이학석사 학위논문

Hall Algebras of Fukaya Categories and Legendrian Skein Algebras

(푸카야 범주의 홀 대수와 르장드리안 스케인 대수)

2023년 8월

서울대학교 대학원

수리과학부

김서연

Hall Algebras of Fukaya Categories and Legendrian Skein Algebras

(푸카야 범주의 홀 대수와 르장드리안 스케인 대수)

지도교수 조 철 현

이 논문을 이학석사 학위논문으로 제출함

2023년 4월

서울대학교 대학원

수리과학부

김서연

김서연의 석사 학위논문을 인준함

2023년 6월

위 원 장	권 재 훈	(인)
부위원장	강 정 수	(인)
위 원	조 철 현	(인)

Abstract

Hall Algebras of Fukaya Categories and Legendrian Skein Algebras

Seoyeon Kim
Department of Mathematical Sciences
The Graduate School
Seoul National University

Given a surface S, it can be considered as a symplectic manifold, so we can think of the Fukaya category of the surface and its Hall algebra. On the other hand, $S \times \mathbb{R}$ has a natural contact structure so that we can think of its Legendrian skein algebra generated by Legendrian links. In this thesis, we examine how to define those algebras show that there is a $\mathbb{Q}$-algebra homomorphism between those two algebras.

Keywords: Fukaya category, Hall algebra, Legendrian Skein algebra

Student ID: 2016-20231

Contents

List of Figures

1 A_∞-categories

1.1 Definitions

There are various convention for defining A_∞-categories. We follow the one from [Hai21]

Definition 1.1 An A_∞*-category* $\mathcal{C}$ *over a field* k consists of the followings:

(a) a set $\mathrm{Ob}(\mathcal{C})$ of objects,

(b) a $\mathbb{Z}$-graded k-vector space $\mathrm{Hom}(X,Y)$ for each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$,

(c) structure maps

$$m_k : \mathrm{Hom}(X_0, X_1) \otimes \cdots \otimes \mathrm{Hom}(X_{k-1}, X_k) \to \mathrm{Hom}(X_0, X_k)$$

of degree $2-k$, for each $k \geq 1$, satisfying A_∞-equations

$$\sum_{r+s=k+1} \sum_i (-1)^\dagger m_r(x_1, \cdots, m_s(x_{i+1}, \cdots, x_{i+s}), \cdots, x_k) = 0 \tag{1.1}$$

where $\dagger = \deg' x_1 + \cdots + \deg' x_i$ and $\deg' x_j = \deg x_j + 1$.

The first three A_∞ equations are

$$m_1(m_1(x)) = 0 \tag{1.2}$$

$$m_2(m_1(x), y) + (-1)^{\deg' x} m_2(x, m_1(y)) + m_1(m_2(x, y)) = 0 \tag{1.3}$$

$$m_2(m_2(x, y), z) + (-1)^{\deg' x} m_2(x, m_2(y, z)) + m_3(m_1(x), y, z)+ \tag{1.4}$$

$$(-1)^{\deg' x} m_3(x, m_1(y), z) + (-1)^{\deg' x + \deg' y} m_3(x, y, m_1(z)) + m_1(m_3(x, y, z)) = 0$$

The equation (1.2) implies that m_1 is a differential operator. The equation (1.3) corresponds to the Leibniz rule for m_1-differential and m_2-product. If $m_k = 0$ for $k \geq 3$, then $\mathcal{C}$ is equivalent to a differential graded category by the following definition:

$$d(x) := (-1)^{\deg x} m_{1(x)}$$

$$x_2 \cdot x_1 := (-1)^{\deg x_1 \deg' x_2} m_2(x_1, x_2)$$

Then $\mathcal{C}$ with the differential d and the product $\cdot$ satisfies the definition of a differential graded category.

For a given A_∞-category $\mathcal{C}$, we can associate its *cohomological category* $H(\mathcal{C})$. It has the same objects as $\mathcal{C}$, and morphism spaces are given by the cohomological group $H(\mathrm{Hom}(X, Y), m_1)$. One can check that the associativity rule for mophisms holds for $H(\mathcal{C})$ up to sign, using the equation (1.4).

Note that an A_∞-category is not a category, since there is no associativity rule for morphisms. However, the equation (1.4) shows that the associativity holds up to homotopy. Also, unlike a category, there need not be the unit morphism in an A_∞-category. This suggests the following definition.

Definition 1.2 Let $\mathcal{C}$ be an A_∞-category.

(a) $\mathcal{C}$ is called *strictly unital* if there exists $1_X \in \mathrm{Hom}(X, X)$ for each object $X \in \mathrm{Ob}(\mathcal{C})$ satisfying the following equations:

$$m_2(1_X, x) = (-1)^{\deg x} m_2(x, 1_X) = x \tag{1.5}$$

$$m_{k+1}(x_1, \cdots, 1_X, \cdots, x_k) = 0 \text{ for } k \neq 2 \tag{1.6}$$

(b) $\mathcal{C}$ is called *cohomologically unital* if $H(\mathcal{C})$ is unital, i.e. there is the identity morphism for every object X of $H(\mathcal{C})$ so that $H(\mathcal{C})$ is a usual category.

There is a notion of functors between A_∞-categories like usual categories.

Definition 1.3 Let $\mathcal{C}, \mathcal{D}$ be A_∞-categories. An A_∞*-functor* from $\mathcal{C}$ to $\mathcal{D}$ consists of a map $\mathcal{F} : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and multilinear maps

$$\mathcal{F}^n : \mathrm{Hom}_{\mathcal{C}}(X_0, X_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{C}}(X_{n-1}, X_n) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X_0), \mathcal{F}(X_n))$$

satisfying the following A_∞-equation:

$$\begin{aligned} &\sum_r \sum_{s_1, \cdots, s_r} m_r(\mathcal{F}^{s_1}(x_1, \cdots, x_{s_1}), \cdots, \mathcal{F}^{s_r}(x_{k-s_r+1}, \cdots, x_k)) \\ &= \sum_{i,j} (-1)^\dagger \mathcal{F}^{k-j+1}(x_1, \cdots, x_i, m_j(x_{i+1}, \cdots, x_{i+j}), \cdots, x_k) \end{aligned} \tag{1.7}$$

where $\dagger = \deg' x_1 + \cdots + \deg' x_{i-1}$.

Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{E}$ be A_∞-functors. Then we can define the composition $\mathcal{G} \circ \mathcal{F} : \mathcal{C} \to \mathcal{E}$ as follows.

$$\begin{aligned} &(\mathcal{G} \circ \mathcal{F})^k(x_1, \cdots, x_k) \\ &= \sum_r \sum_{s_1, \cdots, s_r} \mathcal{G}^r(\mathcal{F}^{s_1}(x_1, \cdots, x_{s_1}), \cdots, \mathcal{F}^{s_r}(x_{k-s_r+1}, \cdots, x_k)) \end{aligned} \tag{1.8}$$

Also, we can define the identity functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ as $\mathcal{F}(X) = X, \mathcal{F}^1 = Id$ on $\mathrm{Hom}(X, X)$, and $\mathcal{F}^k = 0$ for $k \geq 2$.

1.2 Curved A_∞-categories and bounding cochains

In some A_∞-categories naturally arising in symplectic geometry, the A_∞-equations do not hold. In particular, m_1 is not a differential, so we cannot define

m_1-cohomology. This occurs because of the existence of the *curvature terms*. We generalize the definition of A_∞-categories to describe this phenomenon.

Definition 1.4 Given a vector space V, a *decreasing* $\mathbb{R}$*-filtration* on V is a collection of subspaces $V_{\geq\beta} \subset V$ for $\beta \in \mathbb{R}$ which satisfies the followings:

(a) $V_{\geq\alpha} \supset V_{\geq\beta}$ for $\alpha \leq \beta$,

(b) $V_{\geq\beta} = \bigcap_{\alpha<\beta} V_{\geq\alpha}$,

(c) the set of $\beta \in \mathbb{R}$ with $V_{\geq\beta}/V_{>\beta}$ is not empty, where $V_{>\beta} := \bigcup_{\alpha>\beta} V_{\geq\alpha}$ is discrete in $\mathbb{R}$,

(d) $\bigcap_\beta V_{\geq\beta} = \emptyset$

(e) $\bigcup_\beta V_{\geq\beta} = V$

(f) $\lim_{\leftarrow} V/V_{\geq\beta} = V$

An $\mathbb{R}$-*filtered vector space* is a vector space V with a decreasing $\mathbb{R}$-filtration. The filtration induces a topology on V in a natural way.

Definition 1.5 A *curved* A_∞*-category* $\mathcal{C}$ *over a field* k consists of the followings:

(a) a set $\mathrm{Ob}(\mathcal{C})$ of objects,

(b) a $\mathbb{Z}$-graded and $\mathbb{R}$-filtered k-vector space $\mathrm{Hom}(X, Y)$ for each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$,

(c) structure maps

$$m_k : \mathrm{Hom}(X_0, X_1) \otimes \cdots \otimes \mathrm{Hom}(X_{k-1}, X_k) \to \mathrm{Hom}(X_0, X_k)$$

of degree $2 - k$, for each $k \geq 0$, satisfying A_∞-equations

$$\sum_{r+s=k+1} \sum_i (-1)^\dagger m_r(x_1, \cdots, m_s(x_i, \cdots, x_{i+s-1}), \cdots, x_k) = 0 \qquad (1.9)$$

where $\dagger = \deg x_1 + \cdots + \deg x_{i-1} + (i-1)$.

In particular, m_0 is a linear map defined on the base field k with the target $\mathrm{Hom}^2(X,X)$. By abuse of notation, we denote the image of 1 under this linear map by m_0 or $m_0(X)$.

It is also required that $m_0 \in \mathrm{Hom}^2(X,X)_{\geq 0}$ and

$$x_i \in \mathrm{Hom}(X_{i-1},X_i)_{\geq \lambda_i} \Rightarrow m_k(x_1,\cdots,x_k) \in \mathrm{Hom}(X_0,X_k)_{\geq \lambda_1+\cdots+\lambda_k}.$$

The first two equations are

$$m_1(m_0) = 0 \tag{1.10}$$

$$m_1(m_1(x)) + m_2(m_0,x) + (-1)^{\deg x+1} m_2(x,m_0) = 0 \tag{1.11}$$

Thus, unless $m_0 = 0$, m_1 is not a differential.

Definition 1.6 An element $b \in \mathrm{Hom}^1(X,X)_{>0}$ is a *bounding cochain for an object* X if b satisfies the following equation:

$$\sum_{k=0}^{\infty} m_k(b,\cdots,b) = 0 \tag{1.12}$$

The equation (1.10) is called the Maurer-Cartan equation. A bounding cochain is called also a Maurer-Cartan element. We denote the set of bounding cochains for an object X by $\mathcal{MC}(X)$.

Using a bounding cochain, we can deform the A_∞-operations. Suppose there exist bounding cochains for every object X in a given curved A_∞-category $\mathcal{C}$. Then we define new structure maps $\tilde{m}_k$ as follows:

$$\tilde{m}_k(x_1,\cdots,x_k) := \sum m_n(b_0,\cdots,b_0,x_1,b_1,\cdots,b_{k-1},x_k,b_k,\cdots,b_k) \tag{1.13}$$

The right-hand side of the equation (1.11) is an infinite sum, but it converges with respect to the topology given by the filtration. Moreover, these new operations satisfy the usual A_∞-equations. In particular, $\tilde{m}_0$ vanishes and $\tilde{m}_1^2 = 0$. This can be checked by direct calculation as follows:

$$
\begin{aligned}
\tilde{m}_0 &= \sum_{k=0}^{\infty} m_k(b, \cdots, b) = 0 \\
\tilde{m}_1^2(x) &= \sum m_k(b_0, \cdots, b_0, \tilde{m}_1(x), b_1, \cdots, b_1) \\
&= \sum m_r(b_0, \cdots, b_0, m_s(b_0, \cdots, b_0, x, b_1, \cdots, b_1), b_1, \cdots, b_1) \\
&= -\sum m_r(b_0, \cdots, m_s(b_0, \cdots, b_0), \cdots, b_0, x, b_1, \cdots, b_1) \\
&\quad -\sum m_r(b_0, \cdots, b_0, x, b_1, \cdots, m_s(b_1, \cdots, b_1), \cdots, b_1) \\
&= 0
\end{aligned}
$$

The third equality of the second equation comes from A_∞ equation, and the last equality holds because b_0 and b_1 are bounding cochains.

Given a curved A_∞-category $\mathcal{C}$, we define a new A_∞ category $\tilde{\mathcal{C}}$. The object of $\tilde{\mathcal{C}}$ is a pair (X, b) where $X \in \mathrm{Ob}(\mathcal{C})$ and $b \in \mathcal{MC}(X)$. Also, morphism spaces are given by

$$\mathrm{Hom}_{\tilde{\mathcal{C}}}((X_0, b_0), (X_1, b_1)) := \mathrm{Hom}_{\mathcal{C}}(X_0, X_1).$$

The structure maps are $\tilde{m}_k$'s we defined above.

1.3 Twisted complexes

Since an A_∞-category is not an abelian category(as mentioned before, it is not even a category unless $m_{\geq 3} = 0$), there is not a notion of exactness. However, by enlarging it appropriately, we can think of a notion of exact triangles.

Definition 1.7 Let $\mathcal{C}$ be an A_∞-category. A *twisted complex* is a pair (X, δ) consisting of

(a) A formal sum $X = \bigoplus_{i=1}^{n} X_i[k_i]$ of shifted objects (where $X_i \in \mathrm{Ob}(\mathcal{C})$ and $k_i \in \mathbb{Z}$)

(b) A strictly upper triangular matrix $\delta = (\delta_{ij})_{1\leq i,j\leq n}$ where δ_{ij} is an element of $\mathrm{Hom}^{k_j-k_i+1}(X_i[k_i], X_j[k_j])$. Also, it satisfies the Maurer-Cartan equation

$$\sum_{k\geq 1} m_k(\delta, \cdots, \delta) = 0 \tag{1.14}$$

or equivalently, for all $1 \leq i < j \leq n$,

$$\sum_{k\geq 1} \sum_{i=i_0<i_1<\cdots<i_k=j} m_k(\delta_{i_0 i_1}, \cdots, \delta_{i_{k-1} i_k}) = 0 \tag{1.15}$$

Note that the left-hand side of the equation (1.12) is a finite sum since δ is a strictly upper triangular matrix.

Given an A_∞-category $\mathcal{C}$, we can define a new A_∞-category $\mathrm{Tw}(\mathcal{C})$ which is "triangulated". The objects are twisted complexes, and

$$\mathrm{Hom}^d_{\mathrm{Tw}\mathcal{C}}((\oplus X_i[k_i], \delta), (\oplus Y_j[l_j], \sigma)) = \oplus_{i,j} \mathrm{Hom}_{\mathcal{C}}^{l_j-k_i+d}(X_i, Y_j).$$

The structure map is defined similarly as the equation (1.13), i.e. it is defined by inserting δ's everywhere. Then A_∞-equation holds due to the fact that $\mathcal{C}$ is an A_∞-category and the equation (1.14). $\mathrm{Tw}(\mathcal{C})$ contains $\mathcal{C}$ as a full subcategory, since an object X of $\mathcal{C}$ can be considered as a twisted complex $(X, 0)$. Also, for any objects X, Y of $\mathcal{C}$ and an m_1-closed morphism $f \in \mathrm{Hom}^0(X, Y)$, we can define a mapping cone $\mathrm{Cone}(f)$ as

$$\mathrm{Cone}(f) = (X[1] \oplus Y, \left(\begin{smallmatrix} 0 & f \\ 0 & 0 \end{smallmatrix}\right))$$

. Then the sequence

$$X \xrightarrow{f} Y \to \mathrm{Cone}(f) \to X[1]$$

or any other sequence isomorphic to the sequence above can be considered as an exact triangle. Moreover, given objects X and Y, we can say that Z is an *extension of X by Y* if the following sequence is an exact triangle.

$$Y \to Z \to X \to Y[1]$$

Remark 1.1 In terms of Fukaya categories, taking cone of two objects corresponds to Lagrangian surgery of two Lagrangian submanifolds. See [Aur14] for further explanations.

1.4 Weighted counting measures

Definition 1.8 Let $\mathcal{C}$ be an A_∞-category over a finite field k. Then $\mathcal{C}$ is called *locally left-finite* if $\mathrm{Ext}^i(X,Y)$ is a finite set for every $i \in \mathbb{Z}$ for every pair of objects X, Y of $\mathcal{C}$ and $\mathrm{Ext}^i(X,Y) = 0$ for i less than some integer depending on X, Y.

Definition 1.9 Let $\mathcal{C}$ be a locally left-finite A_∞-category over a finite field k. The *weighted counting measure* $\mu_\mathcal{C}$ is a measure assigning to an object X of $\mathcal{C}$ a rational number

$$\mu_\mathcal{C}(X) := |\mathrm{Aut}(X)|^{-1} \prod_{k=1}^{\infty} |\,\mathrm{Ext}^{-k}(X,X)|^{(-1)^{k+1}}$$

where $\mathrm{Aut}(X) \subset \mathrm{Ext}^0(X,X)$ is the group of automorphisms of X.

Consider locally left-finite A_∞-categories $\mathcal{C}$ and $\mathcal{D}$ over a finite field $\mathbb{F}_q$ and a A_∞-functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$. Then $\mathcal{F}$ induces a linear map

$$\mathcal{F}_* : \mathbb{Q}\mathrm{Iso}(\mathcal{C}) \to \mathbb{Q}\mathrm{Iso}(\mathcal{D}), \quad \mathcal{F}_*([X]) = [\mathcal{F}(X)]$$

where $\mathbb{Q}\mathrm{Iso}(\mathcal{C})$ is the $\mathbb{Q}$-vector space generated by the isomorphism classes of objects of $\mathcal{C}$. If $\mathcal{F}$ has the property that for any $[Y] \in \mathrm{Iso}(\mathcal{D})$ the preimage $(\mathcal{F}_*)^{-1}([Y])$ is finite, then we can define a linear map

$$\mathcal{F}^! : \mathbb{Q}\mathrm{Iso}(\mathcal{D}) \to \mathbb{Q}\mathrm{Iso}(\mathcal{C})$$

$$\mathcal{F}^!([Y]) = \sum_{[X]\in(\mathcal{F}_*)^{-1}([Y])} \frac{\mu_{\mathcal{C}(X)}}{\mu_{\mathcal{D}(Y)}}[X].$$

Now we consider the functor $\mathcal{T} : \mathcal{C} \to *$, where $*$ is the final A_∞-category with a single object $\star$ and zero morphism spaces. If we assume that $\mathrm{Iso}(\mathcal{C})$ is finite, then

$$\mathcal{T}^!([\star]) = \sum_{[X]\in\mathrm{Iso}(\mathcal{C})} \mu_{\mathcal{C}}(X)[X]$$

which can be understood as a 'dual' of $\mu_{\mathcal{C}}$. We write $\mathcal{T}^!([\star]) = \mu_{\mathcal{C}}$ as an abuse of notation.

Consider the case $\mathcal{T} = \mathcal{G} \circ \mathcal{F}$ where $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to *$ are A_∞-functors. Then we can easily check the following equation:

$$\mu_{\mathcal{C}} = \mathcal{T}^!([\star]) = (\mathcal{G} \circ \mathcal{F})^!([\star]) = \mathcal{F}^!\mathcal{G}^!([\star]) = \mathcal{F}^!(\mu_{\mathcal{D}})$$

This equation will be used in chapter 4.

2 Hall Algebras

2.1 Hall algebras of abelian categories

Definition 2.1 Let $\mathcal{A}$ be a category. Then $\mathcal{A}$ is called *locally finite* if $\mathrm{Hom}(A, B)$ is a finite set for every pair of objects A, B of $\mathcal{A}$

Definition 2.2 Let $\mathcal{A}$ be a locally finite k-linear abelian category where k is a finite field. The *Hall algebra* $\mathrm{Hall}(\mathcal{A})$ of $\mathcal{A}$ is a $\mathbb{Q}$-vector space generated by isomorphism classes of objects in $\mathcal{A}$ equipped with multiplication

$$[A] \cdot [C] = \sum_{B \in \mathrm{Ext}^1(A,C)} \frac{|\mathrm{Ext}^1(A, C)_B|}{|\mathrm{Hom}(A, C)|}[B]$$

where $\mathrm{Ext}^1(A, C)_B$ is a subset of $\mathrm{Ext}^1(A, C)$ corresponding to short exact sequences of the form $0 \to C \to B \to A \to 0$.

Theorem 2.1 *The Hall algebra* $\mathrm{Hall}(\mathcal{A})$ *is an associative algebra.*

Proof. See section 2.3 of [Bri13] $\square$

Example 2.1 Let $Vect_k$ be the category of k-vector spaces where k is a finite field with q elements. Denote an isomorphism class of n-dimensional k-vector spaces by $[n]$. Then $\mathrm{Hall}(Vect_k)$ is isomorphic to the algebra $\mathbb{Q}[x]$ of polynomials in one variable via the isomorphism $[n] \to \frac{x^n}{[n]_q!}$ where $[n]_q = q^{n-1} + q^{n-2} + \cdots + q + 1$ and $[n]_q! = [1]_q[2]_q \cdots [n]_q$. The explicit multiplication in the Hall algebra is given as follows:

$$[n]\cdot[m]=\frac{[n+m]_q!}{[n]_q![m]_q!}[n+m]$$

. (See example 4.3 from [Kir16] for further explanation of this example.)

2.2 Hall algebras of A_∞-categories

The definition of Hall algebras should be modified in case of A_∞-categories, since the notion of exact sequences is replaced by exact triangles.

Definition 2.3 Let $\mathcal{C}$ be a locally left-finite k-linear A_∞-category where k is a finite field. Also, assume that $\mathcal{C}$ is closed under extensions and has a zero object. The *Hall algebra* $\mathrm{Hall}(\mathcal{C})$ of $\mathcal{C}$ is a $\mathbb{Q}$-vector space generated by isomorphism classes of objects in $\mathcal{C}$ equipped with multiplication

$$[X]\cdot[Z]=\left(\prod_{i=0}^{\infty}|\operatorname{Ext}^{-i}(Z,X)|^{(-1)^{i+1}}\right)\sum_{f\in\operatorname{Ext}^1(Z,X)}\left[\operatorname{Cone}(Z[-1]\xrightarrow{f}X)\right]$$

.

Theorem 2.2 *The Hall algebra* $\mathrm{Hall}(\mathcal{C})$ *is an associative algebra.*

Proof. We may assume that $\operatorname{Hom}^i(X,Y)$ is a finite set for every $i\in\mathbb{Z}$ and vanishes for $i\ll 0$ by considering $H(\mathcal{C})$ instead of $\mathcal{C}$. It is obvious from the definition of Hall algebra that $\mathrm{Hall}(H(\mathcal{C}))$ is the same as $\mathrm{Hall}(\mathcal{C})$). Then we have

$$[X]\cdot[Z]=\left(\prod_{i=0}^{\infty}|\operatorname{Hom}^{-i}(Z,X)|^{(-1)^{i+1}}\right)\sum_{\substack{f\in\operatorname{Hom}^1(Z,X)\\ m_1(f)=0}}\left[\operatorname{Cone}(Z[-1]\xrightarrow{f}X)\right]$$

For $A_1,A_2,A_3\in\mathrm{Ob}(\mathcal{C})$, let X_{123} be a set of twisted complexes $(A_1\oplus A_2\oplus A_3,\delta=(a_{ij})$. Note that $a_{ij}=0\in\operatorname{Hom}^1(A_i,A_j)$ except when $(i,j)=(1,2),(2,3),(1,3)$.

Also, the Maurer-Cartan equation gives $m_1(a_{12}) = 0, m_1(a_{23}) = 0$ and $m_1(a_{13}) + m_2(a_{12}, a_{23}) = 0$. Since $\mathcal{C}$ is closed under extensions, for example, a twisted complex $A_2 \xrightarrow{a_{12}} A_1$ can be regarded as an object in $\mathcal{C}$. Now we have

$$
\begin{aligned}
&([A_1] \cdot [A_2]) \cdot [A_3] = \\
&\quad = \left(\prod_{i=0}^{\infty} |\operatorname{Hom}^{-i}(A_2, A_1)|^{(-1)^{i+1}} \right) \sum_{\substack{a_{12} \in \operatorname{Hom}^1(A_2, A_1) \\ m_1(a_{12}) = 0}} [(A_2 \xrightarrow{a_{12}} A_1)] \cdot [A_3] \\
&\quad = \left(\prod_{i=0}^{\infty} |\operatorname{Hom}^{-i}(A_2, A_1)|^{(-1)^{i+1}} \right) \left(\prod_{i=0}^{\infty} |\operatorname{Hom}^{-i}(A_3, A_2 \oplus A_1)|^{(-1)^{i+1}} \right) \\
&\qquad \sum_{m_1(a_{12}) = 0} \; \sum_{\tilde{m}_1(a_{13}, a_{23}) = 0} \left[A_3 \xrightarrow{(a_{13}, a_{23})} (A_2 \xrightarrow{a_{12}} A_1) \right] \\
&\quad = \left(\prod_{k=0}^{\infty} \prod_{\substack{i,j \in \{1,2,3\} \\ i<j}} |\operatorname{Hom}^{-k}(A_j, A_i)|^{(-1)^{k+1}} \right) \sum_{C \in X_{123}} [C]
\end{aligned}
$$

Also,

$$
\begin{aligned}
&[A_1] \cdot ([A_2] \cdot [A_3]) = \\
&\quad = \left(\prod_{i=0}^{\infty} |\operatorname{Hom}^{-i}(A_3, A_2)|^{(-1)^{i+1}} \right) \sum_{\substack{a_{23} \in \operatorname{Hom}^1(A_3, A_2) \\ m_1(a_{23}) = 0}} [A_1] \cdot [(A_3 \xrightarrow{a_{23}} A_2)] \\
&\quad = \left(\prod_{i=0}^{\infty} |\operatorname{Hom}^{-i}(A_3, A_2)|^{(-1)^{i+1}} \right) \left(\prod_{i=0}^{\infty} |\operatorname{Hom}^{-i}(A_2 \oplus A_3, A_1)|^{(-1)^{i+1}} \right) \\
&\qquad \sum_{m_1(a_{23}) = 0} \; \sum_{\tilde{m}_1(a_{12}, a_{13}) = 0} \left[(A_3 \xrightarrow{a_{23}} A_2) \xrightarrow{(a_{12}, a_{13})} A_1 \right] \\
&\quad = \left(\prod_{k=0}^{\infty} \prod_{\substack{i,j \in \{1,2,3\} \\ i<j}} |\operatorname{Hom}^{-k}(A_j, A_i)|^{(-1)^{k+1}} \right) \sum_{C \in X_{123}} [C]
\end{aligned}
$$

Thus, the associativity holds. $\square$

Example 2.2 Let $\mathcal{C} = \mathrm{Perf}(\mathbb{F}_q)$ be the category of finite-dimensional complexes of vector spaces over a finite field $\mathbb{F}_q$. Then $\mathrm{Hall}(\mathcal{C})$ is generated by $x_k := [\mathbb{F}_q[-k]]$ with relations

$$x_{k+1}x_k - q^{-1}x_kx_{k+1} = q-1, \qquad k \in \mathbb{Z} \tag{2.1}$$

$$x_{k+m}x_k = q^{(-1)^m}x_kx_{k+m}, \qquad k \in \mathbb{Z}, \quad m \geq 2. \tag{2.2}$$

These relations can be obtained from the following computation:

We have

$$\mathrm{Hom}^i(\mathbb{F}_q[-k], \mathbb{F}_q[-k-1]) = \begin{cases} \mathbb{F}_q & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Note that every morphism in $\mathrm{Hom}^1(\mathbb{F}_q[-k], \mathbb{F}_q[-k-1])$ is an isomorphism except the zero morphism. Thus, $\mathrm{Cone}(\mathbb{F}[-k][-1] \xrightarrow{f} \mathbb{F}[-k-1])$ is zero for nonzero f. If $f = 0$, the cone is the direct sum $\mathbb{F}[-k-1] \oplus \mathbb{F}[-k]$. Then we get

$$[\mathbb{F}_q[-k-1]] \cdot [\mathbb{F}_q[-k]] = [\mathbb{F}_q[-k-1] \oplus [\mathbb{F}_q[-k]] + (q-1)[0]. \tag{2.3}$$

Also we have

$$\mathrm{Hom}^i(\mathbb{F}_q[-k-1], \mathbb{F}_q[-k]) = \begin{cases} \mathbb{F}_q & i = -1 \\ 0 & \text{otherwise} \end{cases}$$

Since $\mathrm{Hom}^1(\mathbb{F}_1[-k-1], \mathbb{F}_q[-k] = 0)$, the cone is the direct sum. So we get

$$\begin{aligned}[\mathbb{F}_q[-k]] \cdot [\mathbb{F}_q[-k-1]] &= |\,\mathrm{Hom}^{-1}(\mathbb{F}_q[-k-1], \mathbb{F}_q[-k])| \cdot [\mathbb{F}_q[-k-1] \oplus \mathbb{F}_q[-k]] \\ &= q[\mathbb{F}_q[-k-1] \oplus \mathbb{F}_q[-k]] \end{aligned} \tag{2.4}$$

Then the relation (2.1) is obtained by the equation (2.3) and (2.4).

Similarly, for $m \geq 2$, we can check that

$$[\mathbb{F}_q[-k-m]] \cdot [\mathbb{F}_q[-k]] = [\mathbb{F}_q[-k-m] \oplus \mathbb{F}_q[-k]] \tag{2.5}$$

$$[\mathbb{F}_q[-k]] \cdot [\mathbb{F}_q[-k-m]] = q^{(-1)^{m+1}}[\mathbb{F}_q[-k-m] \oplus \mathbb{F}_q[-k]]. \tag{2.6}$$

Then the relation (2.2) is obtained by the equation (2.5) and (2.6).

3 Fukaya Categories of Surfaces

3.1 Definitions

There are various versions of the Fukaya category. Basically, the Fukaya cateogry is an A_∞-category defined for a given symplectic manifold. Its objects are Lagrangian submanifolds (possibly with some additional structures) and morphisms are intersections between two Lagrangian submanifolds. The A_∞-operations correspond to counting pseudoholomorphic curves with boundary condition given by Lagrangian submanifolds. However, it is not easy to define it precisely, because there are some issues concerning transversality and regularity.

For example, we want the intersecting set of two Lagrangian submanifolds to be finite, but if two Lagrangian submanifolds are not meeting transversally, there might be infinitely many intersection points. This issue becomes critical when we consider the morphism from some Lagrangian submanifold to itself. One can use particulary chosen Lagrangian submanifolds so that any pair of Lagrangian submanifolds meet transversally. In this case, we cannot consider morphism spaces from some object to itself. Another solution is to perturb Lagrangian submanifolds using Hamiltonian functions so that they meet each other transversally after perturbation. Then we can think of morphism spaces for any two objects, but we always have to consider the perturbation data and their compatibility.

Also, we need to be able to 'count' pseudoholomophic curves with some boundary conditions to define A_∞-operations, which implies that the set of certain pseudoholomophic curves are finite. In most cases, we should resolve the regularity issue to achieve this. Pseudoholomorphic curves are given by the solutions of some partial differential equation, and we need to check several conditions to guarantee that the solution set is 'nice'. If the linearized differnential operator derived form the partial differential equation satisfies the regularity condition, then we can count how many solutions there exist and use this number to define the A_∞-operations.

We will use the simplified version of the Fukaya category so that we can avoid regularity issue. We will only consider 2-dimensional symplectic manifolds i.e. any orientable surfaces(possibly with boundary components). Then the lagrangian submaniolds are just curves. In this case, counting pseudoholomophic curves is just counting polygons whose edges are Lagrangians and vertices are intersection points of two lagrangians. Also, we will assume the perturbation data is given appropriately and we will not delve into the transversallity issue deeply(See [Sei08] for details).

Let S be a compact surface with boundary. Instead of dealing with curves on S directly, we will consider $S \times \mathbb{R}$ and a certain class of curves in $S \times \mathbb{R}$.

3.1.1 Contact manifolds and Legendrian submanifolds

Definition 3.1 Let M be a smooth manifold. A *contact form* on M is a 1-form α such that $d\alpha$ is nondegenerate on $\ker \alpha$. M is called *contact manifold* if such an α is given.

The nondegeneracy of $d\alpha$ on $\ker \alpha$ implies that $\ker \alpha$ is even-dimensional, and thus every contact manifold must be odd-dimensional.

Example 3.1 Consider $\mathbb{R}^{2n+1}$ with coordinates $\{x_1, \cdots, x_n, y_1, \cdots, y_n, z\}$. It becomes a contact manifold with a contact form $\alpha = \sum_i x_i dy_i + dz$. The nondegeneracy of α comes from the fact that $\alpha \wedge (d\alpha)^n = dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n \wedge dz \neq 0$.

Example 3.2 Let M be a symplectic manifold with a symplectic form $\omega = d\theta$. Then $M \times \mathbb{R}$ becomes a contact manifold with a contact form $\alpha = p^*\theta + dz$ where z is a coordinate on $\mathbb{R}$ and $p : M \times \mathbb{R} \to M$ is a natural projection. The nondegeneracy of $d\alpha$ comes from the nondegeneracy of ω.

The second example shows that symplectic manifolds correspond to some contact manifolds in some sense. One might expect that there is an analogue of lagrangian submanifolds in contact geometry, which is true.

Proposition 3.1 *Let M be a $(2n+1)$-dimensional contact manifold with a contact form α. If L is a submanifold of M such that $TL \subset \ker \alpha$, then $\dim L \leq n$.*

Proof. Let $\iota : L \to M$ be an inclusion. Then $\iota^*\alpha = 0$. Hence $\iota_* TL$ is an isotropic subspace of the symplectic vector space $(\ker \alpha, d\alpha)$, which implies $\dim L \leq n$. $\square$

Definition 3.2 Let M be a $(2n+1)$-dimensional contact manifold with a contact form α. A submanifold L is called an *isotropic submanifold* if $TL \subset \ker \alpha$. If $\dim L = n$, then L is called a *legendrian submanifold.*

Example 3.3 Let S be a surface with a symplectic form $\omega = d\theta$ and $L \subset M$ be a Lagrangian submanifold, which is a curve. Then there is a Legendrian curve $L' \subset M \times \mathbb{R}$ such that its projection to M is L.

3.1.2 Grading

Let S be a compact surface with boundary. We want the morphism spaces of Fukaya category of S to be $\mathbb{Z}$-graded. This $\mathbb{Z}$-grading comes from a certain structure on S.

Definition 3.3 A *grading structure on* S is a section η of the projectivized tangent bundle $\mathbb{P}(TS)$.

Suppose that a grading structure η is given. Consider a universal cover $p : \widetilde{\mathbb{P}(TS)} \to \mathbb{P}(TS)$. We may assume that the restriction $p : p^{-1}(\mathbb{P}(TS)_x) \to \mathbb{P}(TS)_x$ is a universal cover for all $x \in S$, since η defines a basepoint of each fiber of $\mathbb{P}(TS)$.

Definition 3.4 A *graded curve* is an immersed curve $\gamma : I \to S$ with a section $\tilde{\gamma}$ of $\gamma^*\widetilde{\mathbb{P}(TS)}$ such that $\tilde{\gamma}(t)$ is a lift of the tangent space $TS_{\gamma(t)}$.

Note that an immersed Legendrian curve in $S \times \mathbb{R}$ projects to an immersed curve in S, so we can think of a graded curve in $S \times \mathbb{R}$.

3.1.3 Definition: objects and morphisms

Consider a tuple (S, N, θ, η, k) where S is a compact surface with boundary, $N \subset \partial S$ is a finite set of marked points, θ is a Liouville 1-form on S (i.e. $d\theta$ is a symplectic form on S), η is a grading structure on S, and k is a coefficient field. We will define two different Fukaya catrgories. First, we denote the partially wrapped Fukaya category by $\mathcal{F}(S, N, \theta, \eta, k)$. An object of $\mathcal{F}(S, N, \theta, \eta, k)$ is a compact graded Legendrian curve L in $S \times \mathbb{R}$ with a local system of finite-dimensional k-vector spaces E such that ∂L is either empty or contained in

$(\partial S \setminus N) \times \mathbb{R}$. Another version of Fukaya category is the infinitesimally wrapped category $\mathcal{F}^{\vee}(S, N, \theta, \eta, k)$. An object (L, E) of $\mathcal{F}^{\vee}(S, N, \theta, \eta, k)$ is the same as an object of $\mathcal{F}(S, N, \theta, \eta, k)$, except that ∂L is contained in $N \times \mathbb{R}$ if nonempty.

Before we define morphisms, we need the notion of the *intersection index*. Let $L_i = (\gamma_i : I \to S \times \mathbb{R}, \tilde{\gamma}_i : I \to \widetilde{\mathbb{P}(TS)})$ be graded immersed curves($i = 0, 1$) with transverse intersection at $x = \gamma_0(t_0) = \gamma_1(t_1)$. Then the intersection index of L_0 and L_1 is defined as follows:

$$\begin{aligned} i_x(L_0, L_1) &:= \lceil \tilde{\gamma}_1(t_1) - \tilde{\gamma}_0(t_0) \rceil \\ &:= \text{the smallest } n \in \mathbb{Z} \text{ with } n > \tilde{\gamma}_1(t_1) - \tilde{\gamma}_0(t_0). \end{aligned}$$

Let us explain the meaning of the expression $\tilde{\gamma}_1(t_1) - \tilde{\gamma}_0(t_0)$. Note that both $\tilde{\gamma}_0(t_0)$ and $\tilde{\gamma}_1(t_1)$ is contained in $\widetilde{\mathbb{P}(TS)}_x$, which is a universal cover of $\mathbb{P}(TS)_x$. The fiber of projectivized tangent bundle of a surface is diffeomorphic to $\mathbb{R}/\mathbb{Z}$. Then the identification $\mathbb{P}(TS)_x \simeq \mathbb{R}/\mathbb{Z}$ gives the identification $\widetilde{\mathbb{P}(TS)}_x \simeq \mathbb{R}$. Thus, the difference of $\tilde{\gamma}_0(t_0)$ and $\tilde{\gamma}_1(t_1)$ is a well-defined real number and does not depend of the choice of the identification.

Let (L_0, E_0) and (L_1, E_1) be objects of $\mathcal{F}(S, N, \theta, \eta, k)$ such that their projections to S intersect transversely and $\partial L_i = \emptyset$ for $i = 0, 1$. We also assume that L_1 is oriented. The morphism space is defined as follows:

$$\mathrm{Hom}((L_0, E_0), (L_1, E_1)) := \bigoplus_{p \in pr(L_0) \cap pr(L_1)} \mathrm{Hom}_k((E_0)_p, (E_1)_p)[-i_p(L_0, L_1)].$$

If the orientation on L_1 is reversed, we identify $x \in \mathrm{Hom}((L_0, E_0), (L_1, E_1))$ with $(-1)^{|x|} x$.

If either $pr(L_0)$ is not transverse to $pr(L_1)$(e.g. $L_0 = L_1$) or both L_0 and L_1 have boundary, we need to perturb L_0 as graded Legendrian curves so that $pr(L_0)$ and $pr(L_1)$ intersects transversely. It is equivalent to perturb $pr(L_0)$ by

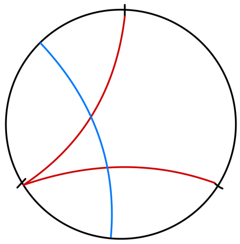


Figure 3.1 Objects of $\mathcal{F}$ (blue arcs) and $\mathcal{F}^\vee$ (red arcs)

a Hamiltonian diffeomorphism. The resulting morphism space depends on the choice of perturbation, but they are all quasi-isomorphic to each other.

Let $p \in S$ be a intersection point of $pr(L_0)$ and $pr(L_1)$. Then the corresponding points of Legendrian curves are $\gamma_0(t_0) = (p, z_0)$ and $\gamma_1(t_1) = (p, z_1)$. We can endow $\mathrm{Hom}((L_0, E_0), (L_1, E_1))$ with the structure of $\mathbb{R}$-filtered vector space by defining $\mathrm{Hom}((L_0, E_0), (L_1, E_1))_{\geq \beta}$ as the morphism space generated by the intersection points with $z_0 - z_1 \geq \beta$.

Remark 3.1 Let us explain relation between the partially wrapped Fukaya category $\mathcal{F}$ and the infinitesimally wrapped Fukaya category $\mathcal{F}^\vee$. As its notation implies, these two category can be understood as a dual of each other. For example, consider a disk D with a set N of 3 marked points in ∂D (See Figure 3.1). A projection on D of object curve of $\mathcal{F}$ is either a closed curve or an arc connecting components of $\partial D \setminus N$. Suppose an object X of $\mathcal{F}^\vee$ whose projection onto D is an arc connecting two marked points is given. This arc meets any curve or arc coming from $\mathcal{F}$ transversally (after some perturbation if necessary), and an integer can be assigned for each intersection, which is an intersection index. In this sense, $\mathcal{F}^\vee$ can be viewed as a dual of $\mathcal{F}$. Similar argument shows that $\mathcal{F}$ is also a dual of $\mathcal{F}^\vee$.

3.1.4 Definition: structure maps

As mentioned before, we will define structure maps by counting polygons with Lagrangian boundaries. Let $L_0, \cdots, L_n$ be graded Legendrian curves such that their projections to S intersect transversely. We choose intersection points $x_k \in pr(L_k) \cap pr(L_{k+1})$ for $k = 1, \cdots, n-1$ and $x_n \in pr(L_0) \cap L_n$. For each intersection points, we have a degree $d_k := i_{x_k}(L_k, L_{k+1})$ for $k = 1, \cdots, n-1$ and $d_n := i_{x_n}(L_0, L_n)$. We assume that

$$d_0 + \cdots d_{n-1} = d_n + n - 2.$$

If this assumption does not hold, the output of structure map is zero.

Consider a smooth $(n+1)$-gon $\phi : D \to S$ up to reparametrization, such that the k-th corner of D is mapped to x_k and the edge from x_{k-1} to x_k is mapped to $pr(L_k)$. Then parallel transport along the edges of $\phi(D)$ defines a map

$$\tau(D) : \mathrm{Hom}((E_{n-1})_{x_{n-1}}, (E_n)_{x_{n-1}}) \otimes \cdots \otimes \mathrm{Hom}((E_0)_{x_0}, (E_1)_{x_0})$$
$$\to \mathrm{Hom}((E_0)_{x_n}, (E_n)_{x_n}).$$

Now we define the structure map as follows:

$$m_n : \mathrm{Hom}(X_{n-1}, X_n) \otimes \mathrm{Hom}(X_0, X_1) \to \mathrm{Hom}(X_0, X_n), m_n := \sum_D \pm\tau(D)$$

where $X_k = (L_k, E_k)$ are objects of the Fukaya category. The sign depends on the orientation of L_k. We omit the details.

3.2 Maurer-Cartan elements

Consider an object (L, E) such that $pr(L)$ has a self-intersection. In this case, there are 1-gons whose vertices are the self-intersection points. In the view of A_∞-category, this corresponds to m_0. Thus, if we allow such objects, Fukaya category becomes a curved A_∞-category.

Let $\delta \in \mathrm{Hom}^1((L, E), (L, E))_{>0}$ be a Maurer-Cartan element. Here, $\mathrm{Hom}((L, E), (L, E))$ is defined by perturbing L slightly to L' so that $pr(L)$ and $pr(L')$ intersect transversely. Then (L, E, δ) is an object of the curved Fukaya category.

Given L, we can define a category $\mathcal{C}(L)_1$ whose objects are rank 1 local system E on L with a Maurer-Cartan element $\delta \in \mathrm{Hom}^1((L, E), (L, E))_{>0}$ and morphisms are

$$\mathrm{Hom}_{\mathcal{C}(L)_1}((L, E_0, \delta_0), (L, E_1, \delta_1)) := \mathrm{Hom}((L, E_0), (L, E_1))_{\geq 0}.$$

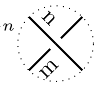
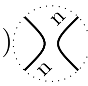
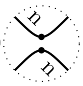
In case that $\partial L \subset N \times \mathbb{R}$, there is a natural functor $F_L : \mathcal{C}(L)_1 \to \mathcal{F}^\vee$. This functor will be used in the next chapter.

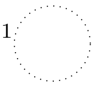
4 Legendrian Skein algebras

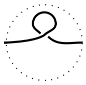
4.1 Definitions

We fix S, N, θ, η, where S is a compact surface with boundary, $N \subset \partial S$ a finite set, θ a Liouville form on S and η a grading on S.

Definition 4.1 The *Legendrian Skein module* $\mathrm{Skein}(S, N, \theta, \eta)$ is the $\mathbb{Z}[t, t^{-1}, (t-1)^{-1}]$-module generated by isotopy classes of graded Legendrian curves(we refer to these curves as links) embedded in $S \times \mathbb{R}$ whose boundary is contained in $N \times \mathbb{R}$, with the relation below:

 $- q^{(-1)^{m-n}}$  $= \delta_{m,n}(q-1)$  $- \delta_{m,n}(1-q^{-1})$  (4.1)

 $= (q-1)^{-1}$  (4.2)

 $= 0$ (4.3)

The *Legendrian Skein algebra* is the Legendrian Skein module $\mathrm{Skein}(S, N, \theta, \eta)$ equipped with a product defined by

$$L_1 \cdot L_2 := \text{stack } L_2 \text{ on top of } L_1.$$

It is obvious from the definition that the product is associative. The unit is a isotopy class of the empty link.

4.2 Legendrian Skein algebras and Hall algebras

Let S, N, θ, η be the ones as before, and $k = \mathbb{F}_q$. We want to define a $\mathbb{Q}$-algebra homomorphism

$$\Phi : \mathrm{Skein}(S, N, \theta, \eta) \otimes_{\mathbb{Z}[t,t^{-1},(t-1)^{-1}]} \mathbb{Q} \to \mathrm{Hall}(\mathcal{F}(S, N, \theta, \eta, k))$$

where $\mathbb{Z}[t, t^{-1}, (t-1)^{-1}]$ acting on $\mathbb{Q}$ as $t \to q$.

Consider a graded Legendrian link L in $S \times \mathbb{R}$ and a category $\mathcal{C}(L)_1)$ from section 3.2. Since our base field is finite, $\mathcal{C}(L)_1$ has finitely many objects and morphism spaces are finite sets. Then we can think of a weighted counting measure

$$\mu_{\mathcal{C}(L)_1}(X) = |\mathrm{Aut(X)}|^{-1} \prod_{k=1}^{\infty} |\operatorname{Ext}^{-k}(X, X)|^{(-1)^{k+1}}.$$

Then we can pushforward this measure using the functor $F_L : \mathcal{C}(L)_1 \to \mathcal{F}^{\vee}(S, N, \theta, \eta, k)$ so that we get a measure on $\mathcal{F}^{\vee}$. This measure assigns a rational number to an isomorphism class of $\mathcal{F}^{\vee}$. Then it can be understood as an element of $\mathrm{Hall}(\mathcal{F})$(see Remark 3.1). Thus, we define

$$\Phi(L) := (F_L)_*(\mu_{\mathcal{C}(L)_1})$$

. We can describe the weighted counting measure $\mu_{\mathcal{C}(L)_1}$ in a more concrete way. Before that, we define an integer $e(L)$ for a graded Legendrian link $L = (\gamma : I \to S \times \mathbb{R}, \tilde{\gamma} : I \to \widetilde{\mathbb{P}(TS)})$. For each self-crossing point $x \in \mathrm{Cr}(L)$ there exists $t_0, t_1 \in I$ such that $pr(\gamma(t_0)) = pr(\gamma(t_1)) = x$ and the z-coordinate of $\gamma(t_0)$ is greater than the one of $\gamma(t_1)$. Now we define

$$\begin{aligned} e(L) =& |\{x \in \mathrm{Cr}(L) : i(L, t_0, L, t_1) \leq 0 \text{ and even}\}| \\ & - |\{x \in \mathrm{Cr}(L) : i(L, t_0, L, t_1) \leq 0 \text{ and odd}\}|. \end{aligned}$$

Proposition 4.1 *Let L be a graded Legendrian link. Then the following formula holds:*

$$\mu_{\mathcal{C}(L)_1} = (q-1)^{-|\pi_0(L)|} q^{-e(L)} \sum_{E} \sum_{\delta \in \mathcal{MC}(L,E)} [(L,E,\delta)]$$

where E ranges over all isomorphism classes of rank one local systems on L.

Proof. Let $\mathcal{C}(L)_{1,0}$ be the category with the same objects as $\mathcal{C}(L)_1$ and morphism spaces $\mathrm{Hom}_{\geq 0} / \mathrm{Hom}_{>0}$. Then we can factor the trivial functor $\mathcal{T}$: $\mathcal{C}(L)_1 \to *$ through $\mathcal{G} : \mathcal{C}(L)_1 \to \mathcal{C}(L)_{1,0}$. As we have seen in section 1.4, we have

$$\mu_{\mathcal{C}(L)_1} = \mathcal{G}^!(\mu_{\mathcal{C}(L)_{1,0}}).$$

Since any Maurer-Cartan element is in $\mathrm{Hom}_{\geq 0}$, two objects (L,E,δ) and $(L,E,\delta'$ are isomorphic. Also, the elements in $\mathrm{Hom}_{\geq 0} / \mathrm{Hom}_{>0}$ are generated by the fundamental class of connected components of L and its Poincare duals. Thus we have $|\mathrm{Aut}((L,E,\delta))| = (q-1)^{|\pi_0(L)|}$ and

$$\mu_{\mathcal{C}(L)_{1,0}} = (q-1)^{-|\pi_0(L)|} \sum_{E} [(L,E)].$$

Note that $e(L)$ is equal to

$$\sum_{i=0}^{\infty} (-1)^i \dim \mathrm{Hom}^{-i}((L,E),(L,E))_{>0}$$

by definition, so the aurgument at the end of section 1.4 implies that

$$\mathcal{G}^!(\mu_{\mathcal{C}(L)_{1,0}}) = (q-1)^{-|\pi_0(L)|} q^{-e(L)} \sum_{E} \sum_{\delta \in \mathcal{MC}(L,E)} [(L,E,\delta)].$$

$\square$

Theorem 4.1 *The assignment $L \mapsto \Phi(L)$ defines a well-defined $\mathbb{Q}$-algebra homomorphism*

$$\Phi : \mathit{Skein}(S, N, \theta, \eta) \otimes_{\mathbb{Z}[t,t^{-1},(t-1)^{-1}]} \mathbb{Q} \to \mathrm{Hall}(\mathcal{F}(S, N, \theta, \eta, k)).$$

Proof. Proving that the Skein relation is preserved by Φ requires the observation of relation between the set of Maurer-Cartan element of different resolutions at intersection points of given links. We omit this part and refer the details to [Hai21]. We only prove the compatibility with product.

Suppose given two Legendrian links L_1, L_2 and the maximum value of z-coordinate of L_1 is less than the minimum value of L_2 so that L_1 is below L_2. Let $L := L_1 \bigsqcup L_2$. We want to show that $\Phi(L) = \Phi(L_1)\Phi(L_2)$. Since L_1 is below L_2, we have

$$\begin{aligned} \mathrm{Hom}((L, E), (L, E))_{\geq 0} = &\mathrm{Hom}((L_1, E_1), (L_1, E_1))_{\geq 0} \oplus \mathrm{Hom}((L_2, E_2), (L_1, E_1)) \\ &\oplus \mathrm{Hom}((L_2, E_2), (L_2, E_2))_{\geq 0}. \end{aligned} \tag{4.4}$$

Also, $\delta = (\delta_{11}, \delta_{21}, \delta_{22}) \in \mathrm{Hom}^1((L, E), (L, E))_{>0}$ is a Maurer-Cartan element if and only if δ_{11} and δ_{22} are Maurer-Cartan elements and δ_{21} is an $\tilde{m}_1$-closed degree 1 morphism. Thus, we have the following:

$$\begin{aligned} &\left(\sum_{\delta_{11} \in \mathcal{MC}(L_1, E_1)} [(L_1, E_1, \delta_{11}] \right) \cdot \left(\sum_{\delta_{22} \in \mathcal{MC}(L_1, E_1)} [(L_2, E_2, \delta_{22}] \right) \\ &\qquad = q^{-e(L_2, L_1)} \sum_{\delta \in \mathcal{MC}(L, E)} [(L, E, \delta)] \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} e(L_2, L_1) &= \sum_{i=0}^{\infty} (-1)^i \dim \mathrm{Ext}^{-i}((L_2, E_2), (L_1, E_1)) \\ &= \sum_{i=0}^{\infty} (-1)^i \dim \mathrm{Hom}^{-i}((L_2, E_2), (L_1, E_1)). \end{aligned}$$

Now we can calculate the product $\Phi(L_1) \cdot \Phi(L_2)$ explicitly. First, we can write $\Phi(L_1)$ and $\Phi(L_2)$ as

$$\Phi(L_1) = (q-1)^{-|\pi_0(L_1)|} q^{-e(L_1)} \sum_{E_1} \sum_{\delta_{11} \in \mathcal{MC}(L_1, E_1)} [(L_1, E_1, \delta_{11})]$$

$$\Phi(L_2) = (q-1)^{-|\pi_0(L_2)|} q^{-e(L_2)} \sum_{E_2} \sum_{\delta_{22} \in \mathcal{MC}(L_2, E_2)} [(L_2, E_2, \delta_{22})]$$

From the equation (4.1), we know that $e(L) = e(L_1) + e(L_2) + e(L_2, L_1)$. Also, it is obvious that $|\pi_0(L)| = |\pi_0(L_1)| + |\pi_0(L_2)|$. Then, using the equation (4.2) we get

$$\begin{aligned}
\Phi(L_1) \cdot \Phi(L_2) &= (q-1)^{-|\pi_0(L)|} q^{e(L_2, L_1) - e(L)} \left(q^{-e(L_2, L_1)} \sum_{\delta \in \mathcal{MC}(L, E)} [(L, E, \delta)] \right) \\
&= (q-1)^{-|\pi_0(L)|} q^{-e(L)} \sum_{\delta \in \mathcal{MC}(L, E)} [(L, E, \delta)] \\
&= \Phi(L)
\end{aligned}$$

□

Bibliography

[Aur14] Denis Auroux. A beginner's introduction to Fukaya categories. *Contact and Symplectic Topology*, pages 85–136, 2014.

[Bri13] Tom Bridgeland. Quantum group via Hall algebras of complexes. *Annals of Mathematics*, (177, issue 2):739–759, 2013.

[Hai21] Fabien Haiden. Legendrian Skein algebras and Hall algebras. *Math. Ann.*, (381):631–684, 2021.

[Kir16] Alexander Kirillov, Jr. Quiver representations and quiver varieties. 2016.

[Sei08] Paul Seidel. Fukaya categories and Picard–Lefschetz theory. 2008.

초 록

곡면 S가 주어졌을 때, 이를 자연스럽게 사교다양체로 이해할 수 있고, 따라서 곡면의 푸카야 범주와 그로부터 정의되는 홀 대수를 생각할 수 있다. 또한, $S \times \mathbb{R}$에는 자연스러운 접촉 구조가 존재하며, 이를 이용해 르장드리안 링크에 의해 생성되는 르장드리안 스케인 대수를 생각할 수 있다. 이 논문에서는, 두 대수를 정의하는 과정을 살펴보고 두 대수 사이에 $\mathbb{Q}$-동형사상이 존재함을 보인다.

주요어: 푸카야 범주, 홀 대수, 르장드리안 스케인 대수

학 번: 2016-20231

감사의 글

짧지 않았던 대학원 생활동안 많은 분들의 도움과 격려가 있었습니다. 덕분에 이렇게 졸업을 할 수 있었습니다. 먼저 학부 시절부터 성심성의껏 지도해주신 조철현 교수님께 감사의 말씀을 전합니다. 부족한 모습을 많이 보여드렸는데도 끝까지 지켜봐주시고, 새로운 길을 찾겠다고 말씀드렸을 때에도 믿고 격려해주셔서 감사합니다. 평생동안 본받고 싶은 어른의 모습으로 삼고 싶습니다. 그리고 바쁘신 와중에 학위 논문 심사를 맡아주신 권재훈 교수님과 강정수 교수님에게도 감사드립니다.

외롭고 힘들게 방황했던 때에 버팀목이 되어준 친구들에게도 감사합니다. 제 곁에 여러분들이 없었다면 지금 이 순간까지 버티지 못했을거라는 생각이 듭니다. 댄스 커뮤니티 루땐에서 만난 친구들 덕분에 세상에 나 혼자 있는 것이 아니라는 것을 깨달았습니다. 영화 안 보는 영화 동아리 조찬 클럽의 친구들 덕분에 삶에서 즐거움을 조금 더 찾을 수 있었습니다. 한 집에 살면서 어려운 시기를 함께 보낸 민주와 소희에게도 고맙고 사랑한다고 말하고 싶습니다.

가족과 여러 일을 겪으며 많이 울기도 웃기도 했습니다. 표현하는 것 이상으로 저를 아끼고 응원해주셨을 것이라는 생각이 듭니다. 언젠가 그 마음을 갚을 수 있는 날이 올 것이라고 믿습니다. 특히 제가 가장 취약해진 순간마다 제 옆을 지켜주신 어머니에게 감사합니다. 엄마 없이는 여기까지 오지 못했을 것 같아요. 사랑합니다.