-\gamma_{ij} = \alpha_{ij} + \sum_{m=1}^{N}\gamma_m S_{ij,I_m}. \quad (4.40)$$

The Fourier transform of the monomial $\prod x_{ij}^{-2\gamma_{ij}}$ can now be evaluated as explained in section 4.2. This yields the simplex integral representation with

$$\hat{f}(\hat{\boldsymbol{u}}) = \left(\prod_{m=1}^{N}\int_{c_m-i\infty}^{c_m+i\infty}\frac{d\gamma_m}{2\pi i}\hat{u}_{I_m}^{\gamma_m}\right)\mathcal{M}_n(\gamma_{ij})\prod_{i<j}^{n}\pi^{d/2}2^{d-2\gamma_{ij}}\Gamma(d/2-\gamma_{ij}). \quad (4.41)$$

Thus, given the Mellin representation for a position-space correlator, we can immediately write down the Mellin representation for $\hat{f}$. For holographic correlators, this formula provides an alternative method of computing $\hat{f}(\hat{\boldsymbol{u}})$ to those we will discuss in chapter 5.

Proof by total derivative

The proof based on the Mellin-Barnes representation is indirect in the sense that it involves an infinite linear combination and potentially susceptible to an order of integration problem. It would be desirable to complement it with an alternative, more direct, proof. We present one such alternative.

The action of the special conformal Ward identity, a second-order differential operator, on the integrand of the simplex integral yields the following form,

$$\tilde{\mathcal{K}}^\kappa\left[\frac{\hat{f}(\hat{\boldsymbol{u}})}{\mathrm{Den}_n(\boldsymbol{\alpha})}\right] = \hat{f}(\hat{\boldsymbol{u}})C^\kappa(\boldsymbol{\alpha}) + \frac{\partial\hat{f}(\hat{\boldsymbol{u}})}{\partial\hat{u}_I}C_I^\kappa(\boldsymbol{\alpha}) + \frac{\partial^2\hat{f}(\hat{\boldsymbol{u}})}{\partial\hat{u}_I\partial\hat{u}_J}C_{IJ}^\kappa(\boldsymbol{\alpha}), \quad (4.42)$$

where $C^\kappa, C_I^\kappa, C_{IJ}^\kappa$ are coefficients depending on the external and internal momenta as well as the parameters $\boldsymbol{\alpha}$. The indices I, J indicate the independent cross ratios which will be summed over any repeated indices. A crucial fact is that the coefficients $C^\kappa, C^\kappa_I, C^\kappa_{IJ}$ are independent of the choice of $\hat{f}$.

The equation above implies that if the simplex integrals with $\hat{f}(\hat{\boldsymbol{u}}) = 1, \hat{u}_I, \hat{u}_I\hat{u}_J$ are conformally invariant, the simplex integrals with any $\hat{f}(\hat{\boldsymbol{u}})$ are conformally invariant. For $\hat{f}(\hat{\boldsymbol{u}}) = 1, \hat{u}_I, \hat{u}_I\hat{u}_J$ to be conformally invariant, the right-hand side of (4.42) must form a total derivative. It follows that that the coefficients $C^\kappa, C^\kappa_I, C^\kappa_{IJ}$ must satisfy the following conditions

$$\hat{f} = 1: \qquad C^\kappa = \sum_{i,j=1,\ i\neq j}^{n-1} \frac{\partial}{\partial q^\mu_{ij}} \Gamma^{\kappa\mu}_{ij} \tag{4.43}$$

$$\hat{f} = \hat{u}_I: \qquad C^\kappa_I = \sum_{i,j=1,\ i\neq j}^{n-1} \left(\Gamma^{\kappa\mu}_{ij} \frac{\partial \hat{u}_I}{\partial q^\mu_{ij}} + \frac{\partial}{\partial q^\mu_{ij}} \Gamma^{\kappa\mu}_{ij,I} \right) \tag{4.44}$$

$$\hat{f} = \hat{u}_I\hat{u}_J: \qquad C^\kappa_{IJ} = \sum_{i,j=1,\ i\neq j}^{n-1} \Gamma^{\kappa\mu}_{ij,J} \frac{\partial \hat{u}_I}{\partial q^\mu_{ij}} \tag{4.45}$$

for some coefficients $\Gamma^{\kappa\mu}_{ij}$ and $\Gamma^{\kappa\mu}_{ij,I}$. Since these coefficients are independent of $\hat{f}(\hat{\boldsymbol{u}})$, the following identity holds for any $\hat{f}(\hat{\boldsymbol{u}})$:

$$\tilde{\mathcal{K}} \left[\frac{\hat{f}(\hat{\boldsymbol{u}})}{\mathrm{Den}_n(\boldsymbol{\alpha})} \right] = \sum_{i,j=1,\ i\neq j}^{n-1} \frac{\partial}{\partial q^\mu_{ij}} \left[\Gamma^{\kappa\mu}_{ij}(\boldsymbol{\alpha}) \hat{f}(\hat{\boldsymbol{u}}) + \sum_{I\in\mathcal{U}} \Gamma^{\kappa\mu}_{ij,I}(\boldsymbol{\alpha}) \frac{\partial \hat{f}(\hat{\boldsymbol{u}})}{\partial \hat{u}_I} \right]. \tag{4.46}$$

We have shown that the action of the special conformal Ward identity operator on the integrand of the simplex integrals yields a total derivative, which proves the conformal invariance of the simplex integrals.

The proof by total derivatives is valid as long as the Γ-coefficients exist and are independent of $\hat{f}$. It would be instructive to compute the coefficients by direct computation. Here, we content ourselves with a short summary of the main results:

$$\begin{aligned}
\Gamma^{\kappa\mu}_{ij}(\boldsymbol{\alpha}) &= (2\alpha_{in} + d) \times \frac{A^{\kappa\mu}_{ij}}{\mathrm{Den}_n(\boldsymbol{\alpha})}\,, \\
\Gamma^{\kappa\mu}_{ij,[pqrs]}(\boldsymbol{\alpha}) &= 2(\delta_{ip}\delta_{rn} + \delta_{iq}\delta_{sn} - \delta_{ip}\delta_{qn} - \delta_{ir}\delta_{sn}) \times \frac{A^{\kappa\mu}_{ij}\hat{u}_{[pqrs]}}{\mathrm{Den}_n(\boldsymbol{\alpha})}\,, \\
A^{\kappa\mu}_{ij} &= (\delta^{\kappa\mu}\delta_{\alpha\beta} + \delta^\kappa_\beta\delta^\mu_\alpha - \delta^\kappa_\alpha\delta^\mu_\beta) \frac{q^\alpha_{ij}\,(\boldsymbol{l}_i - \boldsymbol{p}_i)^\beta}{(\boldsymbol{l}_i - \boldsymbol{p}_i)^2}\,.
\end{aligned} \tag{4.47}$$

The explicit computation can be found in lengthy appendices of [10].

Chapter 5

HOLOGRAPHIC CFTS

As an application of the methods developed in previous chapters, we attempt to rewrite the correlators for holographic CFTs in simplex form. Diagrams for holographic correlators consist of exchange diagrams and contact diagrams. We will focus on contact diagrams. The omission of exchange diagrams is not a fatal loss, since any exchange diagram can be decomposed into a sum of contact diagrams [14, 15].

5.1 Star-Mesh Duality

In momentum space, the n-point contact diagram consists of n bulk-to-boundary propagators interacting at a common bulk point with radial coordinate z over which we integrate. Each propagator is constructed from a modified Bessel function, and with the standard holographic normalization we find

$$\mathcal{I}_n \equiv \langle\!\langle \phi_1(\boldsymbol{p}_1)\cdots\phi_n(\boldsymbol{p}_n)\rangle\!\rangle_{\text{contact}} = \int_0^\infty \frac{dz}{z^{d+1}} \prod_{j=1}^{n} \frac{2^{1-\beta_j}}{\Gamma(\beta_j)} z^{d/2} p_j^{\beta_j} K_{\beta_j}(p_j z), \tag{5.1}$$

where $\beta_j = \Delta_j - \frac{d}{2}$. Using the Schwinger parametrization of Bessel functions as

$$p_j^{\beta_j} K_{\beta_j}(p_j z) = \frac{1}{2} z_j^{\beta} \int_0^\infty dZ_j Z_j^{\beta_j - 1} \exp\left[-\frac{1}{2}\left(\frac{{p_j}^2}{2Z_j} + z^2 Z_j\right)\right] \tag{5.2}$$

and doing the z integral we find

$$\mathcal{I} = \hat{C}_n \left(\prod_{j=1}^{n} \int_0^\infty dZ_j Z_j^{\beta_j - 1} \right) Z_t^{(d-\Delta_t)/2} \exp\left(-\sum_{j=1}^{n} \frac{p_j^2}{2Z_j} \right) , \tag{5.3}$$

where

$$\hat{C}_n = 2^{(n-1)d/2 - \Delta_t/2 - 1} \Gamma\left(\frac{\Delta_t - d}{2}\right) \prod_{j=1}^{n} \frac{1}{\Gamma(\beta_j)}, \qquad Z_t = \sum_{j=1}^{n} Z_j. \tag{5.4}$$

At this point, we can relate the variables in (5.3) to the elements of electrical circuit theory. If we regard the Schwinger parameters Z_j as conductivities and the momentum p_j as incoming currents, then the exponent corresponds to the power dissipation in the star-shaped electrical network as illustrated for $n = 3, 4$ in Fig. 5.1.

A well-known result from electrical circuit theory states that this n-star network is equivalent to a corresponding $(n-1)$-simplex or 'mesh' network. This is called the 'star-mesh transform'. On the simplex side, we assign a current i_{jk} (momentum) flowing from vertex j to k and a conductivity (Schwinger parameter) z_{jk} between the vertices. The current and conductivity are subject to the conditions

$$i_{jk} = -i_{kj} , \quad z_{jk} = z_{kj} . \tag{5.5}$$

The dual variables (i_{jk}, z_{jk}) are fixed uniquely by the original variables (p_k, Z_k) through the 'star-mesh relation':

$$z_{jk} = \frac{Z_j Z_k}{Z_t}, \qquad \boldsymbol{i}_{jk} = \frac{1}{Z_t}(\boldsymbol{p}_j Z_k - \boldsymbol{p}_k Z_j) . \tag{5.6}$$

It is instructive to verify that the star-mesh duality is consistent with both Kirchhoff's Current Law and Kirchhoff's Voltage Law. The momentum conservation at each vertex of the simplex corresponds to the current conservation as

$$\sum_k \boldsymbol{i}_{jk} = \frac{1}{Z_t} \left(\boldsymbol{p}_j \sum_k Z_k - Z_j \sum_k \boldsymbol{p}_k \right) = \boldsymbol{p}_j , \tag{5.7}$$

where we used conservation of the external momenta. The vanishing of the voltage drop around every closed loop requires that

$$0 = \frac{\boldsymbol{i}_{jk}}{z_{jk}} + \frac{\boldsymbol{i}_{kl}}{z_{kl}} + \frac{\boldsymbol{i}_{lj}}{z_{lj}}, \qquad \forall j, k, l . \tag{5.8}$$

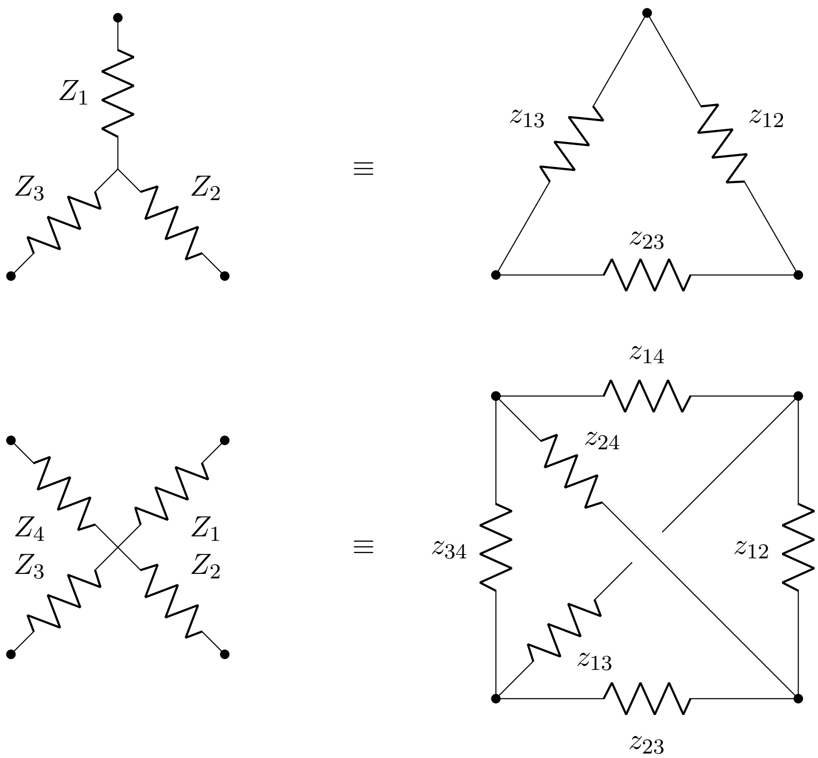

Figure 5.1: Equivalent electrical networks of resistors under star-mesh duality. The conductivities and currents are related as explained in the text. The external currents flowing into the nodes and the overall power dissipation are equal.

This condition is satisfied by the star-mesh relation (5.6) which implies that the 'voltage drop' from vertex j to k is

$$\frac{\boldsymbol{i}_{jk}}{z_{jk}} = \frac{\boldsymbol{p}_j}{Z_j} - \frac{\boldsymbol{p}_k}{Z_k}\,. \tag{5.9}$$

Another interesting feature of the star-mesh duality is that the power dissipated in both networks is the same:

$$\begin{aligned}
\sum_{j<k}\frac{i_{jk}^2}{z_{jk}} &= \frac{1}{Z_t}\sum_{j<k}\left(\frac{Z_k}{Z_j}p_j^2 + \frac{Z_j}{Z_k}p_k^2 - 2\boldsymbol{p}_j\cdot\boldsymbol{p}_k\right) \\
&= \frac{1}{Z_t}\left(\sum_t \frac{1}{Z_j}p_j^2\left(\sum_{k\neq j} Z_k\right) - \sum_j \boldsymbol{p}_j\cdot\left(\sum_{k=\neq j}\boldsymbol{p}_k\right)\right) \\
&= \frac{1}{Z_t}\sum_j\left(\frac{Z_t - Z_j}{Z_j}p_j^2 + p_j^2\right) = \sum_j \frac{p_j^2}{Z_j}.
\end{aligned} \tag{5.10}$$

Before we move on to the next section, we give the following interesting relation among the currents.

$$\boldsymbol{i}_{j[k}\cdot\boldsymbol{i}_{lm]} = \boldsymbol{i}_{jk}\cdot\boldsymbol{i}_{lm} + \boldsymbol{i}_{jm}\cdot\boldsymbol{i}_{kl} + \boldsymbol{i}_{jl}\cdot\boldsymbol{i}_{mk} = 0 \qquad \forall j,k,l,m. \tag{5.11}$$

5.1.1 3-point function

We illustrate how the star-mesh duality works by applying it to the 3-point function. To convert $\mathcal{I}_3$ to 2-simplex (triangle) form, we express the star conductivities Z_j in terms of those of the triangle, z_{jk}, as follows.

$$Z_1 = \frac{\mu}{z_{23}}, \quad Z_2 = \frac{\mu}{z_{13}}, \quad Z_3 = \frac{\mu}{z_{12}}, \qquad \mu = z_{12}z_{23} + z_{23}z_{13} + z_{13}z_{12}. \tag{5.12}$$

By using the (5.10) and the following Jacobian factor

$$\prod_{j=1}^{3} dZ_j = \frac{\mu^3}{(z_{12}z_{23}z_{13})^2}dz_{12}dz_{23}dz_{13}, \tag{5.13}$$

we find

$$\mathcal{I}_3 = \hat{C}_3\left(\prod_{j<k}\int_0^\infty dz_{jk}\,z_{jk}^{\Delta_j+\Delta_k-\Delta_t/2-1}\right)\mu^{-d/2}\exp\left(-\sum_{j<k}\frac{i_{jk}^2}{2z_{jk}}\right). \tag{5.14}$$

Then, we introduce an internal loop current $\boldsymbol{j}$ and define

$$\boldsymbol{i}'_{12} = \boldsymbol{i}_{12} + \boldsymbol{j}, \qquad \boldsymbol{i}'_{23} = \boldsymbol{i}_{23} + \boldsymbol{j}, \qquad \boldsymbol{i}'_{13} = \boldsymbol{i}_{13} - \boldsymbol{j}, \tag{5.15}$$

which leaves all the external current the same,

$$\sum_k \boldsymbol{i}'_{jk} = \sum_k \boldsymbol{i}_{jk} = \boldsymbol{p}_j. \tag{5.16}$$

Now we integrate this internal current to find

$$\int d^d\boldsymbol{j} \exp\left(-\sum_{j<k} \frac{{i'}^2_{jk}}{2z_{jk}}\right) = \left(\frac{2\pi z_{12}z_{23}z_{13}}{\mu}\right)^{d/2} \exp\left(-\sum_{j<k} \frac{i^2_{jk}}{2z_{jk}}\right) \tag{5.17}$$

and, by using (5.8), the vanishing of the voltage drop around closed loops, all $\boldsymbol{j} \cdot \boldsymbol{i}_{jk}$ cross-terms in the expansion of the exponent vanish:

$$\begin{aligned} \sum_{j<k} \frac{(i'_{jk})^2}{2z_{jk}} &= \frac{\mu}{2z_{12}z_{23}z_{13}} j^2 + \boldsymbol{j} \cdot \left(\frac{\boldsymbol{i}_{12}}{z_{12}} + \frac{\boldsymbol{i}_{23}}{z_{23}} - \frac{\boldsymbol{i}_{13}}{z_{13}}\right) + \sum_{j<k} \frac{i^2_{jk}}{2z_{jk}} \\ &= \frac{\mu}{2z_{12}z_{23}z_{13}} j^2 + \sum_{j<k} \frac{i^2_{jk}}{2z_{jk}}. \end{aligned} \tag{5.18}$$

So, we can exchange the factor of $\mu^{-d/2}$ in (5.14) for an integral over the internal current:

$$\mathcal{I}_3 = (2\pi)^{-d/2}\hat{C}_3 \int d^d\boldsymbol{j} \left(\prod_{j<k} \int_0^\infty dz_{jk} z_{jk}^{\Delta_j+\Delta_k-\Delta_t/2-d/2-1}\right) \exp\left(-\sum_{j<k} \frac{i^2_{jk}}{2z_{jk}}\right). \tag{5.19}$$

By shifting the integration variable by

$$\boldsymbol{q} = \boldsymbol{j} + \boldsymbol{i}_{12}, \quad \text{such that} \quad \boldsymbol{i}'_{12} = \boldsymbol{q}\,, \quad \boldsymbol{i}'_{23} = \boldsymbol{p}_2 + \boldsymbol{q}\,, \quad \boldsymbol{i}'_{13} = \boldsymbol{p}_1 - \boldsymbol{q}\,, \tag{5.20}$$

and then performing the integration over the z_{jk}, we can get the following simplex representation of the triple-K integral,

$$\mathcal{I}_3 = \tilde{C}_3 \int \frac{d^d\boldsymbol{q}}{(2\pi)^d} \frac{1}{|\boldsymbol{q}|^{2\alpha_{12}+d}|\boldsymbol{q}-\boldsymbol{p}_1|^{2\alpha_{13}+d}|\boldsymbol{q}+\boldsymbol{p}_2|^{2\alpha_{23}+d}}, \tag{5.21}$$

where

$$\tilde{C}_3 = 2^{2d-\Delta_t\pi^{d/2}}\hat{C}_3 \prod_{j<k} \Gamma(\alpha_{jk} + d/2), \quad \alpha_{jk} = -\Delta_j - \Delta_k + \frac{\Delta_t}{2}. \tag{5.22}$$

This result agrees with (4.11) as expected.

5.1.2 4-point function

We now discuss the 4-point contact diagram. The procedure follows what we have done in the 3-point function above. First, we convert the star form to the corresponding mesh form. We then introduce internal loop currents running around all the faces of the tetrahedron. By integrating over these currents we can get the desired result (4.13).

Note that for $n \geq 4$, the mesh network has more resistors than the corresponding star network, since the number of resistors in the mesh network is $n(n-1)/2$, which is larger than that of the star network, n. This means that the mapping of resistors between them is no longer one-to-one. For $n = 4$, we have six resistors for each edge of the tetrahedron, but only four of these are independent, since from (5.6) all cross ratios of z_{jk} are unity. We can eliminate this freedom by choosing a parametrization centered around a particular vertex of the tetrahedron. Taking this as the fourth vertex, we select independent variables as the set z_{14}, z_{24}, z_{34} and

$$\lambda = z_{12}z_{34} = z_{13}z_{24} = z_{14}z_{23}. \tag{5.23}$$

The remaining conductivities are then

$$z_{12} = \frac{\lambda}{z_{34}}, \quad z_{13} = \frac{\lambda}{z_{24}}, \quad z_{23} = \frac{\lambda}{z_{14}}\,, \tag{5.24}$$

while the conductivities of the original 4-star network are

$$Z_4 = \frac{\rho}{\lambda}, \qquad Z_i = \frac{\rho z_{i4}}{z_{14}z_{24}z_{34}}, \qquad i = 1, 2, 3, \tag{5.25}$$

where

$$\rho = z_{14}z_{24}z_{34} + \lambda(z_{14} + z_{24} + z_{34}). \tag{5.26}$$

Evaluating the Jacobian

$$\prod_{j=1}^{4} dZ_j = \frac{\rho^4}{\lambda^2(z_{14}z_{24}z_{34})^3} d\lambda \prod_{i=1}^{3} dz_{i4}, \tag{5.27}$$

the star form of the 4-point contact diagram (5.3) can be rewritten in the corresponding tetrahedral form

$$\begin{aligned}\mathcal{I}_4 = \hat{C}_4 \int_0^\infty d\lambda\, \lambda^\delta \left(\prod_{i=1}^3 \int_0^\infty dz_{i4} z_{i4}^{\delta_i} \right) \rho^{-d} \\ \times \exp \left[-\frac{1}{2} \left(\frac{i_{14}^2}{z_{14}} + \frac{i_{24}^2}{z_{24}} + \frac{i_{34}^2}{z_{34}} + \frac{z_{14}}{\lambda} i_{23}^2 + \frac{z_{24}}{\lambda} i_{13}^2 + \frac{z_{34}}{\lambda} i_{12}^2 \right) \right] ,\end{aligned} \tag{5.28}$$

where we replaced the exponent using (5.10) and defined for convenience

$$\delta = \frac{\Delta_t}{2} - \Delta_4 - 1, \qquad \delta_i = \Delta_i + \Delta_4 - \frac{\Delta_t}{2} + \frac{d}{2} - 1. \tag{5.29}$$

Now we introduce a set of internal currents running around the faces of the tetrahedron:

$$\begin{aligned} &\boldsymbol{i}'_{12} = \boldsymbol{i}_{12} - \boldsymbol{j}_3 + \boldsymbol{j}_4, \qquad \boldsymbol{i}'_{23} = \boldsymbol{i}_{23} - \boldsymbol{j}_1 + \boldsymbol{j}_4, \qquad \boldsymbol{i}'_{13} = \boldsymbol{i}_{13} - \boldsymbol{j}_2 + \boldsymbol{j}_4 \,, \\ &\boldsymbol{i}'_{14} = \boldsymbol{i}_{14} - \boldsymbol{j}_2 + \boldsymbol{j}_3, \qquad \boldsymbol{i}'_{24} = \boldsymbol{i}_{24} - \boldsymbol{j}_1 + \boldsymbol{j}_3, \qquad \boldsymbol{i}'_{34} = \boldsymbol{i}_{34} - \boldsymbol{j}_1 + \boldsymbol{j}_2 \,. \end{aligned} \tag{5.30}$$

Since these currents are purely internal, all the external currents $\boldsymbol{p}_j$ remain unchanged. Then we integrate out these internal currents and obtain

$$\begin{aligned} &\int \prod_{k=1}^3 d^d \boldsymbol{j}_k \, \exp \left[-\frac{1}{2} \sum_{k=1}^3 \left(\frac{i'^2_{k4}}{z_{k4}} + \frac{z_{k4}}{\lambda} (\hat{i}')_k^2 \right) \right] \\ &\qquad = \left(\frac{8\pi^3 z_{14} z_{24} z_{34} \lambda^3}{\rho^2} \right)^{d/2} \exp \left[-\frac{1}{2} \sum_{k=1}^3 \left(\frac{i_{k4}^2}{z_{k4}} + \frac{z_{k4}}{\lambda} \hat{i}_k^2 \right) \right] , \end{aligned} \tag{5.31}$$

where we introduced the shorthand notation

$$\hat{i}_1 = i_{23} \,, \quad \hat{i}_2 = i_{31} \,, \quad \hat{i}_3 = i_{12} \,. \tag{5.32}$$

In (5.31), all the $\boldsymbol{j}_k \cdot \boldsymbol{i}_{lm}$ cross-terms cancel, because each $\boldsymbol{j}_k$ is dotted with the sum of the 'voltage drop' around a closed loop, which vanishes. The three Gaussian integrals over the $\boldsymbol{j}_k$ then generate the prefactor shown above.

We can use these partial results to replace ρ^{-d} in (5.28) by an integration over internal currents:

$$\begin{aligned}\mathcal{I}_4 =& (2\pi)^{-3d/2} \hat{C}_4 \int_0^\infty d\lambda \lambda^{\delta - 3d/2} \left(\prod_{i=1}^3 \int d^d \boldsymbol{j}_i \int_0^\infty dz_{i4} z_{i4}^{\delta_i - d/2} \right) \\ &\times \exp \left[-\frac{1}{2} \left(\frac{i'^2_{14}}{z_{14}} + \frac{i'^2_{24}}{z_{24}} + \frac{i'^2_{34}}{z_{34}} + \frac{z_{14}}{\lambda} i'^2_{23} + \frac{z_{24}}{\lambda} i'^2_{13} + \frac{z_{34}}{\lambda} i'^2_{12} \right) \right] .\end{aligned} \tag{5.33}$$

We can further simplify the integral by shifting the currents as

$$\begin{aligned}
\boldsymbol{q}_1 &= \boldsymbol{i}_{23} - \boldsymbol{j}_1 + \boldsymbol{j}_4, \\
\boldsymbol{q}_2 &= -\boldsymbol{i}_{13} - \boldsymbol{j}_2 + \boldsymbol{j}_4, \\
\boldsymbol{q}_3 &= \boldsymbol{i}_{12} - \boldsymbol{j}_3 + \boldsymbol{j}_4,
\end{aligned} \tag{5.34}$$

which leads to

$$\begin{aligned}
\mathcal{I}_4 = (2\pi)^{-3d/2}\hat{C}_4 \int_0^\infty d\lambda\, \lambda^{\delta-3d/2} \left(\prod_{i=1}^{3} \int d^d\boldsymbol{q}_i \int_0^\infty dz_{i4} z_{i4}^{\delta_i - d/2} \right) \\
\times \exp\left[-\frac{1}{2} \left(\frac{1}{z_{14}} |\boldsymbol{p}_1 + \boldsymbol{q}_2 - \boldsymbol{q}_3|^2 + \frac{1}{z_{24}} |\boldsymbol{p}_2 + \boldsymbol{q}_3 - \boldsymbol{q}_1|^2 + \frac{1}{z_{34}} |\boldsymbol{p}_3 + \boldsymbol{q}_1 - \boldsymbol{q}_2|^2 \right.\right. \\
\left.\left. + \frac{z_{14}}{\lambda} q_1^2 + \frac{z_{24}}{\lambda} q_2^2 + \frac{z_{34}}{\lambda} q_3^2 \right) \right].
\end{aligned} \tag{5.35}$$

The remaining procedures to reach the desired simplex representation are straightforward. The first is to generate the denominator (4.14) and the second is to verify that the rest of the integral depends only on the momentum-space cross ratios in (5.61). To achieve both, first we simply rescale z_{24} and replace λ as follows

$$z_{24} \to z_{24} |\boldsymbol{p}_2 + \boldsymbol{q}_3 - \boldsymbol{q}_1|, \tag{5.36}$$

$$\lambda = q_2^2 |\boldsymbol{p}_2 + \boldsymbol{p}_3 - \boldsymbol{q}_1| / z^2. \tag{5.37}$$

Then by performing the z_{i4} integrals using (5.2), we obtain

$$\begin{aligned}
\mathcal{I}_4 = 2^4 (2\pi)^{-3d/2} \hat{C}_4 \left(\prod_{i=1}^{3} \int d^d\boldsymbol{q} \right) q_1^{\alpha_{14}-\alpha_{23}} q_2^{-2\alpha_{24}-4\alpha_{13}} q_3^{-\alpha_{34}-4\alpha_{12}} \\
\times |\boldsymbol{p}_1 + \boldsymbol{q}_2 - \boldsymbol{q}_3|^{\alpha_{23}-\alpha_{14}} |\boldsymbol{p}_2 + \boldsymbol{q}_3 - \boldsymbol{q}_1|^{-2\alpha_{13}-4\alpha_{24}-3d} |\boldsymbol{p}_3 + \boldsymbol{q}_1 - \boldsymbol{q}_2|^{\alpha_{12}-\alpha_{34}} \\
\times \int_0^\infty dz z^{3d-\Delta/2-1} K_{\alpha_{23}-\alpha_{14}}(z\sqrt{\hat{u}}) K_{\alpha_{13}-\alpha_{24}}(z) K_{\alpha_{12}-\alpha_{14}}(z/\sqrt{\hat{v}}),
\end{aligned} \tag{5.38}$$

where we replaced the δ_i with

$$2\alpha_{ij} = \frac{\Delta_t}{3} - \Delta_i - \Delta_j, \tag{5.39}$$

which is a solution of (5.73). Compared with the form of a simplex integral (4.1), (5.38) has the form a simplex integral as

$$\mathcal{I}_4 = \int \frac{d^d\boldsymbol{q}_1}{(2\pi)^d}\frac{d^d\boldsymbol{q}_2}{(2\pi)^d}\frac{d^d\boldsymbol{q}_3}{(2\pi)^d}\frac{\hat{f}(\hat{u},\hat{v})}{\mathrm{Den}_4(\boldsymbol{q}_j,\boldsymbol{p_k})}, \tag{5.40}$$

where the momentum running each edge of the tetrahedron is as illustrated in Fig. 4.1 and the denominator is given in (4.14). The function of momentum-space cross ratios $\hat{f}(\hat{u},\hat{v})$ is

$$\begin{aligned}\hat{f}(\hat{u},\hat{v}) = \tilde{C}_4 &\left(\frac{\hat{u}}{\hat{v}}\right)^{(\alpha_{12}+\alpha_{34}+d)/2}\\ &\times \int_0^\infty dz z^{3(\alpha_{12}+\alpha_{34}+d)-1} K_{\alpha_{23}-\alpha_{14}}(z\sqrt{\hat{u}})K_{\alpha_{13}-\alpha_{24}}(z)K_{\alpha_{12}-\alpha_{34}}(z/\sqrt{\hat{v}}),\end{aligned} \tag{5.41}$$

where

$$\tilde{C}_4 = 2^4(2\pi)^{3d/2}\hat{C}_4 = 2^{3d-\Delta_t/2+3}\pi^{3d/2}\Gamma\left(\frac{\Delta_t-d}{2}\right)\prod_{j=1}^4\frac{1}{\Gamma(\Delta_j-d/2)}. \tag{5.42}$$

This is the specific function of momentum-space cross ratios appearing in the simplex representation for the 4-point contact Witten diagram. This $\hat{f}(\hat{u},\hat{v})$ involves the same integral of three Bessel functions, the triple-K integral, as appears in the 3-point function $\mathcal{I}_3$, though the arguments and parameters are now different. Specifically,

$$\hat{f}(\hat{u},\hat{v}) = \tilde{C}_4\frac{\hat{u}^{\alpha_{12}+d/2}}{\hat{v}^{\alpha_{34}+d/2}}I_{3(\alpha_{12}+\alpha_{34}+d)-1,\{\alpha_{23}-\alpha_{14},\alpha_{13}-\alpha_{24},\alpha_{12}-\alpha_{34}\}}(\sqrt{\hat{u}},1,\frac{1}{\sqrt{\hat{v}}}) \tag{5.43}$$

where the triple-K integral [16] is

$$I_{\alpha,\{\beta_1,\beta_2,\beta_3\}}(p_1,p_2,p_3) = \int_0^\infty \mathrm{d}z z^\alpha\prod_{j=1}^3 p_j^{\beta_j}K_{\beta_j}(p_j z). \tag{5.44}$$

5.2 Recursive Convolutions for n-point Functions

In this subsection, we will evaluate the general n-point contact diagram to calculate the specific function of momentum-space cross ratios appearing in the simplex representation (4.1) for this diagram.

In position space, the n-point contact diagram is given by

$$\mathcal{I}_n = \int \frac{dz}{z^{d+1}} \int d^d \boldsymbol{x}_0 \prod_{i=1}^{n} C_{\Delta_i} \left(\frac{z}{z^2 + x_{i0}^2} \right)^{\Delta_i}, \tag{5.45}$$

where $\boldsymbol{x}_{ij} = \boldsymbol{x}_i - \boldsymbol{x}_j$ and the holographic normalization is

$$C_{\Delta_i} = \frac{\Gamma(\Delta_i)}{\pi^{d/2}\Gamma(\Delta_i - d/2)} \tag{5.46}$$

By using Schwinger parametrization, we can parametrize all denominators and perform the z integral. This gives

$$\begin{aligned} \mathcal{I}_n = \frac{1}{2}\Gamma\left(\frac{\Delta_t - d}{2}\right) \left(\prod_{i=1}^{n} \frac{C_{\Delta_i}}{\Gamma(\Delta_i)} \int_0^\infty ds_i s_i^{(\Delta_i - 1)} \right) \\ \times s_t^{(d-\Delta_t)/2} \int d^d \boldsymbol{x}_0 \exp\left(-\sum_i s_i x_{i0}^2 \right). \end{aligned} \tag{5.47}$$

Then we complete the square as

$$\begin{aligned} \frac{1}{s_t}\left(\sum_i s_i \boldsymbol{x}_i\right)^2 - \sum_i s_i x_i^2 &= \frac{1}{s_t}\left(\sum_i s_i(s_i - s_t)x_i^2 + 2\sum_{i<j} s_i s_j \boldsymbol{x}_i \cdot \boldsymbol{x}_j \right) \\ &= \frac{1}{s_t}\left(-\sum_{i<j} s_i s_j (x_i^2 + x_j^2) + 2\sum_{i<j} s_i s_j \boldsymbol{x}_i \cdot \boldsymbol{x}_j \right) \\ &= -\frac{1}{s_t}\sum_{i<j} s_i s_j x_{ij}^2. \end{aligned} \tag{5.48}$$

By performing the $\boldsymbol{x}_0$ integral, we find

$$\mathcal{I}_n = C_n \left(\prod_{i=1}^{n} \int_0^\infty ds_i s_i^{\Delta_i - 1} \right) s_t^{-\Delta_t/2} \exp\left(-\frac{1}{s_t} \sum_{i<j} s_i s_j x_{ij}^2 \right) \tag{5.49}$$

where

$$C_n = \frac{\pi^{d/2}}{2}\Gamma\left(\frac{\Delta_t - d}{2}\right) \prod_{i=1}^{n} \frac{C_{\Delta_i}}{\Gamma(\Delta_i)} \tag{5.50}$$

By using the Symanzik trick, we can make the replacement

$$s_t = \sum_{i=1}^{n} s_i \to \sum_{i=1}^{n} \kappa_i s_i, \tag{5.51}$$

without changing the value of the integral (5.49), for any arbitrary set of $\kappa_i \geq 0$ not all zero. We use this trick to replace $s_t \to s_1$ so that the integrand of (5.49) becomes a product of exponential factors with a recursive structure:

$$\mathcal{I}_n = C_n \left(\prod_{i=1}^{n} \int_0^\infty ds_i s_i^{\Delta_i - 1} \right) s_1^{-\Delta_t/2} g_n \tag{5.52}$$

where

$$g_n = \prod_{1 \leq i < j}^{n} \exp\left(-\frac{s_i s_j}{s_1} x_{ij}^2 \right) = g_{n-1} \times \prod_{i=1}^{n-1} \exp\left(-\frac{s_i s_n}{s_1} x_{in}^2 \right). \tag{5.53}$$

Since we performed the replacement $s_t \to s_1$, the integrand g_{n-1} has no dependence on either $\boldsymbol{x}_n$ or s_n. By Fourier transforming, the recursive product (5.53) becomes a convolution

$$\mathcal{F}[g_n](\boldsymbol{p_1}, \ldots, \boldsymbol{p_n}) = \Big(\mathcal{F}[g_{n-1}](2\pi)^d \delta(\boldsymbol{p_n}) \Big) * \mathcal{F}\left[\prod_{i=1}^{n-1} \exp\left(-\frac{s_i s_n}{s_1} x_{in}^2 \right) \right], \tag{5.54}$$

where the $\delta(\boldsymbol{p_n})$ arises since g_{n-1} is independent of $\boldsymbol{x_n}$. Expressing explicitly, we find

$$\begin{aligned}
\mathcal{F}[g_n](\boldsymbol{p_1}, \ldots, \boldsymbol{p_n}) &= \left(\prod_{k=1}^{n} \int \frac{d^d \boldsymbol{q}_k}{(2\pi)^d} \right) \mathcal{F}[g_{n-1}](\boldsymbol{p}_1 - \boldsymbol{q}_1, \ldots, \boldsymbol{p}_{n-1} - \boldsymbol{q}_{n-1}) \\
&\quad \times (2\pi)^d \delta(\boldsymbol{p}_n - \boldsymbol{q}_n)(2\pi)^d \delta(\sum_{j=1}^{n} \boldsymbol{q}_j) \prod_{j=1}^{n-1} \left(\frac{\pi s_1}{s_n s_i} \right)^{d/2} \exp(-\frac{s_1 q_i^2}{4 s_i s_n}) \\
&= \prod_{i=1}^{n-1} \left(\int \frac{d^d \boldsymbol{q}_{in}}{(2\pi)^d} \left(\frac{\pi s_1}{s_n s_i} \right)^{d/2} \exp\left(-\frac{s_1 q_{in}^2}{4 s_i s_n} \right) \right) \\
&\quad \times (2\pi)^d \delta\left(\boldsymbol{p}_n + \sum_{j=1}^{n-1} \boldsymbol{q}_{jn} \right) \mathcal{F}[g_{n-1}](\boldsymbol{p}_1 - \boldsymbol{q}_{1n}, \ldots, \boldsymbol{p}_{n-1} - \boldsymbol{q}_{n-1,n}).
\end{aligned} \tag{5.55}$$

In the second equation, note that the integration variables are the momenta $\boldsymbol{q}_{in}$ running from the vertex i to vertex n. By using the recursive structure, for $g_1 = 1$,

$$\begin{aligned}
\mathcal{F}[g_1](\boldsymbol{p}_1) &= (2\pi)^d \delta(\boldsymbol{p}_1), \\
\mathcal{F}[g_2](\boldsymbol{p}_1, \boldsymbol{p}_2) &= \int \frac{d^d \boldsymbol{q}_{12}}{(2\pi)^d} \left(\frac{\pi}{s_2} \right)^{d/2} \exp\left(-\frac{s_1 q_{12}^2}{4 s_2} \right) \\
&\qquad \times (2\pi)^d \delta(\boldsymbol{p}_1 + \boldsymbol{q}_{21})(2\pi)^d \delta(\boldsymbol{p}_2 + \boldsymbol{q}_{12})
\end{aligned} \tag{5.56}$$

repeating n times,

$$\mathcal{F}[g_n](\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n)$$
$$=\left(\prod_{1\le i<j}\int\frac{d^d\boldsymbol{q}_{ij}}{(2\pi)d}\left(\frac{\pi s_1}{s_i s_j}\right)^{d/2}\exp\left(-\frac{s_1 q_{ij}^2}{4s_i s_j}\right)\right)\prod_{k=1}^{n}(2\pi)^d\delta\left(\boldsymbol{p}_k+\sum_{l=1}^{n}\boldsymbol{q}_{lk}\right). \tag{5.57}$$

If we restore the Schwinger integration from (5.52), we get the momentum-space contact diagram $\mathcal{I}_n$ which has the expected structure of a simplex integral (4.1) with

$$\hat{f}_n(\hat{\boldsymbol{u}})$$
$$=C_n\left(\prod_{k=1}^{n}\int_0^\infty ds_k s_k^{\Delta_k-1}\right)s_1^{-\Delta_t/2}\prod_{1\le i<j}^{n}\left(\frac{\pi s_1}{s_i s_j}\right)^{d/2}\exp\left(-\frac{s_1 q_{ij}^2}{4s_i s_j}\right)q_{ij}^{2\alpha_{ij}+d}, \tag{5.58}$$

where α_{ij} satisfies (5.73). Before we show this result is indeed a function of only the momentum-space cross ratios $\hat{\boldsymbol{u}}$, we transform further by substituting $s_i=1/t_i$ yielding

$$\hat{f}_n(\hat{\boldsymbol{u}})$$
$$=C_n\left(\prod_{k=1}^{n}\int_0^\infty dt_k t_k^{-\Delta_k-1}\right)t_1^{\Delta_t/2}\prod_{1\le i<j}^{n}\left(\frac{\pi t_i t_j}{t_1}\right)^{d/2}\exp\left(-\frac{t_i t_j q_{ij}^2}{4t_1}\right)q_{ij}^{2\alpha_{ij}+d}. \tag{5.59}$$

Now we will show that this $\hat{f}(\hat{\boldsymbol{u}})$ is indeed a function of only the momentum-space cross ratios $\hat{\boldsymbol{u}}$. For this purpose, we select new independent variables corresponding to the subset of n legs shown in figure 5.2. This parametrization corresponds to

$$t_1=\frac{z_{12}z_{13}}{z_{23}}\frac{q_{23}^2}{q_{12}^2 q_{13}^2},\qquad t_i=\frac{z_{1i}}{q_{1i}^2},\qquad i=2,\ldots,n \tag{5.60}$$

and introduce the $n(n-3)/2$ independent momentum-space cross ratios

$$\hat{u}_{2a}=\frac{q_{2a}^2 q_{13}^2}{q_{1a}^2 q_{23}^2},\qquad \hat{u}_{3a}=\frac{q_{3a}^2 q_{12}^2}{q_{1a}^2 q_{23}^2},\qquad \hat{u}_{ab}=\frac{q_{ab}^2 q_{23}^2}{q_{2a}^2 q_{3b}^2} \tag{5.61}$$

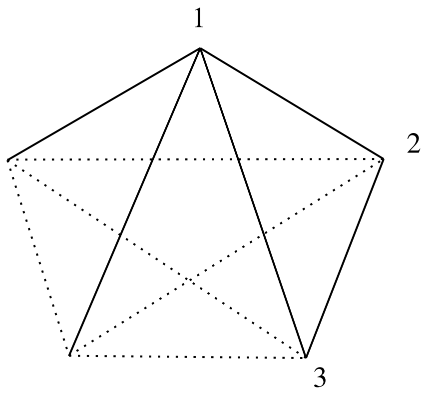

Figure 5.2: Change of integration variables. In (5.60), we exchange the Schwinger parameters t_i where $i = 1, \ldots, n$ for a new set consisting of z_{23} and z_{1i} for $i = 2, \ldots, n$. These correspond to the solid legs on the diagram above, shown for the case $n = 5$.

where $a, b = 4, \ldots, n$ and in the last equation $a < b$ with no sum implied. Converting (5.59) into these new variables, we find

$$\hat{f}_n(\hat{\boldsymbol{u}}) = C_n \pi^{n(n-1)d/4} \left(\prod_{1 \le k < l}^{n} q_{kl}^{2\alpha_{kl}+d} \right) \left(\prod_{i=2}^{n} \int_0^\infty \frac{dz_{1i}}{z_{1i}} \left(\frac{z_{1i}}{q_{1i}^2} \right)^{-\Delta_i + (n-1)d/2} \right)$$
$$\times \int_0^\infty \frac{dz_{23}}{z_{23}} \left(\frac{z_{12} z_{13}}{z_{23}} \frac{q_{23}^2}{q_{12}^2 q_{13}^2} \right)^{\Delta_t/2 - \Delta_1 - (n-1)(n-2)d/4}$$
$$\exp\left[-\frac{1}{4} \left(z_{12} + z_{23} + z_{13} + \sum_{a=4}^{n} z_{1a} (1 + \frac{z_{23}}{z_{13}} \hat{u}_{2a} + \frac{z_{23}}{z_{12}} \hat{u}_{3a}) \right.\right.$$
$$\left.\left. + \sum_{4 \le a < b}^{n} \frac{z_{1a} z_{1b} z_{23}}{z_{12} z_{13}} \hat{u}_{ab} \hat{u}_{2a} \hat{u}_{3b} \right) \right]. \quad (5.62)$$

Then, the remaining q_{ij}^2 can be canceled by choosing α

$$\begin{aligned}
\alpha_{12} &= -\Delta_1 - \Delta_2 + \frac{\Delta_t}{2} - (n-2)(n-3)\frac{d}{4}, \\
\alpha_{13} &= -\Delta_1 - \Delta_3 + \frac{\Delta_t}{2} - (n-2)(n-3)\frac{d}{4}, \\
\alpha_{23} &= \Delta_1 - \frac{\Delta_t}{2} + n(n-3)\frac{d}{4}, \\
\alpha_{1a} &= -\Delta_a + (n-2)\frac{d}{4}, \\
\alpha_{2a} &= \alpha_{3a} = -\frac{d}{2}, \\
\alpha_{ab} &= -\frac{d}{2},
\end{aligned} \tag{5.63}$$

and the same goes for $a, b = 4, \ldots, n$ and $a < b$. Actually, this choice satisfies the constraint (5.73). So finally we get the following expression for $\hat{f}(\hat{\boldsymbol{u}})$ as a function of the momentum-space cross ratios only,

$$\begin{aligned}
\hat{f}_n(\hat{\boldsymbol{u}}) = C_n \pi^{n(n-1)/4} &\left(\prod_{i=2}^{n} \int_0^\infty dz_{1i} z_{1i}^{\alpha_{1i}+d/2-1} \right) \int_0^\infty dz_{23} z_{23}^{\alpha_{23}+d/2-1} \\
&\times \exp\Bigg[-\frac{1}{4}\Bigg(z_{12} + z_{23} + z_{13} + \sum_{a=4}^{n} z_{1a}(1 + \frac{z_{23}}{z_{13}}\hat{u}_{2a} + \frac{z_{23}}{z_{12}}\hat{u}_{3a}) \\
&\qquad + \sum_{4\leq a<b}^{n} \frac{z_{1a}z_{1b}z_{23}}{z_{12}z_{13}} \hat{u}_{ab}\hat{u}_{2a}\hat{u}_{3b} \Bigg) \Bigg].
\end{aligned} \tag{5.64}$$

We can simplify this expression further by evaluating the z_{23} integral as

$$\begin{aligned}
\hat{f}_n(\hat{\boldsymbol{u}}) &= C_n 2^{\alpha_{23}+d} \pi^{n(n-1)d/4} \Gamma(\alpha_{23} + \frac{d}{2}) \\
&\qquad \times \left(\prod_{i=2}^{n} \int_0^\infty dz_{1i} z_{1i}^{\alpha_{1i}+d/2-1} e^{-z_{1i}/4} \right) (z_{12}z_{13})^{\alpha_{23}+d/2} \\
&\times \left(z_{12}z_{13} + \sum_{a=4}^{n} z_{1a}(z_{12}\hat{u}_{2a} + z_{13}\hat{u}_{3a}) + \sum_{4\leq a<b}^{n} z_{1a}z_{1b}\hat{u}_{ab}\hat{u}_{2a}\hat{u}_{3b} \right)^{-\alpha_{23}-d/2}.
\end{aligned} \tag{5.65}$$

Now we set as

$$z_{1i} = \sigma y_{1i}, \qquad \text{for} \quad i = 2, \ldots, n, \tag{5.66}$$

$$\text{with constraint} \quad \sum_{i=2}^{n} y_{1i} = 1, \tag{5.67}$$

so that the exponential reduces to $e^{-\sigma/4}$. By evaluating the Jacobian as discussed in appendix B of [10] and performing the σ integral, we get

$$\begin{aligned} \hat{f}_n(\hat{\boldsymbol{u}}) = \hat{C}_n \left(\prod_{i=2}^{n} \int_0^1 dy_{1i} y_{1i}^{\alpha_{1i}+d/2-1} \right) (y_{12}y_{13})^{\alpha_{23}+d/2} \delta \left(1 - \sum_{i=2}^{n} y_{1i} \right) \\ \times \left(y_{12}y_{13} + \sum_{a=4}^{n} y_{1a}(y_{12}\hat{u}_{2a} + y_{13}\hat{u}_{3a}) + \sum_{4\leq a<b}^{n} y_{1a}y_{1b}\hat{u}_{ab}\hat{u}_{2a}\hat{u}_{3b} \right)^{-\alpha_{23}-d/2}, \end{aligned} \tag{5.68}$$

where the α_{ij} are given by (5.63). The normalization is

$$\hat{C}_n = C_n \pi^{n(n-1)d/4} 4^{nd/2+\alpha_{23}-\Delta_1} \Gamma\left(\alpha_{23} + \frac{d}{2}\right) \Gamma\left((n-1)\frac{d}{2} - \Delta_1\right), \tag{5.69}$$

where (5.73) was used to replace $\sum_{i=2}^{n} \alpha_{1i} = -\Delta_1$. If the overall delta function is removed in (5.68), only $(n-2)$ integrations remain. This seems to be the optimal representation for $\hat{f}_n(\hat{\boldsymbol{u}})$. For comparison, the Mellin-Barnes representation obtained following Symanzik's procedure in [17] has $n(n-3)/2$ Mellin integrations which is larger than $n-2$ for any $n > 4$.

For the 4-point function, (5.68) reduces to

$$\begin{aligned} \hat{f}_4(\hat{\boldsymbol{u}}) = \hat{C}_4 \int_0^1 dy_{12} y_{12}^{\alpha_{12}+\alpha_{23}+d-1} \int_0^1 dy_{13} y_{13}^{\alpha_{13}+\alpha_{23}+d-1} \int_0^1 dy_{14} y_{14}^{\alpha_{14}+d/2-1} \\ \times \delta(1 - y_{12} - y_{13} - y_{14}) \left(y_{12}y_{13} + y_{12}y_{14}\hat{u}_{24} + y_{13}y_{14}\hat{u}_{34}\right)^{-\alpha_{23}-d/2}. \end{aligned} \tag{5.70}$$

This is reminiscent of the Feynman parametrization of the 1-loop triangle integral (see, appendix A.3 of [16]), with the difference that the momentum-space cross ratios $\hat{u}_{24}$ and $\hat{u}_{34}$ take the place of ratios of the squared external momenta. This is not coincidence, since in section 5.1.1 we showed that the 1-loop triangle is equivalent to a triple-K integral and from (5.41) we know that $\hat{f}_4(\hat{\boldsymbol{u}})$ can be written as a triple-K integral where the arguments are given by the momentum-space cross ratios. (5.70) and (5.41) are exactly same and this can be verified by using equation (A.3.23) of [16].

In short, we showed that the n-point contact diagram can be written as a simplex integral and (5.68) expresses $\hat{f}_n(\hat{\boldsymbol{u}})$ as an $(n-2)$-fold Feynman parametric integral over a quadratic denominator. Before finishing this chapter, we present some remarkable connections between this result and other Feynman integrals.

Firstly, equation (5.68), the function $\hat{f}_n(\hat{\boldsymbol{u}})$ of momentum-space cross ratios appearing in the simplex representation, has a close similarity with the corresponding representation for $f_n(\boldsymbol{u})$, ordinary cross ratios describing the contact diagram in position space.

To see this, we start from (5.49) and repeat the above steps from (5.60) to (5.68). We then find the position-space contact diagram is

$$\mathcal{I}_n = \prod_{1\leq i<j<n} x_{ij}^{-2\alpha_{ij}-d} f_n(\boldsymbol{u}), \tag{5.71}$$

where $f_n(\boldsymbol{u})$ is given by exactly the right-hand side of (5.68). The α_{ij} are now given by (5.63) after making the following replacements:

$$\Delta_i \to -\Delta_i, \qquad\qquad n \to 1. \tag{5.72}$$

Instead of (5.73), these new α_{ij} satisfy

$$\Delta_m = -\sum_{j=1}^{n}\left(\alpha_{mj}+\frac{d}{2}\right), \qquad\qquad m = 1,2,\dots,n, \tag{5.73}$$

with $\alpha_{ij}=\alpha_{ji}$ and $\alpha_{ii}=0$. In addition, by sending $q_{ij}^2 \to x_{ij}^2$ in (5.61), we replace

$$\hat{C}_n \to C_n\Gamma(\Delta_1)\Gamma(\alpha_{23}+d/2), \qquad\qquad \hat{\boldsymbol{u}} \to \boldsymbol{u}. \tag{5.74}$$

Thus, with these replacements, both $\hat{f}_n(\hat{\boldsymbol{u}})$ and $f_n(\boldsymbol{u})$ have exactly the same parametrization (5.68). That is, the function $\hat{f}_n(\hat{\boldsymbol{u}})$ of momentum-space cross ratios appearing in the simplex representation has the same form as the function $f_n(\boldsymbol{u})$ of ordinary cross ratios describing the contact diagram in position space.

This equivalence is not a coincidence, as can be seen by comparing (5.49) with (5.68). The Fourier transform of a product of Gaussians is a convolution of Gaussians. If we rewrite this convolution as a simplex, the resulting $\hat{f}_n(\hat{\boldsymbol{u}})$ in (5.68) is identical to our starting point (5.49) up to a change of parameters.

Chapter 6

CONCLUSION

In this thesis, we found the general solution of the conformal Ward identities in momentum space, the scalar correlators. The general CFT scalar n-point function can be written as a simplex integral. It involves an arbitrary function $\hat{f}(\hat{\boldsymbol{u}})$ of the momentum-space cross ratios which play a same role as the cross ratios in position space. But in momentum space, the cross ratios are subject to integration inside the simplex integral.

Then we discussed how we can find the form of $\hat{f}(\hat{\boldsymbol{u}})$ for particular correlators in holographic theory. We saw for the contact diagram we can use the trick for electrical circuit theory. This is the star-mesh transformation to rewrite a contact diagram which has a star topology as a mesh integral which has a simplex topology. And we introduced another useful trick which is just using the convolution theorem and the recursive structure of these simplex integrals. These two methods can be applicable to a wider class of examples. For the star-mesh transformation trick, whenever correlators have a bulk vertex which is integrated over in Witten diagram, that becomes a sort of internal node in electrical circuit which is integrated out. So this method is very generally applicable. And for a recursive application of the convolution theorem, we can use this method whenever the correlators exhibit a recursive product structure in space. We can generally transform the correlators into the recursive form through a suitable Schwinger of Feynman parametrization.

For further investigation, one can consider the simplex representation for a wider range of diagrams like exchange Witten diagrams, loops, particularly those with cosmological relevance etc.

Other very interesting topic is to understand the singularities of these simplex integrals. Because these simplex integrals are generic Feynman integrals, we have all the standard tools to extract the singularities of these correlators. The singularities give a lot of information. The flat-space limit takes us from CFT correlators to scattering amplitudes. We need to find the energy poles in the correlators and extract residue for the general simplex integral. And we would like to understand how these arbitrary function of cross ratios $\hat{f}(\hat{\boldsymbol{u}})$ relate to the scattering amplitudes we obtain in the flat-space limit. Also there are special values of the operator and space-time dimension for which divergences occur and we need to renormalize the correlators. So there would be anomalies and beta functions associated with this. Therefore, to know the singularity structure is very interesting topic. The cases for 3-point [16] and 4-point function [9] was studied but the detailed structure of anomalies and the renormalization of higher-point function remains to be explored.

Next, understanding how to find the simplex representation for more general correlators involving tensorial external operators is also an open question. The case for 3-point functions was studied in [16] and 4-point functions in [18–21].

Another key area of interest of this simplex representation is to do the conformal bootstrap in momentum space. There has been a lot of progress on understanding bootstrap conditions in CFT in position space. But if we can do something similar in momentum space, in particular tensorial correlators, there could be a progress in understanding conformal bootstrap in momentum space. And the simplex representation would be a good way for this.

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초 록

이 논문은 임의의 차원에서 등각장론의 스칼라 n점함수에 대한 등각 워드 항등식의 일반해를 운동량 공간에서 다룬다. 2019년에 Bzowski, McFadden, Skenderis는 일반해가 운동량 공간에서 단순체 적분이라 불리는, (n-1)단순체에 대한 적분으로 표현됨을 발견하였다. 이때 단순체의 n개의 꼭짓점은 n개의 연산자 삽입에 대응된다. 꼭짓점 사이를 움직이는 운동량들이 적분 변수가 되며, 각각의 꼭짓점에서 운동량 보존을 만족해야 한다. 적분의 피적분함수는 운동량 공간에서의 교차비들에 대한 임의의 함수를 포함한다. 우리는 단순체 적분의 등각 불변성을, 교차비들의 함수가 교차비들의 단항식으로 이루어져 있을 때 가장 명확하게 보이는 재귀 구조를 이용하여 증명한다. 이러한 논의를 적용하여, 우리는 홀로그래피 등각 장론에서의 n점 접촉 위튼 다이어그램의 단순체 표현을 유도한다.

주요어: 등각장론, 단순체 적분

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