이학 박사 학위논문

Aleksandrov-Bakelman-Pucci estimate for nonlocal partial differential equation on manifold

(다양체 위에서 비국소 편미분 방정식의 Aleksandrov-Bakelman-Pucci 근사)

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서울대학교 대학원

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이 논문을 이학 박사 학위논문으로 제출함

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Abstract

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This thesis consists of three papers concerning nonlocal elliptic equations on the manifold. In the first paper, we establish the Alexander-Bekelman-Pucci estimate, which is the maximum principle, for fully nonlinear nonlocal equations in a nondivergence form on the manifold with nonnegative sectional curvature. Our approach is based on the control of normal map, and the direct comparison from the sectional curvature condition. The second paper deals with the ABP estimate on hyperbolic space. In hyperbolic space, the behavior of the heat kernel is different from that on Euclidean space. Hence, in the ABP estimate, there is nonhomogeneous behavior. The heart of the analysis lies in capturing the qualitative property for the integral values related to the jump kernel. From these ABP estimates, we obtain Krylov-Safonov Harnack inequality. The third paper discusses the equivalent definitions of fractional p-Laplacian on hyperbolic space. Especially, we establish Caffarelli's extension problem. As a remark, we get the coefficient of fractional Laplacian on hyperbolic space and the robustness of Harnack inequality and Hölder regularity.

Key words: ABP estimate, manifold, nonlocal operator, hyperbolic space, fractional Laplacian, extension problem
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Contents

Chapter 1

Introduction

Analyzing the property of the elliptic partial differential equation is a historically important topic. Especially, Analysis for the nonlocal operator like fractional Laplacian is a complex and notable subject in progress. On the other hand, the elliptic partial differential equation on the manifold is also a historical problem. There are great results on both elliptic divergent form and nondivergent form in regularity theory. On an extension, we will concentrate on an analysis of nonlocal nondivergent elliptic operators on some Riemannian manifolds through comparison methods.

In the first part of this thesis, we will concern with the interior regularity of integro-differential operators on certain manifolds. To illustrate the issues, let us explain the classical problem. Let Ω be an open and bounded subset in $\mathbb{R}^n$, and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a supersolution of $Lu = a_{ij}(x)u(x) = f$ where a_{ij}, f is continuous and $f/\mathcal{D}^* \in L^n(\Omega)$ for $\mathcal{D}*$ is the geometric mean of the eigenvalues of a_{ij}. By area formula, matrix inequality, and the estimate of measure of gradient mapping, we get the following maximum principle

$$\sup_{\Omega} u^- \leq \sup_{\partial\Omega} u^- + \left\| \frac{f^+}{\mathcal{D}^*} \right\|_{L^n(\{u=\Gamma u\})}, \tag{1.0.1}$$

where $\Gamma_u = \sup_L \{L \leq v \text{ in } \Omega, L \text{ an affine function}\}$ is the (convex) envelope and $\{u = \Gamma_u\}$ is the contact set. This is called Aleksandrov-Bakelman-Pucci

maximum principle (Shortly, we will call it an ABP estimate from now on). After that, we analyze the distribution of u roughly as follows.
Let u be a supersolution as before. There is universal constants ε, ε_0 and C such that if

$$u \geq 0 \text{ in } B_{2\sqrt{n}}, \quad \int_{Q_3} \inf u \leq 1, \quad \|f\|_{L^n(B_{2\sqrt{n}})} \leq \varepsilon_0,$$

there holds

$$|\{u \geq t\} \cap Q_1| \leq Ct^{-\varepsilon} \text{ for } t > 0.$$

From this information, we eventually get the Krylov-Safonov Harnack inequality and Hölder regularity. Roughly speaking, for positive solution on B_1,

$$\sup_{B_{\frac{1}{2}}} u \leq C(\inf_{B_{\frac{1}{2}}} u + \|f\|_{L^n(B_1)}),$$

and $u \in C^\alpha(B_1)$ for some universal $\alpha \in (0,1)$. In other words, there holds

$$|u(x) - u(y)| \leq C|x-y|^\alpha(\sup_{B_1} |u| + \|f\|_{L^n(B_1)}) \quad \forall x, y \in B_{\frac{1}{2}}.$$

This type of theory naturally extends to the nondivergent type operator. In this context, we establish Krylov-Safonov theory for nonlocal operators on certain Riemannian manifolds on an extension.

Let me briefly review the history. In the 1960s, Aleksandrov [1], Bakelman [6] and Pucci [94], independently established a maximum principle for linear elliptic equations in nondivergent form with bounded measurable coefficients. Their results were crucial in the proof of the Krylov-Safonov Harnack inequality and Hölder estimate for elliptic nondivergent linear operators with bounded measurable coefficients [96]. Since these estimates for linear operators depend on ellipticity constants and the geometry of the domain, it is naturally extended to the uniformly elliptic fully nonlinear equations; see [18], [19] and the references therein. The main idea of the extension is that one can consider the Pucci extremal operators and solution class so that the viscosity solution in the class is represented by two inequalities of Pucci extremal operators. Then the certain property of Pucci operator makes it possible to generalize linear theory to fully nonlinear equations. There are also other generalizations. For parabolic equation, see [83], [84], [103], [105]. For L^p viscosity solution, see [13], [35]. Other notable results are [11], [45], [43], [44], [85]. Recently, there are also generalizations for degenerate or singular cases [36], [37], [63], [5].

On the other hand, on manifold, Cabré established ABP estimate and Krylov-Safonov theory for nondivergent form on manifold with sectional curvature bounded below by 0 [12]. In this paper, to overcome a missing concept of the hyperplane, he suggested a generalized envelope that is derived from the distance squared function. Also, the geometric comparison principle was an important factor. Later, Wang and Zhang extended the ABP estimate on the manifold with Ricci curvature bounded below by -κ [106]. There are also other types of generalization for nondivergent type [74], [75], [77]. I would like to mention that there is a famous Li and Yau estimate for divergent form [88]. They used the logarithm of a solution to get a heat kernel estimate, which easily leads to a parabolic and elliptic Harnack inequality.

For the regularity of fractional Laplacian or more generally, nonlocal integro-differential operator, Caffarelli and Silvestre [15] first established ABP estimates for the nondivergent integro-differential operator. Due to the nonlocality, they couldn't use integration to deal with matrix inequality terms. Instead, they estimated a gradient of Γ on the annulus centered at the contact point. Interestingly, since the information on annulus was a measure value estimate, they didn't simply sum up those on annuli. Instead, by using the convexity(concavity) of an envelope, they established a gradient estimate on a ball inside of the inner complement of the annulus. The distinctive feature is that the forcing term appears as $\|f\|_{L^\infty}$. Guillen and Schwab [60] took a slightly different kernel and derived ABP estimates with not only $\|f\|_{L^\infty}$ but also $\|f\|_{L^n}$. Recently, there were several improvements in this direction [81], [82]. On the other hand, for divergent form, Kassmann proved in the spirit of De Giorgi-Nash-Moser theory the Harnack inequality and Hölder estimate in an appropriate form which included the nonlocality [67], [68]. Castro, Kuusi, and Palatucci overcame nonlinearity and succeeded to achieve similar results for fractional p Laplacian [40], [41]. There are also lots of improvements in this direction [23], [22], [69], [73], [90].

For more general spaces, Banica, González, and Sáez established a singular integral definition of fractional Laplacian on hyperbolic space. They also achieved Hölder estimates and extension problems. [7]. For sphere, Alonso-Orán, Córdoba, and Martínez derived integral representation[3]. On the other hand, in Dirichlet form theory, while searching for the heat kernel bounds, Grigor'yan, Hu, and Hu derived Hölder estimate for the jump type kernel in analytical method[57]. As a similar result, Chen, Kumagai, and Wang achieved Harnack inequality while they used a more probabilistic method[28].

In chapter 2, We will deal with ABP estimate for nondivergent nonlocal integro-differential operator, which is the main result of [72], on Riemannian manifold with sectional curvature bounded below by 0 and some minor assumptions. Although the nonlocality and nonsymmetric bring difficulties, with rather strong comparison principles and smoothness of manifold, we can achieve robust ABP estimate and Krylov-Safonov theory. In Chapter 3, We will achieve similar results on hyperbolic space, which is the main result of [70]. In hyperbolic space, as we mentioned before, there is a singular kernel representation due to the Fourier transform. Interesting properties such as inhomogeneity of scaling and exponential volume growth, which are closely connected, make analysis hard. However, we can derive enough properties such as the decaying property of integral values related to the jump kernel to achieve regularity.

The second part deals with equivalent definition of fractional p Laplacian, which will be the chapter 4 based on [71]. As the Laplcian operator can be characterized by many different methods, the fractional Laplacian operator also has many equivalent definitions. In Euclidean space, such a result is very well formulated in the survey paper [86]. Furthermore, del Teso, Gómez-Castro, and Vázquez, despite nonlinearity, prove the equivalent definition of fractional p Laplacian operator in 4 forms; via heat kernel, extension, Bochner integral, and Balakrishnan integral[39]. In a similar context, we derive three equivalent definitions of fractional p Laplacian on hyperbolic space.

Chapter 2

Harnack inequality for Nonlocal operators on Manifolds with nonnegative curvature

2.1 Introduction

This paper is concerned with the Harnack inequalities and Hölder estimates for nonlocal equations on Riemannian manifolds with nonnegative curvature. The Harnack inequalities and Hölder estimates for second order local operators have been studied extensively on Riemannian manifolds. We refer the reader to [108, 29, 97] for second order operators of divergence form and [12, 74, 106, 75, 77] for second order operators of non-divergence form. As nonlocal operators have attracted the attention, some of these results have been extended to nonlocal operators in various contexts. For example, the Harnack inequalities and Hölder estimates were established [28] in the framework of Dirichlet form theory on metric measure spaces with the volume doubling property, which include Riemannian manifolds with nonnegative curvature as a special case. Note that this result is appropriate for linear nonlocal operators of divergence form.

The operators under consideration in this paper are nonlinear and of non-divergence form. To the best of author's knowledge, the Krylov–Safonov Harnack inequalities for nonlocal operators were not available on Riemannian manifolds, while they are well-known in the Euclidean spaces [15, 60]. The aim of this work is to establish the Krylov–Safonov Harnack inequalities and Hölder estimates for fully nonlinear nonlocal operators of non-divergence form on Riemannian manifolds with nonnegative sectional curvatures. Since the underlying space is not flat, we focus on how the curvatures affect the regularity properties of solutions to the equations on manifolds.

2.1.1 Nonlocal operators on Riemannian manifolds

There are several ways of understanding nonlocal operators on the Euclidean spaces—via infinitesimal generators of stochastic processes, semigroup and heat kernels, the Dirichlet-to-Neumann map, or generators of Dirichlet forms; each of which has been applied to obtain nonlocal operators on Riemannian manifolds or more abstract spaces in different contexts. Applebaum and Estrade [4] suggested the operators of the form

$$Lu(x) = \int_{T_xM\setminus\{0\}} (u(\exp_x \xi) - u(x))\, \nu_x(\mathrm{d}\xi),$$

as infinitesimal generators of isotropic horizontal Lévy processes on Riemannian manifold M with some symmetry assumption on it, where T_xM is the tangent space at $x \in M$, exp is the exponential map, and ν_x is the Lévy measure.

On the other hand, Banica, González, and Sáez [7] provided the representation of the fractional Laplacian

$$-(-\Delta_{\mathbb{H}^n})^s u(x) = \text{p.v.} \int_{\mathbb{H}^n} (u(z) - u(x))\, \mathcal{K}\left(d_{\mathbb{H}^n}(z,x)\right) \mathrm{d}z \tag{2.1.1}$$

on the hyperbolic spaces $\mathbb{H}^n$ with negative constant curvature, where p.v. denotes the Cauchy principal value, by using the Fourier transform [14]. See [7] for the precise definition of the kernel $\mathcal{K}$ in (2.1.1). For more general compact manifolds and non-compact manifolds with Ricci curvature and injectivity radius bounded below, Alonso-Orán, Córdoba, and Martínez [3] provided an integral representation of the fractional Laplace–Beltrami operator with an error term using the well known formua

$$(-\Delta_g)^s u(x) = \int_0^\infty \left(e^{-t\Delta_g}u(x) - u(x)\right)\frac{\mathrm{d}t}{t^{1+s}}, \quad s \in (0,1,$$

and the heat kernel bounds. However, we do not take the operators in [7] and [3] as our definition because we are going to consider operators in a more specific form.

In the Dirichlet form theory, it is standard to assume that metric measure space (M, d, μ) satisfies the volume doubling property. In this setting, the fractional Laplacian-type Dirichlet form

$$\mathcal{E}(u,v) = \iint_{M\times M\setminus \mathrm{diag}} (u(x)-u(z))(v(x)-v(z))J(x,z)\,\mu(\mathrm{d}x)\mu(\mathrm{d}z)$$

with

$$\frac{\lambda}{\mu(B(x,d(x,z)))d(x,z)^{2s}} \le J(x,z) \le \frac{\Lambda}{\mu(B(x,d(x,z)))d(x,z)^{2s}}, \quad 0<\lambda\le\Lambda,$$

gives rise to the generator of the fractional Laplacian-type [28]. Motivated by the fact that the Riemannian manifolds with nonnegative curvatures are contained within this framework, we are going to modify this generator in order to define non-divergence form operator.

Let (M, g) be a smooth, complete, connected n-dimensional Riemannian manifold with nonnegative sectional curvatures. Let $d_x(z) = d(x,z)$ be the Riemannian distance between two points x and z in M, and μ_g be the Riemannian measure induced by g. The operator considered in this paper is modeled on the linear operator of the form

$$Lu(x) = (2-2s)\mathrm{p.v.}\int_M \frac{u(z)-u(x)}{\mu_g(B(x,d_x(z)))d_x(z)^{2s}}\,\mathrm{d}V(z), \tag{2.1.2}$$

where $s \in (0,1)$ is a constant. The choice of the factor $(2-2s)$ in (2.1.2) is now standard to obtain regularity estimates that are robust in the sense that the constants in the estimates remain uniform as $2s$ approaches 2 (see Section 2.1.2). Note that the operator above satisfies $\lim_{s\to 1} Lu = \Delta u$ as one can consider rotationally symmetric measure.

To define nonlinear operators, let us consider a class $\mathcal{L}_0$ of linear operators of the form

$$Lu(x) = \text{p.v.} \int_M (u(z)-u(x))\nu_x(z)\,\mathrm{d}V(z),$$

with density functions ν_x satisfying

$$\nu_x(z) = \nu_x(\mathcal{T}_x(z)) \quad \text{whenever } d_x(z) < \text{inj}(x), \tag{2.1.3}$$

where $\text{inj}(x)$ is the injectivity radius of x and $\mathcal{T}_x : B(x, \text{inj}(x)) \to B(x, \text{inj}(x))$ is a map given by $\mathcal{T}_x(z) = \exp_x(-\exp_x^{-1}(z))$, and

$$\lambda \frac{2-2s}{\mu_g(B(x,d_x(z)))d_x(z)^{2s}} \le \nu_x(z) \le \Lambda \frac{2-2s}{\mu_g(B(x,d_x(z)))d_x(z)^{2s}}. \tag{2.1.4}$$

Whenever we evaluate Lu at x, we split the integral as follows: for $R < \text{inj}(x)$,

$$\begin{aligned} Lu(x) = \text{p.v.} &\int_{B_R(x)} (u(z)-u(x))\nu_x(z)\,\mathrm{d}V(z) \\ &+ \int_{M\setminus B_R(x)} (u(z)-u(x))\nu_x(z)\,\mathrm{d}V(z). \end{aligned} \tag{2.1.5}$$

In contrast to the case of Euclidean spaces, the expression

$$\begin{aligned} Lu(x) = &\int_{B_R(x)} \delta(u,x,z)\nu_x(z)\,\mathrm{d}V(z) \\ &+ \int_{M\setminus B_R(x)} (u(z)-u(x))\nu_x(z)\,\mathrm{d}V(z), \end{aligned} \tag{2.1.6}$$

where $\delta(u,x,z) = (u(z)+u(\mathcal{T}_x(z))-2u(x))/2$ is the second order incremental quotients, is not available in general because M is not a symmetric manifold. Nevertheless, we will see in Lemma 2.2.3 that for $L \in \mathcal{L}_0$, (2.1.5) is well-defined when u is bounded in M and C^2 in a neighborhood of x. Throughout the paper this observation will be used frequently, especially for the squared distance function $d_x^2(z)$.

The extremal operators and elliptic operators are defined in the standard way as follows. To impose ellipticity on operators, we define the *maximal* and *minimal operators* by

$$\mathcal{M}^+_{\mathcal{L}_0} u(x) = \sup_{L \in \mathcal{L}_0} Lu(x) \quad \text{and} \quad \mathcal{M}^-_{\mathcal{L}_0} u(x) = \inf_{L \in \mathcal{L}_0} Lu(x).$$

We say that an operator $\mathcal{I}$ is *elliptic with respect to* $\mathcal{L}_0$ if

$$\mathcal{M}^-_{\mathcal{L}_0}(u-v)(x) \leq \mathcal{I}(u,x) - \mathcal{I}(v,x) \leq \mathcal{M}^+_{\mathcal{L}_0}(u-v)(x)$$

for every point $x \in M$ and for all bounded functions u and v which are C^2 near x.

We point out that the usual explicit expressions of extremal operators in the Euclidean spaces

$$\begin{aligned}
\mathcal{M}^+_{\mathcal{L}_0} u(x) &= (2-2s) \int_{\mathbb{R}^n} \frac{\Lambda \delta_+(u,x,y) - \lambda \delta_-(u,x,y)}{\omega_n |y|^{n+2s}} \, \mathrm{d}y \quad \text{and} \\
\mathcal{M}^-_{\mathcal{L}_0} u(x) &= (2-2s) \int_{\mathbb{R}^n} \frac{\lambda \delta_+(u,x,y) - \Lambda \delta_-(u,x,y)}{\omega_n |y|^{n+2s}} \, \mathrm{d}y,
\end{aligned}$$

where ω_n is the volume of the n-dimensional unit ball and $\delta(u,x,y) = (u(x+y) + u(x-y) - 2u(x))/2$, are not available on manifolds in general. Thus, whenever we evaluate Lu or $\mathcal{M}^\pm_{\mathcal{L}_0} u$ at x, we have to split the integral as (2.1.5) or (2.2.8) to compute each integral.

2.1.2 Main results

The main results are the Krylov–Safonov Harnack inequality and interior Hölder estimates for fully nonlinear nonlocal operators of non-divergence form on Riemannian manifolds with nonnegative sectional curvatures. Throughout the paper we assume that (M, g) is a smooth, complete, connected Riemannian manifold with nonnegative sectional curvatures, satisfying the reverse volume doubling property (RVD) with constant a_1 and the volume comparability (Comp) with constant a_2. See Section 2.2 for the assumptions (RVD) and (Comp). Let us begin with the Krylov–Safonov Harnack inequality.

Theorem 2.1.1 (Harnack inequality)**.** *Let $s_0 \in (0,1)$ and assume $s \in [s_0, 1)$. For $z_0 \in M$, let $K = K_{\max}(B(z_0, \mathrm{inj}(z_0)))$ be the supremum of the sectional curvatures in $B(z_0, \mathrm{inj}(z_0))$ and let $R > 0$ be such that $2R < \mathrm{inj}(z_0) \wedge \frac{\pi}{\sqrt{K}}$. If $u \in C^2(B_{2R}(z_0)) \cap L^\infty(M)$ is a nonnegative function on M satisfying*

$$\mathcal{M}^-_{\mathcal{L}_0} u \le C_0 \quad \text{and} \quad \mathcal{M}^+_{\mathcal{L}_0} u \ge -C_0 \quad \text{in } B_{2R}(z_0),$$

then

$$\sup_{B_R(z_0)} u \le C \left(\inf_{B_R(z_0)} u + C_0 R^{2s} \right)$$

for some universal constant $C > 0$, depending only on n, λ, Λ, a_1, a_2, and s_0.

The next result is the interior Hölder estimate for fully nonlinear nonlocal operators of non-divergence form. In contrast to the case of local operators, it does not immediately follow from the Harnack inequality. In the sequel, $\|\cdot\|'$ denotes the non-dimensional norm.

Theorem 2.1.2 (Hölder estimates)**.** *Let $s_0 \in (0,1)$ and assume $s \in [s_0, 1)$. For $z_0 \in M$,let $K = K_{\max}(B(z_0, \mathrm{inj}(z_0)))$ and let $R > 0$ be such that $2R < \mathrm{inj}(z_0) \wedge \frac{\pi}{\sqrt{K}}$. If $u \in C^2(B_{2R}(z_0)) \cap L^\infty(M)$ is a function on M satisfying*

$$\mathcal{M}^-_{\mathcal{L}_0} u \le C_0 \quad \text{and} \quad \mathcal{M}^+_{\mathcal{L}_0} u \ge -C_0 \quad \text{in } B_{2R}(z_0),$$

then $u \in C^\alpha(\overline{B_R(z_0)})$ and

$$\|u\|'_{C^\alpha(\overline{B_R(z_0)})} \le C \left(\|u\|_{L^\infty(M)} + C_0 R^{2s} \right)$$

for some universal constants $\alpha \in (0,1)$ and $C > 0$, depending only on n, λ,

Λ, a_1, a_2, and s_0.

It is noticeable that the universal constants in Theorem 2.1.1 and Theorem 2.1.2 do not depend on nearby curvature upper bound although they depend on the lower bound 0. This means that, in particular, when $M = \mathbb{R}^n$, Theorem 2.1.1 and Theorem 2.1.2 provide the results on the Krylov–Safonov Harnack inequality and Hölder estimates as in [15] without any restriction on R. More generally, the restriction on R disappears when M is a manifold with $\mathrm{inj}(M) = \infty$. In this case, Theorem 2.1.1 extends the global Harnack inequality for local operators [12] to nonlocal operators.

Another important feature of Theorem 2.1.1 and Theorem 2.1.2 is the robustness of the estimates. Since the universal constants in the results depend only on s_0, not on $s \in (0,1)$ itself, we could get the local Harnack inequality and Hölder estimates for the second order local operators as limit $s \to 1$, so this result gives unified estimates up to second order elliptic operators.

Let us make some remarks on the results. It would be the best if we get ABP-type estimate with L^n-norm as Cabré proved in [12]. However, we will establish the ABP estimate with Riemann sums of L^∞-norm as Caffarelli and Silvestre showed in the Euclidean space [15]. To the best of author's knowledge, the full ABP estimate with L^n-norm for fully nonlinear operators are not available even in the Euclidean spaces. For the class of operators with additional assumptions, Guillen and Schwab [60] provided the ABP estimates using both L^n and L^∞ norms in Euclidean spaces. For this type of estimate, we believe it would be applicable to our case.

For curvature bound and imposed radius condition, we refer to Cabré's observation in the last paragraph of [12]. So, we used the injectivity radius and imposed the condition $15R < \mathrm{inj}(z_0) \wedge \frac{\pi}{\sqrt{K}}$, for $K = K_{\max}(B(z_0, \mathrm{inj}(z_0)))$, on the radius of the ball. However, it might be more convenient to consider a strongly convex region (or a strongly convexity radius) instead of the injectivity radius: we call $U \subseteq M$ is strongly convex if every ball $B_\rho(x) \subseteq U$ is convex. This is because our operators are nonlocal and we need to consider the relation between nearby points. Nevertheless, we will use the injectivity radius because it is more general.

If we assume the global upper bound of the sectional curvature such as $\mathrm{Sect}(g) < K$ on M, the radius condition would be reduced to $5R < \frac{\pi}{\sqrt{K}}$. Moreover, the additional assumptions—the reverse volume doubling property (RVD) and comparability of volumes (Comp)—on manifold are naturally satisfied.

Since manifold is not symmetric space in general, a nonlocal antisymmetric part in the operator appears naturally. Because it has no second order incremental quotient of function, we cannot expect the integrability of operators as usual. However, due to the smoothness of volume element which exists inherently, we figure out the antisymmetric part has the same order as the traditional symmetric part.

Moreover, we want to emphasize that we do not use affine functions and cone technique as usual because affine functions with arbitrary directions do not exist on manifold in general. Thus, we use the squared distance function to solve this difficulty. Typically, when we control the gradient of the envelope Γ (defined in Section 2.3) with the squared distance function, we might consider the coarea formula as in [12]. However, since the order of differentiability of nonlocal operators is strictly less than 2, we cannot simply use the coarea formula. At this part, we will directly estimate gradient with Jacobi fields.

Lastly, for further researches, we are expecting that we can get a global Harnack inequality for restricted manifolds. In general, in this paper, we could not stretch the radius of ball due to the injectivity issue. We are also expecting that we could get similar regularity properties for nonlocal operator with kernels of variable orders, which are studied in [76, 73, 8] on Euclidean spaces.

2.1.3 Outline

This paper is organized as follows. In Section 2.2, we ensure integrability of operators. We also bound second difference of squared distance function.

Mainly we need this bound for gradient estimate of the solution. Furthermore, we introduce some definitions and collect some result on dyadic cubes for the analysis on manifold. In Section 2.3, we introduce an envelope defined by squared distance function and estimate its gradient so that we get a (weak type) Aleksandrov–Bakelman–Pucci estimate. Section 2.4 is devoted to the construction of a barrier function. In Section 2.5, L^ε-estimate is established by using the ABP estimate and the barrier function obtained in the previous sections. The proofs for the Harnack inequality and Hölder estimate are provided in Section 2.6 and Section 2.7, respectively.

2.2 Preliminaries

This section is devoted to the basic knowledge on Riemannian geometry that will be useful in the rest of the paper. For more details, the reader may consult [66, 30, 104].

Let (M, g) be a smooth, complete manifold of dimension n. Let us denote by $R(\xi, \eta)\zeta$ the curvature tensor, then the sectional curvature of the plane determined by linearly independent tangent vectors $\xi, \eta \in T_xM$ is given by

$$\mathrm{Sect}(\xi, \eta) = \frac{g(R(\xi, \eta)\xi, \eta)}{|\xi|_g^2|\eta|_g^2 - g(\xi, \eta)^2}.$$

Let $d_y(\cdot) := d(\cdot, y)$ be the distance function. We will see that the distance squared function $\frac{1}{2}d_y^2$ will play an important role in the regularity results. Let us collect and study some useful properties of this function. First of all, it is continuous in M and smooth in $M \setminus \mathrm{Cut}_y$. For any $x \notin \mathrm{Cut}_y$, the Gauss lemma implies that

$$\nabla(d_y^2/2)(x) = -\exp_x^{-1} y.$$

Moreover, it is well-known that if $K_1 \le \mathrm{Sect} \le K_2$ in $B_{\mathrm{inj}(y)}(y)$ with $K_1 \le 0$

and $K_2 \geq 0$, then the Hessian of $d_y^2/2$ has upper and lower bounds

$$\begin{aligned}\sqrt{K_2} d_y(x) \cot\left(\sqrt{K_2} d_y(x)\right) |\xi|_g^2 &\leq D^2(d_y^2/2)(x)(\xi, \xi) \\ &\leq \sqrt{-K_1} d_y(x) \coth\left(\sqrt{-K_1} d_y(x)\right) |\xi|_g^2,\end{aligned} \tag{2.2.1}$$

for $x \in B_\rho(y)$ and $\xi \in T_x M$, where $\rho < \frac{\pi}{2\sqrt{K_2}}$ in case $K_2 > 0$ and $\rho < \mathrm{inj}(y)$ otherwise (see, for example, [66, Theorem 6.6.1]). Since we are assuming that $\mathrm{Sect} \geq 0$, the bounds (2.2.1) read as

$$0 \leq \sqrt{K} d_y(x) \cot\left(\sqrt{K} d_y(x)\right) |\xi|_g^2 \leq D^2(d_y^2/2)(x)(\xi, \xi) \leq |\xi|_g^2, \tag{2.2.2}$$

where $K = K_{\max}(B_{\mathrm{inj}(y)}(y))$ is the supremum of the sectional curvatures in $B_{\mathrm{inj}(y)}(y)$. Using (2.2.2) and the mean value theorem for integrals, we obtain the following lemma.

Lemma 2.2.1. *For any $y \in M$ and $x \in B_\rho(y)$, let $\xi \in T_x M$ be such that $\exp_x(s\xi) \in B_\rho(y)$ for all $s \in (-1, 1)$, where*

$$\rho < \begin{cases} \frac{\pi}{2\sqrt{K}} & \text{if } K := K_{\max}(B_{\mathrm{inj}(y)}(y)) > 0, \\ \mathrm{inj}(y) & \text{if } K = 0. \end{cases} \tag{2.2.3}$$

Then,

$$0 \leq (1-t) d_y^2(\exp_x(t\xi)) + t d_y^2(\exp_x((1-t)(-\xi))) - d_y^2(x) \leq t(1-t)|\xi|_g^2$$

for any $t \in (0, 1)$.

Let us now recall Gromov's theorem in a manifold with a nonnegative Ricci curvature. Since we assume that $\mathrm{Sect} \geq 0$, the Ricci curvature is also nonnegative. The Gromov's theorem says that

$$\frac{\mu_g(B(x, R))}{|B_R|} \quad \text{is nonincreasing in } R$$

for any $x \in M$, where $|B_R|$ is the volume in $\mathbb{R}^n$ of a ball of radius R. It is known that the ratio approaches 1 as R goes to zero, so together with the monotonicity it implies that $\mu_g(B(x,R)) \leq |B_R|$. Moreover, the Gromov's theorem also gives rise to the volume doubling property

$$\frac{\mu_g(B(x,R))}{\mu_g(B(x,r))} \leq \left(\frac{R}{r}\right)^n, \quad 0 < r \leq R. \tag{VD}$$

The volume doubling property provides the following integrability of kernels ν_x.

Lemma 2.2.2. *Let $s_0 \in (0,1)$ and assume $s \in [s_0, 1)$. Then,*

$$(2-2s)\int_M \left(R^2 \wedge d_x(z)^2\right) \frac{\mathrm{d}V(z)}{\mu_g(B(x,d_x(z)))d_x(z)^{2s}} \leq CR^{2-2s} \tag{2.2.4}$$

for some constant $C = C(n, s_0) > 0$.

Proof. By the volume doubling property (3.2.8), we have

$$\begin{aligned}
&\int_{B_R(x)} \frac{(2-2s)d_x(z)^{2-2s}}{\mu_g(B(x,d_x(z)))}\,\mathrm{d}V(z) \\
&\quad = \sum_{k=0}^{\infty} \int_{B(x,2^{-k}R)\setminus B(x,2^{-(k+1)}R)} \frac{(2-2s)d_x(z)^{2-2s}}{\mu_g(B(x,d_x(z)))}\,\mathrm{d}V(z) \\
&\quad \leq \sum_{k=0}^{\infty} \frac{\mu_g(B(x,2^{-k}R))}{\mu_g(B(x,2^{-(k+1)}R))}(2-2s)2^{-k(2-2s)}R^{2-2s} \\
&\quad \leq 2^n \frac{2-2s}{1-2^{-(2-2s)}}R^{2-2s} \leq C(n)R^{2-2s},
\end{aligned} \tag{2.2.5}$$

where we observed in the last inequality that the function $t/(1-2^{-t})$ is bounded in $[0,2]$ from above. Similarly, by the volume doubling property

(3.2.8) again, we obtain

$$\begin{aligned}
&\int_{M\setminus B_R(x)} \frac{(2-2s)R^2}{\mu(B(x,d_x(z)))d_x(z)^{2s}}\,\mathrm{d}V(z)\\
&\qquad = \sum_{k=0}^{\infty} \int_{B(x,2^{k+1}R)\setminus B(x,2^kR)} \frac{(2-2s)R^2}{\mu(B(x,d_x(z)))d_x(z)^{2s}}\,\mathrm{d}V(z)\\
&\qquad \le \sum_{k=0}^{\infty} \frac{\mu_g(B(x,2^{k+1}R))}{\mu_g(B(x,2^kR))}(2-2s)2^{-2ks}R^{2-2s}\\
&\qquad \le 2^n\frac{2-2s}{1-2^{-2s}}R^{2-2s} \le \frac{2^{n+1}}{1-2^{-s_0}}R^{2-2s}.
\end{aligned} \tag{2.2.6}$$

Therefore, (2.2.4) follows by combining the inequalities (2.2.5) and (2.2.6). □

Using Lemma 2.2.2, we show that Lu is well-defined.

Lemma 2.2.3. *Let $s_0 \in (0,1)$ and assume $s \in [s_0,1)$. For $x \in M$, let K be the supremum of the sectional curvatures in $B_{\mathrm{inj}(x)}(x)$ and let $2R < \mathrm{inj}(x) \wedge \frac{\pi}{\sqrt{K}}$. Then, for $L \in \mathcal{L}_0$ and for $u \in C^2(\overline{B_R(x)}) \cap L^\infty(M)$,*

$$|Lu(x)| \le C\Lambda\left(\|u\|'_{C^2(\overline{B_R(x)})} + \|u\|_{L^\infty(M)}\right)R^{-2s}, \tag{2.2.7}$$

where $C = C(n,s_0) > 0$ is a universal constant. Therefore, the value of Lu at x is well-defined.

Proof. By assuming R sufficiently small, we may assume that u is $C^2(\overline{B_R(x)})$ and bounded in M.

Let us decompose the measure into the symmetric and antisymmetric parts with respect to x, that is, $\mathrm{d}V(z) = \mathrm{d}V_s(z) + \mathrm{d}V_a(z)$, where $\mathrm{d}V_s(z) := \frac{1}{2}(\mathrm{d}V(z)+\mathrm{d}V(\mathcal{T}_x(z)))$ and $\mathrm{d}V_a(z) := \frac{1}{2}(\mathrm{d}V(z)-\mathrm{d}V(\mathcal{T}_x(z)))$. Then, for $L \in \mathcal{L}_0$

we have

$$\begin{aligned}Lu(x) &= (2-2s)\int_{B_R(x)} \delta(u,x,z)\nu_x(z)\,\mathrm{d}V_s(z)\\&\quad+(2-2s)\int_{B_R(x)}(u(z)-u(x))\nu_x(z)\,\mathrm{d}V_a(z)\\&\quad+(2-2s)\int_{M\setminus B_R(x)}(u(z)-u(x))\nu_x(z)\,\mathrm{d}V(z) =: I_1+I_2+I_3.\end{aligned} \tag{2.2.8}$$

We may apply Lemma 2.2.2 for $\mathrm{d}V_s$ and $\mathrm{d}V$ to obtain $|I_1| \le C\Lambda\|u\|'_{C^2(\overline{B_R(x)})}R^{-2s}$ and $|I_3| \le C\Lambda\|u\|_{L^\infty(M)}R^{-2s}$, respectively. For I_2, we observe that

$$\begin{aligned}&|I_2|\\&\le \Lambda(2-2s)\|u\|_{L^\infty(B_R)}\int_{B_R(x)}\frac{1}{\mu_g(B(x,d_x(z)))d_x(z)^{2s}}\,|\mathrm{d}V_a|(z)\\&\le \Lambda(2-2s)\|u\|_{L^\infty(B_R)}\int_0^R\int_{\partial B_1}\frac{1}{\mu_{g^*}(B(0,t))t^{2s}}\left(t^{n-1}-\left(\frac{\sin(\sqrt{K}t)}{\sqrt{K}}\right)^{n-1}\right)\mathrm{d}v\,\mathrm{d}t,\end{aligned}$$

where g^* is the induced metric. Note that the inequalities

$$t^{n-1}-\left(\frac{\sin(\sqrt{K}t)}{\sqrt{K}}\right)^{n-1} \le \frac{n-1}{3!}(\sqrt{K}t)^2t^{n-1}, \quad t\sqrt{K}\le\sqrt{6},$$

and

$$t \le \frac{\pi}{2}\frac{\sin(\sqrt{K}t)}{\sqrt{K}}, \quad t\sqrt{K}\le\frac{\pi}{2},$$

can be applied to obtain that

$$t^{n-1}-\left(\frac{\sin(\sqrt{K}t)}{\sqrt{K}}\right)^{n-1} \le CKt^2\left(\frac{\sin(\sqrt{K}t)}{\sqrt{K}}\right)^{n-1},$$

since we have assumed that $2R<\pi/\sqrt{K}$. Therefore,

$$|I_2| \le C\Lambda(2-2s)\|u\|_{L^\infty(B_R)}\int_{B_R(x)}\frac{Kd_x^2(z)}{\mu_g(B(x,d_x(z)))d_x(z)^{2s}}\,\mathrm{d}V(z).$$

By Lemma 2.2.2, we arrive at $|I_2| \leq C\Lambda \|u\|_{L^\infty(B_R)} K R^{2-2s}$. Again, by using $2R < \pi/\sqrt{K}$,

$$|I_2| \leq C\Lambda \|u\|_\infty R^{-2s}. \tag{2.2.9}$$

The estimate (2.2.9) together with estimates for I_1 and I_3 finishes the proof.

□

Here are two assumptions on manifold we are going to use throughout the paper.

- (Reversed volume doubling property) Let us assume that there is a constant $a_1 \in (0, 1]$ such that

$$\frac{\mu_g(B_R(x))}{\mu_g(B_r(x))} \geq a_1 \left(\frac{R}{r}\right)^n, \quad 0 < r \leq R < \operatorname{inj}(x). \tag{RVD}$$

- (Comparability of volumes of balls with different centers) Let us assume that there is a constant $a_2 \geq 1$ such that

$$a_2^{-1} \leq \frac{\mu_g(B_R(x_1))}{\mu_g(B_R(x_2))} \leq a_2, \quad 0 < R < \operatorname{inj}(x_1) \wedge \operatorname{inj}(x_2). \tag{Comp}$$

Let us close this section with the following generalization of Euclidean dyadic cubes that will be used in the decomposition of the contact set and in the Calderón–Zygmund technique.

Theorem 2.2.4 (Christ [31]). *There is a countable collection $\mathcal{D} := \{Q^j_\alpha \subset M : j \in \mathbb{Z}, \alpha \in I_j\}$ of open sets and constants $c_1, c_2 > 0$ (with $2c_1 \leq c_2$), and $\delta_0 \in (0, 1)$, depending only on n, such that*

(i) $\mu(M \setminus \cup_\alpha Q^j_\alpha) = 0$ *for each* $j \in \mathbb{Z}$,

(ii) if $i \geq j$, *then either* $Q^i_\beta \subset Q^j_\alpha$ *or* $Q^i_\beta \cap Q^j_\alpha = \emptyset$,

(iii) for each (j, α) *and each* $i < j$, *there is a unique* β *such that* $Q^j_\alpha \subset Q^i_\beta$,

(iv) $\operatorname{diam}(Q^j_\alpha) \leq c_2 \delta_0^j$, *and*

(v) each Q^j_α *contains some ball* $B(z^j_\alpha, c_1\delta_0^j)$.

2.3 Discrete ABP-type estimates

We begin with a discrete version of the ABP-type estimate which will play a key role in the estimates of sub-level sets of u in Section 2.5. Cabré suggested in [12] the use of distance squared functions instead of affine functions as touching functions due to the fact that there is no non-constant affine functions in general. This leads us to the smooth map

$$y = \exp_x \nabla(R^2 u)(x). \tag{2.3.1}$$

That is, if u is a smooth function satisfying $u \geq 0$ in $M \setminus B_{5R}$ and $\inf_{B_{2R}} u \leq 1$, then for any point $y \in B_R$, the minimum of the function $R^2 u + \frac{1}{2} d_y^2$ in $\overline{B}_{5R}$ is achieved at some point $x \in B_{5R}$, leading us to the smooth map (2.3.1). For the second order operators, the Jacobian of this smooth map is controlled by the determinant of $D^2 u$, which is in turn controlled by f through the equations. However, since nonlocal operators have order strictly less than two, we cannot go through the determinant of $D^2 u$.

Motivated by the idea of proof of the discrete ABP estimates in [15], we therefore find a small ring around each contact point, in which u stays quadratically close to the envelope. The main difference is that we need to construct the envelope using the distance squared functions instead of affine functions. For each $y \in B_R$, there is a unique paraboloid

$$P_y(z) = c_y - \frac{1}{2R^2} d_y(z)^2$$

that touches u from below, with a contact point $x \in B_{5R}$. We define the *envelope* Γ *of* u by

$$\Gamma(z) = \sup_{y \in B_R} P_y(z),$$

and the *contact set* $A = \{x \in B_{5R} : u(x) = \Gamma(x)\}$. In the sequel, let us fix the universal constants

$$\rho_1 = 2\left(\frac{1}{a_1}\right)^{1/n} \vee \frac{1}{\delta_0} > 1 \quad \text{and} \quad \rho_0 < \frac{2c_1}{(3+4/\rho_1)c_2}\delta_0 < 1, \tag{2.3.2}$$

where c_1, c_2, and δ_0 are constants, depending only on n, in Theorem 2.2.4, and a_1 is the constant in the reverse volume doubling property (RVD). This section is devoted to the following nonlocal ABP-type estimate on a Riemannian manifold with nonnegative sectional curvature that generalizes the result in [12]. Recall that $\mathcal{D}$ is a family of dyadic cubes in Theorem 2.2.4.

Lemma 2.3.1. *Let $s_0 \in (0,1)$ and assume $s \in [s_0, 1)$. For $z_0 \in M$, let K be the supremum of the sectional curvatures in $B_{\mathrm{inj}(z_0)}(z_0)$ and let $15R < \mathrm{inj}(z_0) \wedge \frac{\pi}{\sqrt{K}}$. Let $u \in C^2(B_{5R}(z_0)) \cap L^\infty(M)$ be a function on M satisfying $u \geq 0$ in $M \setminus B_{5R}(z_0)$ and $\inf_{B_{2R}(z_0)} u \leq 1$, and let Γ be the envelope of u. If $\mathcal{M}^-_{\mathcal{L}_0} u \leq f$ in $B_{5R}(z_0)$, then*

$$\mu_g(B_R(z_0)) \leq C \sum_{\mathcal{D}_1} \left(\Lambda + R^{2s} \max_{\overline{Q}^j_\alpha} f\right)^n_+ \mu_g(Q^j_\alpha), \tag{2.3.3}$$

where $\mathcal{D}_1 = \{Q^j_\alpha\}$ is a finite subcollection of $\mathcal{D}$ of dyadic cubes, with $\mathrm{diam}(Q^j_\alpha) \leq \rho_0 \rho_1^{-1/(2-2s)} R$, that intersect with the contact set A and satisfy $A \subset \cup_j \overline{Q}^j_\alpha$. The constant C depends only on n, λ, a_1, a_2, and s_0.

It is known [60] that the estimates (2.3.3) with the Riemann sums in the right hand side replaced by $\|\Lambda + f\|_{L^n(A)}$ fails to hold even in the case of the Euclidean space. Instead, as in [15], the Riemann sums of $\Lambda + f$ over the set $\cup_j \overline{Q}^j_\alpha$ need to be considered. Thus, we need information not only on the contact set, but also on $\cup_j \overline{Q}^j_\alpha \setminus A$. The map (2.3.1) is not appropriate as a normal map since u and Γ do not coincide outside the contact set. Instead, we will make use of the map ϕ, which assigns each point $x \in M$ the vertex point y of the paraboloid P_y, where P_y is some paraboloid such that $\Gamma(x) = P_y(x)$. Note that the map ϕ may be multivalued since P_y may not be unique.

By using the map ϕ, we prove the following discrete ABP-type estimates.

Lemma 2.3.2. *Assume the same assumptions as in Lemma 2.3.1. There is a finite subcollection $\mathcal{D}_1 \subset \mathcal{D}$ of dyadic cubes Q^j_α, with diameters $d_{j,\alpha} \le \rho_0 \rho_1^{-1/(2-2s)} R$, such that the following holds:*

(i) *Any two different dyadic cubes in $\mathcal{D}_1$ do not intersect.*

(ii) $A \subset \bigcup_{\mathcal{D}_1} \overline{Q}^j_\alpha$.

(iii) $\mu_g(\phi(\overline{Q}^j_\alpha)) \le C \left(\Lambda + R^{2s} \max_{\overline{Q}^j_\alpha} f\right)_+^n \mu_g(Q^j_\alpha)$.

(iv) $\gamma \mu_g(Q^j_\alpha)$
$\le \mu_g \left(B(z^j_\alpha, (1+4\rho_1) c_2 \delta^j_0) \cap \{ u \le \Gamma + C R^{-2} (\Lambda + R^{2s} \max_{\overline{Q}^j_\alpha} f)_+ d^2_{j,\alpha} \} \right)$.

The constants $C > 0$ and $\gamma > 0$ depend only on n, λ, a_1, a_2, and s_0.

It is easy to see that Lemma 2.3.1 follows from Lemma 2.3.2. Indeed, since we have tested all distance squared function centered on B_R, we have $B_R \subset \phi(A)$. Hence, for the family $\mathcal{D}_1$ of dyadic cubes constructed in Lemma 2.3.2 we obtain

$$\mu_g(B_R) \le \mu_g(\phi(A)) \le \sum_{\mathcal{D}_1} \mu_g(\phi(\overline{Q}^j_\alpha)) \le C \sum_{\mathcal{D}_1} \left(\Lambda + R^{2s} \max_{\overline{Q}^j_\alpha} f \right)_+^n \mu_g(Q^j_\alpha).$$

We postpone the proof of Lemma 2.3.2 until the end of this section because we need a series of lemmas in order to prove it.

The next lemma finds a ring around a contact point, where u is quadratically close to the paraboloid in a large portion of the ring. Note that if x is a contact point, then Γ is touched by u from above and by some paraboloid P_y from below at x, which shows that the paraboloid P_y is uniquely determined and Γ is differentiable at x. Moreover, in this case, we have

$$y = \phi(x) = \exp_x \nabla(R^2 \Gamma)(x) = \exp_x \nabla(R^2 u)(x),$$

and hence $\phi(x)$ is also uniquely determined.

Lemma 2.3.3. *Assume the same assumptions as in Lemma 2.3.1, and let* $r_k = \rho_0\rho_1^{-1/(2-2s)-k}R$. *Then, there exists a universal constant* $C_0 > 0$, *depending only on* n, λ, a_1, *and* s_0, *such that for any* $x \in A$ *and any* $M_0 > 0$, *there is an integer* $k \geq 0$ *such that*

$$\mu_g(G_k) \leq \frac{C_0}{M_0}\left(\Lambda + R^{2s}f(x)\right)_+ \mu_g(R_k), \tag{2.3.4}$$

where $R_k = B(x, r_k)\setminus B(x, r_{k+1})$, $G_k = \{z \in R_k : u(z) > P_y(z) + M_0(r_k/R)^2\}$, *and* $y = \phi(x)$.

Proof. Let $x \in A$. By [107], we have

$$\mathrm{inj}(x) \geq (\mathrm{inj}(z_0) \wedge \mathrm{conj}(x)) - d(x, z_0).$$

Using $\mathrm{inj}(z_0) > 15R$, $d(x, z_0) < 5R$, and

$$\mathrm{conj}(x) \geq \frac{\pi}{\sqrt{K}} \wedge (\mathrm{inj}(z_0) - d(x, z_0)) > 10R,$$

we obtain that

$$\mathrm{inj}(x) > 5R. \tag{2.3.5}$$

Let us compute $\mathcal{M}^-_{\mathcal{L}_0}u(x) = \inf_{L\in\mathcal{L}_0}(I_1 + I_2 + I_3)$, where

$$\begin{aligned}
I_1 &= \int_{B_R(x)\cup B_{5R}(z_0)} \left(u(z) + \frac{1}{2R^2}d_y^2(z) - \left(u(x) + \frac{1}{2R^2}d_y^2(x)\right)\right)\nu_x(z)\,\mathrm{d}V(z),\\
I_2 &= -\int_{B_R(x)\cup B_{5R}(z_0)} \left(\frac{1}{2R^2}d_y^2(z) - \frac{1}{2R^2}d_y^2(x)\right)\nu_x(z)\,\mathrm{d}V(z), \quad \text{and}\\
I_3 &= \int_{M\setminus(B_R(x)\cup B_{5R}(z_0))} (u(z) - u(x))\nu_x(z)\,\mathrm{d}V(z).
\end{aligned}$$

By the fact (2.3.5), the symmetry (2.1.3) of density functions $\nu_x(z)$, Lemma 2.2.1, and Lemma 2.2.3, we have

$$\begin{aligned} I_2 = & -\frac{1}{R^2}\int_{B_R(x)} \delta(d_y^2/2, x, z)\nu_x(z)\,\mathrm{d}V_s(z) \\ & -\frac{1}{2R^2}\int_{B_R(x)} \left(d_y^2(z) - d_y^2(x)\right)\nu_x(z)\,\mathrm{d}V_a(z) \\ & -\frac{1}{2R^2}\int_{B_{5R}(z_0)\setminus B_R(x)} \left(d_y^2(z) - d_y^2(x)\right)\nu_x(z)\,\mathrm{d}V(z) \\ \geq & -C\Lambda R^{-2s}, \end{aligned}$$

where $C = C(n, s_0)$ is some universal constant.

On the other hand, we know that $u(x) \leq u(x) + \frac{1}{2R^2}d_y^2(x) \leq \inf_{B_{2R}}(u + \frac{1}{2R^2}d_y^2) \leq 11/2 < 6$. This fact together with the assumption that $u \geq 0$ in $M \setminus B_{5R}(z_0)$ leads us to

$$I_3 \geq -\int_{M\setminus B_R(x)} \frac{6\Lambda(2-2s)}{\mu_g(B(x, d_x(z)))d_x(z)^{2s}}\,\mathrm{d}V(z) \geq -C\Lambda R^{-2s}$$

by the similar argument.

Let us now estimate I_1. Since the contact point x minimizes the function $u + \frac{1}{2R^2}d_y^2$, the integrand in I_1 is nonnegative. Thus, we have

$$I_1 \geq \lambda(2-2s)\sum_{k=0}^{\infty}\int_{G_k} \frac{u(z) + \frac{1}{2R^2}d_y^2(z) - \left(u(x) + \frac{1}{2R^2}d_y^2(x)\right)}{\mu_g(B(x, d_x(z)))d_x(z)^{2s}}\,\mathrm{d}V(z).$$

Let us assume to the contrary that (2.3.4) does not hold for all $k \geq 0$, that is,

$$\mu_g(G_k) > \frac{C_0}{M_0}(\Lambda + R^{2s}f(x))_+\mu_g(R_k) \quad \text{for all } k \geq 0, \tag{2.3.6}$$

for some $C_0 > 0$ that will be chosen at the end of the proof. If $z \in G_k$, then $u(z) + \frac{1}{2R^2}d_y^2(z) - \left(u(x) + \frac{1}{2R^2}d_y^2(x)\right) \geq M_0(r_k/R)^2$. Thus, using (2.3.6) and the reverse volume doubling property (RVD), we obtain the lower bound of

I_1 as

$$
\begin{aligned}
I_1 &\geq \lambda(2-2s)C_0 \sum_{k=0}^{\infty} \frac{(r_k/R)^2}{\mu_g(B(x,r_k))r_k^{2s}} \left(\Lambda + R^{2s} f(x)\right)_+ \mu_g(R_k) \\
&= \lambda \frac{2-2s}{\rho_1} C_0 \sum_{k=0}^{\infty} \rho_0^{2-2s} \rho_1^{-k(2-2s)} R^{-2s} \left(1 - \frac{\mu_g(B(x,r_{k+1}))}{\mu_g(B(x,r_k))}\right) \left(\Lambda + R^{2s} f(x)\right)_+ \\
&\geq \lambda \frac{\rho_0^2}{\rho_1} C_0 \frac{2-2s}{1-\rho_1^{-(2-2s)}} \left(1 - \frac{1}{a_1 \rho_1^n}\right) \left(\Lambda R^{-2s} + f(x)\right)_+ .
\end{aligned}
$$

Recalling (2.3.2) and observing that the function $t/(1-\rho_1^{-t})$ is bounded away from 0 in $[0,2]$, we arrive at

$$
I_1 \geq c_1 C_0 \left(\Lambda R^{-2s} + f(x)\right)_+ ,
$$

where $c_1 = c_1(n, \lambda, a_1) > 0$.

We have obtained that

$$
f(x) \geq \mathcal{M}^-_{\mathcal{L}_0} u(x) \geq \inf_{L \in \mathcal{L}_0} (I_1 + I_2 + I_3) \geq c_1 C_0 \left(\Lambda R^{-2s} + f(x)\right)_+ - C\Lambda R^{-2s}.
$$

Therefore, by taking C_0 sufficiently large, we arrive at a contradiction. $\square$

The next lemma shows that the function $\Gamma - P_y$ is $-R^{-2}$-convex in the sense of second order incremental quotients.

Lemma 2.3.4. *Let $x \in A$, $y \in \phi(x)$, $K = K_{\max}(B_{\mathrm{inj}(y)}(y))$, and let $\rho > 0$ satisfy (2.2.3). For $z \in B_\rho(y)$, let $\xi \in T_zM$ be such that $\exp_z(s\xi) \in B_\rho(y)$ for all $s \in (-1,1)$. Then,*

$$
(\Gamma - P_y)(z) \leq (1-t)(\Gamma - P_y)(z_1) + t(\Gamma - P_y)(z_2) + \frac{1}{2R^2} t(1-t)|\xi|_g^2 \quad (2.3.7)
$$

for all $t \in (0,1)$, where $z_1 = \exp_z(t\xi)$ and $z_2 = \exp_z((1-t)(-\xi))$.

Proof. By the definition of Γ, there is a paraboloid $P_* := P_{y_*}$ with some point

$y_* \in B_R$, such that $\Gamma(z) = P_*(z)$. Then we have

$$(\Gamma - P_*)(z) = 0 \leq (1-t)(\Gamma - P_*)(z_1) + t(\Gamma - P_*)(z_2),$$

and hence

$$\begin{aligned}(\Gamma - P_y)(z) \leq (1-t)&(\Gamma - P_y)(z_1) + t(\Gamma - P_y)(z_2) \\ &- ((1-t)P_*(z_1) + tP_*(z_2) - P_*(z)) \\ &+ ((1-t)P_y(z_1) + tP_y(z_2) - P_y(z)).\end{aligned} \tag{2.3.8}$$

Using Lemma 2.2.1, we obtain

$$\begin{aligned}-((1-t)P_*(z_1) + tP_*(z_2) - P_*(z)) &= \frac{1}{2R^2}\left((1-t)d^2_{y_*}(z_1) + td^2_{y_*}(z_2) - d^2_{y_*}(z)\right) \\ &\leq \frac{1}{2R^2} t(1-t)|\xi|^2_g\end{aligned} \tag{2.3.9}$$

and

$$(1-t)P_y(z_1) + tP_y(z_2) - P_y(z) = -\frac{1}{2R^2}\left((1-t)d^2_y(z_1) + td^2_y(z_2) - d^2_y(z)\right) \leq 0. \tag{2.3.10}$$

Therefore, (2.3.7) follows from (2.3.8), (2.3.9), and (2.3.10). □

By means of Lemma 2.3.3 and Lemma 2.3.4, we will show that in a small ball near a contact point the envelope is captured by two paraboloids that are quadratically close to each other. Recall that the convex envelope constructed by affine functions in the case of Euclidean spaces [15] is captured by two affine planes. The idea in [15] is to carry information from the "good ring" to the ball enclosed by the ring, by using the convexity of the function Γ. In our setting, we use $-R^{-2}$-convexity of the function $\Gamma - P_y$ instead.

Lemma 2.3.5. *Assume the same assumptions as in Lemma 2.3.1. Let $x \in A$, $y = \phi(x)$, and let $r = r_k$ be the radius in Lemma 2.3.3. There is a small* *constant* $\varepsilon_0 = \varepsilon_0(n) \in (0,1)$ *such that if*

$$\mu_g(\{z \in B_r(x) \setminus B_{r/2}(x) : \Gamma(z) > P_y(z) + h\}) \leq \varepsilon_0 \mu_g(B_r(x) \setminus B_{r/2}(x)), \tag{2.3.11}$$

then

$$\Gamma(z) \leq P_y(z) + h + \frac{1}{2}\left(\frac{r}{R}\right)^2$$

for all $z \in B_{r/2}(x)$.

Proof. Let us fix $z \in B_{r/2}(x)$ and claim that there are two points $w_1, w_2 \in B_r(x) \setminus B_{r/2}(x)$ such that three points w_1, z, and w_2 are joined by a geodesic, and that

$$\Gamma(w_i) \leq P_y(w_i) + h, \quad i = 1, 2. \tag{2.3.12}$$

Once we find such points, we may write $w_1 = \exp_z(t\xi)$ and $w_2 = \exp_z((1-t)(-\xi))$ for some $\xi \in T_zM$ with $|\xi|_g$ being the length of the line segment between w_1 and w_2, and $t \in (0,1)$. Then, we have $|\xi|_g \leq 2r$ and $t(1-t) \leq 1/4$. Thus, by Lemma 2.3.4 and (2.3.12), we obtain that

$$(\Gamma - P_y)(z) \leq h + \frac{1}{2R^2} t(1-t)|\xi|_g^2 \leq h + \frac{1}{2}\left(\frac{r}{R}\right)^2,$$

finishing the proof.

To prove the claim, we first extend the line segment between x and z in both directions to find two points z_1 and z_2 on $\partial B_{3r/4}(x)$. We call the farther one from z as z_1 and the closer one from z as z_2. Let $D = \{z : \Gamma(z) \leq P_y(z) + h\}$, then it follows from (2.3.11) and (3.2.8) that

$$\begin{aligned} \mu_g(B_{r/8}(z_1) \cap D^c) &\leq \mu_g((B_r(x) \setminus B_{r/2}(x)) \cap D^c) \\ &\leq \varepsilon_0 \mu_g(B_r(x) \setminus B_{r/2}(x)) \\ &\leq \varepsilon_0 \mu_g(B_{2r}(z_1)) \leq \varepsilon_0 16^n \mu_g(B_{r/8}(z_1)). \end{aligned}$$

Assuming $\varepsilon_0 < 2^{-4n-1}$, we obtain

$$\mu_g(D \cap B_{r/8}(z_1)) \geq \frac{1}{2}\mu_g(B_{r/8}(z_1)). \tag{2.3.13}$$

Let us write $d = d(z, x) \in [0, r/2)$ and define a map $F = \exp_z \circ T \circ \exp_z^{-1} : B_{\mathrm{inj}(z)}(z) \to M$, where $T : T_zM \to T_zM$ is a linear map given by

$$T(\xi) = -\frac{\frac{3r}{4} - d}{\frac{3r}{4} + d}\,\xi.$$

By [107], we know that $\mathrm{inj}(z) \geq (\mathrm{inj}(z_0) \wedge \mathrm{conj}(z)) - d(z, z_0)$. It follows from $\mathrm{inj}(z_0) > 15R$, $d(z, z_0) \leq d(z, x) + d(x, z_0) < r/2 + 5R$, and

$$\mathrm{conj}(z) \geq \frac{\pi}{\sqrt{K}} \wedge (\mathrm{inj}(z_0) - d(z, z_0)) \geq 10R - r/2,$$

that $\mathrm{inj}(z) \geq 5R - r$. Let us recall that we have $r = r_k = \rho_0 \rho_1^{-1/(2-2s)-k} R < R$ from the choice (2.3.2). Thus, we obtain $\mathrm{inj}(z) > 4r$, and hence F is well-defined in $B_{r/8}(z_1) \subset B_{\mathrm{inj}(z)}(z)$.

What we only need to show is that $(B_r(x) \setminus B_{r/2}(x)) \cap D \cap F(D \cap B_{r/8}(z_1))$ is not empty. Let us assume to the contrary that

$$F(D \cap B_{r/8}(z_1)) \subset (B_r(x) \setminus B_{r/2}(x)) \cap D^c \tag{2.3.14}$$

and find a contradiction by estimating the volume change by F. Let $D_1 = \exp_z^{-1}(D \cap B_{r/8}(z_1))$, $D_2 = T(D_1)$, and $D_3 = \exp_z(D_2)$. Due to the lower bound of curvature, we estimate

$$\mu_g(D \cap B_{r/8}(z_1)) = \iint_{D_1} \det(D\exp_z)(tv) t^{n-1} \,\mathrm{d}v\,\mathrm{d}t \leq \iint_{D_1} t^{n-1} \,\mathrm{d}v\,\mathrm{d}t = |D_1|. \tag{2.3.15}$$

by means of the polar coordinates (see, for instance, [66]). Since $d \in [0, r/2)$, we have

$$|D_2| = \left(\frac{\frac{3r}{4} - d}{\frac{3r}{4} + d} \right)^n |D_1| \geq 5^{-n} |D_1|. \tag{2.3.16}$$

Moreover, since the curvature in D_3 is bounded from above by $K_{\max}(D_3)$, which is less than or equal to $K = K_{\max}(B_{\mathrm{inj}(z_0)}(z_0))$, we obtain

$$\mu_g(D_3) = \iint_{D_2} \det(D\exp_z)(tv)t^{n-1}\,\mathrm{d}v\,\mathrm{d}t \geq \iint_{D_2} \left(\frac{\sin(\sqrt{K}t)}{\sqrt{K}t}\right)^{n-1} t^{n-1}\,\mathrm{d}v\,\mathrm{d}t.$$

Note that the function $\sin(\sqrt{K}t)/(\sqrt{K}t)$ is nonnegative and decreasing in $(0, \frac{\pi}{\sqrt{K}}]$. If $(t,v) \in D_2$, then

$$t \leq \frac{\frac{3r}{4} - d}{\frac{3r}{4} + d}\left(\frac{7r}{8} + d\right) \leq \frac{3r}{2} < \frac{3R}{2} < \frac{\pi}{10\sqrt{K}},$$

and hence

$$\mu_g(D_3) \geq \left(\frac{\sin\frac{\pi}{10}}{\frac{\pi}{10}}\right)^{n-1} |D_2|. \tag{2.3.17}$$

Combining (2.3.13) and (2.3.15)–(2.3.17), we have $\mu_g(B_{r/8}(z_1)) \leq C(n)\mu_g(D_3)$. Moreover, by using (2.3.14), (2.3.11), and (3.2.8), we obtain

$$\begin{aligned}\mu_g(B_{r/8}(z_1)) &\leq C\mu_g((B_r(x) \setminus B_{r/2}(x)) \cap D^c) \\ &\leq C\varepsilon_0\mu_g(B_r(x) \setminus B_{r/2}(x)) \\ &\leq C\varepsilon_0\mu_g(B_{2r}(z_1)) \leq 16^n C\varepsilon_0\mu_g(B_{r/8}(z_1)).\end{aligned}$$

Therefore, we arrive at a contradiction by taking $\varepsilon_0 < 16^{-n}C^{-1}$. $\square$

The flatness of Γ in a small region, obtained in Lemma 2.3.5, allows us to control the gradient of Γ in a smaller region, where the gradient of Γ is understood as the gradient of touching paraboloid. This is done by estimating the image of the map ϕ.

Lemma 2.3.6. *Assume the same assumptions as in Lemma 2.3.3, and let ε_0 be the constant in Lemma 2.3.5. For any $x \in A$ there is an $r = r_k \leq r_0$ such that*

$$\mu_g\left(\left\{z \in R_k : u(z) > P_y(z) + C\left(\Lambda + R^{2s}f(x)\right)_+ (r_k/R)^2\right\}\right) \leq \varepsilon_0\mu_g(R_k) \tag{2.3.18}$$

and

$$\mu_g\left(\phi\left(\overline{B_{r/4}(x)}\right)\right) \leq C\left(\Lambda + R^{2s}f(x)\right)_+^n \mu_g(B_{r/4}(x)), \tag{2.3.19}$$

where $C > 0$ is a universal constant depending only on n, λ, a_1, a_2, and s_0.

Proof. Let $x \in A$ and $y = \phi(x)$. By applying Lemma 2.3.3 to u with $M_0 = \frac{C_0}{\varepsilon_0}(\Lambda + R^{2s}f(x))_+$, we find $r = r_k \leq r_0$ such that (2.3.18) holds. Moreover, since $\Gamma \leq u$, we have

$$\mu_g\left(\left\{z \in R_k : \Gamma(z) > P_y(z) + C\left(\Lambda + R^{2s}f(x)\right)_+ (r_k/R)^2\right\}\right) \leq \varepsilon_0 \mu_g(R_k).$$

Thus, Lemma 2.3.5 shows that

$$P_y(z) \leq \Gamma(z) \leq P_y(z) + C\left(\Lambda + R^{2s}f(x)\right)_+ \left(\frac{r}{R}\right)^2 \tag{2.3.20}$$

for all $z \in B_{r/2}(x)$.

We first claim that there is a constant $C_1 > 0$ such that

$$|\nabla P_{y_*}(z) - \nabla P_y(z)|_{g(z)} \leq \frac{C_1}{R^2}\left(\Lambda + R^{2s}f(x)\right)_+ r \tag{2.3.21}$$

for all $z \in B_{r/4}(x)$ and $y_* \in \phi(z)$. It is enough to show that

$$\left|\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(P_{y_*} - P_y\right)(\gamma(t))\right| \leq \frac{C_1}{R^2}\left(\Lambda + R^{2s}f(x)\right)_+ r \tag{2.3.22}$$

for all geodesics γ, with unit speed, starting from $\gamma(0) = z$. Suppose that there is a geodesic γ such that (2.3.22) does not hold. We may assume that

$$\frac{C_1}{R^2}\left(\Lambda + R^{2s}f(x)\right)_+ r \leq \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(P_{y_*} - P_y\right)(\gamma(t)),$$

by considering $\tilde{\gamma}(t) = \gamma(-t)$ instead of $\gamma(t)$ if necessary. Let $\varepsilon > 0$, then there is $\delta > 0$ such that if $|t| < \delta$, we have

$$\frac{C_1}{R^2}\left(\Lambda + R^{2s}f(x)\right)_+ r - \varepsilon \leq \frac{(P_{y_*} - P_y)(\gamma(t)) - (P_{y_*} - P_y)(\gamma(0))}{t} \leq \frac{h(t) - h(0)}{t}, \tag{2.3.23}$$

where $h(t) = (\Gamma - P_y)(\gamma(t))$. Let $T > 0$ be the first time when γ reaches the boundary of $B_{3r/8}(x)$, namely, $\gamma(T) \in \partial B_{3r/8}(x)$. Let N be the least integer not smaller than T/δ, and let $0 = t_0 < t_1 < \cdots < t_N = T$ be equally distributed times. Then we have $t_{i+1} - t_i = T/N \le \delta$. We observe that Lemma 2.3.4 shows

$$\frac{h(t_i) - h(t_{i-1})}{t_i - t_{i-1}} \le \frac{h(t_{i+1}) - h(t_i)}{t_{i+1} - t_i} + \frac{1}{2R^2}(t_{i+1} - t_{i-1}), \quad i = 1, 2, \cdots, N-1. \tag{2.3.24}$$

Thus, it follows from (2.3.23) and (2.3.24) that

$$\begin{aligned}\frac{C_1}{R^2}\left(\Lambda + R^{2s} f(x)\right)_+ r - \varepsilon \le \frac{h(t_1) - h(t_0)}{T/N} &\le \frac{h(t_2) - h(t_1)}{T/N} + \frac{1}{2R^2}\frac{2T}{N} \\ &\le \cdots \le \frac{h(t_N) - h(t_{N-1})}{T/N} + \frac{1}{2R^2}\frac{2T}{N}(N-1).\end{aligned}$$

Therefore, we obtain that

$$N\left(\frac{C_1}{R^2}\left(\Lambda + R^{2s} f(x)\right)_+ r - \varepsilon\right) \le \frac{h(t_N) - h(t_0)}{T/N} + \frac{1}{2R^2}\frac{2T}{N}\frac{N(N-1)}{2}.$$

Since γ has a unit speed, we have $r/8 < T < r$, and hence

$$\begin{aligned}&\frac{C_1}{R^2}\left(\Lambda + R^{2s} f(x)\right)_+ r - \varepsilon \\ &\quad \le \frac{(\Gamma - P_y)(\gamma(T)) - (\Gamma - P_y)(z)}{r/8} + \frac{r}{2R^2} \le \frac{(\Gamma - P_y)(\gamma(T))}{r/8} + \frac{r}{2R^2}.\end{aligned}$$

Recalling that ε was arbitrary, we have

$$\frac{C_1}{8}\left(\Lambda + R^{2s} f(x)\right)_+ \left(\frac{r}{R}\right)^2 - \frac{r^2}{16R^2} \le (\Gamma - P_y)(\gamma(T)). \tag{2.3.25}$$

Since $\gamma(T) \in \partial B_{3r/8}(x) \subset B_{r/2}(x)$, the inequality (2.3.25) with sufficiently large constant $C_1 > 0$ contradicts to (2.3.20). Therefore, we have proved the claim (2.3.21).

Let us next prove (2.3.19) using (2.3.21). It is enough to show that

$$\phi\left(\overline{B(x,r/4)}\right) \subset B\Big(\phi(x), C(\Lambda + R^{2s}f(x))_+ r\Big). \tag{2.3.26}$$

Indeed, once (2.3.26) is proved, then by (Comp) and (3.2.8) (or (RVD)) we have

$$\begin{aligned}&\mu_g\left(\phi\left(\overline{B(x,r/4)}\right)\right)\\&\quad\le \mu_g\Big(B(\phi(x), C(\Lambda + R^{2s}f(x))_+ r)\Big) \le C\left(\Lambda + R^{2s}f(x)\right)_+^n \mu_g(B(x,r/4)).\end{aligned}$$

To verify (2.3.26), let us fix $z \in \overline{B(x,r/4)}$ and $y_* \in \phi(z)$. Then we know from (2.3.21) that

$$\left|\exp_z^{-1} y_* - \exp_z^{-1} y\right|_{g(z)} = R^2|\nabla P_{y_*}(z) - \nabla P_y(z)|_{g(z)} \le C_1\left(\Lambda + R^{2s}f(x)\right)_+ r.$$

Thus, it only remains to show that

$$d(y_*, y) \le \left|\exp_z^{-1} y_* - \exp_z^{-1} y\right|_{g(z)}. \tag{2.3.27}$$

Let $\xi_1 = \exp_z^{-1} y_*$ and $\xi_2 = \exp_z^{-1} y$. Let us consider a family of geodesics

$$\gamma(s,t) = \exp_z(t(\xi_1 + s(\xi_2 - \xi_1))),$$

and the Jacobi field J along γ. Then, by [53, Equation (1.9)] (or see, e.g. [66]), we have

$$|J(1)|_{g(y_*)} \le |J'(0)|_{g(z)} = |\xi_2 - \xi_1|_{g(z)}.$$

Therefore, (2.3.27) follows by considering the curve $s \mapsto \gamma(s,1)$ and observing that

$$d(y_*, y) \le \int_0^1 |\gamma'(s,1)|_{g(y_*)}\,\mathrm{d}s \le \int_0^1 |\xi_2 - \xi_1|_{g(z)}\,\mathrm{d}s = |\xi_2 - \xi_1|_{g(z)}.$$

We have proved (2.3.27), from which we deduce (2.3.26). $\square$

We are now ready to prove Lemma 2.3.2 by using the previous lemmas and the dyadic cubes in Theorem 2.2.4.

Proof of Lemma 2.3.2. In order to construct such a family, we are going to use Theorem 2.2.4. Let us first fix the smallest integer $N \in \mathbb{Z}$ such that $c_2\delta_0^N \leq r_0$. Then there are finitely many dyadic cubes Q_α^N of generation N such that $\overline{Q}_\alpha^N \cap A \neq \emptyset$ and $A \subset \cup_\alpha \overline{Q}_\alpha^N$. Whenever a dyadic cube Q_α^N does not satisfy (iii) and (iv), we consider its successors $Q_\beta^{N+1} \subset Q_\alpha^N$ instead of Q_α^N. Among these successors of $N+1$ generation, we only keep those whose closures intersect A, and discard the rest. We iterate the process in the same way. The only part we need to prove is that the process finishes in a finite number of steps.

Assume that the process produces an infinite sequence of nested dyadic cubes $\{Q_\alpha^j\}_{j=N}^\infty$ with $\alpha = \alpha_j \in I_j$. Then the intersection of their closures is some point x_0 which is contained in the contact set A. By Lemma 2.3.6, we there is an $r = r_k \leq r_0$ such that (2.3.18) and (2.3.19) hold. The condition $\rho_1 \geq 1/\delta_0$ in (2.3.2) allows us to find $j \geq N$ satisfying $r/(4\rho_1) \leq c_2\delta_0^j < r/4$. Then it follows from Theorem 2.2.4 (iv) that

$$\overline{Q}_\alpha^j \subset B_{r/4}(x_0). \tag{2.3.28}$$

Moreover, by Theorem 2.2.4 (v), Q_α^j contains some ball $B(z_\alpha^j, c_1\delta_0^j)$. If $z \in B_r(x_0)$, then $d(z, z_\alpha^j) \leq d(z, x_0) + d(x_0, z_\alpha^j) < (1 + 4/\rho_1)c_2\delta_0^j$, which shows that

$$B_r(x_0) \subset B(z_\alpha^j, (1 + 4\rho_1)c_2\delta_0^j). \tag{2.3.29}$$

Therefore, it follows from (2.3.19), (2.3.28), (Comp), and the volume doubling property (3.2.8) that

$$\begin{aligned}\mu_g(\phi(\overline{Q}_\alpha^j)) \leq \mu_g(\phi(\overline{B_{r/4}(x_0)})) &\leq C\left(\Lambda + R^{2s}f(x_0)\right)_+^n \mu_g(B_{r/4}(x_0)) \\ &\leq C\left(\Lambda + R^{2s}\max_{\overline{Q}_\alpha^j} f\right)_+^n \mu_g(Q_\alpha^j).\end{aligned}$$

Furthermore, (2.3.18), (Comp), (3.2.8), and (RVD) show that

$$\begin{aligned}
&\mu_g\left(B(z_\alpha^j,(1+4\rho_1)c_2\delta_0^j)\cap\left\{u\le\Gamma+CR^{-2}\left(\Lambda+R^{2s}\max_{\overline{Q}_\alpha^j}f\right)_+d_{j,\alpha}^2\right\}\right)\\
&\ge\mu_g\left(B_r(x_0)\cap\left\{u\le P_y+C\left(\Lambda+R^{2s}\max_{\overline{Q}_\alpha^j}f\right)_+\left(\frac{r}{R}\right)^2\right\}\right)\\
&\ge(1-\varepsilon)\mu_g(R_k(x_0))\ge\gamma\mu_g(Q_\alpha^j)
\end{aligned}$$

for some universal constant $\gamma>0$. We have shown that Q_α^j satisfy (iii) and (iv), which yields a contradiction. Therefore, the process must stop in a finite number of steps. □

2.4 A barrier function

In this section we construct a barrier function, which is one of the key ingredients for the Krylov–Safonov Harnack inequality. We use the function of the form $(d_{z_0}^2(\cdot))^{-\alpha}$ with $\alpha>0$ large, which have been used as a barrier function both for nonlocal operators in the Euclidean spaces [15] and local operators on Riemannian manifolds [12]. In [12], the Hessian bound of distance squared function at one point is enough to evaluate the operator's value. However, for nonlocal operators on Riemannian manifolds, the curvatures near the given point have to be taken into account to evaluate the operator. In order to make the universal constants independent of the curvatures, we need to look at a small region.

Lemma 2.4.1. *For $z_0\in M$, let K be the supremum of the sectional curvatures in $B_{\mathrm{inj}(z_0)}(z_0)$ and let $15R<\mathrm{inj}(z_0)\wedge\frac{\pi}{\sqrt{K}}$. There are universal constants $\alpha>0$ and $s_0\in(0,1)$, depending only on n, λ, and Λ, such that the function*

$$v(x)=\max\left\{-\left(\frac{\rho_0}{20}\right)^{-2\alpha},-\left(\frac{d_{z_0}(x)}{5R}\right)^{-2\alpha}\right\}$$

is a supersolution to

$$R^{2s}\mathcal{M}_{\mathcal{L}_0}^+v(x)+\Lambda\le 0,$$

for every $s_0 \le s < 1$ *and* $x \in B_{5R}(z_0) \setminus \overline{B}_{\rho_0 R}(z_0)$.

Proof. Fix x and let $R_0 := d_{z_0}(x) \in (\rho_0 R, 5R)$. Let us consider normal coordinates centered at x, then by (2.3.5) the point z_0 is included in the normal coordinates. Thus, we may assume that $\exp_x^{-1} z_0 = R_0 e_1$. Let $\xi := \exp_x^{-1} z$. By the Toponogov's triangle comparison (see, e.g., [93]), we have $d_{z_0}(z) \le |R_0 e_1 - \xi|_{\mathbb{R}^n}$ and $d_{z_0}(\mathcal{T}_x(z)) \le |R_0 e_1 + \xi|_{\mathbb{R}^n}$, and hence,

$$\delta(v, x, z) \le -\left(\frac{1}{5R}\right)^{-2\alpha} \left(|R_0 e_1 + \xi|_{\mathbb{R}^n}^{-2\alpha} + |R_0 e_1 - \xi|_{\mathbb{R}^n}^{-2\alpha} - 2R_0^{-2\alpha}\right)$$

for $d(z, x) \le R_0/2$. As $T_x M$ being identified as $\mathbb{R}^n$, a simple algebraic inequality shows that

$$\delta(v, x, z) \le 2\alpha \left(\frac{R_0}{5R}\right)^{-2\alpha} \left(\frac{|\xi|^2}{R_0^2} - (2\alpha+2)\frac{\xi_1^2}{R_0^2} + (2\alpha+2)(\alpha+2)\frac{\xi_1^2|\xi|^2}{R_0^4}\right) \tag{2.4.1}$$

as in the proof of [15, Lemma 9.1].

Let us take $\alpha = \alpha(n, \lambda, \Lambda) > 0$ sufficiently large so that

$$\lambda(2\alpha+2)\left(\frac{3}{\pi}\right)^{n-1} \int_{\partial B_1} v_1^2 \,\mathrm{d}v - \Lambda|\partial B_1| > C_1 \Lambda, \tag{2.4.2}$$

for some universal constant $C_1 > 0$ to be determined later, where $\mathrm{d}v$ is the usual spherical measure on ∂B_1. Then,

$$\begin{aligned} R^{2s}\mathcal{M}_{\mathcal{L}_0}^+ v(x) \le{}& (2-2s)R^{2s} \int_{B(0,R_0/2)} \frac{\Lambda\delta(v,x,z)_+ - \lambda\delta(v,x,z)_-}{\mu_{g^*}(B(0,|\xi|))|\xi|^{2s}} \,\mathrm{d}V_s^*(\xi) \\ &+ R^{2s} \sup_{L\in\mathcal{L}_0} \int_{B(x,R_0/2)} (v(z)-v(x))\nu_x(z)\,\mathrm{d}V_a(z) \\ &+ R^{2s} \sup_{L\in\mathcal{L}_0} \int_{M\setminus B(x,R_0/2)} (v(z)-v(x))\nu_x(z)\,\mathrm{d}V(z) =: I_1 + I_2 + I_3. \end{aligned} \tag{2.4.3}$$

For I_1, we use (2.4.1) to obtain

$$I_1 \leq 2\alpha(2-2s)\frac{R^{2s}}{R_0^2}\left(\frac{5R}{R_0}\right)^{2\alpha} \cdot \int_{B_{R_0/2}} \frac{\Lambda|\xi|^2 - \lambda(2\alpha+2)\xi_1^2 + \Lambda(2\alpha+2)(\alpha+2)\xi_1^2|\xi|^2R_0^{-2}}{\mu_{g^*}(B(0,|\xi|))|\xi|^{2s}}\,\mathrm{d}V_s^*(\xi). \tag{2.4.4}$$

Since the sectional curvatures on $B(x, R_0/2)$ are bounded by K from above and 0 from below, we have

$$\begin{aligned}
I_{1,1} &:= \int_{B_{R_0/2}} \frac{\Lambda|\xi|^2 - \lambda(2\alpha+2)\xi_1^2}{\mu_{g^*}(B(0,|\xi|))|\xi|^{2s}}\,\mathrm{d}V_s^*(\xi) \\
&= \int_0^{R_0/2}\int_{\partial B_1} \frac{\Lambda - \lambda(2\alpha+2)v_1^2}{\mu_{g^*}(B(0,t))t^{2s}} \det(D\exp_x)(tv)t^{n+1}\,\mathrm{d}v\,\mathrm{d}t \\
&\leq \int_0^{R_0/2}\int_{\partial B_1} \left(\Lambda - \lambda(2\alpha+2)\left(\frac{\sin(\sqrt{K}t)}{\sqrt{K}t}\right)^{n-1} v_1^2\right)\mathrm{d}v\frac{t^{n+1}}{\mu_{g^*}(B(0,t))t^{2s}}\,\mathrm{d}t.
\end{aligned}$$

As in the proof of Lemma 2.3.5, we observe from $t \leq R_0/2 < \frac{5}{2}R < \frac{\pi}{6\sqrt{K}}$ that

$$\frac{\sin(\sqrt{K}t)}{\sqrt{K}t} \geq \frac{\sin(\pi/6)}{\pi/6} = \frac{3}{\pi}. \tag{2.4.5}$$

Thus, (2.4.2), (2.4.5), and the Gromov's theorem yield that

$$I_{1,1} \leq -C_1\Lambda\int_0^{R_0/2}\frac{t^{n+1}}{|B_t|t^{2s}}\,\mathrm{d}t = -\frac{C_1\Lambda}{(2-2s)\omega_n}(R_0/2)^{2-2s}. \tag{2.4.6}$$

Similarly, we obtain that

$$\begin{aligned} I_{1,2} &:= \int_{B_{R_0/2}} \frac{\Lambda(2\alpha+2)(\alpha+2)\xi_1^2|\xi|^2 R_0^{-2}}{\mu_{g^*}(B(0,|\xi|))|\xi|^{2s}} \,\mathrm{d}V_s^*(\xi) \\ &\leq C\frac{\Lambda(2\alpha+2)(\alpha+2)}{R_0^2}\int_0^{R_0/2} \frac{t^{n+3}}{\mu_{g^*}(B(0,t))t^{2s}}\,\mathrm{d}t \\ &\leq C\frac{\Lambda(2\alpha+2)(\alpha+2)}{R_0^2}\left(\frac{3}{\pi}\right)^{1-n}\int_0^{R_0/2} \frac{t^{n+3}}{\omega_n t^{n+2s}}\,\mathrm{d}t \\ &\leq C\frac{\Lambda(2\alpha+2)(\alpha+2)}{R_0^2}\frac{(R_0/2)^{4-2s}}{4-2s}. \end{aligned} \tag{2.4.7}$$

Therefore, combining (2.4.4), (2.4.6), and (2.4.7), and using $R_0 \in (\rho_0, 5R)$, we estimate I_1 as

$$I_1 \leq 2\alpha\Lambda\left(-cC_1 + C\frac{2-2s}{4-2s}(2\alpha+2)(\alpha+2)\right), \tag{2.4.8}$$

for some universal constants $c, C > 0$, with c independent of α.

For I_2, we use a similar computation in Lemma 2.2.3 to obtain

$$I_2 \leq \frac{C}{2s}\Lambda. \tag{2.4.9}$$

On the other hand, since v is bounded, by following the proof of Lemma 2.2.2 we have

$$I_3 \leq C\Lambda\left(\frac{R}{r}\right)^{2s}\frac{2-2s}{1-2^{-2s}} \tag{2.4.10}$$

for some $C = C(n, \lambda, \Lambda, \alpha) > 0$. Thus, (2.4.3), (2.4.8), (2.4.9), and (2.4.10) yield

$$\begin{aligned} &R^{2s}\mathcal{M}^+_{\mathcal{L}_0}v(x) \\ &\quad\leq 2\alpha\Lambda\left(-cC_1 + C\frac{2-2s}{4-2s}(2\alpha+2)(\alpha+2)\right) + \frac{C}{2s}\Lambda + C\Lambda\left(\frac{R}{r}\right)^{2s}\frac{2-2s}{1-2^{-2s}}. \end{aligned}$$

We now choose s_0 close to 1 so that the terms containing $(2-2s)$ become small. Then we obtain

$$R^{2s}\mathcal{M}^+_{\mathcal{L}_0}v(x) \leq -cC_1\Lambda + \frac{C}{s_0}\Lambda,$$

which finishes the proof by assuming that we have taken α sufficiently large so that (2.4.2) holds with $-cC_1 + \frac{C}{s_0} < -1$. □

Lemma 2.4.2. *For $z_0 \in M$, let K be the supremum of the sectional curvatures in $B_{\mathrm{inj}(z_0)}(z_0)$ and let $15R < \mathrm{inj}(z_0) \wedge \frac{\pi}{\sqrt{K}}$. Given any $s_0 \in (0,1)$, there are universal constants $\alpha > 0$ and $\kappa \in (0, 1/4]$, depending only on n, λ, Λ, and s_0, such that the function*

$$v(x) = \max\left\{ -\left(\frac{\kappa\rho_0}{5}\right)^{-2\alpha}, -\left(\frac{d_{z_0}(x)}{5R}\right)^{-2\alpha} \right\}$$

is a supersolution to

$$R^{2s}\mathcal{M}^+_{\mathcal{L}_0}v(x) + \Lambda \leq 0,$$

for every $s_0 < s < 1$ and $x \in B_{5R}(z_0) \setminus \overline{B}_{\rho_0 R}(z_0)$.

Proof. Let s_1 and α_0 be the s_0 and α in Lemma 2.4.1, respectively. If $s \geq s_1$, then the desired result holds with α_0 and $\kappa = 1/4$. Now for $s_0 < s < s_1$, we will show the result still holds if we choose κ small enough.

Let $\alpha = \max(\alpha_0, n/2)$. For x with $R_0 = d_{z_0}(x) \in (\rho_0 R, 5R)$, let us consider normal coordinates centered at x. Then, as in Lemma 2.4.1 we have

$$\begin{aligned} R^{2s}\mathcal{M}^+_{\mathcal{L}_0}v(x) &\leq (2-2s)R^{2s}\int_{B(0,R_0/2)} \frac{\Lambda\delta(v,x,z)_+}{\mu_{g^*}(B(0,|\xi|))|\xi|^{2s}}\,\mathrm{d}V^*_s(\xi) \\ &\quad - (2-2s)R^{2s}\int_{B(0,R_0/2)} \frac{\lambda\delta(v,x,z)_-}{\mu_{g^*}(B(0,|\xi|))|\xi|^{2s}}\,\mathrm{d}V^*_s(\xi) + C \\ &=: I_1 + I_2 + C. \end{aligned}$$

Since $v \in C^2$ in $B(x, R_0/2)$ and v is bounded above, δ_+ is bounded above. Hence, $I_1 \leq C$ for some universal constant. On the other hand, since δ_- is not integrable and $s < s_1 < 2$, we choose κ small enough so that $I_1 + I_2 + C < -\Lambda$. □

Corollary 2.4.3. *For $z_0 \in M$, let K be the supremum of the sectional curvatures in $B_{\mathrm{inj}(z_0)}(z_0)$ and let $15R < \mathrm{inj}(z_0) \wedge \frac{\pi}{\sqrt{K}}$. There is a function v such that*

$$\begin{cases} v \geq 0 & \text{in } M \setminus B_{5R}(z_0), \\ v \leq 0 & \text{in } B_{2R}(z_0), \\ R^{2s}\mathcal{M}^+_{\mathcal{L}_0} v + \Lambda \leq 0 & \text{in } B_{5R} \setminus \overline{B}_{\rho_0 R}(z_0), \\ R^{2s}\mathcal{M}^+_{\mathcal{L}_0} v \leq C & \text{in } B_{5R}(z_0), \\ v \geq -C & \text{in } B_{5R}(z_0), \end{cases}$$

for some universal constant $C > 0$, depending only on n, λ, Λ, and s_0.

Proof. Let α and κ be constants given in Lemma 2.4.2. We define $v(x) = \psi(d^2_{z_0}(x)/R^2)$, where ψ is a smooth and increasing function on $[0, \infty)$ such that

$$\psi(t) = \left(\frac{3^2}{5^2}\right)^{-\alpha} - \left(\frac{t}{5^2}\right)^{-\alpha} \quad \text{if } t \geq (\kappa\rho_0)^2.$$

By Lemma 2.4.2, $R^{2s}\mathcal{M}^+_{\mathcal{L}_0} v + \Lambda \leq 0$ in $B_{5R} \setminus B_{\rho_0 R}$. Thus, it only remains to show that $R^{2s}\mathcal{M}^+_{\mathcal{L}_0} v \leq C$ in $\overline{B}_{\rho_0 R}$. Indeed, for $x \in \overline{B}_{\rho_0 R}$, we have that $|\delta(v, x, z)| \leq C d_x(z)^2/R^2$ for $z \in B_R(x)$, and that v is bounded by a uniform constant. Therefore, we obtain $R^{2s}\mathcal{M}^+_{\mathcal{L}_0} v(x) \leq C$ by Lemma 2.2.3. $\square$

2.5 L^ε-estimate

This section is devoted to the so-called L^ε-estimate, which is the main ingredient in the proof of the Harnack inequality. It will follow from the following lemma, which connects a pointwise estimate to an estimate in measure, and the standard Calderón–Zygmund technique in [12].

Lemma 2.5.1. *Let $s_0 \in (0, 1)$ and assume $s \in [s_0, 1)$. Let c_1, c_2, and δ_0 be the constants in Theorem 2.2.4, and let $\delta = \frac{2c_1}{c_2}\delta_0$. For $z_0 \in M$, let K be the supremum of the sectional curvatures in $B_{\mathrm{inj}(z_0)}(z_0)$ and let $15R < \mathrm{inj}(z_0) \wedge \frac{\pi}{\sqrt{K}}$. If $u \in C^2(B_{7R}(z_0))$ is a nonnegative function on M satisfying*

$R^{2s}\mathcal{M}^-_{\mathcal{L}_0}u \le \varepsilon_0$ in $B_{7R}(z_0)$ and $\inf_{B_{2R}} u \le 1$, then

$$\frac{\mu_g(\{u \le M_0\} \cap B_{\delta R}(z_0))}{\mu_g(B_{7R}(z_0))} \ge c_0,$$

where $\varepsilon_0 > 0$, $c_0 \in (0,1)$, and $M_0 > 1$ are constants depending only on n, λ, Λ, a_1, a_2, and s_0.

Proof. Let v be the barrier function constructed in Corollary 2.4.3 and define $w = u+v$. Then w satisfies $w \ge 0$ in $M \setminus B_{5R}$, $\inf_{B_{2R}} w \le 1$, and $R^{2s}\mathcal{M}^-_{\mathcal{L}_0}w \le \varepsilon_0 + R^{2s}\mathcal{M}^+_{\mathcal{L}_0}v$ in B_{5R}. By applying Lemma 2.3.1 to w with its envelope Γ_w, we have

$$\mu_g(B_R) \le C \sum_j \left(\varepsilon_0 + R^{2s} \max_{\overline{Q}^j_\alpha} \mathcal{M}^+_{\mathcal{L}_0}v + \Lambda\right)^n_+ \mu_g(Q^j_\alpha).$$

Since $R^{2s}\mathcal{M}^+_{\mathcal{L}_0}v + \Lambda \le 0$ in $B_{5R} \setminus \overline{B}_{\rho_0 R}$ and $R^{2s}\mathcal{M}^+_{\mathcal{L}_0}v \le C$ in B_{5R}, we obtain

$$\mu_g(B_R) \le C\varepsilon_0^n \mu_g(B_{5R}) + C \sum_{\overline{Q}^j_\alpha \cap \overline{B}_{\rho_0 R} \neq \emptyset} \mu_g(Q^j_\alpha).$$

We use the volume doubling property (3.2.8) and then take $\varepsilon_0 > 0$ sufficiently small so that we have

$$\mu_g(B_{7R}) \le C \sum_{\overline{Q}^j_\alpha \cap \overline{B}_{\rho_0 R} \neq \emptyset} \mu_g(Q^j_\alpha).$$

By using Lemma 2.3.2 (iv), we obtain

$$\mu_g(B_{7R}) \le C \sum_{\overline{Q}^j_\alpha \cap \overline{B}_{\rho_0 R} \neq \emptyset} \mu_g\left(B(z^j_\alpha, (1+4/\rho_1)c_2\delta^j_0) \cap \{w \le \Gamma + C\}\right).$$

We claim that $B(z^j_\alpha, (1+4/\rho_1)c_2\delta^j_0) \subset B_{\delta R}(z_0)$ whenever $\overline{Q}^j_\alpha \cap \overline{B}_{\rho_0 R} \neq \emptyset$. Indeed, let $z_* \in \overline{Q}^j_\alpha \cap \overline{B}_{\rho_0 R}$, then for any $z \in B(z^j_\alpha, (1+4/\rho_1)c_2\delta^j_0)$, we have

$$d(z, z_0) \le d(z, z^j_\alpha) + d(z^j_\alpha, z_*) + d(z_*, z_0) < (1+4/\rho_1)c_2\delta^j_0 + c_2\delta^j_0 + \rho_0 R.$$

We recall from the construction of Q_α^j and (2.3.2), that $c_2\delta_0^j \le r_0/4 < \rho_0 R$ and

$$d(z, z_0) < (3 + 4/\rho_1)\rho_0 R \le \delta R,$$

which proves the claim. Thus, by taking a subcover of $\{B(z_\alpha^j, (1+4/\rho_1)c_2\delta_0^j)\}$ with finite overlapping and using $v \ge -C$ in B_{5R}, we arrive at

$$\mu_g(B_{7R}) \le C\mu_g(\{u \le M_0\} \cap B_{\delta R})$$

for some $M_0 > 1$. Therefore, we obtain the desired result by letting $c_0 = 1/C$. □

Let $\delta_1 = \delta_0(1 - \delta_0)/2 \in (0, 1)$. Let k_R be the integer satisfying

$$c_2\delta_0^{k_R-1} < R \le c_2\delta_0^{k_R-2},$$

which is the generation of a dyadic cube whose size is comparable to that of some ball of radius R. Lemma 2.5.1, together with the Calderón–Zygmund technique developed in [12], provides the following L^ε-estimate.

Lemma 2.5.2. *Let $s_0 \in (0, 1)$ and assume $s \in [s_0, 1)$. Let ε_0, c_0, and M_0 be the constants in Lemma 2.5.1. For $z_0 \in M$, let $R > 0$ be such that $15R < \operatorname{inj}(z_0) \wedge \frac{\pi}{\sqrt{K}}$. Let $u \in C^2(B_{7R}(z_0))$ be a nonnegative function on M satisfying $R^{2s}\mathcal{M}^-_{\mathcal{L}_0} u \le \varepsilon_0$ in $B_{7R}(z_0)$ and $\inf_{B(z_0,\delta_1 R)} u \le 1$. If Q_1 is a dyadic cube of generation k_R such that $d(z_0, Q_1) \le \delta_1 R$, then*

$$\frac{\mu_g(\{u > M_0^i\} \cap Q_1)}{\mu_g(Q_1)} \le (1 - c_0)^i.$$

for all $i = 1, 2, \cdots$. As a consequence, we have

$$\frac{\mu_g(\{u > t\} \cap Q_1)}{\mu_g(Q_1)} \le Ct^{-\varepsilon}, \quad t > 0,$$

for some universal constants $C > 0$ and $\varepsilon > 0$.

A simple chaining argument and Lemma 2.5.2 prove the following weak

Harnack inequality.

Theorem 2.5.3 (Weak Harnack inequality)**.** *Let $s_0 \in (0,1)$ and assume $s \in [s_0, 1)$. For $z_0 \in M$, let K be the supremum of the sectional curvatures in $B_{\mathrm{inj}(z_0)}(z_0)$ and let $R > 0$ be such that $2R < \mathrm{inj}(z_0) \wedge \frac{\pi}{\sqrt{K}}$. If $u \in C^2(B_{2R}(z_0))$ is a nonnegative function satisfying $\mathcal{M}^-_{\mathcal{L}_0} u \leq C_0$ in $B_{2R}(z_0)$, then*

$$\left(\frac{1}{\mu_g(B_R)} \int_{B_R} u^p \, \mathrm{d}V(z) \right)^{1/p} \leq C \left(\inf_{B_R} u + C_0 R^{2s} \right),$$

where $p > 0$ and $C > 0$ are universal constants depending only on n, λ, Λ, a_1, a_2, and s_0.

See, for instances, [12, Theorem 8.1] for the proof of Theorem 2.5.3.

2.6 Harnack inequality

In this section we prove the following theorem, from which Theorem 2.1.1 will follow. Let us recall that $\delta_1 = \delta_0(1-\delta_0)/2 \in (0,1)$.

Theorem 2.6.1. *Let $s_0 \in (0,1)$ and assume $s \in [s_0, 1)$. For $z_0 \in M$, let K be the supremum of the sectional curvatures in $B_{\mathrm{inj}(z_0)}(z_0)$ let $R > 0$ be such that $15R < \mathrm{inj}(z_0) \wedge \frac{\pi}{\sqrt{K}}$. If a nonnegative function $u \in C^2(B_{7R}(z_0))$ satisfies*

$$R^{2s} \mathcal{M}^-_{\mathcal{L}_0} u \leq \varepsilon_0 \quad \textit{and} \quad R^{2s} \mathcal{M}^+_{\mathcal{L}_0} u \geq -\varepsilon_0 \quad \textit{in } B_{7R}(z_0)$$

and $\inf_{B(z_0, \delta_1 R)} u \leq 1$, then

$$\sup_{B(z_0, \delta_1 R/4)} u \leq C,$$

where $\varepsilon_0 > 0$ and $C > 0$ are universal constants depending only on n, λ, Λ, a_1, a_2, and s_0.

Proof. Let ε and ε_0 be the constants as in Lemma 2.5.2 and let $\gamma = n/\varepsilon$. Let us consider the minimal value of $t > 0$ such that

$$u(x) \leq h_t(x) := t\left(1 - \frac{d_{z_0}(x)}{\delta_1 R}\right)^{-\gamma} \quad \text{for all } x \in B(z_0, \delta_1 R). \tag{2.6.1}$$

Then by (2.6.1), we have $\sup_{B(z_0,\delta_1 R/4)} u \leq t(3/4)^{-\gamma}$, from which we can conclude the theorem once we show that $t \leq C$ for some universal constant.

There exists a point $x_0 \in B(z_0, \delta_1 R)$ satisfying $u(x_0) = h_t(x_0)$. Let $d = \delta_1 R - d_{z_0}(x_0)$, $r = d/2$, and $A = \{u > u(x_0)/2\}$, then we have $u(x_0) = h_t(x_0) = t(\delta_1 R/d)^{\gamma}$. Let Q_1 be the unique dyadic cube of generation k_R that contains the point x_0, which clearly satisfies $d(z_0, Q_1) \leq \delta_1 R$. Then, by applying Lemma 2.5.2 to u with Q_1, we obtain

$$\mu_g(A \cap Q_1) \leq C\left(\frac{u(x_0)}{2}\right)^{-\varepsilon} \mu_g(Q_1) \leq Ct^{-\varepsilon}\left(\frac{r}{R}\right)^n \mu_g(Q_1). \tag{2.6.2}$$

We will show that there is a small constant $\theta > 0$ such that

$$\mu_g(\{u \leq u(x_0)/2\} \cap Q_2) \leq \frac{1}{2}\mu_g(Q_2), \tag{2.6.3}$$

where $Q_2 \subset Q_1$ is the dyadic cube of generation $k_{\theta r/14}$ containing the point x_0, provided that t is sufficiently large. Recalling that Q_2 contains some ball $B(z, c_1\delta_0^{k_{\theta r/14}})$, we have from (2.6.2) and (3.2.8) that

$$\begin{aligned} \mu_g(A \cap Q_2) \leq \mu_g(A \cap Q_1) &\leq \frac{C}{t^{\varepsilon}}\left(\frac{r}{R}\right)^n \mu_g(B(z, c_2\delta_0^{k_R})) \\ &\leq \frac{C}{t^{\varepsilon}}\mu_g(B(z, c_1\delta_0^{k_{r\theta/14}})) \leq \frac{C}{t^{\varepsilon}}\mu_g(Q_2). \end{aligned}$$

However, when t is large, we also obtain

$$\mu_g(A \cap Q_2) < \frac{1}{2}\mu_g(Q_2),$$

which contradicts to (2.6.3). Therefore, the rest of the proof is dedicated to proving (2.6.3).

For every $x \in B(x_0, \theta r)$, we have

$$u(x) \leq h_t(x) \leq t\left(\frac{d-\theta r}{\delta_1 R}\right)^{-\gamma} = \left(1-\frac{\theta}{2}\right)^{-\gamma} u(x_0).$$

Let us define the functions

$$v(x) := \left(1-\frac{\theta}{2}\right)^{-\gamma} u(x_0) - u(x)$$

and $w := v_+$. We will apply Lemma 2.5.2 to w in $B(x_0, 7(\theta r/14))$. For $x \in B(x_0, 7(\theta r/14))$, since v is nonnegative in $B(x_0, \theta r)$, we have

$$\begin{aligned} \mathcal{M}^-_{\mathcal{L}_0} w(x) &\leq \mathcal{M}^-_{\mathcal{L}_0} v(x) + \mathcal{M}^+_{\mathcal{L}_0} v_-(x) \\ &\leq -\mathcal{M}^+_{\mathcal{L}_0} u(x) + \Lambda(2-2s) \int_{M\setminus B(x_0,\theta r)} \frac{v_-(z)}{\mu_g(B(x, d_x(z)))d_x(z)^{2s}} \, \mathrm{d}V(z) \\ &\leq R^{-2s}\varepsilon_0 + \Lambda(2-2s) \int_{M\setminus B(x_0,\theta r)} \frac{(u(z)-(1-\theta/2)^{-\gamma}u(x_0))_+}{\mu_g(B(x, d_x(z)))d_x(z)^{2s}} \, \mathrm{d}V(z). \end{aligned} \tag{2.6.4}$$

Let us define a function

$$g_\beta(x) := \beta\left(1-\frac{d_{z_0}(x)^2}{R^2}\right)_+,$$

and consider the largest number $\beta > 0$ such that $u \geq g_\beta$. From the assumption $\inf_{B(z_0,\delta_1 R)} u \leq 1$, we have $(1-\delta_1^2)\beta \leq 1$. Let $x_1 \in B(z_0, R)$ be a point where $u(x_1) = g_\beta(x_1)$. Then, since

$$\begin{aligned} &(2-2s)\text{p.v.} \int_M (u(z)-u(x_1))_- \nu_{x_1}(z) \, \mathrm{d}V(z) \\ &\leq (2-2s)\text{p.v.} \int_M (g_\beta(z)-g_\beta(x_1))_- \nu_{x_1}(z) \, \mathrm{d}V(z) \leq CR^{-2s}, \end{aligned}$$

we obtain that

$$
\begin{aligned}
&\varepsilon_0 \\
&\geq R^{2s}\mathcal{M}^-_{\mathcal{L}_0}u(x_1) \\
&= R^{2s}\inf\left((2-2s)\text{p.v.}\int_M \big((u(z)-u(x_1))_+ - (u(z)-u(x_1))_-\big)\nu_{x_1}(z)\,\mathrm{d}V(z)\right) \\
&\geq R^{2s}\lambda(2-2s)\text{p.v.}\int_M \frac{(u(z)-u(x_1))_+}{\mu_g(B(x_1,d_{x_1}(z)))d_{x_1}(z)^{2s}}\,\mathrm{d}V(z) - C.
\end{aligned}
$$

It follows from $u(x_1) \leq \beta \leq 1/(1-\delta_1^2) =: c$ that

$$
\begin{aligned}
&(2-2s)\text{p.v.}\int_M \frac{(u(z)-c)_+}{\mu_g(B(x_1,d_{x_1}(z)))d_{x_1}(z)^{2s}}\,\mathrm{d}V(z) \\
&\leq (2-2s)\text{p.v.}\int_M \frac{(u(z)-u(x_1))_+}{\mu_g(B(x_1,d_{x_1}(z)))d_{x_1}(z)^{2s}}\,\mathrm{d}V(z) \leq CR^{-2s}.
\end{aligned}
\tag{2.6.5}
$$

Let us now estimate the integral in (2.6.4) by using (2.6.5). If $u(x_0) \leq c$, then we have $t = u(x_0)(\delta_1 R/d)^{-\gamma} \leq c\delta_1^{-\gamma}$ and we are done. Otherwise, we obtain that

$$
\begin{aligned}
&\mathcal{M}^-_{\mathcal{L}_0}w(x) \\
&\leq R^{-2s}\varepsilon_0 + \Lambda(2-2s)\int_{M\setminus B(x_0,\theta r)} \frac{(u(z)-c)_+}{\mu_g(B(x,d_x(z)))d_x(z)^{2s}}\,\mathrm{d}V(z) \\
&= R^{-2s}\varepsilon_0 \\
&+ \Lambda(2-2s)\int_{M\setminus B(x_0,\theta r)} \frac{(u(z)-c)_+}{\mu_g(B(x_1,d_{x_1}(z)))d_{x_1}(z)^{2s}}\frac{\mu_g(B(x_1,d_{x_1}(z)))d_{x_1}(z)^{2s}}{\mu_g(B(x,d_x(z)))d_x(z)^{2s}}\,\mathrm{d}V(z).
\end{aligned}
$$

For $x \in B(x_0,\theta r/2)$, $x_1 \in B(z_0,R)$, and $z \in M \setminus B(x_0,\theta r)$, we have

$$
\begin{aligned}
\frac{d_{x_1}(z)}{d_x(z)} &\leq 1 + \frac{d(x,x_1)}{d(x,z)} \leq 1 + \frac{d(x,x_0)+d(x_0,z_0)+d(z_0,x_1)}{d(x_0,z)-d(x_0,x)} \\
&\leq 1 + \frac{\theta r/2 + \delta_1 R + R}{\theta r/2} \leq C\frac{R}{\theta r},
\end{aligned}
$$

and hence, by (3.2.8),

$$\frac{\mu_g(B(x_1, d_{x_1}(z)))d_{x_1}(z)^{2s}}{\mu_g(B(x, d_x(z)))d_x(z)^{2s}} \le C\left(\frac{R}{\theta r}\right)^{n+2s}.$$

Therefore, we have shown that

$$\left(\frac{\theta r}{14}\right)^{2s} \mathcal{M}^-_{\mathcal{L}_0} w \le C\left(\frac{R}{\theta r}\right)^n$$

in $B(x_0, 7(\theta r/14))$.

Let $Q_2 \subset Q_1$ be the dyadic cube of generation $k_{\theta r/14}$ containing the point x_0. Then by Lemma 2.5.2, we have

$$\begin{aligned}
&\mu_g(\{u < u(x_0)/2\} \cap Q_2) \\
&\quad = \mu_g(\{w > \left((1-\theta/2)^{-\gamma} - 1/2\right) u(x_0)\} \cap Q_2) \\
&\quad \le \frac{C\mu_g(Q_2)}{\left((1-\theta/2)^{-\gamma} - 1/2\right)^{\varepsilon} u(x_0)^{\varepsilon}} \left(\inf_{B(x_0, \delta_1\theta r/14)} w + C\left(\frac{R}{\theta r}\right)^n\right)^{\varepsilon}.
\end{aligned}$$

We can make the quantity $(1-\theta/2)^{-\gamma} - 1/2$ bounded away from 0 by taking $\theta > 0$ sufficiently small. Recalling that $u(x_0) = t(\delta_1 R/2r)^\gamma$, $w(x_0) = ((1 - \theta/2)^{-\gamma} - 1)u(x_0)$, and $\gamma = n/\varepsilon$, we obtain

$$\mu_g(\{u < u(x_0)/2\} \cap Q_2) \le C\mu_g(Q_2)\left(((1-\theta/2)^{-\gamma} - 1)^{\varepsilon} + t^{-\varepsilon}\theta^{-n\varepsilon}\right).$$

We choose a constant $\theta > 0$ sufficiently small so that

$$C\left((1-\theta/2)^{-\gamma} - 1\right)^{\varepsilon} \le \frac{1}{4}.$$

If $t > 0$ is sufficiently large so that $Ct^{-\varepsilon}\theta^{-n\varepsilon} \le 1/4$, then we arrive at (2.6.3). Therefore, t is uniformly bounded and the desired result follows. $\square$

2.7 Hölder estimates

This section is devoted to the proof of Theorem 2.1.2, which will follow from Lemma 2.7.1.

Lemma 2.7.1. *Let $s_0 \in (0,1)$ and assume $s \in [s_0, 1)$. For $z_0 \in M$, let K be the supremum of the sectional curvatures in $B_{\mathrm{inj}(z_0)}(z_0)$ let $R > 0$ be such that $15R < \mathrm{inj}(z_0) \wedge \frac{\pi}{\sqrt{K}}$. If $u \in C^2(B(z_0, 7R))$ is a function such that $|u| \leq \frac{1}{2}$ in $B(z_0, 7R)$ and*

$$R^{2s}\mathcal{M}^+_{\mathcal{L}_0} u \geq -\varepsilon_0 \quad \text{and} \quad R^{2s}\mathcal{M}^-_{\mathcal{L}_0} u \leq \varepsilon_0 \quad \text{in } B(z_0, 7R),$$

then $u \in C^\alpha$ at z_0 with an estimate

$$|u(x) - u(z_0)| \leq CR^{-\alpha} d(x, z_0)^\alpha,$$

where $\alpha \in (0,1)$ and $C > 0$ are constants depending only on n, λ, Λ, a_1, a_2, and s_0.

Proof. Let $R_k := 7 \cdot 4^{-k} R$ and $B_k := B(z_0, R_k)$. It is enough to find an increasing sequence $\{m_k\}_{k\geq 0}$ and a decreasing sequence $\{M_k\}_{k\geq 0}$ satisfying $m_k \leq u \leq M_k$ in B_k and $M_k - m_k = 4^{-\alpha k}$. We initially choose $m_0 = -1/2$ and $M_0 = 1/2$ for the case $k = 0$. Let us assume that we have sequences up to m_k and M_k. We want to show that we can continue the sequences by finding m_{k+1} and M_{k+1}.

Let Q_1 be a dyadic cube of generation $k_{R_{k+1}/7}$ that contains the point x. In Q_1, either $u > (M_k + m_k)/2$ or $u \leq (M_k + m_k)/2$ in at least half of the points in measure. We suppose that

$$\mu_g\left(\{u > (M_k + m_k)/2\} \cap Q_1\right) \geq \mu_g(Q_1)/2. \tag{2.7.1}$$

Let us define a function

$$v(x) := \frac{u(x) - m_k}{(M_k - m_k)/2},$$

then $v \geq 0$ in B_k by the induction hypothesis. For $w := v_+$, (2.7.1) is read as

$$\mu_g(\{w > 1\} \cap Q_1) \geq \mu_g(Q_1)/2. \tag{2.7.2}$$

To apply Lemma 2.5.2 to w, we need to estimate $\mathcal{M}^-_{\mathcal{L}_0} w \leq \mathcal{M}^-_{\mathcal{L}_0} v + \mathcal{M}^+_{\mathcal{L}_0} v_-$. We know that $R^{2s}\mathcal{M}^-_{\mathcal{L}_0} v \leq 2\varepsilon_0/(M_k - m_k)$ in B_{7R}. For $\mathcal{M}^+_{\mathcal{L}_0} v^-$, we use the bound

$$v(z) \geq -2((d_{z_0}(z)/R_k)^\alpha - 1) \quad \text{for } z \in M \setminus B_k,$$

which follows from the definition of v and the properties of sequences M_k and m_k. Then we have, for $x \in B(z_0, 3R_{k+1})$,

$$\begin{aligned}
&\mathcal{M}^-_{\mathcal{L}_0} w(x) \\
&\quad \leq \frac{2\varepsilon_0}{M_k - m_k} R^{-2s} + \Lambda(2-2s) \int_{M \setminus B_k} \frac{v_-(z)}{\mu_g(B(x, d_x(z)))d_x(z)^{2s}} \, \mathrm{d}V(z) \\
&\quad \leq \frac{2\varepsilon_0}{M_k - m_k} R^{-2s} + 2\Lambda(2-2s) \int_{M \setminus B(x, R_{k+1})} \frac{(d_{z_0}(z)/R_k)^\alpha - 1}{\mu_g(B(x, d_x(z)))d_x(z)^{2s}} \, \mathrm{d}V(z).
\end{aligned}$$

Note that $d_{z_0}(z) \leq d_x(z) + d_x(z_0) \leq d_x(z) + 3R_{k+1} \leq 4d_x(z)$. Thus, by using

(3.2.8) and assuming $\alpha < 2s_0$, we obtain that

$$\begin{aligned}
&\int_{M\setminus B(x,R_{k+1})} \frac{(d_{z_0}(z)/R_k)^\alpha - 1}{\mu_g(B(x,d_x(z)))d_x(z)^{2s}}\,\mathrm{d}V(z) \\
&\le \int_{M\setminus B(x,R_{k+1})} \frac{(d_x(z)/R_{k+1})^\alpha - 1}{\mu_g(B(x,d_x(z)))d_x(z)^{2s}}\,\mathrm{d}V(z) \\
&\le \sum_{j=0}^{\infty} \int_{B(x,2^{j+1}R_{k+1})\setminus B(x,2^jR_{k+1})} \frac{2^{(j+1)\alpha} - 1}{\mu_g(B(x,2^jR_{k+1}))(2^jR_{k+1})^{2s}}\,\mathrm{d}V(z) \\
&\le \sum_{j=0}^{\infty} \frac{2^{(j+1)\alpha} - 1}{(2^jR_{k+1})^{2s}} \frac{\mu_g(B(x,2^{j+1}R_{k+1}))}{\mu_g(B(x,2^jR_{k+1}))} \\
&\le \frac{2^n}{R_{k+1}^{2s}} \sum_{j=0}^{\infty} \left(2^\alpha 2^{j(\alpha-2s)} - 2^{-j\alpha}\right) \\
&= \frac{2^n}{R_{k+1}^{2s}} \frac{2^\alpha - 1}{(1-2^{\alpha-2s})(1-2^{-2s})} \\
&\le \frac{2^n}{R_{k+1}^{2s}} \frac{2^\alpha - 1}{(1-2^{\alpha-2s_0})(1-2^{-2s_0})} =: \frac{c_1(n,\alpha,s_0)}{R_{k+1}^{2s}}.
\end{aligned}$$

The constant $c_1(n,\alpha,s_0)$ can be made arbitrarily small by taking small α. We have estimated

$$\left(\frac{R_{k+1}}{7}\right)^{2s} \mathcal{M}^-_{\mathcal{L}_0} w \le C\left(\varepsilon_0 + c_1\right)$$

in $B(x, 7(R_{k+1}/7))$ for $x \in B(z_0, 2R_{k+1})$. Therefore, by Lemma 2.5.2 and (2.7.2), we have

$$\frac{1}{2}\mu_g(Q_1) \le \mu_g(\{w > 1\} \cap Q_1) \le C\mu_g(Q_1)\left(w(x) + C(\varepsilon_0 + c_1)\right)^\varepsilon,$$

or equivalently,

$$\theta \le w(x) + C(\varepsilon_0 + c_1)$$

for some universal constant $\theta > 0$. We take ε_0 and α sufficiently small so that $C\varepsilon_0 < \theta/4$ and $Cc_1 < \theta/4$, then we have $w \ge \theta/2$ in $B(z_0, 2R_{k+1})$. Thus, if we set $M_{k+1} = M_k$ and $m_{k+1} = M_k - 4^{-\alpha(k+1)}$, then

$$M_{k+1} \geq u \geq m_k + \frac{M_k - m_k}{4}\theta = M_k - \left(1 - \frac{\theta}{4}\right) 4^{-\alpha k} \geq m_{k+1}$$

in B_{k+1}.

On the other hand, if $\mu_g(\{u \leq (M_k + m_k)/2\} \cap Q_1) \geq \mu_g(Q_1)/2$, we define

$$v(x) = \frac{M_k - u(x)}{(M_k - m_k)/2}$$

and continue in the same way using that $R^{2s}\mathcal{M}^+_{\mathcal{L}_0} u \geq -\varepsilon_0$. $\square$

Chapter 3

Harnack inequality for fractional Laplacian-type operators on hyperbolic spaces

3.1 Introduction

In a celebrated series of papers [15, 16, 17] by Caffarelli and Silvestre, the regularity theories such as Krylov–Safonov, Cordes–Nirenberg, and Evans–Krylov theory are established for fractional-order operators on Euclidean spaces. The most important feature is that the constants in regularity estimates do not blow up and remain uniform as the order of operator approaches 2. It means that the regularity theories for fractional-order and second-order operators are unified. It has also been extended to the parabolic cases [78, 79, 80].

On the other hand, the regularity theory for the local operators on Riemannian manifolds has been studied. In particular, the studies on the Harnack inequalities, initiated by Yau [108] and Cheng–Yau [29], have been extended to second-order divergence and non-divergence form operators. These are extensions of the De Giorgi–Nash–Moser [97] and Krylov–Safonov Harnack inequalities [12, 74, 106], respectively. See also [75, 77] for the parabolic

Harnack inequalities.

The natural extension is to obtain the regularity results for fractional-order operators on Riemannian manifolds. Indeed, the Harnack inequality for nonlocal operators on metric measure spaces with volume doubling property, which includes Riemannian manifolds with nonnegative curvature, has been studied via the Dirichlet form theory [28]. However, this approach does not provide the unified regularity theory and is not appropriate for non-divergence form operators. For nonlocal non-divergence form operators, the Krylov–Safonov Harnack inequality on Riemannian manifolds with nonnegative curvature has recently been established by the authors [72]. The result in [72] unifies the Krylov–Safonov Harnack inequalities for local and nonlocal operators on manifolds with nonnegative curvature as in the works of Caffarelli and Silvestre.

In this paper, we continue to pursue the unified regularity theory for fully nonlinear nonlocal operators of order $2s \in (0,2)$ on the hyperbolic spaces $\mathbb{H}^n_\kappa$ that have constant negative curvatures $-\kappa < 0$. We establish the Alexandrov–Bakelman–Pucci (or ABP for short) estimates, Krylov–Safonov Harnack inequality, and Hölder estimates, which are robust as $s \to 1$ and $\kappa \to 0$ in the sense that the regularity estimates recover the classical regularity estimates for second-order operators on the hyperbolic spaces as $s \to 1$ and for fractional-order operators on the Euclidean spaces as $\kappa \to 0$.

The operators considered in this work are modeled on the fractional Laplacian on the hyperbolic spaces. Since the hyperbolic geometry is not distinguished from the Euclidean geometry when $n = 1$, we assume $n \geq 2$ throughout the paper. Let us recall the fractional Laplacian $-(-\Delta_{\mathbb{H}^n_\kappa})^s$, $s \in (0,1)$, $\kappa > 0$, on the hyperbolic spaces $\mathbb{H}^n_\kappa$. Let K_ν be the modified Bessel function of the second kind and define $\mathscr{K}_{\nu,a}(\rho) = \rho^{-\nu} K_\nu(a\rho)$ for notational convenience. Then, the fractional Laplacian $-(-\Delta_{\mathbb{H}^n_\kappa})^s$ is given by

$$-(-\Delta_{\mathbb{H}^n_\kappa})^s u(x) = \text{ P.V.} \int_{\mathbb{H}^n_\kappa} (u(z) - u(x)) \mathcal{K}_{n,s,\kappa}(d_{\mathbb{H}^n_\kappa}(x,z)) \,\mathrm{d}\mu_{\mathbb{H}^n_\kappa}(z), \quad (3.1.1)$$

where the kernel $\mathcal{K}_{n,s,\kappa}$ is given by

$$\mathcal{K}_{n,s,\kappa}(\rho) = c_{n,s}\sqrt{\kappa}^{1+2s}\left(\frac{-\sqrt{\kappa}\,\partial_\rho}{\sinh(\sqrt{\kappa}\rho)}\right)^{\frac{n-1}{2}}\mathscr{K}_{\frac{1+2s}{2},\frac{n-1}{2}}\left(\sqrt{\kappa}\rho\right) \tag{3.1.2}$$

when $n \geq 3$ is odd and

$$\begin{aligned}&\mathcal{K}_{n,s,\kappa}(\rho)\\ &= c_{n,s}\int_\rho^\infty \frac{\sqrt{\kappa}^{1+2s}\sinh(\sqrt{\kappa}r)}{\sqrt{\pi}\sqrt{\cosh(\sqrt{\kappa}r)-\cosh(\sqrt{\kappa}\rho)}}\left(\frac{-\sqrt{\kappa}\,\partial_r}{\sinh(\sqrt{\kappa}r)}\right)^{\frac{n}{2}}\mathscr{K}_{\frac{1+2s}{2},\frac{n-1}{2}}\left(\sqrt{\kappa}r\right)\,\mathrm{d}r\end{aligned} \tag{3.1.3}$$

when $n \geq 2$ is even, and

$$c_{n,s} = \frac{(n-1)^{\frac{1+2s}{2}}}{2^{\frac{n-1}{2}}\pi^{\frac{n}{2}}}\frac{1}{|\Gamma(-s)|}.$$

See Section 4.2 for details.

Remark 3.1.1. (i) Note that the normalizing constant $c_{n,s}$ has the same asymptotic behavior with $1-s$ as $s \to 1$ up to a dimensional constant. This is a crucial fact for the robust regularity estimates as in [15, 16, 17].

(ii) It is natural to expect that $\mathcal{K}_{n,s,\kappa}$ converges to the kernel of the fractional Laplacian on the Euclidean space as curvature $-\kappa$ approaches zero. Indeed, we have from [71, Theorem 1.2]

$$\mathcal{K}_{n,s,\kappa}(\rho) \to \rho^{-n-2s},$$

as $\kappa \to 0$ up to some constant depending on n and s.

(iii) The kernel $\mathcal{K}_{n,s,\kappa}$ decays exponentially as $\rho \to \infty$, which is different from the behavior of the kernel on the Euclidean spaces. The difference comes from the exponential growth of the volume of balls in the hyperbolic spaces, and this is why the regularity theories on manifolds with negative and nonnegative curvatures are dealt with separately.

Modeled on the fractional Laplacian (3.1.1), fully nonlinear operators of the fractional Laplacian-type can be defined in the standard way. For a class $\mathcal{L}_0$ of linear operators of the form

$$Lu(x) = \text{P.V.} \int_{\mathbb{H}^n_\kappa} (u(z) - u(x))\mathcal{K}(x,z)\, \mathrm{d}\mu_{\mathbb{H}^n_\kappa}(z), \quad x \in \mathbb{H}^n_\kappa,$$

with measurable kernels $\mathcal{K}$ satisfying

$$\lambda \mathcal{K}_{n,s,\kappa}(d_{\mathbb{H}^n_\kappa}(x,z)) \le \mathcal{K}(x,z) \le \Lambda \mathcal{K}_{n,s,\kappa}(d_{\mathbb{H}^n_\kappa}(x,z)), \quad 0 < \lambda \le \Lambda,$$

the *maximal* and *minimal operators* are defined by

$$\mathcal{M}^+ u(x) := \mathcal{M}^+_{\mathcal{L}_0} u(x) := \sup_{L \in \mathcal{L}_0} Lu(x),$$

and

$$\mathcal{M}^- u(x) := \mathcal{M}^-_{\mathcal{L}_0} u(x) := \inf_{L \in \mathcal{L}_0} Lu(x),$$

respectively. It is easy to see that these extremal operators are well defined at $x \in \mathbb{H}^n_\kappa$ for any bounded function u that is C^2 near x, see (3.3.3). An operator $\mathcal{I}$ is said to be *elliptic with respect to* $\mathcal{L}_0$ if

$$\mathcal{M}^-_{\mathcal{L}_0}(u-v)(x) \le \mathcal{I}(u,x) - \mathcal{I}(v,x) \le \mathcal{M}^+_{\mathcal{L}_0}(u-v)(x)$$

for every point $x \in \mathbb{H}^n_\kappa$ and for all bounded functions u and v which are C^2 near x.

The first step towards the Krylov–Safonov Harnack inequality and Hölder estimate is the ABP-type estimate, which provides an estimate on the distribution function of supersolutions to fully nonlinear nonlocal operators. To state this result, we define functions

$$\mathcal{S}_\kappa(t) = \frac{\sinh(\sqrt{\kappa} t)}{\sqrt{\kappa} t}, \quad \mathcal{H}_\kappa(t) = \sqrt{\kappa} t \coth(\sqrt{\kappa} t),$$

and

$$\mathcal{T}_\kappa(t) = \frac{\sqrt{\kappa}t}{\tanh^{-1}(\frac{1}{2}\tanh(\sqrt{\kappa}t))}.$$

Theorem 3.1.2 (ABP-type estimate)**.** *Let $s_0 \in (0,1)$ and assume $s \in [s_0, 1)$. Let $u \in C^2(B_{5R}) \cap L^\infty(\mathbb{H}^n_\kappa)$ be a function on $\mathbb{H}^n_\kappa$ satisfying $\mathcal{M}^- u \le f$ in B_{5R}, $u \ge 0$ in $\mathbb{H}^n_\kappa \setminus B_{5R}$, and $\inf_{B_{2R}} u \le 1$. Let $\mathscr{C}$ be a contact set defined by* (3.4.1)*, then there is a finite collection $\{Q^j_\alpha\}$ of dyadic cubes, with $\operatorname{diam}(Q^j_\alpha) \le r_0$, such that $Q^j_\alpha \cap \mathscr{C} \neq \emptyset$, $\mathscr{C} \subset \cup_j \overline{Q}^j_\alpha$, and*

$$|B_R| \le \sum_j cF^n |Q^j_\alpha|, \tag{3.1.4}$$

where r_0, c, and F are given by (3.3.6)*,*

$$F = \mathcal{S}_\kappa(7R)\left(\Lambda\mathcal{H}_\kappa(7R) + \frac{R^2}{\mathcal{I}_{0,\kappa}(R)} \max_{\overline{Q}^j_\alpha} f\right)_+,$$

and

$$c = C\cosh^{n-1}\left(C\sqrt{\kappa}r_0\mathcal{T}^2_\kappa(r_0)F\right)\left(C\mathcal{T}^2_\kappa(r_0)F\right)^{(n-1)\log\cosh(C\sqrt{\kappa}r_0\mathcal{T}^2_\kappa(r_0)F)}\mathcal{T}^{2n}_\kappa(r_0),$$

respectively. See (3.3.1) *for the definition of $\mathcal{I}_{0,\kappa}(R)$. The universal constant $C > 0$ depends only on n, λ, Λ, and s_0.*

Remark 3.1.3. (i) The Riemann sum in (3.1.4) converges as $s \to 1$ to the integral

$$C\int_{\mathscr{C}} \mathcal{S}^n_\kappa(7R)\left(\Lambda\mathcal{H}_\kappa(7R) + R^2 f(x)\right)^n_+ \, \mathrm{d}\mu_{\mathbb{H}^n_\kappa}(x),$$

which is of the form appearing in [106, Theorem 1.2]. This implies that Theorem 3.1.2 recovers the ABP estimate for second-order operators on the hyperbolic spaces as a limit $s \to 1$. Indeed, it will be proved in Proposition 3.3.4 and Lemma 3.3.6 that $\mathcal{I}_{0,\kappa}(R) \to C(n)$ and $r_0 \to 0$ as $s \to 1$. Moreover, since $\lim_{t\to 0}\mathcal{T}_\kappa(t) = 2$, the dependence of c on κ and R disappears in the limit $s \to 1$.

(ii) Theorem 3.1.2 provides a new result even for second-order operators because it covers fully nonlinear operators.

(iii) Theorem 3.1.2 also recovers the ABP estimate on the Euclidean spaces [15] as $\kappa \to 0$.

(iv) The function $\frac{R^2}{\mathcal{I}_{0,\kappa}(R)}$ corresponds to R^{2s} in the cases of the Euclidean spaces [15] and manifolds with nonnegative curvature [72]. However, this function exhibits qualitatively different behavior than R^{2s} because it involves the kernels $\mathcal{K}_{n,s,\kappa}(\rho)$ for the fractional Laplacian $-(-\Delta_{\mathbb{H}^n_\kappa})^s$ which decay exponentially as $\rho \to \infty$ while those in the case of manifolds with nonnegative curvature decay polynomially.

We next establish the Krylov–Safonov Harnack inequality and Hölder estimates for solutions of fully nonlinear nonlocal equations on the hyperbolic spaces $\mathbb{H}^n_\kappa$.

Theorem 3.1.4 (Harnack inequality). *Let $s_0 \in (0,1)$ and assume $s \in [s_0, 1)$. If a nonnegative function $u \in C^2(B_{7R}) \cap L^\infty(\mathbb{H}^n_\kappa)$ satisfies*

$$\mathcal{M}^- u \le C_0 \quad \text{and} \quad \mathcal{M}^+ u \ge -C_0 \quad \text{in } B_{7R}, \tag{3.1.5}$$

then

$$\sup_{B_{\delta_1 R}} u \le C \left(\inf_{B_{\delta_1 R}} u + C_0 \frac{(7R)^2}{\mathcal{I}_{0,\kappa}(7R)} \right)$$

for some universal constants $\delta_1 \in (0,1)$ and $C > 0$ depending only on n, λ, Λ, $\sqrt{\kappa}R$, and s_0.

Let us denote by $\|\cdot\|'$ the non-dimensional norm in the following theorem.

Theorem 3.1.5 (Hölder estimates). *Let $s_0 \in (0,1)$ and assume $s \in [s_0, 1)$. If $u \in C^2(B_{7R}) \cap L^\infty(\mathbb{H}^n_\kappa)$ satisfies (3.1.5), then*

$$\|u\|'_{C^\alpha(\overline{B_R})} \le C \left(\|u\|_{L^\infty(\mathbb{H}^n_\kappa)} + C_0 \frac{(7R)^2}{\mathcal{I}_{0,\kappa}(7R)} \right)$$

for some universal constants $\alpha \in (0,1)$ and $C > 0$ depending only on n, λ, Λ, $\sqrt{\kappa}R$, and s_0.

Since $\mathcal{I}_{0,\kappa}(R) \to C(n)$ as $s \to 1$ and the universal constants in Theorem 3.1.4 and Theorem 3.1.5 do not depend on s, the regularity estimates in Theorem 3.1.4 and Theorem 3.1.5 recover the classical estimates for second-order operators on the hyperbolic space as limits.

The main difficulties in establishing regularity results, Theorem 3.1.2, Theorem 3.1.4, and Theorem 3.1.5, arise from the effect of negative curvatures. The volume of a ball in the hyperbolic spaces behaves like that in the Euclidean spaces when a radius is small, while it grows exponentially as a radius gets bigger. Due to this non-homogeneity of the volume, the scaling property does not hold, making the standard arguments for regularity results break. This kind of difficulty also appears on the heat kernel estimates [26, 56, 57]. Hence, our result may provide some hints about the heat kernel estimation on non-homogenous spaces. To overcome this difficulty, we introduce new scale functions that take non-homogeneity into account and provide some monotonicity properties in Section 3.3.

Another difficulty arising from the non-homogeneity of the volumes lies in the dyadic ring argument in the ABP estimates. In the ABP estimates, we find a dyadic ring around a given contact point in which a supersolution is quadratically close to a tangent paraboloid in a large portion of the ring. However, the standard dyadic rings $B_{2^{-k}r} \setminus B_{2^{-(k+1)}r}$ no longer work in the framework of the hyperbolic spaces. It leads to introducing a *hyperbolic dyadic ring* whose radii are determined by the volume of balls (see Section 3.4). The hyperbolic dyadic ring turns out to be the natural "dyadic" ring in the hyperbolic geometry.

After the ABP estimates, we prove the regularity estimates by constructing a barrier function. It is standard to use the distance function for the construction, but the computation is significantly different from the standard one because of the hyperbolic structure. We observe in Section 3.5 how the negative curvature of the hyperbolic spaces affects the computations.

Let us also emphasize some applications. As mentioned in [15], the fully nonlinear operators considered in this paper are naturally related to the stochastic optimal control theory [99]. On the other hand, it is too hard to miss how important the hyperbolic space is in mathematics. For example, the hyperbolic spaces appear on the context of the uniformization theorem and the special relativity. We also want to mention that hyperbolic spaces can be used for computer scientific applications [92].

This paper is organized as follows. In Section 4.2, we recall several models for the hyperbolic spaces and define the fractional Laplacian $-(-\Delta_{\mathbb{H}^n_\kappa})^s$ on the hyperbolic spaces $\mathbb{H}^n_\kappa$. In Section 3.3, new scale functions are introduced and some monotonicity properties are studied. The regularity theory begins with ABP-type estimates in Section 3.4. In this section Theorem 3.1.2 is proved. The next step is the construction of a barrier function, and this is presented in Section 3.5. This barrier function, together with the ABP estimates, is used to obtain the so-called L^ε-estimates in Section 3.6. Theorem 3.1.4 and Theorem 3.1.5 are proved in Section 3.7 and Section 3.8, respectively. In Section 3.9.1, some properties of special functions are collected.

3.2 Preliminaries

In this section, we recall several models for the n-dimensional hyperbolic spaces, revisit the definition of the fractional Laplacian on these spaces, and collect some well-known results on the hyperbolic spaces.

3.2.1 Hyperbolic spaces

Let us recall the hyperboloid model and the Poincaré ball model (see, e.g., [52, 95, 102]).

We first recall the hyperboloid model

$$\mathbb{H}^n_\kappa = \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} : x_0^2 - x_1^2 - \cdots - x_n^2 = \kappa^{-1}, x_0 > 0\}$$

with the metric induced by the Lorentzian metric $-\mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2$ in

$\mathbb{R}^{n+1}$. The space $\mathbb{H}^n_\kappa$ has a constant curvature $-\kappa < 0$. The internal product induced by the Lorentzian metric is denoted by $[x, x'] = x_0 x'_0 - x_1 x'_1 - \cdots - x_n x'_n$, and the distance between two points x and x' is given by

$$d_{\mathbb{H}^n_\kappa}(x, x') = \frac{1}{\sqrt{\kappa}} \cosh^{-1}\left(\kappa[x, x']\right).$$

Using the polar coordinates, $\mathbb{H}^n_\kappa$ can also be realized as

$$\mathbb{H}^n_\kappa = \left\{ x = \left(\frac{\cosh r}{\sqrt{\kappa}}, \frac{\sinh r}{\sqrt{\kappa}} \omega \right) \in \mathbb{R}^{n+1} : r \geq 0, \omega \in \mathbb{S}^{n-1} \right\}.$$

Then, the metric and volume element are given by $\mathrm{d}s^2 = \frac{1}{\kappa}(\mathrm{d}r^2 + \sinh^2 r \, \mathrm{d}\omega^2)$ and $\mathrm{d}\mu_{\mathbb{H}^n_\kappa} = \frac{1}{\sqrt{\kappa^n}} \sinh^{n-1} r \, \mathrm{d}r \, \mathrm{d}\omega$, respectively.

Let us also consider the Poincaré ball model $\mathbb{B}^n_{t,\kappa} = \{y \in \mathbb{R}^n : |y| < t\}$ with the metric

$$\mathrm{d}s^2 = \frac{4b^2}{(t^2 - |y|^2)^2} \, \mathrm{d}y^2$$

and the volume measure

$$\mathrm{d}\mu_{\mathbb{B}^n_{t,\kappa}}(y) = \left(\frac{2b}{t^2 - |y|^2} \right)^n \mathrm{d}y, \tag{3.2.1}$$

where $t/b = \sqrt{\kappa}$. Note that the measure (3.2.1) tends to the Lebesgue measure $\mathrm{d}y$ as $\kappa \to 0$, provided that $\sqrt{\kappa} = 2/t$ and $b = t^2/2$.

The map defined by

$$\phi : (x_0, x_1, \cdots x_n) \in \mathbb{H}^n_\kappa \mapsto \frac{\sqrt{\kappa} t}{1 + \sqrt{\kappa} x_0}(x_1, \cdots, x_n) \in \mathbb{B}^n_{t,\kappa}, \tag{3.2.2}$$

or equivalently, by

$$\phi : \left(\frac{\cosh r}{\sqrt{\kappa}}, \frac{\sinh r}{\sqrt{\kappa}} \omega \right) \in \mathbb{H}^n_\kappa \mapsto t \frac{\sinh \frac{r}{2}}{\cosh \frac{r}{2}} \omega \in \mathbb{B}^n_{t,\kappa},$$

is an isometry and its inverse is given by

$$\phi^{-1}: y \in \mathbb{B}^n_{t,\kappa} \mapsto \left(\frac{t^2+|y|^2}{\sqrt{\kappa}(t^2-|y|^2)}, \frac{2ty_1}{\sqrt{\kappa}(t^2-|y|^2)}, \cdots, \frac{2ty_n}{\sqrt{\kappa}(t^2-|y|^2)} \right) \in \mathbb{H}^n_\kappa.$$

See [52, Chapter 8] for the proof. Therefore, there are several ways of describing the distance function as follows:

$$\begin{aligned} d_{\mathbb{B}^n_{t,\kappa}}(0,y) = d_{\mathbb{H}^n_\kappa}(0_\kappa, x) &= \frac{1}{\sqrt{\kappa}} \cosh^{-1}(\kappa[0_\kappa, x]) \\ &= \frac{1}{\sqrt{\kappa}} \cosh^{-1}\left(\frac{t^2+|y|^2}{t^2-|y|^2} \right) = \frac{1}{\sqrt{\kappa}} \log\left(\frac{t+|y|}{t-|y|} \right), \end{aligned}$$

where $0_\kappa = (\frac{1}{\sqrt{\kappa}}, 0, \cdots, 0) \in \mathbb{H}^n_\kappa$. We shall write $0_\kappa = 0$ if there is no confusion.

3.2.2 Fractional Laplacian on the hyperbolic spaces

The fractional Laplacian on $\mathbb{H}^n$ is defined in [7] by using the Helgason Fourier transform [51, 62, 101] (see also [52, 48]), and its normalizing constant is computed in [71] by using the heat kernel [59]. However, these works are built on the hyperbolic space with curvature -1. Since we are working on $\mathbb{H}^n_\kappa$ with curvature $-\kappa < 0$, we define the fractional Laplacian $-(-\Delta_{\mathbb{H}^n_\kappa})^s$ on $\mathbb{H}^n_\kappa$ and deduce the representation of its kernel from that of $-(-\Delta_{\mathbb{H}^n})^s$.

We recall the Helgason Fourier transform on the hyperbolic spaces. The interested reader may consult [62, 51, 101, 52, 48]. By means of the isometry (3.2.2), we may work on the Poincaré ball model instead of the hyperboloid model. Let $t = 1$ be fixed and denote $\mathbb{B}^n_\kappa = \mathbb{B}^n_{1,\kappa}$. The Helgason Fourier transform is defined for $u \in C^\infty_c(\mathbb{B}^n_\kappa)$ by

$$\widehat{u}(\lambda, \xi; \kappa) = \int_{\mathbb{B}^n_\kappa} u(x) e_{-\lambda,\xi;\kappa}(x) \, \mathrm{d}\mu_{\mathbb{B}^n_\kappa}(x), \quad \lambda \in \mathbb{R}, \ \xi \in \mathbb{S}^{n-1},$$

where

$$e_{\lambda,\xi;\kappa}(x) = \left(\frac{1-|x|^2}{|\xi - x|^2} \right)^{\frac{n-1}{2} + i\frac{\lambda}{\sqrt{\kappa}}}$$

is the eigenfunction with eigenvalue $-(\lambda^2+\kappa\frac{(n-1)^2}{4})$. It is well known that the following inversion formula holds:

$$u(x)=\int_{-\infty}^{\infty}\int_{\mathbb{S}^{n-1}}\widehat{u}(\lambda,\xi;\kappa)e_{\lambda,\xi;\kappa}(x)\frac{\sqrt{\kappa}^{n-1}}{|c_\kappa(\lambda)|^2}\,\mathrm{d}\sigma(\xi)\,\mathrm{d}\lambda, \tag{3.2.3}$$

where

$$c_\kappa(\lambda)=\sqrt{2}(2\pi)^{n/2}\frac{\Gamma(i\frac{\lambda}{\sqrt{\kappa}})}{\Gamma(\frac{n-1}{2}+i\frac{\lambda}{\sqrt{\kappa}})}$$

is the Harish-Chandra coefficient. Moreover, the Plancherel formula holds:

$$\int_{\mathbb{B}^n_\kappa}|u(x)|^2\,\mathrm{d}\mu_{\mathbb{B}^n_\kappa}(x)=\int_{-\infty}^{\infty}\int_{\mathbb{S}^{n-1}}|\widehat{u}(\lambda,\xi;\kappa)|^2\frac{\sqrt{\kappa}^{n-1}}{|c_\kappa(\lambda)|^2}\,\mathrm{d}\sigma(\xi)\,\mathrm{d}\lambda. \tag{3.2.4}$$

Since

$$\begin{aligned}&\widehat{-\Delta_{\mathbb{B}^n_\kappa}u}(\lambda,\xi;\kappa)\\&=-\int_{\mathbb{B}^n_\kappa}\Delta_{\mathbb{B}^n_\kappa}u(x)e_{-\lambda,\xi;\kappa}(x)\,\mathrm{d}\mu_{\mathbb{B}^n_\kappa}(x)\\&=-\int_{\mathbb{B}^n_\kappa}u(x)\Delta_{\mathbb{B}^n_\kappa}e_{-\lambda,\xi;\kappa}(x)\,\mathrm{d}\mu_{\mathbb{B}^n_\kappa}(x)=\left(\lambda^2+\kappa\frac{(n-1)^2}{4}\right)\widehat{u}(\lambda,\xi;\kappa),\end{aligned}$$

it is natural to define the fractional Laplacian $-(-\Delta_{\mathbb{B}^n_\kappa})^s$ by

$$\widehat{(-\Delta_{\mathbb{B}^n_\kappa})^s u}(\lambda,\xi;\kappa)=\left(\lambda^2+\kappa\frac{(n-1)^2}{4}\right)^s\widehat{u}(\lambda,\xi;\kappa), \tag{3.2.5}$$

which coincides with the definition given in [7, 71] when $\kappa=1$. By using

(3.2.4) and (3.2.5), we have

$$\begin{aligned}
\widehat{(-\Delta_{\mathbb{B}^n_\kappa})^s u}(\lambda,\xi;1) &= \int_{\mathbb{B}^n_1} (-\Delta_{\mathbb{B}^n_\kappa})^s u(x) e_{-\lambda,\xi;1}(x)\,\mathrm{d}\mu_{\mathbb{B}^n_1}(x) \\
&= \int_{\mathbb{B}^n_1} u(x)(-\Delta_{\mathbb{B}^n_\kappa})^s e_{-\sqrt{\kappa}\lambda,\xi;\kappa}(x)\,\mathrm{d}\mu_{\mathbb{B}^n_1}(x) \\
&= \kappa^s \left(\lambda^2 + \frac{(n-1)^2}{4}\right)^s \int_{\mathbb{B}^n_1} u(x) e_{-\sqrt{\kappa}\lambda,\xi;\kappa}(x)\,\mathrm{d}\mu_{\mathbb{B}^n_1}(x) \\
&= \kappa^s \left(\lambda^2 + \frac{(n-1)^2}{4}\right)^s \widehat{u}(\lambda,\xi;1) \\
&= \kappa^s \widehat{(-\Delta_{\mathbb{B}^n})^s u}(\lambda,\xi;1),
\end{aligned}$$

and hence $(-\Delta_{\mathbb{B}^n_\kappa})^s u = \kappa^s(-\Delta_{\mathbb{B}^n})^s u$ by the inversion formula (3.2.3). Since

$$d_{\mathbb{B}^n}(x,z) = \sqrt{\kappa} d_{\mathbb{B}^n_\kappa}(x,z) \quad \text{and} \quad \mathrm{d}\mu_{\mathbb{B}^n}(z) = \sqrt{\kappa}^n\,\mathrm{d}\mu_{\mathbb{B}^n_\kappa}(z), \tag{3.2.6}$$

we have

$$\begin{aligned}
(-\Delta_{\mathbb{B}^n_\kappa})^s u(x) &= \kappa^s \int_{\mathbb{B}^n} (u(x)-u(z))\mathcal{K}_{n,s,1}(d_{\mathbb{B}^n}(x,z))\,\mathrm{d}\mu_{\mathbb{B}^n}(z) \\
&= \kappa^s \int_{\mathbb{B}^n_\kappa} (u(x)-u(z))\mathcal{K}_{n,s,1}(\sqrt{\kappa}\, d_{\mathbb{B}^n_\kappa}(x,z))\sqrt{\kappa}^n\,\mathrm{d}\mu_{\mathbb{B}^n_\kappa}(z),
\end{aligned}$$

from which we conclude

$$\mathcal{K}_{n,s,\kappa}(\rho) = \sqrt{\kappa}^{n+2s}\mathcal{K}_{n,s,1}(\sqrt{\kappa}\rho). \tag{3.2.7}$$

Therefore, (3.1.2) and (3.1.3) follow from [71, Theorem 1.2] and (3.2.7).

Proposition 3.2.1. *$\mathcal{K}_{n,s,\kappa}$ is strictly decreasing.*

Proof. It suffices to prove that $\mathcal{K}_s := \mathcal{K}_{n,s,1}$ is non-increasing. It is known [71] that the kernel $\mathcal{K}_s$ is represented as

$$\mathcal{K}_s(\rho) = C\int_0^\infty p(t,\rho)\frac{\mathrm{d}t}{t^{1+s}}$$

for some $C = C(n,s) > 0$, where $p(t,\rho)$ is the heat kernel of the Laplacian

$\Delta_{\mathbb{H}^n}$ on the hyperbolic space $\mathbb{H}^n$. Moreover, it is known [2] that $p(t,\rho)$ is strictly decreasing with respect to ρ. Therefore, $\mathcal{K}_s$ is strictly decreasing. $\square$

3.2.3 Hyperbolic spaces revisited

We collect some well-known results on the hyperbolic spaces. The first one is the volume doubling property that will be used frequently throughout the paper.

Lemma 3.2.2. *For any $B_r \subset B_R \in \mathbb{H}^n_\kappa$,*

$$\left(\frac{R}{r}\right)^n \le \frac{|B_R|}{|B_r|} \le \mathcal{D}\left(\frac{R}{r}\right)^{\log_2 \mathcal{D}}, \tag{3.2.8}$$

where $\mathcal{D} = 2^n \cosh^{n-1}(2\sqrt{\kappa}R)$.

Lemma 3.2.2 is a direct consequence of the Bishop–Gromov inequality. See [104, Theorem 18.8 and Corollary 18.11] for the first inequality and the second inequality with $R = 2r$ in (3.2.8), respectively. For the full inequality, we find a $k \in \mathbb{N}$ satisfying $R \in [2^{k-1}r, 2^k r)$, and then iterate the inequality.

The next result is a bound of the Hessian of the squared distance. See [33, Lemma 3.12] for instance.

Lemma 3.2.3. *Fix a point $y \in \mathbb{H}^n_\kappa$ and consider the distance function $d_{\mathbb{H}^n_\kappa}(\cdot, y)$. Then,*

$$D^2\left(d^2_{\mathbb{H}^n_\kappa}(x,y)/2\right)(\xi,\xi) \le \mathcal{H}_\kappa(d_{\mathbb{H}^n_\kappa}(x,y))|\xi|^2$$

for all $\xi \in T_x\mathbb{H}^n_\kappa$.

Let us close this section with the following generalization of Euclidean dyadic cubes that will be used in the decomposition of the contact set and in the Calderón–Zygmund technique.

Theorem 3.2.4 (Christ [31]). *There is a countable collection $\{Q^j_\alpha \subset \mathbb{H}^n_\kappa : j \in \mathbb{Z}, \alpha \in I_j\}$ of open sets and constants $c_1, c_2 > 0$ (with $2c_1 \le c_2$), and $\delta_0 \in (0,1)$, depending only on n, such that* *(i)* $|\mathbb{H}^n_\kappa \setminus \cup_\alpha Q^j_\alpha| = 0$ *for each* $j \in \mathbb{Z}$,

(ii) if $i \geq j$, *then either* $Q^i_\beta \subset Q^j_\alpha$ *or* $Q^i_\beta \cap Q^j_\alpha = \emptyset$,

(iii) for each (j, α) *and each* $i < j$, *there is a unique* β *such that* $Q^j_\alpha \subset Q^i_\beta$,

(iv) $\mathrm{diam}(Q^j_\alpha) \leq c_2 \delta_0^j$, *and*

(v) each Q^j_α *contains some ball* $B(z^j_\alpha, c_1 \delta_0^j)$.

The original statement of [31, Theorem 11] consists of six properties. As mentioned in [31], the first five properties concern only the quasi-metric space structure and the last property requires the space to be of homogeneous type. Since the hyperbolic spaces are not of homogeneous type, the last property—which is not needed in this work—cannot be included.

3.3 Scale functions

Recall that, in the Euclidean spaces, it is sufficient to obtain regularity estimates in B_1 because the estimates in B_R can be recovered from those in B_1 by using the scale invariance of the equations. The scale invariance heavily relies on the homogeneity of the underlying spaces. However, the hyperbolic spaces are not homogeneous. Indeed, the volume of a ball grows exponentially as the radius goes to infinity in the hyperbolic spaces, and hence the kernel $\mathcal{K}_{n,s,\kappa}(\rho)$ of the fractional Laplacian $(-\Delta_{\mathbb{H}^n_\kappa})^s$ decays exponentially as $\rho \to \infty$. Therefore, it is crucial to find appropriate scale functions that capture the correct behavior at every scale. For this purpose, we define

$$\begin{aligned} \mathcal{I}_{0,\kappa}(R) &:= \int_{B_R(x)} d^2_{\mathbb{H}^n_\kappa}(z,x) \mathcal{K}_{n,s,\kappa}(d_{\mathbb{H}^n_\kappa}(z,x)) \,\mathrm{d}\mu_{\mathbb{H}^n_\kappa}(z), \\ \mathcal{I}_{\infty,\kappa}(R) &:= \int_{\mathbb{H}^n_\kappa \setminus B_R(x)} R^2 \mathcal{K}_{n,s,\kappa}(d_{\mathbb{H}^n_\kappa}(z,x)) \,\mathrm{d}\mu_{\mathbb{H}^n_\kappa}(z), \end{aligned} \tag{3.3.1}$$

and $\mathcal{I}_\kappa(R) := \mathcal{I}_{0,\kappa}(R) + \mathcal{I}_{\infty,\kappa}(R)$. They play a role of scale functions in the regularity estimates on the hyperbolic spaces as the homogeneous polynomial R^{2-2s} does on the Euclidean spaces.

We omit the subscript κ when $\kappa = 1$, i.e., $\mathcal{I}_0 = \mathcal{I}_{0,1}$, $\mathcal{I}_\infty = \mathcal{I}_{\infty,1}$ and $\mathcal{I} = \mathcal{I}_1$. Then, it follows from (3.2.6) and (3.2.7)

$$\begin{aligned} \mathcal{I}_{0,\kappa}(R) &= \sqrt{\kappa}^{-2+2s}\mathcal{I}_0(\sqrt{\kappa}R), \\ \mathcal{I}_{\infty,\kappa}(R) &= \sqrt{\kappa}^{-2+2s}\mathcal{I}_\infty(\sqrt{\kappa}R), \\ \mathcal{I}_{\kappa}(R) &= \sqrt{\kappa}^{-2+2s}\mathcal{I}(\sqrt{\kappa}R). \end{aligned} \tag{3.3.2}$$

It is known that $\mathcal{K}_{n,s,1}(\rho) \sim \rho^{-n-2s}$ as $\rho \to 0^+$ and $\mathcal{K}_{n,s,1}(\rho) \sim \rho^{-1-s}e^{-(n-1)\rho}$ as $\rho \to \infty$, see [7, Theorem 2.4] and [71, Theorem 1.2]. Thus, the scale functions in (3.3.1) are well defined. Moreover, we observe

$$|Lu(x)| \leq \Lambda \|u\|_{C^2(\overline{B_R(x)})}\mathcal{I}_{0,\kappa}(R) + 2\Lambda\|u\|_{L^\infty(\mathbb{H}^n_\kappa)}\mathcal{I}_{\infty,\kappa}(R) < +\infty \tag{3.3.3}$$

for any operator $L \in \mathcal{L}_0$ and any function $u \in C^2(\overline{B_R(x)}) \cap L^\infty(\mathbb{H}^n_\kappa)$.

Let us now investigate some properties of the scale functions that are useful for the upcoming regularity theory. Although the scale functions do not have scaling properties, they satisfy some monotonicity properties. For instance, $\mathcal{I}_{0,\kappa}$ is increasing and $\mathcal{I}_{\infty,\kappa}$ is decreasing by definition. Moreover, some variations of these scale functions satisfy *almost monotonicity*. We say that a function f is *almost decreasing* if there exists a constant $C \geq 1$ such that $f(R) \leq Cf(r)$ for all $R > r > 0$.

Proposition 3.3.1. *The functions $R^{-2+s}\mathcal{I}_{0,\kappa}(R)$ and $R^{-2}\mathcal{I}_{0,\kappa}(R)$ are almost decreasing.*

The almost monotonicity of $R^{-2}\mathcal{I}_{0,\kappa}(R)$ follows from that of $R^{-2+s}\mathcal{I}_{0,\kappa}(R)$. Furthermore, it is enough to show that $R^{-2+s}\mathcal{I}_0(R)$ is almost decreasing by the relation (3.3.2). Indeed, it is a direct consequence of the following lemma.

Lemma 3.3.2. *There exist constants $0 < C_1(n) \leq C_2(n)$ and a non-increasing function F such that $C_1F(R) \leq R^{-2+s}\mathcal{I}_0(R) \leq C_2F(R)$ for all $R > 0$.*

Proof. It is enough to show

$$C_3 f(R) \leq \frac{\mathcal{I}_0'(R)}{R^{1-s}} \leq C_4 f(R) \tag{3.3.4}$$

for some constants $0 < C_3(n) \leq C_4(n)$ and some non-increasing function $f(R)$. Indeed, once we prove (3.3.4), we then have

$$\frac{C_3}{2}F(R) \leq \frac{\mathcal{I}_0(R)}{R^{2-s}} = \frac{\int_0^R \mathcal{I}_0'(\rho)\,\mathrm{d}\rho}{\int_0^R (2-s)\rho^{1-s}\,\mathrm{d}\rho} \leq C_4 F(R),$$

where

$$F(R) = \frac{\int_0^R \rho^{1-s} f(\rho)\,\mathrm{d}\rho}{\int_0^R \rho^{1-s}\mathrm{d}\rho}$$

is a non-increasing function.

To prove the claim (3.3.4), we observe

$$\frac{\mathcal{I}_0'(R)}{R^{1-s}} = |\mathbb{S}^{n-1}| R^{1+s}\mathcal{K}_{n,s,1}(R)\sinh^{n-1} R.$$

By Lemma 3.9.2, it is comparable to

$$f(R) := R I_{\frac{n}{2}-1}\left(\frac{n-1}{2}R\right) K_{\frac{n}{2}+s}\left(\frac{n-1}{2}R\right)$$

up to dimensional constants, where I_ν is the modified Bessel function of the first kind. It only remains to show that f is non-increasing. We note that it is sufficient to prove that $g(R) = R I_{\frac{n}{2}-1}(R) K_{\frac{n}{2}+s}(R)$ is non-increasing. Indeed, by using (3.9.1) and [98, Theorem 3] we obtain

$$\begin{aligned}
g'(R) &= -s I_{\frac{n}{2}-1} K_{\frac{n}{2}+s} + R I_{\frac{n}{2}} K_{\frac{n}{2}+s} - R I_{\frac{n}{2}-1} K_{\frac{n}{2}+s-1} \\
&\leq \left(-s + \frac{R^2}{\sqrt{R^2+a^2}+a} - \frac{R^2}{\sqrt{R^2+(a+s)^2}+a+s}\right) I_{\frac{n}{2}-1} K_{\frac{n}{2}+s} \\
&= \left(\sqrt{R^2+a^2} - \sqrt{R^2+(a+s)^2}\right) I_{\frac{n}{2}-1} K_{\frac{n}{2}+s} \\
&\leq 0,
\end{aligned}$$

where $a = (n-1)/2$. □

The following result shows a relation between two scale functions $\mathcal{I}_0$ and $\mathcal{I}_\infty$. Let us mention that the function $\mathcal{H}(t) = t\coth t$ naturally appears when the negative curvature is involved as in Lemma 3.2.3. The relation between $\mathcal{I}_{0,\kappa}$ and $\mathcal{I}_{\infty,\kappa}$ follows from Proposition 3.3.3 and (3.3.2).

Proposition 3.3.3. *There exists a constant $C = C(n) > 0$ such that*

$$\mathcal{I}_\infty(R) \le C\frac{1-s}{s}\mathcal{H}(R)\mathcal{I}_0(R)$$

for all $R > 0$.

Proof. By Lemma 3.9.2, we have

$$\begin{aligned}
\mathcal{I}_\infty(R) &= |\mathbb{S}^{n-1}| \int_R^\infty R^2 \mathcal{K}_{n,s,1}(\rho) \sinh^{n-1}\rho \,\mathrm{d}\rho \\
&\le CR^2 \int_R^\infty \rho^{-s} I_{\frac{n}{2}-1}(a\rho) K_{\frac{n}{2}+s}(a\rho)\,\mathrm{d}\rho \\
&\le CR^2 \int_{aR}^\infty \rho^{-s} I_{\frac{n}{2}-1}(\rho) K_{\frac{n}{2}+s}(\rho)\,\mathrm{d}\rho,
\end{aligned}$$

where $a = (n-1)/2$. Similarly, we also have

$$\mathcal{I}_0(R) \ge C\int_0^{aR} \rho^{2-s} I_{\frac{n}{2}-1}(\rho) K_{\frac{n}{2}+s}(\rho)\,\mathrm{d}\rho.$$

Thus, it is enough to show that there is a constant $C = C(n) > 0$ such that

$$R^2 \int_{aR}^\infty \rho^{-s} I_{\frac{n}{2}-1}(\rho) K_{\frac{n}{2}+s}(\rho)\,\mathrm{d}\rho \le C\frac{1-s}{s}\mathcal{H}(R)\int_0^{aR} \rho^{2-s} I_{\frac{n}{2}-1}(\rho) K_{\frac{n}{2}+s}(\rho)\,\mathrm{d}\rho.$$

By Lemma 3.9.4, the problem is reduced to finding a constant $C = C(n) > 0$ such that

$$\begin{aligned}
&I_{\frac{n}{2}-1}(aR)K_{\frac{n}{2}+s}(aR) + I_{\frac{n}{2}}(aR)K_{\frac{n}{2}+s-1}(aR) \\
&\le C\mathcal{H}(R)\Bigg[I_{\frac{n}{2}-1}(aR)K_{\frac{n}{2}+s}(aR) + I_{\frac{n}{2}}(aR)K_{\frac{n}{2}+s-1}(aR) \\
&\qquad - \frac{1}{2-s}\left(I_{\frac{n}{2}}(aR)K_{\frac{n}{2}+s-1}(aR) + I_{\frac{n}{2}+1}(aR)K_{\frac{n}{2}+s-2}(aR)\right)\Bigg].
\end{aligned} \tag{3.3.5}$$

Indeed, it is easy to check that (3.3.5) holds by comparing the asymptotic behavior of the functions on both sides. □

Since the limit behavior of the scale functions as $s \to 0$ are of interest in the unified regularity theory, we recall the following results.

Proposition 3.3.4. *There exists a constant $C = C(n) > 0$ such that*

$$\lim_{s \to 1} \mathcal{I}_0(R) = \lim_{s \to 1} \mathcal{I}_{0,\kappa}(R) = C$$

for any $R > 0$.

Proof. The assertion follows from [71, Lemma 5.4] and (3.3.2). □

Proposition 3.3.5. *For any $R > 0$,*

$$\lim_{s \to 1} \mathcal{I}_\infty(R) = \lim_{s \to 1} \mathcal{I}_{\infty,\kappa}(R) = 0.$$

Proof. The desired result follows from Proposition 3.3.3, Proposition 3.3.4, and (3.3.2). See also [71, Lemma 5.3]. □

In the work of Caffarelli and Silvestre [15], the quantity $r_0 = \rho_0 2^{-1/(2-2s)} R$, which is characterized by the relation $(r_0/\rho_0)^{2-2s} = R^{2-2s}/2$, plays a fundamental role. The most important feature of this quantity is that it converges to 0 as $s \to 1$.

We define such a quantity in a similar way in our framework. Since the scale function $\mathcal{I}_0$ is strictly increasing, its inverse exists. Thus, for a given $R > 0$, we define $r_0 \in (0, R)$ by

$$r_0 = \rho_0 \mathcal{I}_{0,\kappa}^{-1}(\mathcal{I}_{0,\kappa}(R)/2) \tag{3.3.6}$$

for some universal constant $\rho_0 \in (0, 1)$ that will be determined later. Let us close the section with the following lemma.

Lemma 3.3.6. *For a fixed $R > 0$, $\lim_{s \to 1} r_0 = 0$.*

Proof. Suppose that $\lim_{s\to 1} r_0 \neq 0$. Then, since $\tilde{r} := \limsup_{s\to 1} r_0 \neq 0$, there is a sequence $s_k \to 1$ such that $r_{0,k} := \rho_0 \mathcal{I}^{-1}_{0,\kappa,s_k}(\mathcal{I}_{0,\kappa,s_k}(R)/2) \in (0, R)$ converges to $\tilde{r}$ as $k \to \infty$, where $\mathcal{I}_{0,\kappa,s_k}$ is the scale function $\mathcal{I}_{0,\kappa}$ with respect to s_k. We have

$$\mathcal{I}_{0,\kappa,s_k}(r_{0,k}/\rho_0) = \mathcal{I}_{0,\kappa,s_k}(R)/2. \tag{3.3.7}$$

By Proposition 3.3.4 and continuity of $\mathcal{I}_0$, the left-hand side of (3.3.7) converges to C as $k \to \infty$ whereas the right-hand side of (3.3.7) converges to $C/2$, which is a contradiction. □

3.4 Discrete ABP-type estimates

In this section, we provide the proof of Theorem 3.1.2. Throughout the section, u is assumed to be a supersolution given in Theorem 3.1.2. On Riemannian manifolds, the distance squared function in construction of envelope was suggested by Cabré in [12] and has been used by many in [74, 75, 77, 106]. More precisely, for each $y \in B_R$, there is a unique paraboloid

$$P_y(z) = c_y - \frac{1}{2R^2} d^2_{\mathbb{H}^n_\kappa}(z, y)$$

that touches u from below, with a contact point $x \in B_{5R}$. The *envelope* Γ *of* u is defined by

$$\Gamma(z) = \sup_{y\in B_R} P_y(z),$$

and the *contact set* is given by

$$\mathscr{C} = \{x \in B_{5R} : u(x) = \Gamma(x)\}. \tag{3.4.1}$$

The first step toward to ABP-type estimates for nonlocal operators is to find a ring around a given contact point in which supersolution u is quadratically close to the paraboloid in a large portion of the ring. In the Euclidean spaces [15], or more generally in Riemannian manifolds with nonnegative curvature [72], the standard dyadic rings $B_{2^{-k}r_0} \setminus B_{2^{-(k+1)}r_0}$ are used. However, these are not appropriate within the framework of hyperbolic spaces due to the lack of homogeneity of the volume of balls. We thus define r_k recursively by

$$\frac{|B_{r_k}|}{|B_{r_{k-1}}|} = 2^{-n}, \quad k = 1, 2, \cdots,$$

and a *hyperbolic dyadic ring* by $R_k = R_k(x) = B_{r_k}(x) \setminus B_{r_{k+1}}(x)$. Note that we have $|B_{r_k}|/|B_{r_{k-1}}| \le (r_k/r_{k-1})^n$ from Lemma 3.2.2, and hence $r_{k+1} \ge r_k/2$. By using the hyperbolic dyadic rings, we will prove a series of lemmas to deduce Theorem 3.1.2. For notational convenience, we shall write

$$\tilde{f}_\kappa(x) := \Lambda \mathcal{H}_\kappa(7R) + \frac{R^2}{\mathcal{I}_{0,\kappa}(R)} f(x),$$

where we recall $\mathcal{H}_\kappa(t) = \sqrt{\kappa} t \coth(\sqrt{\kappa} t)$.

Lemma 3.4.1. *Let u be a supersolution given in Theorem 3.1.2. Then, there exists a universal constant $C_0 > 0$, independent of s, κ, and R, such that for each $x \in \mathscr{C}$ and $M_0 > 0$, there is an integer $k \ge 0$ satisfying*

$$|G_k| \le \frac{C_0}{M_0} \tilde{f}_\kappa(x) |R_k|, \tag{3.4.2}$$

where $G_k = R_k \cap \{u > P_y + M_0 (r_k/R)^2\}$.

Proof. Since x minimizes the function $u + \frac{1}{2R^2} d^2_{\mathbb{H}^n_\kappa}(\cdot, y)$, we have $\mathcal{M}^- u(x) \ge I_1 + I_2 + I_3$, where

$$\begin{aligned}
I_1 &= \lambda \int_{B_R(x) \cup B_{5R}} \delta\left(u + \frac{1}{2R^2} d^2_{\mathbb{H}^n_\kappa}(\cdot, y), x, z\right) \mathcal{K}_{n,s,\kappa}(d_{\mathbb{H}^n_\kappa}(z, x)) \, \mathrm{d}\mu_{\mathbb{H}^n_\kappa}(z), \\
I_2 &= -\Lambda \int_{B_R(x) \cup B_{5R}} \delta^+\left(\frac{1}{2R^2} d^2_{\mathbb{H}^n_\kappa}(\cdot, y), x, z\right) \mathcal{K}_{n,s,\kappa}(d_{\mathbb{H}^n_\kappa}(z, x)) \, \mathrm{d}\mu_{\mathbb{H}^n_\kappa}(z), \\
I_3 &= -\Lambda \int_{\mathbb{H}^n_\kappa \setminus (B_R(x) \cup B_{5R})} \delta^-(u, x, z) \mathcal{K}_{n,s,\kappa}(d_{\mathbb{H}^n_\kappa}(z, x)) \, \mathrm{d}\mu_{\mathbb{H}^n_\kappa}(z),
\end{aligned}$$

and $\delta(v, x, z) = (v(z) + v(\exp_x(-\exp_x^{-1} z)) - 2v(x))/2$ is the second order incremental quotients. By the mean value theorem for integrals and

Lemma 3.2.3, we obtain

$$I_2 \geq -C\Lambda \mathcal{H}_\kappa(7R)\frac{\mathcal{I}_{0,\kappa}(R)}{R^2}. \tag{3.4.3}$$

Since $u(x) \leq u(x) + \frac{1}{2R^2}d^2_{\mathbb{H}^n_\kappa}(x,y) \leq \inf_{B_{2R}}(u + \frac{1}{2R^2}d^2_{\mathbb{H}^n_\kappa}(x,y) \leq 11/2 < 6$ and $u \geq 0$ in $\mathbb{H}^n_\kappa \setminus B_{5R}$, we also have

$$I_3 \geq -C\Lambda\frac{\mathcal{I}_{\infty,\kappa}(R)}{R^2}. \tag{3.4.4}$$

Let us now focus on I_1. Assume that (3.4.2) does not hold for all k. Then, since

$$\delta\left(u + \frac{1}{2R^2}d^2_{\mathbb{H}^n_\kappa}(\cdot,y),x,z\right) \geq M_0\left(\frac{r_k}{R}\right)^2 \quad \text{on } G_k$$

and $\mathcal{K}_{s,\kappa}$ is decreasing by Proposition 3.2.1, we have

$$\begin{aligned}
I_1 &\geq \lambda M_0 \sum_{k=1}^\infty \int_{G_k}\left(\frac{r_k}{R}\right)^2 \mathcal{K}_{n,s,\kappa}(d_{\mathbb{H}^n_\kappa}(z,x))\,\mathrm{d}\mu_{\mathbb{H}^n_\kappa}(z) \\
&\geq \lambda C_0 \frac{\tilde{f}(x)}{R^2}\sum_{k=1}^\infty r_k^2\mathcal{K}_{n,s,\kappa}(r_k)|R_k|.
\end{aligned}$$

Since $r_{k+1} \geq r_k/2$ and $|R_k| = 2^n|R_{k+1}|$, we obtain

$$\begin{aligned}
I_1 &\geq \lambda C_0 \frac{\tilde{f}(x)}{R^2}\sum_{k=0}^\infty r_{k+1}^2\mathcal{K}_{n,s,\kappa}(r_{k+1})|R_{k+1}| \\
&\geq \lambda 2^{-2-n}C_0\frac{\tilde{f}(x)}{R^2}\sum_{k=0}^\infty r_k^2\mathcal{K}_{n,s,\kappa}(r_{k+1})|R_k| \\
&\geq \lambda 2^{-2-n}C_0\frac{\tilde{f}(x)}{R^2}\sum_{k=0}^\infty \int_{R_k} d^2_{\mathbb{H}^n_\kappa}(z,x)\mathcal{K}_{n,s,\kappa}(d_{\mathbb{H}^n_\kappa}(z,x))\,\mathrm{d}\mu_{\mathbb{H}^n_\kappa}(z) \\
&= \lambda 2^{-2-n}C_0\frac{\tilde{f}(x)}{R^2}\mathcal{I}_{0,\kappa}(r_0).
\end{aligned}$$

Furthermore, by using Proposition 3.3.1 we have

$$I_1 \geq \lambda 2^{-2-n}C_0\frac{\tilde{f}(x)}{R^2}\rho_0^2\mathcal{I}_{0,\kappa}(r_0/\rho_0) = \lambda 2^{-3-n}C_0\frac{\tilde{f}(x)}{R^2}\rho_0^2\mathcal{I}_{0,\kappa}(R). \tag{3.4.5}$$

By combining (3.4.3), (3.4.4), and (3.4.5), and then using (3.3.6) and Proposition 3.3.3, we obtain

$$f(x) \geq \lambda 2^{-3-n} C_0 \frac{\tilde{f}(x)}{R^2} \rho_0^2 \mathcal{I}_{0,\kappa}(R) - C\left(1 + s_0^{-1}\right) \Lambda \mathcal{H}_\kappa(7R) \frac{\mathcal{I}_{0,\kappa}(R)}{R^2}.$$

By taking C_0 sufficiently large, we arrive at a contradiction. □

The next lemma shows that the function $\Gamma - P_y$ is c-convex with an appropriate function c. See [50, 89] for the definition of c-convex function. The proof is exactly the same with that of [71, Lemma 3.4] except for the Hessian bound of the distance squared function. That is, we use Lemma 3.2.3 instead of [71, Lemma 2.1].

Lemma 3.4.2. *Let $x \in \mathscr{C}$, $z \in \mathbb{H}^n_\kappa$, and $y \in B_R$ be a vertex point of a paraboloid P_y. Then,*

$$(\Gamma - P_y)(z) \leq (1-t)(\Gamma - P_y)(z_1) + t(\Gamma - P_y)(z_2) + \frac{1}{2R^2} t(1-t) \mathcal{H}_\kappa(d_{\mathbb{H}^n_\kappa}(y,z) + |\xi|)|\xi|^2$$

for all $t \in (0,1)$, where $z_1 = \exp_z(t\xi)$ and $z_2 = \exp_z((1-t)(-\xi))$.

Using Lemma 3.4.2, we show that the envelope is captured in a small ball near a contact point by two paraboloids that are quadratically close to each other.

Lemma 3.4.3. *Under the setting of Lemma 3.4.1, there is a universal constant $\varepsilon_0 \in (0,1)$ such that if*

$$|\{z \in R_k : \Gamma(z) > P_y(z) + h\}| \leq \varepsilon_0 |R_k|,$$

then

$$\Gamma \leq P_y + h + C\mathcal{H}_\kappa(7R) \left(\frac{r_k}{R}\right)^2$$

in $B_{\tilde{r}_{k+1}}(x)$, where $\tilde{r}_{k+1} = \frac{1}{\sqrt{\kappa}} \tanh^{-1}(\frac{1}{2} \tanh(\sqrt{\kappa} r_{k+1}))$.

Proof. Let us fix $z \in B_{\tilde{r}_{k+1}}(x)$ and set $D = \{z \in R_k : \Gamma(z) \leq P_y(z) + h\}$. For $w \in R_k$, let us consider a geodesic $c : \mathbb{R} \to \mathbb{H}^n_\kappa$ passing through w and z.

Then, $c(\mathbb{R}) \cap R_k$ consists of two connected components $c(t_1, t_2)$ and $c(t_3, t_4)$, where $t_1 < t_2 < t_3 < t_4$ satisfy $t_4 - t_3 = t_2 - t_1$. We may assume that $w = c(t) \in c(t_1, t_2)$. We define a map $\varphi_z : R_k \to R_k$ by $\varphi_z(w) = c\left(-t + t_1 + t_4\right)$, which is clearly one-to-one and onto.

Among all the geodesics passing through the point z, let us consider geodesics $c_\perp$ that are perpendicular to the geodesic joining x and z. Then $\cup\, c_\perp$ divides R_k into two regions: let A_1 be the smaller one and A_2 the bigger one. We claim

$$|E| \leq |\varphi_z(E)| \quad \text{for any Borel set } E \subset A_1. \tag{3.4.6}$$

Indeed, we may assume that $z = 0_\kappa \in \mathbb{H}^n_\kappa$ by using a global isometry. Then the map $\varphi := \varphi_z$ can be represented by

$$\varphi(w) = \frac{1}{\sqrt{\kappa}}(\cosh(r + C_\theta), \sinh(r + C_\theta)(-\theta)), \quad w = \frac{1}{\sqrt{\kappa}}(\cosh r, \sinh r\theta),$$

where $C_\theta = d_{\mathbb{H}^n_\kappa}(\varphi(w^*), 0) - d_{\mathbb{H}^n_\kappa}(w^*, 0)$, with w^*, the intersection point of $\partial B_{r_{k+1}}(x)$ and the geodesic segment joining 0 and w. Note that φ is a smooth map because it is a composition of smooth maps. Clearly, $C_\theta \geq 0$ if and only if $w \in A_1$. Thus, we obtain

$$\begin{aligned}
|E| &= \iint \mathbf{1}_E(w) \frac{\sinh^{n-1}(\sqrt{\kappa} r)}{\sqrt{\kappa}^{n-1}} \,\mathrm{d}r \,\mathrm{d}\theta \\
&\leq \iint \mathbf{1}_{\varphi(E)}(\varphi(w)) \frac{\sinh^{n-1}(\sqrt{\kappa}(r + C_\theta))}{\sqrt{\kappa}^{n-1}} \,\mathrm{d}r \,\mathrm{d}\theta \\
&= \iint \mathbf{1}_{\varphi(E)} \left(\frac{\cosh \tilde{r}}{\sqrt{\kappa}}, \frac{\sinh \tilde{r}}{\sqrt{\kappa}} \tilde{\theta} \right) \frac{\sinh^{n-1}(\sqrt{\kappa}\tilde{r})}{\sqrt{\kappa}^{n-1}} \,\mathrm{d}\tilde{r} \,\mathrm{d}\tilde{\theta} = |\varphi(E)|,
\end{aligned}$$

where we have used change of variables $\tilde{r} = r + C_\theta$ and $\tilde{\theta} = -\theta$. This proves (3.4.6).

We next claim that

$$|R_k| \leq C|A_1| \text{ with } C > 0 \text{ a universal constant.} \tag{3.4.7}$$

Let us first deduce the lemma assuming that (3.4.7) is true. If we show that $\varphi_z(A_1 \cap D) \cap D \neq \emptyset$, then there are points $w_i \in A_i \cap D$, $i = 1, 2$, such that $\varphi_z(w_1) = w_2$. Since $\Gamma(w_i) \leq P_y(w_i) + h$, for $i = 1, 2$, the desired result follows from Lemma 3.4.2. Assume to the contrary that the set $\varphi(A_1 \cap D) \cap D$ is empty. By (3.4.7), we have

$$|A_1 \cap D^c| \leq |R_k \cap D^c| \leq \varepsilon_0 |R_k| \leq C\varepsilon_0 |A_1|.$$

By taking $\varepsilon_0 = (2C)^{-1}$, we obtain $|A_1 \cap D| > |A_1|/2$. Since $\varphi_z(A_1 \cap D) \subset A_2 \cap D^c$, it follows that

$$\frac{1}{2}|A_1| < |A_1 \cap D| \leq |\varphi_z(A_1 \cap D)| \leq |A_2 \cap D^c| \leq |R_k \cap D^c| \leq \frac{1}{2}|A_1|,$$

which is a contradiction.

From now on, we focus on the proof of (3.4.7). To this end, it is convenient to use the Poincaré ball model $\mathbb{B}^n_\kappa = \mathbb{B}^n_{1,\kappa}$. Let $\tilde{A}_1 = \phi(A_1)$ and $\tilde{R}_k = \phi(R_k)$, where ϕ is the isometry given by (3.2.2). Since we are concerned with volumes, we may assume $\phi(z) = |\phi(z)|e_1$ so that $\tilde{A}_1$ is rotationally symmetric with respect to x_1-axis. Let ρ_k be such that $r_k = d_{\mathbb{B}^n_\kappa}(0, \rho_k e_1)$. We observe that

$$\{y \in \mathbb{B}^n_\kappa : \rho_{k+1} < |y| < \rho_k,\ e_1 \cdot y/|y| > 1/2\} \subset \tilde{A}_1. \tag{3.4.8}$$

Indeed, if we define $\tilde{A}_1'$ in the same way as $\tilde{A}_1$ with $z' \in \partial B_{\tilde{r}_{k+1}}(x)$ instead of $z \in B_{\tilde{r}_{k+1}}(x)$, then $\tilde{A}_1 \supset \tilde{A}_1'$. Moreover, any geodesic that is perpendicular to x_1-axis and passes through $\tilde{\rho}_{k+1}e_1$ is contained in the sphere

$$\left(x_1 - \frac{1+\tilde{\rho}_{k+1}^2}{2\tilde{\rho}_{k+1}}\right)^2 + x_2^2 + x_3^2 + ... + x_n^2 = \left(\frac{1-\tilde{\rho}_{k+1}^2}{2\tilde{\rho}_{k+1}}\right)^2. \tag{3.4.9}$$

The x_1-coordinate of the intersection of the spheres (3.4.9) and $x_1^2 + x_2^2 + ... + x_n^2 = \rho_{k+1}^2$ is given by

$$x_1 = \frac{\tilde{\rho}_{k+1}}{1+\tilde{\rho}_{k+1}^2}(1+\rho_{k+1}^2) = \frac{\tanh(\sqrt{\kappa}\tilde{r}_{k+1})}{\tanh(\sqrt{\kappa}r_{k+1})}\rho_{k+1} = \frac{1}{2}\rho_{k+1},$$

where we used

$$d_{\mathbb{H}^n_\kappa}(0,\rho e_1) = \frac{1}{\sqrt{\kappa}}\cosh^{-1}\frac{1+\rho^2}{1-\rho^2} = \frac{1}{\sqrt{\kappa}}\tanh^{-1}\frac{2\rho}{1+\rho^2}$$

in the second equality. Note that the radius $\tilde{r}_{k+1} = \frac{1}{\sqrt{\kappa}}\tanh^{-1}(\frac{1}{2}\tanh(\sqrt{\kappa}r_{k+1}))$ is chosen so that the last equality holds. Therefore, (3.4.8) holds.

We now compute

$$\begin{aligned}|A_1| &= |\tilde{A}_1| \\ &\geq \int_0^{2\pi}\int_{\rho_{k+1}}^{\rho_k}\int_0^{\frac{\pi}{3}}\cdots\int_0^{\frac{\pi}{3}}\left(\frac{2}{\sqrt{\kappa}(1-\rho^2)}\right)^n \\ &\qquad\cdot\rho^{n-1}\sin^{n-2}\varphi_1\cdots\sin\varphi_{n-2}\,\mathrm{d}\varphi_1\cdots\mathrm{d}\varphi_{n-2}\,\mathrm{d}\rho\,\mathrm{d}\theta \\ &\geq C(n)\int_{\rho_{k+1}}^{\rho_k}\left(\frac{2}{\sqrt{\kappa}(1-\rho^2)}\right)^n\rho^2\,\mathrm{d}\rho.\end{aligned}$$

Since

$$|R_k| = |\tilde{R}_k| = |S^{n-1}|\int_{\rho_{k+1}}^{\rho_k}\left(\frac{2}{\sqrt{\kappa}(1-\rho^2)}\right)^n\rho^{n-1}\,\mathrm{d}\rho,$$

(3.4.7) is proved with some $C(n) > 0$. □

We define $\phi : \mathbb{H}^n_\kappa \to B_R$ by a map assigning each point $x \in \mathbb{H}^n_\kappa$ a vertex point y of the paraboloid P_y, where P_y is a paraboloid such that $\Gamma(x) = P_y(x)$, which is not necessarily unique. Then, the flatness of Γ obtained in Lemma 3.4.3 allows us to control the image of ϕ, which can be understood as the image of gradient of Γ.

Lemma 3.4.4. *Under the setting of Lemma 3.4.1, let $x \in \mathscr{C}$ and let k be such that* (3.4.2) *holds, and let ε_0 be the constant in Lemma 3.4.3. Then,*

$$\left|\left\{z \in R_k : u(z) > P_y(z) + C\tilde{f}_\kappa(x)(r_k/R)^2\right\}\right| \leq \varepsilon_0|R_k| \tag{3.4.10}$$

and

$$\phi\left(\overline{B(x,\tilde{r}_{k+1}/2)}\right) \subset B\left(y, C\mathcal{S}_\kappa(7R)\mathcal{T}_\kappa(r_{k+1})\tilde{f}_\kappa(x)r_k\right), \tag{3.4.11}$$

where $C > 0$ is a universal constant depending only on n, λ, Λ, and s_0.

Proof. By taking $M_0 = C_0 \tilde{f}_\kappa(x)/\varepsilon_0$ in Lemma 3.4.1, we obtain (3.4.10). Moreover, by Lemma 3.4.3 we have

$$P_y \leq \Gamma \leq P_y + C\tilde{f}(x)\left(\frac{r_k}{R}\right)^2 \tag{3.4.12}$$

in $B_{\tilde{r}_{k+1}}(x)$, with a universal constant $C > 0$.

To prove (3.4.11), let $z \in \overline{B(x, \tilde{r}_{k+1}/2)}$ and $y_* \in \phi(z)$. We need to find a upper bound of $d_{\mathbb{H}^n_\kappa}(y_*, y)$. Let $\xi_1 = \exp_z^{-1} y_*$ and $\xi_2 = \exp_z^{-1} y$. Let us consider a family of geodesics

$$c(s,t) = \exp_z(t(\xi_1 + s(\xi_2 - \xi_1))),$$

and the Jacobi field J along c. Then, by [53, Equation (1.8b)] (or see, e.g. [66]), we have

$$|J(1)|_{g(y_*)} \leq \mathcal{S}_\kappa(|\xi_1|)|J'(0)|_{g(z)} \leq \mathcal{S}_\kappa(7R)|\xi_2 - \xi_1|_{g(z)}.$$

Considering the curve $s \mapsto c(s,1)$, we obtain

$$d_{\mathbb{H}^n_\kappa}(y_*, y) \leq \int_0^1 |\partial_s c(s,1)|_{g(y_*)}\,\mathrm{d}s \leq \mathcal{S}_\kappa(7R)|\exp_z^{-1} y_* - \exp_z^{-1} y|_{g(z)}.$$

By the Gauss lemma, we know that $|\exp_z^{-1} y_* - \exp_z^{-1} y|_{g(z)} = R^2|\nabla P_{y_*}(z) - \nabla P_y(z)|_{g(z)}$. Thus, it only remains to show that

$$R^2|\nabla P_{y_*}(z) - \nabla P_y(z)|_{g(z)} \leq C\mathcal{T}_\kappa(r_{k+1})\tilde{f}(x) r_k \tag{3.4.13}$$

for some universal constant $C > 0$.

To this end, we prove that

$$\left|\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (P_{y_*} - P_y)(c(t))\right| \leq C\mathcal{T}_\kappa(r_{k+1})\tilde{f}_\kappa(x)\frac{r_k}{R^2} \tag{3.4.14}$$

for all geodesics c, with unit speed, starting from $c(0) = z$. Suppose that

(3.4.14) does not hold for some c. We may assume that

$$C\mathcal{T}_\kappa(r_{k+1})\tilde{f}_\kappa(x)\frac{r_k}{R^2} \leq \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (P_{y_*} - P_y)\,(c(t)),$$

by considering $\tilde{c}(t) = c(-t)$ instead of $c(t)$ if necessary. Let $\varepsilon > 0$, then there is a $\delta > 0$ such that if $|t| < \delta$, we have

$$C\mathcal{T}_\kappa(r_{k+1})\tilde{f}_\kappa(x)\frac{r_k}{R^2} - \varepsilon \leq \frac{(P_{y_*} - P_y)(c(t)) - (P_{y_*} - P_y)(c(0))}{t} \leq \frac{h(t) - h(0)}{t}, \tag{3.4.15}$$

where $h(t) = (\Gamma - P_y)(c(t))$. Let $T > 0$ be the first time such that $c(T) \in \partial B_{3\tilde{r}_{k+1}/4}(x)$. Let N be the least integer not smaller than T/δ, and let $0 = t_0 < t_1 < \cdots < t_N = T$ be equally distributed times. Then, $t_{i+1} - t_i = T/N \leq \delta$. By Lemma 3.4.2, we have

$$\frac{h(t_i) - h(t_{i-1})}{t_i - t_{i-1}} \leq \frac{h(t_{i+1}) - h(t_i)}{t_{i+1} - t_i} + \frac{\mathcal{H}_\kappa(7R)}{2R^2}(t_{i+1} - t_{i-1}), \quad i = 1, 2, \cdots, N-1. \tag{3.4.16}$$

Thus, it follows from (3.4.15) and (3.4.16) that

$$C\mathcal{T}_\kappa(r_{k+1})\tilde{f}_\kappa(x)\frac{r_k}{R^2} - \varepsilon \leq \frac{h(t_{i+1}) - h(t_i)}{T/N} + \frac{\mathcal{H}_\kappa(7R)}{2R^2}\frac{2Ti}{N}, \quad i = 1, 2, \cdots, N-1. \tag{3.4.17}$$

Summing up (3.4.17) for $i = 1, 2, \cdots, N-1$, we obtain

$$N\left(C\mathcal{T}_\kappa(r_{k+1})\tilde{f}_\kappa(x)\frac{r_k}{R^2} - \varepsilon\right) \leq \frac{h(t_N) - h(t_0)}{T/N} + \frac{\mathcal{H}_\kappa(7R)}{2R^2}\frac{2T}{N}\frac{N(N-1)}{2}.$$

Since c has a unit speed, we have $\tilde{r}_{k+1}/4 < T < \tilde{r}_{k+1}$, and hence

$$\begin{aligned} C\mathcal{T}_\kappa(r_{k+1})\tilde{f}_\kappa(x)\frac{r_k}{R^2} - \varepsilon &\leq \frac{(\Gamma - P_y)(c(T)) - (\Gamma - P_y)(z)}{\tilde{r}_{k+1}/4} + \frac{\mathcal{H}_\kappa(7R)}{2R^2}\tilde{r}_{k+1} \\ &\leq \frac{(\Gamma - P_y)(c(T))}{\tilde{r}_{k+1}/4} + \frac{\mathcal{H}_\kappa(7R)}{2R^2}\tilde{r}_{k+1}. \end{aligned}$$

Recalling that $\mathcal{T}_\kappa(r_{k+1}) = r_{k+1}/\tilde{r}_{k+1}$ and $r_{k+1} \geq r_k/2$, and that ε was arbitrary, we have

$$C\tilde{f}_\kappa(x)\frac{r_k^2}{8R^2} \le (\Gamma - P_y)(c(T)) + \mathcal{H}_\kappa(7R)\frac{r_k^2}{8R^2}. \tag{3.4.18}$$

Since $c(T) \in \partial B_{3\tilde{r}_{k+1}/4}(x) \subset B_{\tilde{r}_{k+1}}(x)$, the inequality (3.4.18) with a sufficiently large constant $C_1 > 0$ contradicts to (3.4.12). Therefore, we have proved (3.4.13), which finishes the proof.

□

We are now ready to prove a discrete ABP-type estimate, from which Theorem 3.1.2 follows.

Lemma 3.4.5. *Assume the same assumptions as in Theorem 3.1.2. There is a finite collection $\mathcal{D}$ of dyadic cubes $\{Q_\alpha^j\}$, with diameters $d_j \le r_0$, such that the following hold:*

(i) Any two different dyadic cubes in $\mathcal{D}$ do not intersect.

(ii) $\mathscr{C} \subset \bigcup \overline{Q}_\alpha^j$.

(iii) $|\phi(\overline{Q}_\alpha^j)| \le cF^n|Q_\alpha^j|$.

(iv) $|B(z_\alpha^j, 2r_0) \cap \{u \le \Gamma + C(\sup_{\overline{Q}_\alpha^j} \tilde{f}(x))(r_0/R)^2\}| \ge \mu|Q_\alpha^j|$.

The constants $C > 0$ and $\mu > 0$ depend only on n, λ, Λ, and s_0.

Proof. Let c_1, c_2, and δ_0 be the constants given in Theorem 3.2.4, which depend only on n. Let us fix the smallest integer N such that $c_2\delta_0^N \le r_0$, then there are finitely many dyadic cubes Q_α^N of generation N such that $\overline{Q}_\alpha^N \cap \mathscr{C} \ne \emptyset$ and $\mathscr{C} \subset \cup_\alpha \overline{Q}_\alpha^N$. Whenever a dyadic cube Q_α^j ($j \ge N$) does not satisfy (iii) and (iv), we consider all of its successors $Q_\beta^{j+1} \subset Q_\alpha^j$ instead of Q_α^j. Among these successors of $j+1$ generation, we only keep those whose closures intersect A and discard the rest. We prove that this process must finish in a finite number of steps.

Assume to the contrary that the process produces an infinite sequence of nested dyadic cubes $\{Q_\alpha^j\}_{j=N}^\infty$. Then, the intersection of their closures is some contact point $x \in \mathscr{C}$. By Lemma 3.4.4, there is a $k \geq 0$ such that (3.4.10) and (3.4.11) hold. Let $j \geq N$ be such that $\delta_0 \tilde{r}_{k+1}/2 \leq c_2 \delta_0^j < \tilde{r}_{k+1}/2 \leq r_0$, then it follows from Theorem 3.2.4 that

$$B(z_\alpha^j, c_1\delta_0^j) \subset Q_\alpha^j \subset \overline{Q}_\alpha^j \subset B(x, \tilde{r}_{k+1}/2). \tag{3.4.19}$$

Thus, it follows from (3.4.11) and (3.4.19) that

$$|\phi(\overline{Q}_\alpha^j)| \leq |\phi(\overline{B(x, \tilde{r}_{k+1}/2)})| \leq |B(y, C\mathcal{S}_\kappa(7R)\mathcal{T}_\kappa(r_{k+1})\tilde{f}_\kappa(x) r_k)|.$$

Since $\mathcal{S}_\kappa(7R)\tilde{f}_\kappa(x) \leq F$ and $r_k \leq 2r_{k+1} = 2\mathcal{T}_\kappa(r_{k+1})\tilde{r}_{k+1} \leq 4\mathcal{T}_\kappa(r_0)c_2\delta_0^{j-1}$, we have

$$|\phi(\overline{Q}_\alpha^j)| \leq |B(z_\alpha^j, C\mathcal{T}_\kappa^2(r_0)Fc_1\delta_0^j)|.$$

Therefore, by Lemma 3.2.2 we obtain

$$|\phi(\overline{Q}_\alpha^j)| \leq \mathcal{D}\left(C\mathcal{T}_\kappa^2(r_0)F\right)^{\log_2 \mathcal{D}} |Q_\alpha^j|$$

where $\mathcal{D} = 2^n \cosh^{n-1}(C\sqrt{\kappa}\mathcal{T}_\kappa^2(r_0)c_1\delta_0^j F)$, which shows that Q_α^j satisfies (iii).

If $z \in B(x, r_k)$, then $d(z, z_\alpha^j) \leq d(z, x) + d(x, z_\alpha^j) < r_k + c_2\delta_0^j \leq 2r_0$, which shows that $B(x, r_k) \subset B(z_\alpha^j, 2r_0)$. Thus, by using (3.4.10), we have

$$\begin{aligned}
|B(z_\alpha^j, 2r_0) \cap \{u \leq \Gamma + C(\sup_{\overline{Q}_\alpha^j} \tilde{f}_\kappa(x))(r_0/R)^2\}| \\
&\geq |R_k \cap \{u \leq P_y + C\tilde{f}_\kappa(x)(r_k/R)^2\}| \\
&\geq (1-\varepsilon_0)|R_k| \\
&= (1-\varepsilon_0)(2^n - 1)|B_{r_{k+1}}| \\
&\geq \mu |Q_\alpha^j|
\end{aligned}$$

for some universal constant $\mu > 0$. This proves that Q_α^j also satisfies (iv), which yields a contradiction. Therefore, the process must stop in a finite number of steps. □

3.5 A barrier function

This section is devoted to a construction of a special barrier function, which is a key ingredient together with the ABP-type estimates for the Krylov–Safonov Harnack inequality. It is standard to use distance function to construct a barrier function, but computations are significantly different from the standard argument. We will observe how the negative curvature of hyperbolic spaces comes into play. Let us begin with some inequalities.

Lemma 3.5.1. *Let $\alpha > 0$ and $R_0 > 0$. Then*

$$(\cosh^{-1}(t\cosh(\sqrt{\kappa}R_0)))^{-2\alpha} - (\sqrt{\kappa}R_0)^{-2\alpha} \geq -2\alpha\frac{\mathcal{H}_\kappa(R_0)}{(\sqrt{\kappa}R_0)^{2\alpha+2}}(t-1) \quad (3.5.1)$$

for all $t > 1/\cosh(\sqrt{\kappa}R_0)$. Moreover,

$$\begin{aligned}&\frac{(\cosh^{-1}(t\cosh(\sqrt{\kappa}R_0)))^{-2\alpha-2}}{t^2\cosh^2(\sqrt{\kappa}R_0)-1} - \frac{(\sqrt{\kappa}R_0)^{-2\alpha-2}}{\sinh^2(\sqrt{\kappa}R_0)}\\ &\geq -\frac{(2\alpha+2+2\mathcal{H}_\kappa(R_0))\mathcal{H}_\kappa(R_0)}{(\sqrt{\kappa}R_0)^{2\alpha+4}\sinh^2(\sqrt{\kappa}R_0)}(t-1)\end{aligned} \quad (3.5.2)$$

and

$$\begin{aligned}&(\cosh^{-1}(t\cosh(\sqrt{\kappa}R_0)))^{-2\alpha-1}\frac{t\cosh(\sqrt{\kappa}R_0)}{(t^2\cosh^2(\sqrt{\kappa}R_0)-1)^{3/2}} - (\sqrt{\kappa}R_0)^{-2\alpha-1}\frac{\cosh(\sqrt{\kappa}R_0)}{\sinh^3(\sqrt{\kappa}R_0)}\\ &\geq -\frac{((2\alpha+1)\mathcal{H}_\kappa(R_0) - (\sqrt{\kappa}R_0)^2 + 3\mathcal{H}_\kappa^2(R_0))\,\mathcal{H}_\kappa(R_0)}{(\sqrt{\kappa}R_0)^{2\alpha+4}\sinh^2(\sqrt{\kappa}R_0)}(t-1)\end{aligned} \quad (3.5.3)$$

for all $t > 1/\cosh(\sqrt{\kappa}R_0)$.

Proof. Since the function

$$f(t) := (\cosh^{-1}(t\cosh(\sqrt{\kappa}R_0)))^{-2\alpha}, \quad t > \frac{1}{\cosh(\sqrt{\kappa}R_0)},$$

is convex, (3.5.1) follows from the inequality $f(t) \geq f(1) + f'(1)(t-1)$. The inequalities (3.5.2) and (3.5.3) can be obtained similarly by considering

$$
\begin{aligned}
g(t) &:= \frac{(\cosh^{-1}(t\cosh(\sqrt{\kappa}R_0)))^{-2\alpha-2}}{t^2\cosh^2(\sqrt{\kappa}R_0)-1} \quad \text{and} \\
h(t) &:= (\cosh^{-1}(t\cosh(\sqrt{\kappa}R_0)))^{-2\alpha-1}\frac{t\cosh(\sqrt{\kappa}R_0)}{(t^2\cosh^2(\sqrt{\kappa}R_0)-1)^{3/2}},
\end{aligned}
$$

which are also convex functions. □

Using Lemma 3.5.1, we first construct a barrier function when s is sufficiently close to 1. Let us denote $\mathcal{K}_{s,\kappa} = \mathcal{K}_{n,s,\kappa}$ in the following lemmas.

Lemma 3.5.2. *Let $\delta \in (0,1)$. There are constants $\alpha > 0$ and $s_0 \in (0,1)$, depending only on n, λ, Λ, δ, and $\sqrt{\kappa}R$, such that the function*

$$
v(x) = \max\left\{ -\left(\frac{\delta}{20}\right)^{-2\alpha}, -\left(\frac{d_{\mathbb{H}^n_\kappa}(x,0)}{5R}\right)^{-2\alpha} \right\}
$$

is a supersolution to

$$
\frac{(7R)^2}{\mathcal{I}_{0,\kappa}(7R)}\mathcal{M}^+ v(x) + \Lambda\mathcal{H}_\kappa(7R) \le 0, \tag{3.5.4}
$$

for every $s_0 < s < 1$ and $x \in B_{5R} \setminus \overline{B}_{\delta R/4}$.

Proof. Fix x and let $R_0 := d_{\mathbb{H}^n_\kappa}(x,0) \in (\delta R/4, 5R)$. We are going to consider the coordinates centered at x. There is an isometry $\varphi \in SO(1,n)$ such that $x = \varphi(0)$ and $0 = \varphi(\frac{1}{\sqrt{\kappa}}\cosh(\sqrt{\kappa}R_0), \frac{1}{\sqrt{\kappa}}\sinh(\sqrt{\kappa}R_0)e_1)$ with $e_1 \in \mathbb{S}^{n-1}$. Notice that 0 denotes $0_\kappa = (\frac{1}{\sqrt{\kappa}}, 0, \cdots, 0) \in \mathbb{H}^n_\kappa$.

Let $z \in B_{R_0/2}(x)$, then $z = \varphi(\frac{1}{\sqrt{\kappa}}\cosh(\sqrt{\kappa}r), \frac{1}{\sqrt{\kappa}}\sinh(\sqrt{\kappa}r)\omega)$ for some $r \in [0, R_0/2)$ and $\omega \in \mathbb{S}^{n-1}$. By the hyperbolic law of cosines, we have

$$
\begin{aligned}
d_{\mathbb{H}^n_\kappa}(z,0) &= d_{\mathbb{H}^n_\kappa}\left(\varphi\left(\frac{\cosh(\sqrt{\kappa}r)}{\sqrt{\kappa}}, \frac{\sinh(\sqrt{\kappa}r)}{\sqrt{\kappa}}\omega\right), \varphi\left(\frac{\cosh(\sqrt{\kappa}R_0)}{\sqrt{\kappa}}, \frac{\sinh(\sqrt{\kappa}R_0)}{\sqrt{\kappa}}e_1\right)\right) \\
&= d_{\mathbb{H}^n_\kappa}\left(\left(\frac{\cosh(\sqrt{\kappa}r)}{\sqrt{\kappa}}, \frac{\sinh(\sqrt{\kappa}r)}{\sqrt{\kappa}}\right), \left(\frac{\cosh(\sqrt{\kappa}R_0)}{\sqrt{\kappa}}, \frac{\sinh(\sqrt{\kappa}R_0)}{\sqrt{\kappa}}\right)\right) \\
&= \frac{1}{\sqrt{\kappa}}\cosh^{-1}(A-B)
\end{aligned}
$$

where $A = \cosh(\sqrt{\kappa}r)\cosh(\sqrt{\kappa}R_0)$ and $B = \sinh(\sqrt{\kappa}r)\sinh(\sqrt{\kappa}R_0)\omega_1$. Similarly, we have

$$d_{\mathbb{H}^n_\kappa}(\exp_x(-\exp_x^{-1} z), 0) = \frac{1}{\sqrt{\kappa}}\cosh^{-1}(A+B).$$

Thus, we obtain

$$\delta(v,x,z) = -(5\sqrt{\kappa}R)^{2\alpha}\frac{(\cosh^{-1}(A-B))^{-2\alpha} + (\cosh^{-1}(A+B))^{-2\alpha} - 2(\sqrt{\kappa}R_0)^{-2\alpha}}{2}.$$

Since $(\cosh^{-1}(\cdot))^{-2\alpha}$ is convex at A, we obtain

$$\begin{aligned}\delta(v,x,z) \leq &-(5\sqrt{\kappa}R)^{2\alpha}\left(\alpha(2\alpha+1)\frac{(\cosh^{-1}A)^{-2\alpha-2}}{(A^2-1)^{1/2}} + \alpha\frac{A(\cosh^{-1}A)^{-2\alpha-1}}{(A^2-1)^{3/2}}\right)B^2\\ &-(5\sqrt{\kappa}R)^{2\alpha}\left((\cosh^{-1}A)^{-2\alpha} - (\sqrt{\kappa}R_0)^{-2\alpha}\right).\end{aligned}$$

Moreover, by applying Lemma 3.5.1 with $t = \cosh(\sqrt{\kappa}r)$, we have

$$\begin{aligned}&\delta(v,x,z)\\ &\leq \alpha(2\alpha+1)c_\delta\left((2\alpha+2+2\mathcal{H}_\kappa(R_0))\mathcal{H}_\kappa(R_0)\frac{\cosh(\sqrt{\kappa}r)-1}{(\sqrt{\kappa}R_0)^2} - 1\right)\frac{\sinh^2(\sqrt{\kappa}r)}{(\sqrt{\kappa}R_0)^2}\omega_1^2\\ &\quad + \alpha c_\delta\left(\left((2\alpha+1)\mathcal{H}_\kappa(R_0) - (\sqrt{\kappa}R_0)^2 + 3\mathcal{H}_\kappa^2(R_0)\right)\frac{\cosh(\sqrt{\kappa}r)-1}{(\sqrt{\kappa}R_0)^2} - 1\right)\\ &\qquad \times \mathcal{H}_\kappa(R_0)\frac{\sinh^2(\sqrt{\kappa}r)}{(\sqrt{\kappa}R_0)^2}\omega_1^2 + 2\alpha c_\delta\mathcal{H}_\kappa(R_0)\frac{\cosh(\sqrt{\kappa}r)-1}{(\sqrt{\kappa}R_0)^2},\end{aligned} \tag{3.5.5}$$

where $c_\delta = (20/\delta)^{2\alpha}$. Let us now compute

$$\begin{aligned}\mathcal{M}^+v(x) &\leq \int_{B(x,\frac{R_0}{2})}\left(\Lambda\delta^+(v,x,z) - \lambda\delta^-(v,x,z)\right)\mathcal{K}_{s,\kappa}(d_{\mathbb{H}^n_\kappa}(z,x))\,\mathrm{d}\mu_{\mathbb{H}^n_\kappa}(z)\\ &\quad + \int_{\mathbb{H}^n\setminus B(x,\frac{R_0}{2})}\left(\Lambda\delta^+(v,x,z) - \lambda\delta^-(v,x,z)\right)\mathcal{K}_{s,\kappa}(d_{\mathbb{H}^n_\kappa}(z,x))\,\mathrm{d}\mu_{\mathbb{H}^n_\kappa}(z)\\ &=: I_1 + I_2.\end{aligned} \tag{3.5.6}$$

We take $\alpha = \alpha(n, \lambda, \Lambda, \sqrt{\kappa}R) > 0$ sufficiently large so that

$$\lambda(2\alpha+1) \fint_{\mathbb{S}^{n-1}} \omega_1^2 \,\mathrm{d}\sigma - \Lambda\mathcal{H}_\kappa(7R) > C_1\Lambda\mathcal{H}_\kappa(7R), \tag{3.5.7}$$

for some universal constant $C_1 > 0$ to be determined later. Then, by (3.5.5) we have

$$\begin{aligned}
&I_1\\
&\le \Lambda\alpha c_\delta \left((2\alpha+1)\mathcal{H}_\kappa(R_0) - (\sqrt{\kappa}R_0)^2 + 3\mathcal{H}_\kappa^2(R_0)\right)\mathcal{H}_\kappa(R_0)\\
&\qquad\times \int_{B_{R_0/2}} \frac{\cosh(\sqrt{\kappa}r)-1}{(\sqrt{\kappa}R_0)^2}\frac{\sinh^2(\sqrt{\kappa}r)}{(\sqrt{\kappa}R_0)^2}\omega_1^2\mathcal{K}_{s,\kappa}(d_{\mathbb{H}^n_\kappa}(z,x))\,\mathrm{d}\mu_{\mathbb{H}^n_\kappa}(z)\\
&\quad + \Lambda\alpha(2\alpha+1)c_\delta\left(2\alpha+2+2\mathcal{H}_\kappa(R_0)\right)\mathcal{H}_\kappa(R_0)\\
&\qquad\times \int_{B_{R_0/2}} \frac{\cosh(\sqrt{\kappa}r)-1}{(\sqrt{\kappa}R_0)^2}\frac{\sinh^2(\sqrt{\kappa}r)}{(\sqrt{\kappa}R_0)^2}\omega_1^2\mathcal{K}_{s,\kappa}(d_{\mathbb{H}^n_\kappa}(z,x))\,\mathrm{d}\mu_{\mathbb{H}^n_\kappa}(z)\\
&\quad + \alpha c_\delta \int_{B_{\frac{R_0}{2}}} \left(2\Lambda\mathcal{H}_\kappa(R_0)\frac{\cosh(\sqrt{\kappa}r)-1}{(\sqrt{\kappa}R_0)^2} - \lambda(2\alpha+1+\mathcal{H}_\kappa(R_0))\frac{\sinh^2(\sqrt{\kappa}r)}{(\sqrt{\kappa}R_0)^2}\omega_1^2\right)\\
&\qquad\times \mathcal{K}_{s,\kappa}(d_{\mathbb{H}^n_\kappa}(z,x))\,\mathrm{d}\mu_{\mathbb{H}^n_\kappa}(z)\\
&= \alpha c_\delta\left(I_{1,1}+I_{1,2}+I_{1,3}\right).
\end{aligned} \tag{3.5.8}$$

We use (3.5.7) to estimate $I_{1,3}$ as follows:

$$\begin{aligned}I_{1,3} &= \int_{B_{R_0/2}} \left(4\Lambda\mathcal{H}_\kappa(R_0) - 4\lambda(2\alpha+1+\mathcal{H}_\kappa(R_0))\cosh^2\left(\frac{\sqrt{\kappa}}{2}\right)\omega_1^2\right) \\ &\qquad\cdot \frac{\sinh^2(\sqrt{\kappa}\frac{r}{2})}{(\sqrt{\kappa}R_0)^2}\mathcal{K}_{s,\kappa}\,\mathrm{d}\mu_{\mathbb{H}^n_\kappa}(z) \\ &\leq \int_0^{R_0/2}\left(4\Lambda|\mathbb{S}^{n-1}|\mathcal{H}_\kappa(7R) - 4\lambda(2\alpha+1)\int_{\mathbb{S}^{n-1}}\omega_1^2\,\mathrm{d}\sigma\right) \\ &\qquad\cdot \frac{\sinh^2(\sqrt{\kappa}\frac{r}{2})}{(\sqrt{\kappa}R_0)^2}\mathcal{K}_{s,\kappa}\frac{\sinh^{n-1}(\sqrt{\kappa}r)}{\sqrt{\kappa}^{n-1}}\,\mathrm{d}r \\ &\leq -4C_1\Lambda\frac{\mathcal{H}_\kappa(7R)}{(\sqrt{\kappa}R_0)^2}\int_0^{R_0/2}|\mathbb{S}^{n-1}|\left(\frac{r}{2}\right)^2\mathcal{K}_{s,\kappa}(r)\frac{\sinh^{n-1}(\sqrt{\kappa}r)}{\sqrt{\kappa}^{n-1}}\,\mathrm{d}r \\ &= -C_1\Lambda\mathcal{H}_\kappa(7R)\frac{\mathcal{I}_{0,\kappa}(R_0/2)}{(\sqrt{\kappa}R_0)^2}.\end{aligned} \tag{3.5.9}$$

For $I_{1,1}$ and $I_{1,2}$, we observe that $\cosh(\sqrt{\kappa}r)-1 \leq C\kappa r^2$ and $\sinh^2(\sqrt{\kappa}r) \leq C\kappa r^2$ for $r \in [0, R_0/2]$, where C is some constant depending on $\sqrt{\kappa}R$. Thus, by using Lemma 3.9.2 and (3.2.7) we obtain

$$\begin{aligned}&I_{1,1}+I_{1,2} \\ &\leq C\Lambda\int_0^{R_0/2}\int_{\mathbb{S}^{n-1}}\frac{\cosh(\sqrt{\kappa}r)-1}{\kappa R_0^2}\frac{\sinh^2(\sqrt{\kappa}r)}{\kappa R_0^2}\omega_1^2\mathcal{K}_{s,\kappa}(r)\frac{\sinh^{n-1}(\sqrt{\kappa}r)}{\sqrt{\kappa}^{n-1}}\,\mathrm{d}\sigma\,\mathrm{d}r \\ &\leq C\Lambda(1-s)\frac{\sqrt{\kappa}^{1+s}}{R_0^4}\int_0^{R_0/2}r^{4-s}I_{\frac{n}{2}-1}\left(\frac{n-1}{2}\sqrt{\kappa}r\right)K_{\frac{n}{2}+s}\left(\frac{n-1}{2}\sqrt{\kappa}r\right)\,\mathrm{d}r \\ &\leq C\Lambda(1-s)\frac{\sqrt{\kappa}^{-4+2s}}{R_0^4}\int_0^{\frac{n-1}{4}\sqrt{\kappa}R_0}r^{4-s}I_{\frac{n}{2}-1}(r)K_{\frac{n}{2}+s}(r)\,\mathrm{d}r \\ &\leq C\Lambda(1-s)\frac{\sqrt{\kappa}^{-4+2s}}{R_0^4}A^{4-s}_{\frac{n}{2}+s,\frac{n}{2}-1}\left(\frac{n-1}{4}\sqrt{\kappa}R_0\right),\end{aligned} \tag{3.5.10}$$

where A is the function defined in (3.9.7).

On the other hand, by using the fact that v is bounded and Proposition 3.3.3, we obtain

$$I_2 \leq C\Lambda \frac{\mathcal{I}_{\infty,\kappa}(R_0/2)}{(R_0/2)^2} \leq C\Lambda \frac{1-s}{s} \mathcal{H}_\kappa(7R) \frac{\mathcal{I}_{0,\kappa}(7R)}{(7R)^2}. \tag{3.5.11}$$

Thus, (3.5.6), (3.5.8), (3.5.9), (3.5.10), (3.5.11), and Lemma 3.3.2 yield

$$\begin{aligned} &\frac{(7R)^2}{\mathcal{I}_{0,\kappa}(7R)} \mathcal{M}^+ v(x) \\ &\leq C\alpha\Lambda \Big[-C_1 \\ &\qquad + C(1-s) \frac{(7R)^2}{\mathcal{I}_{0,\kappa}(7R)} \frac{\sqrt{\kappa}^{-4+2s}}{R_0^4} A^{4-s}_{\frac{n}{2}+s,\frac{n}{2}-1} \left(\frac{n-1}{4} \sqrt{\kappa} R_0 \right) + \frac{1-s}{s} \Big] \mathcal{H}_\kappa(7R). \end{aligned} \tag{3.5.12}$$

Recall from Proposition 3.3.4 that $\mathcal{I}_0(7R) \to C$ as $s \to 1$. Moreover, the function $A^{4-s}_{\frac{n}{2}+s,\frac{n}{2}-1}$ does not blow up as $s \to 1$ by Lemma 3.9.4. Thus, the second and the third terms in (3.5.12) can be made as small as we want by choosing s_0 close to 1. Therefore, the proof is finished by assuming that we have taken α sufficiently large so that (3.5.4) holds. □

In the following lemma, we construct a barrier function for any $s \in (s_0, 1)$ for given $s_0 \in (0, 1)$.

Lemma 3.5.3. *Given $s_0 \in (0,1)$ and $\delta \in (0,1)$, there exist universal constants $\alpha > 0$ and $\eta \in (0, 1/4]$, depending only on n, λ, Λ, δ, $\sqrt{\kappa}R$, and s_0, such that the function*

$$v(x) = \max\left\{ -\left(\frac{\eta\delta}{20}\right)^{-2\alpha}, -\left(\frac{d_{\mathbb{H}^n_\kappa}(x,0)}{5R}\right)^{-2\alpha} \right\}$$

is a supersolution to

$$\frac{(7R)^2}{\mathcal{I}_{0,\kappa}(7R)} \mathcal{M}^+ v(x) + \Lambda \mathcal{H}_\kappa(7R) \leq 0, \tag{3.5.13}$$

for every $s_0 < s < 1$ and $x \in B_{5R} \setminus \overline{B}_{\delta R/4}$.

Proof. Let s_1 and α_1 be the s_0 and α in Lemma 3.5.2, respectively. When $s \in [s_1, 1)$, the desired result holds with α_1 and $\eta = 1/4$.

Let us now assume $s \in (s_0, s_1)$. For x with $R_0 := d_{\mathbb{H}^n_\kappa}(x, 0) \in (\delta R/4, 5R)$, we know that $v \in C^2(B(x, R_0/2))$ and that $\delta^+(v, x, z)$ is bounded for $z \in \mathbb{H}^n_\kappa \setminus B(x, R_0/2)$. Thus, we have

$$\frac{(7R)^2}{\mathcal{I}_{0,\kappa}(7R)}\mathcal{M}^+ v(x) \leq C\mathcal{H}_\kappa(7R) - \lambda \int_{\mathbb{H}^n_\kappa} \delta^-(v, x, z)\mathcal{K}_{s,\kappa}(d_{\mathbb{H}^n_\kappa}(z, x))\, \mathrm{d}\mu_{\mathbb{H}^n_\kappa}(z). \tag{3.5.14}$$

If we take $\alpha = \max\{\alpha_1, n/2\}$, then the function $\delta^-(-(d_{\mathbb{H}^n_\kappa}(\cdot, 0)/5R)^{-2\alpha}, x, z)$ is not integrable. Therefore, the last integral in (3.5.14) can be made arbitrarily large, by taking η small. In particular, we choose η so that (3.5.13) holds. $\square$

Corollary 3.5.4. *Let $\delta \in (0, 1)$ and assume $0 < s_0 \leq s < 1$. Then, there is a function v_δ such that*

$$\begin{cases} v_\delta \geq 0 & \text{in } \mathbb{H}^n_\kappa \setminus B_{5R}, \\ v_\delta \leq 0 & \text{in } B_{2R}, \\ \frac{(7R)^2}{\mathcal{I}_{0,\kappa}(7R)}\mathcal{M}^+ v_\delta + \Lambda\mathcal{H}_\kappa(7R) \leq 0 & \text{in } B_{5R} \setminus \overline{B}_{\delta R/4}, \\ \frac{(7R)^2}{\mathcal{I}_{0,\kappa}(7R)}\mathcal{M}^+ v_\delta \leq C\Lambda\mathcal{H}_\kappa(7R) & \text{in } B_{5R}, \\ v \geq -C & \text{in } B_{5R}, \end{cases}$$

for some universal constant $C > 0$, depending only on n, λ, Λ, δ, $\sqrt{\kappa}R$, and s_0.

Proof. Let α and η be the constants given in Lemma 3.5.3, and define a function $v_\delta(x) = \psi(d^2_{\mathbb{H}^n_\kappa}(x, 0)/R^2)$, where ψ is a smooth and increasing function on $[0, \infty)$ such that

$$\psi(t) = \left(\frac{3^2}{5^2}\right)^{-\alpha} - \left(\frac{t}{5^2}\right)^{-\alpha} \quad \text{if } t \geq (\eta\delta)^2.$$

We already know from Lemma 3.5.3 that $\frac{(7R)^2}{\mathcal{I}_{0,\kappa}(7R)}\mathcal{M}^+ v_\delta + \Lambda\mathcal{H}_\kappa(7R) \leq 0$ in

$B_{5R} \setminus \overline{B}_{\delta R/4}$. Finally, for $x \in \overline{B}_{\delta R/4}$, we have $|\delta(v_\delta, x, z)| \leq C\mathcal{H}_\kappa(7R) d_{\mathbb{H}^n_\kappa}(x,z)^2/R^2$ for $z \in B_R(x)$ and $|\delta(v_\delta, x, z)| \leq C$ for $z \in \mathbb{H}^n_\kappa \setminus B_R(x)$ with a uniform constant $C > 0$. Therefore, we conclude $\frac{(7R)^2}{\mathcal{I}_{0,\kappa}(7R)} \mathcal{M}^+ v_\delta \leq C\Lambda \mathcal{H}_\kappa(7R)$ in B_{5R}, with the help of Proposition 3.3.3. □

3.6 L^ε-estimate

In this section, we prove the so-called L^ε-estimate, which connects a pointwise estimate to an estimate in measure. Such a result forms a basis for the proofs of the Harnack inequality and Hölder estimate. From now on, we will prove the results only on $\mathbb{H}^n$ since Theorem 3.1.4 and Theorem 3.1.5 can be derived from the results on $\mathbb{H}^n$ by using a simple scaling argument. Moreover, since the essential results in the previous sections have been proved on $\mathbb{H}^n_\kappa$, one may easily reprove forthcoming results on $\mathbb{H}^n_\kappa$. We write $\mathcal{K}_s = \mathcal{K}_{n,s,1}$, $\mathcal{H} = \mathcal{H}_1$, $\mathcal{S} = \mathcal{S}_1$, and $\mathcal{T} = \mathcal{T}_1$ for simplicity in the sequel.

Lemma 3.6.1. *Assume* $0 < s_0 \leq s < 1$, *and let* $\delta \in (0,1)$. *If* $u \in C^2(B_{7R})$ *is a nonnegative function on* $\mathbb{H}^n$ *satisfying* $\frac{(7R)^2}{\mathcal{I}_0(7R)} \mathcal{M}^- u \leq \varepsilon_\delta$ *in* B_{7R} *and* $\inf_{B_{2R}} u \leq 1$, *then*

$$\frac{|\{u \leq M_\delta\} \cap B_{\delta R}|}{|B_{7R}|} \geq \mu_\delta,$$

where $\varepsilon_\delta > 0$, $\mu_\delta \in (0,1)$, *and* $M_\delta > 1$ *are universal constants depending only on* n, λ, Λ, δ, R *and* s_0.

Proof. Let v_δ be the barrier function constructed in Corollary 3.5.4 and define $w = u + v_\delta$. Then w satisfies $w \geq 0$ in $\mathbb{H}^n \setminus B_{5R}$, $\inf_{B_{2R}} w \leq 1$, and $\mathcal{M}^- w \leq \frac{\mathcal{I}_0(7R)}{(7R)^2} \varepsilon_\delta + \mathcal{M}^+ v_\delta$ in B_{5R}. By applying Theorem 3.1.2 to w with its envelope Γ_w, we have

$$|B_R| \leq \sum_j cF^n |Q^j_\alpha|,$$

where

$$F = \mathcal{S}(7R) \left(\Lambda \mathcal{H}(7R) + \frac{R^2}{\mathcal{I}_0(R)} \left(\frac{\mathcal{I}_0(7R)}{(7R)^2} \varepsilon_\delta + \max_{\overline{Q}^j_\alpha} \mathcal{M}^+ v_\delta \right) \right)_+$$

and $c = C\cosh^{n-1}(C\mathcal{T}^2(r_0)r_0F)(C\mathcal{T}^2(r_0)F)^{(n-1)\log\cosh(C\mathcal{T}^2(r_0)r_0F)}\mathcal{T}^{2n}(r_0)$. We obtain by Proposition 3.3.1

$$F \leq \mathcal{S}(7R)\left(\varepsilon_\delta + \Lambda\mathcal{H}(7R) + \frac{(7R)^2}{\mathcal{I}_0(7R)}\max_{\overline{Q}_\alpha^j}\mathcal{M}^+v_\delta\right)_+.$$

Since $\Lambda\mathcal{H}(7R)+\frac{(7R)^2}{\mathcal{I}_0(7R)}\mathcal{M}^+v_\delta \leq 0$ in $B_{5R}\backslash\overline{B}_{\delta R/4}$ and $\frac{(7R)^2}{\mathcal{I}_0(7R)}\mathcal{M}^+v_\delta \leq C\Lambda\mathcal{H}(7R)$ in B_{5R}, we have

$$|B_R| \leq C\varepsilon_\delta^n \sum_{\overline{Q}_\alpha^j\cap\overline{B}_{\delta R/4}=\emptyset}|Q_\alpha^j| + C\sum_{\overline{Q}_\alpha^j\cap\overline{B}_{\delta R/4}\neq\emptyset}|Q_\alpha^j|$$

for some universal constant $C > 0$, depending on R. By taking $\varepsilon_\delta > 0$ sufficiently small, we have

$$|B_{7R}| \leq C\sum_{\overline{Q}_\alpha^j\cap\overline{B}_{\delta R/4}\neq\emptyset}|Q_\alpha^j|.$$

By using Lemma 3.4.5 (iv), we obtain

$$|B_{7R}| \leq C\sum_{\overline{Q}_\alpha^j\cap\overline{B}_{\delta R/4}\neq\emptyset}|B(z_\alpha^j, 2r_0)\cap\{w \leq \Gamma_w + C\}|.$$

Whenever $\overline{Q}_\alpha^j\cap\overline{B}_{\delta R/4} \neq \emptyset$, the ball $B(z_\alpha^j, 2r_0)$ is contained in $B_{\delta R}$ if we have taken $\rho_0 = \delta/4$. Indeed, for $z \in B(z_\alpha^j, 2r_0)$

$$d_{\mathbb{H}^n}(z, 0) \leq d_{\mathbb{H}^n}(z, z_\alpha^j) + d_{\mathbb{H}^n}(z_\alpha^j, z_*) + d_{\mathbb{H}^n}(z_*, 0) \leq 2r_0 + r_0 + \delta R/4 < \delta R,$$

where z_* is a point in $\overline{Q}_\alpha^j\cap\overline{B}_{\delta R}$. By taking a subcover of $\{B(z_\alpha^j, 2r_0)\}$ with finite overlapping and using $v_\delta \geq -C$ in B_{5R}, we arrive at

$$|B_{7R}| \leq C|\{u \leq M_\delta\}\cap B_{\delta R}|$$

for some $M_\delta > 1$. Taking $\mu_\delta = 1/C$ finishes the proof. □

Lemma 3.6.1, together with the Calderón–Zygmund technique developed in [12], provides the following L^ε-estimate. As in [12], we fix $\delta = \frac{2c_1}{c_2}\delta_0$ and $\delta_1 = \delta_0(1-\delta_0)/2 \in (0,1)$. Let k_R be the integer satisfying

$$c_2\delta_0^{k_R-1} < R \leq c_2\delta_0^{k_R-2},$$

which is the generation of a dyadic cube whose size is comparable to that of some ball of radius R.

Lemma 3.6.2. *Assume* $0 < s_0 \leq s < 1$. *Let* ε_δ, μ_δ, *and* M_δ *be the constants in Lemma 3.6.1. Let* $u \in C^2(B_{7R})$ *be a nonnegative function on* $\mathbb{H}^n$ *satisfying* $\frac{(7R)^2}{\mathcal{I}_0(7R)}\mathcal{M}^- u \leq \varepsilon_\delta$ *in* B_{7R} *and* $\inf_{B_{\delta_1 R}} u \leq 1$. *If* Q_1 *is a dyadic cube of generation* k_R *such that* $\inf_{x\in Q_1} d_{\mathbb{H}^n}(x,0) \leq \delta_1 R$, *then*

$$|\{u > M_\delta^i\} \cap Q_1| \leq (1-c_\delta)^i |Q_1|.$$

for all $i = 1, 2, \cdots$. *As a consequence, we have*

$$|\{u > t\} \cap Q_1| \leq Ct^{-\varepsilon}|Q_1|, \quad t > 0,$$

for some universal constants $C > 0$ *and* $\varepsilon > 0$.

Corollary 3.6.3 (Weak Harnack inequality)**.** *Assume* $0 < s_0 \leq s < 1$. *If* $u \in C^2(B_{2R})$ *is a nonnegative function satisfying* $\mathcal{M}^- u \leq C_0$ *in* B_{2R}, *then*

$$\left(\fint_{B_R} u^p \,\mathrm{d}\mu_{\mathbb{H}^n} \right)^{1/p} \leq C \left(\inf_{B_R} u + C_0 \frac{R^2}{\mathcal{I}_0(R)} \right),$$

where $p > 0$ *and* $C > 0$ *are universal constants depending only on* n, λ, Λ, R, *and* s_0.

See, e.g., [12, Theorem 8.1] for the proof of Corollary 3.6.3.

3.7 Harnack inequality

The purpose of this section is to prove the Krylov–Safonov Harnack inequality by using Lemma 3.6.2. A simple scaling argument will provide Theorem 3.1.4.

Theorem 3.7.1. *Assume $0 < s_0 \leq s < 1$. If a nonnegative function $u \in C^2(B_{7R})$ satisfies*

$$\frac{(7R)^2}{\mathcal{I}_0(7R)}\mathcal{M}^- u \leq \varepsilon_0 \quad \textit{and} \quad \frac{(7R)^2}{\mathcal{I}_0(7R)}\mathcal{M}^+ u \geq -\varepsilon_0 \quad \textit{in } B_{7R}$$

and $\inf_{B_{\delta_1 R}} u \leq 1$, *then*

$$\sup_{B_{\delta_1 R/4}} u \leq C,$$

where $\varepsilon_0 > 0$ and $C > 0$ are universal constants depending only on n, λ, Λ, R, and s_0.

Proof. Let ε and ε_δ be the constants given in Lemma 3.6.2, and let $t > 0$ be the minimal value such that the following holds:

$$u(x) \leq h_t(x) := t\left(\frac{1}{\mathcal{D}}\left(1 - \frac{d_{\mathbb{H}^n}(x, z_0)}{\delta_1 R}\right)^{\log_2 \mathcal{D}}\right)^{-1/\varepsilon} \qquad \text{for all } x \in B_{\delta_1 R},$$

where for $\mathcal{D} = 2^n \cosh^{n-1}(2\delta_0 R)$. Since $\sup_{B_{\delta_1 R/4}} u \leq t\mathcal{D}^{-\frac{1}{\varepsilon}\log_2(3/8)}$, we can conclude the theorem once we show that $t \leq C$ for some universal constant C.

Let $x_0 \in B_{\delta_1 R}$ be a point such that $u(x_0) = h_t(x_0)$. Let $d = \delta_1 R - d_{\mathbb{H}^n}(x_0, 0)$, $r = d/2$, and $A = \{u > u(x_0)/2\}$, then we have

$$u(x_0) = h_t(x_0) = t\mathcal{D}^{1/\varepsilon}\left(\frac{2r}{\delta_1 R}\right)^{-\frac{1}{\varepsilon}\log_2 \mathcal{D}}.$$

We apply Lemma 3.6.2 to u to obtain

$$|A \cap Q_1| \leq C\left(\frac{u(x_0)}{2}\right)^{-\varepsilon}|Q_1| \leq Ct^{-\varepsilon}\frac{1}{\mathcal{D}}\left(\frac{r}{R}\right)^{\log_2 \mathcal{D}}|Q_1|,$$

where Q_1 is the unique dyadic cube of generation k_R that contains the point x_0.

We will show that there is a small constant $\theta > 0$ such that

$$|A^c \cap Q_2| \le \frac{1}{2}|Q_2|, \tag{3.7.1}$$

where $Q_2 \subset Q_1$ is the dyadic cube of generation $k_{\theta r/14}$ containing the point x_0, provided that t is large. However, when t is sufficiently large, we also have

$$\begin{aligned}|A \cap Q_2| \le |A \cap Q_1| &\le \frac{C}{t^\varepsilon \mathcal{D}} \left(\frac{r}{R}\right)^{\log_2 \mathcal{D}} |B(z, c_2 \delta_0^{k_R})| \\ &\le \frac{C}{t^\varepsilon} |B(z, c_1 \delta_0^{k_{r\theta/14}})| \le \frac{C}{t^\varepsilon}|Q_2| < \frac{1}{2}|Q_2|,\end{aligned}$$

where $B(z, c_1\delta_0^{k_{\theta r/14}})$ is a ball contained in Q_2. This contradicts to (3.7.1) and will lead us to a conclusion that t is uniformly bounded.

Let us now focus on proving (3.7.1). For every $x \in B(x_0, \theta r)$, we have

$$u(x) \le h_t(x) \le t\left(\frac{1}{\mathcal{D}}\left(\frac{d-\theta r}{\delta_1 R}\right)^{\log_2 \mathcal{D}}\right)^{-1/\varepsilon} = \left(1 - \frac{\theta}{2}\right)^{-\frac{1}{\varepsilon}\log_2 \mathcal{D}} u(x_0).$$

We define a function

$$v(x) := \left(1 - \frac{\theta}{2}\right)^{-\frac{1}{\varepsilon}\log_2 \mathcal{D}} u(x_0) - u(x).$$

Since we will apply Lemma 3.6.2, we need a function which is nonnegative on the whole space. Thus, we apply Lemma 3.6.2 to $w := v^+$ in $B(x_0, 7(\theta r/14))$.

For $x \in B(x_0, 7(\theta r/14))$, we have

$$
\begin{aligned}
\mathcal{M}^- w(x) &\le \mathcal{M}^- v(x) + \mathcal{M}^+ v_-(x) \\
&\le -\mathcal{M}^+ u(x) + \Lambda \int_{\mathbb{H}^n \setminus B(x_0,\theta r)} v^-(z) \mathcal{K}_s(d_{\mathbb{H}^n}(z,x))\,\mathrm{d}\mu_{\mathbb{H}^n}(z) \\
&\le \frac{\mathcal{I}_0(7R)}{(7R)^2}\varepsilon_0 \\
&\quad + \Lambda \int_{\mathbb{H}^n \setminus B(x_0,\theta r)} (u(z) - (1-\theta/2)^{-\frac{\log_2 \mathcal{D}}{\varepsilon}} u(x_0))^+ \mathcal{K}_s(d_{\mathbb{H}^n}(z,x))\,\mathrm{d}\mu_{\mathbb{H}^n}(z).
\end{aligned}
\tag{3.7.2}
$$

To compute the last integral in (3.7.2), we introduce another auxiliary function

$$
g_\beta(x) := \beta \left(1 - \frac{d_{\mathbb{H}^n}(x,0)^2}{R^2}\right)^+,
$$

with the largest number $\beta > 0$ satisfying $u \ge g_\beta$. From the assumption $\inf_{B_{\delta_1 R}} u \le 1$, we have $(1-\delta_1^2)\beta \le 1$. Let $x_1 \in B_R$ be a point where $u(x_1) = g_\beta(x_1)$. Since

$$
\begin{aligned}
\int_{\mathbb{H}^n} \delta^-(u,x_1,z)\mathcal{K}_s(d_{\mathbb{H}^n}(z,x_1))\,\mathrm{d}\mu_{\mathbb{H}^n}(z) &\le \int_{\mathbb{H}^n} \delta^-(g_\beta,x_1,z)\mathcal{K}_s(d_{\mathbb{H}^n}(z,x_1))\,\mathrm{d}\mu_{\mathbb{H}^n}(z) \\
&\le C\mathcal{H}(7R)\frac{\mathcal{I}_0(7R)}{(7R)^2},
\end{aligned}
$$

we obtain that

$$
\begin{aligned}
\varepsilon_0 &\ge \frac{(7R)^2}{\mathcal{I}_0(7R)} \mathcal{M}^- u(x_1) \\
&\ge \lambda \frac{(7R)^2}{\mathcal{I}_0(7R)} \int_{\mathbb{H}^n} \delta^+(u,x_1,z)\mathcal{K}_s(d_{\mathbb{H}^n}(z,x_1))\,\mathrm{d}\mu_{\mathbb{H}^n}(z) - C\Lambda\mathcal{H}(7R).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\int_{\mathbb{H}^n} (u(z)-c)^+ \mathcal{K}_s(d_{\mathbb{H}^n}(z,x_1))\,\mathrm{d}\mu_{\mathbb{H}^n}(z) \\
&\le \int_{\mathbb{H}^n} \delta^+(u,x_1,z)\mathcal{K}_s(d_{\mathbb{H}^n}(z,x_1))\,\mathrm{d}\mu_{\mathbb{H}^n}(z) \le C\mathcal{H}(7R)\frac{\mathcal{I}_0(7R)}{(7R)^2},
\end{aligned}
\tag{3.7.3}
$$

where $c := 1/(1-\delta_1^2)$.

If $u(x_0) \le c$, then we find an upper bound $t = u(x_0)(\delta_1 R/d)^{-\frac{1}{\varepsilon}\log_2 \mathcal{D}} \le c\delta_1^{-\frac{1}{\varepsilon}\log_2 \mathcal{D}}$, which finishes the proof. Otherwise, it follows from (3.7.2) and (3.7.3) that

$$\begin{aligned}
\mathcal{M}^- w(x) &\le \frac{\mathcal{I}_0(7R)}{(7R)^2}\varepsilon_0 + \Lambda \int_{\mathbb{H}^n \setminus B(x_0,\theta r)} (u(z)-c)_+ \mathcal{K}_s(d_{\mathbb{H}^n}(z,x))\, \mathrm{d}\mu_{\mathbb{H}^n}(z) \\
&\le \frac{\mathcal{I}_0(7R)}{(7R)^2}\varepsilon_0 + \Lambda M \int_{\mathbb{H}^n \setminus B(x_0,\theta r)} (u(z)-c)_+ \mathcal{K}_s(d_{\mathbb{H}^n}(z,x_1))\, \mathrm{d}\mu_{\mathbb{H}^n}(z) \\
&\le \frac{\mathcal{I}_0(7R)}{(7R)^2}\varepsilon_0 + CM\mathcal{H}(7R)\frac{\mathcal{I}_0(7R)}{(7R)^2},
\end{aligned}$$

where

$$M = \sup\left\{ \frac{\mathcal{K}_s(d_{\mathbb{H}^n}(z,x))}{\mathcal{K}_s(d_{\mathbb{H}^n}(z,x_1))} : x \in B(x_0,\theta r/2), x_1 \in B_R, z \in \mathbb{H}^n \setminus B(x_0,\theta r) \right\}.$$

Let $d = d_{\mathbb{H}^n}(z,x)$ and $d_1 = d_{\mathbb{H}^n}(z,x_1)$ for the sake of brevity. We recall from Lemma 3.9.2 that the kernel $\mathcal{K}_s$ is comparable with the function

$$R^{s-\frac{1}{2}} \sinh^{-\frac{n-1}{2}}(R) K_{\frac{n}{2}+s}\left(\frac{n-1}{2}R\right).$$

If $d \ge d_1$, then by [87, Chapter 4] we have

$$\begin{aligned}
\frac{\mathcal{K}_s(d)}{\mathcal{K}_s(d_1)} &\le C\left(\frac{d_1}{d}\right)^{1/2+s} \left(\frac{\sinh d_1}{\sinh d}\right)^{\frac{n-1}{2}} \frac{K_{n/2+s}(\frac{n-1}{2}d)}{K_{n/2+s}(\frac{n-1}{2}d_1)} \\
&\le C\left(\frac{\sinh d_1}{\sinh d}\right)^{\frac{n-1}{2}} e^{\frac{n-1}{2}(d_1-d)} \le C.
\end{aligned}$$

If $d < d_1$, by [87, Theorem 3.1] we have

$$\begin{aligned}
\frac{\mathcal{K}_s(d)}{\mathcal{K}_s(d_1)} &\le C\left(\frac{d_1}{d}\right)^{\frac{n+1}{2}+2s} \left(\frac{\sinh d_1}{\sinh d}\right)^{\frac{n-1}{2}} e^{\frac{n-1}{2}(d_1-d)} \\
&< \left(\frac{d_1}{d}\right)^{\frac{n+5}{2}} \left(\frac{\sinh d_1}{\sinh d}\right)^{\frac{n-1}{2}} e^{\frac{n-1}{2}(d_1-d)}.
\end{aligned}$$

Since $d_1 - d \le d(x, x')$, we can bound $\mathcal{K}_s(d)/\mathcal{K}_s(d_1)$ by a constant depending on R. Thus, for any cases the ratio $\mathcal{K}_s(d)/\mathcal{K}_s(d_1)$ is bounded by a universal constant depending on R which is independent of s. By using Lemma 3.3.2, we arrive at

$$\frac{(\theta r/2)^2}{\mathcal{I}_0(\theta r/2)} \mathcal{M}^-_{\mathcal{L}_0} w \le C \frac{\mathcal{I}_0(R)/R^2}{\mathcal{I}_0(\theta r/2)/(\theta r/2)^2} \mathcal{H}(7R) \le C$$

in $B(x_0, 7(\theta r/14))$.

Let $Q_2 \subset Q_1$ be the dyadic cube of generation $k_{\theta r/14}$ containing the point x_0. Then by Lemma 3.6.2, we have

$$\begin{aligned}|\{u < u(x_0)/2\} \cap Q_2| &= |\{w > \left((1-\theta/2)^{-s} - 1/2\right) u(x_0)\} \cap Q_2| \\ &\le \frac{C|Q_2|}{\left((1-\theta/2)^{-s} - 1/2\right)^{\varepsilon} u(x_0)^{\varepsilon}} \left(\inf_{B(x_0, \delta_1 \theta r/14)} w + C \right)^{\varepsilon}.\end{aligned}$$

We can make the quantity $(1-\theta/2)^{-s} - 1/2$ bounded away from 0 by taking $\theta > 0$ sufficiently small. Recalling that $w(x_0) = ((1-\theta/2)^{-s} - 1)u(x_0)$, we obtain

$$|\{u < u(x_0)/2\} \cap Q_2| \le C|Q_2| \left(((1-\theta/2)^{-s} - 1)^{\varepsilon} + \left(\frac{C}{u(x_0)} \right)^{\varepsilon} \right).$$

We choose a constant $\theta > 0$ sufficiently small so that

$$C\left((1-\theta/2)^{-s} - 1\right)^{\varepsilon} \le \frac{1}{4}.$$

If $t > 0$ is sufficiently large so that $C(C/u(x_0))^{\varepsilon} < 1/4$, then we arrive at (3.7.1). Therefore, t is uniformly bounded and the desired result follows. □

3.8 Hölder estimates

In this section, the following Hölder regularity result is proved. Theorem 3.1.5 follows from simple scaling and covering arguments.

Lemma 3.8.1. *Assume $0 < s_0 \le s < 1$. There is a universal constant ε_0 such that if $u \in C^2(B_{7R})$ is a function such that $|u| \le \frac{1}{2}$ in B_{7R} and*

$$\frac{(7R)^2}{\mathcal{I}_0(7R)}\mathcal{M}^+ u \ge -\varepsilon_0 \quad \textit{and} \quad \frac{(7R)^2}{\mathcal{I}_0(7R)}\mathcal{M}^- u \le \varepsilon_0 \quad \textit{in } B_{7R},$$

then $u \in C^\alpha$ at $0 \in \mathbb{H}^n$ with an estimate

$$|u(x) - u(0)| \le CR^{-\alpha} d_{\mathbb{H}^n}(x, 0)^\alpha,$$

where $\alpha \in (0,1)$ and $C > 0$ are universal constants depending only on n, λ, Λ, R, and s_0.

Proof. Let $R_k := 7 \cdot 4^{-k} R$ and $B_k := B_{R_k}$. It suffices to construct an increasing sequence $\{m_k\}_{k \ge 0}$ and a decreasing sequence $\{M_k\}_{k \ge 0}$ such that $m_k \le u \le M_k$ in B_k and $M_k - m_k = 4^{-\alpha k}$. We initially choose $m_0 = -1/2$ and $M_0 = 1/2$ for the case $k = 0$. Let us assume that we have sequences up to m_k and M_k and find m_{k+1} and M_{k+1}.

For $x \in B_{2R_{k+1}}$, let Q_1 be a dyadic cube of generation $k_{R_{k+1}/7}$. In Q_1, either $u > (M_k + m_k)/2$ or $u \le (M_k + m_k)/2$ in at least half of the points in measure. We assume

$$|\{u > (M_k + m_k)/2\} \cap Q_1| \ge \frac{1}{2}|Q_1|. \tag{3.8.1}$$

A function defined by

$$v(x) := \frac{u(x) - m_k}{(M_k - m_k)/2}$$

satisfies $v \ge 0$ in B_k by the induction hypothesis. To apply Lemma 3.6.2, let us consider a function $w := v^+$, which satisfies

$$|\{w > 1\} \cap Q_1| \ge \frac{1}{2}|Q_1| \tag{3.8.2}$$

by (3.8.1). Since $\frac{(7R)^2}{\mathcal{I}_0(7R)}\mathcal{M}^- v \leq 2\varepsilon_0/(M_k - m_k)$ in B_{7R}, we have

$$\begin{aligned}\frac{R_{k+1}^2}{\mathcal{I}_0(R_{k+1})}\mathcal{M}^- w &\leq \frac{R_{k+1}^2}{\mathcal{I}_0(R_{k+1})}(\mathcal{M}^- v + \mathcal{M}^+ v^-)\\ &\leq \frac{2\varepsilon_0}{M_k - m_k}\frac{R_{k+1}^2}{\mathcal{I}_0(R_{k+1})}\frac{\mathcal{I}_0(7R)}{(7R)^2} + \frac{R_{k+1}^2}{\mathcal{I}_0(R_{k+1})}\mathcal{M}^+ v^-\end{aligned}$$

in $B_{3R_{k+1}}$. By Lemma 3.3.2, we have

$$\frac{R_{k+1}^2}{\mathcal{I}_0(R_{k+1})}\frac{\mathcal{I}_0(7R)}{(7R)^2} \leq \left(\frac{R_{k+1}}{7R}\right)^s = 4^{-(k+1)s} < 4^{-ks_0}.$$

Thus, we obtain

$$\frac{R_{k+1}^2}{\mathcal{I}_0(R_{k+1})}\mathcal{M}^- w \leq 2\varepsilon_0 + \frac{R_{k+1}^2}{\mathcal{I}_0(R_{k+1})}\mathcal{M}^+ v^-,$$

by assuming $\alpha < s_0$.

For $\mathcal{M}^+ v^-$, we use an inequality $v(z) \geq -2((d_{\mathbb{H}^n}(z,0)/R_k)^\alpha - 1)$, $z \in \mathbb{H}^n \setminus B_k$, which follows from the definition of v and the properties of sequences M_k and m_k. Then, for any $x_0 \in B_{3R_{k+1}}$, we have

$$\begin{aligned}\mathcal{M}^+ v^-(x_0) &\leq \Lambda \int_{\mathbb{H}^n \setminus B_k} v^-(z)\mathcal{K}_s(d_{\mathbb{H}^n}(x_0,z))\,\mathrm{d}\mu_{\mathbb{H}^n}(z)\\ &\leq 2\Lambda \int_{\mathbb{H}^n \setminus B_k}\left(\left(\frac{d_{\mathbb{H}^n}(z,0)}{R_k}\right)^\alpha - 1\right)\mathcal{K}_s(d_{\mathbb{H}^n}(x_0,z))\,\mathrm{d}\mu_{\mathbb{H}^n}(z).\end{aligned}$$

Since $d_{\mathbb{H}^n}(z,0) \leq 4d_{\mathbb{H}^n}(z,x_0)$, we obtain

$$\begin{aligned}&\frac{R_{k+1}^2}{\mathcal{I}_0(R_{k+1})}\mathcal{M}^+ v^-\\ &\leq \frac{2\Lambda R_{k+1}^2}{\mathcal{I}_0(R_{k+1})}\int_{\mathbb{H}^n \setminus B(x_0,R_{k+1})}\left(\left(\frac{d_{\mathbb{H}^n}(z,x_0)}{R_{k+1}}\right)^\alpha - 1\right)\mathcal{K}_s(d_{\mathbb{H}^n}(x_0,z))\,\mathrm{d}\mu_{\mathbb{H}^n}(z).\end{aligned} \tag{3.8.3}$$

Let I be the right-hand side of (3.8.3). By the dominated convergence theorem, we know that I converges to 0 as $\alpha \to 0$ for each s. Let $\alpha_s > 0$ be the constant such that $I \le \varepsilon_0$ whenever $\alpha \le \alpha_s$. Since I is continuous with respect to α and s, α_s is chosen continuously. Thus, the quantity $\min_{s\in[s_0,1]} \alpha_s$ is positive and depends on s_0 (not on s). By choosing $\alpha = \min_{s\in[s_0,1]} \alpha_s$, we obtain

$$\frac{R_{k+1}^2}{\mathcal{I}_0(R_{k+1})}\mathcal{M}^- w \le 3\varepsilon_0$$

in $B(x, 7(R_{k+1}/7))$ for $x \in B_{2R_{k+1}}$. Therefore, by Lemma 3.6.2 and (3.8.2), we have

$$\frac{1}{2}|Q_1| \le |\{w > 1\} \cap Q_1| \le C|Q_1| \left(w(x) + 3\varepsilon_0\right)^{\varepsilon},$$

or equivalently, $\theta \le w(x) + 3\varepsilon_0$ for some universal constant $\theta > 0$. By taking $\varepsilon_0 < \theta/6$, we arrive at $w \ge \theta/2$ in $B_{2R_{k+1}}$. Thus, if we set $M_{k+1} = M_k$ and $m_{k+1} = M_k - 4^{-\alpha(k+1)}$, then

$$M_{k+1} \ge u \ge m_k + \frac{M_k - m_k}{4}\theta = M_k - \left(1 - \frac{\theta}{4}\right) 4^{-\alpha k} \ge m_{k+1}$$

in B_{k+1}.

When (3.8.1) does not hold, a similar proof can be made by using $\frac{(7R)^2}{\mathcal{I}_0(7R)}\mathcal{M}^+ u \ge -\varepsilon_0$ instead of $\frac{(7R)^2}{\mathcal{I}_0(7R)}\mathcal{M}^- u \le \varepsilon_0$. □

3.9 Appendix

3.9.1 Special functions

The equation

$$\rho^2 \frac{\mathrm{d}^2 y}{\mathrm{d}\rho^2} + \rho \frac{\mathrm{d}y}{\mathrm{d}\rho} - (\rho^2 + \nu^2) y = 0$$

is called the modified Bessel's equation, and its solutions are given by

$$I_\nu(\rho) = \sum_{j=0}^{\infty} \frac{1}{j!\Gamma(\nu + j + 1)} \left(\frac{\rho}{2}\right)^{2j+\nu} \quad \text{and} \quad K_\nu(\rho) = \frac{\pi}{2}\frac{I_{-\nu}(\rho) - I_\nu(\rho)}{\sin \nu\pi}.$$

They are called modified Bessel functions of the first and second kind, respectively. They satisfy the recurrence relations

$$K_{\nu+1} - K_{\nu-1} = \frac{2\nu}{R} K_\nu, \quad I_{\nu-1} - I_{\nu+1} = \frac{2\nu}{R} I_\nu,$$

and the following system of first-order differential equations:

$$\begin{cases} I'_\nu &= I_{\nu-1} - \frac{\nu}{R} I_\nu, \\ I'_\nu &= I_{\nu+1} + \frac{\nu}{R} I_\nu, \end{cases} \quad \text{and} \quad \begin{cases} K'_\nu &= -K_{\nu-1} - \frac{\nu}{R} K_\nu, \\ K'_\nu &= -K_{\nu+1} + \frac{\nu}{R} K_\nu. \end{cases} \tag{3.9.1}$$

Moreover, the following asymptotic behavior is well known. For further properties of special functions, the reader may consult the book [91].

Lemma 3.9.1. *The asymptotic behavior of the modified Bessel functions are given by*

$$\begin{aligned} I_\nu(\rho) &\sim \frac{1}{\Gamma(\nu+1)} \left(\frac{\rho}{2}\right)^\nu, \quad \nu \neq -1, -2, \cdots, \\ K_\nu(\rho) &\sim \frac{1}{2}\Gamma(\nu) \left(\frac{\rho}{2}\right)^{-\nu}, \quad \operatorname{Re}\nu > 0, \end{aligned}$$

as $\rho \to 0$, *and*

$$\begin{aligned} I_\nu(\rho) &\sim \frac{e^\rho}{\sqrt{2\pi\rho}}, \\ K_\nu(\rho) &\sim \sqrt{\frac{\pi}{2\rho}} e^{-\rho}, \end{aligned}$$

as $\rho \to \infty$.

In this paper, some special functions involving the modified Bessel functions appear. Let us first study the kernel of the fractional Laplacian on the hyperbolic spaces.

Lemma 3.9.2. *There exist constants* $C_1, C_2 > 0$, *depending only on* n, *such that*

$$C_1 \leq \frac{\sinh^{n-1}(R) \mathcal{K}_{n,s,1}(R)}{s(1-s) R^{-s} I_{n/2-1}(\frac{n-1}{2}R) K_{n/2+s}(\frac{n-1}{2}R)} \leq C_2.$$

The proof of Lemma 3.9.2 is divided into two parts: the odd and even dimensional cases. For the even dimensional case, we need the following lemma.

Lemma 3.9.3. *Let $a > 0$ and $\nu > -\frac{n-1}{2}$. Then*

$$\int_R^\infty \frac{\sinh^{-n/2+1} r}{\sqrt{\cosh r - \cosh R}} r^{-\nu} K_{n/2+\nu}(ar)\,\mathrm{d}r \sim \sqrt{\frac{\pi}{2}} \frac{\Gamma(\nu + \frac{n-1}{2})}{\Gamma(\nu + \frac{n}{2})} R^{-\nu} \sinh^{-n/2+1}(R) K_{n/2+\nu}(aR)$$

as $R \to 0^+$ up to dimensional constants.

Proof. By the change of variables $r = Rt$, we have

$$\begin{aligned}
&\int_R^\infty \frac{1}{\sqrt{\cosh r - \cosh R}} \frac{r^{-\nu}}{R^{-\nu}} \frac{\sinh^{-n/2+1} r}{\sinh^{-n/2+1} R} \frac{K_{n/2+\nu}(ar)}{K_{n/2+\nu}(aR)}\,\mathrm{d}r \\
&= \int_1^\infty \frac{Rt^{-\nu}}{\sqrt{\cosh(Rt) - \cosh(R)}} \frac{\sinh^{-n/2+1}(Rt)}{\sinh^{-n/2+1}(R)} \frac{K_{n/2+\nu}(aRt)}{K_{n/2+\nu}(aR)}\,\mathrm{d}t.
\end{aligned}$$

We define for each $R \in (0, 1)$ a function f_R by

$$f_R(t) = \frac{Rt^{-\nu}}{\sqrt{\cosh(Rt) - \cosh(R)}} \frac{\sinh^{-n/2+1}(Rt)}{\sinh^{-n/2+1}(R)} \frac{K_{n/2+\nu}(aRt)}{K_{n/2+\nu}(aR)}, \quad t \in (1, \infty).$$

Note that

$$\frac{\cosh(Rt) - \cosh(R)}{R^2} \ge \frac{1}{2}(t^2 - 1) \quad \text{and} \quad \sinh(Rt) \ge \sinh(R)t.$$

Moreover, by [65, Equation (2.17)], we have

$$\frac{K_{n/2+\nu}(aRt)}{K_{n/2+\nu}(aR)} \le t^{-\frac{n}{2}-\nu}.$$

Thus, f_R is bounded from above by a function

$$f(t) = \frac{t^{-n-2\nu+1}}{\sqrt{(t^2-1)/2}},$$

which is integrable on $(0, \infty)$. Indeed, by the change of variables $t^2 - 1 = \tau$, we obtain

$$\begin{aligned}\int_1^\infty f(t)\,\mathrm{d}t &= \frac{1}{\sqrt{2}}\int_0^\infty \frac{\tau^{-1/2}}{(1+\tau)^{n/2+\nu}}\mathrm{d}\tau = \frac{1}{\sqrt{2}} B\left(\frac{1}{2}, \nu + \frac{n-1}{2}\right) \\ &= \sqrt{\frac{\pi}{2}}\frac{\Gamma(\nu + \frac{n-1}{2})}{\Gamma(\nu + \frac{n}{2})} < \infty,\end{aligned}$$

where B is Euler's Beta Integral (see [91, 5.12.3]).

For fixed $t \in (1, \infty)$, we have

$$\frac{\cosh(Rt) - \cosh(R)}{R^2} \to \frac{1}{2}(t^2 - 1), \quad \frac{\sinh(Rt)}{\sinh(R)} \to t \quad \text{and} \quad \frac{K_{n/2+\nu}(aRt)}{K_{n/2+\nu}(aR)} \to t^{-n/2-\nu}$$

as $R \to 0^+$. Hence, we obtain $\lim_{R\to 0} f_R(t) = f(t)$. Therefore, the Lebesgue dominated convergence theorem concludes the lemma. $\square$

Proof of Lemma 3.9.2. Observe that the function $\sqrt{R} I_{n/2-1}(\frac{n-1}{2}R)$ is comparable to the function $\sinh^{\frac{n-1}{2}}(R)$ up to dimensional constants by Lemma 3.9.1. Thus, it suffices to prove that $\sinh^{\frac{n-1}{2}}(R)\mathcal{K}_{n,s,1}(R)$ is comparable to $s(1 - s)R^{-1/2-s}K_{n/2+s}(\frac{n-1}{2}R)$ up to dimensional constants.

Let us first consider the odd dimensional case $n = 2m + 1$. It is sufficient to prove

$$\begin{aligned}C_3 \le G(R, s) &:= \frac{R^{1/2+s}\sinh^m R}{K_{m+1/2+s}(mR)}\left(\frac{-\partial_R}{\sinh R}\right)^m \mathscr{K}_{1/2+s,m}(R) \\ &\le C_4, \quad R > 0, s \in [0, 1], \end{aligned} \tag{3.9.2}$$

by recalling (3.1.2) and observing $c_{n,s} \le C(n)s(1 - s)$. By Lemma 3.9.1, the modified Bessel function $K_\nu(\rho)$ is asymptotic to $\sqrt{\frac{\pi}{2\rho}}e^{-\rho}$ as $\rho \to \infty$ uniformly with respect to $\nu \in [1/2, n/2 + 1]$. Moreover, its i-th derivative is asymptotic to $\rho^{-1/2}e^{-\rho}$ up to constants depending only on n and i by (3.9.1) in the same range of ν. Therefore, G is bounded from above and below near $R = \infty$ by positive constants depending only on n.

On the other hand, G is also bounded near $R = 0$ by a dimensional constant since $K_\nu(\rho)$ is asymptotic to $2^{\nu-1}\Gamma(\nu)\rho^{-\nu}$ as $\rho\to 0$ and $2^{\nu-1}\Gamma(\nu)$ is bounded from above and below by dimensional constants when $\nu\in[1/2, n/2+1]$. Since G is continuous, we conclude (3.9.2).

Let us next consider the even dimensional case $n=2m$. In this case, we consider

$$H(R,s) := \int_R^\infty \frac{\sinh r}{\sqrt{\cosh r-\cosh R}}\left(\frac{-\partial_r}{\sinh r}\right)^m \mathscr{K}_{\frac{1+2s}{2},\frac{n-1}{2}}(r)\,\mathrm{d}r$$

We first prove that $R^{1+s}e^{(n-1)R}H(R,s)$ is bounded from above and below near $R=\infty$ by positive constants depending only on n. As R is sufficiently close to ∞, we have

$$\begin{aligned}
H(R,s) &\le C\int_R^\infty \frac{e^r}{\sqrt{\sinh(\frac{r-R}{2})}\sqrt{\sinh(\frac{r+R}{2})}} r^{-1-s}e^{(-n+1/2)r}\,\mathrm{d}r\\
&\le C\frac{1}{\sqrt{\sinh R}}\int_0^\infty \frac{1}{\sqrt{\sinh\frac{t}{2}}}(t+R)^{-1-s}e^{(-n+3/2)(t+R)}\,\mathrm{d}t\\
&\le CR^{-1-s}e^{-(n-1)R}\int_0^\infty \frac{1}{\sqrt{\sinh\frac{t}{2}}}dt\\
&\le CR^{-1-s}e^{-(n-1)R}
\end{aligned}$$

and

$$\begin{aligned}
H(R,s) &= \int_R^\infty 2\sinh r\sqrt{\cosh r-\cosh R}\left(\frac{-\partial_r}{\sinh r}\right)^{\frac{n+2}{2}} \mathscr{K}_{\frac{1+2s}{2},\frac{n-1}{2}}(r)\,\mathrm{d}r\\
&\ge C\int_R^\infty e^r\sqrt{\sinh\frac{r-R}{2}}\sqrt{\sinh\frac{r+R}{2}}r^{-1-s}e^{-(n+1/2)r}\,\mathrm{d}r\\
&\ge C\sqrt{\sinh R}\int_0^\infty \sqrt{\sinh\frac{t}{2}}(t+R)^{-1-s}e^{-(n-1/2)(t+R)}\,\mathrm{d}t\\
&\ge CR^{-1-s}e^{-(n-1)R}\int_0^\infty \sqrt{\sinh\frac{t}{2}}(1+t)^{-2}e^{-(n-1/2)t}\,\mathrm{d}t\\
&\ge CR^{-1-s}e^{-(n-1)R}
\end{aligned}$$

for some dimensional constants $C > 0$.

Finally, we prove that $R^{n+2s}H(R,s)$ is bounded from above and below near $R = 0$ by positive dimensional constants. By similar arguments as in the odd dimensional case, the function H is comparable to

$$\int_R^\infty \frac{\sinh^{-m+1} r}{\sqrt{\cosh r - \cosh R}} r^{-1/2-s} K_{m+1/2+s}\left(\frac{2m-1}{2} r\right) \mathrm{d}r,$$

and hence to

$$R^{-1/2-s} \sinh^{-m+1}(R) K_{m+1/2+s}\left(\frac{2m-1}{2} R\right)$$

by Lemma 3.9.3, up to dimensional constants. The desired result now follows from Lemma 3.9.1. □

Another special function involving the modified Bessel functions used in this paper is given as follows: we define the definite integral

$$A^\beta_{\mu,\nu} = \int \rho^\beta I_\mu K_\nu \,\mathrm{d}\rho. \tag{3.9.3}$$

Lemma 3.9.4. *Let $k \in \mathbb{N}$ and $\beta = \mu - \nu + 2k + 1 \neq 0, 1, \dots, k$. Then*

$$A^\beta_{\mu,\nu} = \sum_{j=0}^k \frac{(-1)^j k!/(k-j)!}{2(\beta-k)\cdots(\beta-k+j)} \rho^{\beta+1} \left(I_{\mu+j}K_{\nu-j} + I_{\mu+j+1}K_{\nu-j-1}\right).$$

Proof. By using (3.9.1) and the integration by parts, we obtain

$$A^\beta_{\mu,\nu} = \frac{1}{\beta+\mu-\nu+1}\left(\rho^{\beta+1} I_\mu K_\nu + A^{\beta+1}_{\mu,\nu-1} - A^{\beta+1}_{\mu+1,\nu}\right) \quad \text{and} \tag{3.9.4}$$

$$A^\beta_{\mu,\nu} = \frac{1}{\beta-\mu+\nu+1}\left(\rho^{\beta+1} I_\mu K_\nu - A^{\beta+1}_{\mu-1,\nu} + A^{\beta+1}_{\mu,\nu+1}\right). \tag{3.9.5}$$

By plugging (3.9.5), with μ, ν replaced by $\mu+1, \nu-1$, into (3.9.4), we obtain

$$A^\beta_{\mu,\nu} = \frac{1}{\beta+\mu-\nu+1}\rho^{\beta+1}(I_\mu K_\nu + I_{\mu+1}K_{\nu-1}) - \frac{\beta-\mu+\nu-1}{\beta+\mu-\nu+1} A^\beta_{\mu+1,\nu-1}. \tag{3.9.6}$$

The desired result follows by iterating (3.9.6). □

We also define the indefinite integral

$$A^{\beta}_{\mu,\nu}(R) = \int_0^R \rho^{\beta} I_\mu K_\nu \, \mathrm{d}\rho. \tag{3.9.7}$$

Note that it is well defined by Lemma 3.9.1, provided that $-\mu \notin \mathbb{N}$, $\mu > 0$, and $\beta + \mu - \nu + 1 > 0$.

Chapter 4

The fractional p-Laplacian on hyperbolic spaces

4.1 Introduction

Operators of fractional-order have been studied extensively not only on the Euclidean spaces [42] but also on various spaces such as Riemannian manifolds [3, 7, 24, 38, 54, 55, 61], metric measure spaces [21, 27, 49, 56, 58], discrete models [32], Lie groups [20, 34, 46, 47], Wiener spaces [20], and so on. On the Euclidean spaces, there are several equivalent definitions of the fractional Laplacian [86] due to the simple structure of the spaces. In contrast to the case of Euclidean spaces, not all definitions are equivalent on general spaces. For instance, one can study a regional-type operator [61] or a spectral-type operator [100] on Riemannian manifolds. Moreover, some definitions, such as the one using the Fourier transform, do not even work on general Riemannian manifolds and metric measure spaces. However, several representations for the fractional Laplacians on some Riemannian manifolds, such as hyperbolic spaces and spheres, have been established [7, 38] by means of rich structures of the spaces.

The aim of this paper is two-fold. We first generalize representation formulas in [7] to the nonlinear regime on the hyperbolic spaces. Precisely, we define the fractional p-Laplacian $(-\Delta_{\mathbb{H}^n})^s_p$ for $n \in \mathbb{N}$, $0 < s < 1$, and $p > 1$ by using the heat semigroup and establish the singular integral representation and the Caffarelli–Silvestre extension. Note that the definition via the Fourier transform is not available because of the nonlinearity of the operator. We next study the pointwise convergence of $(-\Delta_{\mathbb{H}^n})^s_p u(x)$ as $s \to 1^-$ using the singular integral representation. For this purpose, we compute the explicit values of the normalizing constants in the singular integral representation. This explicit value was available only when $n = 3$ and $p = 2$, see [70].

Let us define the fractional p-Laplacian on the hyperbolic spaces. We adopt the definition proposed in [39, Section 8.2], which is a nonlinear extension of the Bochner's definition [9]. See also [100]. To this end, let $\{e^{t\Delta_{\mathbb{H}^n}}\}_{t\geq 0}$ be the heat semigroup generated by the Laplacian $\Delta_{\mathbb{H}^n}$ on hyperbolic spaces. That is, for a given function $f : \mathbb{H}^n \to \mathbb{R}$ we denote by $e^{t\Delta_{\mathbb{H}^n}}[f](x)$ the solution $w(x,t)$ of a Cauchy problem

$$\begin{cases} \partial_t w(x,t) - \Delta_{\mathbb{H}^n} w(x,t) = 0, & x \in \mathbb{H}^n, t > 0, \\ w(x,0) = f(x), & x \in \mathbb{H}^n. \end{cases} \tag{4.1.1}$$

We define $C^2_b(\mathbb{H}^n)$ by the space of bounded C^2 functions on $\mathbb{H}^n$.

Definition 4.1.1. Let $n \in \mathbb{N}$, $s \in (0,1)$, and $p > 1$. Let $u \in C^2_b(\mathbb{H}^n)$ and $x \in \mathbb{H}^n$. If $p \in (1, \frac{2}{2-s}]$, assume in addition $\nabla u(x) \neq 0$. The *fractional p-Laplacian on $\mathbb{H}^n$* is defined by

$$(-\Delta_{\mathbb{H}^n})^s_p u(x) = C_1 \int_0^\infty e^{t\Delta_{\mathbb{H}^n}}[\Phi_p(u(x) - u(\cdot))](x) \frac{\mathrm{d}t}{t^{1+\frac{sp}{2}}},$$

where

$$C_1 = \frac{p}{2} \frac{\sqrt{\pi}/2}{\Gamma(\frac{p+1}{2})} \frac{2^{s(2-p)}}{|\Gamma(-s)|} \tag{4.1.2}$$

and $\Phi_p(r) = |r|^{p-2} r$.

The constant C_1 in (4.1.2) is chosen so that the pointwise convergence $\lim_{s\nearrow 1}(-\Delta_{\mathbb{H}^n})^s_p u(x) = (-\Delta_{\mathbb{H}^n})_p u(x)$ holds, see Theorem 4.1.4. The same constant is used in the case of Euclidean spaces [39]. Moreover, this choice is in accordance with the constant in the case $p = 2$, see [7].

The first result is the pointwise integral representation of the fractional p-Laplacian with singular kernels. Note that the hyperbolic geometry is distinguished from the Euclidean geometry only when $n \geq 2$.

Theorem 4.1.2. *Let $n \geq 2$, $s \in (0,1)$, and $p > 1$. Let $u \in C_b^2(\mathbb{H}^n)$ and $x \in \mathbb{H}^n$. If $p \in (1, \frac{2}{2-s}]$, assume in addition $\nabla u(x) \neq 0$. Then, the fractional p-Laplacian on $\mathbb{H}^n$ has the pointwise representation*

$$(-\Delta_{\mathbb{H}^n})_p^s u(x) = c_{n,s,p}\,\mathrm{P.V.}\int_{\mathbb{H}^n} |u(x)-u(\xi)|^{p-2}(u(x)-u(\xi))\mathcal{K}_{n,s,p}(d(x,\xi))\,\mathrm{d}\xi \tag{4.1.3}$$

with the kernel $\mathcal{K}_{n,s,p}$ given by

$$\mathcal{K}_{n,s,p}(\rho) = C_2\left(\frac{-\partial_\rho}{\sinh\rho}\right)^{\frac{n-1}{2}}\left(\rho^{-\frac{1+sp}{2}}K_{\frac{1+sp}{2}}\left(\frac{n-1}{2}\rho\right)\right)$$

when $n \geq 3$ is odd and

$$\mathcal{K}_{n,s,p}(\rho) = C_2\int_\rho^\infty \frac{\sinh r}{\sqrt{\pi}\sqrt{\cosh r - \cosh\rho}}\left(\frac{-\partial_r}{\sinh r}\right)^{\frac{n}{2}}\left(r^{-\frac{1+sp}{2}}K_{\frac{1+sp}{2}}\left(\frac{n-1}{2}r\right)\right)\mathrm{d}r$$

when $n \geq 2$ is even, where

$$c_{n,s,p} = \frac{p}{2}\frac{\sqrt{\pi}/2}{\Gamma(\frac{p+1}{2})}\frac{2^{2s}\Gamma(\frac{n+sp}{2})}{\pi^{\frac{n}{2}}|\Gamma(-s)|}, \quad C_2 = \frac{1}{2^{\frac{n-2+sp}{2}}\Gamma(\frac{n+sp}{2})}\left(\frac{n-1}{2}\right)^{\frac{1+sp}{2}},$$

and K_ν is the modified Bessel function of the second kind. Moreover, the kernel $\mathcal{K}_{n,s,p}$ is positive and has the asymptotic behavior

$$\mathcal{K}_{n,s,p}(\rho) \sim \rho^{-n-sp}$$

as $\rho \to 0^+$ and

$$\mathcal{K}_{n,s,p}(\rho) \sim \rho^{-1-\frac{sp}{2}}e^{-(n-1)\rho}$$

as $\rho \to +\infty$, up to constants.

In the linear case $p = 2$, the pointwise integral representation with singular kernel is provided in [7, Theorem 2.4 and 2.5] without constants. The novelty of Theorem 4.1.2 is that it generalizes the representation formula to the nonlinear regime with the normalizing constant. We emphasize that the normalizing constant plays a crucial role in some contexts. For instance, it is used in the convergence result (Theorem 4.1.4) and the robust regularity theory (see [15, 70]).

The main tool in [7, 70] is the Fourier transform on the hyperbolic spaces. Since the Fourier transform is not available in the nonlinear setting, we use the heat kernel for the Laplace operator $\Delta_{\mathbb{H}^n}$ on hyperbolic spaces to prove Theorem 4.1.2. The explicit formula for the heat kernel with a normalizing constant given in [59] enables us to obtain the exact values of the constants in Theorem 4.1.2.

Let us proceed to another representation for the fractional p-Laplacian on $\mathbb{H}^n$. We recall that the fractional Laplacian on $\mathbb{R}^n$ is obtained by a Dirichlet-to-Neumann map via the Caffarelli–Silvestre extension [14]. Later, the article [100] relates the heat semigroup to this extension. Moreover, this relation is extended to the nonlinear framework [39] in $\mathbb{R}^n$. We further investigate this relation on the hyperbolic spaces. Let us consider the extension problem

$$\begin{cases} \Delta_x U(x,y) + \dfrac{1-sp}{y} U_y(x,y) + U_{yy}(x,y) = 0, & x \in \mathbb{H}^n, y > 0, \\ U(x,0) = f(x), & x \in \mathbb{H}^n, \end{cases} \tag{4.1.4}$$

and define an extension operator $E_{s,p}$ by $E_{s,p}[f] := U$. The following theorem is our next main result.

Theorem 4.1.3. *Let $n \in \mathbb{N}$, $s \in (0,1)$, and $p > 1$. Let $u \in C_b^2(\mathbb{H}^n)$ and $x \in \mathbb{H}^n$. If $p \in (1, \frac{2}{2-s}]$, assume $\nabla u(x) \neq 0$ additionally. Then*

$$\begin{aligned} (-\Delta_{\mathbb{H}^n})_p^s u(x) &= C_3 \lim_{y \searrow 0} \frac{E_{s,p}[\Phi_p(u(x) - u(\cdot))](x,y)}{y^{sp}} \\ &= \frac{C_3}{sp} \lim_{y \searrow 0} y^{1-sp} \partial_y \Big(E_{s,p}[\Phi_p(u(x) - u(\cdot))] \Big)(x,y), \end{aligned} \tag{4.1.5}$$

where

$$C_3 = \frac{p}{2}\frac{\sqrt{\pi}/2}{\Gamma(\frac{p+1}{2})}\frac{2^{2s}\Gamma(\frac{sp}{2})}{|\Gamma(-s)|}.$$

To prove Theorem 4.1.3, we represent the solution U of the extension problem (4.1.4) by using the heat semigroup. Then, the formula for the heat kernel [59] leads to the Poisson formula for U and the representation (4.1.5).

The last result is the pointwise convergence of the fractional p-Laplacian on $\mathbb{H}^n$ as $s \to 1^-$. As one can expect, the fractional p-Laplacian converges to the p-Laplacian as a limit. Recall that the p-Laplacian on $\mathbb{H}^n$ is defined by $(-\Delta_{\mathbb{H}^n})_p u(x) = -\mathrm{div}(|\nabla u(x)|^{p-2}\nabla u(x))$.

Theorem 4.1.4. *Let $n \in \mathbb{N}$, $p > 1$, and $u \in C_b^2(\mathbb{H}^n)$. For $x \in \mathbb{H}^n$ such that $\nabla u(x) \neq 0$,*

$$\lim_{s \nearrow 1} (-\Delta_{\mathbb{H}^n})_p^s u(x) = (-\Delta_{\mathbb{H}^n})_p u(x).$$

The pointwise convergence of the fractional p-Laplacian on the Euclidean spaces is well known [10, 42, 64]. Recall that the proof uses Taylor's theorem and the following computations:

$$\begin{aligned} \int_{\mathbb{S}^{n-1}}\int_R^\infty K(\rho)\rho^{n-1}\,\mathrm{d}\rho\,\mathrm{d}\omega &= \frac{|\mathbb{S}^{n-1}|}{sp}R^{-sp}, \\ \int_{\mathbb{S}^{n-1}}\int_0^R K(\rho)\rho^{p+n-1}\,\mathrm{d}\rho\,\mathrm{d}\omega &= \frac{|\mathbb{S}^{n-1}|}{p(1-s)}R^{p(1-s)}, \\ \int_{\mathbb{S}^{n-1}}\int_0^R K(\rho)\rho^{\beta+p+n-1}\,\mathrm{d}\rho\,\mathrm{d}\omega &= \frac{|\mathbb{S}^{n-1}|}{\beta+p(1-s)}R^{\beta+p(1-s)}, \end{aligned} \tag{4.1.6}$$

where $\beta > 0$ and $K(\rho) = \rho^{-n-sp}$ is the kernel for the fractional p-Laplacian on $\mathbb{R}^n$. However, in our framework we need the integrals in (4.1.6) with the kernel K and the volume element $\rho^{n-1}\,\mathrm{d}\rho\,\mathrm{d}\omega$ replaced by $\mathcal{K}_{n,s,p}$ and $\sinh^{n-1}\rho\,\mathrm{d}\rho\,\mathrm{d}\omega$, respectively. These integrals do not seem to be of a form that is easily computed. Instead, we compute the limits of these integrals as $s \to 1^-$, which are sufficient to establish Theorem 4.1.4. This is still not straightforward, but can be obtained by using the asymptotic behavior of modified Bessel functions.

The paper is organized as follows. In Section 4.2 we recall the hyperboloid model and study the modified Bessel function and its properties. Section 4.3

is devoted to the proof of Theorem 4.1.2, which provides the pointwise integral representation of the fractional p-Laplacian with singular kernels. In Section 4.4, we relate the heat semigroup to the extension problem (4.1.4) and find the Poisson formula. Using the Poisson formula and the representation of the fractional p-Laplacian, we prove Theorem 4.1.3. Finally, we prove the pointwise convergence result, Theorem 4.1.4, in Section 4.5. An auxiliary result can be found in Section 4.6.1.

4.2 Preliminaries

In this section, we recall the basics of the hyperbolic spaces and collect some facts about the modified Bessel function.

4.2.1 The hyperbolic space

There are several models for the hyperbolic spaces, but let us focus on the hyperboloid model in this paper. The hyperboloid model is given by

$$\mathbb{H}^n = \left\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : x_0^2 - x_1^2 - \dots - x_n^2 = 1, x_0 > 0\right\}$$

with the Lorentzian metric $-\mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \dots + \mathrm{d}x_n^2$ in $\mathbb{R}^{n+1}$. The Lorentzian metric induces the natural internal product

$$[x, \xi] = x_0\xi_0 - x_1\xi_1 - \dots - x_n\xi_n$$

on $\mathbb{H}^n$. Moreover, the distance between two points x and ξ is given by

$$d(x, \xi) = \cosh^{-1}([x, \xi]).$$

Using the polar coordinates, $\mathbb{H}^n$ can also be realized as

$$\mathbb{H}^n = \left\{x = (\cosh r, \sinh r\, \omega) \in \mathbb{R}^{n+1} : r \geq 0, \omega \in \mathbb{S}^{n-1}\right\}.$$

Then, the metric and the volume element are given by $\mathrm{d}r^2 + \sinh^2 r\, \mathrm{d}\omega^2$ and $\sinh^{n-1} r\, \mathrm{d}r\, \mathrm{d}\omega$, respectively.

4.2.2 The modified Bessel function

The modified Bessel functions naturally appear in the study of hyperbolic geometry. In this paper, they are used to describe the kernel of the fractional p-Laplacian and the Poisson kernel. For this purpose, we recall the definition and some properties of the modified Bessel functions.

We call the ordinary differential equation

$$\rho^2 \frac{\mathrm{d}^2 y}{\mathrm{d}\rho^2} + \rho \frac{\mathrm{d}y}{\mathrm{d}\rho} - (\rho^2 + \nu^2) y = 0$$

the modified Bessel equation. The solutions are given by

$$I_\nu(\rho) = \sum_{j=0}^{\infty} \frac{1}{j!\Gamma(\nu + j + 1)} \left(\frac{\rho}{2}\right)^{2j+\nu} \quad \text{and} \quad K_\nu(\rho) = \frac{\pi}{2} \frac{I_{-\nu}(\rho) - I_\nu(\rho)}{\sin \nu\pi},$$

and they are called the *modified Bessel functions of the first and the second kind*, respectively. Since only K_ν appears in this work, we focus on the properties of K_ν. This function has the following integral representation (see [91, 10.32.10]):

$$K_\nu(\rho) = \frac{1}{2} \left(\frac{1}{2}\rho\right)^{\nu} \int_0^\infty e^{-t-\frac{\rho^2}{4t}} t^{-\nu-1}\, \mathrm{d}t. \tag{4.2.1}$$

The asymptotic behavior of K_ν is given by

$$\begin{aligned} K_\nu(\rho) &\sim \frac{1}{2}\Gamma(\nu) \left(\frac{\rho}{2}\right)^{-\nu} \quad \text{as } \rho \to 0^+, \text{ for } \nu > 0, \text{ and} \\ K_\nu(\rho) &\sim \sqrt{\frac{\pi}{2\rho}} e^{-\rho} \quad \text{as } \rho \to +\infty. \end{aligned} \tag{4.2.2}$$

Moreover, K_ν satisfies the following recurrence relations:

$$K_\nu' = -K_{\nu-1} - \frac{\nu}{R} K_\nu \quad \text{and} \quad K_\nu' = -K_{\nu+1} + \frac{\nu}{R} K_\nu. \tag{4.2.3}$$

We also recall that K_ν is increasing with respect to $\nu > 0$. For further properties of the modified Bessel functions, the reader may consult the handbook [91].

In the sequel, functions of the form $\rho^{-\nu} K_\nu(a\rho)$ with $\nu \in \mathbb{R}$ and $a > 0$ will appear frequently. For notational convenience, we define

$$\mathscr{K}_{\nu,a}(\rho) := \rho^{-\nu} K_\nu(a\rho). \tag{4.2.4}$$

Then, it follows from (4.2.3)

$$-\partial_\rho(\mathscr{K}_{\nu,a}(f(\rho))) = af'(\rho)f(\rho)\mathscr{K}_{\nu+1,a}(f(\rho))$$

for any differentiable function $f : (0, +\infty) \to (0, +\infty)$.

4.3 Pointwise representation with singular kernel

The singular kernel ρ^{-n-sp} for the fractional p-Laplacian on the Euclidean space $\mathbb{R}^n$ is homogeneous of degree $-n-sp$. This is a natural property coming from the scale invariance of the operator. However, this cannot be expected in the case of hyperbolic spaces because the hyperbolic geometry comes into play. Indeed, we will see that the kernel $\mathcal{K}_{n,s,p}(\rho)$ behaves like ρ^{-n-sp} near $\rho = 0$ whereas it decays like $\rho^{-1-\frac{sp}{2}} e^{-(n-1)\rho}$ as $\rho \to +\infty$, up to constants, by providing the explicit form of the kernel $\mathcal{K}_{n,s,p}$. Moreover, we investigate the pointwise integral representation of the fractional p-Laplacian on $\mathbb{H}^n$.

It is well known that the Cauchy problem (4.1.1) has the unique solution

$$w(x,t) = \int_{\mathbb{H}^n} p(t, d(x,\xi)) f(\xi)\,\mathrm{d}\xi,$$

provided that f is a bounded continuous function, where the heat kernel

$p(t,\rho)$ is given [59] by

$$p(t,\rho) = \frac{1}{(2\pi)^m}\frac{1}{(4\pi t)^{1/2}}\left(\frac{-\partial_\rho}{\sinh\rho}\right)^m e^{-m^2t-\frac{\rho^2}{4t}} \tag{4.3.1}$$

when $n = 2m+1 \geq 1$ is odd and

$$p(t,\rho) = \frac{1}{2(2\pi)^{m+1/2}} t^{-3/2} e^{-\frac{(2m-1)^2}{4}t}\left(\frac{-\partial_\rho}{\sinh\rho}\right)^{m-1}\int_\rho^\infty \frac{re^{-\frac{r^2}{4t}}}{\sqrt{\cosh r - \cosh\rho}}\,\mathrm{d}r \tag{4.3.2}$$

when $n = 2m \geq 2$ is even. We use these explicit formulas for the heat kernels not only in the computation of the singular kernels $\mathcal{K}_{n,s,p}$ but also in the next sections. For this purpose, we first prove the following lemma, which is useful especially in the even dimensional case.

Lemma 4.3.1. *Let $\nu > 1/2$, $a \geq 1/2$, and $y \geq 0$. For $m \in \mathbb{N}\cup\{0\}$ define*

$$F_m(r) := \frac{\sinh r}{\sqrt{\cosh r - \cosh\rho}}\left(\frac{-\partial_r}{\sinh r}\right)^m \mathscr{K}_{\nu,a}\left(\sqrt{r^2+y^2}\right),$$

where $\mathscr{K}_{\nu,a}$ is the function given in (4.2.4)*. Then, F_m is integrable on $(\rho,+\infty)$ and satisfies*

$$\left(\frac{-\partial_\rho}{\sinh\rho}\right)\int_\rho^\infty F_m(r)\,\mathrm{d}r = \int_\rho^\infty F_{m+1}(r)\,\mathrm{d}r \tag{4.3.3}$$

for all $m \in \mathbb{N}\cup\{0\}$.

Proof. Note that for any $j \geq 1$

$$-\partial_r\left(\frac{(e^r+e^{-r})^{j-1}}{(e^r-e^{-r})^j}\right) = j\frac{(e^r+e^{-r})^j}{(e^r-e^{-r})^{j+1}} - (j-1)\frac{(e^r+e^{-r})^{j-2}}{(e^r-e^{-r})^{j-1}}.$$

Therefore, all derivatives of $\frac{1}{\sinh r}$ (and $\frac{r}{\sinh r}$) have the same asymptotic behavior as e^{-r} (and re^{-r}, respectvely) as $r \to +\infty$. Hence, $F_m(r) \sim r^{-\nu-1/2}e^{(1/2-m-a)r}$ as $r \to +\infty$, which shows that the function F_m is integrable.

Using the integration by parts, we have

$$\begin{aligned}\int_\rho^\infty F_m(r)\,\mathrm{d}r &= \int_\rho^\infty 2\partial_r\left(\sqrt{\cosh r - \cosh\rho}\right)\left(\frac{-\partial_r}{\sinh r}\right)^m \mathscr{K}_{\nu,a}\left(\sqrt{r^2+y^2}\right)\mathrm{d}r\\ &= \int_\rho^\infty 2\sinh r\sqrt{\cosh r - \cosh\rho}\left(\frac{-\partial_r}{\sinh r}\right)^{m+1} \mathscr{K}_{\nu,a}\left(\sqrt{r^2+y^2}\right)\mathrm{d}r.\end{aligned}$$

Thus, the recurrence relation (4.3.3) follows by applying the Leibniz integral rule. □

Let us now prove Theorem 4.1.2 using the heat kernel and Lemma 4.3.1.

Proof of Theorem 4.1.2. Let $\varepsilon > 0$ and define $g_\varepsilon(x,\xi) = \Phi_p(u(x)-u(\xi))\chi_{d(x,\xi)>\varepsilon}$. The heat semigroup associated to $g_\varepsilon(x,\cdot)$ is given by

$$e^{t\Delta_{\mathbb{H}^n}}[g_\varepsilon(x,\cdot)](x) = \int_{\mathbb{H}^n} \frac{1}{(2\pi)^m}\frac{1}{(4\pi t)^{1/2}}\left(\left(\frac{-\partial_\rho}{\sinh\rho}\right)^m e^{-m^2t-\frac{\rho^2}{4t}}\right) g_\varepsilon(x,\xi)\,\mathrm{d}\xi$$

when $n = 2m+1 \geq 3$ is odd and

$$\begin{aligned}&e^{t\Delta_{\mathbb{H}^n}}[g_\varepsilon(x,\cdot)](x)\\ &\quad= \int_{\mathbb{H}^n} \frac{t^{-3/2}e^{-\frac{(2m-1)^2}{4}t}}{2(2\pi)^{m+1/2}}\left(\frac{-\partial_\rho}{\sinh\rho}\right)^{m-1}\int_\rho^\infty \frac{re^{-\frac{r^2}{4t}}\,\mathrm{d}r}{\sqrt{\cosh r - \cosh\rho}}\, g_\varepsilon(x,\xi)\,\mathrm{d}\xi\end{aligned}$$

when $n = 2m \geq 2$ is even, where $\rho = d(x,\xi)$. We will prove

$$C_1\int_0^\infty e^{t\Delta_{\mathbb{H}^n}}[g_\varepsilon(x,\cdot)](x)\frac{\mathrm{d}t}{t^{1+\frac{sp}{2}}} = c_{n,s,p}\int_{d(x,\xi)>\varepsilon}\Phi_p(u(x)-u(\xi))\mathcal{K}_{n,s,p}(d(x,\xi))\,\mathrm{d}\xi \tag{4.3.4}$$

in both cases.

Let us first consider the odd dimensional case. We fix $\delta > 0$ and integrate the heat semigroup with respect to the singular measure $t^{-1-\frac{sp}{2}}\,\mathrm{d}t$ over the interval (δ,∞) to obtain

$$\begin{aligned}
&C_1\int_\delta^\infty e^{t\Delta_{\mathbb{H}^n}}[g_\varepsilon(x,\cdot)](x)\frac{\mathrm{d}t}{t^{1+\frac{sp}{2}}}\\
&=C_1\int_\delta^\infty\int_{\mathbb{H}^n}\frac{1}{(2\pi)^m}\frac{1}{(4\pi t)^{1/2}}\left(\left(\frac{-\partial_\rho}{\sinh\rho}\right)^m e^{-m^2t-\frac{\rho^2}{4t}}\right)g_\varepsilon(x,\xi)\,\mathrm{d}\xi\,\frac{\mathrm{d}t}{t^{1+\frac{sp}{2}}}\\
&=\frac{C_1}{(2\pi)^m(4\pi)^{1/2}}\int_{\mathbb{H}^n}\left(\frac{-\partial_\rho}{\sinh\rho}\right)^m\left(\int_\delta^\infty e^{-m^2t-\frac{\rho^2}{4t}}t^{-\frac{3+sp}{2}}\,\mathrm{d}t\right)g_\varepsilon(x,\xi)\,\mathrm{d}\xi.
\end{aligned}\tag{4.3.5}$$

Note that the function $e^{-m^2t-\frac{\rho^2}{4t}}t^{-\frac{3+sp}{2}}$ is integrable on $(0,\infty)$. Indeed, the formula (4.2.1) and the change of variables show

$$\int_0^\infty e^{-m^2t-\frac{\rho^2}{4t}}t^{-\frac{3+sp}{2}}\,\mathrm{d}t=m^{1+sp}\int_0^\infty e^{-t-\frac{(m\rho)^2}{4t}}t^{-\frac{3+sp}{2}}\,\mathrm{d}t=2(2m)^{\frac{1+sp}{2}}\mathscr{K}_{\frac{1+sp}{2},m}(\rho).\tag{4.3.6}$$

Thus, (4.3.4) in the odd dimensional case follows by combining (4.3.5)–(4.3.6) and passing the limit $\delta\searrow 0$.

We next consider the even dimensional case. Similarly as in the odd dimensional case, we obtain

$$\begin{aligned}
&C_1\int_\delta^\infty e^{t\Delta_{\mathbb{H}^n}}[g_\varepsilon(x,\cdot)](x)\frac{\mathrm{d}t}{t^{1+\frac{sp}{2}}}\\
&=C_1\int_\delta^\infty\int_{\mathbb{H}^n}\frac{t^{-3/2}e^{-\frac{(2m-1)^2}{4}t}}{2(2\pi)^{m+1/2}}\left(\frac{-\partial_\rho}{\sinh\rho}\right)^{m-1}\\
&\qquad\int_\rho^\infty\frac{re^{-\frac{r^2}{4t}}}{\sqrt{\cosh r-\cosh\rho}}\,\mathrm{d}r\,g_\varepsilon(x,\xi)\,\mathrm{d}\xi\,\frac{\mathrm{d}t}{t^{1+\frac{sp}{2}}}\\
&=\frac{C_1}{2(2\pi)^{m+1/2}}\int_{\mathbb{H}^n}\left(\frac{-\partial_\rho}{\sinh\rho}\right)^{m-1}\\
&\qquad\int_\rho^\infty\left(\int_\delta^\infty e^{-\frac{(2m-1)^2}{4}t-\frac{r^2}{4t}}t^{-\frac{5+sp}{2}}\,\mathrm{d}t\right)\frac{r\,\mathrm{d}r\,g_\varepsilon(x,\xi)\,\mathrm{d}\xi}{\sqrt{\cosh r-\cosh\rho}}.
\end{aligned}$$

Moreover, we have from (4.2.1) and (4.2.3)

$$\int_0^\infty e^{-\frac{(2m-1)^2}{4}t-\frac{r^2}{4t}}t^{-\frac{5+sp}{2}}\,\mathrm{d}t = 2(2m-1)^{\frac{3+sp}{2}}\mathscr{K}_{\frac{3+sp}{2},\frac{2m-1}{2}}(r)$$
$$= 4(2m-1)^{\frac{1+sp}{2}}\left(\frac{-\partial_r}{r}\right)\mathscr{K}_{\frac{1+sp}{2},\frac{2m-1}{2}}(r).$$

Thus, we deduce

$$\begin{aligned}
&C_1\int_0^\infty e^{t\Delta_{\mathbb{H}^n}}[g_\varepsilon(x,\cdot)](x)\frac{\mathrm{d}t}{t^{1+\frac{sp}{2}}}\\
&= c_{n,s,p}C_2\int_{\mathbb{H}^n}\left(\frac{-\partial_\rho}{\sinh\rho}\right)^{m-1}\int_\rho^\infty\frac{(-\partial_r)\mathscr{K}_{\frac{1+sp}{2},\frac{2m-1}{2}}(r)}{\sqrt{\pi}\sqrt{\cosh r-\cosh\rho}}\,\mathrm{d}r\,g_\varepsilon(x,\xi)\,\mathrm{d}\xi\\
&= c_{n,s,p}C_2\int_{\mathbb{H}^n}\int_\rho^\infty\frac{\sinh r}{\sqrt{\pi}\sqrt{\cosh r-\cosh\rho}}\left(\frac{-\partial_r}{\sinh r}\right)^m\mathscr{K}_{\frac{1+sp}{2},\frac{2m-1}{2}}(r)\,\mathrm{d}r\,g_\varepsilon(x,\xi)\,\mathrm{d}\xi,
\end{aligned}$$

where we used Lemma 4.3.1 with $\nu=\frac{1+sp}{2}$, $a=\frac{2m-1}{2}$, and $y=0$ in the last equality. This proves (4.3.4) in the even dimensional case.

On the one hand, the integral in the right-hand side of (4.3.4) converges to the Cauchy principal value

$$\mathrm{P.V.}\int_{\mathbb{H}^n}\Phi_p(u(x)-u(\xi))\mathcal{K}_{n,s,p}(d(x,\xi))\,\mathrm{d}\xi$$

as $\varepsilon\searrow 0$. For the left-hand side of (4.3.4), on the other hand, we need to estimate

$$A:=\int_0^\infty e^{t\Delta_{\mathbb{H}^n}}[\Phi_p(u(x)-u(\cdot))](x)\frac{\mathrm{d}t}{t^{1+\frac{sp}{2}}}-\int_0^\infty e^{t\Delta_{\mathbb{H}^n}}[g_\varepsilon(x,\cdot)](x)\frac{\mathrm{d}t}{t^{1+\frac{sp}{2}}}.$$

Proceeding as above, we have

$$|A|\lesssim\left|\mathrm{P.V.}\int_{d(x,\xi)\le\varepsilon}\Phi_p(u(x)-u(\xi))\mathcal{K}_{n,s,p}(d(x,\xi))\,\mathrm{d}\xi\right|.$$

Thus, applying Lemma 4.6.1 to $K = \mathcal{K}_{n,s,p}$ yields

$$|A| \lesssim \int_{d(x,\xi)\le\varepsilon} \rho^{\alpha}\mathcal{K}_{n,s,p}(\rho)\,\mathrm{d}y \lesssim \int_0^{\varepsilon} \rho^{\alpha}\mathcal{K}_{n,s,p}(\rho)\sinh^{n-1}\rho\,\mathrm{d}\rho, \tag{4.3.7}$$

where $\alpha = 2p-2$ when $p \in (\frac{2}{2-s}, 2)$ and $\alpha = p$ when $p \in (1, \frac{2}{2-s}] \cup [2,\infty)$. Note that $\mathcal{K}_{n,s,p}$ is positive, which will be proved later in Corollary 4.5.2. Since $\mathcal{K}_{n,s,p} \sim \rho^{-n-sp}$ as $\rho \to 0^+$ up to constants, the function $\rho^{\alpha}\mathcal{K}_{n,s,p}(\rho)\sinh^{n-1}\rho$ is integrable near zero and hence the right-hand side of (4.3.7) converges to zero as $\varepsilon \searrow 0$. Therefore, the left-hand side of (4.3.4) converges to that of (4.1.3) as $\varepsilon \searrow 0$. □

4.4 Extension problem

In this section, we prove Theorem 4.1.3, which provides another representation of the fractional p-Laplacian on the hyperbolic spaces. We first relate the heat semigroup to the extension problem (4.1.4) and find the Poisson formula.

Lemma 4.4.1. *Let $n \ge 2$, $s \in (0,1)$, and $p > 1$. If $f \in C_b(\mathbb{H}^n)$, then the solution $U = E_{s,p}[f]$ of the extension problem* (4.1.4) *is given by*

$$U(x,y) = \frac{y^{sp}}{2^{sp}\Gamma(\frac{sp}{2})}\int_0^{\infty} e^{t\Delta_{\mathbb{H}^n}}[f](x)e^{-\frac{y^2}{4t}}\frac{\mathrm{d}t}{t^{1+\frac{sp}{2}}}. \tag{4.4.1}$$

Moreover, the solution can be represented by using the Poisson kernel:

$$U(x,y) = \int_{\mathbb{H}^n} P(d(x,\xi),y)f(\xi)\,\mathrm{d}\xi. \tag{4.4.2}$$

The Poisson kernel $P(\rho,y)$ is given by

$$P(\rho,y) = C_4\, y^{sp}\left(\frac{-\partial_\rho}{\sinh\rho}\right)^{\frac{n-1}{2}} \mathscr{K}_{\frac{1+sp}{2},\frac{n-1}{2}}\left(\sqrt{\rho^2+y^2}\right)$$

when $n \geq 3$ odd and

$$P(\rho, y) = C_4\, y^{sp} \int_\rho^\infty \frac{\sinh r}{\sqrt{\pi}\sqrt{\cosh r - \cosh \rho}} \left(\frac{-\partial_r}{\sinh r}\right)^{\frac{n}{2}} \mathscr{K}_{\frac{1+sp}{2}, \frac{n-1}{2}} \left(\sqrt{r^2+y^2}\right) \mathrm{d}r$$

when $n \geq 2$ even, where

$$C_4 = \frac{1}{2^{\frac{n-3}{2}} \pi^{\frac{n}{2}} \Gamma(\frac{sp}{2})} \left(\frac{n-1}{4}\right)^{\frac{1+sp}{4}}$$

and $\mathscr{K}_{\nu,a}$ is the function given in (4.2.4).

Proof. For each $x \in \mathbb{H}^n$ and $y > 0$, we define $V(x, y)$ by the function given in the right-hand side of (4.4.1). Then, we have

$$V(x, y) = \frac{y^{sp}}{2^{sp}\Gamma(\frac{sp}{2})} \int_0^\infty \int_{\mathbb{H}^n} p(t, \rho) f(\xi)\, \mathrm{d}\xi\, e^{-\frac{y^2}{4t}} \frac{\mathrm{d}t}{t^{1+\frac{sp}{2}}},$$

where $\rho = d(x, \xi)$. Recalling the expression (4.3.1) for the heat kernel $p(t, \rho)$ and using (4.2.1), we obtain

$$\begin{aligned}
&V(x, y) \\
&= \frac{y^{sp}}{2^{sp}\Gamma(\frac{sp}{2})} \int_0^\infty \int_{\mathbb{H}^n} \frac{1}{(2\pi)^m} \frac{1}{(4\pi t)^{1/2}} \left(\left(\frac{-\partial_\rho}{\sinh \rho}\right)^m e^{-m^2 t - \frac{\rho^2+y^2}{4t}} \right) f(\xi)\, \mathrm{d}\xi \frac{\mathrm{d}t}{t^{1+sp/2}} \\
&= \int_{\mathbb{H}^n} \frac{y^{sp}}{2^{sp}\Gamma(\frac{sp}{2})} \frac{1}{(2\pi)^m} \frac{1}{(4\pi)^{1/2}} \left(\frac{-\partial_\rho}{\sinh \rho}\right)^m \left(\int_0^\infty e^{-m^2 t - \frac{\rho^2+y^2}{4t}} t^{-\frac{3+sp}{2}}\, \mathrm{d}t \right) f(\xi)\, \mathrm{d}\xi \\
&= \int_{\mathbb{H}^n} P(d(x, \xi), y) f(\xi)\, \mathrm{d}\xi
\end{aligned}$$

when $n = 2m + 1$ is odd. If $n = 2m$ is even, then we use (4.3.2) instead of

(4.3.1) to have

$$\begin{aligned}
&V(x,y)\\
&= \frac{y^{sp}}{2^{sp}\Gamma(\frac{sp}{2})}\int_0^\infty\int_{\mathbb{H}^n}\frac{t^{-\frac{5+sp}{2}}e^{-\frac{(2m-1)^2}{4}t}}{2(2\pi)^{m+1/2}}\left(\frac{-\partial_\rho}{\sinh\rho}\right)^{m-1}\int_\rho^\infty\frac{re^{-\frac{r^2+y^2}{4t}}\,\mathrm{d}r}{\sqrt{\cosh r-\cosh\rho}}\,f(\xi)\,\mathrm{d}\xi\,\mathrm{d}t\\
&= \frac{y^{sp}}{2^{sp}\Gamma(\frac{sp}{2})}\int_{\mathbb{H}^n}\left(\frac{-\partial_\rho}{\sinh\rho}\right)^{m-1}\int_\rho^\infty\int_0^\infty\frac{t^{-\frac{5+sp}{2}}e^{-\frac{(2m-1)^2}{4}t}}{2(2\pi)^{m+1/2}}\frac{re^{-\frac{r^2+y^2}{4t}}\,\mathrm{d}t}{\sqrt{\cosh r-\cosh\rho}}\,\mathrm{d}r\,f(\xi)\,\mathrm{d}\xi.
\end{aligned}$$

Moreover, using (4.2.1) we compute

$$\begin{aligned}
\int_0^\infty e^{-\frac{(2m-1)^2}{4}t-\frac{r^2+y^2}{4t}}t^{-\frac{5+sp}{2}}\,\mathrm{d}t &= 2(2m-1)^{\frac{3+sp}{2}}\mathscr{K}_{\frac{3+sp}{2},\frac{2m-1}{2}}\left(\sqrt{r^2+y^2}\right)\\
&= 4(2m-1)^{\frac{1+sp}{2}}\left(\frac{-\partial_r}{r}\right)\mathscr{K}_{\frac{1+sp}{2},\frac{2m-1}{2}}\left(\sqrt{r^2+y^2}\right).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
V(x,y) &= C_4\,y^{sp}\int_{\mathbb{H}^n}\left(\frac{-\partial_\rho}{\sinh\rho}\right)^{m-1}\int_\rho^\infty\frac{(-\partial_r)\mathscr{K}_{\frac{1+sp}{2},\frac{2m-1}{2}}\left(\sqrt{r^2+y^2}\right)}{\sqrt{\pi}\sqrt{\cosh r-\cosh\rho}}\,\mathrm{d}r\,f(\xi)\,\mathrm{d}\xi\\
&= \int_{\mathbb{H}^n}P(d(x,\xi),y)f(\xi)\,\mathrm{d}\xi
\end{aligned}$$

in the even dimensional case as well, where we used Lemma 4.3.1 in the last equality.

It only remains to prove the equality in (4.4.1) to conclude lemma. Note that (4.4.2) will follow from (4.4.1) and the representations of V above. To prove the equality in (4.4.1), we check that the function V solves the extension problem (4.1.4). Since the heat semigroup $e^{t\Delta_{\mathbb{H}^n}}[f]$ solves (4.1.1), V satisfies

$$\Delta_x V = \frac{y^{sp}}{2^{sp}\Gamma(\frac{sp}{2})}\int_0^\infty\partial_t\left(e^{t\Delta_{\mathbb{H}^n}}[f](x)\right)e^{-\frac{y^2}{4t}}t^{-1-\frac{sp}{2}}\,\mathrm{d}t.$$

Using the integration by parts and the fact that $|e^{t\Delta_{\mathbb{H}^n}}[f](x)|\le\|f\|_{L^\infty}$, we obtain

$$\begin{aligned}
&\Delta_x V \\
&= \frac{y^{sp}}{2^{sp}\Gamma(\frac{sp}{2})}\left(\left[e^{t\Delta_{\mathbb{H}^n}}[f](x)e^{-\frac{y^2}{4t}}t^{-1-\frac{sp}{2}}\right]_0^\infty - \int_0^\infty e^{t\Delta_{\mathbb{H}^n}}[f](x)\partial_t\left(e^{-\frac{y^2}{4t}}t^{-1-\frac{sp}{2}}\right)\mathrm{d}t\right) \\
&= -\frac{y^{sp}}{2^{sp}\Gamma(\frac{sp}{2})}\int_0^\infty e^{t\Delta_{\mathbb{H}^n}}[f](x)\left(\frac{y^2}{4}e^{-\frac{y^2}{4t}}t^{-3-\frac{sp}{2}} - \left(1+\frac{sp}{2}\right)e^{-\frac{y^2}{4t}}t^{-2-\frac{sp}{2}}\right)\mathrm{d}t.
\end{aligned}$$

Since

$$\begin{aligned}
V_y = &\frac{spy^{sp-1}}{2^{sp}\Gamma(\frac{sp}{2})}\int_0^\infty e^{t\Delta_{\mathbb{H}^n}}[f](x)e^{-\frac{y^2}{4t}}t^{-1-\frac{sp}{2}}\,\mathrm{d}t \\
&- \frac{y^{sp+1}}{2^{sp+1}\Gamma(\frac{sp}{2})}\int_0^\infty e^{t\Delta_{\mathbb{H}^n}}[f](x)e^{-\frac{y^2}{4t}}t^{-2-\frac{sp}{2}}\,\mathrm{d}t
\end{aligned}$$

and

$$\begin{aligned}
V_{yy} = &\frac{sp(sp-1)y^{sp-2}}{2^{sp}\Gamma(\frac{sp}{2})}\int_0^\infty e^{t\Delta_{\mathbb{H}^n}}[f](x)e^{-\frac{y^2}{4t}}t^{-1-\frac{sp}{2}}\,\mathrm{d}t \\
&- \frac{2sp+1}{2^{sp+1}\Gamma(\frac{sp}{2})}y^{sp}\int_0^\infty e^{t\Delta_{\mathbb{H}^n}}[f](x)e^{-\frac{y^2}{4t}}t^{-2-\frac{sp}{2}}\,\mathrm{d}t \\
&+ \frac{y^{sp+2}}{2^{sp+1}\Gamma(\frac{sp}{2})}\int_0^\infty e^{t\Delta_{\mathbb{H}^n}}[f](x)e^{-\frac{y^2}{4t}}t^{-3-\frac{sp}{2}}\,\mathrm{d}t,
\end{aligned}$$

one can easily compute

$$\Delta_x V(x,y) + \frac{1-sp}{y}V_y(x,y) + V_{yy}(x,y) = 0.$$

Finally, we prove $V(x,0) = f(x)$. Indeed, we have $P(\rho,y) \to 0$ as $y \searrow 0$ if $\rho \neq 0$ by definition. Moreover, since the heat kernel $p(t,\rho)$ satisfies

$$\int_{\mathbb{H}^n} p(t, d(x,\xi))\,\mathrm{d}\xi = 1,$$

we obtain

$$\begin{aligned}\int_{\mathbb{H}^n} P(d(x,\xi),y)\,\mathrm{d}\xi &= \frac{y^{sp}}{2^{sp}\Gamma(\frac{sp}{2})}\int_0^\infty \left(\int_{\mathbb{H}^n} p(t,d(x,\xi))\,\mathrm{d}\xi\right) e^{-\frac{y^2}{4t}}\frac{\mathrm{d}t}{t^{1+\frac{sp}{2}}}\\ &= \frac{y^{sp}}{2^{sp}\Gamma(\frac{sp}{2})}\int_0^\infty e^{-\frac{y^2}{4t}}\frac{\mathrm{d}t}{t^{1+\frac{sp}{2}}} = 1.\end{aligned}$$

This concludes that V solves the extension problem (4.1.4). □

Let us now prove Theorem 4.1.3 by using the Poisson formula in Lemma 4.4.1.

Proof of Theorem 4.1.3. We have the kernel representation of $(-\Delta_{\mathbb{H}^n})_p^s u(x)$ from Theorem 4.1.2 and the Poisson kernel representation of $E_{s,p}[\Phi_p(u(x)-u(\cdot))](x,y)$ from Lemma 4.4.1. Since $c_{n,s,p}C_2 = C_3C_4$, it is enough to show

$$\left|\int_{\mathbb{H}^n}\Phi_p(u(x)-u(\xi))K(d(x,\xi))\,\mathrm{d}\xi\right| \to 0$$

as $y \searrow 0$, where

$$K(\rho) = \left(\frac{-\partial_\rho}{\sinh\rho}\right)^{\frac{n-1}{2}}\left(\mathscr{K}_{\frac{1+sp}{2},\frac{n-1}{2}}(\rho) - \mathscr{K}_{\frac{1+sp}{2},\frac{n-1}{2}}\left(\sqrt{\rho^2+y^2}\right)\right)$$

when n is odd and

$$\begin{aligned}&K(\rho)\\ &= \int_\rho^\infty \frac{\sinh r}{\sqrt{\pi}\sqrt{\cosh r-\cosh\rho}}\left(\frac{-\partial_r}{\sinh r}\right)^{\frac{n}{2}}\left(\mathscr{K}_{\frac{1+sp}{2},\frac{n-1}{2}}(r) - \mathscr{K}_{\frac{1+sp}{2},\frac{n-1}{2}}\left(\sqrt{r^2+y^2}\right)\right)\mathrm{d}r\end{aligned}$$

when n is even.

We first split the integral as follows:

$$\begin{aligned}&\left|\int_{\mathbb{H}^n}\Phi_p(u(x)-u(\xi))K(d(x,\xi))\,\mathrm{d}\xi\right|\\ &\le \left|\int_{d(x,\xi)\le 1}\Phi_p(u(x)-u(\xi))K(d(x,\xi))\,\mathrm{d}\xi\right| + \left|\int_{d(x,\xi)>1}\Phi_p(u(x)-u(\xi))K(d(x,\xi))\,\mathrm{d}\xi\right|\\ &= J_1+J_2.\end{aligned}$$

For J_1, we apply Lemma 4.6.1 to K to obtain

$$J_1 \lesssim \int_{d(x,\xi)\leq 1} d(x,\xi)^\alpha |K(d(x,\xi))| \,\mathrm{d}\xi,$$

where $\alpha = 2p-2$ when $p \in (\frac{2}{2-s}, 2)$ and $\alpha = p$ when $p \in (1, \frac{2}{2-s}] \cup [2,\infty)$. For J_2, we have

$$J_2 \lesssim \|u\|^{p-1}_{L^\infty(\mathbb{H}^n)} \int_{d(x,\xi)>1} |K(d(x,\xi))| \,\mathrm{d}\xi.$$

By the dominated convergence theorem, we conclude $J_1 + J_2 \to 0$ as $y \searrow 0$. □

4.5 Pointwise convergence

This section is devoted to the proof of Theorem 4.1.4, which uses the pointwise representation (4.1.3) of the fractional p-Laplacian on $\mathbb{H}^n$. As mentioned in Section 4.1, the limits of the integrals

$$c_{n,s,p} \int_R^\infty \mathcal{K}_{n,s,p}(\rho) \sinh^{n-1}\rho \,\mathrm{d}\rho, \quad c_{n,s,p} \int_0^R \rho^p \mathcal{K}_{n,s,p}(\rho) \sinh^{n-1}\rho \,\mathrm{d}\rho, \quad (4.5.1)$$

and

$$c_{n,s,p} \int_0^R \rho^{\beta+p} \mathcal{K}_{n,s,p} \sinh^{n-1}\rho \,\mathrm{d}\rho, \quad \beta > 0, \quad (4.5.2)$$

as $s \to 1^-$, play a key role in the proof of Theorem 4.1.4. Let us begin with the following lemma.

Lemma 4.5.1. *Let $\nu > 1/2$, $a \geq 1/2$, and $m \in \mathbb{N} \cup \{0\}$. Then, the function*

$$\rho \mapsto \left(\frac{-\partial_\rho}{\sinh\rho}\right)^m \mathscr{K}_{\nu,a}(\rho)$$

is positive, where $\mathscr{K}_{\nu,a}$ is the function given in (4.2.4).

Proof. Using the formula (4.2.1) and change of variables, we have

$$\mathscr{K}_{\nu,a}(\rho) = \frac{a^\nu}{2^{\nu+1}} \int_0^\infty e^{-t-\frac{(a\rho)^2}{4t}} t^{-\nu-1}\,\mathrm{d}t = \frac{1}{2(2a)^\nu} \int_0^\infty \frac{1}{t^{1/2}} e^{-a^2 t - \frac{\rho^2}{4t}} \frac{\mathrm{d}t}{t^{\nu+1/2}}.$$

Thus, recalling the expression of the heat kernel (4.3.1) for odd dimensional case, we obtain

$$\left(\frac{-\partial_\rho}{\sinh\rho}\right)^m \mathscr{K}_{\nu,a}(\rho) = \int_0^\infty e^{(m^2-a^2)t} p(t,\rho) \frac{\mathrm{d}t}{t^{\nu+1/2}}. \tag{4.5.3}$$

It is known [25, Lemma 2.3] that the heat kernel $p(t,\rho)$ is strictly decreasing with respect to ρ. Since $p(t,\rho) \to 0$ as $\rho \to \infty$, we deduce that $p(t,\rho)$ is positive. Therefore, the conclusion follows from (4.5.3). □

As a consequence of Lemma 4.5.1, we obtain the positivity of the kernel $\mathcal{K}_{n,s,p}$.

Corollary 4.5.2. *Let $n \in \mathbb{N}$, $s \in (0,1)$, and $p > 1$. The kernel $\mathcal{K}_{n,s,p}$ is positive.*

In the following series of lemmas, we compute limits of the integrals in (4.5.1) and (4.5.2) with the help of Lemma 4.5.1.

Lemma 4.5.3. *Let $n \geq 2$ and $p > 1$. For any $R > 0$,*

$$\lim_{s \nearrow 1} c_{n,s,p} \int_R^\infty \mathcal{K}_{n,s,p}(\rho) \sinh^{n-1}\rho\,\mathrm{d}\rho = 0.$$

Proof. Let us first consider the case $n = 2m+1$ with $m \geq 1$. Since $c_{n,s,p}C_2 \leq C(1-s)$ for some $C = C(n,p) > 0$, by using Lemma 4.5.1 we have

$$\begin{aligned} 0 &\leq c_{n,s,p} \int_R^\infty \mathcal{K}_{n,s,p}(\rho) \sinh^{n-1}\rho\,\mathrm{d}\rho \\ &\lesssim (1-s) \int_R^\infty \sinh^{2m}\rho \left(\frac{-\partial_\rho}{\sinh\rho}\right)^m \mathscr{K}_{\frac{1+sp}{2},m}(\rho)\,\mathrm{d}\rho. \end{aligned} \tag{4.5.4}$$

Thus, it is enough to show that the right-hand side of (4.5.4) converges to zero as $s \to 1^-$. We actually prove the following stronger statement:

$$\lim_{s \nearrow 1}(1-s)\int_R^\infty \sinh^{m+a}\rho \left(\frac{-\partial_\rho}{\sinh\rho}\right)^m \mathscr{K}_{\frac{1+sp}{2},a}(\rho)\,\mathrm{d}\rho = 0 \quad \text{for each } a>0. \tag{4.5.5}$$

We use the induction on m. When $m=1$, using (4.2.3) and the fact that K_ν is increasing with respect to $\nu > 0$, we have

$$\begin{aligned}\int_R^\infty \sinh^{1+a}\rho \left(\frac{-\partial_\rho}{\sinh\rho}\right) \mathscr{K}_{\frac{1+sp}{2},a}(\rho)\,\mathrm{d}\rho &= a\int_R^\infty (\sinh^a\rho)\rho^{-\frac{1+sp}{2}} K_{\frac{3+sp}{2}}(a\rho)\,\mathrm{d}\rho \\ &\le a\int_R^\infty (\sinh^a\rho)\rho^{-\frac{1+sp}{2}} K_{\frac{3+p}{2}}(a\rho)\,\mathrm{d}\rho.\end{aligned}$$

By (4.2.2), there exists $M = M(p) > 1$ such that

$$K_{\frac{3+p}{2}}(\rho) \le \sqrt{\frac{\pi}{\rho}}e^{-\rho} \quad \text{for } \rho > M. \tag{4.5.6}$$

The inequalities $\rho^{-\frac{1+sp}{2}} \le \max\{\rho^{-\frac{1}{2}}, \rho^{-\frac{1+p}{2}}\}$ and $\sinh\rho < e^\rho$, together with (4.5.6), yield

$$\begin{aligned}&\int_R^\infty (\sinh^a\rho)\rho^{-\frac{1+sp}{2}} K_{\frac{3+p}{2}}(a\rho)\,\mathrm{d}\rho \\ &\le \int_R^{M/a} (\sinh^a\rho)\max\left\{\rho^{-\frac{1}{2}}, \rho^{-\frac{1+p}{2}}\right\} K_{\frac{3+p}{2}}(a\rho)\,\mathrm{d}\rho + \sqrt{\frac{\pi}{a}}\int_{M/a}^\infty \rho^{-1-\frac{sp}{2}}\,\mathrm{d}\rho.\end{aligned}$$

Note that the first integral in the right-hand side of the inequality above is a constant depending on a, p, and R only. For the second integral, we estimate

$$\sqrt{\frac{\pi}{a}}\int_{M/a}^\infty \rho^{-1-\frac{sp}{2}}\,\mathrm{d}\rho = \frac{2}{sp}\sqrt{\frac{\pi}{a}}\left(\frac{a}{M}\right)^{\frac{sp}{2}} \le \frac{2}{sp}\sqrt{\frac{\pi}{a}}\max\left\{\left(\frac{a}{M}\right)^{\frac{p}{2}}, 1\right\}.$$

Thus, we arrive at

$$\lim_{s \nearrow 1}(1-s)\int_R^\infty \sinh^{1+a}\rho \left(\frac{-\partial_\rho}{\sinh\rho}\right) \mathscr{K}_{\frac{1+sp}{2},a}(\rho)\,\mathrm{d}\rho = 0,$$

which proves (4.5.5) for $m=1$.

Assume now that (4.5.5) is true for m and prove it for $m+1$. Using integration by parts, we have

$$\begin{aligned}
&\lim_{s\nearrow 1}(1-s)\int_R^\infty \sinh^{m+1+a}\rho\left(\frac{-\partial_\rho}{\sinh\rho}\right)^{m+1}\mathscr{K}_{\frac{1+sp}{2},a}(\rho)\,\mathrm{d}\rho\\
&=\lim_{s\nearrow 1}(1-s)(m+a)\int_R^\infty \sinh^{m+a-1}\rho\cosh\rho\left(\frac{-\partial_\rho}{\sinh\rho}\right)^{m}\mathscr{K}_{\frac{1+sp}{2},a}(\rho)\,\mathrm{d}\rho.
\end{aligned}$$

Thus, by an inequality

$$\cosh\rho\le\coth R\sinh\rho\quad\text{for }\rho\ge R, \tag{4.5.7}$$

Lemma 4.5.1, and the induction hypothesis, we conclude

$$\begin{aligned}
&\lim_{s\nearrow 1}(1-s)\int_R^\infty \sinh^{m+1+a}\rho\left(\frac{-\partial_\rho}{\sinh\rho}\right)^{m+1}\mathscr{K}_{\frac{1+sp}{2},a}(\rho)\,\mathrm{d}\rho\\
&\le(m+a)(\coth R)\lim_{s\nearrow 1}(1-s)\int_R^\infty \sinh^{m+a}\rho\left(\frac{-\partial_\rho}{\sinh\rho}\right)^{m}\mathscr{K}_{\frac{1+sp}{2},a}(\rho)\,\mathrm{d}\rho=0.
\end{aligned}$$

This finishes the proof of the lemma in the odd dimensional case.

Let us next consider the even dimensional cases $n=2m$ with $m\ge 1$. Similarly as in the odd dimensional case, since

$$\begin{aligned}
0&\le\int_R^\infty \mathcal{K}_{n,s,p}(\rho)\sinh^{n-1}\rho\,\mathrm{d}\rho\\
&\lesssim(1-s)\int_R^\infty\sinh^{2m-1}\rho\int_\rho^\infty\frac{\sinh r}{\sqrt{\cosh r-\cosh\rho}}\left(\frac{-\partial_r}{\sinh r}\right)^m\mathscr{K}_{\frac{1+sp}{2},\frac{2m-1}{2}}(r)\,\mathrm{d}r\,\mathrm{d}\rho,
\end{aligned}$$

the desired result will follow once we prove the following:

$$\begin{aligned}
\lim_{s\nearrow 1}(1-s)\int_R^\infty\sinh^{\frac{2m-1}{2}+a}\rho\int_\rho^\infty\frac{\sinh r}{\sqrt{\cosh r-\cosh\rho}}\left(\frac{-\partial_r}{\sinh r}\right)^m\mathscr{K}_{\frac{1+sp}{2},a}(r)\,\mathrm{d}r\,\mathrm{d}\rho&\\
=0\quad\text{for each }a\ge 1/2.&
\end{aligned}\tag{4.5.8}$$

If $m = 1$, then

$$
\begin{aligned}
&\int_R^\infty \sinh^{\frac{1}{2}+a}\rho \int_\rho^\infty \frac{\sinh r}{\sqrt{\cosh r - \cosh\rho}} \left(\frac{-\partial_r}{\sinh r}\right) \mathscr{K}_{\frac{1+sp}{2},a}(r)\,\mathrm{d}r\,\mathrm{d}\rho \\
&\le a\int_R^{M/a} \sinh^{\frac{1}{2}+a}\rho \int_\rho^\infty \frac{1}{\sqrt{\cosh r - \cosh\rho}} r^{-\frac{1+sp}{2}} K_{\frac{3+p}{2}}(ar)\,\mathrm{d}r\,\mathrm{d}\rho \\
&\quad + a\int_{M/a}^\infty \sinh^{\frac{1}{2}+a}\rho \int_\rho^\infty \frac{1}{\sqrt{\cosh r - \cosh\rho}} r^{-\frac{1+sp}{2}} K_{\frac{3+p}{2}}(ar)\,\mathrm{d}r\,\mathrm{d}\rho =: J_1 + J_2.
\end{aligned}
$$

For J_2, we use (4.5.6) to obtain

$$
J_2 \le \sqrt{\pi a}\int_{M/a}^\infty \frac{\sinh^{\frac{1}{2}+a}\rho}{\rho^{1+\frac{sp}{2}}e^{a\rho}} \int_\rho^\infty \frac{1}{\sqrt{\cosh r - \cosh\rho}}\,\mathrm{d}r\,\mathrm{d}\rho.
$$

Since

$$
\begin{aligned}
\int_\rho^\infty \frac{1}{\sqrt{\cosh r - \cosh\rho}}\,\mathrm{d}r &= \frac{1}{\sqrt{2}}\int_\rho^\infty \frac{1}{\sqrt{\sinh\frac{r+\rho}{2}\sinh\frac{r-\rho}{2}}}\,\mathrm{d}r \\
&\le \frac{1}{\sqrt{2\sinh\rho}}\int_\rho^\infty \frac{1}{\sqrt{\sinh\frac{r-\rho}{2}}}\,\mathrm{d}r \\
&= \frac{1}{\sqrt{2\sinh\rho}}\int_0^\infty \frac{1}{\sqrt{\sinh\frac{r}{2}}}\,\mathrm{d}r = \frac{\Gamma(1/4)}{\Gamma(3/4)}\sqrt{\frac{\pi}{\sinh\rho}}
\end{aligned}
\tag{4.5.9}
$$

and $\sinh^a\rho \le e^{a\rho}$, we have

$$
J_2 \le \frac{\Gamma(1/4)}{\Gamma(3/4)}\pi\sqrt{a}\int_{M/a}^\infty \rho^{-1-\frac{sp}{2}}\,\mathrm{d}\rho = \frac{\Gamma(1/4)}{\Gamma(3/4)}\frac{\pi\sqrt{a}}{sp}\left(\frac{a}{M}\right)^{\frac{sp}{2}}.
$$

On the other hand, for J_1 we observe

$$
J_1 \le a\int_R^{M/a} \sinh^{\frac{1}{2}+a}\rho \int_\rho^\infty \frac{\max\{r^{-\frac{1+p}{2}}, r^{-\frac{1}{2}}\}}{\sqrt{\cosh r - \cosh\rho}} K_{\frac{3+p}{2}}(ar)\,\mathrm{d}r\,\mathrm{d}\rho.
$$

Since the inner integral is continuous and integrable on $[R, M/a]$, J_1 is controlled by some constant $C = C(a,p,R) > 0$. Therefore, we conclude

$\lim_{s\nearrow 1}(1-s)(J_1+J_2)=0$, which proves (4.5.8) for $m=1$.

Finally, let us assume that (4.5.8) holds for m and prove it for $m+1$. By Lemma 4.3.1, we have

$$\begin{aligned}
&\lim_{s\nearrow 1}(1-s)\int_R^\infty \sinh^{\frac{2m+1}{2}+a}\rho\int_\rho^\infty \frac{\sinh r}{\sqrt{\cosh r-\cosh\rho}}\left(\frac{-\partial_r}{\sinh r}\right)^{m+1}\mathscr{K}_{\frac{1+sp}{2},a}(r)\,\mathrm{d}r\,\mathrm{d}\rho\\
&=\lim_{s\nearrow 1}(1-s)\int_R^\infty \sinh^{\frac{2m+1}{2}+a}\rho\left(\frac{-\partial_\rho}{\sinh\rho}\right)\\
&\qquad\int_\rho^\infty \frac{\sinh r}{\sqrt{\cosh r-\cosh\rho}}\left(\frac{-\partial_r}{\sinh r}\right)^{m}\mathscr{K}_{\frac{1+sp}{2},a}(r)\,\mathrm{d}r\,\mathrm{d}\rho.
\end{aligned}$$

Using integration by parts, (4.5.7), and Lemma 4.5.1, we deduce

$$\begin{aligned}
&\lim_{s\nearrow 1}(1-s)\int_R^\infty \sinh^{\frac{2m+1}{2}+a}\rho\int_\rho^\infty \frac{\sinh r}{\sqrt{\cosh r-\cosh\rho}}\left(\frac{-\partial_r}{\sinh r}\right)^{m+1}\mathscr{K}_{\frac{1+sp}{2},a}(r)\,\mathrm{d}r\,\mathrm{d}\rho\\
&\le C\lim_{s\nearrow 1}(1-s)\int_R^\infty \sinh^{\frac{2m-1}{2}+a}\rho\int_\rho^\infty \frac{\sinh r}{\sqrt{\cosh r-\cosh\rho}}\left(\frac{-\partial_r}{\sinh r}\right)^{m}\mathscr{K}_{\frac{1+sp}{2},a}(r)\,\mathrm{d}r\,\mathrm{d}\rho
\end{aligned}$$

for some $C=C(m,a,R)$. Therefore, the statement (4.5.8) for $m+1$ follows by the induction hypothesis. □

Lemma 4.5.4. *Let $n\ge 2$ and $p>1$. For any $R>0$,*

$$\lim_{s\nearrow 1}c_{n,s,p}\int_0^R \rho^p\mathcal{K}_{n,s,p}(\rho)\sinh^{n-1}\rho\,\mathrm{d}\rho=\frac{1}{\pi^{\frac{n-1}{2}}}\frac{\Gamma(\frac{p+n}{2})}{\Gamma(\frac{p+1}{2})}. \tag{4.5.10}$$

The proof of Lemma 4.5.4 for the even dimensional case needs the following lemma.

Lemma 4.5.5. *Let $a>0$ and $\nu>-1/2$. Then,*

$$\int_\rho^\infty \frac{r^{-\nu}K_{\nu+1}(ar)}{\sqrt{\cosh r-\cosh\rho}}\,\mathrm{d}r\sim\sqrt{\frac{\pi}{2}}\frac{\Gamma(\nu+\frac12)}{\Gamma(\nu+1)}\rho^{-\nu}K_{\nu+1}(a\rho)$$

as $\rho\to 0^+$.

Proof. By the change of variables $r = \rho t$, we have

$$\int_\rho^\infty \frac{1}{\sqrt{\cosh r - \cosh \rho}} \frac{r^{-\nu} K_{\nu+1}(ar)}{\rho^{-\nu} K_{\nu+1}(a\rho)} \,\mathrm{d}r$$
$$= \int_1^\infty \frac{t^{-\nu}}{\sqrt{\cosh(\rho t) - \cosh \rho}} \frac{K_{\nu+1}(a\rho t)}{K_{\nu+1}(a\rho)} \rho \,\mathrm{d}t.$$

We define for each $\rho \in (0,1)$ a function f_ρ by

$$f_\rho(t) = \frac{t^{-\nu}}{\sqrt{\cosh(\rho t) - \cosh \rho}} \frac{K_{\nu+1}(a\rho t)}{K_{\nu+1}(a\rho)} \rho$$

on $(1,\infty)$. Note that we have

$$\frac{\cosh(\rho t) - \cosh \rho}{\rho^2} \geq \frac{1}{2}(t^2 - 1).$$

Moreover, by [65, Equation (2.17)], we have

$$\frac{K_{\nu+1}(a\rho t)}{K_{\nu+1}(a\rho)} \leq t^{-\nu-1}.$$

Thus, f_ρ is bounded from above by a function

$$f(t) := \frac{t^{-2\nu-1}}{\sqrt{(t^2-1)/2}},$$

which is integrable on $(1,\infty)$. Indeed, by the change of variables $t^2 - 1 = \tau$, we obatin

$$\int_1^\infty f(t)\,\mathrm{d}t = \frac{1}{\sqrt{2}} \int_1^\infty \frac{\tau^{-1/2}}{(1+\tau)^{1+\nu}} \,\mathrm{d}\tau = \frac{1}{\sqrt{2}} B\left(\frac{1}{2}, \nu + \frac{1}{2}\right)$$
$$= \sqrt{\frac{\pi}{2}} \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu+1)} < +\infty,$$

where B is Euler's Beta Integral (see [91, Section 5.12]).

For fixed $t \in (1, \infty)$, we have

$$\frac{\cosh(\rho t) - \cosh \rho}{\rho^2} \to \frac{1}{2}(t^2 - 1) \quad \text{and} \quad \frac{K_{\nu+1}(a\rho t)}{K_{\nu+1}(a\rho)} \to t^{-\nu-1}$$

as $\rho \to 0^+$. Hence, we obtain $\lim_{\rho \searrow 0} f_\rho(t) = f(t)$. Therefore, the Lebesgue dominated convergence theorem concludes the lemma. □

We are in a position to prove Lemma 4.5.4 by using Lemma 4.5.5.

Proof of Lemma 4.5.4. Let us first consider the odd dimensional case $n = 2m + 1$ with $m \geq 1$. One can easily check that (4.5.10) is equivalent to

$$\begin{aligned} &\lim_{s \nearrow 1}(1 - s) \int_0^R \rho^p \sinh^{2m} \rho \left(\frac{-\partial_\rho}{\sinh \rho} \right)^m \mathscr{K}_{\frac{1+sp}{2}, m}(\rho)\, \mathrm{d}\rho \\ &= \frac{2^{m-1}}{p} \left(\frac{2}{m} \right)^{\frac{p+1}{2}} \Gamma \left(\frac{p + 2m + 1}{2} \right) \end{aligned} \tag{4.5.11}$$

by using $\lim_{s \nearrow 1}(1 - s)|\Gamma(-s)| = 1$. Actually, we will prove the following statement, which is slightly stronger than (4.5.11):

$$\begin{aligned} &\lim_{s \nearrow 1}(1 - s) \int_0^R \rho^p \sinh^{2m} \rho \left(\frac{-\partial_\rho}{\sinh \rho} \right)^m \mathscr{K}_{\frac{1+sp}{2}, a}(\rho)\, \mathrm{d}\rho \\ &= \frac{2^{m-1}}{p} \left(\frac{2}{a} \right)^{\frac{p+1}{2}} \Gamma \left(\frac{p + 2m + 1}{2} \right) \quad \text{for each } a \geq 1. \end{aligned} \tag{4.5.12}$$

Let $\varepsilon \in (0, 1)$, then there exists $\delta_0 \in (0, 1)$ such that

$$1 - \varepsilon \leq \frac{\sinh \rho}{\rho} \leq 1 + \varepsilon \tag{4.5.13}$$

for all $\rho \in (0, \delta_0)$. Moreover, using the asymptotic behavior (4.2.2) of the modified Bessel function, for each $s \in [0, 1]$ we find $\delta_s > 0$ such that

$$\frac{1 - \varepsilon}{2} \Gamma \left(\frac{3 + sp}{2} \right) \left(\frac{\rho}{2} \right)^{-\frac{3+sp}{2}} \leq K_{\frac{3+sp}{2}}(\rho) \leq \frac{1 + \varepsilon}{2} \Gamma \left(\frac{3 + sp}{2} \right) \left(\frac{\rho}{2} \right)^{-\frac{3+sp}{2}} \tag{4.5.14}$$

for all $\rho \in (0, \delta_s)$. Furthermore, since K_ν is uniformly continuous with respect to ν, we may assume that δ_s has been chosen continuously on s. Let us take $\delta = \delta_0 \wedge \min_{s\in[0,1]} \delta_s \wedge R$, then $\delta = \delta(\varepsilon, p, R) > 0$, and (4.5.13) and (4.5.14) hold for all $\rho \in (0, \delta)$.

We fix $a \geq 1$ and denote by $G_{s,p,m,a}(\rho)$ the integrand in the left-hand side of (4.5.12). Then, $|G_{s,p,m,a}(\rho)|$ is bounded by the function $\sup_{0\leq s\leq 1} |G_{s,p,m,a}(\rho)|$, which is independent of s and bounded on a compact interval $[\delta/a, R]$. Thus, we have

$$\lim_{s\nearrow 1}(1-s)\int_{\delta/a}^{R} G_{s,p,m,a}(\rho)\,\mathrm{d}\rho = 0,$$

and hence

$$\lim_{s\nearrow 1}(1-s)\int_{0}^{R} G_{s,p,m,a}(\rho)\,\mathrm{d}\rho = \lim_{s\nearrow 1}(1-s)\int_{0}^{\delta/a} G_{s,p,m,a}(\rho)\,\mathrm{d}\rho.$$

Let us now prove (4.5.12) by induction. When $m = 1$, we first use (4.2.3) to have

$$G_{s,p,1,a}(\rho) = a\rho^{p-\frac{1+sp}{2}} K_{\frac{3+sp}{2}}(a\rho)\sinh\rho.$$

If $\rho < \delta/a$, then $\rho \leq a\rho < \delta$ since $a \geq 1$. Thus, we utilize (4.5.13) and (4.5.14) to obtain

$$\begin{aligned}(1-\varepsilon)^2\left(\frac{2}{a}\right)^{\frac{1+sp}{2}}\Gamma\left(\frac{3+sp}{2}\right)\rho^{p(1-s)-1} &\leq G_{s,p,1,a}(\rho)\\ &\leq (1+\varepsilon)^2\left(\frac{2}{a}\right)^{\frac{1+sp}{2}}\Gamma\left(\frac{3+sp}{2}\right)\rho^{p(1-s)-1}.\end{aligned}$$

This leads us to the inequalities

$$\begin{aligned}
&\lim_{s\nearrow 1}(1-s)\int_0^R G_{s,p,1,a}(\rho)\,\mathrm{d}\rho \\
&= \lim_{s\nearrow 1}(1-s)\int_0^{\delta/a} G_{s,p,1,a}(\rho)\,\mathrm{d}\rho \\
&\le \lim_{s\nearrow 1}(1-s)(1+\varepsilon)^2\left(\frac{2}{a}\right)^{\frac{1+sp}{2}}\Gamma\left(\frac{3+sp}{2}\right)\int_0^{\delta/a}\rho^{p(1-s)-1}\,\mathrm{d}\rho \\
&= (1+\varepsilon)^2\frac{1}{p}\left(\frac{2}{a}\right)^{\frac{p+1}{2}}\Gamma\left(\frac{p+3}{2}\right)
\end{aligned}$$

and

$$\lim_{s\nearrow 1}(1-s)\int_0^R G_{s,p,1,a}(\rho)\,\mathrm{d}\rho \ge (1-\varepsilon)^2\frac{1}{p}\left(\frac{2}{a}\right)^{\frac{p+1}{2}}\Gamma\left(\frac{p+3}{2}\right).$$

Therefore, the statement (4.5.12) for $m=1$ follows by taking $\varepsilon \to 0$.

Assume now that (4.5.12) holds for $m \ge 1$. Then, a similar argument shows

$$\begin{aligned}
&\lim_{s\nearrow 1}(1-s)\int_0^R G_{s,p,m+1,a}(\rho)\,\mathrm{d}\rho \\
&= \lim_{s\nearrow 1}(1-s)\int_0^{\delta/a}\rho^p\sinh^{2m+2}\rho\left(\frac{-\partial_\rho}{\sinh\rho}\right)^{m+1}\mathscr{K}_{\frac{1+sp}{2},a}(\rho)\,\mathrm{d}\rho \\
&\le \lim_{s\nearrow 1}(1-s)(1+\varepsilon)^{2m+1}\int_0^{\delta/a}\rho^{p+2m+1}(-\partial_\rho)\left(\frac{-\partial_\rho}{\sinh\rho}\right)^{m}\mathscr{K}_{\frac{1+sp}{2},a}(\rho)\,\mathrm{d}\rho,
\end{aligned}$$

where nonnegativity of the integrands follows from Lemma 4.5.1. Using the integration by parts, (4.5.13), and the induction hypothesis, we arrive at

$$\begin{aligned}
&\lim_{s\nearrow 1}(1-s)\int_0^R G_{s,p,m+1,a}(\rho)\,\mathrm{d}\rho \\
&\le (1+\varepsilon)^{2m+1}(p+2m+1)\lim_{s\nearrow 1}(1-s)\int_0^{\delta/a}\rho^{p+2m}\left(\frac{-\partial_\rho}{\sinh\rho}\right)^m \mathscr{K}_{\frac{1+sp}{2},a}(\rho)\,\mathrm{d}\rho \\
&\le \frac{(1+\varepsilon)^{2m+1}}{(1-\varepsilon)^{2m}}(p+2m+1)\lim_{s\nearrow 1}(1-s)\int_0^{\delta/a}\rho^{p}\sinh^{2m}\rho\left(\frac{-\partial_\rho}{\sinh\rho}\right)^m \mathscr{K}_{\frac{1+sp}{2},a}(\rho)\,\mathrm{d}\rho \\
&= \frac{(1+\varepsilon)^{2m+1}}{(1-\varepsilon)^{2m}}(p+2m+1)\frac{2^{m-1}}{p}\left(\frac{2}{a}\right)^{\frac{p+1}{2}}\Gamma\left(\frac{p+2m+1}{2}\right) \\
&= \frac{(1+\varepsilon)^{2m+1}}{(1-\varepsilon)^{2m}}\frac{2^{m}}{p}\left(\frac{2}{a}\right)^{\frac{p+1}{2}}\Gamma\left(\frac{p+2m+3}{2}\right).
\end{aligned}$$

Similarly, we obtain

$$\lim_{s\nearrow 1}(1-s)\int_0^R G_{s,p,m+1,a}(\rho)\,\mathrm{d}\rho \ge \frac{(1-\varepsilon)^{2m+1}}{(1+\varepsilon)^{2m}}\frac{2^{m}}{p}\left(\frac{2}{a}\right)^{\frac{p+1}{2}}\Gamma\left(\frac{p+2m+3}{2}\right),$$

from which (4.5.12) for $m+1$ follows by taking $\varepsilon \to 0$. The statement (4.5.12) has been proved for all $m \in \mathbb{N}$, finishing the proof of (4.5.10) for the odd dimensional case.

Let us next consider the even dimensional case $n = 2m$ with $m \ge 1$. In this case, (4.5.10) is equivalent to

$$\begin{aligned}
&\lim_{s\nearrow 1}(1-s)\int_0^R \rho^p\sinh^{2m-1}\rho\int_\rho^\infty \frac{\sinh r}{\sqrt{\cosh r-\cosh\rho}}\left(\frac{-\partial_r}{\sinh r}\right)^m \mathscr{K}_{\frac{1+sp}{2},\frac{2m-1}{2}}(r)\,\mathrm{d}r\,\mathrm{d}\rho \\
&= \sqrt{\frac{\pi}{2}}\frac{2^{m-1}}{p}\left(\frac{2}{m-1/2}\right)^{\frac{p+1}{2}}\Gamma\left(\frac{p+2m}{2}\right).
\end{aligned}$$

As in the odd dimensional case, we will prove a stronger statement:

$$\lim_{s\nearrow 1}(1-s)\int_0^R \rho^p \sinh^{2m-1}\rho \int_\rho^\infty \frac{\sinh r}{\sqrt{\cosh r-\cosh\rho}}\left(\frac{-\partial_r}{\sinh r}\right)^m \mathscr{K}_{\frac{1+sp}{2},a}(r)\,\mathrm{d}r\,\mathrm{d}\rho$$
$$=\sqrt{\frac{\pi}{2}}\frac{2^{m-1}}{p}\left(\frac{2}{a}\right)^{\frac{p+1}{2}}\Gamma\left(\frac{p+2m}{2}\right)\quad \text{for each } a\geq 1/2. \tag{4.5.15}$$

Recall that we have taken δ so that (4.5.13) and (4.5.14) hold for all $\rho\in(0,\delta)$. Let us fix $a\geq 1/2$. By Lemma 4.5.5, for each $s\in[0,1]$ we find $\tilde{\delta}_s>0$ such that

$$\begin{aligned}(1-\varepsilon)\sqrt{\frac{\pi}{2}}\frac{\Gamma(\frac{2+sp}{2})}{\Gamma(\frac{3+sp}{2})}\rho^{-\frac{1+sp}{2}}K_{\frac{3+sp}{2}}(a\rho) &\leq \int_\rho^\infty \frac{r^{-\frac{1+sp}{2}}K_{\frac{3+sp}{2}}(ar)}{\sqrt{\cosh r-\cosh\rho}}\,\mathrm{d}r\\ &\leq (1+\varepsilon)\sqrt{\frac{\pi}{2}}\frac{\Gamma(\frac{2+sp}{2})}{\Gamma(\frac{3+sp}{2})}\rho^{-\frac{1+sp}{2}}K_{\frac{3+sp}{2}}(a\rho)\end{aligned} \tag{4.5.16}$$

for all $\rho\in(0,\tilde{\delta}_s)$. Moreover, we may assume that $\tilde{\delta}_s$ has been chosen continuously on s. Let $\tilde{\delta}=\delta\wedge\min_{s\in[0,1]}\tilde{\delta}_s$, then $\delta=\delta(\varepsilon,p,R,a)>0$ and (4.5.16) holds for all $\rho\in(0,\tilde{\delta})$.

We denote by $H_{s,p,m,a}(\rho)$ the integrand in the left-hand side of (4.5.15). Then, the same argument as in the odd dimensional case shows

$$\lim_{s\nearrow 1}(1-s)\int_0^R H_{s,p,m,a}(\rho)\,\mathrm{d}\rho=\lim_{s\nearrow 1}(1-s)\int_0^{\frac{\delta}{2a}} H_{s,p,m,a}(\rho)\,\mathrm{d}\rho.$$

We argue by induction again to prove (4.5.15). If $m=1$, then

$$H_{s,p,1,a}(\rho)=a\rho^p\sinh\rho\int_\rho^\infty \frac{1}{\sqrt{\cosh r-\cosh\rho}}r^{-\frac{1+sp}{2}}K_{\frac{3+sp}{2}}(ar)\,\mathrm{d}r.$$

Since $a\geq 1/2$, we have $\rho<\delta$ and $a\rho<\delta$ for $\rho<\frac{\delta}{2a}$. Thus, by (4.5.13), (4.5.16), and (4.5.14), we obtain

$$
\begin{aligned}
(1-\varepsilon)^3 \sqrt{\frac{\pi}{2}} \left(\frac{2}{a}\right)^{\frac{1+sp}{2}} \Gamma\left(\frac{2+sp}{2}\right) \rho^{p(1-s)-1} & \\
\leq H_{s,p,1,a}(\rho) & \\
\leq (1+\varepsilon)^3 \sqrt{\frac{\pi}{2}} \left(\frac{2}{a}\right)^{\frac{1+sp}{2}} \Gamma\left(\frac{2+sp}{2}\right) \rho^{p(1-s)-1}. &
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
(1-\varepsilon)^3 \sqrt{\frac{\pi}{2}} \frac{1}{p} \left(\frac{2}{a}\right)^{\frac{p+1}{2}} \Gamma\left(\frac{p+3}{2}\right) &\leq \lim_{s\nearrow 1} (1-s) \int_0^R H_{s,p,1,a}(\rho)\,\mathrm{d}\rho \\
&\leq (1+\varepsilon)^3 \sqrt{\frac{\pi}{2}} \frac{1}{p} \left(\frac{2}{a}\right)^{\frac{p+1}{2}} \Gamma\left(\frac{p+3}{2}\right),
\end{aligned}
$$

from which we deduce (4.5.15) for $m = 1$ by taking $\varepsilon \to 0$.

Suppose that (4.5.15) is true for $m \geq 1$. Then, by (4.5.13), Lemma 4.3.1, and Lemma 4.5.1, we have

$$
\begin{aligned}
&\lim_{s\nearrow 1} (1-s) \int_0^R H_{s,p,m+1,a}(\rho)\,\mathrm{d}\rho \\
&= \lim_{s\nearrow 1} (1-s) \int_0^{\frac{\delta}{2a}} \rho^p \sinh^{2m+1}\rho \\
&\qquad \int_\rho^\infty \frac{\sinh r}{\sqrt{\cosh r - \cosh\rho}} \left(\frac{-\partial_r}{\sinh r}\right)^{m+1} \mathscr{K}_{\frac{1+sp}{2},a}(r)\,\mathrm{d}r\,\mathrm{d}\rho \\
&\leq \lim_{s\nearrow 1} (1-s)(1+\varepsilon)^{2m} \int_0^{\frac{\delta}{2a}} \rho^{p+2m}(-\partial_\rho) \\
&\qquad \int_\rho^\infty \frac{\sinh r}{\sqrt{\cosh r - \cosh\rho}} \left(\frac{-\partial_r}{\sinh r}\right)^{m} \mathscr{K}_{\frac{1+sp}{2},a}(r)\,\mathrm{d}r\,\mathrm{d}\rho.
\end{aligned}
$$

Using the integration by parts, (4.5.13), and the induction hypothesis, we arrive at

$$
\begin{aligned}
&\lim_{s\nearrow 1}(1-s)\int_0^R H_{s,p,m+1,a}(\rho)\,\mathrm{d}\rho \\
&= (1+\varepsilon)^{2m}(p+2m) \\
&\quad\times \lim_{s\nearrow 1}(1-s)\int_0^{\frac{\delta}{2a}} \rho^{p+2m-1}\int_\rho^\infty \frac{\sinh r}{\sqrt{\cosh r-\cosh\rho}}\left(\frac{-\partial_r}{\sinh r}\right)^m \mathscr{K}_{\frac{1+sp}{2},a}(r)\,\mathrm{d}r\,\mathrm{d}\rho \\
&\le \frac{(1+\varepsilon)^{2m}}{(1-\varepsilon)^{2m-1}}(p+2m)\lim_{s\nearrow 1}(1-s)\int_0^{\frac{\delta}{2a}} H_{s,p,m,a}(\rho)\,\mathrm{d}\rho \\
&= \frac{(1+\varepsilon)^{2m}}{(1-\varepsilon)^{2m-1}}\sqrt{\frac{\pi}{2}}\frac{2^m}{p}\left(\frac{2}{a}\right)^{\frac{p+1}{2}}\Gamma\left(\frac{p+2m+2}{2}\right).
\end{aligned}
$$

The inequality

$$
\lim_{s\nearrow 1}(1-s)\int_0^R H_{s,p,m+1,a}(\rho)\,\mathrm{d}\rho \ge \frac{(1-\varepsilon)^{2m}}{(1+\varepsilon)^{2m-1}}\sqrt{\frac{\pi}{2}}\frac{2^m}{p}\left(\frac{2}{a}\right)^{\frac{p+1}{2}}\Gamma\left(\frac{p+2m+2}{2}\right)
$$

can be obtained in the same way. Thus, we conclude that (4.5.15) for $m+1$ holds by taking $\varepsilon\to 0$. This finishes the proof for the even dimensional case. □

Lemma 4.5.6. *Let $n\ge 2$ and $p>1$. For any $R>0$ and $\beta>0$,*

$$
\lim_{s\nearrow 1} c_{n,s,p}\int_0^R \rho^{p+\beta}\mathcal{K}_{n,s,p}(\rho)\sinh^{n-1}\rho\,\mathrm{d}\rho = 0. \tag{4.5.17}
$$

Proof. We proceed as in the previous lemma to prove (4.5.17). When $n=2m+1$ with $m\ge 1$, we show

$$
\lim_{s\nearrow 1}\int_0^R \rho^{p+\beta}\sinh^{2m}\rho\left(\frac{-\partial_\rho}{\sinh\rho}\right)^m \mathscr{K}_{\frac{1+sp}{2},a}(\rho)\,\mathrm{d}\rho = 0 \quad \text{for each } a\ge 1
$$

by induction. Indeed, for $\varepsilon\in(0,1)$ let $\delta>0$ be the constant given in the proof of Lemma 4.5.4. Then, by using (4.5.13) and (4.5.14) we prove

$$\begin{aligned}
&\lim_{s\nearrow 1}(1-s)\int_0^R \rho^\beta G_{s,p,1,a}(\rho)\,\mathrm{d}\rho \\
&\leq \lim_{s\nearrow}(1-s)(1+\varepsilon)^2\left(\frac{2}{a}\right)^{\frac{1+sp}{2}}\Gamma\left(\frac{3+sp}{2}\right)\int_0^{\delta/a}\rho^{p(1-s)+\beta-1}\,\mathrm{d}\rho \\
&= \lim_{s\nearrow}(1-s)(1+\varepsilon)^2\left(\frac{2}{a}\right)^{\frac{1+sp}{2}}\Gamma\left(\frac{3+sp}{2}\right)\frac{1}{p(1-s)+\beta}\left(\frac{\delta}{a}\right)^{p(1-s)+\beta} = 0
\end{aligned}$$

for the case $m = 1$, where $G_{s,p,m,a}$ is the function defined in the proof of Lemma 4.5.4. Moreover, one can follow the steps in the proof of Lemma 4.5.4 to obtain

$$\begin{aligned}
&\frac{(1-\varepsilon)^{2m+1}}{(1+\varepsilon)^{2m}}(p+\beta+2m+1)\lim_{s\nearrow 1}(1-s)\int_0^R \rho^\beta G_{s,p,m,a}(\rho)\,\mathrm{d}\rho \\
&\leq \lim_{s\nearrow 1}(1-s)\int_0^R \rho^\beta G_{s,p,m+1,a}(\rho)\,\mathrm{d}\rho \\
&\leq \frac{(1+\varepsilon)^{2m+1}}{(1-\varepsilon)^{2m}}(p+\beta+2m+1)\lim_{s\nearrow 1}(1-s)\int_0^R \rho^\beta G_{s,p,m,a}(\rho)\,\mathrm{d}\rho,
\end{aligned}$$

which proves the induction step.

The even dimensional case $n = 2m$ with $m \geq 1$ can also be verified by proving

$$\lim_{s\nearrow 1}(1-s)\int_0^R \rho^{p+\beta}\sinh^{2m-1}\rho\int_\rho^\infty \frac{\sinh r}{\sqrt{\cosh r-\cosh\rho}}\left(\frac{-\partial_r}{\sinh r}\right)^m \mathscr{K}_{\frac{1+sp}{2},a}(r)\,\mathrm{d}r\,\mathrm{d}\rho = 0$$

for each $a \geq 1/2$. This can be proved by the induction as in the previous lemma, so we omit the proof. □

Let us provide the proof of Theorem 4.1.4 by using the pointwise representation (4.1.3) and Taylor's theorem, and gathering pieces of limits in the preceding lemmas.

Proof of Theorem 4.1.4. Let $u \in C_b^2(\mathbb{H}^n)$ and let $x \in \mathbb{H}^n$ be such that

$\nabla u(x) \neq 0$. Let $R > 0$, then by Lemma 4.5.3 we first have

$$\left| c_{n,s,p} \int_{d(x,\xi)\geq R} \Phi_p(u(x)-u(\xi)) \mathcal{K}_{n,s,p}(d(x,\xi)) \,\mathrm{d}\xi \right| \\ \lesssim c_{n,s,p} \int_R^\infty \mathcal{K}_{n,s,p}(\rho) \sinh^{n-1}\rho \,\mathrm{d}\rho \to 0$$

as $s \to 1^-$. Thus, by the pointwise representation (4.1.3) of the fractional p-Laplacian, we obtain

$$\lim_{s\nearrow 1} (-\Delta_{\mathbb{H}^n})_p^s u(x) = \lim_{s\nearrow 1} c_{n,s,p} \mathrm{P.V.} \int_{d(x,\xi)<R} \Phi_p(u(x)-u(\xi)) \mathcal{K}_{n,s,p}(d(x,\xi)) \,\mathrm{d}\xi. \tag{4.5.18}$$

Let $v = \exp_x^{-1} \xi$ be a tangent vector in $T_x\mathbb{H}^n$ and denote by $\mathcal{T}_x\xi$ the point $\exp_x(-v) \in \mathbb{H}^n$. Since $\mathcal{K}_{n,s,p}(d(x,\xi)) = \mathcal{K}_{n,s,p}(d(x,\mathcal{T}_x\xi))$, we write

$$\begin{aligned}
&\int_{d(x,\xi)<R} \Phi_p(u(x)-u(\xi)) \mathcal{K}_{n,s,p}(d(x,\xi)) \,\mathrm{d}\xi \\
&= \frac{1}{2} \int_{d(x,\xi)<R} |u(x)-u(\xi)|^{p-2} (2u(x)-u(\xi)-u(\mathcal{T}_x\xi)) \mathcal{K}_{n,s,p}(d(x,\xi)) \,\mathrm{d}\xi \\
&\quad + \frac{1}{2} \int_{d(x,\xi)<R} \left(|u(x)-u(\mathcal{T}_x\xi)|^{p-2} - |u(x)-u(\xi)|^{p-2} \right) \\
&\qquad\qquad \cdot (u(x)-u(\mathcal{T}_x\xi)) \mathcal{K}_{n,s,p}(d(x,\xi)) \,\mathrm{d}\xi \\
&=: J_1 + J_2.
\end{aligned}$$

By Taylor's theorem, we have

$$u(x)-u(\xi) = -\langle \nabla u(x), v\rangle + O(|v|^2), \quad u(x)-u(\mathcal{T}_x\xi) = \langle \nabla u(x), v\rangle + O(|v|^2),$$

and

$$2u(x)-u(\xi)-u(\mathcal{T}_x\xi) = -\langle D^2u(x)v, v\rangle + O(|v|^3).$$

If we write $\omega = v/|v|$, then

$$|u(x)-u(\xi)|^{p-2} = |v|^{p-2} |\langle \nabla u(x), \omega\rangle|^{p-2} + O(|v|^{p-1}).$$

Thus, we obtain

$$
\begin{aligned}
&|u(x)-u(\xi)|^{p-2}(2u(x)-u(\xi)-u(\mathcal{T}_x\xi)) \\
&\qquad\qquad = -|v|^p|\langle\nabla u(x),\omega\rangle|^{p-2}\langle D^2u(x)\omega,\omega\rangle + O(|v|^{p+1}).
\end{aligned}
$$

Therefore, we deduce

$$
\begin{aligned}
J_1 = &-\frac{1}{2}\int_0^R\int_{\mathbb{S}^{n-1}}\rho^p|\langle\nabla u(x),\omega\rangle|^{p-2}\langle D^2u(x)\omega,\omega\rangle\mathcal{K}_{n,s,p}(\rho)\sinh^{n-1}\rho\,\mathrm{d}\omega\,\mathrm{d}\rho \\
&+\frac{1}{2}\int_{d(x,\xi)<R}O(d(x,\xi)^{p+1})\mathcal{K}_{n,s,p}(d(x,\xi))\,\mathrm{d}\xi.
\end{aligned}
\tag{4.5.19}
$$

For J_2, since

$$
\begin{aligned}
&|u(\mathcal{T}_x\xi)-u(x)|^{p-2}-|u(x)-u(\xi)|^{p-2} \\
&= (p-2)|v|^{p-1}\langle\nabla u(x),\omega\rangle|\langle\nabla u(x),\omega\rangle|^{p-4}\langle D^2u(x)\omega,\omega\rangle + O(|v|^p),
\end{aligned}
$$

we have

$$
\begin{aligned}
&\left(|u(\mathcal{T}_x\xi)-u(x)|^{p-2}-|u(x)-u(\xi)|^{p-2}\right)(u(x)-u(\mathcal{T}_x\xi)) \\
&= -(p-2)|v|^p|\langle\nabla u(x),\omega\rangle|^{p-2}\langle D^2u(x)\omega,\omega\rangle + O(|v|^{p+1}).
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
J_2 = &-\frac{p-2}{2}\int_0^R\int_{\mathbb{S}^{n-1}}\rho^p|\langle\nabla u(x),\omega\rangle|^{p-2}\langle D^2u(x)\omega,\omega\rangle\mathcal{K}_{n,s,p}(\rho)\sinh^{n-1}\rho\,\mathrm{d}\omega\,\mathrm{d}\rho \\
&+\frac{1}{2}\int_{d(x,\xi)<R}O(d(x,\xi)^{p+1})\mathcal{K}_{n,s,p}(d(x,\xi))\,\mathrm{d}\xi.
\end{aligned}
\tag{4.5.20}
$$

Combining (4.5.18), (4.5.19), and (4.5.20), and using Lemma 4.5.4 and Lemma 4.5.6, we arrive at

$$
\lim_{s\nearrow 1}(-\Delta_{\mathbb{H}^n})_p^s u(x) = -\frac{p-1}{2}\frac{1}{\pi^{\frac{n-1}{2}}}\frac{\Gamma(\frac{p+n}{2})}{\Gamma(\frac{p+1}{2})}\int_{\mathbb{S}^{n-1}}|\langle\nabla u(x),\omega\rangle|^{p-2}\langle D^2u(x)\omega,\omega\rangle\,\mathrm{d}\omega.
$$

The argument as in the proof of [10, Theorem 2.8] shows

$$\int_{\mathbb{S}^{n-1}} |\langle \nabla u(x), \omega\rangle|^{p-2} \langle D^2 u(x)\omega, \omega\rangle \,\mathrm{d}\omega = \gamma_p (\Delta_{\mathbb{H}^n})_p$$

when $\nabla u(x) \neq 0$, where

$$\gamma_p = \int_{\mathbb{S}^{n-1}} |\omega_n|^{p-2} \omega_1^2 \,\mathrm{d}\omega = \pi^{\frac{n-1}{2}} \frac{\Gamma(\frac{p-1}{2})}{\Gamma(\frac{p+n}{2})}. \tag{4.5.21}$$

See [64, Lemma 2.1] for the computation of (4.5.21). This finishes the proof. □

4.6 Appendix

4.6.1 Auxiliary result

In this section, we recall an auxiliary result from [39] that helps proving Theorem 4.1.2 in Section 4.3.

Lemma 4.6.1. *Let $p > 1$, $r > 0$, $u \in C_b^2(\mathbb{H}^n)$, and $x \in \mathbb{H}^n$. If $p \in (1, \frac{2}{2-s}]$, assume $\nabla u(x) \neq 0$ additionally. If $K : \mathbb{H}^n \to \mathbb{R}$ is rotationally symmetric with respect to x, that is, $K(\xi) = K(d(x, \xi))$ for all $\xi \in \mathbb{H}^n$, then*

$$\left| \mathrm{P.V.} \int_{d(x,\xi)<r} \Phi_p(u(x) - u(\xi)) K(d(x, \xi)) \,\mathrm{d}\xi \right| \leq C \int_{d(x,\xi)<r} d(x, \xi)^\alpha |K(d(x, \xi))| \,\mathrm{d}\xi$$

for some constant $C = C(n, p, \|u\|_{C^2(\mathbb{H}^n)}) > 0$, where $\alpha = 2p - 2$ when $p \in (\frac{2}{2-s}, 2)$ and $\alpha = p$ otherwise.

The cases $p \in (1, \frac{2}{2-s}]$, $p \in (\frac{2}{2-s}, 2)$, and $p \in [2, \infty)$ are proved in [39, Lemma A.1, A2, and A3], respectively, for the case of Euclidean spaces. We omit the proof of Lemma 4.6.1 because the same proofs work in our framework.

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국문초록

본 학위논문은 비국소 타원형 방정식을 다룬 세 편의 연구논문으로 구성된다. 첫 번째 논문에서, 우리는 최대값 정리의 일종인 알렉산드로브-베켈만-푸찌 정리를 완전 비선형 비국소 작용소에 대해 0 이상의 단면 곡률을 가지는 다양체 위에서 확립한다. 우리의 접근 방법은 정규 함수의 조절과 단면 곡률로 부터 오는 직접 비교에 기인한다. 두 번째 논문은 쌍곡선 공간에서 에이비피 정리를 다루는 것이다. 쌍곡선 공간에서는 열 핵의 움직임이 유클리드 공간과는 다르다. 따라서 에이비피 정리를 얻는 과정에서 그러한 비동차 현상을 관찰할 수 있다. 분석의 핵심은 도약 핵과 관련된 적분 값들의 질적 특징을 얻는 것에 있다. 이러한 에이비피 정리들로 부터 우리는 크릴로브-사포노브 하낙 부등식을 얻을 수 있다. 세 번째 논문은 부분 쌍곡선 공간에서 피 라플라시안의 대등 정리들에 대한 것이다. 특히, 우리는 카파렐리의 확장 문제 또한 정립할 수 있었다. 특징으로, 우리는 쌍곡선 공간에서 부분 라플라시안의 계수와 하낙 부등식과 홀더 정칙성의 단단함을 구할 수 있었다.

주요어휘: ABP 근사, 다양체, 비국소 작용소, 하이퍼볼릭 공간, 파편 라플라시안, 확장 문제

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감사의 글

다사다난했던 박사과정을 이렇게 성공적으로 마무리할 수 있게 되어 정말 감사합니다. 무엇인가 새로운 것을 만든다는 것이 참 어려운 일이지만 또 한편으론 보람찬 일이라는 것을 배우게 된 박사과정이었습니다. 힘든 길이었지만 무사히 마무리할 수 있게 된 데에는 많은 분의 도움이 있었습니다. 우선 지도교수님이신 이기암 교수님께 감사드립니다. 수학을 공부한다는 게 어떤 것인지 교수님을 통해 배울 수 있었습니다. 결코 완벽히 따라 할 순 없겠지만 교수님께 보여주신 수학에 대한 놀라운 통찰력과 열정적인 자세를 배우고자 정말 많이 노력하였습니다. 또한 제가 힘들 때마다 따듯한 격려와 긍정적인 조언을 해주셔서 정말 힘이 많이 되었습니다.

바쁘신 와중에도 박사학위논문 심사에 참여해 주신 변순식 교수님, 김성훈 교수님, 김민현 교수님, 김판기 교수님께도 감사의 말씀을 드립니다. 박사 학위심사를 준비하는 과정에서 저의 주제에 대해 깊은 관심을 두시고 세심한 조언을 해주셔서 박사논문을 완성하는 데, 그리고 앞으로 좋은 수학자가 되는 데 많은 도움이 되었습니다.

박사학위 과정 동안 정말 많은 시간을 함께 보낸 연구실 동료들에게도 감사합니다. 효석이형, 상필이형, 형성이형, 성하형, 성한이형, 민현이형, 태훈이형, 탁원이형, 그리고 진제와 성은이, 애솔이에게 정말 많은 도움을 받았습니다. 비단 수학 공부, 수학 이야기를 한 것이 도움이 되었을 뿐만 아니라 이분들과 좋은 추억을 함께 할 수 있었기 때문에 행복하게 박사과정을 끝마칠 수 있었습니다.

짧게나마 지인들에게도 감사의 말을 전하고 싶습니다. 고향 친구들, 학부 친구들 그리고 함께 운동한 동아리 친구들에게 정말 감사합니다. 그들과 함께했기 때문에 숱한 어려움에도 좌절하지 않고 좋은 선택을 내릴 수 있었던 것 같습니다.

소중한 가족들에게 감사를 표하고 싶습니다. 힘들 때 항상 저를 믿어주시고 지지하여 주신 부모님. 항상 좋은 조언을 해준 형과 동생. 이처럼 가족 모두가 항상 무한한 사랑과 지지를 보내주어서 어떻게 이 은혜를 보답할 수 있을지 모르겠습니다. 어린 시절부터 부모님께서 베풀어 주신 사랑과 관심이 없었더라면 제가 이렇게 수학을 좋아하는 바른 사람으로 성장하지 못했을 겁니다.

설명할 순 없지만 힘들 때마다 가족들과의 추억이 정말 저에겐 많은 힘이 되었습니다. 오랜 기간이 걸렸던 학위과정이었지만 가족들이 언제나 저의 꿈을 묵묵히 지지해 주었기 때문에 박사학위를 마무리할 수 있었습니다.
미처 언급하지 못한 다른 모든 분에게도 감사를 표합니다. 학위과정은 제 인생에서 정말 행복했던 소중한 경험이었습니다. 이만 줄이도록 하겠습니다.