이학박사 학위논문

Affine RSK correspondence and crystals of level zero extremal weight modules

(아핀 RSK 대응과 레벨 0 극단 무게 가군의 결정)

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이 논문을 이학박사 학위논문으로 제출함

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Affine RSK correspondence and crystals of level zero extremal weight modules

A dissertation
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Abstract

The Robinson-Schensted-Knuth (RSK) correspondence is a bijection that maps a matrix of non-negative integers to a pair of semistandard tableaux of the same shape. The correspondence has deep connections to algebraic combinatorics and representation theory, serving as the combinatorial counterpart of the Howe duality on a symmetric algebra over the space of matrices. From the viewpoint of crystal theory, the correspondence preserves the crystal structures on the set of matrices and the set of pairs of tableaux.

Recently, Chmutov-Pylyavskyy-Yudovina extended the correspondence to affine permutations using diagrammatic method which is called the matrix-ball construction. In this thesis, we introduce an affine analogue of the RSK correspondence, which generalizes the result of Chmutov-Pylyavskyy-Yudovina via standardization. The affine RSK maps an affine matrix to a pair of tableaux of the same shape, where one of the pair belongs to a tensor product of perfect crystals of level one, and the other belongs to a crystal of a level zero extremal weight module. We prove that the affine RSK preserves the affine crystal structures of type A. We give a brief comparison of our result with another affine generalization of RSK introduced by Imamura-Mucciconi-Sasamoto. We also introduce a dual affine RSK correspondence.

Key words: affine RSK correspondence, extremal weight crystals, matrix-ball construction
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Contents

Chapter 1

Introduction

1.1 Backgrounds

The Robinson-Schensted-Knuth correspondence (RSK for short) is a fundamental bijection in algebraic combinatorics and representation theory that associates a matrix of non-negative integers with a pair of semistandard tableaux of the same shape. It has rich applications in a wide range of areas, including representation theory, geometry, and statistical mechanics. In representation theory, the RSK can be regarded as the combinatorial counterpart of the Howe duality [11]

$$\mathcal{S}(\mathcal{M}_{m\times n}(\mathbb{C})) \cong \bigoplus_{\lambda} V_m(\lambda) \otimes V_n(\lambda), \tag{1.1.1}$$

that decompose the symmetric algebra over the space of matrices into irreducible $GL_m(\mathbb{C})\times GL_n(\mathbb{C})$-modules. The RSK describes the bijection between the bases of the spaces on both sides of (1.1.1), and it preserves the crystal structures on the bases [24].

The origins of the Robinson-Schensted-Knuth correspondence can be traced back to the work of Robinson [29], which associates a permutation with a pair of standard Young tableaux of the same shape. This correspondence was also discovered independently by Schensted [31], as an insertion algorithm of tableaux, which is called the Robinson-Schensted correspondence (RS for short). It provides a combinatorial description of the left and right cells of the symmetric group in Kazhdan-Lusztig theory [20]. By Knuth [22], the insertion algorithm is generalized to the RSK correspondence.

Unlike the insertion algorithm, Viennot [37] introduced a diagrammatic method that describes the RS correspondence. It naturally respects the symmetry of RS correspondence. That is, if a permutation w corresponds to (P,Q), then the inverse w^{-1} corresponds to (Q,P). Viennot's method works for the RSK after applying standardization, which is called the matrix-ball construction (see [7, Chapter 4]).

An affine analogue of the RS correspondence was introduced by Shi [32, 33] in the study of affine Kazhdan-Lusztig cells. Shi associates an affine permutation w with a pair of tabloids $(P(w), Q(w))$, where each tabloid determines a left or right cell of the Hecke algebras of affine symmetric groups. However, the map $w \mapsto (P(w), Q(w))$ is not injective. Recently, Chmutov-Pylyavskyy-Yudovina [5] constructed a bijection $w \mapsto (P(w), Q(w), \rho(w))$ using an affine generalization of matrix-ball construction. Here $\rho(w)$ is an integral vector satisfying a condition called dominance. It is a natural question that how to extend the affine RS correspondence for a affine matrix or an affine matrix.

1.2 Main Results

Let m and n be positive integers. Let $\widehat{\mathcal{M}}_{m\times n}$ be the set of matrices $A = (a_{ij})_{i,j\in\mathbb{Z}}$ of non-negative integers such that $a_{i+m,j+n} = a_{ij}$ for all $i, j \in Z$, and for each $j \in \mathbb{Z}$, $a_{ij} = 0$ except for finitely many i's. Let λ be a partition with length not greater than m and n. Let $CSST_{[m]}(\lambda)$ be the set of column semistandard tableaux of shape λ with entries from 1 to m. Let $\mathcal{B}_n(\lambda)$ be the set of tableaux of shape λ with entries in $\mathbb{Z}$ such that each pair of adjacent columns of the same column length is form a semistandard tableau.

The first main result of this thesis is to construct a bijection

$$\begin{array}{rccc} \kappa : & \widehat{\mathcal{M}}_{m\times n} & \longrightarrow & \displaystyle\bigsqcup_{\lambda\in\mathscr{P}_m\cap\mathscr{P}_n} CSST_{[m]}(\lambda)\times\mathcal{B}_n(\lambda) , \\ & A & \longmapsto & (P_0, Q) \end{array} \tag{1.2.1}$$

where $\mathscr{P}_n$ is the set of all partition with length not greater than n. The main ingredient is the affine RS correspondence in [5], and the standardizations of matricies and tableaux. A key observation is that the dominance condition of the vector $\rho(w)$ is compatiable with the description for $\mathcal{B}(\lambda)$.

Let us consider the crystal structures on both sides. If $m \geq 2$, $\widehat{\mathcal{M}}_{m\times n}$ possess a natural $U_q(\widehat{\mathfrak{sl}}_m)$-crystals structure which seems to be *tensor-product-like* of its columns, and $CSST_{[m]}(\lambda)$ has a $U_q'(\widehat{\mathfrak{sl}}_m)$-crystal structure isomorphic to a tensor product of perfect

Kirillov-Reshetikhin crystals of level 1.

On the other hand, if $n \geq 2$, $\widehat{\mathcal{M}}_{m\times n}$ again possess a natural $U_q(\widehat{\mathfrak{sl}}_n)$-crystals structure which seems to be *tensor-product-like* of its rows, and $\mathcal{B}_n(\lambda)$ has a $U_q(\widehat{\mathfrak{sl}}_n)$-crystal structure isomorhpic to the crystal base of a level zero extremal weight module. The $U_q(\widehat{\mathfrak{sl}}_m)$-crystal (or $U_q'(\widehat{\mathfrak{sl}}_m)$-crystal) on both sides are compatible with the $U_q(\widehat{\mathfrak{sl}}_n)$-crystal (or $U_q'(\widehat{\mathfrak{sl}}_n)$-crystal). We expect that the crystal structure of $\widehat{\mathcal{M}}_{m\times n}$ coincides with the ones in [25].

The second result of this thesis is that the bijection κ preserves the crystal structures on both sides. Indeed we show that κ commutes with the Kashiwara operators except for the $\widetilde{e}_0$ and $\widetilde{f}_0$ for $U_q(\widehat{\mathfrak{sl}}_m)$-crystals. As a corollary, we have an isomorphism of $(U_q(\mathfrak{sl}_m) \times U_q(\widehat{\mathfrak{sl}}_n))$-crystals

$$\widehat{\mathcal{M}}_{m\times n} \cong \bigoplus_{\lambda\in\mathscr{P}_n\cap\mathscr{P}_n} CSST_{[m]}(\lambda) \times \mathcal{B}_n(\lambda).$$

Let $\widehat{\mathcal{N}}_{m\times n}$ be the set of $\{0,1\}$-matrices satisfying the similar relations. Let $RSST_{[m]}(\lambda)$ be the set of row semistandard tableaux of shape λ with entries from 1 to m. If $m \geq 1$, $RSST_{[m]}(\lambda)$ has a $U_q'(\widehat{\mathfrak{sl}}_m)$-crystal structure isomorphic to a tensor product of perfect Kirillov-Reshetikhin crystals of level ≥ 1. Then we have a dual analogue of (1.2.1)

$$\begin{array}{rccc} \kappa' : & \widehat{\mathcal{N}}_{m\times n} & \longrightarrow & \displaystyle\bigsqcup_{\lambda\in\mathscr{P}_n} RSST_{[m]}(\lambda') \times \mathcal{B}_n(\lambda)\,, \\ & A & \longmapsto & (P_0^t, Q) \end{array} \tag{1.2.2}$$

where λ' is the conjugate partition of λ, and P_0' is the conjugate tableau of P_0. We show also that κ' is an isomorphism of $(U_q(\mathfrak{sl}_m) \times U_q(\widehat{\mathfrak{sl}}_n))$-crystals. We expect that κ' gives a level zero analogue of the decomposition of the crystals [8,9] associated to the higher level q-deformed Fock space [36].

We remark that another affine generaliziation of RSK correspondence is given by Imamura-Mucciconi-Sasamoto [13]. The algorithm uses the dynamics of Sagan-Stanley's skew RSK correspondence [30]. We give an expository example which compares two algorithms. A representation theoretic interpretation of the identity corresponding to the bijection in [13] is also recently given using representations of current Lie algebras [6].

1.3 Organization

The remainder of this thesis is organized as follows.

- **Chapter 2** We review preliminaries on the combinatorics of tableaux, and introduce offset vectors and rectangular decomposition of tableaux which have crucial roles in defining κ.
- **Chapter 3** We adopt the notions and results of [3, 5] on the affine RS correspondence, which is needed for the rest of our thesis.
- **Chapter 4** We give a brief review on the representations of quantum groups and crystal theory introduced by [17]. We focus on the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$ and define affine crystal structure on the set of tableaux introduced in Chapter 2.
- **Chapter 5** Using the ingredients give in Chapter 2-4, we construct the affine RSK correspondence κ and show that it is a bijection. We describe the natural $U_q(\widehat{\mathfrak{sl}}_m)$-crystal and $U_q(\widehat{\mathfrak{sl}}_n)$-crystal structure on $\widehat{\mathcal{M}}_{m\times n}$ and state that κ preserves the crystal structures.
- **Chapter 6** We give a dual analogue κ' of κ for $\widehat{\mathcal{N}}_{m\times n}$.
- **Chapter 7** We prove that the crystal equivalences of κ and κ', which needs more technical works.

Chapter 2

Semistandard tableaux

In this chapter, we review the definitions of tableaux and introduce rectangular semistandard tableaux, which are the main objects of this thesis. Throughout this thesis, let $\mathbb{Z}_{\geq 0}$ denote the set of non-negative integers and $[n] = \{ 1, 2, \dots, k \}$ for $n \geq 1$.

A *partition* is a weakly increasing sequence of integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$. We call $l = \ell(\lambda)$ the *length* of λ, and call $|\lambda| = \lambda_1 + \cdots + \lambda_l$ the *size* of λ. We identify a partition λ with its *Young diagram* (cf. [7]) and we denote it by the same notation λ. We denote by $\mu = (\mu_1, \mu_2, \dots, \mu_{\lambda_1})$ the *conjugate partition* of λ where $\mu_j = \#\{ i \mid \lambda_i \geq j \}$. We denote by $\mathscr{P}_k$ the set of partitions of length less than or equal to k.

Let $\mathcal{A}$ denote either $\mathbb{Z}$ or $[n]$, equipped with their usual linear orders. An $\mathcal{A}$-*tableau* of shape λ is a filling T of the Young diagram λ with entries taken from $\mathcal{A}$. We may drop the prefix $\mathcal{A}$ when there is no ambiguity on $\mathcal{A}$. A tableau T of shape λ is said to be *bijective* if its entries are distinct and range from 1 to $|\lambda|$.

Throughout this chapter, let n be a positive integer.

2.1 Column and row semistandard tableaux

In this section, we define column semistandard tableaux and row semistandard tableaux with their standardizations. We also define descents and ascents for bijective tableaux.

Definition 2.1.1. Let T be a tableau of shape λ. We say that T is:

(1) *column semistandard* if the entries in each column are increasing from top to bottom,

(2) *row semistandard* if the entries in each row are weakly increasing from left to right,

(3) *semistandard* if it is both column semistandard and row semistandard.

We denote the set of column semistandard $\mathcal{A}$-tableaux of shape λ by $CSST_{\mathcal{A}}(\lambda)$. When $\mathcal{A} = [n]$, we denote the set of bijective semistandard tableaux of shape λ by $CST_{[n]}(\lambda)$. Similarly, we use the notations $RSST_{\mathcal{A}}(\lambda)$ and $RST_{[n]}(\lambda)$ for row semistandard tableaux.

The *content* of an $[n]$-tableau T is $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ where α_t is the number of occurrences of the entry t in T.

Let T be a column semistandard tableau T with content $\alpha \in \mathbb{Z}^n_{\geq 0}$. We define the *column standardization* $T^{\mathtt{st}}$ of T to be the tableau obtained by replacing each entry $t \in [n]$ in T with $\alpha_t \neq 0$ by the consecutive numbers

$$\alpha_1 + \cdots + \alpha_{t-1} + 1, \dots, \alpha_1 + \cdots + \alpha_t$$

from left to right. Here we understand the empty sum as 0. By definition, $T^{\mathtt{st}}$ is a bijective column semistandard tableau of the same shape.

Let us provide a more explicit description of the image of the column standardization. Let S be a bijective column semistandard $[k]$-tableau of shape λ, where k is the size of λ. We say that $i \in [k]$ is a *(column) descent* of S if the entry $i+1$ appears to be the right of i in S. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ be given with $|\alpha| = \alpha_1 + \cdots + \alpha_n = k$. We say that S is α-*descending* if for any $t \in [n]$ and i with

$$\alpha_1 + \cdots + \alpha_{t-1} + 1 < i < \alpha_1 + \cdots + \alpha_t,$$

i is a descent of S.

Let a partition λ and $\alpha \in \mathbb{Z}^n_{\geq 0}$ be given with $|\alpha| = |\lambda|$. We denote by $CSST_{[n]}(\lambda)_\alpha$ the set of column semistandard tableaux of shape λ with content α, and denote by $CST_{[k],\alpha}(\lambda)$ the set of α-descending bijective column semistandard tableaux of shape λ, where k is the size of λ.

Example 2.1.2. Let $\lambda = (3, 2, 2)$, $n = 5$, $k = 7$, and let

$$T = \begin{array}{|c|c|c|} \hline 2 & 1 & 4 \\ \hline 3 & 4 \\ \cline{1-2} 4 & 5 \\ \cline{1-2} \end{array} \in CSST_{[5]}(\lambda).$$

The content of T is $\alpha = (1, 1, 1, 3, 1)$. We see that the column standardization $T^{\mathtt{st}}$ given by

$$T^{\mathtt{st}} = \begin{array}{|c|c|c|}\hline 2 & 1 & 6 \\ \hline 3 & 5 \\ \cline{1-2} 4 & 7 \\ \cline{1-2}\end{array} \in CST_{[7]}(\lambda),$$

and $T^{\mathtt{st}}$ is α-descending.

The following lemma can be easily checked using the definition of standardization.

Lemma 2.1.3. *Under the above hypothesis, we have a bijection*

$$\begin{array}{ccc} CSST_{[n]}(\lambda)_\alpha & \longrightarrow & CST_{[k],\alpha}(\lambda) \; . \\ T & \longmapsto & T^{\mathtt{st}} \end{array}$$

We can similarly define the *row standardization* $T^{\mathtt{st}'}$ of a row semistandard tableau T with content $\alpha \in \mathbb{Z}^n_{\geq 0}$ to be the tableau obtained by replacing each entry $t \in [n]$ in T with $\alpha_t \neq 0$ by the consecutive numbers

$$\alpha_1 + \cdots + \alpha_{t-1} + 1, \ldots, \alpha_1 + \cdots + \alpha_t$$

from the bottom row to top row, and from left to right in each row.

Let S be a bijective row semistandard $[k]$-tableau of shape λ. We say that $i \in [k]$ is a *(row) ascent* of S if the entry $i+1$ does not appears below i in S. For $\alpha \in \mathbb{Z}^n_{\geq 0}$ with $|\alpha| = k$, we say that S is *α-ascending* if for any $t \in [n]$ and i with

$$\alpha_1 + \cdots + \alpha_{t-1} + 1 < i < \alpha_1 + \cdots + \alpha_t,$$

i is an ascent of S. Let us denote by $RSST_{[n]}(\lambda)_\alpha$ and $RST_{[k],\alpha}(\lambda)$ for the row semistandard tableaux similarly.

Example 2.1.4. Let $\lambda = (3,3,1,1)$, $n = 3$, $k = 8$, and let

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 1 & 1 & 3 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \in RSST_{[3]}(\lambda).$$

The content of T is $\alpha = (3,2,3)$, and its row standardization $T^{\mathtt{st}'}$ is α-ascending as follows

$$T^{\mathtt{st}'} = \begin{array}{|c|c|c|}\hline 3 & 5 & 8 \\ \hline 1 & 2 & 7 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array} \in RST_{[8]}(\lambda),$$

We have the row analogous to Lemma 2.1.3

Lemma 2.1.5. *Under the above hypothesis, we have a bijection*

$$\begin{array}{ccc} RSST_{[n]}(\lambda)_\alpha & \longrightarrow & RST_{[k],\alpha}(\lambda) \\ T & \longmapsto & T^{\mathtt{st}'} \end{array}.$$

2.2 Offset vectors

In this section, we introduce maps τ and τ' on tableaux that shift the entries in each column and row of tableaux respectively. We define offset vectors for tableaux to be the minimal vector that make the given tableau to be semistandard via the map τ and τ'.

Let

$$\mathcal{B}_n((1^b)) = \left\{ T \in CSST_{\mathbb{Z}}((1^b)) \,\middle|\, T(b) - T(1) < n \right\}$$

where $T(i)$ denotes the entry in T at the i-th row from the top. Note that $\mathcal{B}_n((1^b))$ is empty unless $b \leq n$. For $T \in \mathcal{B}_n((1^b))$, let $\tau_n(T)$ be the tableau obtained by replacing its entries

$$T(1) < T(2) < \cdots < T(b-1) < T(b)$$

with

$$T(2) < T(3) < \cdots < T(b) < T(1) + n.$$

Then τ_n is a bijection on $\mathcal{B}_n((1^b))$. We may write simply $\mathcal{B}_n((1^b)) = \mathcal{B}((1^b))$ and $\tau_n = \tau$ if there is no ambiguity on n. In general, for $\alpha = (\alpha_1, \ldots, \alpha_a) \in \mathbb{Z}^a$, we define the bijection τ^α on $\mathcal{B}((1^b))^a$ which acts on the j-th factor by τ^{α_j}.

Let $R = (a^b) = (\underbrace{a, \ldots, a}_{b})$ be a Young diagram of rectangular shape. Let us regard

$CSST_{[n]}(R)$ as the subset of $\mathcal{B}((1^b))^a$ via the embedding

$$\begin{array}{rcl} CSST_{[n]}(R) & \longrightarrow & \mathcal{B}((1^b))^a \quad , \\ T & \longmapsto & (T^a \ldots, T^1) \end{array}$$

where T^j is the j-th column of T from the right.

Definition 2.2.1. Let T be a column semistandard $[n]$-tableau of shape $R = (a^b)$ with $b \leq n$. For $1 \leq j < a$, let r_j be the minimal non-negative integer such that the tableau

$$(T^{j+1}, \tau^{r_j}(T^j))$$

of shape (2^b) is semistandard, and let $\eta_j = r_j + r_{j+1} + \cdots r_{a-1}$. We call $(r_1, \ldots, r_{a-1})$ the *offset vector* for T, and $\eta = (\eta_1, \ldots, \eta_{a-1})$ the *symmetrized offset vector* for T.

It is obvious that $\eta_{\mathtt{rev}} = (0, \eta_{a-1}, \ldots, \eta_1)$ is the unique vector in $\mathbb{Z}^a_{\geq 0}$ of minimal size such that $\tau^{\eta_{\mathtt{rev}}}(T)$ is semistandard.

We can similarly define the offset vector for row semistandard tableaux. Let

$$\mathcal{B}'((a)) = \left\{ T \in RSST_{\mathbb{Z}}((a)) \,\middle|\, T(a) - T(1) \leq n \right\}$$

where $T(j)$ denotes the entry in T at the j-th column from the left. We define the bijection τ' on $\mathcal{B}'((a))$ by $\tau'(T)$ to be the tableau obtained by replacing the entries

$$T(1) \leq T(2) \leq \cdots \leq T(a-1) \leq T(a)$$

of T with

$$T(2) \leq T(3) \leq \cdots \leq T(a) \leq T(1) + n,$$

and define τ'^{α} on $\mathcal{B}'((a))^b$ similarly for $\alpha \in \mathbb{Z}^b$ in general.

We regard $RSST_{[n]}(R)$ as the subset of $\mathcal{B}'((a))^b$ via the embedding

$$\begin{array}{rcl} RSST_{[n]}(R) & \longrightarrow & \mathcal{B}'((a))^b \quad , \\ T & \longmapsto & (T^b \ldots, T^1) \end{array}$$

where T^i is the i-th row of T from the bottom.

We have row analogous to Definition 2.2.1.

Definition 2.2.2. Let $T \in RSST_{[n]}(R)$ be given. For $1 \leq j < a$, let r_j be the minimal non-negative integer such that the tableau

$$(T^{j+1}, \tau'^{r_j}(T^j))$$

of shape $(a)^2$ is semistandard, and let $\eta_j = r_j + r_{j+1} + \cdots r_{a-1}$. We call $(r_1, \ldots, r_{a-1})$ the *offset vector* for T, and $\eta = (\eta_1, \ldots, \eta_{a-1})$ the *symmetrized offset vector* for T.

Lemma 2.2.3. *Let T be a row semistandard $[n]$-tableau of shape $R = (a^b)$. Then the offset vectors for T and $T^{\mathsf{st}'}$ coincide.*

Example 2.2.4.

(1) Let $n = 5$ and let

$$T = \begin{array}{|c|c|c|}\hline 2 & 1 & 2 \\ \hline 3 & 4 & 3 \\ \hline 4 & 5 & 5 \\ \hline \end{array} \in CSST_{[5]}((3^3)).$$

We have $r_1 = 1$ since

$$(T^2, T^1) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline 5 & 5 \\ \hline \end{array}$$

is not semistandard, while

$$(T^2, \tau^1(T^1)) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 & 5 \\ \hline 5 & 7 \\ \hline \end{array}$$

is semistandard. We see that the offset vector for T is $(1, 1)$, the symmetrized offset vector for T is $\eta = (2, 1)$ so that

$$\tau^{\eta_{\mathsf{rev}}}(T) = (T^3, \tau^1(T^2), \tau^2(T^1)) = \begin{array}{|c|c|c|}\hline 2 & 4 & 5 \\ \hline 3 & 5 & 7 \\ \hline 4 & 6 & 10 \\ \hline \end{array}$$

is semistandard.

(2) Let $m = 3$ and let

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 1 & 2 & 2 \\ \hline 2 & 2 & 3 \\ \hline \end{array} \in RSST_{[3]}((3^3)).$$

Then the symmetrized offset vector for T is $\eta = (3, 2)$ so that

$$\tau'^{\eta_{\mathtt{rev}}}(T) = (T^3, \tau^2(T^2), \tau^3(T^1)) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 4 & 5 \\ \hline 5 & 5 & 6 \\ \hline \end{array}$$

is semistandard.

2.3 Rectangular semistandard tableaux

In this section, we introduce rectangular semistandard tableaux, which will play a crucial role in this thesis. Let $R = (a^b)$ of rectangular shape with $b \leq n$.

Definition 2.3.1. Let $R = (a^b)$ be a partition with $b \leq n$. We say that a semistandard $\mathbb{Z}$-tableau $T = (T^a, \ldots, T^1)$ of shape R is *rectangular semistandard* if $T^j \in \mathcal{B}((1^b))$ for each $1 \leq j \leq a$ where T^J is the j-th column of T from the right. We denote the set of rectangular semistandard tableaux of shape R by $\mathcal{B}(R)$.

Regarding $\mathcal{B}(R)$ as a subset of $\mathcal{B}((1^b))^a$, we define τ^α on $\mathcal{B}(R)$ for $\alpha \in \mathbb{Z}^a$, similar to Section 2.2. Let

$$\mathcal{B}(R)_0 = \left\{ T \in \mathcal{B}(R) \;\middle|\; (T^{j+1}, \tau^{-1}(T^j)) \text{ is not semistandard for } 1 \leq j \leq a-1 \right\}.$$

Recall that $\mathscr{P}_{a-1}$ is the set of partitions of length less than a. For $\nu = (\nu_1, \ldots, \nu_{a-1}) \in \mathscr{P}_{a-1}$, we write $\nu_{\mathtt{rev}} = (0, \nu_{a-1}, \ldots, \nu_1) \in \mathbb{Z}^a_{\geq 0}$. Then we have a bijection

$$\begin{array}{ccc} \mathcal{B}(R)_0 \times \mathscr{P}_{a-1} & \longrightarrow & \mathcal{B}(R) \\ (T, \nu) & \longmapsto & \tau^{\nu_{\mathtt{rev}}}(T) \end{array} . \tag{2.3.1}$$

Lemma 2.3.2. *Let $T \in CSST_{[n]}(R)$ and $\alpha = (\alpha_1, \ldots, \alpha_a) \in \mathbb{Z}^a$ be given and let $\alpha_{\mathtt{rev}} = (\alpha_a, \ldots, \alpha_1)$. Then $\tau^{\alpha_{\mathtt{rev}}}(T) \in \mathcal{B}(R)$ if and only if*

$$\alpha_a \leq \alpha_{a-1} - \eta_{a-1} \leq \cdots \leq \alpha_1 - \eta_1,$$

where η is the symmetrized offset vector for T.

Proof. It follows immediately from that η is the unique partition of length less than a such that $\tau^{\eta_{\mathtt{rev}}}(T) \in \mathcal{B}(R)_0$. $\square$

By Lemma 2.3.2, we have a bijection

$$\begin{aligned} CSST_{[n]}(R) \times \mathcal{P}_a &\longrightarrow \mathcal{B}(R) , \\ (T,\nu) &\longmapsto \tau^{\nu_{\mathrm{rev}}+\eta_{\mathrm{rev}}}(T) \end{aligned} \tag{2.3.2}$$

where $\mathcal{P}_a = \{\, \nu = (\nu_1, \dots, \nu_a) \in \mathbb{Z}^a \,|\, \nu_1 \geq \cdots \geq \nu_a \,\}$ is the set of generalized partitions of length a and η is the symmetrized offset vector for T.

Let

$$\begin{aligned} \boldsymbol{\tau} : \mathcal{B}(R) &\longrightarrow \mathcal{B}(R) \\ T = (T^a, \dots, T^1) &\longmapsto (\tau(T^a), \dots, \tau(T^1)) \end{aligned} \tag{2.3.3}$$

be the bijection given by applying τ to each column of the tableaux in $\mathcal{B}(R)$, which induces a $\mathbb{Z}$-action on $\mathcal{B}(R)$ and $\mathcal{B}(R)_0$. Let $\mathcal{B}(R)_0/\mathbb{Z}$ denote the set of equivalence classes under this $\mathbb{Z}$-action. We identify $\mathcal{B}(R)_0/\mathbb{Z}$ with the set of $T \in \mathcal{B}(R)_0$ such that the first column of T has entries in $[n]$. Hence, we have another bijection

$$\begin{aligned} CSST_{[n]}(R) &\longrightarrow \mathcal{B}(R)_0/\mathbb{Z} , \\ T &\longmapsto [\tau^{\eta_{\mathrm{rev}}}(T)] \end{aligned} \tag{2.3.4}$$

where $[T]$ denotes the equivalence class of T and η is the symmetrized offset vector for T.

The following lemma implies that the symmetrized offset vector of T is invariant under standardization.

Lemma 2.3.3. *Let $T \in CSST_{[n]}(R)$ be given where $R = (a^b) \in \mathscr{P}_n$ for some $a, b \geq 1$. For $\nu \in \mathscr{P}_{a-1}$, we have*

$$\tau_n^{\nu_{\mathrm{rev}}}(T) \in \mathcal{B}_n(R)_0 \quad \textit{if and only if} \quad \tau_k^{\nu_{\mathrm{rev}}}(T^{\mathtt{st}}) \in \mathcal{B}_k(R)_0,$$

where $k = ab$.

Proof. Let $T = (T^a, \dots, T^1)$ and $T^{\mathtt{st}} = S = (S^a, \dots, S^1)$. For $1 \leq j \leq a-1$, let r_j be the smallest integer satisfying $(T^{j+1}, \tau_n^{r_j}(T^j))$ is semistandard. It suffices to show that $(S^{j+1}, \tau_K^{r_j}(S^j))$ is semistandard but $(S^{j+1}, \tau_K^{r_j-1}(S^j))$ is not. It is straightforward to see from the definition of $T^{\mathtt{st}}$. $\square$

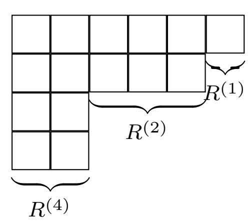
Closing this section, we note that while the results of this section have row counterparts of rectangular semistandard tableaux, they do not have interesting implications in terms of representation or crystal theory (cf. 4.2.5). Therefore, we only focus on the column semistandard case in the rest of the thesis.

2.4 Rectangular decomposition

Let λ be a partition. We decompose λ into its subdiagrams of rectangular shapes $R^{(i)}$ defined by

$$R^{(i)} = (\underbrace{m_i, \dots, m_i}_{i}) \quad (1 \leq i \leq l),$$

where m_i is the number of occurrences of i in μ and $l = \ell(\lambda)$. Here we assume that $R^{(i)}$ is empty when $m_i = 0$. For example, if $\lambda = (6, 4, 1, 1)$, then we see that $R^{(1)} = (1)$, $R^{(2)} = (3^2)$, $R^{(3)} = \emptyset$ and $R^{(4)} = (2^4)$ as illustrated in the following figure.



For a tableau T of shape λ, we denote by $T^{(i)}$ the subtableau of T corresponding to the subdiagram $R^{(i)}$ of λ. We call $(R^{(1)}, \dots, R^{(l)})$ and $(T^{(1)}, \dots, T^{(l)})$ the *rectangular decompositions* of λ and T respectively.

Definition 2.4.1. Let λ be a partition of length less than or equal to n, with the rectangular decomposition $(R^{(1)}, \dots, R^{(l)})$. We say that a tableau T of shape λ is *rectangular semistandard* if $T^{(i)} \in \mathcal{B}(R^{(i)})$ for each $1 \leq i \leq l$. We denote the set of rectangular semistandard tableaux of shape λ by $\mathcal{B}(\lambda)$.

Note that $\mathcal{B}(\lambda)$ is empty when $\ell(\lambda) > n$. Let us identify

$$\begin{aligned} \mathcal{B}(\lambda) &= \mathcal{B}(R^{(1)}) \times \cdots \times \mathcal{B}(R^{(l)}), \\ CSST_{[n]}(\lambda) &= CSST_{[n]}(R^{(1)}) \times \cdots \times CSST_{[n]}(R^{(l)}) \end{aligned} \tag{2.4.1}$$

via rectangular decompositions and define

$$\mathcal{B}(\lambda)_0 = \mathcal{B}(R^{(1)})_0 \times \cdots \times \mathcal{B}(R^{(l)})_0.$$

If we put $\mathscr{P}(\lambda) = \mathscr{P}_{m_l-1} \times \cdots \times \mathscr{P}_{m_2-1} \times \mathscr{P}_{m_1-1}$, where we take the product over $m_i \geq 1$, then we have a bijection

$$\begin{array}{ccc} \mathcal{B}(\lambda)_0 \times \mathscr{P}(\lambda) & \longrightarrow & \mathcal{B}(\lambda) \\ \Big(\big(T^{(i)}\big)_{1\leq i\leq l}, (\nu^{(i)})_{1\leq i\leq l} \Big) & \longmapsto & \Big(\tau^{\nu^{(i)}_{\mathtt{rev}}}(T^{(i)}) \Big)_{1\leq i\leq l} \end{array}, \tag{2.4.2}$$

by applying (2.3.1) to each component, where $T^{(i)} \in \mathcal{B}(R^{(i)})_0$ and $\nu^{(i)} \in \mathscr{P}_{m_i-1}$.

Similarly, if we let $\mathcal{P}(\lambda) = \mathcal{P}_{m_l} \times \cdots \times \mathcal{P}_{m_1}$, where we take the product over $m_i \geq 1$, and regard

$$CSST_{[n]}(\lambda) = CSST_{[n]}(R^{(l)}) \times \cdots \times CSST_{[n]}(R^{(1)}),$$

then by (2.3.3) we have a bijection

$$\begin{array}{ccc} CSST_{[n]}(\lambda) \times \mathcal{P}(\lambda) & \longrightarrow & \mathcal{B}(\lambda) \\ \Big(\big(T^{(i)}\big)_{1\leq i\leq l}, (\nu^{(i)})_{1\leq i\leq l} \Big) & \longmapsto & \Big(\tau^{\nu^{(i)}_{\mathtt{rev}}+\eta^{(i)}_{\mathtt{rev}}}(T^{(i)}) \Big)_{1\leq i\leq l} \end{array}, \tag{2.4.3}$$

where $\eta^{(i)} \in \mathscr{P}_{m_i-1}$ is the symmetrized offset vector for $T^{(i)}$

Let

$$\mathcal{B}(\lambda)_0/\mathbb{Z}^l = \mathcal{B}(R^{(1)})_0/\mathbb{Z} \times \cdots \times \mathcal{B}(R^{(l)})_0/\mathbb{Z},$$

where each $\mathcal{B}(R^{(i)})_0/\mathbb{Z}$ is the set of equivalence classes under the $\mathbb{Z}$-action (2.3.3). Then we also have a bijection

$$\begin{array}{ccc} CSST_{[n]}(\lambda) & \longrightarrow & \mathcal{B}(\lambda)_0/\mathbb{Z}^l \\ \big(T^{(i)}\big)_{1\leq i\leq l} & \longmapsto & \Big(\big[\tau^{\eta^{(i)}_{\mathtt{rev}}}(T^{(i)})\big] \Big)_{1\leq i\leq l} \end{array}.$$

Example 2.4.2. Let $\lambda = (5,5,2)$, $n = 4$, and let

$$T = \begin{array}{|c|c|c|c|c|} \hline 2 & 1 & 2 & 1 & 2 \\ \hline 3 & 2 & 3 & 3 & 3 \\ \hline 4 & 4 \\ \cline{1-2} \end{array} \in CSST_{[4]}(\lambda).$$

Then λ is decomposed into $R^{(2)} = (3,3)$ and $R^{(3)} = (2,2,2)$, and the corresponding decompositions of T are

$$T^{(2)} = \begin{array}{|c|c|c|} \hline 2 & 1 & 2 \\ \hline 3 & 3 & 3 \\ \hline \end{array}, \quad T^{(3)} = \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 3 & 2 \\ \hline 4 & 4 \\ \hline \end{array}$$

We see that $\eta^{(2)} = (1,1)$, $\eta^{(3)} = (1)$ are the symmetrized offset vectors for $T^{(2)}$, $T^{(3)}$ respectively. For $\nu = ((1,-1),((2,1,0)) \in \mathcal{P}(\lambda)$, the image of (T,ν) under the bijection (2.4.3) is

$$\begin{array}{|c|c|c|c|c|} \hline 0 & 4 & 2 & 5 & 7 \\ \hline 2 & 5 & 3 & 7 & 10 \\ \hline 3 & 6 \\ \cline{1-2} \end{array} \in \mathcal{B}(\lambda).$$

Chapter 3

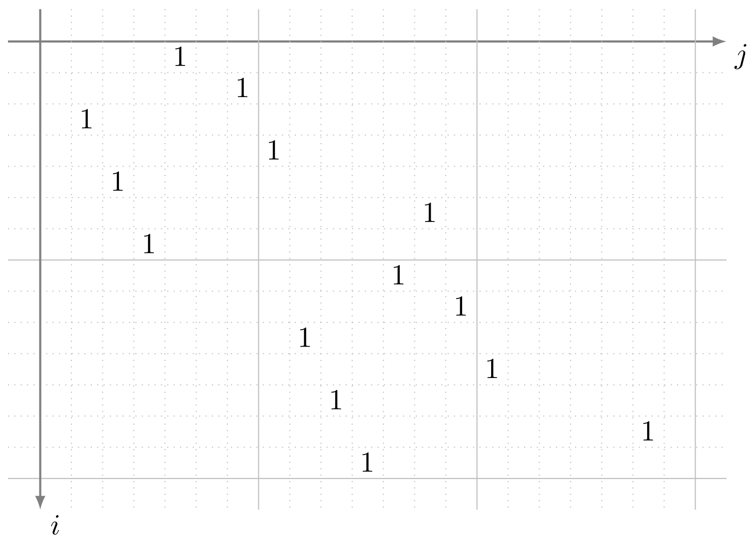
Affine RS correspondence and matrix-ball construction

In this chapter, we provide a brief review of the affine RS correspondence described in [5], while adopting the necessary notations and terminologies with slight modifications. We refer to the statements in the literature by their precise numbers and omit their proofs.

Let k be a fixed positive integer. An *(extended) affine permutation* of k is a bijection $w : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that

$$w(i + k) = w(i) + k$$

for each i. We denote by $\widehat{\mathcal{W}}_k$ the set of affine permutations of k. If $k \geq 2$, it is an extended affine Weyl group of type $A(1)_{k-1}$. We may represent an affine permutation $w \in \widehat{\mathcal{W}}_k$ by the *window notation* $w = [w_1, \ldots, w_k]$, where $w_i = w(i)$ for each i, or by the *matrix representation* $w = (w_{ij})_{i,j\in\mathbb{Z}}$, where $w_{ij} = \delta_{w(i)j}$ and $\delta_{w(i)j}$ is the Kronecker delta. For example, we visualize the matrix representation of $w = [5, 7, 2, 8, 3, 13, 4] \in \widehat{\mathcal{W}}_7$ as follows.



Here, we only draw nonzero entries in $1 \le i \le 14$ and $1 \le j \le 21$. Note that a matrix representation w satisfies $\sum_{i\in\mathbb{Z}} w_{ij} = 1$ and $\sum_{j\in\mathbb{Z}} w_{ij} = 1$ for each i, j. Generally, a matrix $w = (w_{ij})_{i,j\in\mathbb{Z}}$ is called a *partial (extended) affine permutation* of k if $w_{ij} \in \mathbb{Z}$, $w_{i+k,\,j+k} = w_{ij}$, $\sum_{i\in\mathbb{Z}} w_{ij} \le 1$, and $\sum_{j\in\mathbb{Z}} w_{ij} \le 1$ for each i, j. We also use the window notation $w = [w_1, \ldots, w_k]$ for a partial affine permutation $w = (w_{ij})$, where we write $w_i = j$ if $w_{ij} = 1$, or $w_i = \cdot$ if $w_{ij} = 0$ for all j. We denote by $\varnothing$ the empty partial permutation.

Consider the lattice $\mathbb{Z} \times \mathbb{Z}$ as the set of matrix co-ordinates. An element $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ is called a *cell*. We use compass directional orders $>_{\mathtt{NW}}$, $\geq_{\mathtt{nw}}$ and $\leq_{\mathtt{ne}}$ on $\mathbb{Z} \times \mathbb{Z}$ as follows:

(1) $c_1 >_{\mathtt{NW}} c_2$ if and only if $i_1 < i_2$ and $j_1 < j_2$,

(2) $c_1 \geq_{\mathtt{nw}} c_2$ if and only if $i_1 \le i_2$ and $j_1 \le j_2$,

(3) $c_1 \leq_{\mathtt{ne}} c_2$ if and only if $i_1 \ge i_2$ and $j_1 \le j_2$,

for $c_1 = (i_1, j_1), c_2 = (i_2, j_2) \in \mathbb{Z} \times \mathbb{Z}$. By convention, we use $\mathtt{N}$ (or $\mathtt{E}, \mathtt{W}, \mathtt{S}$) to emphasize strict inequality, while $\mathtt{n}$ (or $\mathtt{e}, \mathtt{w}, \mathtt{s}$) allows equality of the co-ordinates of cells.

Throughout this chapter, we assume that $w = (w_{ij})$ is a non-empty partial affine permutation of k.

3.1 Southwest channel numberings

To perform the matrix-ball construction in the affine case, a certain ordering of the positions of 1's is required. One of the notable results of [5] is that they single out a numbering on the positions of 1's which exhibits desirable properties.

Let

$$\operatorname{supp}(w) = \{ (i,j) \in \mathbb{Z} \times \mathbb{Z} \mid w_{ij} = 1 \}$$

be the support of w. It is invariant under the translation $\tau = \tau_{k,k}$ on $\mathbb{Z} \times \mathbb{Z}$ given by

$$\tau(i,j) = (i+k, j+k) \quad ((i,j) \in \mathbb{Z} \times \mathbb{Z}).$$

A *numbering on w* is a function $d : \operatorname{supp}(w) \longrightarrow \mathbb{Z}$.

Definition 3.1.1 (cf. [5, Definition 3.1])**.** A numbering d on w is called *proper* if

(1) $d(c_2) < d(c_1)$ if $c_2 >_{\texttt{NW}} c_1$,

(2) for any $c_1 \in \operatorname{supp}(w)$, there exists $c_2 \in \operatorname{supp}(w)$ such that $c_2 >_{\texttt{NW}} c_1$ and $d(c_2) = d(c_1) - 1$.

The conditions (1) and (2) are called *monotone* and *continuous*, respectively. We remark that any proper numbering can be shifted by any integer.

Lemma 3.1.2 (cf. [5, Proposition 3.4])**.**

(1) For any proper numbering d on w, there exists a positive integer ℓ, which we call the period*, such that $d(\tau(c)) = d(c) + \ell$ for $c \in \operatorname{supp}(w)$.*

(2) If ℓ and ℓ' are the periods of any two proper numberings d and d' on A, respectively, then we have $\ell = \ell'$, which we call the width *of w.*

Definition 3.1.3 (cf. [5, Definitions 3.6, 3.20])**.** A *stream* is an infinite collection of cells $\mathbf{s} = \{c_i\}_{i \in \mathbb{Z}}$, which is invariant under τ and forms a chain with respect to $>_{\texttt{NW}}$, that is, $c_i >_{\texttt{NW}} c_{i+1}$ for all i. A *flow of a stream* $\mathbf{s}$ is the number ℓ such that $\tau(c_i) = c_{i+\ell}$ for all i. A *defining data of a stream* $\mathbf{s} = \{c_i = (a_i, b_i)\}_{i \in \mathbb{Z}}$ of flow ℓ is a triple $(\mathbf{a}, \mathbf{b}, r)$, where

(1) $\mathbf{a} = (a_{1+r_1}, \dots, a_{\ell+r_1}) \in [k]^l$ with $1 \leq a_{1+r_1} < \cdots < a_{\ell+r_1} \leq k$,

(2) $\mathbf{b} = (b_{1+r_2}, \dots, b_{\ell+r_2}) \in [k]^l$ with $1 \leq b_{1+r_2} < \cdots < b_{\ell+r_2} \leq k$,

(3) $r = r_1 - r_2$.

A stream $\mathbf{s}$ is called a *stream of* w if $\mathbf{s} \subset \operatorname{supp}(w)$. A stream $\mathbf{s}$ of w is called a *channel of* w if its flow is maximal among the streams of w.

Let $C = \{c_i\}_{i\in\mathbb{Z}}$ be a channel of w and let $c \in \operatorname{supp}(w)$. Let m be the maximal integer such that $c_m >_{\mathtt{NW}} c$. The maximal property of channel ensures that $c \not>_{\mathtt{NW}} c_{m+1}$. This implies either $c_{m+1} \leq_{\mathtt{ne}} c$ or $c \leq_{\mathtt{ne}} c_{m+1}$. In other words, we have

$$\operatorname{supp}(w) = C_{\mathtt{ne}} \cup C_{\mathtt{sw}},$$

where

$$\begin{aligned} C_{\mathtt{ne}} &= \{\, c \in \operatorname{supp}(w) \,|\, c' \leq_{\mathtt{ne}} c \text{ for some } c' \in C \,\}, \\ C_{\mathtt{sw}} &= \{\, c \in \operatorname{supp}(w) \,|\, c \leq_{\mathtt{ne}} c' \text{ for some } c' \in C \,\}, \end{aligned} \tag{3.1.1}$$

and $C_{\mathtt{ne}} \cap C_{\mathtt{sw}} = C$. Let $\mathcal{C}_w$ denote the set of channels of w. We define a partial order $\succcurlyeq_{\mathtt{sw}}$ on $\mathcal{C}_w$ by

$$C_1 \succcurlyeq_{\mathtt{sw}} C_2 \text{ if and only if } C_1 \subset (C_2)_{\mathtt{sw}}. \tag{3.1.2}$$

for $C_1, C_2 \in \mathcal{C}_w$.

Proposition 3.1.4 (cf. [5, Proposition 3.14])**.** *The set $\mathcal{C}_w$ has a greatest element with respect to $\succcurlyeq_{\mathtt{sw}}$, which we denote by $C_w^{\mathtt{sw}}$.*

We call $C_w^{\mathtt{sw}}$ the *southwest channel of* w.

Let $C = \{c_i\}_{i\in\mathbb{Z}}$ be a channel of w, and let d_0 be the numbering on C defined by $d_0(c_i) = i$. For $c \in \operatorname{supp}(w)$, we define

$$d_w^C(c) = \sup\left\{ d_0(c'_k) + l \;\middle|\; \begin{array}{c} c'_l >_{\mathtt{NW}} \cdots >_{\mathtt{NW}} c'_0 \text{ is a chain in } \operatorname{supp}(w) \\ (k \geq 0) \text{ such that } c'_0 = c \text{ and } c'_l \in C \end{array} \right\}. \tag{3.1.3}$$

We call d_w^C the *channel numbering* on w with respect to C.

Proposition 3.1.5 (cf. [5, Proposition 3.10])**.** *The numbering d_w^C on w is a well-defined proper numbering. Moreover, we have $d_w^C(c) = d_0(c)$ for $c \in C$.*

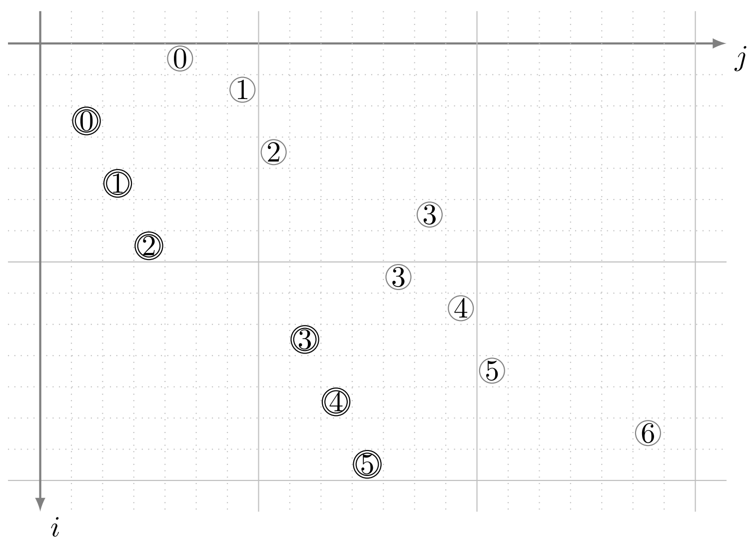
Note that the width of w or the period of the channel numbering d_w^C is equal to the flow of a channel C of w.

Definition 3.1.6. For the southwest channel $C_w^{\mathtt{sw}}$ of w, we write $d_w^{\mathtt{sw}} = d_w^{C_w^{\mathtt{sw}}}$ for short, and call it the *southwest channel numbering on* w.

Example 3.1.7. Let $w = [5, 7, 2, 8, 3, 13, 4] \in \widehat{\mathcal{W}}_7$. The southwest channel of w is given by

$$C_w^{\mathtt{sw}} = \left\{ \cdots >_{\mathtt{NW}} \tau^{-1}(7,4) >_{\mathtt{NW}} (3,2) >_{\mathtt{NW}} (5,3) >_{\mathtt{NW}} (7,4) >_{\mathtt{NW}} \tau(3,2) >_{\mathtt{NW}} \cdots \right\}.$$

In the following diagram, we draw a ball for each $c \in \operatorname{supp}(w)$ and fill in the value of the southwest channel numbering $d_w^{\mathtt{sw}}(c)$ for each ball. The balls that form the southwest channel are doubly circled.



Remark 3.1.8. Consider the southwest channel $C_{w^{-1}}^{\mathtt{sw}}$ of the (partial) inverse permutation W^{-1} of w. Then the channel

$$C_w^{\mathtt{ne}} = \{ (i,j) \mid (j,i) \in C_{w^{-1}}^{\mathtt{sw}} \}$$

is the minimal element in $\mathcal{C}_w$ with respect to $\succcurlyeq_{\mathtt{sw}}$. We call $C_w^{\mathtt{ne}}$ the *northeast channel* of w. Let $d_w^{\mathtt{ne}}$ be the channel numbering on w with respect to $C_A^{\mathtt{ne}}$. Then it follows from definition that $d_{w^{-1}}^{\mathtt{sw}}(j,i) = d_w^{\mathtt{ne}}(i,j)$ for $(j,i) \in \operatorname{supp}(w^{-1})$.

The following lemma gives a characterization of channel numberings (cf. [5, Remark 11.8]).

Lemma 3.1.9. *Let C be a channel of w. Let d be a proper numbering on w such that $d(c) = d_w^C(c)$ for $c \in C$. Then the following are equivalent:*

(1) $d = d_w^C$,

(2) for $c \in \operatorname{supp}(w)$, *there exists a chain* $c_l >_{\mathtt{NW}} \cdots >_{\mathtt{NW}} c_0$ *in* $\operatorname{supp}(w)$ *such that* $c_0 = c$, $c_l \in C$ *and* $d(c_i) = d(c) - i$ *for* $0 \leq i \leq l$,

(3) if d' *is a proper numbering such that* $d'(c) = d_w^C(c)$ *for* $c \in C$, *then we have* $d(c) \leq d'(c)$ *for every* $c \in \operatorname{supp}(w)$.

Proof. Suppose that (1) holds. Let $c_l >_{\mathtt{NW}} \cdots >_{\mathtt{NW}} c_0 = c$ be a chain which gives the maximum value $d_0(c_l) + l$ in (3.1.3). Since d is monotone, we have

$$d(c_l) + l \leq d(c_{l-1}) + l - 1 \leq \cdots \leq d(c) = d_0(c_l) + l. \tag{3.1.4}$$

Since $c_l \in C$, we have $d(c_l) = d_0(c_l)$ by Proposition 3.1.5. Thus all the inequalities in (3.1.4) are in fact equalities and hence, $d(c_i) = d(c) - i$ for $0 \leq i \leq l$. This implies (2).

Suppose that (2) holds. For $c \in \operatorname{supp}(w)$, let $c_l >_{\mathtt{NW}} \cdots >_{\mathtt{NW}} c_0 = c$ be a chain satisfying the condition in (2). Let d' be a proper numbering such that $d' = d_w^C$ on C. Along this chain, we have

$$d'(c_l) + l \leq d'(c_{l-1}) + l - 1 \leq \cdots \leq d'(c).$$

from the monotonicity of d'. Since $d'(c_l) = d(c_l)$, we conclude that $d(c) = d(c_l) + l = d'(c_l) + k \leq d'(c)$. This implies (3).

Suppose that (3) holds. Then, in particular, we have $d(c) \leq d_w^C(c)$ for $c \in \operatorname{supp}(w)$ by letting $d' = d_w^C$. Let $c_l >_{\mathtt{NW}} \cdots >_{\mathtt{NW}} c_0 = c$ be a chain which gives the maximal value $d_w^C(c)$. We have $d(c_l) + l \leq d(c_{l-1}) + l - 1 \leq \cdots \leq d(c)$ from the monotonicity of d. Then we see that $d_w^C(c) = d_w^C(c_l) + l = d(c_l) + l \leq d(c)$. Hence $d(c) = d_w^C(c)$. $\square$

By Lemma 3.1.9, we regard the channel numbering d_w^C as the proper numbering with minimal values among the proper numberings d which coincide with d_0 on w.

3.2 Matrix-ball construction for affine permutations

Definition 3.2.1. A *zig-zag* is an infinite collection $\mathbf{z} = \{c_i\}_{i \in \mathbb{Z}}$ of cells such that the following hold:

- for each $i \in \mathbb{Z}$, c_{i+1} is the adjacent east or north cell of c_i,
- c_{i+1} is the adjacent east cell of c_i if $i \gg 0$,
- c_{i-1} is the adjacent south cell of c_i if $i \ll 0$,

For a zig-zag $\mathbf{z} = \{c_i\}_{i\in\mathbb{Z}}$, we say that

- the *inner corners* are the cells c_i such that c_{i-1} is located to the south and c_{i+1} is located to the east of c_i,
- the *outer corners* are the cells c_i such that c_{i-1} is located to the west and c_{i+1} is located to the north of c_i,
- the *back-post corner* is the cell $c = (i_r, j_l)$ when $c_l = (i_l, j_l)$ and $c_r = (i_r, j_r))$ are the leftmost and the rightmost inner corners of $\mathbf{z}$, respectively.

Let d be a proper numbering on a partial permutation w of k. We associate a set of zig-zags $Z_d = \{\mathbf{z}_i\}_{i\in\mathbb{Z}}$ to d, where $\mathbf{z}_i$ is the unique zig-zag whose inner corners form the level set $d^{-1}(i)$. It is straightforward to see that $\{\mathbf{z}_i\}_{i\in\mathbb{Z}}$ satisfies

(z.1) $\tau(\mathbf{z}_i) = \mathbf{z}_{i+\ell}$ where ℓ is the period of d,

(z.2) the inner corners of each $\mathbf{z}_i$ are contained in $\mathrm{supp}(w)$,

(z.3) $\mathbf{z}_i$'s are mutually disjoint and $\mathrm{supp}(w) \subset \bigsqcup_{i\in\mathbb{Z}} \mathbf{z}_i$,

(z.4) $\mathbf{z}_i$ is located to the southeast of $\mathbf{z}_{i-1}$ for $i \in \mathbb{Z}$ in the sense that

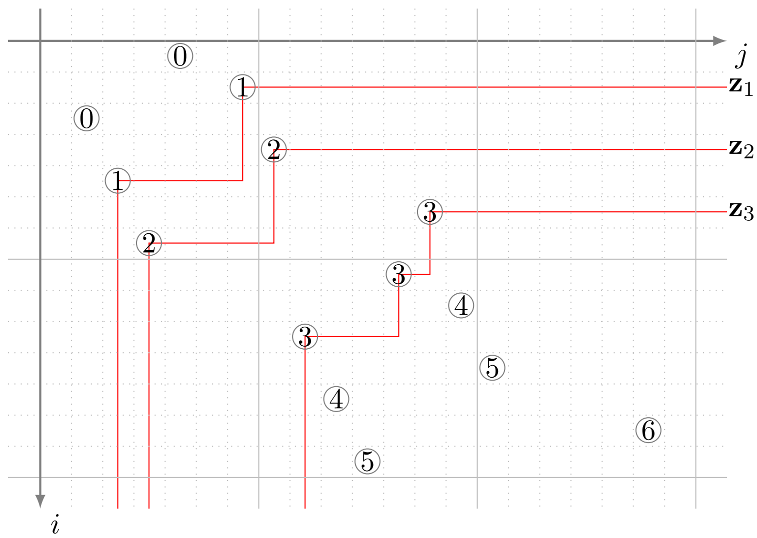
$$\text{for each } c_1 \in \mathbf{z}_i, \text{ there exists } c_2 \in \mathbf{z}_{i-1} \text{ such that } c_2 >_{\mathtt{NW}} c_1. \tag{3.2.1}$$

Conversely, a set of zig-zags $Z = \{\mathbf{z}_i\}_{i\in\mathbb{Z}}$ satisfying (z.1)-(z.4) determines a unique proper numbering d^Z on w given by

$$d^Z(c) = i \quad \text{if } c \in \mathrm{supp}(w) \cap \mathbf{z}_i. \tag{3.2.2}$$

whose associated set of zig-zags is Z. Note that the set of back-corner posts of $\mathbf{z}_i$'s form a stream of flow ℓ.

Example 3.2.2. Let w and $d_w^{\mathtt{sw}}$ be as in Example 3.1.7. The period of $d_w^{\mathtt{sw}}$ is 3. The zig-zags $\mathbf{z}_k$ corresponding to the level sets $(d_w^{\mathtt{sw}})^{-1}(i)$ for $i = 1, 2, 3$ are given as red lines below.



Let w be a non-empty partial permutation of k, and let $\{\mathbf{z}_i\}_{i\in\mathbb{Z}}$ be the set of zig-zags associated to $d_w^{\mathtt{sw}}$. We define

- $w^\flat$: the unique partial permutation of k such that $\mathrm{supp}(w^\flat)$ consists of outer corners of $\mathbf{z}_i$'s,
- $\mathbf{s}(w)$: the stream consisting of the back-post corners of $\mathbf{z}_i$'s.

The matrix-ball construction for affine permutations can be described as follows:

- $w^{(0)} = w$,
- $w^{(t)} = (w^{(t-1)})^\flat$,
- $\mathbf{s}^{(t)} = \mathbf{s}(w^{(t-1)})$,
- μ_t: the flow of $\mathbf{s}^{(t)}$, or equivalently, the width of $w^{(t-1)}$,
- $(\mathbf{a}_t, \mathbf{b}_t, \rho_t)$: the defining data of $\mathbf{s}^{(t)}$,

for $t \geq 0$. It is obvious that there exists $s \geq 1$ such that

$$w^{(s-1)} \neq \varnothing, \quad w^{(s)} = \varnothing.$$

Lemma 3.2.3 (cf. [5, Proposition 3.10])**.** *We have* $\mu_1 \geq \cdots \geq \mu_s > 0$.

Now we let

- $\lambda = \mu'$: the conjugate partition of $\mu = (\mu_1, \ldots, \mu_s)$,
- P_0 : the tableau of shape λ, whose t-th column from the left is $\mathbf{a}_t$ $(1 \leq t \leq s)$,
- Q_0 : the tableau of shape λ, whose t-th column from the left is $\mathbf{b}_t$ $(1 \leq t \leq s)$,
- $\rho = (\rho_1, \ldots, \rho_s) \in \mathbb{Z}^s$.

Note that if w is an extended affine permutation of k, then λ is a partition of k, and the tableaux P_0 and Q_0 are bijective column semistandard $[k]$-tableaux.

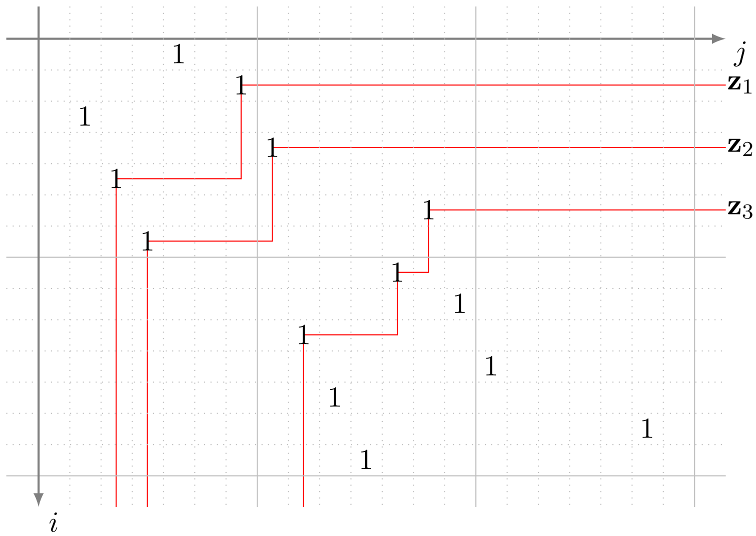
Definition 3.2.4. Let w be an extended affine permutation of k. We define a map Φ on $\widehat{\mathcal{W}}_k$ by

$$\begin{aligned} \Phi: \ \widehat{\mathcal{W}}_k &\longrightarrow \bigsqcup_{\lambda} CST_{[k]}(\lambda) \times CST_{[k]}(\lambda) \times \mathbb{Z}^{\lambda_1}, \\ w &\longmapsto (P_0, Q_0, \rho) \end{aligned} \tag{3.2.3}$$

Theorem 3.2.5 (cf. [5, Theorem 5.1])**.** *The map* Φ *is injective.*

Example 3.2.6. Let w be as in Example 3.1.7.

$w =$

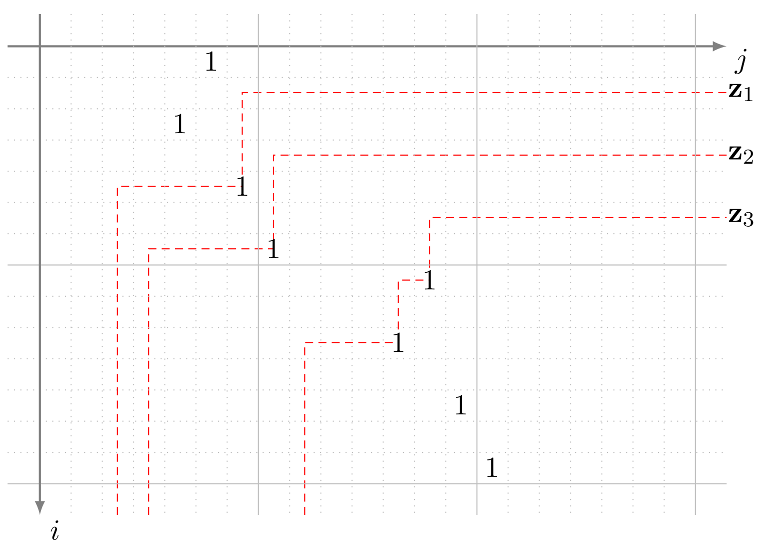


The stream consisting of the back-post corners of $\mathbf{z}_k$ $(k \in \mathbb{Z})$ is

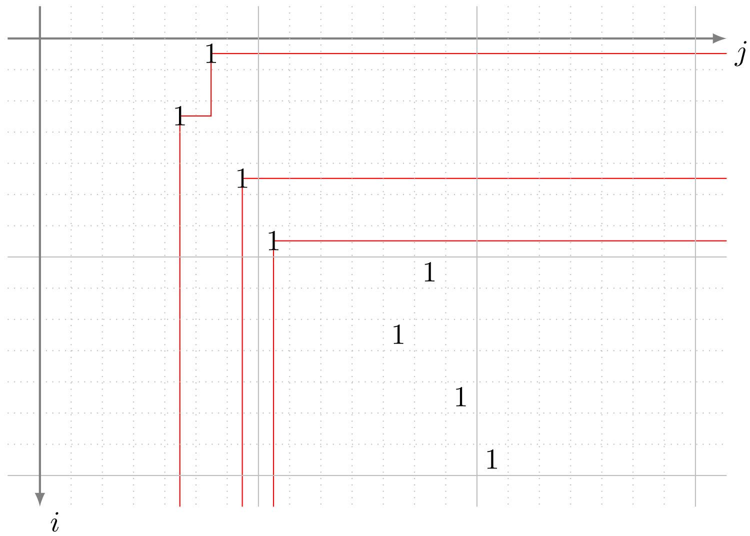
$$\mathbf{s}^{(1)} = \{\cdots >_{\mathtt{NW}} (2,3) >_{\mathtt{NW}} (4,4) >_{\mathtt{NW}} (6,9) >_{\mathtt{NW}} \cdots\}.$$

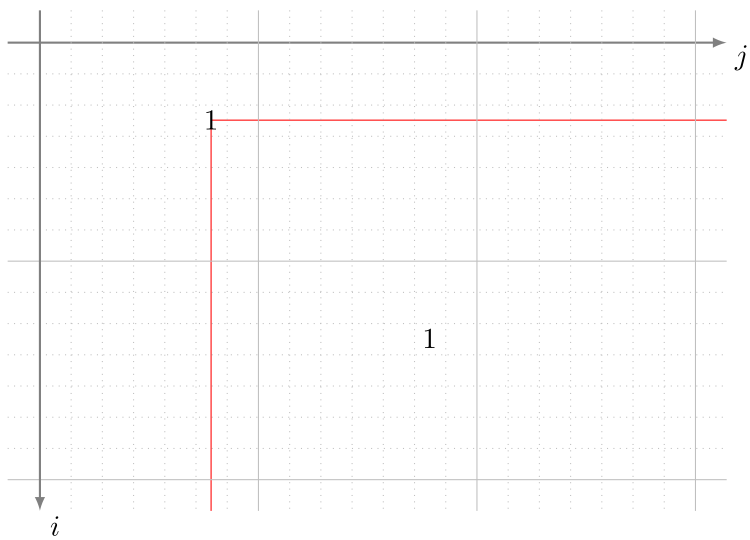
We obtain $w^\flat$ as follows:

$w^\flat =$



Repeating this process, we see that

$w^{(1)} =$ 

$w^{(2)} =$ 

with $w^{(3)} = \varnothing$, and

$$\mathbf{s}^{(2)} = \{\cdots >_{\texttt{NW}} (1,5) >_{\texttt{NW}} (5,7) >_{\texttt{NW}} (7,8) >_{\texttt{NW}} \cdots\},$$
$$\mathbf{s}^{(3)} = \{\cdots >_{\texttt{NW}} (3,6) >_{\texttt{NW}} \cdots\},$$

Then we have

$$P_0 = \begin{array}{|c|c|c|}\hline 2 & 1 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2}\end{array}, \quad Q_0 = \begin{array}{|c|c|c|}\hline 2 & 1 & 6 \\ \hline 3 & 5 \\ \cline{1-2} 4 & 7 \\ \cline{1-2}\end{array}, \quad \rho = (1,1,0)$$

3.3 Affine RS correspondence

Suppose that two stream $\mathbf{s} = \{c_i\}_{i\in\mathbb{Z}}$ and $\mathbf{s}' = \{c'_i\}_{i\in\mathbb{Z}}$ of the same flow l are invariant under the translation $\tau_{k,k}$ with $2l \leq k$. Let $(\mathbf{a}, \mathbf{b}, \rho)$, $(\mathbf{a}', \mathbf{b}', \rho')$ be the defining data of $\mathbf{s}$, $\mathbf{s}'$ respectively. We may assume that

$$\begin{aligned} \mathbf{a} &= (a_1, \dots, a_l), \quad \mathbf{b} = (b_{1-\rho}, \dots, b_{l-\rho}), \\ \mathbf{a}' &= (a'_1, \dots, a'_l), \quad \mathbf{b}' = (b'_{1-\rho'}, \dots, b'_{l-\rho'}). \end{aligned} \tag{3.3.1}$$

Let m be the smallest integer such that $c_i \geq_{\mathtt{nw}} c'_{i+m}$ for all $i \in \mathbb{Z}$, and consider the following two streams

$$\mathbf{t} = \{\, d_i = (a'_{i+m}, b_i) \,\}_{i\in\mathbb{Z}}, \quad \mathbf{t}' = \{\, d'_i = (a_i, b'_{i+m}) \,\}_{i\in\mathbb{Z}}.$$

Then there is a partial permutation w of k such that

$$\mathrm{supp}(w) = \mathbf{t} \cup \mathbf{t}'. \tag{3.3.2}$$

Then it is straightforward to see that l is the width of w, and $\mathbf{t}$ is the southwest channel of w. Let d be a proper numbering on w defined by $d(d_i) = d(d'_i) = i$ for $i \in \mathbb{Z}$.

Proposition 3.3.1 (cf. [5, Proposition 5.6])**.** *There exists unique integer r such that $d = d_w^{\mathtt{sw}}$ if and only if $\rho - \rho' \geq r$.*

Definition 3.3.2.

- The number r in 3.3.1 is called the *offset constant* of a pair $(\mathbf{s}, \mathbf{s}')$,
- A pair $(\mathbf{s}, \mathbf{s}')$ is called *dominant* if $\rho - \rho' \geq r$.
- More generally, for a triple $(P_0, Q_0, \rho) \in CST_{[k]}(\lambda) \times CST_{[k]}(\lambda) \times \mathbb{Z}^{\lambda_1}$, let $\mathbf{s}^{(t)}$ be the stream with defining data $(\mathbf{a}_t, \mathbf{b}_t, \rho_t)$, where $\mathbf{a}_t$, $\mathbf{b}_t$ are the t-th column of P_0, Q_0 respectively. Then (P_0, Q_0, ρ) is called *dominant* if $(\mathbf{s}^{(t)}, \mathbf{s}^{(t+1)})$ is dominant for $1 \leq t < \lambda_1$ such that $\mu_t = \mu_{t+1}$.

Theorem 3.3.3 (cf. [5, Proposition 5.12])**.** *The image of Φ is*

$$\Phi(\widehat{\mathcal{W}}_k) = \{\, (P_0, Q_0, \rho) \mid (P_0, Q_0, \rho) \textit{ is domininat} \}.$$

Recall that we say $i \in [k]$ is a descent of $S \in CST_{[k]}(\lambda)$ if $i+1$ appears to be the right of i in S. For an extended affine permutation w of k, we say that $i \in [k]$ is a *descent* of w if $w(i) > w(i+1)$. The following lemma generalizes a well-known property of the usual RS correspondence (cf. [7, Section 1.1]).

Lemma 3.3.4 (cf. [3, Proposition 3.6]). *Suppose that $\Phi(w) = (P_0, Q_0, \rho)$. We have*

(1) i is a descent of w if and only if i is a descent of P_0

(2) j is a descent of w^{-1} if and only if j is a descent of Q_0

We can interpret the dominant condition in terms of rectangular semistand tableaux.

Lemma 3.3.5. *Suppose that $(P_0, Q_0, \rho) \in CST_k(\lambda) \times CST_k(\lambda) \times \mathbb{Z}^{\lambda_1}$ is given and let η and θ be the symmetrized offset vectors for P_0 and Q_0, respectively. Then (P_0, Q_0, ρ) is dominant if and only if*

$$\tau_k^{\rho+\eta_{\mathrm{rev}}}(Q_0) \in \mathcal{B}_k(\lambda)$$

Proof. It is enough to show when $\lambda = (2^l)$. Let $(\mathbf{a}, \mathbf{b}, \rho)$, $(\mathbf{a}', \mathbf{b}', \rho')$ be the defining data of $\mathbf{s}^{(1)}$, $\mathbf{s}^{(2)}$ respectively. We adopt the notation given in 3.3.1. By Lemma 2.3.2, the condition 3.3.5 is equivalent to that

$$\rho' - \rho \geq \theta - \eta.$$

Hence we claim that $\theta - \eta$ is the offset constant.

We observe that in order to show $d = d_w^{\mathtt{sw}}$, where w is given by (3.3.2), it suffices to find i such that $d_i >_{\mathtt{NW}} d'_{i+1}$ by the condition (2) in Lemma 3.1.9. Furthermore, $d_{i-1} >_{\mathtt{NW}} d'_i$ for some i is equivalent to $a'_{i-1+m} < a_i$ since $b_{i-1} < b'_{i+m}$ is redundant by our choice of m.

The constants η and θ are the smallest integers such that

$$a_i \leq a'_{i+\eta}, \quad b_i \leq b'_{i+\rho-\rho'+\theta} \quad (i \in \mathbb{Z}),$$

respectively. By the minimality of η, θ, we have $\eta \leq m$ and $\theta \leq \rho_2 - \rho_1 + m$.

Suppose that $d = d_w^{\mathtt{sw}}$. Then by the above observation, there exists i with $a_i > a'_{i+m-1}$. In this case, we have $m - 1 < \eta \leq m$ by the minimality of η. Hence $\eta = m$ and

$$\rho' - \rho \geq \theta - m = \theta - \eta = r.$$

Note that we have $\eta \geq \rho - \rho' + \theta$, which implies that $b_i \leq b'_{i+\eta}$.

Conversely, suppose that $\rho' - \rho \geq \theta - \eta$. Since $\rho' - \rho + \eta \geq \theta$, we have

$$b_i \leq b'_{i+\rho-\rho'+(\rho'-\rho_1+\eta)} = b'_{i+\eta} \quad (i \in \mathbb{Z}).$$

Since $a_i \leq a'_{i+\eta}$, we have $m \leq \eta$ by the minimality of m, which implies $a'_{i+m-1} < a_i$ for some i. Hence $d = d^{\texttt{sw}}_w$. □

Chapter 4

Affine Crystals

In this chapter, we provide background information on quantum affine algebras, extremal weight modules, and their crystal bases, which are essential to our study. The chapter is structured as follows:

- In Section 4.1, we review the general theory of representations of quantum groups, based on the work of [17, 18]. We adopt the notations introduced in [10].
- In Section 4.2, we focus on the affine case where $\mathfrak{g} = \widehat{\mathfrak{sl}}_n$. Following the results of [1, 19], we introduce an affine crystal structure on the sets of tableaux presented in Chapter 2.

Throughout this chapter, we assume that n is an integer greater than 1, and work over the field of rational polynomials $\mathbb{Q}(q)$.

4.1 Crystals bases

4.1.1 Quantum groups and their representations

Let I be a finite index set, and let $A = (a_{ij})_{i,j \in I}$ be a generalized Cartan matrix, which is a square matrix with entries in $\mathbb{Z}$ that satisfies the following conditions:

(1) $a_{ii} = 2$ for all $i \in I$,

(2) $a_{ij} \leq 0$ for all $i \neq j \in I$,

(3) $a_{ij} = 0$ if and only if $a_{ji} = 0$ for all $i, j \in I$.

A generalized Cartan matrix is called symmetrizable if there exists a diagonal matrix $D = \operatorname{diag}(s_i)_{i \in I}$ such that DA is a symmetric matrix. In this thesis, we assume that the generalized Cartan matrix A is symmetrizable, with the diagonal entries s_i being coprime integers. Let $P^\vee$ be a finitely generated abelian group which we call a *dual weight lattice*, and let $P = \operatorname{Hom}_{\mathbb{Z}}(P^\vee, \mathbb{Z})$ be the *weight lattice* with the natural pairing $\langle \cdot , \cdot \rangle : P^\vee \times P \longrightarrow \mathbb{Z}$. We fix subsets $\Pi^\vee = \{h_i\}_{i \in I} \subset P^\vee$ and $\Pi = \{\alpha_i\}_{i \in I} \subset P$ sastisfying $\langle h_i, \alpha_j \rangle = a_{ij}$. The elements h_i and α_i are called *simple coroots* and *simple roots* respectively. The quintuple $(A, P^\vee, P, \Pi^\vee, \Pi)$ is called a *Cartan datum* associated with A. A Cartan datum corresponds to a symmetrizable Kac-Moody algebra $\mathfrak{g}$ (see [15, Chapter 1] for details).

Definition 4.1.1. Let $(A, P^\vee, P, \Pi^\vee, \Pi)$ be a Cartan datum, and let $\mathfrak{g}$ be the corresponding Kac-Moody algebra. The *quantum group* or *quantized universal enveloping algebra* $U_q(\mathfrak{g})$ is an associative $\mathbb{Q}(q)$-algebra with 1, generated by the symbols e_i, f_i $(i \in I)$ and q^h $(h \in P^\vee)$ with the following defining relations:

(1) $q^0 = 1$, $q^{h+h'} = q^h q^{h'}$ for $h, h' \in P^\vee$,

(2) $q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i$ for $h \in P^\vee$,

(3) $q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i$ $h \in P^\vee$,

(4) $e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ for $i, j \in I$,

(5) $\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(1-a_{ij}-k)} e_j e_i^{(k)} = 0$ for $i \neq j$,

(6) $\sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(1-a_{ij}-k)} f_j f_i^{(k)} = 0$ for $i \neq j$,

where $q_i = q^{s_i}$, $K_i = q^{s_i h_i}$, and

$$
\begin{aligned}
&[k]_x = (x^k - x^{-k})/(x - x^{-1}), && [k]_x! = [1]_x [2]_x \cdots [k]_x, \\
&e_i^{(k)} = e_i^k / [k]_{q_i}!, && f_i^{(k)} = f_i^k / [k]_{q_i}!.
\end{aligned}
$$

Let $\Delta : U_q(\mathfrak{g}) \longrightarrow U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ be the $\mathbb{Q}(q)$-algebra homomorphism defined by

$$\Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i, \quad \Delta(q^h) = q^h \otimes q^h,$$

for $i \in I$ and $h \in P^\vee$. The map Δ is called a *comultiplication* on $U_q(\mathfrak{g})$. For $U_q(\mathfrak{g})$-modules V and W, the tensor product $V \otimes_{\mathbb{Q}(q)} W$ admits an $U_q(\mathfrak{g})$-module structure via Δ. We denote the resulting module by $V \otimes W$.

An element $\lambda \in P$ is called a *weight*. Let V be a $U_q(\mathfrak{g})$-module. A non-zero vector $v \in V$ is called a *weight vector* of weight $\mu \in P$ if $q^h v = q^{\langle h,\mu \rangle} v$ for all $h \in P^\vee$. We call V a *weight module* if it admits a *weight space decomposition* $V = \oplus_{\mu \in P} V_\mu$ where

$$V_\mu = \{ v \in V \mid q^h v = q^{\langle h,\mu \rangle} v \text{ for all } h \in P^\vee \}.$$

A weight module V is called *integrable* if all e_i and f_i are locally nilpotent on V, i.e., for every $v \in V$, there exists non-negative integer N such that $e_i^N v = f_i^N v = 0$ for all $i \in I$. A weight λ is called *dominant* if $\langle h_i, \lambda \rangle \geq 0$ for all $i \in I$.

Definition 4.1.2. For a dominant weight λ, let $V(\lambda)$ be a $U_q(\mathfrak{g})$-module generated by the single element u_λ with reltations

$$q^h u_\lambda = q^{\langle h,\lambda \rangle} u_\lambda, \quad e_i u_\lambda = 0, \quad f_i^{1+\langle h_i,\lambda \rangle} u_\lambda = 0$$

for $h \in P^\vee$ and $i \in I$. It is known that $V(\lambda)$ is irreducible and integrable. The generator u_λ is called a *highest weight vector* and λ is a *highest weight*. We call $V(\lambda)$ a *irreducible highest weight module* of weight λ.

4.1.2 Weyl groups and extremal weight modules

For $i \in I$, let $s_i : P \longrightarrow P$ be the simple reflection defined by

$$s_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i.$$

We denote by W the subgroup of $GL(P)$ generated by s_i $(i \in I)$, which we call the *Weyl group* of $\mathfrak{g}$. For $w \in W$ and $\lambda \in P$, we simply write $w\lambda$ rather than $w(\lambda)$.

Let V be an integrable module and $u \in V$ be a weight vector of weight λ. We call u an *extremal vector* if there exists a family of vectors $\{u_w\}_{w \in W}$ such that

- $u_e = u$ for the identity $e \in W$,
- if $\langle h_i, w\lambda \rangle \geq 0$, then $e_i u_w = 0$ and $f_i^{(\langle h_i, w\lambda \rangle)} u_w = u_{s_i w}$,
- if $\langle h_i, w\lambda \rangle \leq 0$, then $f_i u_w = 0$ and $e_i^{(-\langle h_i, w\lambda \rangle)} u_w = u_{s_i w}$.

Note that u_w is a weight vector of weight $w\lambda$. We denote u_w by $S_w u$.

Definition 4.1.3. For $\lambda \in P$, let $V(\lambda)$ be a $U_q(\mathfrak{g})$-module generated by u_λ with the defining relations that u_λ is an extremal weight vector of weight λ. We call it an *extremal weight module* of weight λ.

For a dominant weight λ, the extremal weight module $V(\lambda)$ is the irreducible highest weight module of highest weight λ. Hence we use the same notation $V(\lambda)$. The map $u_\lambda \mapsto S_{w^{-1}} u_{w\lambda}$ induces an $U_q(\mathfrak{g})$-module isomorphism

$$V(\lambda) \xrightarrow{\sim} V(w\lambda) \tag{4.1.1}$$

for any $w \in W$.

4.1.3 Crystal bases and crystal graphs

Let V be an integrable $U_q(\mathfrak{g})$-module. For $i \in I$, every weight vector $u \in V$ of weight λ can be written uniquely in the form

$$u = \sum_{k=0}^{N} f_i^{(k)} u_k, \tag{4.1.2}$$

where $N \in \mathbb{Z}_{\geq} 0$ and $u_k \in \ker e_i \cap V_{\lambda + k\alpha_i}$ for $k \geq 0$. Here $u_k \neq 0$ only if $\lambda(h_i) + k \geq 0$. We define operators $\widetilde{e}_i$, $\widetilde{f}_i$ $(i \in I)$ by

$$\widetilde{e}_i u = \sum_{k=1}^{N} f_i^{(k-1)} u_k, \quad \widetilde{f}_i u = \sum_{k=0}^{N} f_i^{(k+1)} u_k.$$

The operators $\widetilde{e}_i$ and $\widetilde{f}_i$ are called the *Kashiwara operators.*

Let $\mathbf{A}_0 \subset \mathbb{Q}(q)$ be the ring of regular functions at $q = 0$. An $\mathbf{A}_0$-*lattice* of $\mathbb{Q}(q)$-module V is an $\mathbf{A}_0$-submodule L of V such that $\mathbb{Q}(q) \otimes_{\mathbf{A}_0} L = V$. Let V be an $U_q(\mathfrak{g})$-module. An $\mathbf{A}_0$-lattice L of V is called a *crystal lattice* if

(1) $L = \oplus_{\mu \in P} L_\mu$, where $L_\mu = L \cap V_\mu$,

(2) $\widetilde{e}_i L \subset L$, $\quad \widetilde{f}_i L \subset L$ for all $i \in I$.

Note that $\widetilde{e}_i$ and $\widetilde{f}_i$ induces $\mathbb{Q}$-linear operators on L/qL, which we shall denote by the same symbols.

Definition 4.1.4. A *crystal base* of a $U_q(\mathfrak{g})$-module V is a pair (L, B) such that

(1) L is a crystal lattice of V,

(2) B is a $\mathbb{Q}$-basis of L/qL,

(3) $B = \cup_{\mu \in P} B_\mu$ where $B_\mu = B \cap (L_\mu / qL_\mu)$,

(4) $\widetilde{e}_i B \subset B \cup \{\mathbf{0}\}$, $\widetilde{f}_i B \subset B \cup \{\mathbf{0}\}$ for $i \in I$ where $\mathbf{0}$ is the zero vector in L/qL,

(5) for any $b, b' \in B$ and $i \in I$, we have $\widetilde{e}_i b = b'$ if and only if $\widetilde{f}_i b' = b$.

We may call B a crystal base if (L, B) is a crystal base of a $U_q(\mathfrak{g})$-module V for some crystal lattice L of V.

Theorem 4.1.5 ([18])**.** *For $\lambda \in P$, the extremal weight module $V(\lambda)$ is integrable and has a crystal basis $B(\lambda)$.*

We regard a crystal base B as a directed I-colored graph, which we call *crystal graph*, whose arrows consist of

$$b \xrightarrow{i} b' \quad \text{if and only if} \quad \widetilde{f}_i b = b'$$

for $b, b' \in B$ and $i \in I$.

Example 4.1.6. Let $A = (2)$ be the 1×1 Cartan matrix with $I = \{1\}$, and let $P^\vee = \mathbb{Z}h$, $P = \mathbb{Z}\Lambda$ with $\langle h, \Lambda \rangle = 1$ and $\alpha = 2\Lambda$. Then $\mathfrak{sl}_2$ is the Kac-Moody algebra corresponding to the Cartan datum $(A, P^\vee, P, \{h\}, \{\alpha\})$, and the quantum group $U_q(\mathfrak{sl}_2)$ is generated by the symbols e, f, and $q^{\pm h}$ under the defining relations

$$q^h q^{-h} = q^{-h} q^h = 1, \quad q^h e q^{-h} = q^2 e, \quad q^h f q^{-h} = q^{-2} f, \quad ef - fe = \frac{q^h - q^{-h}}{q - q^{-1}}.$$

For a non-negative integer ℓ, let $V(\ell)$ be an $(\ell + 1)$-dimensional vector space with a basis $\{ u_0^{(\ell)}, \ldots, u_\ell^{(\ell)} \}$. We define the $U_q(\mathfrak{sl}_2)$-action on $V(\ell)$ by

$$e u_k^{(\ell)} = [\ell - k + 1]_q u_{k-1}^{(\ell)}, \quad f u_k^{(\ell)} = [k + 1]_q u_{k+1}^{(\ell)}, \quad q^h u_k^{(\ell)} = q^{\ell - 2k} u_k^{(\ell)},$$

where we understand that $u_{-1}^{(\ell)} = u_{\ell+1}^{(\ell)} = 0$. It is straightforward to see that $V(\ell)$ is the irreducible highest weight $U_q(\mathfrak{sl}_2)$-module of weight $\ell\Lambda$.

Let $L(\ell)$ be the $\mathbf{A}_0$-submodule of $V(\ell)$ spanned by $u_0^{(\ell)}, \dots, u_\ell^{(\ell)}$, and let $B(\ell) = \{\, \overline{u}_0^{(\ell)}, \overline{u}_1^{(\ell)}, \dots, \overline{u}_\ell^{(\ell)} \,\}$ where $\overline{u}_k^{(\ell)} = u_k^{(\ell)} + qL(\ell) \in L(\ell)/qL(\ell)$. Then the pair $(L(\ell), B(\ell))$ is a crystal base of $V(\ell)$, and the crystal graph is

$$\overline{u}_0^{(\ell)} \longrightarrow \overline{u}_1^{(\ell)} \longrightarrow \cdots \longrightarrow \overline{u}_\ell^{(\ell)}.$$

For a crystal basis B, we define $\mathrm{wt}(b) = \lambda$ for $b \in B_\lambda$, and define $\varepsilon_i, \varphi_i : B \longrightarrow \mathbb{Z}_{\geq 0}$ $(i \in I)$ by

$$\varepsilon_i(b) = \max\{\, k \in \mathbb{Z}_{\geq 0} \,|\, \widetilde{e}_i^k b \neq \mathbf{0} \,\}, \quad \varphi_i(b) = \max\{\, k \in \mathbb{Z}_{\geq 0} \,|\, \widetilde{f}_i^k b \neq \mathbf{0} \,\}, \tag{4.1.3}$$

for $b \in B$.

Theorem 4.1.7. *Let V_1, V_2 be $U_q(\mathfrak{g})$-modules with crystal bases (L_1, B_1), (L_2, B_2), respectively. We have*

(1) $V_1 \oplus V_2$ has a crystal basis $(L_1 \oplus L_2, B_1 \sqcup B_2)$,

(2) $V_1 \otimes V_2$ has a crystal basis $(L_1 \otimes_{\mathbf{A}_0} L_2, B_1 \otimes B_2)$.

Here, we understand $B_1 \otimes B_2$ as the set of images of $b_1 \otimes b_2$ under the natural isomorphism

$$L_1/qL_1 \otimes_{\mathbb{Q}} L_2/qL_2 \xrightarrow{\sim} \big(L_1 \otimes_{\mathbf{A}_0} L_2\big)/q\big(L_1 \otimes_{\mathbf{A}_0} L_2\big)$$

for $b_1 \in B_1$, $b_2 \in B_2$. Moreover, the action of Kashiwara operators on $B_1 \otimes B_2$ are given by

$$\begin{aligned} \widetilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \widetilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \widetilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\ \widetilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \widetilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \widetilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases} \end{aligned} \tag{4.1.4}$$

Hence, we have

$$\begin{aligned} &\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2), \\ &\varepsilon_i(b_1 \otimes b_2) = \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \mathrm{wt}(b_1)(h_i)\}, \\ &\varphi_i(b_1 \otimes b_2) = \max\{\varphi_i(b_1) + \mathrm{wt}(b_2)(h_i), \varphi_i(b_2)\}. \end{aligned} \tag{4.1.5}$$

Here, we understand that $\mathbf{0} \otimes b_2 = b_1 \otimes \mathbf{0} = \mathbf{0}$.

The equations (4.1.4) and (4.1.5) together are called the *tensor product rule*. The tensor product rule gives a combinatorial description of the action of Kashiwara operators on the multifold tensor product of crystal bases. Let $B_1, \ldots, B_r$ be crystals bases. Let $\sigma = (\ldots, \sigma_{-1}, \sigma_0, \sigma_1, \ldots)$ be a sequence with $\sigma_k \in \{+, -, \cdot\}$ such that $\sigma_k = \cdot$ except for finitely many $k \in \mathbb{Z}$. We replace $(\sigma_s, \sigma_t) = (+, -)$ with $(\cdot, \cdot)$ if $s < t$ and $\sigma_k = \cdot$ for $s < k < t$. Repeating this as far as possible, we get a reduced sequence $\widetilde{\sigma}$ where no $+$ precedes $-$. Note that $\widetilde{\sigma}$ is independent on the order of the replacements, and $\widetilde{\sigma}$ is a sequence of $-$'s followed by $+$'s if we neglect $\cdot$'s. For $b_k \in B_k$ $(k = 1, \ldots, r)$ and $i \in I$, let

$$\sigma = (\underbrace{-\cdots-}_{\varepsilon_i(b_1)}, \underbrace{+\cdots+}_{\varphi_i(b_1)}, \cdots, \underbrace{-\cdots-}_{\varepsilon_i(b_r)}, \underbrace{+\cdots+}_{\varphi_i(b_r)}).$$

The reduced sequence $\widetilde{\sigma}$ is called the *i-signature* of $b = b_1 \otimes \cdots \otimes b_r$. Then $\widetilde{e}_i b$ and $\widetilde{f}_i b$ are given by

$$\begin{aligned} \widetilde{e}_i b &= \begin{cases} b_1 \otimes \cdots \otimes \widetilde{e}_i b_s \otimes \cdots \otimes b_r & \text{if } \widetilde{\sigma} \text{ has } - \text{ and the rightmost } - \\ & \text{in } \widetilde{\sigma} \text{ comes from the } s\text{-th factor,} \\ \mathbf{0} & \text{otherwise,} \end{cases} \\ \widetilde{f}_i b &= \begin{cases} b_1 \otimes \cdots \otimes \widetilde{f}_i b_t \otimes \cdots \otimes b_r & \text{if } \widetilde{\sigma} \text{ has } + \text{ and the leftmost } + \\ & \text{in } \widetilde{\sigma} \text{ comes from the } t\text{-th factor,} \\ \mathbf{0} & \text{otherwise.} \end{cases} \end{aligned} \tag{4.1.6}$$

Example 4.1.8. Let $B(4) = \{\overline{u}_0^{(4)}, \overline{u}_1^{(4)}, \overline{u}_2^{(4)}, \overline{u}_3^{(4)}, \overline{u}_4^{(4)}\}$ be the crystal base of the $U_q(\mathfrak{sl}_2)$-module $V(4)$ given in Example 4.1.6. Let $b = \overline{u}_3^{(4)} \otimes \overline{u}_1^{(4)} \otimes \overline{u}_3^{(4)} \in B(4)^{\otimes 3}$. We see that

$$\begin{aligned} \sigma &= (-,-,-,+,-,+,+,+,-,-,-,+), \\ \widetilde{\sigma} &= (-,-,-,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,+). \end{aligned}$$

since $\varepsilon(\overline{u}_k^{(4)}) = k$ and $\varphi(\overline{u}_k^{(4)}) = 4 - k$. Hence we have

$$\begin{aligned} \widetilde{e}\, b &= \widetilde{e}\, \overline{u}_3^{(4)} \otimes \overline{u}_1^{(4)} \otimes \overline{u}_3^{(4)} = \overline{u}_2^{(4)} \otimes \overline{u}_1^{(4)} \otimes \overline{u}_3^{(4)}, \\ \widetilde{f}\, b &= \overline{u}_3^{(4)} \otimes \overline{u}_1^{(4)} \otimes \widetilde{f}\, \overline{u}_3^{(4)} = \overline{u}_3^{(4)} \otimes \overline{u}_1^{(4)} \otimes \overline{u}_4^{(4)}. \end{aligned}$$

Let B be a crystal basis. It is straightforward to see that the maps $\widetilde{e}_i, \widetilde{f}_i, \varepsilon_i, \varphi_i$ $(i \in I)$ and wt satisfy

$$\begin{aligned}
&\mathrm{wt}(b)(h_i) = \varphi_i(b) - \varepsilon_i(b),\\
&\varepsilon_i(\widetilde{e}_i b) = \varepsilon_i(b) - 1, \quad \varphi_i(\widetilde{e}_i b) = \varphi_i(b) + 1, \quad \mathrm{wt}(\widetilde{e}_i b) = \mathrm{wt}(b) + \alpha_i \quad \text{if } \widetilde{e}_i b \neq \mathbf{0},\\
&\varepsilon_i(\widetilde{f}_i b) = \varepsilon_i(b) + 1, \quad \varphi_i(\widetilde{f}_i b) = \varphi_i(b) - 1, \quad \mathrm{wt}(\widetilde{f}_i b) = \mathrm{wt}(b) - \alpha_i \quad \text{if } \widetilde{f}_i b \neq \mathbf{0}.
\end{aligned}$$

We define the abstract notion of *crystals* by characterizing these maps.

Definition 4.1.9. Let I be a finite index set and $(A, P, P^\vee, \Pi, \Pi^\vee)$ be a Cartan datum. A $U_q(\mathfrak{g})$-*crystal* is a set B together with the maps $\widetilde{e}_i, \widetilde{f}_i : B \longrightarrow B \sqcup \{\mathbf{0}\}$, $\varepsilon_i, \varphi_i : B \longrightarrow \mathbb{Z} \sqcup \{-\infty\}$ $(i \in I)$ and $\mathrm{wt} : B \longrightarrow P$ satisfying

$$\begin{aligned}
&\mathrm{wt}(b)(h_i) = \varphi_i(b) - \varepsilon_i(b),\\
&\varepsilon_i(\widetilde{e}_i b) = \varepsilon_i(b) - 1, \quad \varphi_i(\widetilde{e}_i b) = \varphi_i(b) + 1, \quad \mathrm{wt}(\widetilde{e}_i b) = \mathrm{wt}(b) + \alpha_i \quad \text{if } \widetilde{e}_i b \neq \mathbf{0},\\
&\varepsilon_i(\widetilde{f}_i b) = \varepsilon_i(b) + 1, \quad \varphi_i(\widetilde{f}_i b) = \varphi_i(b) - 1, \quad \mathrm{wt}(\widetilde{f}_i b) = \mathrm{wt}(b) - \alpha_i \quad \text{if } \widetilde{f}_i b \neq \mathbf{0},\\
&\widetilde{f}_i b = b' \quad \text{if and only if} \quad b = \widetilde{e}_i b',\\
&\widetilde{e}_i b = \widetilde{f}_i b = \mathbf{0} \quad \text{if } \varphi_i(b) = -\infty.
\end{aligned}$$

for $b, b' \in B$ where $\mathbf{0}$ is a formal symbol. Here, we understand $-\infty + k = -\infty$ for $k \in \mathbb{Z}$.

We omit the prefix $U_q(\mathfrak{g})$ if there is no confusion. We regard B as a directed I-colored graph, similar to the crystal base. Let B_1 and B_2 be crystals. A *morphism* $\psi : B_1 \longrightarrow B_2$ is a map from $B_1 \sqcup \{\mathbf{0}\}$ to $B_2 \sqcup \{\mathbf{0}\}$ such that

$$\begin{aligned}
&\psi(\mathbf{0}) = \mathbf{0},\\
&\mathrm{wt}(\psi(b)) = \mathrm{wt}(b), \quad \varepsilon_i(\psi(b)) = \varepsilon_i(b), \quad \varphi_i(\psi(b)) = \varphi_i(b) \quad \text{if} \quad \psi(b) \neq \mathbf{0},\\
&\psi(\widetilde{e}_i b) = \widetilde{e}_i \psi(b) \quad \text{if } \psi(b) \neq \mathbf{0} \text{ and } \psi(\widetilde{e}_i b) \neq \mathbf{0},\\
&\psi(\widetilde{f}_i b) = \widetilde{f}_i \psi(b) \quad \text{if } \psi(b) \neq \mathbf{0} \text{ and } \psi(\widetilde{f}_i b) \neq \mathbf{0}
\end{aligned}$$

for $b \in B_1$ and $i \in I$. A morphism ψ is called *embedding* if ψ is injective, and is called *strict* if ψ commutes with $\widetilde{e}_i$, $\widetilde{f}_i$ for all $i \in I$. Here, we understand $\widetilde{e}_i \mathbf{0} = \widetilde{f}_i \mathbf{0} = \mathbf{0}$. We call ψ a isomorphism if ψ is bijective. For $b_1 \in B_1$ and $b_2 \in B_2$, we say that b_1 is *crystal equivalent* to b_2 if there exists an isomorphism of crystals $C(b_1) \longrightarrow C(B_2)$ sending b_1 to b_2, where $C(b_k)$ denote the connected component of B_k containing b_k for $k = 1, 2$.

For example, the isomorphism (4.1.1) of $U_q(\mathfrak{g})$-module from the extremal weight modules $V(\lambda)$ to $V(w\lambda)$ induces the isomorphism of $U_q(\mathfrak{g})$-crystals from $B(\lambda)$ to $B(w\lambda)$.

We remark that crystal bases of integrable $U_q(\mathfrak{g})$-modules are $U_q(\mathfrak{g})$-crystals indeed. However, there exists a crystal which is not isomorphic to any crystal basis.

Example 4.1.10. For a weight $\lambda \in P$, the set $T_\lambda = \{t_\lambda\}$ with the maps

$$\mathrm{wt}(t_\lambda) = \lambda, \quad \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty, \quad \widetilde{e}_i(t_\lambda) = \widetilde{f}_i(t_\lambda) = \mathbf{0}$$

is a crystal. But it is not a crystal bases of a $U_q(\mathfrak{g})$-module since t_λ fails to satisfy (4.1.3).

A crystal B satisfying (4.1.3) is called *semi-normal*. For a semi-normal crystal B and $b \in B$, let $C_i(b)$ be the i-string containing b, that is, the set of $b' \in B$ connected to b by i-arrows. Then $C_i(b)$ is isomorphic to the $U_q(\mathfrak{sl}_2)$-crystal $B(\ell)$, which is given in Example 4.1.6, where $\ell = \varphi_i(b) + \varepsilon_i(b)$ is the *length* of $C_i(b)$.

Let B_1, B_2 be crystals. The disjoint union $B_1 \sqcup B_2$ has a crystal structure in obvious manner. Let $B_1 \otimes B_2$ be the set of symbols $b_1 \otimes b_2$ for $b_1 \in B_1$, $b_2 \in B_2$. We define a crystal structure on $B_1 \otimes B_2$ by the tensor product rule (4.1.4), (4.1.5). Note that, for another crystal B_3, we have

$$(B_1 \otimes B_2) \otimes B_3 \cong B_1 \otimes (B_2 \otimes B_3)$$

as crystals. For semi-normal crystals $B_1, \ldots, B_r$, the action of the Kashiwara operators on $B_1 \otimes \cdots \otimes B_r$ enjoy the same combinatorial rule described in (4.1.6).

4.2 Crystals of quantum affine algebras

4.2.1 Quantum affine algebras

Let $I = \{0, 1, \ldots, n-1\}$ be the index set and let $(A, P^\vee, P, \Pi^\vee, \Pi)$ be the Cartan datum defined as follows:

(1) $A = (a_{ij})_{i,j \in I}$, the Cartan matrix where

$$a_{ij} = \begin{cases} 2 & \text{if i = j}, \\ -1 & \text{if } i - j = \pm 1 \pmod{n > 2}, \\ -2 & \text{if } i - j = \pm 1 \pmod{n = 2}, \\ 0 & \text{otherwise}, \end{cases}$$

(2) $P^\vee = \mathbb{Z}h_0 \oplus \cdots \oplus \mathbb{Z}h_{n-1} \oplus \mathbb{Z}d$: the dual weight lattice,

(3) $P = \mathbb{Z}\Lambda_0 \oplus \cdots \oplus \mathbb{Z}\Lambda_{n-1} \oplus \mathbb{Z}\delta$: the weight lattice, where $\{ \Lambda_0, \ldots \Lambda_{n-1}, \delta \}$ is the dual basis of $\{ h_0, \ldots, h_{n-1}, d \}$ with respect to $\langle \cdot , \cdot \rangle$,

(4) $\Pi^\vee = \{ h_i \}_{i \in I}$: the simple coroots,

(5) $\Pi = \{ \alpha_j = \sum_{i=0}^{n-1} a_{ij}\Lambda_i + \delta_{j0}\delta \}_{j \in I}$: the simple roots.

Here, δ_{j0} is the Kronecker delta. The Kac-Moody algebra corresponding to the Cartan datum $(A, P^\vee, P, \Pi^\vee, \Pi)$ is the affine Lie algebra $\widehat{\mathfrak{sl}}_n$, and we call the quantum group $U_q(\widehat{\mathfrak{sl}}_n)$ a *quantum affine algebra*.

We call the element $c = h_0 + \ldots + h_{n-1}$ the *canonical central element*. The value $\langle c, \lambda \rangle$ for $\lambda \in P$ is called the *level* of λ. Note that the simple roots α_i and δ have level zero. We denote by P^0 the set of level zero weights.

Let us introduce other families of level zero weights as follows:

(1) $\epsilon_i = \Lambda_i - \Lambda_{i-1}$ for $i = 1, \ldots, n-1$ and $\epsilon_n = \Lambda_0 - \Lambda_{n-1}$,

(2) $\varpi_i = \Lambda_i - \Lambda_0$ for $i = 1, \ldots, n-1$.

We call the elements ϖ_i the *i-th level zero fundamental weights*. Then we have the following relations:

(1) $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i = 1, \ldots, n-1$ and $\alpha_0 = \epsilon_n - \epsilon_1 + \delta$,

(2) $\varpi_i = \epsilon_1 + \cdots \epsilon_i$ for $i = 1, \ldots, n-1$ and $0 = \epsilon_1 + \cdots + \epsilon_n$.

Let $P^\vee_{\mathrm{cl}} = \bigoplus_{i=0}^{n-1} \mathbb{Z}h_i$ and let $P^\vee_{\mathrm{fin}} = \bigoplus_{i=1}^{n-1} \mathbb{Z}h_i$. Then we have the following identifications

$$\mathrm{Hom}_{\mathbb{Z}}(P^\vee_{\mathrm{cl}}, \mathbb{Z}) = \mathrm{cl}(P), \quad \mathrm{Hom}_{\mathbb{Z}}(P^\vee_{\mathrm{fin}}, \mathbb{Z}) = \mathrm{cl}(P^0)$$

where $\mathrm{cl} : P \longrightarrow P/\mathbb{Z}\delta$ is the canonical projection.

The subalgebra $U_q'(\widehat{\mathfrak{sl}}_n)$ of $U_q(\widehat{\mathfrak{sl}}_n)$ generated by e_i, f_i and K_i $(i = 0, \dots, n-1)$ is called also the *quantum affine algebra*. It is the quantum group $U_q(\mathfrak{g})$ corresponding to the Cartan datum $(A, P_{\mathrm{cl}}^{\vee}, \mathrm{cl}(P), \Pi^{\vee}, \mathrm{cl}(\Pi))$, where $\mathfrak{g}$ is the derived subalgebra $[\widehat{\mathfrak{sl}}_n, \widehat{\mathfrak{sl}}_n]$. Note that $\mathrm{cl}(\Pi)$ is linearly dependent. Similarly, the subalgebra of $U_q(\widehat{\mathfrak{sl}}_n)$ generated by e_i, f_i and K_i $(i = 1, \dots, n-1)$ is the quantum group $U_q(\mathfrak{sl}_n)$, which corresponds to the Cartan datum $((a_{ij})_{i,j=1}^{n-1}, P_{\mathrm{fin}}^{\vee}, \mathrm{cl}(P^0), \{h_i\}_{i=1}^{n-1}, \{\mathrm{cl}(\alpha_i)\}_{i=1}^{n-1})$. Hence we have a chain of subalgebras

$$U_q(\mathfrak{sl}_n) \subset U_q'(\widehat{\mathfrak{sl}}_n) \subset U_q(\widehat{\mathfrak{sl}}_n),$$

and any $U_q(\widehat{\mathfrak{sl}}_n)$-crystals B can be regarded as $U_q'(\widehat{\mathfrak{sl}}_n)$-crystals or $U_q(\mathfrak{sl}_n)$-crystals.

Example 4.2.1.

(1) For $1 \le b \le n$, let

$$B^{b,1} = \left\{ (x_1, \dots, x_n) \,\middle|\, x_1, \dots, x_n \in \{0,1\}, \sum_{i=1}^{n} x_i = b \right\}.$$

with the $U_q'(\widehat{\mathfrak{sl}}_n)$-crystal structure

$$\begin{aligned}
\widetilde{e}_i(x_1, \dots, x_n) &= \begin{cases} (x_1, \dots, x_i + 1, x_{i+1} - 1, \dots, x_n) & \text{if } x_i = 0 \text{ and } x_{i+1} = 1, \\ \mathbf{0} & \text{otherwise,} \end{cases} \\
\widetilde{f}_i(x_1, \dots, x_n) &= \begin{cases} (x_1, \dots, x_i - 1, x_{i+1} + 1, \dots, x_n) & \text{if } x_i = 1 \text{ and } x_{i+1} = 0, \\ \mathbf{0} & \text{otherwise,} \end{cases} \\
\varepsilon_i(x_1, \dots, x_n) &= \max\{x_{i+1} - x_i, 0\}, \\
\varphi_i(x_1, \dots, x_n) &= \max\{x_i - x_{i+1}, 0\}, \\
\mathrm{wt}(x_1, \dots, x_n) &= \sum_{i=0}^{n-1} x_i \mathrm{cl}(\epsilon_i),
\end{aligned}$$

for $i \in I$, where the indices are understood modulo n. Indeed, $B^{b,1}$ is a well-defined semi-normal $U_q'(\widehat{\mathfrak{sl}}_n)$-crystal. Let

$$W^{b,1} = \bigoplus_{x \in B^{b,1}} \mathbb{Q}(q) x$$

with the $U_q'(\widehat{\mathfrak{sl}}_n)$-action

$$e_i x = \begin{cases} \widetilde{e}_i x & \text{if } \widetilde{e}_i x \neq \mathbf{0}, \\ 0 & \text{otherwise} \end{cases}, \quad f_i x = \begin{cases} \widetilde{f}_i x & \text{if } \widetilde{f}_i x \neq \mathbf{0}, \\ 0 & \text{otherwise} \end{cases}$$

and $K_i x = q^{x_i - x_{i+1}} x$. We see that $W^{b,1}$ is an irreducible $U_q'(\widehat{\mathfrak{sl}}_n)$-module and has a crystal basis $B^{b,1}$. Note that $W^{b,1}$ is a one-dimensional trivial module when $b = 1$.

(2) Similarly, let

$$B^{1,a} = \left\{ (x_1, \ldots, x_n) \,\middle|\, x_1, \ldots, x_n \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^{n} x_i = a \right\}.$$

with the $U_q'(\widehat{\mathfrak{sl}}_n)$-crystal structure

$$\begin{aligned}
\widetilde{e}_i(x_1, \ldots, x_n) &= \begin{cases} (x_1, \ldots, x_i + 1, x_{i+1} - 1, \ldots, x_n) & \text{if } x_{i+1} > 0, \\ \mathbf{0} & \text{otherwise}, \end{cases} \\
\widetilde{f}_i(x_1, \ldots, x_n) &= \begin{cases} (x_1, \ldots, x_i - 1, x_{i+1} + 1, \ldots, x_n) & \text{if } x_i > 0, \\ \mathbf{0} & \text{otherwise}, \end{cases} \\
\varepsilon_i(x_1, \ldots, x_n) &= x_{i+1}, \\
\varphi_i(x_1, \ldots, x_n) &= x_i, \\
\mathrm{wt}(x_1, \ldots, x_n) &= \sum_{i=0}^{n-1} x_i \mathrm{cl}(\epsilon_i),
\end{aligned}$$

for $i \in I$ and $a \geq 1$. Let

$$W^{1,a} = \bigoplus_{x \in B^{1,a}} \mathbb{Q}(q) x$$

with the $U_q'(\widehat{\mathfrak{sl}}_n)$-actions

$$e_i x = \begin{cases} [x_i + 1]_q \widetilde{e}_i x & \text{if } \widetilde{e}_i x \neq \mathbf{0}, \\ 0 & \text{otherwise} \end{cases}, \quad f_i x = \begin{cases} [x_{i+1} + 1]_q \widetilde{f}_i x & \text{if } \widetilde{f}_i x \neq \mathbf{0}, \\ 0 & \text{otherwise} \end{cases}$$

and $K_i x = q^{x_i - x_{i+1}} x$. It is straightforward to see that $W^{1,a}$ is an irreducible $U_q'(\widehat{\mathfrak{sl}}_n)$-

module and has a crystal base $B^{1,a}$.

Note that the set the set of column semistandard tableaux $CSST_{[n]}((1^b))$ can be identified with the set $B^{b,1}$ by reading the content of a tableau. Similarly, we identify $RSST_{[n]}((a))$ with $B^{1,a}$. In general, there is a falimy of finite-dimensional irreducible $U_q'(\widehat{\mathfrak{sl}}_n)$-module $W^{b,a}$ for $1 \le b < n$ and $1 \le a$ whose crystal base $B^{b,a}$ can be identified with the set of semistandard $[n]$-tableaux of rectangular shape (a^b). The module $W^{b,a}$ is called the *Kirillov-Reshetikhin (KR) module* and the $B^{b,a}$ is called *KR crystal*. As a $U_q(\mathfrak{sl}_n)$-module, $W^{b,a}$ is the irreducible highest weight module $V(a\mathrm{cl}(\varpi_b))$. The corresponding $U_q(\mathfrak{sl}_n)$-crystal structure on $B^{b,a} \cong B(a\mathrm{cl}(\varpi_b))$ is described in [21] in terms of semistandard tableaux. Chari and Presseley [4] realized $W^{b,a}$ as the minimal affinization of $V(a\mathrm{cl}(\varpi_b))$. Another realization of $W^{b,a}$ can be obtained from the fusion construction [16]. It follows that the crystal basis $B^{b,a}$ is a perfect crystal of level a (see [16, Definition 1.1.1.]).

Definition 4.2.2. Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition with $l \le n$, and let $\mu = (\mu_1, \ldots, \mu_{\lambda_1})$ be the conjugate partition of λ. We define the $U_q'(\widehat{\mathfrak{sl}}_n)$-crystal structure on $CSST_{[n]}(\lambda)$ by identifying

$$\begin{aligned} CSST_{[n]}(\lambda) &\longrightarrow B^{\mu_{\lambda_1},1} \otimes \cdots \otimes B^{\mu_1,1}, \\ T &\longmapsto T^{\lambda_1} \otimes \cdots \otimes T^1 \end{aligned}$$

where T^j is the j-th column of T from the right. Similarly, we define the $U_q'(\widehat{\mathfrak{sl}}_n)$-crystal structure on $RSST_{[n]}(\lambda)$ by identifying

$$\begin{aligned} RSST_{[n]}(\lambda) &\longrightarrow B^{1,\lambda_1} \otimes \cdots \otimes B^{1,\lambda_l}, \\ T &\longmapsto T^{l} \otimes \cdots \otimes T^1 \end{aligned}$$

where T^i is the i-th row of T from the bottom.

4.2.2 Affinization

Recall the map $\mathrm{cl} : P \longrightarrow P/\mathbb{Z}\delta$ is the canonical projection. We define a map $\mathrm{aff} : \mathrm{cl}(P) \longrightarrow P$ by

$$\mathrm{aff}(\mathrm{cl}(\Lambda_i)) = \Lambda_i$$

for $i \in I$. Note that $\mathrm{cl} \circ \mathrm{aff} = \mathrm{id}_{\mathrm{cl}(P)}$ and $\mathrm{aff}(\mathrm{cl}(\alpha_i)) = \alpha_i$ except for $i = 0$.

Let W be a finite-dimensional $U_q'(\widehat{\mathfrak{sl}}_n)$-module with a crystal base B. For an indeterminate z, let

$$W^{\mathrm{aff}} = \mathbb{Q}(q)[z^{\pm 1}] \otimes_{\mathbb{Q}(q)} W$$

and define the $U_q(\widehat{\mathfrak{sl}}_n)$-actions

$$e_i(z^t \otimes v) = z^{t+\delta_{i0}} \otimes e_i v, \quad f_i(z^t \otimes v) = z^{t-\delta_{i0}} \otimes f_i v, \quad K_i(z^t \otimes v) = z^t \otimes K_i v, \quad q^d(z^t \otimes v) = q^t z^t \otimes v,$$

where δ_{i0} is the Kronecker delta. Then W^{aff} is a well-defined $U_q(\widehat{\mathfrak{sl}}_n)$-module, and has the crystal base $B^{\mathrm{aff}} = \mathbb{Z} \times B$ where the Kashiwara operators $\widetilde{e}_i$ and $\widetilde{f}_i$ act on $(t, b) \in B^{\mathrm{aff}}$ by

$$\widetilde{e}_i(t, b) = (t + \delta_{i0}, \widetilde{e}_i b), \quad \widetilde{f}_i(t, b) = (t - \delta_{i0}, \widetilde{f}_i b).$$

We call W^{aff} and B^{aff} the *affinization* of W and B, respectively. Let z be the automorphism of the $U_q'(\widehat{\mathfrak{sl}}_n)$-module W^{aff} given by the multiplication by z. By specializing W^{aff} at $z = 1$, we recover the $U_q'(\widehat{\mathfrak{sl}}_n)$-module

$$W \cong W^{\mathrm{aff}}/(z-1)W^{\mathrm{aff}}.$$

Similarly, let z be the automorphism of $U_q'(\widehat{\mathfrak{sl}}_n)$-crystal on B^{aff} defined by $z(t, b) = (t+1, b)$. Then we recover the crystal B by identifying

$$B = B^{\mathrm{aff}}/\mathbb{Z}$$

where $B^{\mathrm{aff}}/\mathbb{Z}$ is understood as the set of orbits of z.

Let $W = W^{b,1}$ be the finite-dimensional irreducible $U_q'(\widehat{\mathfrak{sl}}_n)$-module given in Example 4.2.1 (1), with the crystal base $B = B^{b,1} = CSST_{[n]}((1^b))$. We define the $U_q(\widehat{\mathfrak{sl}}_n)$-crystal structure on $\mathcal{B}((1^b))$ by identifying $\tau^t(T) \in \mathcal{B}((1^b))$ with $(-t, T) \in \mathbb{Z} \times CSST_{[n]}((1^b)) = B^{\mathrm{aff}}$ (cf. (2.3.2))

More explicitly, let $T \in \mathcal{B}((1^b))$ be given with entries $T(1) < T(2) < \cdots < T(b)$. Then $\widetilde{e}_i T$ is given as follows:

(1) if $T(k) \equiv i + 1 \pmod{n}$ and $T(k-1) < T(k) - 1$ for some $2 \le k \le b$, then $\widetilde{e}_i T$ is the tableau obtained by replacing $T(k)$ with $T(k) - 1$,

(2) if $T(1) \equiv i + 1 \pmod{n}$ and $T(b) < T(1) - 1 + n$, then $\widetilde{e}_i T$ is the tableau obtained by replacing $T(1)$ with $T(1) - 1$,

(3) otherwise, $\widetilde{e}_i T = 0$.

We see that $\widetilde{f}_i T$ is given in a similar manner. Note that every integer k can be uniquely written in the form

$$k = sn + r \tag{4.2.1}$$

with $s \in \mathbb{Z}$ and $1 \leq r \leq n$. Then the weight of $T \in \mathcal{B}((1^b))$ is given by

$$\mathrm{wt}(T) = \sum_{k=1}^{b} \mathrm{wt}(T(i)),$$

where $\mathrm{wt}(k) = \epsilon_r - s\delta \in P^0$ according to (4.2.1) By definition, we have the following lemma.

Lemma 4.2.3. *Let $T \in \mathcal{B}((1^b))$ be given. We have*

$$\widetilde{e}_i \tau(T) = \tau(\widetilde{e}_i T), \quad \widetilde{f}_i \tau(T) = \tau(\widetilde{f}_i T), \quad \mathrm{wt}(\tau(T)) = \mathrm{wt}(T) - \delta.$$

We note that $\mathcal{B}((1^n))$ can be identified with the crystal $\mathbb{Z}$ whose crystal structure given by

$$\mathrm{wt}(t) = -t\delta, \quad \varepsilon_i(t) = \varphi_i(t) = 0, \quad \widetilde{e}_i t = \widetilde{f}_i t = \mathbf{0}$$

for $t \in \mathbb{Z}$, which corresponds to the tableau $\tau^t(T_{(1^n)})$ whose entry from the i-th row from the top is $t + i$.

Proposition 4.2.4. *For $b < n$, the $U_q(\widehat{\mathfrak{sl}}_n)$-crystal $\mathcal{B}((1^b))$ is isomorphic to the crystal base $B(\varpi_b)$ of the extremal weight module $V(\varpi_b)$.*

Proof. Let W^{aff} be the affinization of $W = W^{b,1}$. According to the realization of $W^{b,1}$ given in Example 4.2.1(1), W has a basis $\{z^{-t} \otimes T \,|\, t \in \mathbb{Z},\, T \in CSST_{[n]}((1^b))\}$ which can be parametrized by $\tau^{-t}(T) \in \mathcal{B}((1^b))$. It follows that the $U_q(\widehat{\mathfrak{sl}}_n)$-actions on W^{aff} is given by

$$e_i T = \begin{cases} \widetilde{e}_i T & \text{if } \widetilde{e}_i T \neq \mathbf{0}, \\ 0 & \text{otherwise} \end{cases}, \quad f_i T = \begin{cases} \widetilde{f}_i T & \text{if } \widetilde{f}_i T \neq \mathbf{0}, \\ 0 & \text{otherwise} \end{cases}$$

for $T \in \mathcal{B}((1^b))$. In particular, we see that W^{aff} is irreducible and is generated by the extremal weight vector $T_{(1^b)}$ of level zero weight ϖ_b, where $T_{(1^b)}$ is the tableau whose i-th entry from the top is i. By [19, Proposition 5.16.], W^{aff} is isomorphic to $V(\varpi_b)$ and the assertion follows. $\square$

Remark 4.2.5. Consider the row analogous case $W = W^{1,a}$. We see also in this case that the affinization W^{aff} is irreducible and is generated by the extremal weight vector $z^0 \otimes T_{(a)}$ of weight $a\varpi_1$, where $T_{(a)}$ is the row semistandard tableau of shape (a) whose all entries are 1. However, the extremal weight module $V(a\varpi_1)$ has a proper submodule if $a > 1$.

4.2.3 Crystals of level zero extremal weight modules

Let $R = (a^b)$ be a rectangular shape with $b \leq n$. We identify $\mathcal{B}(R)$ as the subset of $\mathcal{B}((1^b)) \otimes \cdots \otimes \mathcal{B}((1^b))$ by the embedding

$$\begin{array}{ccc} \mathcal{B}(R) & \longrightarrow & \mathcal{B}((1^b)) \otimes \cdots \otimes \mathcal{B}((1^b)) \,, \\ T & \longmapsto & T^a \otimes \cdots \otimes T^1 \end{array}$$

where T^j is the j-th column of T from the right. By the following lemma, we regard $\mathcal{B}(R)$ as a subcrystal of $\mathcal{B}((1^b)) \otimes \cdots \otimes \mathcal{B}((1^b))$.

Lemma 4.2.6. *The subset $\mathcal{B}(R)$ of $\mathcal{B}((1^b)) \otimes \cdots \otimes \mathcal{B}((1^b))$ is invariant under $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in I$.*

Proof. It is enough to show that $\widetilde{e}_i T$ and $\widetilde{f}_i T$ are semistandard for $T \in \mathcal{B}(R)$. The proof is similar to the case of $U_q(\mathfrak{sl}_n)$-crystal (cf. [10, Chapter 7.]). □

Recall the definition of $T \in \mathcal{B}(R)_0$ is that $(T^{j+1}, \tau^{-1}(T^j))$ is not semistandard for any subtableau (T^{j+1}, T^j) of T. Let T_R be the semistandard tableaux all whose entries in the i-th row of T_R from the top is i.

Proposition 4.2.7. *The subset $\mathcal{B}(R)_0$ of $\mathcal{B}(R)$ is invariant under $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in I$. Moreover, $\mathcal{B}(R)_0$ is the connected component of $\mathcal{B}(R)$ containing T_R if $b < n$.*

Proof. Let $T = (T^a, \ldots, T^1) \in \mathcal{B}(R)_0$ be given.

Suppose that $\widetilde{e}_i T \notin \mathcal{B}(R)_0 \cup \{\mathbf{0}\}$ for some i. Then $\widetilde{e}_i T = (\ldots, \widetilde{e}_i T^j, \ldots)$ for some $1 \leq j \leq a$, and $\widetilde{e}_i T^j$ is obtained from T^j by replacing $T^j(k)$ with $T^j(k) - 1$ for some $1 \leq k \leq b$. Since $T \in \mathcal{B}(R)_0$ but $\widetilde{e}_i T \notin \mathcal{B}(R)_0$, at least one of $(\widetilde{e}_i T^j, \tau^{-1}(T^{j-1}))$ or $(T^{j+1}, \tau^{-1}(\widetilde{e}_i T^j))$ is semistandard. Suppose that $(\widetilde{e}_i T^j, \tau^{-1}(T^{j-1}))$ is semistandard. Considering $\widetilde{e}_i T \in \mathcal{B}(R)$, it is straightforward to see that $(T^j, \tau^{-1}(T^{j-1}))$ is also semistandard, which is a contradiction. For the other cases, we have similar contradiction. By the same arguments, we also have $\widetilde{f}_i T \in \mathcal{B}(R)_0 \cup \{\mathbf{0}\}$. Hence $\mathcal{B}(R)_0$ is closed under $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in I$.

We now claim that any $T \in \mathcal{B}(R)_0$ is connected to T_R. Let $t = T^a(1)$ be the entry in the first row of the first column of T. We first show that $T \in \mathcal{B}(R)_0$ is connected to $T_0 = T_R^{(t)}$, where $T_R^{(t)}$ is the element of $\mathcal{B}(R)$ such that the i-th row from the top is filled with $t+i-1$ for $1 \leq i \leq b$. Suppose that $T \neq T_0$. Let $d(T) = \sum_{j=1}^{a}\sum_{k=1}^{b}(T^j(k) - T_0^j(k)) \geq 0$. Since $T \in \mathcal{B}(R)_0$ and $\mathcal{B}(R)_0$ is closed under $\widetilde{e}_i$ and $\widetilde{f}_i$, we have $0 \leq d(\widetilde{e}_i T) < d(T)$ for each i such that $\widetilde{e}_i T \neq \mathbf{0}$. Note that there exists at least one i such that $\widetilde{e}_i T \neq \mathbf{0}$. For example, choose the smallest k such that $T^1(k) \neq T_0^1(k)$ and then $i = T^1(k) - 1$. By induction on $d(T)$, we conclude that T is connected to T_0.

Next, we have

$$\begin{cases} \widetilde{e}^{a}_{t+b-2} \dots \widetilde{e}^{a}_{t}\widetilde{e}^{a}_{t-1} T_0 = T_R^{(t-1)} & (t > 1), \\ \widetilde{f}^{a}_{t} \dots \widetilde{f}^{a}_{t+b-2}\widetilde{f}^{a}_{t+b-1} T_0 = T_R^{(t+1)} & (t < 1). \end{cases}$$

Repeating this step, we conclude that T_0 is connected to $T_R = T_R^{(0)}$. Therefore, $\mathcal{B}(R)_0$ is the connected component of $\mathcal{B}(R)$ containing T_R. $\square$

Remark 4.2.8.

(1) Recall the bijection

$$\begin{array}{ccc} \mathcal{B}(R)_0 \times \mathscr{P}_{a-1} & \longrightarrow & \mathcal{B}(R) \\ (T,\nu) & \longmapsto & \tau^{\nu_{\mathrm{rev}}}(T) \end{array}, \tag{4.2.2}$$

which is given in (2.3.1). If we regard $\mathscr{P}_{a-1}$ as a $U_q(\widehat{\mathfrak{sl}}_n)$-crystal defined by

$$\widetilde{e}_i \nu = \widetilde{f}_i = \mathbf{0}, \quad \mathrm{wt}(\nu) = -|\nu|\delta$$

for $\nu \in \mathscr{P}_{a-1}$, the bijection is a crystal isomorphism from $\mathcal{B}(R)_0 \otimes \mathscr{P}_{a-1}$ to $\mathcal{B}(R)$. This isomorphism is also proved in [27, Theorem 2] for an affine Lie algebra $\mathfrak{g}$ in terms of Lakshmibai-Seshadri paths.

(2) Suppose that $R = (a^b)$ with $b = n$. In this case, $\mathcal{B}(R)_0 = \{\,\boldsymbol{\tau}^t(u_R) \,|\, t \in \mathbb{Z}\,\} = \bigoplus_{t\in\mathbb{Z}}\{\,\boldsymbol{\tau}^t(u_R)\,\}$, where each $\{\,\boldsymbol{\tau}^t(u_R)\,\}$ forms a trivial crystal of weight $-at\delta$.

Proposition 4.2.9. *For $R = (a^b)$ $b < n$, $\mathcal{B}(R)$ is isomorphic to the crystal base $B(a\varpi_b)$ of the extremal weight $U_q(\widehat{\mathfrak{sl}}_n)$-module $V(a\varpi_b)$.*

Proof. It follows from Proposition 4.2.4, Lemma 4.2.7, Remark 4.2.8 (1), and [1, Theorem 4.16(a)]. $\square$

Let λ be a partition with $l = \ell(\lambda) < n$, and $(R^{(1)}, \dots, R^{(l)})$ be the rectangular decomposition of λ. Following the rectangular decomposition (2.4.1), we regard $\mathcal{B}(\lambda)$ as the crystal

$$\mathcal{B}(\lambda) = \mathcal{B}(R^{(1)}) \otimes \cdots \otimes \mathcal{B}(R^{(l)})$$

Let us define $\varpi_\lambda = m_1\varpi_1 + \cdots m_l\varpi_l$, where i is the multiplicity of i in μ.

Proposition 4.2.10. *For a partition λ of length less than n, $\mathcal{B}(\lambda)$ is isomorphic to the crystal base $B(\varpi_\lambda)$ of the extremal weight $U_q(\widehat{\mathfrak{sl}}_n)$-module $V(\varpi_\lambda)$.*

Proof. It follows from [1, Theorem 4.16] and [19, Conjectures 13.1, 13.2] (see also [1, Remark 4.17]). □

Remark 4.2.11.

(1) In [12], another proof of Proposition 4.2.10 is given using the standard monomial theory for semi-infinite Lakshmibai-Seshadri paths [14].

(2) Let T_λ be the tableau of shape λ all whose entries in the i-th row from the top are i. Then $\mathcal{B}(\lambda)_0 = \mathcal{B}(R^{(1)})_0 \otimes \cdots \otimes \mathcal{B}(R^{(l)})_0$ is the connected component of T_λ in $\mathcal{B}(\lambda)$ (see [1, Remark 4.17.]). We can prove it directly using the crystal structure described here.

(3) Suppose that the length of λ is n. We have $\mathcal{B}(\lambda) = \mathcal{B}(\mu) \otimes \mathcal{B}(R^{(n)})$, where $\mu = (\lambda_1, \dots, \lambda_{n-1})$ (see Remark 4.2.8 (2)).

Chapter 5

Affine RSK correspondence

In this chapter, we use the results of Chapter 2 and 3 to get a generalization of affine RS. Let m and n be positive integers.

5.1 Affine matrix and standardization

Let

$$\widehat{\mathcal{M}}_{m\times n} = \left\{ A = (a_{ij})_{i,j\in\mathbb{Z}} \;\middle|\; \begin{array}{l} (1)\ a_{ij} \in \mathbb{Z}_{\geq 0} \text{ and } a_{i+mj+n} = a_{ij} \text{ for all } i,j \in \mathbb{Z}, \\ (2) \text{ for each } j,\ a_{ij} = 0 \text{ except for finitely many } i\text{'s} \end{array} \right\}.$$

We call an element $A \in \widehat{\mathcal{M}}_{m\times n}$ a *affine matrix* or an *affine matrix* for short. For an affine matrix $A = (a_{ij})_{i,j\in\mathbb{Z}}$, we denote by

$$\mathrm{row}(A) = \left(\sum_{j\in\mathbb{Z}} a_{1j},\ \ldots,\ \sum_{j\in\mathbb{Z}} a_{mj} \right), \quad \mathrm{col}(A) = \left(\sum_{i\in\mathbb{Z}} a_{i1},\ \ldots,\ \sum_{i\in\mathbb{Z}} a_{in} \right)$$

the row and column contents of A, respectively. We also denote by

$$|A| = \sum_{i=1}^{m} \sum_{j\in\mathbb{Z}} a_{ij} = \sum_{j=1}^{n} \sum_{i\in\mathbb{Z}} a_{ij}.$$

the size of A.

Let $a = (a_j)_{j\in\mathbb{Z}}$ be a single row matrix with $a_j \in \mathbb{Z}_{\geq 0}$. If $r = \sum_{j\in\mathbb{Z}} a_j < \infty$, we define

r-row matrix $a^\circ = (a^\circ_{ij})_{i\in[r],j\in\mathbb{Z}}$ by

$$a^\circ_{ij} = \begin{cases} 1 & \sum_{s=j+1}^{\infty} a_s < i \le \sum_{s=j}^{\infty} a_s, \\ 0 & \text{otherwise.} \end{cases}$$

For example, if $a = (\dots, 0, 1, 0, 3, 2, 0, \dots)$, then

$$a^\circ = \begin{pmatrix} \cdots & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix},$$

For an affine matrix $A = (a_{ij})_{i,j\in\mathbb{Z}}$, we define A° to be the matrix obtained from A by replacing each row $A_i = (a_{ij})_{j\in\mathbb{Z}}$ with A_i° for $i \in \mathbb{Z}$. Similary, we define $A^{\circ'}$ with respect to the columns of A, that is, $A^{\circ'} = ((A^t)^\circ)^t$, where A^t denotes the transpose of A. Then we define the *standardization of* A to be

$$A^{\mathtt{st}} = (A^\circ)^{\circ'}.$$

By definition, we have $A^{\mathtt{st}} = (A^\circ)^{\circ'} = (A^{\circ'})^\circ$. If A is non-zero, then $A^{\mathtt{st}}$ is an extended affine permutation of $k = |A|$.

Remark 5.1.1. Let $a = (a_j)_{j\in\mathbb{Z}}$ with $a_j \in \mathbb{Z}_{\ge 0}$ and assume that $r = \sum_{j\in\mathbb{Z}} a_j < \infty$. We define r-row matrix $a^\bullet = (a^\bullet_{ij})_{i\in[r],j\in\mathbb{Z}}$ by

$$a^\bullet_{ij} = \begin{cases} 1 & \sum_{s=-\infty}^{j-1} a_s < i \le \sum_{s=-\infty}^{j} a_s, \\ 0 & \text{otherwise.} \end{cases}$$

For $a = (\dots, 0, 1, 0, 3, 2, 0, \dots)$, we have

$$a^\bullet = \begin{pmatrix} \cdots & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \end{pmatrix},$$

We define $A^\bullet$ for an affine matrix A similarly. This version of standardization will be used in Chapter 6.

Let us describe the standardization of affine matrices more explicitly. Assume $k = |A| > 0$ and write $\mathrm{row}(A) = (\alpha_1, \dots, \alpha_m)$, $\mathrm{col}(A) = (\beta_1, \dots, \beta_n)$. For $i \in [m]$ and $j \in [n]$, let

$$\begin{aligned} I_i &= \{\, r \in [k] \mid \alpha_1 + \cdots + \alpha_{i-1} < r \le \alpha_1 + \cdots + \alpha_{i-1} + \alpha_i \,\}, \\ J_j &= \{\, s \in [k] \mid \beta_1 + \cdots + \beta_{j-1} < s \le \beta_1 + \cdots + \beta_{j-1} + \beta_j \,\}, \end{aligned}$$

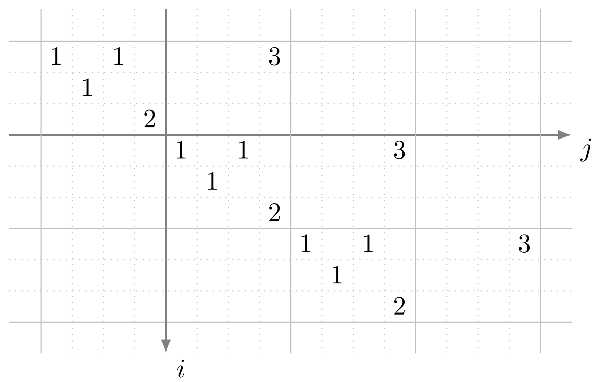
where we understand the empty sum is 0, and let

$$I_{i+tm} = I_i + tk, \quad J_{j+tn} = J_j + tk \quad (t \in \mathbb{Z}).$$

Then we have

$$\begin{aligned} [k] &= \bigsqcup_{i \in [m]} I_i = \bigsqcup_{j \in [n]} J_j, \\ \mathbb{Z} &= \bigsqcup_{i \in Z} I_i = \bigsqcup_{j \in Z} J_j. \end{aligned} \tag{5.1.1}$$

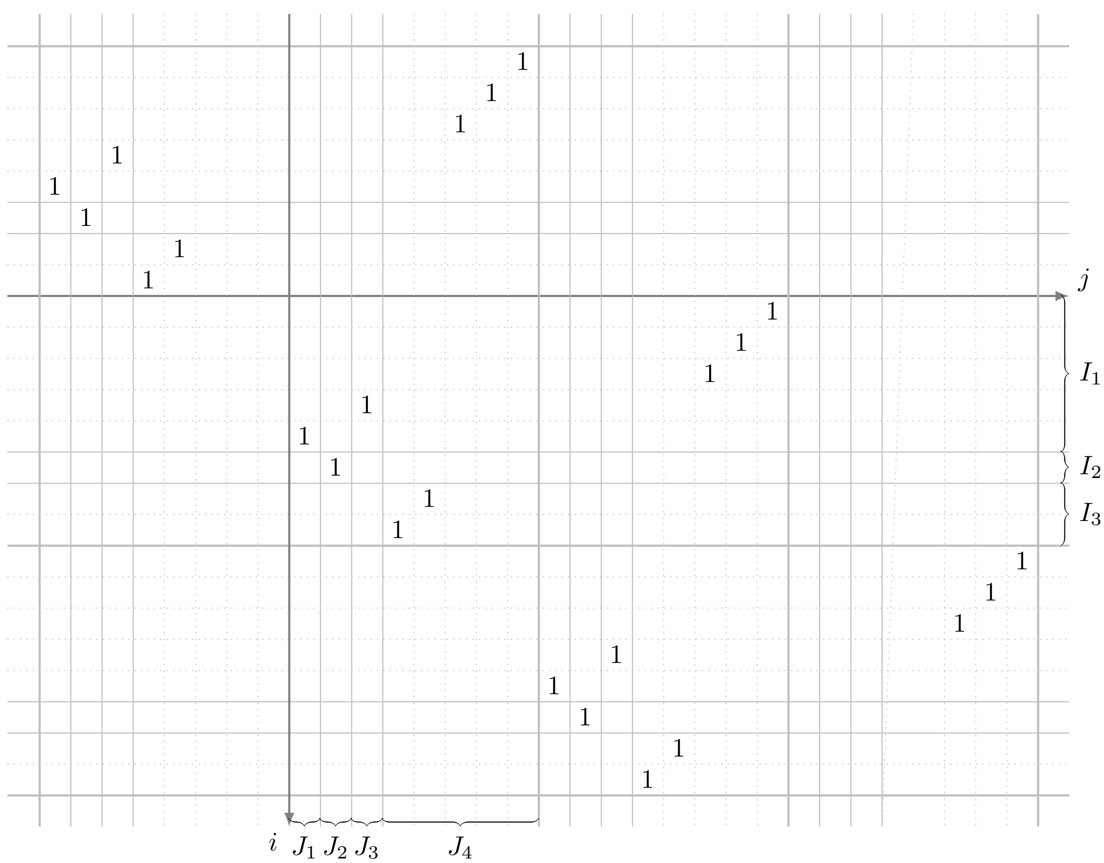
Example 5.1.2. Let $m = 3$, $n = 4$, and let $A \in \widehat{\mathcal{M}}_{3\times 4}$ be an affine matrix given by



then we have

$$I_1 = \{1,2,3,4,5\}, \quad I_2 = \{6\}, \quad I_3 = \{7,8\},$$
$$J_1 = \{1\}, \quad J_2 = \{2\}, \quad J_3 = \{3\}, \quad J_4 = \{4,5,6,7,8\},$$

and A^{st} is



Suppose that $\alpha \in \mathbb{Z}_+^m$ and $\beta \in \mathbb{Z}_+^n$ are given with $k = |\alpha| = |\beta|$. Let $\widehat{\mathcal{M}}_{m\times n}(\alpha, \beta)$ be the set of affine matrix whose row and column contents are α, β respectively. Recall that $i \in [k]$ is a descent of an affine permutation w if $w(i) > w(i+1)$. We say that w is *α-descending* if for any $t \in [n]$ and i with

$$\alpha_1 + \cdots + \alpha_{t-1} + 1 < i < \alpha_1 + \cdots + \alpha_t,$$

i is a descent of w. Let $\widehat{\mathcal{W}}_{k,(\alpha,\beta)}$ denote the set of affine permutation w of k such that w is α-descending and w^{-1} is β-descending. The following lemma is the matrix counterpart of Lemma 2.1.3

Lemma 5.1.3. *Under the hypothesis, we have a bijection*

$$\begin{array}{ccc} \widehat{\mathcal{M}}_{m\times n}(\alpha, \beta) & \longrightarrow & \widehat{\mathcal{W}}_{k,(\alpha,\beta)} \ . \\ A & \longmapsto & A^{\mathtt{st}} \end{array}$$

Let

$$\operatorname{supp}(A) = \{\, (i,j) \in \mathbb{Z} \times \mathbb{Z} \,|\, a_{ij} \neq 0 \,\}$$

be the support of A. It is invariant under the transllation $\tau = \tau_{m,n}$ on $\mathbb{Z} \times \mathbb{Z}$ given by

$$\tau(i,j) = (i+m, j+n) \quad ((i,j) \in \mathbb{Z} \times \mathbb{Z}).$$

Remark 5.1.4. For $c = (i,j) \in \operatorname{supp}(A)$, we denote by $A_c^{\mathtt{st}}$ the matrix in $\mathcal{M}_{\mathbb{Z}\times\mathbb{Z}}$, which is equal to $A^{\mathtt{st}}$ at the positions of $(k,l) \in I_i \times J_j$ and has zero entries elsewhere. Then $A_c^{\mathtt{st}}$ has an $a_{ij} \times a_{ij}$ block submatrix at $I_i \times J_j$ with 1 on the antidiagonal, and zero entries elsewhere.

The following lemmas follows from the remark above immediately.

Lemma 5.1.5. *Let $c_1, c_2 \in \operatorname{supp}(A^{\mathtt{st}})$ be given with $c_1 = (i_1, j_1)$ and $c_2 = (i_2, j_2)$.*

(1) If $i_1 < i_2$ and $c_1, c_2 \in I_i \times \mathbb{Z}$ for some $i \in \mathbb{Z}$, then $c_2 \leq_{\mathtt{ne}} c_1$.

(2) If $j_1 < j_2$ and $c_1, c_2 \in \mathbb{Z} \times J_j$ for some $j \in \mathbb{Z}$, then $c_1 \leq_{\mathtt{ne}} c_2$.

Lemma 5.1.6. *Let $c_1, c_2 \in \operatorname{supp}(A^{\mathtt{st}})$ be given with $c_i \in \operatorname{supp}\left(A_{c_i'}^{\mathtt{st}}\right)$ for some $c_i' \in \operatorname{supp}(A)$ $(i = 1, 2)$. Then we have*

(1) $c_2 >_{\mathtt{NW}} c_1$ if and only if $c_2' >_{\mathtt{NW}} c_1'$,

(2) $c_2 = \tau_{k,k}(c_1)$ implies $c_2' = \tau_{m,n}(c_1')$.

where $k = |A|$.

Following Definitions 3.1.3, 3.1.1, we define streams, channels, and proper numberings on affine matrices.

Definition 5.1.7. A *stream* is an infinite collection of cells $\mathbf{s} = \{c_i\}_{i\in\mathbb{Z}}$, which is invariant under $\tau = \tau_{m,n}$ and forms a chain with respect to $>_{\mathtt{NW}}$. A stream $\mathbf{s}$ is called a *stream of* A if $\mathbf{s} \subset \mathrm{supp}(A)$. A stream $\mathbf{s}$ of w is called a *channel of* A if its flow is maximal among the streams of A.

Definition 5.1.8. A numbering $d : \mathrm{supp}(A) \longrightarrow \mathbb{Z}$ on A is called *proper* if

(1) $d(c_2) < d(c_1)$ if $c_2 >_{\mathtt{NW}} c_1$,

(2) for any $c_1 \in \mathrm{supp}(w)$, there exists $c_2 \in \mathrm{supp}(w)$ such that $c_2 >_{\mathtt{NW}} c_1$ and $d(c_2) = d(c_1) - 1$.

Let d be a numbering on A. Let $d^{\mathtt{st}}$ be the numbering on $\mathrm{supp}(A^{\mathtt{st}})$ given by

$$d^{\mathtt{st}}(c) = d(c') \quad \text{if } c \in \mathrm{supp}(A^{\mathtt{st}}_{c'}) \text{ for some } c' \in \mathrm{supp}(A).$$

Lemma 5.1.9. *We have the following:*

(1) d is a proper numbering on A if and only if $d^{\mathtt{st}}$ is a proper numbering on $A^{\mathtt{st}}$,

(2) any proper numbering on $A^{\mathtt{st}}$ is given by $d^{\mathtt{st}}$ for a unique proper numbering d on A.

Proof. (1) Since no two cells corresponding to non-zero entries in $A^{\mathtt{st}}_c$ ($c \in \mathrm{supp}(A)$) are comparable with respect to $>_{\mathtt{NW}}$ (see Remark 5.1.4), it follows from Lemma 5.1.6(1) that d satisfies the conditions Definition 5.1.8(1) and (2) if and only if $d^{\mathtt{st}}$ does so.

(2) Let d' be a proper numbering on $A^{\mathtt{st}}$. We claim that $d' = d^{\mathtt{st}}$ for some proper numbering d on A. By Lemma 5.1.6(1), it suffices to show that d' is constant on $A^{\mathtt{st}}_c$ for each $c \in \mathrm{supp}(A)$. Suppose that it does not hold. Then there exist $c_1, c_2 \in A^{\mathtt{st}}_c$ for some $c \in \mathrm{supp}(A)$ such that $d'(c_1) < d'(c_2)$. By Definition 5.1.8(1), there exists $c_3 \in A^{\mathtt{st}}_{c'}$ for some $c' \in \mathrm{supp}(A)$ such that $c' >_{\mathtt{NW}} c$ and $d'(c_3) = d'(c_1)$. Since $c_3 >_{\mathtt{NW}} c_1$ by Lemma 5.1.6,

it is a contradiction. This proves the claim. $\square$

Hence any proper numbering d on A enjoys properties given in Section 3.1. We denote by $d_A^{\mathtt{sw}}$ the unique numbering such that $(d_A^{\mathtt{sw}})_{\mathtt{st}} = d_{A^{\mathtt{st}}}^{\mathtt{sw}}$ and call it the *southwest channel numbering on A*.

5.2 Matrix-ball construction for affine matrices

In this section, we define matrix-ball construction for affine matrices and construct the bijection (1.2.1). Throughout this section, we assume that $A \in \widehat{\mathcal{M}}_{m\times n}$ is a non-zero affine matrix.

We will define a matrix analogous of the notions given in Section 3.2

Definition 5.2.1. Let A be an affine matrix and let $\{\mathbf{z}_k\}_{k\in\mathbb{Z}}$ be the set of zig-zags associated to $d_A^{\mathtt{sw}}$. Here $\mathbf{z}_k$ is the unique zig-zag whose inner corners form the level set $(d_A^{\mathtt{sw}})^{-1}(k)$.

We define

- $A^\flat$: the matrix in $\widehat{\mathcal{M}}_{m\times n}$ obtained from A by

  (i) subtracting one at the inner corners in $\mathbf{z}_k$ $(k \in \mathbb{Z})$,

  (ii) adding one at the outer corners in $\mathbf{z}_k$ $(k \in \mathbb{Z})$.

  We remark that
  $$(A^{\mathtt{st}})^\flat = (A^\flat)^{\mathtt{st}},$$
  in the sense that the right-hand side is obtained by removing all zero rows and columns on the left-hand side.

- $\mathbf{s}(A)$: the stream consisting of the back-post corners of $\mathbf{z}_i$'s,
- $A^{(t)} = (A^{(t-1)})^\flat$, $A^{(0)} = A$,
- $\mathbf{s}^{(t)} = \mathbf{s}(A^{(t-1)})$,
- μ_t : the flow of $\mathbf{s}^{(t)}$ or the width of $A^{(t-1)}$,
- $(\mathbf{a}_t, \mathbf{b}_t, \rho_t)$: the defining data of $\mathbf{s}^{(t)}$.

for $t \geq 0$. It is obvious that there exists $s \geq 1$ such that

$$A^{(s-1)} \neq \varnothing, \quad A^{(s)} = \varnothing,$$

and we have $\mu_1 \geq \cdots \geq \mu_s > 0$.

Now we let

- $\lambda = \mu'$: the conjugate partition of $\mu = (\mu_1, \ldots, \mu_s)$,
- P_0 : the tableau of shape λ, whose t-th column from the left is $\mathbf{a}_t$ $(1 \leq t \leq s)$,
- Q_0 : the tableau of shape λ, whose t-th column from the left is $\mathbf{b}_t$ $(1 \leq t \leq s)$,
- $\rho = (\rho_1, \ldots, \rho_s) \in \mathbb{Z}^s$.

We define a map κ_0 on $\widehat{\mathcal{M}}_{m\times n}$ by

$$\begin{aligned} \kappa_0 :\ \widehat{\mathcal{M}}_{m\times n} &\longrightarrow \bigsqcup_{\lambda} CSST_{[m]}(\lambda) \times CSTS_{[n]}(\lambda) \times \mathbb{Z}^{\lambda_1}, \\ A &\longmapsto (P_0, Q_0, \rho) \end{aligned} \tag{5.2.1}$$

We understand that $\kappa_0(\mathbb{O}) = (\varnothing, \varnothing, 0)$ where $\mathbb{O}$ is the zero matrix, $\varnothing$ is the empty tableau of shape (0).

It is clear that κ_0 preserves contents, that is, if $A \in \widehat{\mathcal{M}}_{m\times n}(\alpha, \beta)$ with $\kappa_0(A) = (P_0, Q_0, \rho)$, then $P_0 \in CSST_{[m]}(\lambda)_\alpha$ and $Q_0 \in CSST_{[n]}(\lambda)_\beta$. The following proposition shows that κ_0 is a generalization of Φ via standardizations.

Proposition 5.2.2. *For $A \in \widehat{\mathcal{M}}_{m\times n}$ with $\kappa_0(A) = (P_0, Q_0, \rho)$, we have*

$$\kappa_0(A^{\mathtt{st}}) = (P_0^{\mathtt{st}}, Q_0^{\mathtt{st}}, \rho).$$

Proof. Assume that $A \in \widehat{\mathcal{M}}_{m\times n}(\alpha, \beta)$ for some $\alpha \in \mathbb{Z}^m_{\geq 0}$ and $\beta \in \mathbb{Z}^n_{\geq 0}$ with $|\alpha| = |\beta| = k$.

Let $\kappa_0(A^{\mathtt{st}}) = (P_1, Q_1, \varrho)$. We have

$$P_1 \in CST_{[k],\alpha}(\lambda), \quad Q_1 \in CST_{[k],\beta}(\lambda)$$

for some $\lambda \in \mathscr{P}$. We claim that P_0 (resp. Q_0) is the image of P_1 (resp. Q_1) under the inverse of the bijection in Lemma 2.1.3 and $\rho = \varrho$. Let $\{\, I_i \times J_j \,\}_{i,j\in\mathbb{Z}}$ be the partition of $\mathbb{Z} \times \mathbb{Z}$ associated to α and β given in (5.1.1). Let $\mathbf{s}$ and $\mathbf{s}^{\mathtt{st}}$ be the stream given from the the back-post corners of the zig-zags $\{\mathbf{z}_t\}_{t\in\mathbb{Z}}$ and $\{\mathbf{z}^{\mathtt{st}}_t\}_{t\in\mathbb{Z}}$ associated to $d^{\mathtt{sw}}_A$ and $d^{\mathtt{sw}}_{A^{\mathtt{st}}}$, respectively. Since $(d^{\mathtt{sw}}_A)^{\mathtt{st}} = d^{\mathtt{sw}}_{A^{\mathtt{st}}}$, we have $(A^\flat)^{\mathtt{st}} = (A^{\mathtt{st}})^\flat$ and the flow of $\mathbf{s}$ is equal to that of $\mathbf{s}^{\mathtt{st}}$, and $\mathbf{s}$ is obtained from $\mathbf{s}^{\mathtt{st}}$ by replacing each (r,s) in $\mathbf{s}^{\mathtt{st}}$ with (i,j) when $(r,s) \in I_i \times J_j$. We use induction on k to prove the claim, hence we have $P^{\mathtt{st}}_0 = P_1$ and $Q^{\mathtt{st}}_0 = Q_1$. $\square$

Let $\kappa_0(A) = (P_0, Q_0, \rho)$, and let λ be the shape of P_0. By the above proposition, we see that $(P^{\mathtt{st}}_0, Q^{\mathtt{st}}_0, \rho)$ is dominant. Let $(R^{(1)}, \ldots, R^{(l)})$ be the rectangular decomposition of λ.

- $P^{(i)}_0$, $Q^{(i)}_0$: the rectangular decompositions of P_0 and Q_0,
- $\rho^{(i)} \in \mathbb{Z}^{m_i}$: the subsequence of ρ corresponding to the rectangular decomposition, where m_i is the occurrence of i in μ.
- $\eta^{(i)} \in \mathscr{P}_{m_i-1}$: the symmetrized offset vector of $P^{(i)}_0$.

Then we define

$$Q = \left(\tau^{\rho^{(l)}+\eta^{(l)}_{\mathtt{rev}}}(Q^{(l)}_0), \ldots, \tau^{\rho^{(1)}+\eta^{(1)}_{\mathtt{rev}}}(Q^{(1)}_0)\right) = \left(Q^{(l)}, \ldots, Q^{(1)}\right). \tag{5.2.2}$$

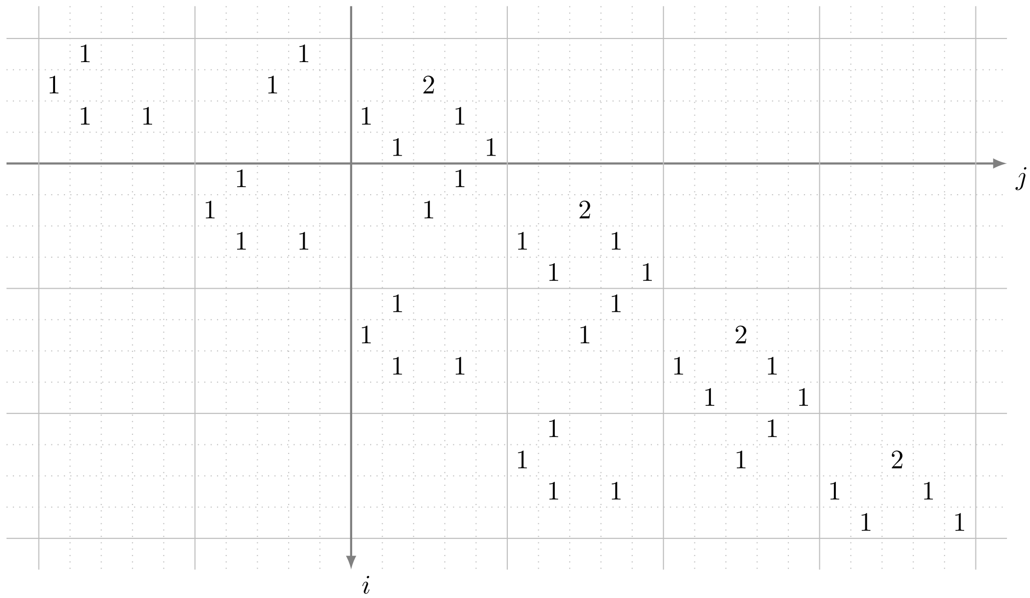
Note that the action of τ on Q_0 should be understood as τ_n.

Since the symmetrized offset vectors for $(P^{(i)}_0)^{\mathtt{st}}$ (resp. $(Q^{(i)}_0)^{\mathtt{st}}$) is also $\eta^{(i)}$ (resp, $\theta^{(i)}$), we have $\tau_k^{\rho^{(i)}+\eta^{(i)}_{\mathtt{rev}}}((Q^{(i)}_0)^{\mathtt{st}}) \in \mathcal{B}_k(R^{(i)})$ by Lemma 2.3.3. Consequently, we have $Q \in \mathcal{B}_n(\lambda)$ by Lemma 3.3.5. Then we have the following bijection, which we call the *affine RSK correspondence*.

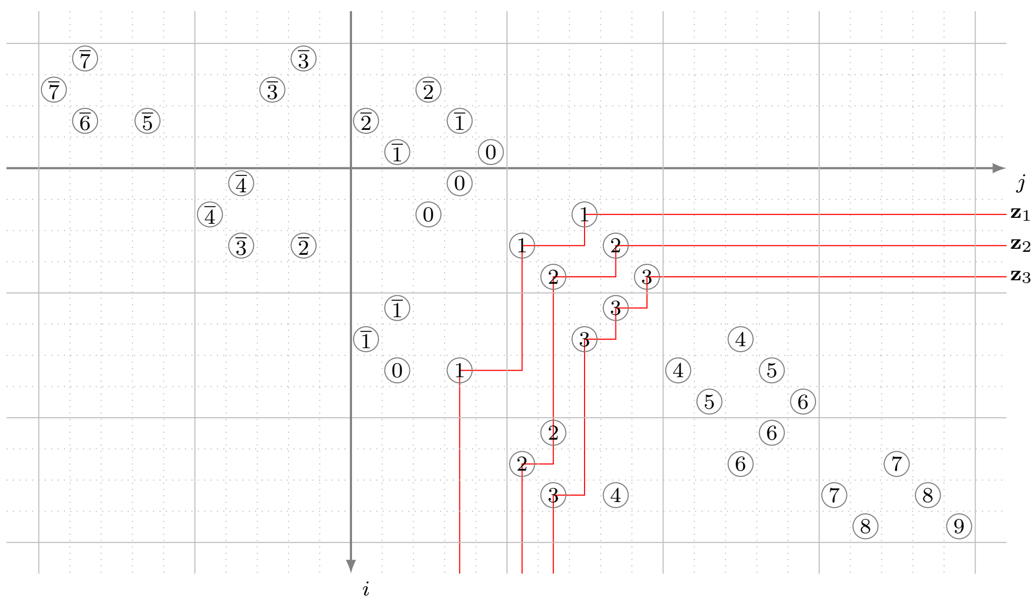
Theorem 5.2.3. *We have a bijection*

$$\begin{aligned} \kappa : \quad \widehat{\mathcal{M}}_{m\times n} &\longrightarrow \bigsqcup_{\lambda \in \mathscr{P}_m \cap \mathscr{P}_n} CSST_{[m]}(\lambda) \times \mathcal{B}_n(\lambda)\,, \\ A &\longmapsto (P_0, Q) \end{aligned} \tag{5.2.3}$$

Example 5.2.4. Let $A \in \widehat{\mathcal{M}}_{4\times 5}$ be an affine matrix as follows



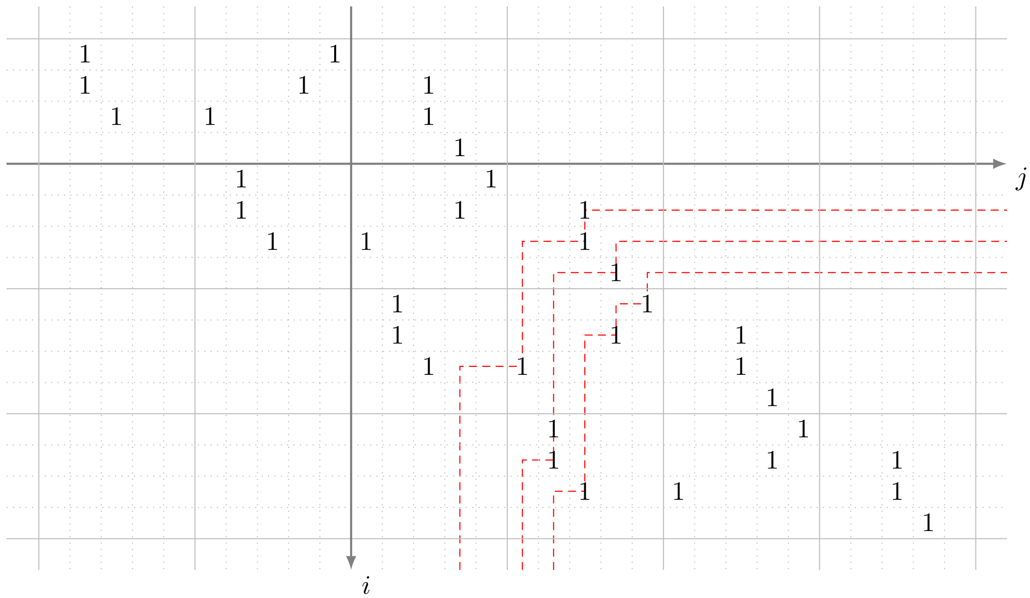
The southwest channel numbering $d_A^{\mathtt{st}}$ on A is given by



where $\mathbf{z}_k$ is the zig-zag corresponding to $d_A^{\mathtt{st}}(k)$ for $k = 1, 2, 3$. Here we denote the negative integers $-s$ by $\overline{s}$. The stream consisting of the back-post corners of $\mathbf{z}_k$ $(k \in \mathbb{Z})$ is

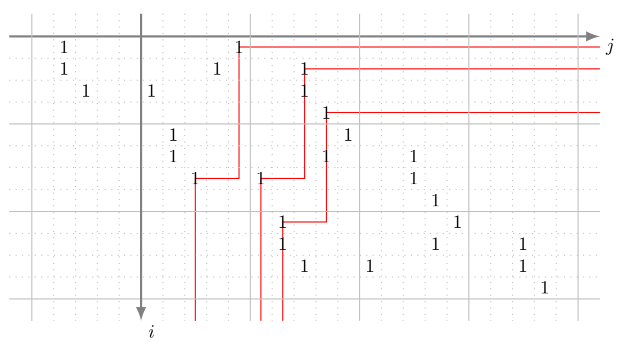
$$\mathbf{s}^{(1)} = \{\cdots >_{\mathtt{NW}} (2,4) >_{\mathtt{NW}} (3,6) >_{\mathtt{NW}} (4,7) >_{\mathtt{NW}} \cdots\}.$$

By subtracting one at each inner corner and adding one at each outer corner of $\mathbf{z}_k$, we obtain $A^\flat$ as follows:

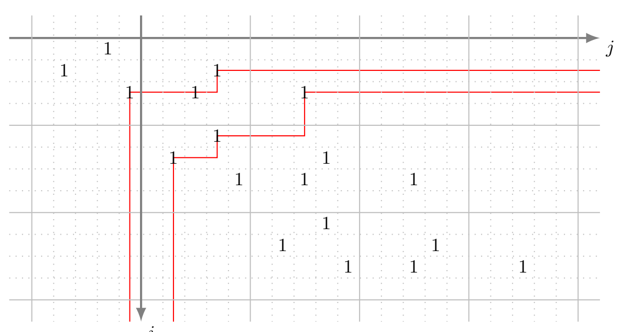


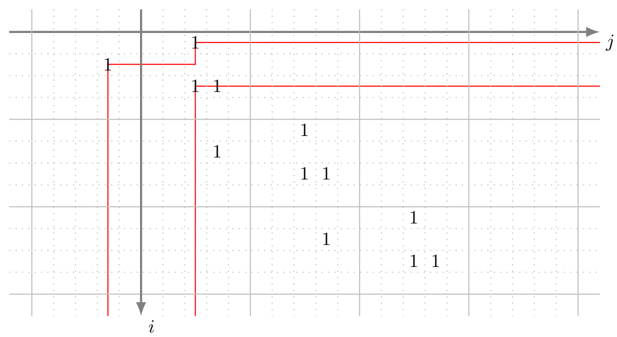
Repeating this process, we have

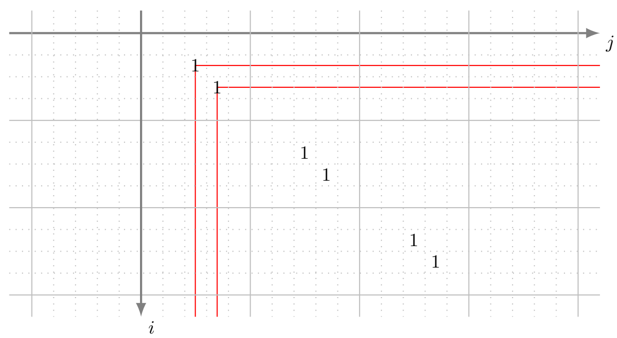
$A^{(1)} =$



$A^{(2)} =$



$A^{(3)} =$ 

$A^{(4)} =$ 

with $A^{(5)} = \mathbb{O}$, and

$$\begin{aligned}
\mathbf{s}^{(2)} &= \{\cdots >_{\mathtt{NW}} (1,3) >_{\mathtt{NW}} (2,6) >_{\mathtt{NW}} (4,7) >_{\mathtt{NW}} \cdots\},\\
\mathbf{s}^{(3)} &= \{\cdots >_{\mathtt{NW}} (2,0) >_{\mathtt{NW}} (3,2) >_{\mathtt{NW}} \cdots\},\\
\mathbf{s}^{(4)} &= \{\cdots >_{\mathtt{NW}} (1,-1) >_{\mathtt{NW}} (3,3) >_{\mathtt{NW}} \cdots\},\\
\mathbf{s}^{(5)} &= \{\cdots >_{\mathtt{NW}} (2,3) >_{\mathtt{NW}} (3,4) >_{\mathtt{NW}} \cdots\}.
\end{aligned}$$

Hence $\kappa_0(A) = (P_0, Q_0, \rho)$, where

$$P_0 = \begin{array}{|c|c|c|c|c|}\hline 2 & 1 & 2 & 1 & 2 \\ \hline 3 & 2 & 3 & 3 & 3 \\ \hline 4 & 4 \\ \cline{1-2}\end{array}, \quad Q_0 = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 3 & 3 \\ \hline 2 & 2 & 5 & 4 & 4 \\ \hline 4 & 3 \\ \cline{1-2}\end{array}, \quad \rho = (2, 2, -1, -1, 0).$$

The rectangular decompositions of P_0, Q_0 and ρ are

$$P_0^{(2)} = \begin{array}{|c|c|c|}\hline 2 & 1 & 2 \\ \hline 3 & 3 & 3 \\ \hline \end{array}, \qquad Q_0^{(2)} = \begin{array}{|c|c|c|}\hline 2 & 3 & 3 \\ \hline 5 & 4 & 4 \\ \hline \end{array}, \qquad \rho^{(2)} = (-1,-1,0),$$

$$P_0^{(3)} = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & 2 \\ \hline 4 & 4 \\ \hline \end{array}, \qquad Q_0^{(3)} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline 4 & 3 \\ \hline \end{array}, \qquad \rho^{(3)} = (2,2),$$

where R_2 and R_3 are the only non-trivial rectangles in this decomposition. The symmetrized offset vectors of $P_0^{(2)}$ and $P_0^{(3)}$ are

$$\eta_{\mathtt{rev}}^{(2)} = (0,1,1), \quad \eta_{\mathtt{rev}}^{(3)} = (0,1),$$

and hence

$$Q = \begin{array}{|c|c|c|c|c|}\hline 4 & 6 & 0 & 3 & 4 \\ \hline 6 & 7 & 2 & 4 & 8 \\ \hline 7 & 8 & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array}.$$

Remark 5.2.5. Let us give some comments on κ_0 and the bijection in [13]. Let $\overline{\mathbb{M}}_{m\times n}$ be the set of $\overline{M} = \big(\overline{M}_{j,i}(k)\big)$ ($i \in [m]$, $j \in [n]$, $k \in \mathbb{Z}$) with $\overline{M}_{j,i}(k) \in \mathbb{Z}_{\geq 0}$ and $\overline{M}_{j,i}(k) = 0$ for $|k| \gg 0$ [13, (2.5)] and let $\overline{\mathbb{M}}^+_{m\times n}$ be the subset of $\overline{\mathbb{M}}_{m\times n}$ consisting of $\overline{M}$ such that $\overline{M}_{j,i}(k) = 0$ for $k < 0$. For $A = (a_{ij})_{i,j\in\mathbb{Z}} \in \widehat{\mathcal{M}}_{m\times n}$, we define $\overline{M}_A = \big(\overline{M}_{j,i}(k)\big)$ by

$$\overline{M}_{j,i}(k) = a_{i-km,\,n+1-j}.$$

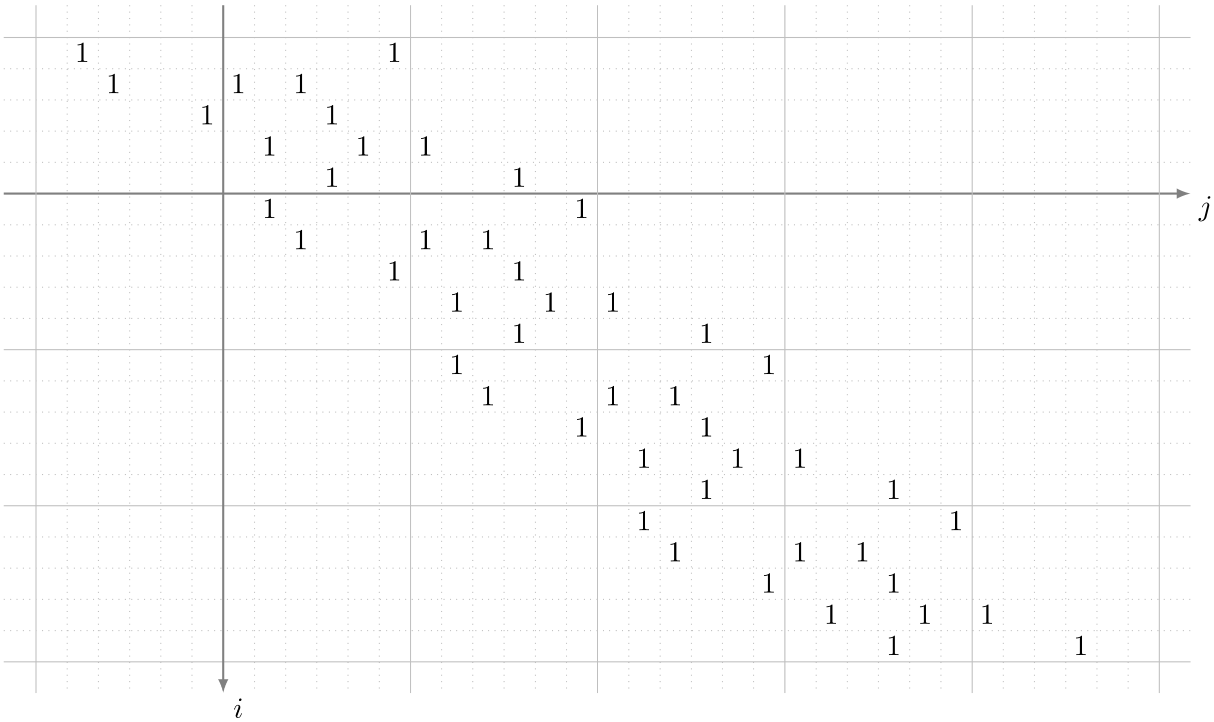
Then the map sending A to $\overline{M}_A$ gives a bijection from $\widehat{\mathcal{M}}_{m\times n}$ to $\overline{\mathbb{M}}_{m\times n}$.

Let $\widetilde{\Upsilon}$ denote the bijection

$$\overline{\mathbb{M}}^+_{m\times n} \longrightarrow \bigsqcup_{\lambda\in\mathscr{P}_m\cap\mathscr{P}_n} CSST_{[m]}(\lambda) \times CSST_{[n]}(\lambda) \times \mathcal{K}(\lambda)\ ,$$

given in [13, Corollary 8.2].

Let $A = (a_{ij})_{i,j\in\mathbb{Z}} \in \widehat{\mathcal{M}}_{m\times n}$ be given such that $\overline{M}_A \in \overline{\mathbb{M}}^+_{m\times n}$. Applying κ_0 and $\widetilde{\Upsilon}$ directly to A and $\overline{M}_A$, respectively, do not seem to give the same result in general. This may happen due to conventions for affine matrixs. For example, let $A \in \widehat{\mathcal{M}}_{5\times 6}$ be as follows:



Then we have $\kappa_0(A) = (P_0, Q_0, \rho)$, where

$$P_0 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 1 \\ \hline 4 & 3 & 4 & 2 \\ \hline 5 & 4 & 5 & 3 \\ \hline \end{array}\,, \quad Q_0 = \begin{array}{|c|c|c|c|}\hline 2 & 1 & 3 & 1 \\ \hline 3 & 2 & 4 & 4 \\ \hline 6 & 4 & 5 & 6 \\ \hline \end{array}\,.$$

On the other hand, we have $\widetilde{\Upsilon}(\overline{M}_A) = (V, W, \xi)$, where

$$V = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 1 \\ \hline 4 & 3 & 4 & 2 \\ \hline 5 & 4 & 5 & 3 \\ \hline \end{array}\,, \quad W = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 1 \\ \hline 3 & 3 & 5 & 4 \\ \hline 6 & 4 & 6 & 5 \\ \hline \end{array}\,.$$

Hence $P_0 = V$ but $Q_0 \neq W$, while we observe that $W^j(i) = 7 - Q_0^{5-j}(4-i)$ for $1 \leq i \leq 3$ and $1 \leq j \leq 4$. Recall that W^j and Q_0^j denote the j-th columns from the right.

In general, one may expect that $P_0 = V$ and $Q_0 = e_n W$, where e_n is an operator on $CSST_{[n]}(\lambda)$ given by

$$(e_n W)^j(i) = n + 1 - W^{a+1-j}(b+1-i) \quad (1 \leq i \leq b, 1 \leq j \leq a),$$

in case of a rectangular shape $\lambda = (a^b)$. The operator e_n can be viewed as a generalization of the affine evacuation in [2]. We do not know yet a precise relation between ρ and ξ.

5.3 Affine RSK and crystals

For $i, j \in \mathbb{Z}$, let E_{ij} denote the elementary matrix with 1 at the (i, j)-position and 0 elsewhere and put

$$\widehat{E}_{ij} = \sum_{k\in\mathbb{Z}} E_{i+km\, j+kn} \in \widehat{\mathcal{M}}_{m\times n}.$$

Let us first describe an $U_q(\widehat{\mathfrak{sl}}_m)$-crystal structure on $\widehat{\mathcal{M}}_{m\times n}$ for $m \geq 2$. Suppose that $A = (a_{ij})_{i,j\in\mathbb{Z}} \in \widehat{\mathcal{M}}_{m\times n}$ is given. For $i \in \{0, 1, \dots, m-1\}$, we define $\widetilde{e}_i A$ and $\widetilde{f}_i A$ as follows:

(1) Let σ be a sequence of $\{+, -\}$ given by

$$\sigma = (\,\cdots,\ \underbrace{-\cdots-}_{a_{i+1\,j}},\ \underbrace{+\cdots+}_{a_{i\,j}},\ \underbrace{-\cdots-}_{a_{i+1\,j+1}},\ \underbrace{+\cdots+}_{a_{i\,j+1}},\ \cdots\,),$$

and let $\widetilde{\sigma}$ be the reduced one, which is well-defined since σ has only finitely many $+$'s and $-$'s.

(2) If $\widetilde{\sigma}$ has at least one $-$, then we define

$$\widetilde{e}_i A = A + \widehat{E}_{ij_0} - \widehat{E}_{i+1 j_0},$$

where j_0 is the column index of A corresponding the rightmost $-$ in $\widetilde{\sigma}$. If $\widetilde{\sigma}$ has no $-$, then we define $\widetilde{e}_i A = \mathbf{0}$. Similarly, if $\widetilde{\sigma}$ has at least one $+$, then we define

$$\widetilde{f}_i A = A - \widehat{E}_{ij_1} + \widehat{E}_{i+1 j_1},$$

where j_1 is the column index of A corresponding the leftmost $+$ in $\widetilde{\sigma}$. If $\widetilde{\sigma}$ has no $+$, then we define $\widetilde{f}_i A = \mathbf{0}$.

Put

$$\mathrm{wt}(A) = \sum_{k\in\mathbb{Z}} \left(\sum_{j=1}^{n} a_{1+km\, j} \right) (\epsilon_1 - k\delta) + \cdots + \sum_{k\in\mathbb{Z}} \left(\sum_{j=1}^{n} a_{m+km\, j} \right) (\epsilon_m - k\delta) \in P^0. \quad (5.3.1)$$

$$\varepsilon_i(A) = \max\{\, k \mid \widetilde{e}_i^{\,k} A \neq \mathbf{0}\,\}, \quad \varphi_i(A) = \max\{\, k \mid \widetilde{f}_i^{\,k} A \neq \mathbf{0}\,\}, \quad (i \in I).$$

Both $\varepsilon_i(A)$ and $\varphi_i(A)$ are finite since σ has only finitely many $+$'s and $-$'s. Moreover, we have

$$\varphi_i(A) - \varepsilon_i(A) = \sum_{j \in \mathbb{Z}} a_{i+1\,j} - \sum_{j \in \mathbb{Z}} a_{ij} = \langle h_i, \mathrm{wt}(A) \rangle.$$

Hence we have the following lemma.

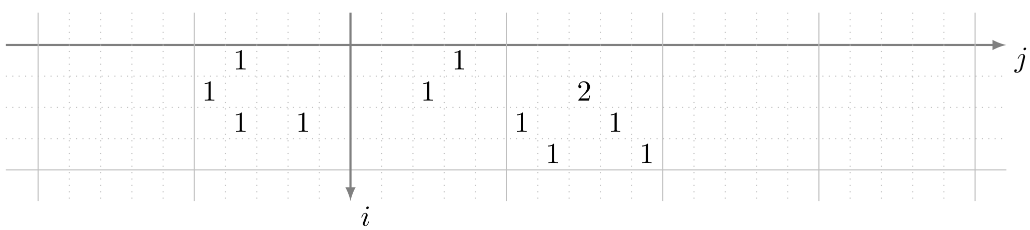
Lemma 5.3.1. *The set* $\widehat{\mathcal{M}}_{m\times n}$ *is a normal* $U_q(\widehat{\mathfrak{sl}}_m)$*-crystal with respect to* $\mathrm{wt}, \widetilde{e}_i, \widetilde{f}_i$ *for* $i \in \{\,0, 1, \dots, m-1\,\}$.

Example 5.3.2. Let A be the affine matrix in Example 5.2.4. For $i = 2$, the associated sequence σ and its reduced one $\widetilde{\sigma}$ are

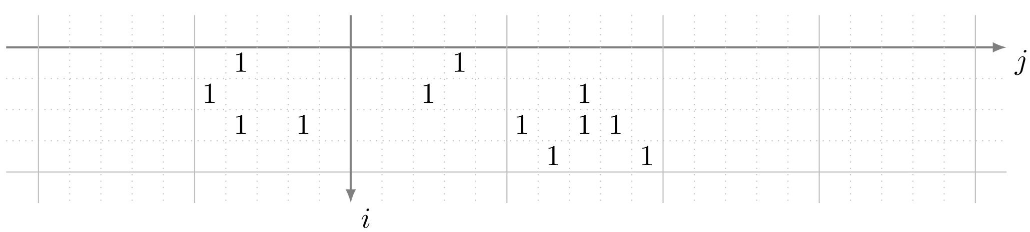
$$\begin{aligned} \sigma &= (+, -\,, -\,, +\,, -\,, +\,, +\,, -\,), \\ \widetilde{\sigma} &= (\,\cdot\,,\ \cdot\,, -,\ \cdot\,,\ \cdot\,, +,\ \cdot\,,\ \cdot\,). \end{aligned}$$

Hence the cell $(2, 8)$ is the position corresponding to the leftmost $+$ in $\widetilde{\sigma}$, and $\widetilde{f}_i A = A - \widehat{E}_{2\,8} + \widehat{E}_{3\,8}$.

$A =$



$\widetilde{f}_2 A =$



Here we present the submatrices of A and $\widetilde{f}_2 A$ with the row indices in $[4]$.

Next, we define an $U_q(\widehat{\mathfrak{sl}}_n)$-crystal structure on $\widehat{\mathcal{M}}_{m\times n}$ for $n \geq 2$, say $\mathrm{wt}^t, \varepsilon_i^t, \varphi_i^t, \widetilde{e}_i^t, \widetilde{e}_i^t$ for $i \in \{\,0, 1, \dots, n-1\,\}$, by applying the $U_q(\widehat{\mathfrak{sl}}_n)$-crystal structure on $\widehat{\mathcal{M}}_{n\times m}$ to the transpose

A^t of $A \in \widehat{\mathcal{M}}_{m\times n}$

$$\begin{gathered}\mathrm{wt}^t(A) = \mathrm{wt}(A^t),\\ \widetilde{e}_i^t A = (\widetilde{e}_i A^t)^t, \quad \widetilde{f}_i^t A = (\widetilde{f}_i A^t)^t,\\ \varepsilon_i^t(A) = \varepsilon_i(A^t), \quad \varphi_i^t(A) = \varphi_i(A^t).\end{gathered}$$

Proposition 5.3.3. *The operators $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in \{0,1,\dots,m-1\}$ commute with $\widetilde{e}_j^t$ and $\widetilde{f}_j^t$ for $j \in \{0,1,\dots,n-1\}$ on $\widehat{\mathcal{M}}_{m\times n} \cup \{\mathbf{0}\}$.*

Proof. Let $A \in \widehat{\mathcal{M}}_{m\times n}$ be given. Suppose that for each $1 \le i \le m$, we have $a_{ij} = 0$ unless $1 \le j \le n$. Then A can be viewed as an $m \times n$ matrix and it is well known that the proposition holds for A when $i, j \neq 0$ (see for examples [23, Lemma 3.4] or [38, Lemma 1.4.7]). For an arbitrary $A \in \widehat{\mathcal{M}}_{m\times n}$ and i, j, we may apply the same argument. $\square$

Remark 5.3.4. We remark that $\widetilde{e}_j^t$ and $\widetilde{f}_j^t$ preserve wt except for $j = 0$, and $\widetilde{e}_i$ and $\widetilde{f}_i$ preserve wt^t except for $i = 0$. $\widetilde{x}_i$ and $\widetilde{y}_j^t$ on $\widehat{\mathcal{M}}_{m\times n} \cup \{\mathbf{0}\}$ are strict morphisms of $U_q(\widehat{\mathfrak{sl}}_n)$-crystals and $U_q(\widehat{\mathfrak{sl}}_m)$-crystals, respectively for $x, y \in \{e, f\}$, $i \in \{0,1,\dots,m-1\}$, and $j \in \{0,1,\dots,n-1\}$. In case of $\widetilde{e}_0^t, \widetilde{f}_0^t$, we have

$$\mathrm{wt}(\widetilde{e}_0^t A) = \mathrm{wt}(A) - \delta, \quad \mathrm{wt}(\widetilde{f}_0^t A) = \mathrm{wt}(A) + \delta,$$

for $A \in \widehat{\mathcal{M}}_{m\times n}$. The same holds for $\widetilde{e}_i$ and $\widetilde{f}_i$. Hence, the we understand the set $\widehat{\mathcal{M}}_{m\times n}$ as a $(U_q'(\widehat{\mathfrak{sl}}_m) \times U_q'(\widehat{\mathfrak{sl}}_n))$-crystal or $(U_q(\widehat{\mathfrak{sl}}_m) \times U_q(\widehat{\mathfrak{sl}}_n))$-crystal.

Let

$$\mathcal{T}_{m\times n} = \bigsqcup_{\lambda \in \mathscr{P}_m \cap \mathscr{P}_n} CSST_{[m]}(\lambda) \times \mathcal{B}_n(\lambda). \tag{5.3.2}$$

We regard it as $(U_q'(\widehat{\mathfrak{sl}}_m) \times U_q(\widehat{\mathfrak{sl}}_n))$-crystal with respect to $\widetilde{e}_i$, $\widetilde{f}_i$, $\widetilde{e}_j^t$, $\widetilde{f}_j^t$ ($i \in \{0,1,\dots,m-1\}, j \in \{0,1,\dots,n-1\}$), where $\widetilde{e}_i$, $\widetilde{f}_i$ are the Kashiwara operators on $CSST_{[m]}(\lambda)$, and $\widetilde{e}_j^t$, $\widetilde{f}_j^t$ are the Kashiwara operators on $\mathcal{B}_n(\lambda)$.

The following is the second main result in this paper. The proof is given in Section 7.1.

Theorem 5.3.5. *The bijection*

$$\kappa : \quad \widehat{\mathcal{M}}_{m\times n} \longrightarrow \mathcal{T}_{m\times n}$$

commutes with $\widetilde{e}_i$, $\widetilde{f}_i$ *for* $i \in \{1, \dots, m-1\}$ *and* $\widetilde{e}^t_j$, $\widetilde{f}^t_j$ *for* $j \in \{0, 1, \dots, n-1\}$.

We remark that the map κ does not commute with $\widetilde{e}_0$ and $\widetilde{f}_0$, but $\kappa_1 := \pi_1 \circ \kappa$ does, where π_1 is the projection of $\mathcal{T}_{m\times n}$ along the first component (see Remark 7.1.10). Since $\mathrm{wt}^0_{\mathrm{cl}}(A) = \mathrm{wt}^0_{\mathrm{cl}}(P_0)$, κ_1 induces the following.

Corollary 5.3.6. *An affine matrix* $A \in \widehat{\mathcal{M}}_{m\times n}$ *is* $U_q(\widehat{\mathfrak{sl}}_m)$*-crystal equivalent to* P_0*, where* $\kappa(A) = (P_0, Q)$.

Note that κ is does not preserve wt^t. More precisely, for $A \in \widehat{\mathcal{M}}_{m\times n}$ with $\kappa(A) = (P_0, Q)$, we see from the definition of κ that

$$\mathrm{wt}^t(A) = \mathrm{wt}(A^t) = \mathrm{wt}(Q) - \left(\sum_{i=1}^{l} |\eta^{(i)}|\right)\delta,$$

where $\eta^{(i)}$ is the symmetrized offset vectors of $P_0^{(i)}$.

So in order to have a morphism of $U_q(\widehat{\mathfrak{sl}}_n)$-crystals, we may modify the weight function on $\widehat{\mathcal{M}}_{m\times n}$ by

$$\mathbf{wt}^t(A) = \mathrm{wt}(A^t) + H_m(A)\delta, \tag{5.3.3}$$

where $H_m(A) = \sum_{i=1}^{l} |\eta^{(i)}|$. Then we have the following.

Corollary 5.3.7. *If we regard* $\widehat{\mathcal{M}}_{m\times n}$ *as an* $U_q(\widehat{\mathfrak{sl}}_n)$*-crystal with respect to* $\mathbf{wt}^t$*, then* κ *is an isomorphism of* $(U_q(\mathfrak{sl}_m) \times U_q(\widehat{\mathfrak{sl}}_n))$*-crystals. In particular, an affine matrix* $A \in \widehat{\mathcal{M}}_{m\times n}$ *is* $U_q(\widehat{\mathfrak{sl}}_n)$*-crystal equivalent to* Q*, where* $\kappa(A) = (P_0, Q)$.

We remark that both m and n do not need to be greater than 1 for Theorem 5.3.5 and its corollaries. In particular, Corollary 5.3.6 holds for $n = 1$ and Corollary 5.3.7 holds for $m = 1$.

Remark 5.3.8. The function $H_m(\cdot)$ in (5.3.3) is related to the intrinsic energy function on $U_q(\widehat{\mathfrak{sl}}_m)$-crystals with P^0_{cl}-weights as follows. Let $T \in CSST_{[m]}(R)$ be given where $R = (a^b)$. Let $r = (r_1, \dots, r_{a-1})$ be the offset vector and $\eta = (\eta_1, \dots, \eta_{a-1})$ the symmetrized offset vector of T. Then we have

$$\mathcal{H}_m(T) = |\eta| - a|r|,$$

where $|r| = r_1 + \cdots + r_{a-1}$ and $\mathcal{H}_m(\cdot)$ is the intrinsic energy function on $CSST_{[n]}(R)$ with $\mathcal{H}_m(u_R) = 0$ (cf. [34] for its definition). Hence for $A \in \widehat{\mathcal{M}}_{m\times n}$ with $\kappa(A) = (P_0, Q)$, we have

$$H_m(A) = \sum_{i=1}^{l} \left(\mathcal{H}_m(P_0^{(i)}) + m_i |r^{(i)}| \right),$$

where $R_i = (a_i^{m_i})$ is the shape and $r^{(i)}$ is the offset vector of $P_0^{(i)}$ respectively.

Chapter 6

Dual affine RSK correspondence

In this chapter, we construct a dual analogue of Theorem 5.2.3 and 5.3.5.

6.1 Matrix-ball construction for dual affine matrices

Let

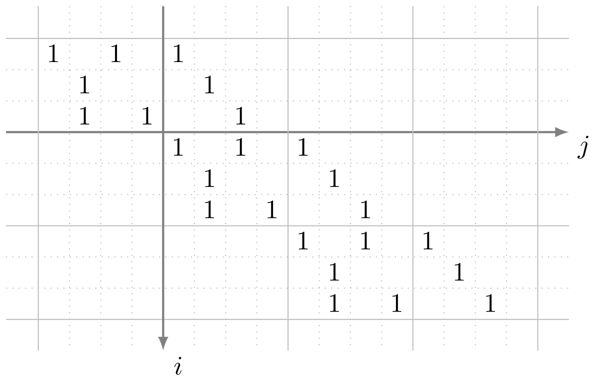
$$\widehat{\mathcal{N}}_{m\times n} = \left\{ B = (b_{ij})_{i,j\in\mathbb{Z}} \;\middle|\; \begin{array}{l} (1)\ b_{ij} \in \{0,1\} \text{ and } b_{i+mj+n} = a_{ij} \text{ for all } i,j\in\mathbb{Z}, \\ (2)\ \text{for each } j,\ b_{ij}=0 \text{ except for finitely many } i\text{'s.} \end{array} \right\}.$$

We call $B \in \widehat{\mathcal{N}}_{m\times n}$ a *generalized dual affine permutation* or a *dual affine matrix* for short. For a dual affine matrix B, we define *dual standardization of B* to be

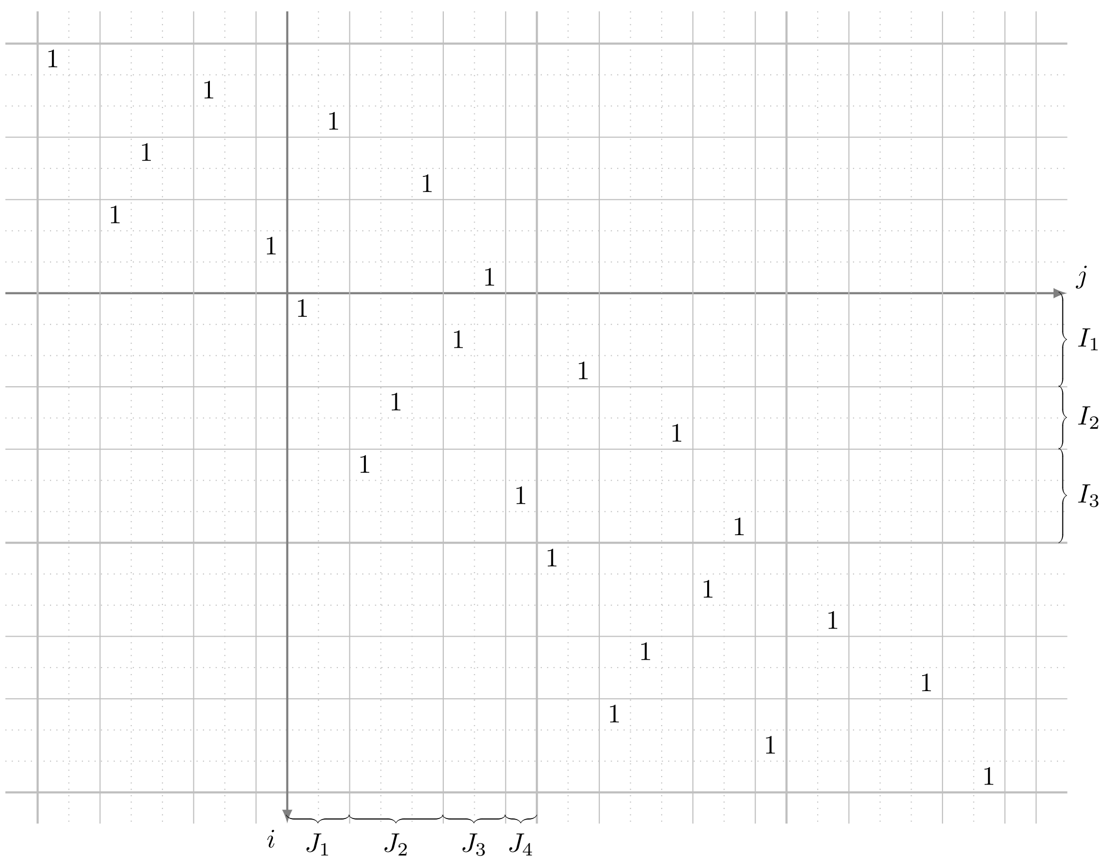
$$B^{\mathtt{st}'} = (B^{\bullet})^{\circ'}.$$

Note that we have $B^{\mathtt{st}'} = (B^{\bullet})^{\circ'} = (B^{\circ'})^{\bullet}$ and $B^{\mathtt{st}'} \in \widehat{\mathcal{W}}_{|B|}$.

Example 6.1.1. Let $B \in \widehat{\mathcal{N}}_{3\times 4}$ be given as follows.



Then $B^{\mathtt{st}'}$ is



Suppose that $\alpha \in \mathbb{Z}_+^m$ and $\beta \in \mathbb{Z}_+^n$ are given with $k = |\alpha| = |\beta|$. Let $\widehat{\mathcal{N}}_{m\times n}(\alpha, \beta)$ be the set of dual affine matrix whose row and column contents are α, β respectively. Let $\widehat{\mathcal{W}}_{k,[\alpha,\beta]}$ denote the set of affine permutations w such that w is α-ascending and w^{-1} is β-descending.

Lemma 6.1.2. *Under the hypothesis, we have a bijection*

$$\begin{array}{ccc} \widehat{\mathcal{N}}_{m\times n}(\alpha,\beta) & \longrightarrow & \widehat{\mathcal{W}}_{k,[\alpha,\beta]} \; . \\ B & \longmapsto & B^{\mathtt{st}'} \end{array}$$

We use another compass directional orders $>_{\mathtt{nW}}$, $<_{\mathtt{Ne}}$ on $\mathbb{Z}\times\mathbb{Z}$ as follows:

(1) $c_1 >_{\mathtt{nW}} c_2$ if and only if $i_1 \le i_2$ and $j_1 < j_2$,

(2) $c_1 <_{\mathtt{Ne}} c_2$ if and only if $i_1 > i_2$ and $j_1 \le j_2$,

for $c_1 = (i_1, j_1), c_2 = (i_2, j_2) \in \mathbb{Z}\times\mathbb{Z}$.

With respect to these partial orders, we have natural dual analogues of the notions and their properties given in Section 3. The proofs are almost parallel to those in the case of $\widehat{\mathcal{M}}_{m\times n}$. Let us summarize them as follows:

Suppose that $B \in \widehat{\mathcal{N}}_{m\times n}$ is given.

- A proper numbering d on B is defined as in Definition 5.1.8 with respect to $>_{\mathtt{nW}}$ instead of $>_{\mathtt{NW}}$. Let $d^{\mathtt{st}'}$ denote the proper numbering on $B^{\mathtt{st}'}$ which naturally correspond to d.

- Streams and channels B are defined in the same way as in Definition 5.1.7 with respect to $>_{\mathtt{nW}}$.

- The southwest channel numbering $d_B^{\mathtt{sw}}$ on B is the unique numbering such that

$$(d_B^{\mathtt{sw}})^{\mathtt{st}'} = d_{B^{\mathtt{st}'}}^{\mathtt{sw}}$$

Let d be a proper numbering on $B \in \widehat{\mathcal{N}}_{m\times n}$. Note that each level set $d^{-1}(k)$ forms a chain with respect to $<_{\mathtt{Ne}}$. Let $\{\mathbf{z}_k\}_{k\in\mathbb{Z}}$ be the set of zig-zags associated to d, where the inner corners of $\mathbf{z}_k$ are the set of elements in $d^{-1}(k)$ maximal with respect to $>_{\mathtt{nW}}$. Then $\{\mathbf{z}_k\}_{k\in\mathbb{Z}}$ satisfies

(z$'$.1) the inner corners of each $\mathbf{z}_k$ are contained in $\mathrm{supp}(A)$,

(z$'$.2) $\mathrm{supp}(A) \subseteq \bigcup_{k\in\mathbb{Z}} \mathbf{z}_k$,

(z$'$.3) $\mathbf{z}_k$ is located to the southeast of $\mathbf{z}_{k-1}$ for $k \in \mathbb{Z}$ with respect to $>_{\mathtt{nW}}$.

Remark 6.1.3. We should remark that no outer cell of $\mathbf{z}_k$ belongs to $\mathrm{supp}(B)$, and $\mathbf{z}_k$'s are not always mutually disjoint. More precisely, two horizontal lines (or line segments) in $\mathbf{z}_k$ and $\mathbf{z}_l$ $(k < l)$ may have non-trivial intersection, while vertical lines (or line segments) in $\mathbf{z}_k$ and $\mathbf{z}_l$ $(k < l)$ are always disjoint.

Suppose that a non-zero $B \in \widehat{\mathcal{N}}_{m\times n}$ is given and let

- $\{\mathbf{z}_k\}_{k\in\mathbb{Z}}$: the set of zig-zags associated to $d_B^{\mathtt{sw}}$,
- $B^{\flat'}$: the matrix obtained from B by the same rule as in $B^{\flat}$ with respect to $\{\mathbf{z}_k\}_{k\in\mathbb{Z}}$,
- $B^{(t)}$: the matrices in $\widehat{\mathcal{N}}_{m\times n}$ defined inductively by

$$B^{(0)} = B, \quad B^{(t)} = \left(B^{(t-1)}\right)^{\flat'} \quad (t \geq 1).$$

Note that $B^{(s-1)} \neq \mathbb{O}$ and $B^{(s)} = \mathbb{O}$ for some $s \geq 1$. For $1 \leq t \leq s$, we let

- $\{\mathbf{z}_k^{(t)}\}_{k\in\mathbb{Z}}$: the set of zig-zags associated to $d_{B^{(t-1)}}^{\mathtt{sw}}$,
- $\mathbf{s}^{(t)} = (\mathbf{a}_t, \mathbf{b}_t, \rho_t)$: the stream of the back-post corners of $\{\mathbf{z}_k^{(t)}\}_{k\in\mathbb{Z}}$ with flow μ_t,

where we can check that $\mu = (\mu_1, \ldots, \mu_s) \in \mathscr{P}_s$.

Now let

- P_0, Q_0 : the tableau of shape $\lambda = \mu'$ defined as in Section 5.2,
- P_0^t: the tableau of shape μ obtained by flipping P_0 with respect to the main diagonal. It follows immediately that $P_0^t \in RSST_{[m]}(\mu)$.
- $\rho = (\rho_1, \ldots, \rho_s) \in \mathbb{Z}^s$.

Let λ be the shape of P_0, and let $(R^{(1)}, \ldots, R^{(l)})$ be the rectangular decomposition of λ.

- $P_0^{(i)}$, $Q_0^{(i)}$: the rectangular decompositions of P_0 and Q_0,
- $\rho^{(i)} \in \mathbb{Z}^{m_i}$: the subsequence of ρ corresponding to the rectangular decomposition, where m_i is the occurrence of i in μ.
- $\eta^{(i)} \in \mathscr{P}_{m_i-1}$: the symmetrized offset vector of $(P_0^{(i)})^t$.

Then we define

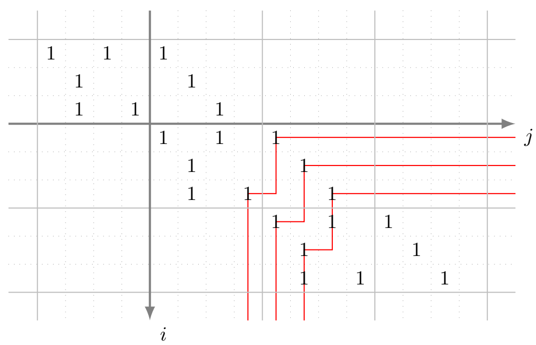
$$Q = \left(\tau^{\rho^{(l)}+\eta^{(l)}_{\mathtt{rev}}}(Q_0^{(l)}), \dots, \tau^{\rho^{(1)}+\eta^{(1)}_{\mathtt{rev}}}(Q_0^{(1)})\right) = \left(Q^{(l)}, \dots, Q^{(1)}\right). \tag{6.1.1}$$

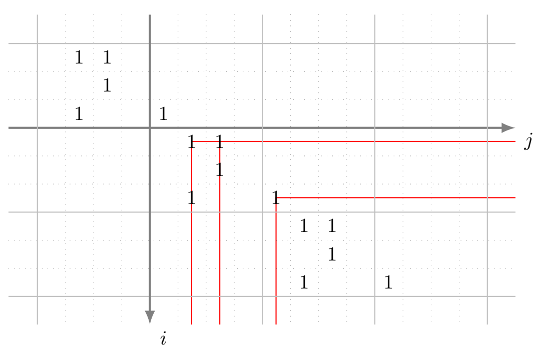
The following is a dual analogue of Theorem 5.2.3.

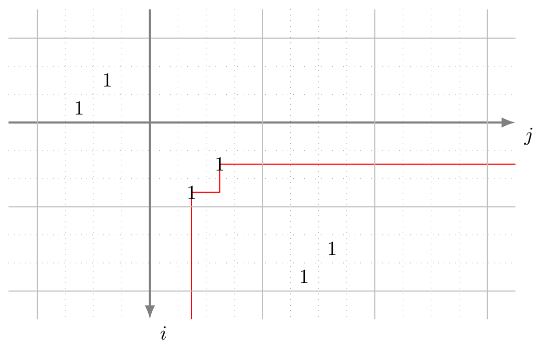
Theorem 6.1.4. *We have a bijection*

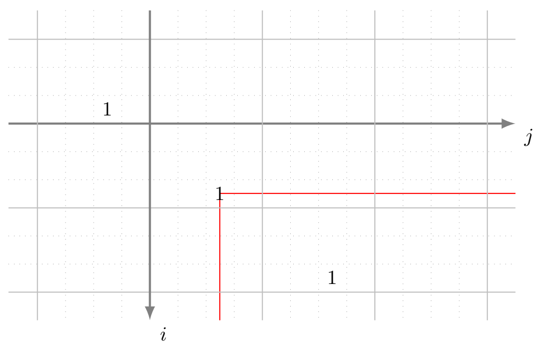
$$\begin{array}{rcccl} \kappa' : & \widehat{\mathcal{N}}_{m\times n} & \longrightarrow & \displaystyle\bigsqcup_{\lambda\in\cap\mathscr{P}_n} RSST_{[m]}(\lambda)\times\mathcal{B}_n(\lambda)\,, \\ & B & \longmapsto & (P_0^t, Q) \end{array} \tag{6.1.2}$$

Example 6.1.5. Let B be the dual affine matrix given in Example 6.1.1. Then

$B^{(0)} =$ 

$B^{(1)} =$ 

$B^{(2)} =$ 

$B^{(3)} =$ 

where the red lines denote the zig-zags associated to $d^{\mathtt{sw}}_{B^{(t-1)}}$ for $1 \leq t \leq 4$, and

$$P_0^t = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 1 & 1 & 3 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}, \quad Q_0 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 \\ \cline{1-2} 4 & 3 \\ \cline{1-2} \end{array}, \quad \rho = (2, 1, 0, 0).$$

In this case, R_1 and R_3 are the only non-trivial rectangles in the decomposition of the shape of P_0 and Q_0. It is easy to see that $\eta^{(3)}_{\mathtt{rev}} = (0, 2)$, $\eta^{(1)}_{\mathtt{rev}} = (0, 0)$, and hence

$$Q = \begin{array}{|c|c|c|c|}\hline 4 & 5 & 2 & 3 \\ \hline 5 & 6 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} \end{array}.$$

6.2 Dual affine RSK and crystals

Let us describe an $U_q(\widehat{\mathfrak{sl}}_m)$-crystal structure, and $U_q(\widehat{\mathfrak{sl}}_n)$-crystal structure on $\widehat{\mathcal{N}}_{m \times n}$ for $m \geq 2$.

Suppose that $B = (b_{ij})_{i,j \in \mathbb{Z}} \in \widehat{\mathcal{N}}_{m \times n}$ is given. For $i \in \{ 0, 1, \ldots, m-1 \}$, we define $\widetilde{e}_i B$ and $\widetilde{f}_i B$ as follows:

(1) Let σ be a sequence of $\{+,-\}$ given by

$$\sigma = (\cdots , \underbrace{+}_{b_{i\,j+1}}, \underbrace{-}_{b_{i+1\,j+1}}, \underbrace{+}_{b_{i\,j}}, \underbrace{-}_{b_{i+1\,j}}, \cdots),$$

and let $\widetilde{\sigma}$ be the reduced one, which is well-defined since σ has only finitely many $+$'s and $-$'s.

(2) If $\widetilde{\sigma}$ has at least one $-$, then we define

$$\widetilde{e}_i B = B + \widehat{E}_{ij_0} - \widehat{E}_{i+1j_0},$$

where j_0 is the column index of B corresponding the rightmost $-$ in $\widetilde{\sigma}$. If $\widetilde{\sigma}$ has no $-$, then we define $\widetilde{e}_i B = \mathbf{0}$. Similarly, if $\widetilde{\sigma}$ has at least one $+$, then we define

$$\widetilde{f}_i B = B - \widehat{E}_{ij_1} + \widehat{E}_{i+1j_1},$$

where j_1 is the column index of B corresponding the leftmost $+$ in $\widetilde{\sigma}$. If $\widetilde{\sigma}$ has no $+$, then we define $\widetilde{f}_i B = \mathbf{0}$.

Similarly, we define $\widetilde{e}^t_j B$ and $\widetilde{f}^t_j B$ as follows for $j \in \{0, 1, \ldots, n-1\}$:

(1) Let σ be a sequence of $\{+,-\}$ given by

$$\sigma' = (\cdots , \underbrace{+}_{b_{i\,j}}, \underbrace{-}_{b_{i\,j+1}}, \underbrace{+}_{b_{i+1\,j}}, \underbrace{-}_{b_{i+1\,j+1}}, \cdots).$$

and let $\widetilde{\sigma'}$ be the reduced one, which is well-defined since σ has only finitely many $+$'s and $-$'s.

(2) If $\widetilde{\sigma}$ has at least one $-$, then we define

$$\widetilde{e}^t_j B = B + \widehat{E}_{i_0\,j} - \widehat{E}_{i_0\,j+1},$$

where i_0 is the row index of B corresponding the rightmost $-$ in $\widetilde{\sigma'}$. If $\widetilde{\sigma'}$ has no $-$, then we define $\widetilde{e}^t_j B = \mathbf{0}$. Similarly, if $\widetilde{\sigma'}$ has at least one $+$, then we define

$$\widetilde{f}^t_j B = B - \widehat{E}_{i_1\,j} + \widehat{E}_{i_1\,j+1},$$

where i_1 is the row index of B corresponding the leftmost $+$ in $\widetilde{\sigma}'$. If $\widetilde{\sigma}'$ has no $+$, then we define $\widetilde{f}_j^t B = \mathbf{0}$.

Note that $\widetilde{e}_j^t$, $\widetilde{f}_j^t$ are not defined in the transpose of matrices for dual case.

Proposition 6.2.1. *The operators $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in \{0, 1, \ldots, m-1\}$ commute with $\widetilde{e}_j^t$ and $\widetilde{f}_j^t$ for $j \in \{0, 1, \ldots, n-1\}$ on $\widehat{\mathcal{N}}_{m\times n} \cup \{\mathbf{0}\}$.*

Let

$$\mathcal{S}_{m\times n} = \bigsqcup_{\lambda \in \mathscr{P}_n} RSST_{[m]}(\lambda') \times \mathcal{B}_n(\lambda).$$

We regard it as $(U_q'(\widehat{\mathfrak{sl}}_m) \times U_q(\widehat{\mathfrak{sl}}_n))$-crystal with respect to $\widetilde{e}_i$, $\widetilde{f}_i$, $\widetilde{e}_j^t$, $\widetilde{f}_j^t$ ($i \in \{0, 1, \ldots, m-1\}, j \in \{0, 1, \ldots, n-1\}$), where $\widetilde{e}_i$, $\widetilde{f}_i$ are the Kashiwara operators on $RSST_{[m]}(\lambda)$, and $\widetilde{e}_j^t$, $\widetilde{f}_j^t$ are the Kashiwara operators on $\mathcal{B}_n(\lambda)$.

Theorem 6.2.2. *The bijection*

$$\kappa' : \quad \widehat{\mathcal{N}}_{m\times n} \longrightarrow \mathcal{S}_{m\times n}$$

commutes with $\widetilde{e}_i$, $\widetilde{f}_i$ for $i \in \{1, \ldots, m-1\}$ and $\widetilde{e}_j^t$, $\widetilde{f}_j^t$ for $j \in \{0, 1, \ldots, n-1\}$.

Let $\kappa_1' = \pi_1 \circ \kappa'$, where π_1 is the projection of $\mathcal{S}_{m\times n}$ along the first component. Then κ_1' commutes with $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in \{0, 1, \ldots, m-1\}$, and preserves $\mathrm{wt}_{\mathrm{cl}}$. Hence we have the following.

Corollary 6.2.3. *A dual affine matrix $B \in \widehat{\mathcal{N}}_{m\times n}$ is $U_q(\widehat{\mathfrak{sl}}_m)$-crystal equivalent to P_0^t, where $\kappa(A) = (P_0^t, Q)$.*

Moreover, if we define $\mathbf{wt}^t$ on $\widehat{\mathcal{N}}_{m\times n}$ in the same way as in (5.3.3) with respect to κ', then we have the following analogue of Corollary 5.3.7.

Corollary 6.2.4. *If we regard $\widehat{\mathcal{N}}_{m\times n}$ as an $U_q(\widehat{\mathfrak{sl}}_n)$-crystal with respect to $\mathbf{wt}^t$, then κ' is an isomorphism of $(U_q(\mathfrak{sl}_m) \times U_q(\widehat{\mathfrak{sl}}_n))$-crystals. In particular, a dual affine matrix $B \in \widehat{\mathcal{N}}_{m\times n}$ is $U_q(\widehat{\mathfrak{sl}}_n)$-crystal equivalent to Q, where $\kappa'(B) = (P_0^t, Q)$.*

We remark that both m and n do not need to be greater than 1 for Theorem 5.3.5 and its corollaries. In particular, when $m = 1$ we have the following multiplicity-free decomposition

$$\widehat{\mathcal{N}}_{1\times n} \cong \bigoplus_{\lambda \in \mathscr{P}_n} \mathcal{B}_n(\lambda),$$

since $RSST_{[1]}(\lambda')$ consists of single element for all $\lambda \in \mathscr{P}_n$.

Chapter 7

Proofs

In this chapter, we prove the main results Theorem 5.3.5 and 6.2.2, namely:

$$\kappa(\widetilde{x}_i A) = \widetilde{x}_i \kappa(A), \quad \kappa(\widetilde{y}^t_j A) = \widetilde{y}^t_j \kappa(A) \tag{7.0.1}$$

$$\kappa'(\widetilde{x}_i A) = \widetilde{x}_i \kappa'(B), \quad \kappa'(\widetilde{y}^t_j B) = \widetilde{y}^t_j \kappa'(B) \tag{7.0.2}$$

for $A \in \widehat{\mathcal{M}}_{m\times n}$, $B \in \widehat{\mathcal{N}}_{m\times n}$, $i \in \{1,\ldots,m-1\}$, $j \in \{0,1,\ldots,n-1\}$ and $x, y \in \{e, f\}$.

7.1 Theorem 5.3.5

7.1.1 Augmented affine matrices

Let $\mathbf{s}$ be a stream of flow l with defining data $(\mathbf{a}, \mathbf{b}, r)$. We regard $\mathbf{s}$ as an element of $\mathcal{T}_{m\times n}$ in (5.3.2) as

$$\mathbf{s} = (\mathbf{a}, \tau^r \mathbf{b}) \in CSST_{[m]}((1^l)) \times \mathcal{B}_n((1^l)) \subset \mathcal{T}_{m\times n}.$$

Generally, let $\mathbf{s}^{(1)}, \ldots, \mathbf{s}^{(s)}$ be the streams corresponding to an affine matrix $A \in \widehat{\mathcal{M}}_{m\times n}$ in Section 5.2. We identify $\kappa(A)$ with $\mathbf{s}^{(s)} \otimes \cdots \otimes \mathbf{s}^{(1)}$ as a tensor product of crystals.

Define a map

$$\begin{array}{rccc} \Psi: & \widehat{\mathcal{M}}_{m\times n} & \longrightarrow & \displaystyle\bigsqcup_{l\geq 0} \widehat{\mathcal{M}}_{m\times n} \otimes \left(CSST_{[m]}((1^l)) \times \mathcal{B}_n((1^l)) \right). \\ & A & \longmapsto & A^\flat \otimes \mathbf{s}^{(1)} \end{array} \tag{7.1.1}$$

Since

$$\left((\Psi \otimes \mathrm{id}^{\otimes s-1}) \circ \cdots \circ (\Psi \otimes \mathrm{id}) \circ \Psi\right)(A) = \mathbb{O} \otimes \mathbf{s}^{(s)} \otimes \cdots \otimes \mathbf{s}^{(1)} = \mathbb{O} \otimes \kappa(A), \qquad (7.1.2)$$

where id is the identity morphism, it suffices to show that Ψ commutes with $\widetilde{x}_i$ and $\widetilde{y}_j^t$ for the proof of (7.0.1).

In order to simplify the description of $\widetilde{x}_i$ and $\widetilde{y}_j^t$ on $A^\flat \otimes \mathbf{s}$ (see (7.1.6)), let us introduce some additional notations and conventions. Let $\mathbb{Z}^* = \mathbb{Z} \cup \{\infty\}$, where we understand that $a < \infty$ and $a + \infty = \infty$ for $a \in \mathbb{Z}$. Let $A \in \widehat{\mathcal{M}}_{m\times n}$ be given. Let $\Psi(A) = A^\flat \otimes \mathbf{s}^{(1)}$, where $A^\flat = (a^\flat_{ij})_{i,j\in\mathbb{Z}}$. We define $A^* = (a^*_{ij})_{i,j\in\mathbb{Z}^*}$ by

$$a^*_{ij} = \begin{cases} a^\flat_{ij} & \text{if } (i,j) \in \mathbb{Z}\times\mathbb{Z}, \\ 1 & \text{if } i = \infty \text{ and } (k,j) \in \mathbf{s}^{(1)} \text{ for some } k \in \mathbb{Z}, \\ 1 & \text{if } j = \infty \text{ and } (i,k) \in \mathbf{s}^{(1)} \text{ for some } k \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, A^* is an augmented matrix obtained from $A^\flat$ by

$$A^* = A^\flat + \sum_{(i,j)\in\mathbf{s}^{(1)}} (E_{i\infty} + E_{\infty j}).$$

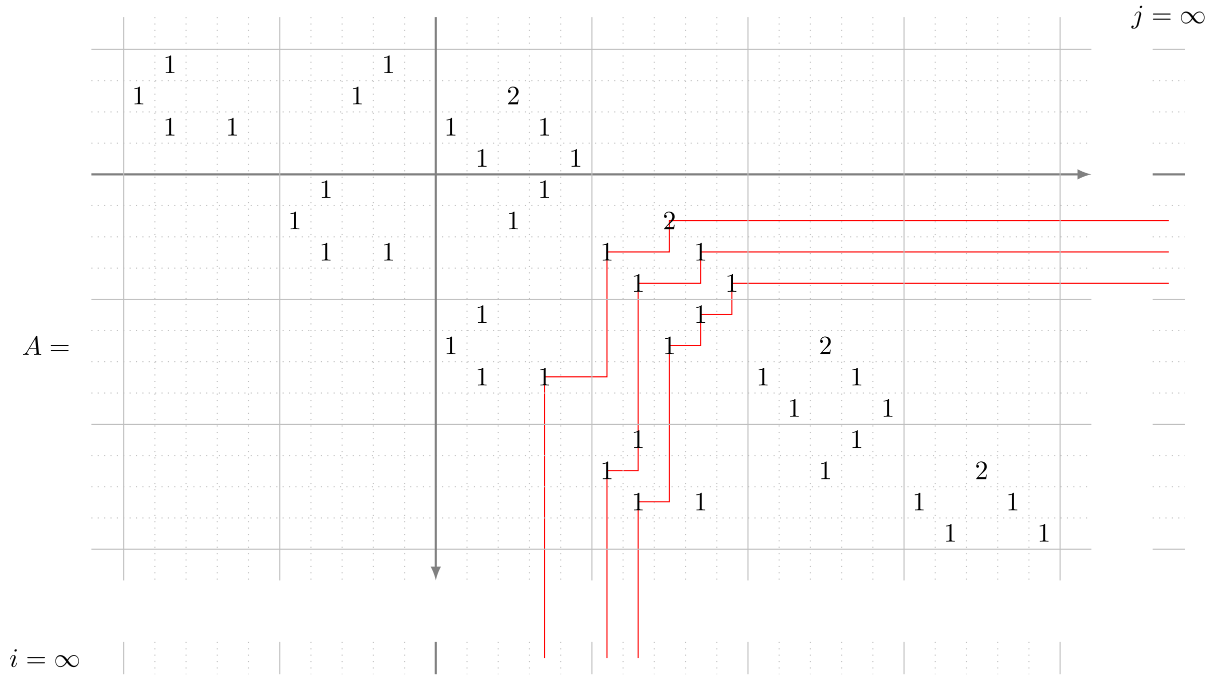
Note that A^* satisfies $a^*_{i+m\,j+n} = a^*_{ij}$ for $(i,j) \in \mathbb{Z}^*\times\mathbb{Z}^*$.

Let $\mathbf{z}$ be a zig-zag of A with the back-post corner (i,j). Let $\mathbf{z}^* = \mathbf{z} \cup \{(\infty,j),(i,\infty)\}$ and regard (∞,j) and (i,∞) as outer corners of $\mathbf{z}^*$. Then we may understand $\Psi(A) = A^*$ as a $\mathbb{Z}^*\times\mathbb{Z}^*$-matrix obtained by

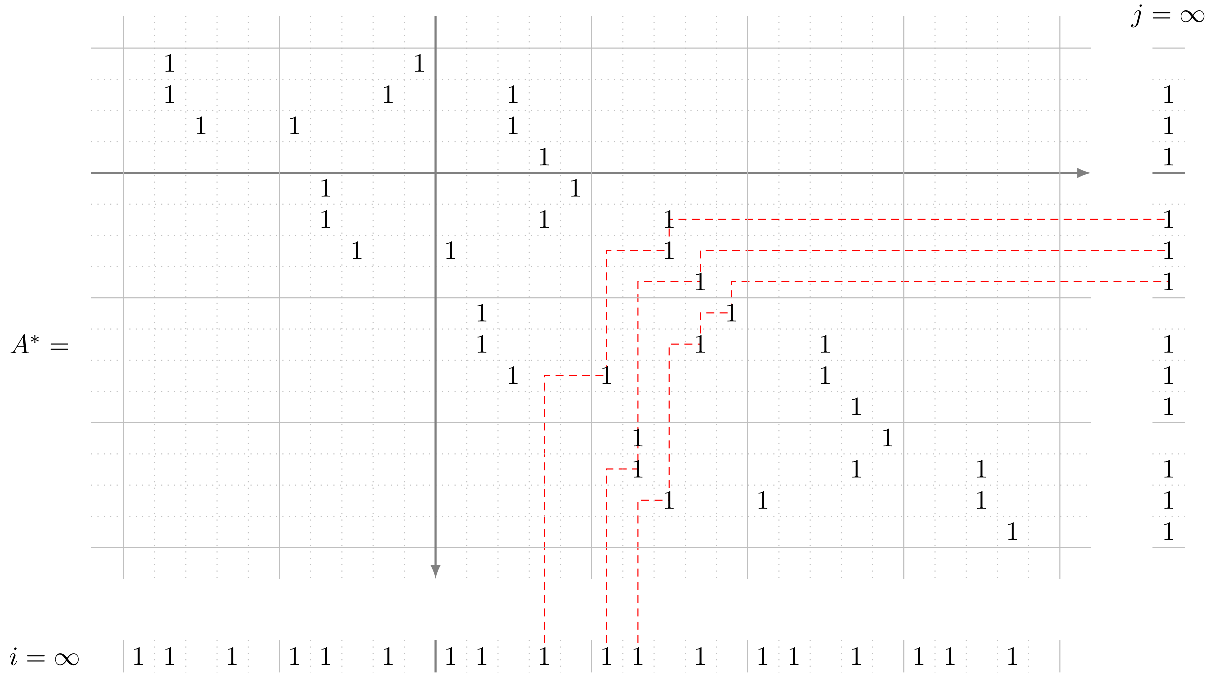
- identifying $A = (a_{ij})_{i,j\in\mathbb{Z}}$ with $(a_{ij})_{i,j\in\mathbb{Z}^*}$ where $a_{\infty j} = a_{i\infty} = 0$ for $i,j \in \mathbb{Z}^*$,
- applying the same rules for $\flat$ in Section 5.2 to A along $\mathbf{z}^*_k$ instead of $\mathbf{z}_k$,

where $\{\mathbf{z}_k\}_{k\in\mathbb{Z}}$ is the set of zig-zags associated to $d^{\mathtt{sw}}_A$. Note that we can recover $A^\flat$ and $\mathbf{s}^{(1)}$ from A^* and $\{\mathbf{z}^*_k\}_{k\in\mathbb{Z}}$. From now on, we assume that a matrix is a $\mathbb{Z}^*\times\mathbb{Z}^*$-matrix and a zig-zag is of the form $\mathbf{z}^*$.

Example 7.1.1. Let A be the affine matrix in Example 5.2.4. We regard A as a $\mathbb{Z}^*\times\mathbb{Z}^*$ matrix as follows:



where the red lines denote the zig-zags $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ associated to $d_A^{\mathtt{sw}}$. Then A^* is given as follows.



7.1.2 Tensor product rule

From now on, we fix $A \in \widehat{\mathcal{M}}_{m\times n}$ and $j \in [n]$. If there is no confusion, let us write $\widetilde{f}_j$, ε_j, and φ_j instead of $\widetilde{f}_j^t$, ε_j^t, and φ_j^t for simplicity. In the remaining of this section, we will focus on the proof of

$$\widetilde{f}_j\Psi(A) = \Psi(\widetilde{f}_j A). \tag{7.1.3}$$

Let

$$\begin{aligned} \sigma &= (\cdots, \underbrace{-\cdots-}_{a_{i-1j+1}}, \underbrace{+\cdots+}_{a_{i-1j}}, \underbrace{-\cdots-}_{a_{ij+1}}, \underbrace{+\cdots+}_{a_{ij}}, \cdots), \\ \sigma^* &= (\cdots, \underbrace{-\cdots-}_{a^*_{i-1j+1}}, \underbrace{+\cdots+}_{a^*_{i-1j}}, \underbrace{-\cdots-}_{a^*_{ij+1}}, \underbrace{+\cdots+}_{a^*_{ij}}, \cdots)\cdot(\underbrace{+}_{a^*_{\infty,j}}, \underbrace{-}_{a^*_{\infty j+1}}), \end{aligned} \tag{7.1.4}$$

where σ^* is a concatenation of two sequences. By tensor product rule (4.1.4), we see that

$$\widetilde{f}_j\left(A^\flat \otimes \mathbf{s}\right) = \begin{cases} A^\flat \otimes \widetilde{f}_j\mathbf{s} & \text{if the leftmost } + \text{ in } \widetilde{\sigma^*} \text{ corresponds to } (\infty, j), \\ \left(\widetilde{f}_j A^\flat\right) \otimes \mathbf{s} & \text{if the leftmost } + \text{ in } \widetilde{\sigma^*} \text{ corresponds to } (i,j) \text{ for some } i < \infty, \\ \mathbf{0} & \text{if } \widetilde{\sigma^*} \text{ has no } +. \end{cases} \tag{7.1.5}$$

In terms of A^*, this can be simplified as

$$\widetilde{f}_j A^* = \begin{cases} A^* - \widehat{E}_{ij} + \widehat{E}_{ij+1} & \text{if the leftmost } + \text{ in } \widetilde{\sigma^*} \text{ corresponds to } (i,j), \\ \mathbf{0} & \text{if } \widetilde{\sigma^*} \text{ has no } +, \end{cases} \tag{7.1.6}$$

where we assume that $\widehat{E}_{\infty j} = \sum_{k\in\mathbb{Z}} E_{\infty\, j+kn}$.

Lemma 7.1.2. *We have $\widetilde{f}_j A \neq \mathbf{0}$ if and only if $\widetilde{f}_j \Psi(A) \neq \mathbf{0}$.*

Proof. We may assume that there exists a non-zero cell in the j-th column. Otherwise, we have $\widetilde{f}_j A = \widetilde{f}_j \Psi(A) = \mathbf{0}$.

Let $\{\mathbf{z}_k\}_{k\in\mathbb{Z}}$ be the set of zig-zags associated to $d_A^{\mathtt{sw}}$. Let k_0 (resp. k_1) be the minimal (resp. maximal) value of $d_A^{\mathtt{sw}}$ in the j-th column. For $k_0 \leq k \leq k_1$, let i_k be the minimal row index with $(i_k, j) \in \mathbf{z}_k$. Note that $(i_k, j+1) \in \mathbf{z}_k$.

Put

$$\begin{aligned} \sigma_k &= (\underbrace{+\cdots+}_{a_{i_k\, j}}, \underbrace{-\cdots-}_{a_{i_k+1\, j+1}}, \cdots, \underbrace{+\cdots+}_{a_{i_{k+1}-1\, j}}, \underbrace{-\cdots-}_{a_{i_{k+1}\, j+1}}), \\ \sigma_k^* &= (\underbrace{+\cdots+}_{a^*_{i_k\, j}}, \underbrace{-\cdots-}_{a^*_{i_k+1\, j+1}}, \cdots, \underbrace{+\cdots+}_{a^*_{i_{k+1}-1\, j}}, \underbrace{-\cdots-}_{a^*_{i_{k+1}\, j+1}}), \end{aligned}$$

for $k_0 \leq k < k_1$, and

$$\begin{aligned} \sigma_{-\infty} &= (\cdots, \underbrace{-\cdots-}_{a_{i_{k_0}-1 j+1}}, \underbrace{-\cdots-}_{a_{i_{k_0} j+1}}), \qquad \sigma_\infty = (\underbrace{+\cdots+}_{a_{i_{k_1} j}}, \underbrace{-\cdots-}_{a_{i_{k_1}+1 j+1}}, \cdots), \\ \sigma^*_{-\infty} &= (\cdots, \underbrace{-\cdots-}_{a^*_{i_{k_0}-1\, j+1}}, \underbrace{-\cdots-}_{a^*_{i_{k_0}\, j+1}}), \qquad \sigma^*_\infty = (\underbrace{+\cdots+}_{a^*_{i_{k_1}\, j}}, \underbrace{-\cdots-}_{a^*_{i_{k_1}+1\, j+1}}, \cdots) \cdot (\underbrace{+}_{a^*_{\infty\, j}}, \underbrace{-}_{a^*_{\infty\, j+1}}). \end{aligned}$$

Then σ and σ^* in (7.1.4) decompose as follows:

$$\sigma = \sigma_{-\infty} \cdot \sigma_{k_0} \cdot \cdots \cdot \sigma_{k_1-1} \cdot \sigma_{\infty},$$
$$\sigma^* = \sigma^*_{-\infty} \cdot \sigma^*_{k_0} \cdot \cdots \cdot \sigma^*_{k_1-1} \cdot \sigma^*_{\infty}.$$

Suppose that k is given with $k_0 \le k < k_1$. Let u be the maximal row index with $(u, j) \in \mathbf{z}_k$, and let v be the minimal row index with $(v, j+1) \in \mathbf{z}_{k+1}$. Note that $i_k \le u$ and $v \le i_{k+1}$. Suppose first that $u < v$. Then we have

$$\sigma_k = (\underbrace{+\cdots+}_{a_{i_k\,j}},\ +\cdots+,\ \underbrace{+\cdots+}_{a_{u\,j}},\ \underbrace{-\cdots-}_{a_{v\,j+1}},\ -\cdots-,\ \underbrace{-\cdots-}_{a_{i_{k+1}\,j+1}}),$$
$$\sigma_k^* = (\underbrace{+\cdots+}_{a_{i_k\,j}-1},\ +\cdots+,\ \underbrace{+\cdots+}_{a_{u\,j}+1},\ \underbrace{-\cdots-}_{a_{v\,j+1}-1},\ -\cdots-,\ \underbrace{-\cdots-}_{a_{i_{k+1}\,j+1}+1}) = \sigma_k,$$

and hence $\widetilde{\sigma_k} = \widetilde{\sigma_k^*}$.

Next, suppose that $u \ge v$. In this case, we have $i_k < v \le u < i_{k+1}$. Therefore, (i_k, j), $(v, j+1)$ are inner corners, and (u, j), $(i_{k+1}, j+1)$ are outer corners. Then we have

$$\sigma_k = (\underbrace{+\cdots+}_{a_{i_k\,j}},\ +\cdots+,\ \underbrace{-\cdots-}_{a_{v\,j+1}},\ \cdots,\ \underbrace{+\cdots+}_{a_{u\,j}},\ -\cdots-,\ \underbrace{-\cdots-}_{a_{i_{k+1}\,j+1}}),$$
$$\sigma_k^* = (\underbrace{+\cdots+}_{a_{i_k\,j}-1},\ +\cdots+,\ \underbrace{-\cdots-}_{a_{v\,j+1}-1},\ \cdots,\ \underbrace{+\cdots+}_{a_{u\,j}+1},\ -\cdots-,\ \underbrace{-\cdots-}_{a_{i_{k+1}\,j+1}+1}).$$

Since one cancelling pair $(+, -)$ of σ_k in

$$(\underbrace{+\cdots+}_{a_{i_k\,j}},\ +\cdots+,\ \underbrace{-\cdots-}_{a_{v\,j+1}})$$

is moved to a pair $(+, -)$ of σ_k^* in

$$(\underbrace{+\cdots+}_{a_{u\,j}+1},\ -\cdots-,\ \underbrace{-\cdots-}_{a_{i_{k+1}\,j+1}+1}),$$

we conclude that $\widetilde{\sigma_k} = \widetilde{\sigma_k^*}$. By similar argument, we see that $(\sigma_{\pm\infty})^{\sim} = (\sigma^*_{\pm\infty})^{\sim}$.

Hence we have

$$\widetilde{\sigma_k} = \widetilde{\sigma_k^*} \quad (-\infty \le k \le \infty). \tag{7.1.7}$$

Since reducing a sequence does not depend on the order of cancelling $(+,-)$, we have

$$\begin{aligned}\widetilde{\sigma} &= (\sigma_{-\infty}\cdot\cdots\cdot\sigma_{\infty})^{\sim} = ((\sigma_{-\infty})^{\sim}\cdot\cdots\cdot(\sigma_{\infty})^{\sim})^{\sim}\\ &= ((\sigma^*_{-\infty})^{\sim}\cdot\cdots\cdot(\sigma^*_{\infty})^{\sim})^{\sim} = (\sigma^*_{-\infty}\cdot\cdots\cdot\sigma^*_{\infty})^{\sim} = \widetilde{\sigma^*}.\end{aligned}$$

This shows that $\varphi_j(A) = \varphi_j(\Psi(A))$ and $\varepsilon_j(A) = \varepsilon_j(\Psi(A))$, and hence the lemma follows. □

From now on, we assume that $\widetilde{f}_j A \neq \mathbf{0}$ and $\widetilde{f}_j \Psi(A) \neq \mathbf{0}$. We also assume the following notations:

- u : the row index corresponding to the leftmost $+$ in $\widetilde{\sigma}$,
- u^* : the row index corresponding to the leftmost $+$ in $\widetilde{\sigma^*}$,
- $s = d^{\mathtt{sw}}_A(u,j)$,
- $\widetilde{A} = \widetilde{f}_j A = (\widetilde{a}_{ij})_{i,j\in\mathbb{Z}^*}$.

We have

$$\widetilde{A} = \widetilde{f}_j A = A - \widehat{E}_{uj} + \widehat{E}_{uj+1}, \quad \widetilde{f}_j A^* = A^* - \widehat{E}_{u^*j} + \widehat{E}_{u^*j+1}. \tag{7.1.8}$$

Note that the leftmost $+$ in $\widetilde{\sigma^*}$ also appears in $(\sigma^*_s)^{\sim}$ by (7.1.7). More explicitly, u^* is the minimal row index such that $u \leq u^*$ and $a^*_{u^*j} \neq 0$. In particular, we have

$$\begin{cases} u < u^* & \text{if } (u,j) \text{ is an inner corner of } \mathbf{z}_s \text{ with } a_{uj} = 1,\\ u = u^* & \text{otherwise.}\end{cases}$$

7.1.3 Southwest channel numberings on A and $\widetilde{A}$

In this subsection, we discuss the relation between the southwest channel numberings on A and $\widetilde{A}$.

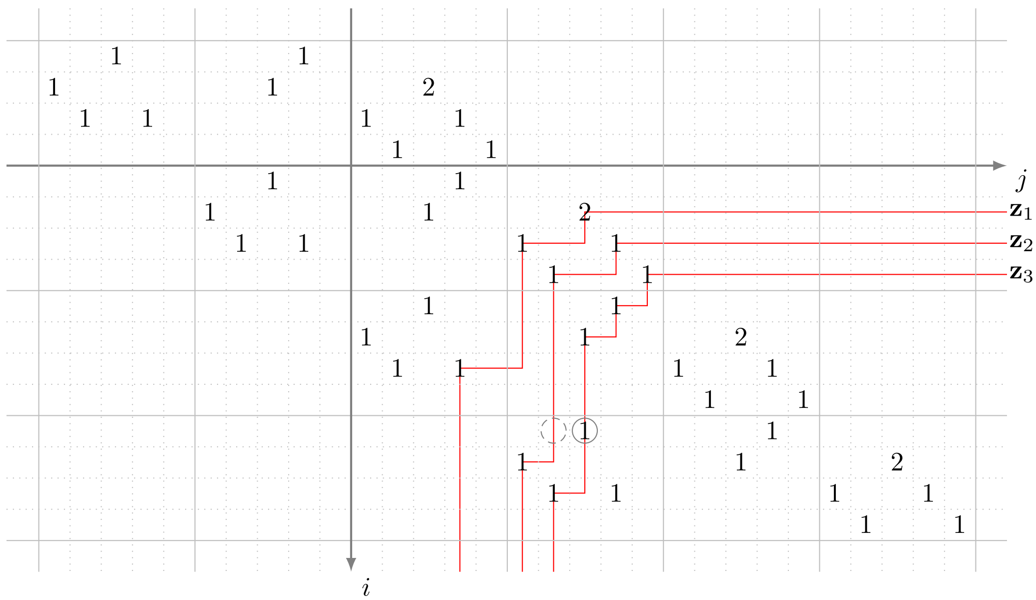
For $(x,y) \in \mathbb{Z}^2$, let $(x,y)^{\wedge} = \{\, \tau^k(x,y) \,|\, k \in \mathbb{Z}\,\}$. We have

$$\mathrm{supp}(\widetilde{A}) - \mathrm{supp}(A) \subseteq (u, j+1)^{\wedge}, \quad \mathrm{supp}(A) - \mathrm{supp}(\widetilde{A}) \subseteq (u,j)^{\wedge},$$

where the equalities hold when $a_{uj+1} = 0$ and $a_{uj} = 1$ respectively.

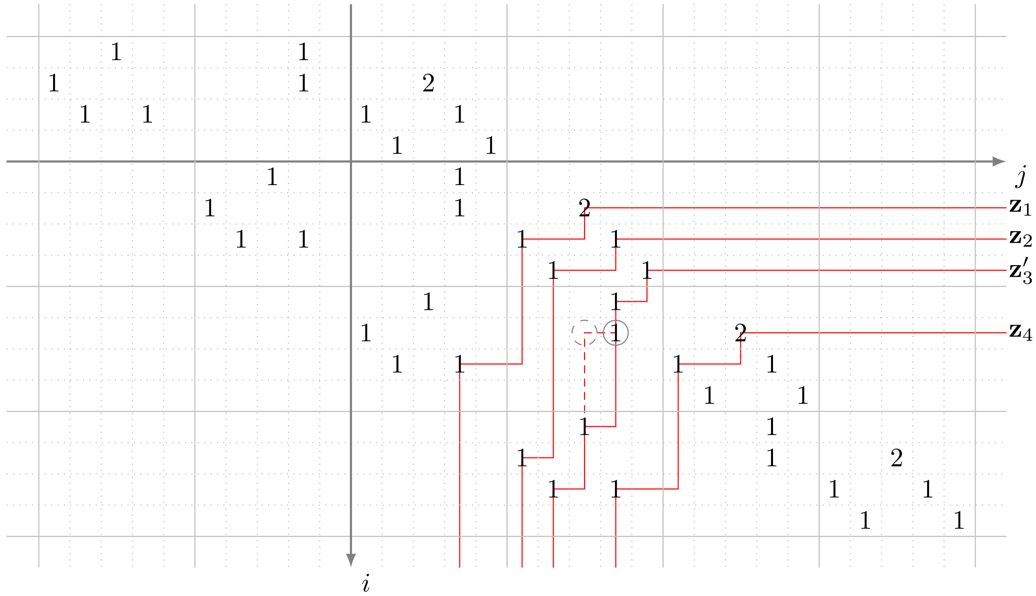
We first give an example, where a proper numbering on $\widetilde{A}$ is induced by a proper numbering on A.

Example 7.1.3. Let A be the affine matrix given in Example 5.2.4. It is easily checked that $\widetilde{f}_2^t A = A - \widehat{E}_{52} + \widehat{E}_{53}$. Consider the set of zig-zags $Z = \{\mathbf{z}_k\}_{k\in\mathbb{Z}}$ corresponding to the southwest channel numbering $d_A^{\mathbf{st}}$ on A. If we draw the zig-zags $\mathbf{z}_1$, $\mathbf{z}_2$ and $\mathbf{z}_3$ over $\widetilde{f}_2^t A$ as follows;



then one can see that Z satisfies the conditions (z.1)-(z.3) in Section 3.2 with respect to $\widetilde{f}_2^t A$. Here, the dashed and solid circles are the positions where $\widetilde{f}_2^t A$ differs from A. We conclude that Z induces a proper numbering on $\widetilde{f}_2^t A$.

It is also easily checked that $\widetilde{f}_3^t \widetilde{f}_2^t A = \widetilde{f}_2^t A - \widehat{E}_{23} + \widehat{E}_{24}$. However, Z does not give a proper numbering on $\widetilde{f}_3^t \widetilde{f}_2^t A$, since an inner corner $(2,3)$ of $\mathbf{z}_3$ is not a non-zero cell of $\widetilde{f}_3^t \widetilde{f}_2^t A$. If we modify a segment $(9,8), (8,8), (7.8).(6.8).(6.9)$ of $\mathbf{z}_3$ by $(9,8), (9,9), (8,9), (7,9), (6,9)$ (see below), it remedies the failure of the condition (z.1). The modified zig-zag, which is denoted by $\mathbf{z}_3'$, does not intersect with $\mathbf{z}_4$ as follow,



where dashed red line is the original part of $\mathbf{z}_3$, and the dashed circle and solid circle are the positions where $\widetilde{f}^t_3\widetilde{f}^t_2A$ differs from $\widetilde{f}^t_2A$. Hence the modified set of zig-zags $Z' = \{\cdots, \mathbf{z}'_0, \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}'_3, \mathbf{z}_4, \cdots\}$ give a proper numbering on $\widetilde{f}^t_3\widetilde{f}^t_2A$.

In the remainder of this section, we will see that the induced numberings on $\widetilde{f}^t_2A$ and $\widetilde{f}^t_3\widetilde{f}^t_2A$ are, in fact, the southwest channel numberings.

The following lemma describes how to construct a proper numbering on $\widetilde{A}$ from a given numbering d on A in general.

Lemma 7.1.4. *We have the following.*

(1) Let d be a proper numbering on A with the associated zig-zags $Z = \{\mathbf{z}_k\}_{k\in\mathbb{Z}}$ and $d(u,j) = s$. Then there exists a proper numbering d^- on $\widetilde{A}$ satisfying

$$d^-(c) = d(c) \qquad \text{if } c \in \operatorname{supp}(A) \cap \operatorname{supp}(\widetilde{A}),$$

$$d^-(u, j+1) = \begin{cases} d(u,j) & \text{if } u \text{ is minimal such that } (u,j) \in \mathbf{z}_s, \\ d(u,j)+1 & \text{otherwise.} \end{cases}$$

(2) Let d be a proper numbering on $\widetilde{A}$ with the associated zig-zags $Z = \{\mathbf{z}_k\}_{k\in\mathbb{Z}}$ and

$d(u, j+1) = t$. Then there exists a proper numbering d^+ on A satisfying

$$d^+(c) = d(c) \quad \textit{if } c \in \mathrm{supp}(A) \cap \mathrm{supp}(\widetilde{A}),$$

$$d^+(u,j) = \begin{cases} d(u,j) - 1 & \textit{if } (u,j) \in \mathbf{z}_{t-1}, \\ d(u,j) & \textit{otherwise.} \end{cases}$$

In particular, the widths of A and $\widetilde{A}$ are the same.

Proof. (1) We construct a set of zig-zags Z^- (by adjusting Z) which satisfies the conditions (z.1)-(z.3) in Section 3.2 with respect to $\widetilde{A}$ and hence gives a proper numbering d^- on $\widetilde{A}$.

Let i_s be the minimal row index with $(i_s, j) \in \mathbf{z}_s$. If $i_s = u$, then $(u, j+1) \in \mathbf{z}_s$ by definition of zig-zag. Suppose that $i_s < u$. Then (i_s, j) is an inner corner of $\mathbf{z}_s$, and $a_{i_s\, j} > 0$. Consider a subsequence of σ in (7.1.4)

$$(\underbrace{+\cdots+}_{a_{i_s j}}, \underbrace{-\cdots-}_{a_{i_s+1\, j+1}}, \cdots, \underbrace{+\cdots+}_{a_{u-1\, j}}, \underbrace{-\cdots-}_{a_{u\, j+1}}). \tag{7.1.9}$$

Since (u, j) is the cell corresponding to the leftmost $+$ in $\widetilde{\sigma}$, there exists no $+$ in the reduced form of (7.1.9). This implies that there exists some $a_{v\, j+1} > 0$ for some v with $i_s < v \le u$ so that $+$ in the cell (i_s, j) is paired with $-$ in $(v, j+1)$. It is easy to see that $d(v, j+1) = s+1$, and hence $(u, j+1) \in \mathbf{z}_{s+1}$.

Hence we see that $\mathrm{supp}(\widetilde{A}) \subseteq \bigsqcup_{k\in\mathbb{Z}} \mathbf{z}_k$ and Z satisfies the conditions in (z.2) and (z.3) for $\widetilde{A}$. Note that the condition (z.1) fails if and only if (u, j) is an inner corner of $\mathbf{z}_s$ with $a_{uj} = 1$.

Case 1. Suppose that (u, j) is not an inner corner of $\mathbf{z}_s$ or (u, j) is an inner corner of $\mathbf{z}_s$ with $a_{uj} > 1$. Then $Z^- := Z$ satisfies the condition (z.1), and induces a proper numbering d^- on $\widetilde{A}$ as given in (3.2.2). Hence d^- satisfies

$$d^-(u, j+1) = \begin{cases} s & \text{if } u = i_s, \\ s+1 & \text{if } i_s < u, \end{cases}$$

and $d^-(c) = d(c)$ for $c \in \mathrm{supp}(A) \cap \mathrm{supp}(\widetilde{A})$.

Case 2. Suppose that (u, j) is an inner corner of $\mathbf{z}_s$ with $a_{uj} = 1$. In this case, the condition (z.1) fails since the inner corner (u, j) of $\mathbf{z}_s$ does not lie in $\mathrm{supp}(\widetilde{A})$. Now let us modify $\mathbf{z}_s$ as follows: Let (v, j) be an outer corner of $\mathbf{z}_s$ and let

$$w = \min\{\, i \in \mathbb{Z} \,|\, u < i \leq v \text{ and } a_{ij} > 0 \,\}.$$

Note that $w = u^*$ when $d = d_A^{\mathtt{sw}}$. Consider a subsequence of σ

$$(\underbrace{+}_{a_{uj}}, \underbrace{-\cdots-}_{a_{u+1\,j+1}}, \cdots \underbrace{+\cdots+}_{a_{w-1\,j}}, \underbrace{-\cdots-}_{a_{w\,j+1}}). \tag{7.1.10}$$

We see that $a_{ij} = 0$ for $u < i \leq w - 1$ by definition of w, and moreover $a_{ij+1} = 0$ for $u < i \leq w$ since the reduced form of (7.1.10) is $(+)$. Indeed the sequence in (7.1.10) is $(+)$. According to this observation, we define a zig-zag $\mathbf{z}_s^-$ by replacing the cells

$$(w - 1, j),\ \cdots,\ (u + 1, j),\ (u, j)$$

in $\mathbf{z}_s$ with the following cells

$$(w, j + 1),\ (w - 1, j + 1),\ \cdots,\ (u + 1, j + 1).$$

Then each inner corner of $\mathbf{z}_s^-$ lies in $\mathrm{supp}(\widetilde{A})$.

Let Z^- be the set of zig-zags obtained from Z by replacing $\{\, \tau^k \mathbf{z}_s \,|\, k \in \mathbb{Z} \,\}$ with $\{\, \tau^k \mathbf{z}_s^- \,|\, k \in \mathbb{Z} \,\}$. Then Z^- satisfies the conditions (z.1)-(z.3) for $\widetilde{A}$, and hence induces a proper numbering d^- on $\widetilde{A}$. It is easy to see that $d^-(u, j + 1) = s$ since $u = i_s$, and that $d^-(c) = d(c)$ for $c \in \mathrm{supp}(A) \cap \mathrm{supp}(\widetilde{A})$.

By definition, the proper numberings d and d^- have the same flow, which implies that A and $\widetilde{A}$ have the same width. This proves (1).

(2) As in (1), we construct a set of zig-zags from Z to which a proper numbering d^+ on A is associated.

Case 1. Suppose first that $(u, j) \in \mathbf{z}_{t-1}$. Then there exists an inner corner (v, j) of $\mathbf{z}_{t-1}$ and an inner corner $(w, j + 1)$ of $\mathbf{z}_t$ with $v < w \leq u$. In particular, we have $\widetilde{a}_{vj} > 0$. Consider a subsequence of σ

$$(\underbrace{+\cdots+}_{\widetilde{a}_{vj}}, \underbrace{-\cdots-}_{\widetilde{a}_{v-1\,j+1}}, \cdots, \underbrace{+\cdots+}_{\widetilde{a}_{u-1\,j}}, \underbrace{-\cdots-}_{\widetilde{a}_{u\,j+1}}).$$

Since $\widetilde{a}_{vj} > 0$ and $(u, j+1)$ is the cell corresponding to the rightmost $-$, we have

$$\widetilde{a}_{w\,j+1} + \widetilde{a}_{w+1\,j+1} + \cdots + \widetilde{a}_{u\,j+1} \geq 2.$$

This shows that $w < u$ and the inner corner $(w, j+1)$ of $\mathbf{z}_t$ lies in $\mathrm{supp}(A)$. Hence $Z^+ := Z$ satisfies the conditions (z.1)-(z.3), and it induces a proper numbering d^+ on A. It is obvious that $d^+(u, j) = t - 1$ and $d^+(c) = d(c)$ for $c \in \mathrm{supp}(A) \cap \mathrm{supp}(\widetilde{A})$.

Case 2. Suppose that $(u, j) \notin \mathbf{z}_{t-1}$. Let i_t be the maximal row index with $(i_t, j+1) \in \mathbf{z}_t$. Since $(u, j) \notin \mathbf{z}_{t-1}$, we have $\widetilde{a}_{ij} = 0$ for $u \leq i < i_t$. Consider a subsequence of σ

$$(\underbrace{-\cdots-}_{\widetilde{a}_{u\,j+1}}, \underbrace{+\cdots+}_{\widetilde{a}_{uj}}, \cdots, \underbrace{-\cdots-}_{\widetilde{a}_{i_t\,j+1}}) = (\underbrace{-\cdots-}_{\widetilde{a}_{u\,j+1}}, \cdots, \underbrace{-\cdots-}_{\widetilde{a}_{i_t\,j+1}}). \tag{7.1.11}$$

Since the rightmost $-$ in $\widetilde{\sigma}$ corresponding to position $(u, j+1)$ is the one in (7.1.11), we see that $\widetilde{a}_{i\,j+1} = 0$ for $u < i \leq i_t$. We define $\mathbf{z}_t^+$ to be a zig-zag by replacing the cells

$$(i_t, j+1),\ (i_t - 1, j+1),\ \cdots,\ (u+1, j+1)$$

in $\mathbf{z}_t$ with the following cells

$$(i_t - 1, j),\ \cdots,\ (u-1, j),\ (u, j).$$

Then (u, j) is an inner corner of $\mathbf{z}_t^+$, and $(u, j) \in \mathrm{supp}(A)$. Let Z^+ be the set of zig-zags obtained from Z by replacing $\{\, \tau^k \mathbf{z}_t \,|\, k \in \mathbb{Z} \,\}$ with $\{\, \tau^k \mathbf{z}_t^+ \,|\, k \in \mathbb{Z} \,\}$. Then Z^+ satisfies the conditions (z.1)-(z.3) for A, and it induces a proper numbering d^+ on A. We have $d^+(u, j) = t$ and $d^+(c) = d(c)$ for $c \in \mathrm{supp}(A) \cap \mathrm{supp}(\widetilde{A})$. This proves (2). □

Remark 7.1.5. Let d_1 be a proper numbering on A. If follows from the construction of $d^{\pm}$ in the proof of Lemma 7.1.4 that

(1) $(d_1^-)^+ = d_1$,

(2) if d_2 is another proper numbering on A such that $d_1(c) \leq d_2(c)$ for $c \in \mathrm{supp}(A)$, then $d_1^-(c) \leq d_2^-(c)$ for $c \in \mathrm{supp}(\widetilde{A})$.

The similar properties hold for a proper numbering on $\widetilde{A}$.

Lemma 7.1.6. *We have the following.*

(1) Let C be a channel of A. Then there exists a channel C^- of $\widetilde{A}$ given by

$$\begin{cases} \big(C - (u,j)^\wedge\big) \cup (u,j+1)^\wedge & \text{if } (u,j) \in C,\ a_{uj} = 1 \text{ and } (i,j+1) \notin C \text{ for all } i, \\ \big(C - (u,j)^\wedge\big) \cup (v,j)^\wedge & \text{if } (u,j) \in C,\ a_{uj} = 1 \text{ and } (i,j+1) \in C \text{ for some } i, \\ C & \text{otherwise,} \end{cases}$$

where v is the minimal row index such that $u < v$ and $a_{vj} > 0$ if it exists.

(2) Let C be a channel of $\widetilde{A}$. Then there exists a channel C^+ of A given by

$$\begin{cases} \big(C - (u,j+1)^\wedge\big) \cup (u,j)^\wedge & \text{if } (u,j+1) \in C,\ a_{u\,j+1} = 1 \text{ and } (i,j) \notin C \text{ for all } i, \\ \big(C - (u,j+1)^\wedge\big) \cup (w,j+1)^\wedge & \text{if } (u,j+1) \in C,\ a_{u\,j+1} = 1 \text{ and } (i,j) \in C \text{ for some } i, \\ C & \text{otherwise,} \end{cases}$$

where w is the maximal row index such that $w < u$ and $a_{w\,j+1} > 0$ if it exists.

Proof. Let us prove (1) only, since the proof of (2) is similar.

First, suppose $(u,j) \notin C$ or $a_{uj} > 1$. Then $C \subseteq \mathrm{supp}(\widetilde{A})$ and C is a channel of $\widetilde{A}$. We put $C^- = C$ in this case.

Now suppose that $(u,j) \in C$ and $a_{uj} = 1$. Then we have $C \not\subseteq \mathrm{supp}(\widetilde{A})$. Let us write

$$C = \{\cdots >_{\texttt{NW}} c_{s-1} >_{\texttt{NW}} c_s >_{\texttt{NW}} c_{s+1} >_{\texttt{NW}} \cdots\}$$

with $c_s = (u,j)$. We have two cases.

Case 1. Suppose that $(i,j+1) \notin C$ for all i. Let C^- be a set obtained from C by replacing $c_s^\wedge = (u,j)^\wedge \subset C$ with $(u,j+1)^\wedge$, which is clearly a stream of $\widetilde{A}$ by assumption.

Case 2. Suppose that $(i,j+1) \in C$ for some i. Then we have $c_{s+1} = (u',j+1)$ for some $u' > u$. Consider a subsequence of σ in (7.1.4)

$$(\underbrace{+}_{a_{uj}}, \underbrace{-\cdots-}_{a_{u+1\,j+1}}, \cdots, \underbrace{+\cdots+}_{a_{u'-1\,j}}, \underbrace{-\cdots-}_{a_{u'\,j+1}}). \tag{7.1.12}$$

Since $-$ in c_{s+1} is paired with $+$ in (7.1.12) other than $+$ in $c_s = (u,j)$, we have $a_{ij} > 0$

for some $u < i < u'$. Let v be the minimal such one. Then we have

$$c_{s-1} >_{\mathtt{NW}} (v, j+1) >_{\mathtt{NW}} c_{s+1}. \tag{7.1.13}$$

Let C^- be a set obtained from C by replacing $c_s^\wedge = (u,j)^\wedge \subset C$ with $(v, j+1)^\wedge$, which is a stream of $\widetilde{A}$ by (7.1.13).

By definition, C^- has the same flow as C. Since A and $\widetilde{A}$ have the same width by Lemma 7.1.4, C^- is a stream of maximal flow, and hence a channel of $\widetilde{A}$ $\square$

Remark 7.1.7. Let C be channel of A. Under the above hypothesis, we have

$$(C^-)^+ - C = \begin{cases} (v,j) & \text{if } (u,j) \in C,\ a_{uj} = 1 \text{ and } (i, j+1) \in C \text{ for some } i, \\ (u, j+1) & \text{if } (u,j) \in C,\ a_{uj} = 1,\ a_{u\,j+1} > 0 \text{ and } (i, j+1) \notin C \text{ for all } i, \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that there exists no cell in $\mathrm{supp}(A)$ between (u,j) and (v,j) and between (u,j) and $(u, j+1)$. Hence it follows that

$$C \succcurlyeq_{\mathtt{sw}} (C^-)^+ \quad \text{or} \quad (C^-)^+ \succcurlyeq_{\mathtt{sw}} C,$$

and there exists no other channel between C and $(C^-)^+$. It is also easy to check that if C' is another channel of A with $C \succcurlyeq_{\mathtt{sw}} C'$, then we have

$$C^- \succcurlyeq_{\mathtt{sw}} (C')^-. \tag{7.1.14}$$

The similar properties also hold with respect to channels of $\widetilde{A}$.

Lemma 7.1.8. *We have the following.*

(1) Let C be a channel of A. If d is the channel numbering on A associated to C, then d^- is the channel numbering on $\widetilde{A}$ associated to C^-.

(2) Let C be a channel of $\widetilde{A}$. If d is the channel numbering on $\widetilde{A}$ associated to C, then d^+ is the channel numbering on A associated to C^+.

Proof. Let us prove (1) only since the proof of (2) is similar.

Let d' be the channel numbering on $\widetilde{A}$ associated to C^-. Since the widths of A and $\widetilde{A}$ coincide, we may assume that d^- coincides with d' on the channel C^- by adding a constant to d^-. Hence we have $d' \leq d^-$ by Lemma 3.1.9, and $(d')^+ \leq (d^-)^+ = d$ by Remark 7.1.5.

Let ℓ be the common width of A and $\widetilde{A}$. If $\ell > 1$, then we see from Lemma 7.1.6(1) that there exists $c \in C \cap C^-$ such that $c \in \mathrm{supp}(A) \cap \mathrm{supp}(\widetilde{A})$. We have

$$d(c) = d^-(c) = d'(c) = (d')^+(c).$$

So $(d')^+$ and d also coincide on C (cf. Remark 7.1.7), and $d \leq (d')^+$ by Lemma 3.1.9. Therefore, we have $d = (d')^+$, and $d^- = d'$ by Remark 7.1.5.

If $\ell = 1$, then it is not possible to have $d^-(u, j+1) = d(u, j)+1$ or $C^- = \big(C - (u, j)^\wedge\big) \cup (v, j)^\wedge$ since we must have another cell $(i, j+1)$ with $(u, j) >_{\texttt{NW}} (i, j+1) >_{\texttt{NW}} (u, j)+(m, n)$. Hence we see directly that $(d')^+(u, j) = d(u, j) = d^-(u, j+1) = d'(u, j+1)$. By similar arguments as in the above case, we conclude that $d^- = d'$. □

Now we can describe the southwest channel numbering on $\widetilde{A}$ in terms of the one on A.

Proposition 7.1.9. *Let d be the southwest channel numbering on A. Then d^- is the southwest channel numbering on $\widetilde{A}$. Equivalently, let d be the southwest channel numbering on $\widetilde{A}$. Then d^+ is the southwest channel numbering on A.*

Proof. Let $C_1 = C_A^{\texttt{sw}}$ and $C_2 = C_{\widetilde{A}}^{\texttt{sw}}$. Let d' be the southwest channel numbering on $\widetilde{A}$. By Lemma 7.1.8, we have

$$d^- = d_{\widetilde{A}}^{C_1^-}, \quad (d')^+ = d_A^{C_2^+}.$$

Thus it suffices to show that either $C_1 = C_2^+$ or $C_1^- = C_2$, which implies that $d = (d')^+$ or $d^- = d'$, respectively (see also Remark 7.1.5).

Since C_1 and C_2 are the southwest channels, we have

$$C_1 \succcurlyeq_{\texttt{sw}} C_2^+, \quad C_2 \succcurlyeq_{\texttt{sw}} C_1^-. \tag{7.1.15}$$

By Remark 7.1.7 (cf. (7.1.14)), we get from (7.1.15)

$$C_1 \succcurlyeq_{\texttt{sw}} C_2^+ \succcurlyeq_{\texttt{sw}} (C_1^-)^+, \quad C_2 \succcurlyeq_{\texttt{sw}} C_1^- \succcurlyeq_{\texttt{sw}} (C_2^+)^-. \tag{7.1.16}$$

We claim that $C_1 = C_2^+$ if $C_2 \succ_{\texttt{sw}} C_1^-$. By (7.1.16), we have $C_2 \succ_{\texttt{sw}} (C_2^+)^-$. By

Remark 7.1.7, we see that $C_2 \succ_{\mathtt{sw}} (C_2^+)^-$ occurs only when

$$C_2^+ = \big(C_2 - (u, j+1)^\wedge\big) \cup (w, j+1)^\wedge.$$

Then we have $C_2^+ = (C_2^+)^-$.

On the other hand, we have $C_1^- = (C_2^+)^-$ since there is no other channel between C_2 and $(C_2^+)^-$. Hence we get $C_1^- = (C_2^+)^- = C_2^+$, and in particular $(w, j+1) \in C_1^-$. Since we have $(u, j) \notin C_1$ by Lemma 7.1.6(1), it follows that $C_1 = C_1^- = C_2^+$. This proves the claim. □

7.1.4 Proof of (7.1.3)

Now we are in a position to prove (7.1.3). Let $d = d_A^{\mathtt{sw}}$ and let $Z = \{\mathbf{z}_k\}_{k\in\mathbb{Z}}$ be the set of zig-zags associated to d. Let Z^- be the set of zig-zags associated to d^- (see the proof of Lemma 7.1.4(1)). Note that $d^- = d_{\widetilde{A}}^{\mathtt{sw}}$ by Proposition 7.1.9.

Case 1. Suppose that (u, j) is not an inner corner of $\mathbf{z}_s$ or (u, j) is an inner corner of $\mathbf{z}_s$ with $a_{uj} > 1$. Since $Z = Z^-$ and the cells corresponding the leftmost $+$ in A and A^* coincide in this case, we have $\widetilde{A^*} = (\widetilde{A})^*$.

Case 2. Suppose that (u, j) is an inner corner of $\mathbf{z}_s$ with $a_{uj} = 1$.

Let us first compare $\mathbf{z}_s$ with the modified zig-zag $\mathbf{z}_s^-$. Let (v_0, j) be an outer corner of $\mathbf{z}_s$ and let v_1 be the minimal row index with $(v_1, j+1) \in \mathbf{z}_s$. Note that the inner and outer corners of Z and Z^- always coincide other than the following cells (more precisely, their orbits under $\tau^{\pm 1}$):

$$(u, j),\ (u^*, j),\ (v_0, j),\ (v_1, j+1),\ (u, j+1),\ (u^*, j+1), \tag{7.1.17}$$

where $v_1 \leq u < u^* \leq v_0 \leq \infty$. For the reader's convenience, we summarize the positions of the cells in (7.1.17) as follows:

	$\mathbf{z}_s$	$\mathbf{z}_s^-$
(u, j)	inner corner	
(u^*, j)	outer corner if $u^* = v_0$	inner corner if $u^* < v_0$
(v_0, j)	outer corner	outer corner if $u^* < v_0$
$(v_1, j+1)$	inner corner if $v_1 < u$	inner corner
$(u, j+1)$	outer corner if $v_1 < u$	inner corner if $v_1 = u$
$(u^*, j+1)$		outer corner

Hence we may write

$$\begin{aligned} A^* &= A - \widehat{E}_{uj} + \widehat{E}_{v_0 j} - \widehat{E}_{v_1\, j+1} + \widehat{E}_{u\, j+1} + B, \\ (\widetilde{A})^* &= \widetilde{A} - \widehat{E}_{u^* j} + \widehat{E}_{v_0 j} - \widehat{E}_{v_1\, j+1} + \widehat{E}_{u^*\, j+1} + B, \end{aligned} \tag{7.1.18}$$

where B is a finite linear combination of $\widehat{E}_{kl}$'s over the cells (k, l) not belonging to (7.1.17). Combining (7.1.8) and (7.1.18), we have

$$\begin{aligned} \widetilde{A^*} &= A^* - \widehat{E}_{u^* j} + \widehat{E}_{u^*\, j+1} \\ &= \Big(A - \widehat{E}_{uj} + \widehat{E}_{v_0 j} - \widehat{E}_{v_1\, j+1} + \widehat{E}_{u\, j+1} + B\Big) - \widehat{E}_{u^* j} + \widehat{E}_{u^*\, j+1} \\ &= \Big(A - \widehat{E}_{uj} + \widehat{E}_{u\, j+1}\Big) - \widehat{E}_{u^* j} + \widehat{E}_{v_0 j} - \widehat{E}_{v_1\, j+1} + \widehat{E}_{u^*\, j+1} + B \\ &= \widetilde{A} - \widehat{E}_{u^* j} + \widehat{E}_{v_0 j} - \widehat{E}_{v_1\, j+1} + \widehat{E}_{u^*\, j+1} + B \\ &= (\widetilde{A})^*. \end{aligned}$$

By *Case 1* and *Case 2*, we have $\widetilde{A^*} = (\widetilde{A})^*$. Let us write $\Psi(\widetilde{f}_j A) = (\widetilde{f}_j A)^\flat \otimes \mathbf{s}'$. From $\widetilde{A^*} = (\widetilde{A})^*$ and Proposition 7.1.9, we see that

$$(\widetilde{f}_j A)^\flat \otimes \mathbf{s}' = \begin{cases} A^\flat \otimes \widetilde{f}_j \mathbf{s} & \text{if } u^* = \infty, \\ (\widetilde{f}_j A^\flat) \otimes \mathbf{s} & \text{if } u^* < \infty. \end{cases}$$

Comparing this with (7.1.5), we have (7.1.3).

By (7.1.1), κ commutes with $\widetilde{f}_j$, and hence κ commutes with $\widetilde{e}_j$ for $j \in \{\,0, 1, \ldots, n-1\,\}$.

7.1.5 Proof of Theorem 5.3.5

We have proved that κ commutes with $\widetilde{e}_j^t$ and $\widetilde{f}_j^t$ for $j \in \{0, 1, \ldots, n-1\}$. Let us finish the proof of Theorem 5.3.5 by showing that κ commutes with $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in \{1, \ldots, m-1\}$.

First, it is not difficult to see that Proposition 7.1.9 still holds if we replace the southwest channel numberings with the northeast channel numberings. Hence by the same arguments as Section 7.1.4 we have

$$\widetilde{x}_i A^* = (\widetilde{x}_i A)^* \tag{7.1.19}$$

for $i \in \{0, 1, \ldots, m-1\}$ and $x \in \{e, f\}$. This implies that Ψ commutes with $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in \{1, \ldots, m-1\}$ (see Remark 7.1.10 for $i = 0$). Hence κ commutes with $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in \{1, \ldots, m-1\}$. This completes the proof of Theorem 5.3.5. $\square$

Remark 7.1.10. We should remark that $\widetilde{e}_0$ and $\widetilde{f}_0$ may not commute with Ψ. Let $A \in \widehat{\mathcal{M}}_{m\times n}$ be given such that $\widetilde{x}_0 A \neq \mathbf{0}$ for $x \in \{e, f\}$. Suppose that $\Psi(A) = A^\flat \otimes \mathbf{s}$ and $\mathbf{s} = (\mathbf{a}, \mathbf{b}, r)$.

If $\widetilde{x}_0(A^\flat \otimes \mathbf{s}) = A^\flat \otimes \widetilde{x}_0\mathbf{s}$, then it follows from (7.1.19) that $\Psi(\widetilde{x}_0 A) = A^\flat \otimes \mathbf{s}'$ and $\mathbf{s}' = (\widetilde{x}_0\mathbf{a}, \mathbf{b}, r')$, where $r' = r + 1$ (resp. $r - 1$) if $x = e$ (resp. $x = f$). Since $\widetilde{x}_0\mathbf{s} = (\widetilde{x}_0\mathbf{a}, \mathbf{b}, r) \neq \mathbf{s}'$, we have $\Psi(\widetilde{x}_0 A) \neq \widetilde{x}_0\Psi(A)$. If $\kappa(A) = (P_0, Q)$, then by applying Ψ repeatedly we have

$$\kappa(\widetilde{x}_0 A) = (\widetilde{x}_0 P_0, Q'),$$

for some $Q' \in \mathcal{B}_n(\lambda)$ with $Q' \neq Q$.

7.2 Theorem 6.2.2

Let $B \in \widehat{\mathcal{N}}_{m\times n}$ be given, and let

$$\kappa'(B) = (P_0^t, Q).$$

In this section, we show that

$$\widetilde{x}_i B = \widetilde{x}_i P_0^t, \quad (i \in \{0, \ldots, m-1\} \tag{7.2.1}$$

and

$$\widetilde{y}_j^t B = \widetilde{y}_j Q, \quad (j \in \{0, \ldots, n-1\} \tag{7.2.2}$$

for $x, y \in \{e, f\}$. We denote by $\mathbf{s}^{(1)}, \ldots, \mathbf{s}^{(s)}$ the streams corresponding to B given in Section 6.1.

7.2.1 Proof of (7.2.2)

$$\begin{array}{rccc} \Psi' : & \widehat{\mathcal{N}}_{m\times n} & \longrightarrow & \displaystyle\bigsqcup_{l\geq 0} \widehat{\mathcal{N}}_{m\times n} \otimes \big(RSST_{[m]}((l)) \times \mathcal{B}_n((1^l))\big) . \\ & B & \longmapsto & B^{\flat'} \otimes \mathbf{s}^{(1)} \end{array}$$

It is enough to see that $\widetilde{y}_j^t$ is compatible with Ψ'. Consider the dual row standardization $B \longmapsto B^{\bullet}$. According to the crystal structures of $\widehat{\mathcal{M}}_{m\times n}$ and $\widehat{\mathcal{N}}_{m\times n}$, we see that the map

$$\begin{array}{rccc} \bullet : & \displaystyle\bigsqcup_{m\geq 0} \widehat{\mathcal{N}}_{m\times n} & \longrightarrow & \displaystyle\bigsqcup_{m\geq 0} \widehat{\mathcal{M}}_{m\times n} . \\ & B & \longmapsto & B^{\bullet} \end{array}$$

is an $U_q(\widehat{\mathfrak{sl}}_n)$-crystal embedding. Hence the composition $\Psi(B^{\bullet})$ is compatible with $\widetilde{y}_j^t$ on B. Meanwhile, it is easy to see that

$$\Psi(B^{\bullet}) = (B^{\flat'})^{\bullet} \otimes (\mathbf{s}^{(1)})^{\bullet}$$

where $(\mathbf{s}^{(1)})^{\bullet}$ is the stream whose row indices are standardized from $\mathbf{s}^{(1)}$ according to B. In particular, we have $\kappa'(B) = (P_0^{\mathtt{st}}, Q)$ where $\kappa(B^{\bullet}) = (P_0, Q)$. Thus κ' commutes with $\widetilde{y}_j^t$.

7.2.2 Proof of (7.2.1)

Recall that the order of tensor product on $RSST_{[m]}(\lambda)$ is defined from the top row to the bottom. Define a map

$$\begin{array}{rccc} \psi' : & \widehat{\mathcal{N}}_{m\times n} & \longrightarrow & \displaystyle\bigsqcup_{l\geq 0} \big(RSST_{[m]}((l)) \times \mathcal{B}_n((1^l))\big) \otimes \widehat{\mathcal{N}}_{m\times n} \\ & B & \longmapsto & \mathbf{s}^{(1)} \otimes B^{\flat'} \end{array}$$

As in Section 7.1, we let $B^* = \mathbf{s}^{(1)} \otimes B^{\flat'}$, and regard it as an augmented matrix.

For the nonational convenience, let $\widetilde{B} = \widetilde{f}_i B$. Let

$$\begin{aligned}\sigma =&(\dots, \underbrace{+}_{b_{i,j+1}}, \underbrace{-}_{b_{i+1,j+1}}, \underbrace{+}_{b_{i,j}}, \underbrace{-}_{b_{i+1,j}}, \dots),\\ \sigma^* =&(\underbrace{-\cdots-}_{b^*_{i+1,\infty}}, \underbrace{+\cdots+}_{b^*_{i,\infty}}) \cdot (\dots, \underbrace{+}_{b^*_{i,j+1}}, \underbrace{-}_{b^*_{i+1,j+1}}, \underbrace{+}_{b^*_{i,j}}, \underbrace{-}_{b^*_{i+1,j}}, \dots),\end{aligned}$$

where $B = (b_{ij})$ and $B^* = (b^*_{ij})$. Let $\mathbf{z}_k$ be a zig-zags associated to the southwest channel numbering $d = d_B^{\mathtt{sw}}$. Since each $\mathbf{z}_k$ has at most one non-zero cell (i,j), we lable each $+$ in σ or σ^* by $+_k$ if $(i,j) \in \mathbf{z}_k$ with $b_{ij} = 1$. We also lable each $-$ in σ or σ^* by $-_k$ similarly.

The following lemma is dual analogue to Lemma 7.1.2.

Lemma 7.2.1. *We have $\widetilde{f}_i B \neq \mathbf{0}$ if and only if $\widetilde{f}_i \psi'(B) \neq \mathbf{0}$.*

Proof. Since the length of σ and σ^* are equal, it suffice to show that the number of cancelling pair $(+,-)$ in σ and σ^* are the same. Let $\tau = (\dots, \tau_{k+1}, \tau_k, \dots)$ be a sequence defined by

$$\tau_k = \begin{cases} \cdot & \text{if } \mathbf{z}_k \text{ has both nonzero cell } (i,j) \text{ and } (i+1,j') \text{ for some } j \text{ and } j', \\ + & \text{if } \mathbf{z}_k \text{ has nonzero cell } (i,j) \text{ for some } j \text{ and has no nonzero cell } (i+1,j') \text{ for all } j', \\ - & \text{if } \mathbf{z}_k \text{ has nonzero cell } (i+1,j') \text{ for some } j' \text{ and has no nonzero cell } (i,j) \text{ for all } j. \end{cases}$$

We claim that $(+_k, -_t)$ is a cancelling pair in σ if and only if $\tau_k = +$ is cancelled in τ with for some $\tau_{t'} = -$ or $\tau_k = \cdot$.

Suppose first that $\tau_k = \cdot$. In σ, $+_k$ precedes $-_k$. If $+_k$ is not cancelled, there must be $+_{k'}$ between $+_k$ and $-_k$ in σ with $k' < k$. Note that for such k', we have $\tau_{k'} = \cdot$. Therefore we have another $-_{k'}$ in σ such that $+_k$ precedes it. Repeating this argument concludes that we have infinitely many nonzero cells in i and $i+1$ row, which is contradiction.

Suppose now that $\tau_k = +$ and (τ_k, τ_t) is a cancelling pair in τ. We see that the number of $\tau_{k'} = +$ with $t < k' < k$ is equal to the number of $\tau_{k'} = -$ with $t < k' < k$. That is, the number of $+_{k'}$ with $t < k' < k$ is equal to the number of $-_{k'}$ with $t < k' < k$. Hence, if $+_k$ is not cancelled in σ, there must be $+_{k''}$ between $+_k$ and $-_t$ in σ with $k'' < t$. This leads a contradiction in a similar way described in the last paragraph. Conversely, suppose $\tau_k = +$ and $(+_k, -_t)$ is a cancelling pair in σ. Note that $t < k$ since $\tau_k = +$. Since $+_k$ is cancelled with $-_t$, there must be eqully many $+$ and $-$ between $+_k$ and $-_t$ in σ. In particular, the number of $+_{k'}$ with $t < k' < k$ is not greater than the number of

$-_{k'}$ with $t < k' < k$. Hence τ_k is cancelled with $\tau_{k''} = -$ for some $t \leq k'' < k$ and this completes the claim.

In a similar argument, we can show that $(+_k, -_t)$ is a cancelling pair in σ^* if and only if $\tau_t = -$ is cancelled in τ or $\tau_t = \cdot$. Hence both the numbers of cancelling pair in σ and σ^* are equal to the number of cancelling pair in τ plus the number of $\cdot$ in τ. □

Let $u_k^{(0)}$ (resp. $u_k^{(1)}$) be the minimal (resp. maximal) column indices with $(i, u_k^{(0)}) \in \mathbf{z}_k$ (resp. $(i, u_k^{(1)}) \in \mathbf{z}_k$) and let $v_k^{(0)}$ be the minimal column indices with $(i+1, v_k^{(0)}) \in \mathbf{z}_k$. Note that if $u_k^{(0)} < u_k^{(1)}$, then $(i, u_k^{(0)})$ is an inner corner and $(i, u_k^{(1)})$ is an outer corner of $\mathbf{z}_k$: in particular, $b_{i,u_k^{(0)}} = 1$ and $b^*_{i,u_k^{(1)}} = 1$.

Suppose that $\widetilde{f}_i B = B - \widehat{E}_{iu} + \widehat{E}_{i+1u}$ and let $d(i,j) = s$. Let $s_0 \geq s$ be the minimal number such that $u_{s_0}^{(1)} < u_{s_0+1}^{(0)}$.

Lemma 7.2.2. *Under the above hypothesis, we have*

$$\widetilde{f}_i(B^*) = B^* - \widehat{E}_{iu_{s_0}^{(1)}} + \widehat{E}_{i+1u_{s_0}^{(1)}}.$$

Proof. For a sign $\mathscr{S} \in \{+, -, \cdot\}$ in σ, we label it as $\mathscr{S}_k$ when the cell (i,j) corresponding to $\mathscr{S}$ is numbered k by d. Consider a connected subsequence $(\cdots, +_s)$ of σ. If $+_s$ is the first sign in σ, it is trivial that $s = s_0$ and $+_s$ is the first non matched $+$ in σ^*. So suppose that $+_s$ is not the first sign. Then the sign just before $+_s$ in σ is $-_t$ with $t > s$. If $t > s+1$, it is also trivial since the zig-zag $\mathbf{z}_{s+1}$ divide σ vertically. Let $t = s+1$. Consider three cases as follows.

(case 1) $+_t$ doesn't exists in σ,

(case 2) $+_t$ exists and $u_s^{(1)} < u_t^{(0)}$,

(case 3) $+_t$ exists and $u_s^{(1)} \geq u_t^{(0)}$,

Note that (case 1) implies $u_s^{(1)} < u_t^{(0)}$. Since $u_t^{(1)}$ is the column index of the position of $-_t$ in σ^*, (case 1) and (case 2) implies $s = s_0$ and $-_t$ is the sign just before $+_s$ in σ^*. Hence we are done in these case. In (case 3), $+_s$ precedes $-_t$ in σ^*. Then we repeat the above reasoning for $t = s+1$ and t', instead of s and t. If (case 1) and (case 2), $s_0 = t$ and $+_t$ is the first non cancelled $+$ in σ^*. If (case 3), we repeat for $t+1$, and so on. This procedure will terminates. □

Next, we compare the southwest channel numbering on B with those on $\widetilde{B}$. We keep the notation $u_k^{(0)}$, $u_k^{(1)}$ and s_0. Define d^- on $\widetilde{B}$ by

$$d^-(c) = \begin{cases} d(i,u) & \text{if } c = (i+1,u), \\ d(c) - 1 & \text{if } c = (i,j) \text{ for some } j \text{ with } s < d(c) \leq s_0, \\ d(c) & \text{otherwise.} \end{cases} \tag{7.2.3}$$

Similarly, let d' be any proper numbering on $\widetilde{B}$ and define d'^+ on B by

$$d'^+(c) = \begin{cases} d'(i+1,u) & \text{if } c = (i,u), \\ d'(c) + 1 & \text{if } c = (i,j) \text{ for some } j \text{ with } s \leq d'(c) < s'_0, \\ d'(c) & \text{otherwise.} \end{cases} \tag{7.2.4}$$

Lemma 7.2.3. *d^-, d'^+ is well defined proper numbering.*

Proof. For d^-, since $+_s$ is not cancelled, there is no $-_s$ as in the proof of Lemma 2.2. And there is such s_0 also as in Lemma 2.2. Hence the definition of d^- is valid. Then it is easy to see that the level set of d^- forms zig-zags satisfying the conditions of zig-zags formed by a proper numbering.

For d'^+, by similar reasoning, there is no $+_s$ and there is such s'_0. Then it is easy to see that d'^+ is a proper numbering □

Moreover, $(d^-)^+ = d$. And if d_2 is another numbering on B such that $d_2(i,u) = 2$ and $d_2 \geq d$, then s_0 of d_2 is smaller than that of d, so $d_2^- \geq d^-$.

Let $C = \{\cdots >_{\mathtt{nW}} c_0 >_{\mathtt{nW}} \cdots >_{\mathtt{nW}} c_l = c_0 + (m,n) >_{\mathtt{nW}} \cdots\}$ be a channel of B with $c_0 = (i,u)$ Let $r \geq 0$ be the maximal number such that $c_0, \ldots, c_r$ is in i-th row. Let $c_0^- = (i+1,u)$. Note that the length r $\mathtt{nW}$-chain between c_0 and c_{r+1} has length at most r, by the maximality of channel. With the signature rules, we conclude that there exists a chain $c_0 >_{\mathtt{nW}} c_1^- >_{\mathtt{nW}} \cdots >_{\mathtt{nW}} c_r^- >_{\mathtt{nW}} c_{r+1}$. with $c_1^-, \cdots, c_r^-$ is in $i+1$-th row. Therefore,

$$C^- = \{\cdots >_{\mathtt{nW}} c_0^- >_{\mathtt{nW}} \cdots >_{\mathtt{nW}} c_r^- >_{\mathtt{nW}} c_{r+1} >_{\mathtt{nW}} \cdots >_{\mathtt{nW}} c_{l-1} >_{\mathtt{nW}} c_l^- = c_0^- + (m,n) >_{\mathtt{nW}} \cdots\}$$

is a channel of $\widetilde{B}$.

Similarly, for a channel C' of $\widetilde{B}$, we may define C'^+ of B by

$$C'^+ = \{\cdots >_{\mathtt{nW}} c_0'^+ >_{\mathtt{nW}} c_1' >_{\mathtt{nW}} \cdots >_{\mathtt{nW}} c_{l-r+1}' >_{\mathtt{nW}} c_{l-r}'^+ >_{\mathtt{nW}} \cdots >_{\mathtt{nW}} c_l'^+ = c_0'^+ + (m,n) >_{\mathtt{nW}} \cdots\}$$

Note that we modify the cell *before* c_0' in C'^{+} case.

In this case, $+$ and $-$ preserves ordering on channels, but there is gap between C and $(C^{-})^{+}$. However, we always have $(i,u) \in C \cap (C^{-})^{+}$ and $(i+1,u) \in C' \cap (C'^{+})^{-}$.

Together with subsection 2.1. The lemma 6.6 in the paper also hold for dual case: $d^{-} = d'$

Let C, C' be the southwest channels of B and $\widetilde{B}$. In this case, C^{-} differs only by (i,u) from C. Hence, $C \geq_{sw} C'^{+} \geq_{sw} (C^{-})^{+} = C$ so $C = C'^{+}$. Thus $d^{-} = d'$ or equivalently $d'^{+} = d$.

Even though d^{-} is modified several zig-zags from d, the set of inner and outer corners are differ by one for each corner. More precisely, the inner corner (i,u) of $\mathbf{z}_s$ of B moves to $(i+1,u)$ and and outer corner $(i, u_{s_0}^{(1)})$ moves to $(i+1, u_{s_0}^{(1)})$. This amounts exactly that $\widetilde{f}_i$ moves a cell $(i, u_{s_0}^{(1)})$ of B^* one row down. $\square$

The remaining part of the proof follows from that $(d_B^{\mathtt{sw}})'$ is the southwest channel numbering on $\widetilde{B}$, which can be proved in similar manner described in Section 7.1.3.

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국문초록

Robinson-Schensted-Knuth (RSK) 대응은 음이 아닌 정수 계수를 갖는 행렬을 같은 모양의 반표준 타블로 쌍에 대응시키는 전단사 함수이다. 이 대응은 대수적 조합론, 표현론과 깊은 관련이 있으며, 행렬 공간 위의 대칭대수의 하우 쌍대성을 조합적으로 설명한다. 결정이론적 관점에서 RSK 대응은 행렬집합과 타블로 쌍들의 집합위에 정의된 결정구조를 보존한다.

최근 Chmutov-Pylyavskyy-Yudovina의 연구에서 행렬-공 구성이라 불리는 도형적 방법으로 RSK 대응을 아핀 순열로까지 확장하였다. 본 학위논문에서는 이들의 결과에 표준화를 사용하여 아핀 행렬로 확장된 아핀 RSK 대응을 소개한다. 이 아핀 RSK는 아핀 행렬을 같은 모양의 타블로 쌍에 대응시키는데, 이 중 하나는 레벨 1 완전 결정의 텐서곱의 원소이고, 다른 하나는 레벨 0 극단 무게 가군 결정의 원소이다. 이 때, 아핀 RSK 대응이 A형 결정 구조를 보존함을 증명하고, Immamura-Mucciconi-Sasamoto가 소개한 또다른 아핀 일반화된 RSK 대응과의 간략한 비교를 제시한다. 끝으로 쌍대 아핀 RSK 대응 또한 소개한다.

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감사의 글

학위 논문을 마무리하며 박사 과정을 돌아보니 정말 많은 분들의 도움으로 졸업을 한다는 생각이 듭니다. 먼저 본 학위 과정 동안 저를 연구자의 길로 이끌어 주신 권재훈 교수님께 진심으로 감사드립니다. 긴 연구 기간 동안 힘든 시기도 많았지만, 그 때마다 천천히 기다려주시고 격려해주신 교수님 덕분에 학위를 잘 마칠수 있었습니다. 교수님과의 연구를 통해 수학 뿐만 아니라 연구자로서 가져야할 자세도 배울 수 있었습니다. 그 가르침과 격려를 잊지 않고, 더 나은 연구자가 되도록 노력하겠습니다.

본 학위 논문 심사를 위해 시간을 내주신 오병권 교수님, 이승진 교수님, 오영탁 교수님, 박의용 교수님께도 감사드립니다. 논문 심사 외에도 강연과 세미나를 통해 많은 가르침을 받을 수 있었습니다. 특히 자격 시험과 수업을 통해 깊은 가르침을 주신 이승진 교수님께 다시 한 번 감사드립니다. 덕분에 낯설었던 조합론에 익숙해질 수 있었으며, 본 학위 논문 연구에 큰 도움이 되었습니다.

표현론을 함께 공부했던 연구실 동료들에게도 고마움을 전합니다. 연구실 선배로서 잘 적응하도록 도와준 장일승 교수, 허태혁 박사, 유정우 박사부터, 어려운 질문에 잘 대답해주던 신명이, 세미나 진행을 도와주던 아끼또와 정이 덕분에 즐겁게 연구할 수 있었습니다. 비록 긴 시간을 같이 보내진 못했지만 좋은 추억 함께한 상준이, 호빈이, 지원이, 수홍씨에게도 감사드립니다. 함께 세미나와 워크샵을 진행했던 최승일 박사님, 황병학 박사님, 김영훈 박사님, 이승재 박사님, 이석형 박사님께도, 그리고 이강산, 송아림, 박민희, 최동준, 윤상원 학생께도 감사의 말씀 드립니다. 덕분에 본 연구에 도움이 되는 많은 것들을 배울 수 있었습니다. 자격 시험 준비로 대학원 생활을 같이 시작한 재성이, 연이, 재훈이, 지혜, 두형이에게도 고맙습니다. 특히 434호에서 보낸 시간은 잊지 못할 것입니다. 마지막으로 본 학위를 마칠 때까지 항상 지지해주고 응원해준 부모님, 누나에게도 고맙습니다.